Essays on Consumption and Portfolio Choice

by

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Abstract

This thesis analyzes optimal consumption and portfolio strategy by considering three different extensions to the classic work by Merton (1971). The first chapter considers consumption and strategic asset allocation when expected returns are predictable for Epstein-Zin preferences. The second chapter focuses on the role of imperfect information in the consumption and portfolio choice problem and presents a tractable solution to the strategic asset allocation problem in incomplete markets. The third chapter considers the role of human capital in consumption and portfolio choice and presents normative evidence of hump-shaped life-cycle investment in risky assets, in line with empirical findings on asset allocation strategies.

In Chapter 1 (co-authored with John Campbell, George Chacko, and Luis Viceira) we derive an approximate solution to a continuous-time intertemporal portfolio and consumption choice problem. The problem is the continuous-time equivalent of the discrete-time problem studied by Campbell and Viceira (1999), in which the expected excess return on a risky asset follows an AR(1) process, while the riskless interest rate is constant. We show also how to obtain continuous-time parameters that are consistent with discrete-time econometric estimates. The continuous-time solution is numerically close to that of Campbell and Viceira and has the property that conservative long-term investors have a large positive intertemporal hedging demand for stocks.

In Chapter 2, we relax the assumption on preferences made in Chapter 1 and consider how imperfect information about expected excess returns on the risky asset shifts the asset allocation strategy. I present a model of consumption and portfolio choice with imperfect information. I solve analytically the consumption and portfolio choice problem for an investor learning about the current value of time-varying expected returns. When prices are the only observables, the investor optimally estimates the current expected returns using the realized returns. Because of this, the market is observationally complete for an imperfectly informed investor. The observational completeness of the market allows me to find analytical, closed-form solutions to the investor's consumption and portfolio choice problem. I show how learning affects both the covariance and the consumption smoothing component of the hedging portfolio. Applying the model to monthly return data, I show a significant reduction in hedging demands due to imperfect information. In contrast to portfolio choice assuming expected returns are observed, in some cases the reduction implies the agent will optimally hold a negative hedging portfolio. I solve in closed-form for the model implied $R^2$ for the return forecast regression, in other words the predictable fraction of return variance, and discuss the relationship between
the reduction in hedging demands and the reduction in the model implied $R^2$ for the return forecast regression.

Little work has been done in regards to the role of labor income when investment opportunities are stochastic. Chapter 3 considers the consumption and portfolio choice problem of an investor when interest rates are time-varying and labor income growth might be sensitive to changes in interest rates. We obtain closed-form solutions to the consumption and portfolio choice for an investor with both inelastic and elastic labor supply. In our calibration, we find the allocation to long-term bonds changes the most when stochastic human capital is introduced. Allocation to equity is greater when human capital is introduced, but human capital seems to be a better substitute for holding long-term bonds. Our calibration suggests the horizon effect in equity and bond holding can be non-monotonic. Once we account for changes in financial wealth and human capital as the investor nears retirement, we find some investors initially increase their holdings of equity, and at some point before retirement reduce their holdings. Long-term bond holdings can follow a similar pattern. The allocation strategies for long-term bonds suggest the sensitivity to interest rate risk in human capital is initially greater than the interest rate risk in total wealth. As the investor nears retirement, the interest rate sensitivity of total wealth decreases at a lower rate than the interest rate sensitivity of human capital. The shift in the magnitude of the sensitivities explain why the horizon effect might not hold once labor income is considered.

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Chapter 1

Strategic Asset Allocation in a Continuous-Time VAR Model (joint work with John Campbell, George Chacko, and Luis Viceira)

1.1 Introduction

Campbell and Viceira (1999) study the impact of predictable variation in stock returns on intertemporal optimal portfolio choice and consumption. They consider an infinitely lived investor who faces a constant riskless interest rate and a time-varying equity premium. They model this time-variation using a discrete-time, homoskedastic VAR (1) process for log excess stock returns and a state variable driving changes in expected returns. This model of investment opportunities implies that the Sharpe ratio is linear in the state variable.

Campbell and Viceira assume that the investor has recursive Epstein-Zin utility (Epstein and Zin 1989, 1991), a generalization of power utility that allows both the coefficient of relative risk aversion and the elasticity of intertemporal substitution in consumption to be constant free parameters. They derive an approximate analytical solution for the optimal portfolio rule, and show that this rule is linear in the state variable. When they calibrate this model to U.S.
stock market data for the postwar period, they find that intertemporal hedging motives greatly increase the average demand for stocks by investors whose relative risk aversion coefficients exceed one.

Because Campbell and Viceira work in discrete time, no exact portfolio solutions are available in their model except in the trivial case of unit risk aversion, which implies myopic portfolio choice. Campbell and Viceira claim, however, that their solution becomes exact in the limit of continuous time when the elasticity of intertemporal substitution equals one. They base this claim on the fact that they use an approximation to the investor’s intertemporal budget constraint which becomes exact as the time interval of their model shrinks.

This paper presents a continuous-time analysis of Campbell and Viceira’s portfolio choice problem. The paper finds a continuous-time representation of the VAR(1) process in their paper, and solves a continuous-time version of their model. The solution is exact when the elasticity of intertemporal substitution equals one, and approximate otherwise. The continuous-time solution has the same qualitative properties as the discrete-time solution in Campbell and Viceira, and is quantitatively similar. However the continuous-time solution is likely to be more appealing and intuitive to finance theorists who are accustomed to working in continuous time.

1.2 Investment Opportunity Set

1.2.1 A continuous-time VAR

We start by assuming that there are two assets available to the investor, a riskless asset with instantaneous return

\[
\frac{dB_t}{B_t} = r dt, \tag{1.1}
\]

and a risky asset ("stocks") whose instantaneous return and expected return follow a continuous-time bivariate process:
\[
\begin{bmatrix}
    \frac{d \left( \log S_t + \frac{1}{2} \sigma_S^2 t - \theta t \right)}{d (\mu_t - \theta)} \\
    \frac{d (\mu_t - \theta)}{d (\mu_t - \theta)}
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    0 & -\kappa
\end{bmatrix} \begin{bmatrix}
    \log S_t + \frac{1}{2} \sigma_S^2 t - \theta t \\
    \mu_t - \theta
\end{bmatrix} dt \\
+ \begin{bmatrix}
    \sigma_S & 0 \\
    \rho \sigma_S & \sqrt{\sigma_S^2 - \rho^2 \sigma_\mu^2}
\end{bmatrix} \begin{bmatrix}
    dZ_{S,t} \\
    dZ_{\mu,t}
\end{bmatrix},
\]

where \( dZ_{S,t} \) and \( dZ_{\mu,t} \) are independent Wiener processes.

Equation (1.2) implies that the instantaneous return on stocks \( (dS_t/S_t) \) follows a Geometric Brownian Motion, whose drift (or instantaneous expected return) \( \mu_t \) follows a mean-reverting process. Section 1.2.3 below shows that this is the continuous-time counterpart of the discrete-time VAR(1) process in Campbell and Viceira (1999).

We can write (1.2) in compact form as

\[
dy_t = Ay_t dt + CdZ_t. \tag{1.3}
\]

Note that the instantaneous variance of \( dy \) is given by \( CC' \):

\[
\text{Var} (dy) = CC' = \begin{bmatrix}
    \sigma_S^2 & \rho \sigma_S \sigma_\mu \\
    \rho \sigma_S \sigma_\mu & \sigma_\mu^2
\end{bmatrix}.
\]

### 1.2.2 Time-aggregation of the continuous-time VAR

Bergstrom (1984) and Campbell and Kyle (1993) show how to derive the discrete-time process implied by a continuous-time VAR when we take point observations of the continuous time process at evenly spaced points \( \{ t_0, t_1, ..., t_n, t_{n+1}, ... \} \), with \( \Delta t = t_n - t_{n-1} \). Direct application of their results shows that the process \( y \) in (1.3) has the following discrete-time VAR(1) representation:

\[
y_{t_n+\Delta t}^p = \exp \{ \Delta t A \} y_{t_n}^p + u_{t_n+1}^p, \tag{1.4}
\]

where

\[
u_{t_n+1}^p = \int_{\tau=0}^{\Delta t} \exp \{ (\Delta t - \tau) A \} C dZ_{t_n+\tau}, \tag{1.5}
\]
and
\[ \exp\{A\} = I + \sum_{r=1}^{\infty} \frac{A^r}{r!}. \quad (1.6) \]

We prove in Appendix A that \( \exp\{sA\} \) is equal to
\[ \exp(sA) = \begin{bmatrix} 1 & \frac{1}{\kappa} (1 - e^{-\kappa s}) \\ 0 & e^{-\kappa s} \end{bmatrix}. \quad (1.7) \]

Thus we can write \((1.4)\) in matrix form as:

\[ \begin{bmatrix} \log S_{t_n + \Delta t} + \frac{\kappa^2}{2} (t_n + \Delta t) - \theta (t_n + \Delta t) \\ \mu_{t_n + \Delta t} - \theta \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\kappa} (1 - e^{-\kappa \Delta t}) \\ 0 & e^{-\kappa \Delta t} \end{bmatrix} \begin{bmatrix} \log S_{t_n} + \frac{\kappa^2}{2} t_n - \theta t_n \\ \mu_{t_n} - \theta \end{bmatrix} + u_{t_n + \Delta t}, \quad (1.8) \]

where

\[ u_{t_n + \Delta t} = \begin{bmatrix} u_{S,t_n + \Delta t} \\ u_{\mu,t_n + \Delta t} \end{bmatrix} = \int_{0}^{\Delta t} \begin{bmatrix} 1 & \frac{1}{\kappa} (1 - e^{-\kappa (\Delta t - \tau)}) \\ 0 & e^{-\kappa (\Delta t - \tau)} \end{bmatrix} \begin{bmatrix} \sigma_S & 0 \\ \rho \sigma_\mu \sqrt{\sigma_\mu^2 - \rho^2 \sigma_\mu^2} \end{bmatrix} dZ_{t_n + \tau}. \quad (1.9) \]

From equation \((1.9)\), it follows that the variance-covariance matrix of the innovations \( u_{t_n + \Delta t} \) in the discrete-time representation of the continuous-time VAR is given by

\[ \text{Var}(u_{t_n + \Delta t}^P) = \int_{\tau=0}^{\Delta t} \exp\{(\Delta t - \tau) A\} CC' \exp\{(\Delta t - \tau) A'\} \, d\tau \quad (1.10) \]

where

\[ B_{11} = \sigma_S^2 + \frac{2 \rho \sigma_S \sigma_\mu}{\kappa} \left(1 - e^{-\kappa (\Delta t - \tau)}\right) + \frac{\sigma_\mu^2}{\kappa^2} \left(1 - e^{-\kappa (\Delta t - \tau)}\right)^2, \]

\[ B_{12} = \rho \sigma_S \sigma_\mu e^{-\kappa (\Delta t - \tau)} + \frac{\sigma_\mu^2}{\kappa} \left(e^{-\kappa (\Delta t - \tau)} - e^{-2\kappa (\Delta t - \tau)}\right), \]

\[ B_{22} = \sigma_\mu^2 e^{-2\kappa (\Delta t - \tau)}. \]
Therefore, given values for the parameters of the continuous-time process (1.2), we can easily aggregate to any frequency $\Delta t$, by using (1.8) and (1.10). The discrete-time representation is especially useful in recovering the parameters of the continuous-time VAR (1.2) from estimates of the equivalent discrete-time VAR (1.8). We do this in the next section.

1.2.3 Recovering continuous-time parameters from a discrete-time VAR

In their analysis of optimal consumption and portfolio choice with time-varying expected returns, Campbell and Viceira (1999, 2000) assume that the log excess returns on stocks is described by the following discrete-time VAR(1):

$$
\begin{bmatrix}
\Delta \log S_{t_n+\Delta t} - r_f \\
\eta_{t_n+\Delta t}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & \Delta \log S_{t_n} - r_f \\
1 & 0 & x_{t_n}
\end{bmatrix} + 
\begin{bmatrix}
\varepsilon_{t_n+\Delta t}
\end{bmatrix},
$$

(1.11)

where $r_f$ is the return (assumed constant) on a T-bill with maturity $\Delta t$.

We now show that the discrete-time VAR given in (1.11) and the continuous-time VAR given in (1.2) are equivalent representations of the same process. To see this, note that we can rewrite the discrete-time aggregation of $y$ in (1.8) as follows:

$$
\begin{bmatrix}
\Delta \log S_{t_n+\Delta t} - r \Delta t \\
\mu_{t_n+\Delta t}
\end{bmatrix} = 
\begin{bmatrix}
\left(\theta - \frac{\sigma^2_S}{2} - r\right)\Delta t - \frac{1}{\kappa} (1 - e^{-\kappa \Delta t}) \theta \\
(1 - e^{-\kappa \Delta t}) \theta
\end{bmatrix} 
+ 
\begin{bmatrix}
1 & \frac{1}{\kappa} (1 - e^{-\kappa \Delta t}) \\
0 & e^{-\kappa \Delta t}
\end{bmatrix} 
\begin{bmatrix}
\log S_{t_n} + \frac{\sigma^2_S}{2} t_n - \theta t_n \\
\mu_{t_n}
\end{bmatrix} + 
\begin{bmatrix}
\varepsilon_{t_n+\Delta t} \\
\eta_{t_n+\Delta t}
\end{bmatrix},
$$

(1.12)

Using the following linear transformation for the process $\mu_t$,

$$
v_t = \left(\theta - \frac{\sigma^2_S}{2} - r\right)\Delta t - \frac{1}{\kappa} (1 - e^{-\kappa \Delta t}) \theta + \frac{1}{\kappa} (1 - e^{-\kappa \Delta t}) \mu_t,
$$
we can further rewrite (1.12) in the same form as (1.11):

\[
\begin{bmatrix}
\Delta \log S_{t_n + \Delta t} - r \Delta t \\
v_{t_n + \Delta t}
\end{bmatrix} =
\begin{bmatrix}
0 \\
(1 - e^{-\kappa \Delta t}) \left( \theta - \frac{\sigma^2}{2} - r \right) \Delta t
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 1 \\
0 & e^{-\kappa \Delta t}
\end{bmatrix}
\begin{bmatrix}
\Delta \log S_{t_n + \Delta t} - r \Delta t \\
v_{t_n}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
1 & 0 \\
0 & \frac{1}{\kappa} (1 - e^{-\kappa \Delta t})
\end{bmatrix}
\begin{bmatrix}
u_S_{t_n + \Delta t} \\
u_{u, t_n + \Delta t}
\end{bmatrix}.
\]

(1.13)

A simple comparison of the coefficients in (1.11) and (1.13) gives us a system of equations that relate the discrete-time parameters of the VAR process in Campbell and Viceira (1999) to the continuous-time parameters of our continuous time VAR process. For the intercept and slope parameters we have the following equivalence relations:

\[
\tau_f = r \Delta t, \quad (1.14)
\]

\[
\mu = \left( \theta - \frac{\sigma^2}{2} - r \right) \Delta t, \quad (1.15)
\]

\[
\phi = e^{-\kappa \Delta t}. \quad (1.16)
\]

Finally, using (1.10), we obtain the following equivalence relations for the variance and
covariance parameters:

\[
\text{Var}_t (\eta_{t_n+\Delta t}) = \frac{1}{\kappa^2} (1 - e^{-\kappa \Delta t})^2 \text{Var}_t (u_{\mu, t_n+\Delta t})
\]

\[
= \frac{\sigma^2_{\mu}}{2\kappa^3} (1 - e^{-\kappa \Delta t})^2 (1 - e^{-2\kappa \Delta t}),
\]

\[
\text{Cov}_t (\varepsilon_{t_n+\Delta t}, \eta_{t_n+\Delta t}) = \frac{1}{\kappa} (1 - e^{-\kappa \Delta t}) \text{Cov}_t (u_{S, t_n+\Delta t}, u_{\mu, t_n+\Delta t})
\]

\[
= \frac{\rho \sigma_S \sigma_{\mu}}{\kappa^2} (1 - e^{-\kappa \Delta t})^2 + \frac{\sigma^2_{\mu}}{\kappa^3} (1 - e^{-\kappa \Delta t})^2
\]

\[
- \frac{\sigma^2_{\mu}}{2\kappa^3} (1 - e^{-2\kappa \Delta t}) (1 - e^{-\kappa \Delta t}),
\]

\[
\text{Var}_t (\varepsilon_{t_n+\Delta t}) = \text{Var}_t (u_{S, t_n+\Delta t})
\]

\[
= \left( \frac{\sigma^2_S}{\kappa^3} + \frac{2\rho \sigma_S \sigma_{\mu}}{\kappa} + \frac{\sigma^2_{\mu}}{\kappa^2} \right) \Delta t - \frac{2\rho \sigma_S \sigma_{\mu}}{\kappa^2} (1 - e^{-\kappa \Delta t})
\]

\[
- \frac{2\sigma^2_{\mu}}{\kappa^3} (1 - e^{-\kappa \Delta t}) + \frac{\sigma^2_{\mu}}{2\kappa^3} (1 - e^{-2\kappa \Delta t}).
\]

Campbell-Viceira (2000) report estimates of the VAR (1.11) based on US quarterly data for the period 1947.Q1-1995.Q4. Table 1.1 shows the value of the parameters of the continuous-time equivalent VAR implied by their estimates.\(^1\)

1.2.4 A common mistake

Anyone used to working with the discrete-time representation of a univariate continuous-time process will find natural and intuitive the relation between the intercept and slope of the continuous-time VAR and its discrete-time representation implied by equations (1.14)-(1.16). However, equations (1.17)-(1.19) show that the equivalence relation for the variance-covariance matrix of innovations is less obvious. Using an intuitive extension of the usual matching rules

---

\(^1\)There is an estimation error in Campbell and Viceira (1999) that results in an underestimation of the degree of predictability in stock returns in their paper. Campbell and Viceira (2000) report correct estimates, and calibration results based on the corrected estimates.
for simple, univariate process, one might be tempted to write:

\[ \text{Var}_t (\varepsilon_{n+1}) \approx \sigma_S^2 \Delta t \]  
\[ \text{Cov}_t (\varepsilon_{n+\Delta t}, \eta_{n+\Delta t}) \approx \rho \sigma_S \sigma_\mu \Delta t \]  
\[ \text{Var}_t (\eta_{n+\Delta t}) \approx \sigma_\mu^2 \Delta t. \]  

(1.20)

It should be apparent from equations (1.17)-(1.19) that this matching is incorrect—though equation (1.20) is a first-order Taylor expansion of the correct expression for \( \text{Var}_t (\varepsilon_{n+1}) \) given in (1.19). The use of (1.20) is particularly dangerous when \( \Delta t \neq 1 \), as might be the case when one is using annualized parameters and quarterly data. In this case portfolio solutions based on (1.20) can be quite different from the correct solutions that we will derive using equations (1.17)-(1.19).

### 1.3 Intertemporal Portfolio Choice

We have shown in Section 1.2, that the investment opportunity set described by equations (1.1) and (1.2) is equivalent to the investment opportunity set that Campbell and Viceira (1999) assume in their discrete-time, intertemporal optimal consumption and portfolio choice model. In this section we solve their model in continuous time, using the techniques described in Chacko and Viceira (2000) and Campbell and Viceira (2002), and we show that the solution is invariant to the choice of discrete-time or continuous-time approximations to solve for the model.

#### 1.3.1 Assumptions on investment opportunities and preferences

We consider an investor who has only two assets available for investment, a riskless bond and stocks, and no labor income. Return dynamics are given by (1.1) and the bivariate system (1.2). For convenience, we rewrite the system (1.2) as

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_S d\tilde{Z}_S,
\]

where

\[
d\mu_t = \kappa (\theta - \mu_t) dt + \sigma_\mu d\tilde{Z}_\mu,
\]
where \( d\bar{Z}_S = dZ_S \), and \( d\bar{Z}_\mu = \rho dZ_S + \sqrt{1 - \rho^2} dZ_\mu \). Note that the instantaneous correlation between \( dS_t/S_t \) and \( d\mu_t \) is \( \rho \).

These assumptions on investment opportunities imply that the wealth dynamics for the investor are given by

\[
dW_t = rW_t dt + \alpha_t W_t [(\mu_t - r) dt + \sigma_S dZ_S] - C_t dt, \tag{1.21}
\]

where \( \alpha_t \) is the fraction of wealth invested in stocks.

Campbell and Viceira (1999) assume that the investor has recursive Epstein-Zin preferences over consumption. Duffie and Epstein (1992a, b) provide an equivalent continuous-time parameterization of recursive preferences:

\[
J_t = \int_t^\infty f(C_s, J_s) ds,
\]

where \( f(C_s, J_s) \) is a normalized aggregator of current consumption and continuation utility that takes the form

\[
f(C, J) = \frac{\beta}{1 - \psi} (1 - \gamma) \left[ \frac{C}{((1 - \gamma) J)^{1-\gamma}} \right]^{(1-\frac{1}{\psi})} - 1. \tag{1.22}
\]

Here \( \beta > 0 \) is the rate of time preference, \( \gamma > 0 \) is the coefficient of relative risk aversion, and \( \psi > 0 \) is the elasticity of intertemporal substitution.

There are two interesting special cases of the normalized aggregator (1.22): \( \psi = 1/\gamma \) and \( \psi = 1 \). The case \( \psi = 1/\gamma \) is interesting because in that case the normalized aggregator (1.22) reduces to the standard, additive power utility function—from which log utility obtains by setting \( \gamma = 1 \). In the second special case, the aggregator \( f(C_s, J_s) \) takes the following form as \( \psi \to 1 \):

\[
f(C_s, J_s) = \beta (1 - \gamma) J \left[ \log(C) - \frac{1}{1 - \gamma} \log((1 - \gamma) J) \right]. \tag{1.23}
\]

The case \( \psi = 1 \) is important because it allows an exact solution to our dynamic optimization problem for investors who are more risk averse than an investor with unit coefficient of relative risk aversion. We now explore this solution, as well as an approximate solution for investors.
with \( \psi \neq 1 \) in the next section.

### 1.3.2 Bellman equation

Duffie and Epstein (1992a, b) show that the standard Bellman principle of optimality applies to recursive utility. The Bellman equation for this problem is

\[
0 = \sup_{\{\alpha_t, C_t\}} \left\{ f(C_t, J_t) + J_W \left[ W_t (r + \alpha_t (\mu_t - r)) - C_t \right] + J_\mu \kappa (\theta - \mu_t) 
+ \frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma_S^2 + J_{WW} W_t \alpha_t \rho \sigma_S \sigma_\mu + \frac{1}{2} J_{\mu\mu} \sigma_\mu^2 \right\},
\]

(1.24)

where \( f(C_t, J_t) \) is given in (1.22) when \( \psi \neq 1 \), or (1.23) when \( \psi = 1 \). \( J_x \) denotes the partial derivative of \( J \) with respect to \( x \), except \( J_t \), which denotes the value of \( J \) at time \( t \).

The first order condition for consumption is given by

\[
C_t = J_W^\psi \left[ (1 - \gamma) J \right]^{\frac{1 - \gamma \psi}{1 - \gamma}} \beta^\psi,
\]

(1.25)

which reduces to \( C_t = (J/J_W(1 - \gamma)) \beta \) when \( \psi = 1 \).

The first order condition for portfolio choice is given by

\[
\alpha_t = \frac{-J_W}{W_t J_{WW}} \left( \frac{\mu_t - r}{\sigma_S^2} \right) - \frac{J_{WW}}{W_t J_{WW}} \left( \frac{\rho \sigma_\mu}{\sigma_S} \right). \tag{1.26}
\]

Substitution of the first order conditions (1.25) and (1.26) into the Bellman equation (1.24) results in the following partial differential equation for the value function \( J \):

\[
0 = \left\{ J_W^{-\psi} \left[ (1 - \gamma) J \right]^{\frac{1 - \gamma \psi}{1 - \gamma}} \beta^\psi, J_t \right\} - J_W \left\{ J_W^{-\psi} \left[ (1 - \gamma) J \right]^{\frac{1 - \gamma \psi}{1 - \gamma}} \beta^\psi \right\}
+ J_W W_t r + J_{\mu\mu} \kappa (\theta - \mu_t) + \frac{1}{2} J_{\mu\mu} \sigma_\mu^2
- \frac{1}{2} \left\{ \frac{J_W^2}{J_{WW}} \frac{(\mu_t - r)^2}{\sigma_S^2} + 2 \frac{J_{WW} J_{WW}}{J_{WW}} \frac{\rho \sigma_\mu (\mu_t - r)}{\sigma_S} + \frac{J_{WW}^2}{J_{WW}} \rho^2 \sigma_\mu^2 \right\}. \tag{1.27}
\]

Of course, the form of this equation depends on whether we consider the case \( \psi \neq 1 \) and use the normalized aggregator in (1.22), or we consider the case \( \psi = 1 \) and use the normalized aggregator (1.23). Appendix B shows the partial differential equation that obtains in each case.
Campbell and Viceira (1999) claim that their discrete-time solution is exact for the case \( \psi = 1 \) up to a discrete-time approximation to the log return on wealth, and note that this approximation becomes exact in continuous-time. The continuous-time model in this paper confirms their claim. We show in Appendix B that (1.27) has an exact analytical solution in the case \( \psi = 1 \). This solution is

\[
J(W_t, \mu_t) = I(\mu_t) \frac{W_t^{1-\gamma}}{1-\gamma},
\]

with

\[
I(\mu_t) = \exp \left\{ A_0 + B_0 \mu_t + \frac{C_0}{2} \mu_t^2 \right\},
\]

where \( A, B, \) and \( C \) are functions of the primitive parameters of the model describing investment opportunities and preferences.

In the more general case \( \psi \neq 1 \), there is no exact analytical solution to (1.27). However, we can still find an approximate analytical solution following the methods described in Campbell and Viceira (2002) and Chacko and Viceira (1999). We start by guessing that the value function in this case also has the form given in (1.28), with

\[
I(\mu_t) = H(\mu_t)^{-\left(\frac{1-\gamma}{1-\psi}\right)}.
\]

Substitution of (1.30) into the Bellman equation (1.27) results in an ordinary differential equation for \( H(\mu_t) \). This equation does not have an exact analytical solution in general. However, we show in Appendix B that taking a loglinear approximation to one of the terms in the equation results in a new equation for \( H(\mu_t) \) that admits an analytical solution. The form of this solution is an exponential-quadratic function similar to (1.29):

\[
H(\mu_t) = \exp \left\{ A_1 + B_1 \mu_t + \frac{C_1}{2} \mu_t^2 \right\}.
\]

The term that we need to approximate in the ordinary differential equation for \( H(\mu_t) \) is \( \beta^\psi H(\mu_t)^{-1} \). Simple substitution of (1.28) and (1.31) into the first order condition (1.25) shows that this term is simply the optimal consumption-wealth ratio \( C_t/W_t \). Thus this loglinearization is equivalent to loglinearizing the optimal consumption-wealth ratio around one particular point.
of the state space. Campbell and Viceira (2002) and Chacko and Viceira (1999) suggest approximating this term around the unconditional mean of the log consumption wealth-ratio. This choice has the advantage that the solution will be accurate if the log consumption-wealth is not too variable around its mean. Appendix B provides full details of this solution procedure.

### 1.3.3 Optimal portfolio choice

The optimal portfolio policy of the investor obtains from substitution of the solution for the value function into the first order condition (1.26). In the case \( \psi = 1 \), substitution of (1.28)-(1.29) into the first order condition (1.26) gives

\[
\alpha_t = \left( \frac{1}{\gamma} \right) \frac{\mu_t - \tau}{\sigma_S^2} + \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_p}{\sigma_S} \rho (B_0 + C_0 \mu_t),
\]  

(1.32)

where \( B_0 = -B_0/(1 - \gamma) \) and \( C_0 = -C_0/(1 - \gamma) \).

In the case \( \psi \neq 1 \), substitution of the approximate solution (1.28)-(1.30)-(1.31) into the first order condition (1.26) gives

\[
\alpha_t = \left( \frac{1}{\gamma} \right) \frac{\mu_t - \tau}{\sigma_S^2} + \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_p}{\sigma_S} \rho (B_1 + C_1 \mu_t),
\]  

(1.33)

where \( B_1 = -B_1/(1 - \psi) \) and \( C_1 = -C_1/(1 - \psi) \). Appendix B shows that \( B_1 \) and \( C_1 \) do not depend on \( \psi \), except through a loglinearization parameter.

Equations (1.32) and (1.33) show that the optimal allocation to stocks is a weighted average (with weights \( 1/\gamma \) and \( 1 - 1/\gamma \)) of two terms, both of them linear in the expected return on stocks \( \mu_t \). The first term is the myopic portfolio allocation to stocks, and the second term is the intertemporal hedging demand for stocks. The myopic portfolio allocation is proportional to \( (1/\gamma) \), so that it approaches zero as we consider increasingly risk averse investors. The intertemporal hedging component is proportional to \( (1 - 1/\gamma) \), so that one might be tempted to conclude that it does not approach zero in the limit as \( \gamma \to 0 \). However, we need to consider that \( B_0 \) (or \( B_1 \)) and \( C_0 \) (or \( C_1 \)) are also functions of \( \gamma \). We can show that the limit of the overall expression approaches zero as \( \gamma \to 0 \).
1.3.4 Numerical calibration

Section 1.3.3 derives the optimal portfolio rule for the continuous-time version of the discrete-time model in Campbell and Viceira (1999). This portfolio rule is similar to the discrete time portfolio rule in their model\(^2\). We can use the parameter values given in Table 1.1 to calibrate the continuous-time portfolio rule (1.33), and compare the resulting mean allocations to those reported in Table III of Campbell and Viceira (2000).\(^3\) Note that the linearity of the optimal portfolio rule (1.32)-(1.33) implies that

\[
E[\alpha_i] = \left(\frac{1}{\gamma}\right) \frac{\theta - r}{\sigma^2_S} + \left(1 - \frac{1}{\gamma}\right) \frac{\sigma_{\mu}}{\sigma_S} \rho (B_i + C_i \theta), \quad i = 0, 1, \tag{1.34}
\]

where the first element of the sum is the mean myopic portfolio allocation, and the second element is the mean intertemporal hedging portfolio allocation.

Table 1.2 has a structure identical to Table III in Campbell and Viceira (2000) to facilitate comparison. Panel A in Table 1.2 shows mean optimal portfolio allocations implied by the parameter values given in Table I. These allocations are similar, but not identical, to those given in panel A of Table III in Campbell and Viceira (2000). This is a direct result of the nonlinearity in the time-aggregation of the variances and covariances of innovations (\(\sigma_S, \sigma_{\mu}, \rho\)), and the persistence parameter \(\kappa\).

Panel B in Table 1.2 shows the percentage that the mean intertemporal hedging portfolio allocation

\[
\left(1 - \frac{1}{\gamma}\right) \frac{\sigma_{\mu}}{\sigma_S} \rho (B_i + C_i \theta)
\]

represents over the total mean allocation (1.34). The numbers in this panel are very similar to those reported in Table III of Campbell and Viceira (2000), and support one of the main conclusions in Campbell and Viceira (1999): That given the historical experience in the US stock market, intertemporal hedging motives greatly increase the average demand for stocks by investors who are more risk averse than a logarithmic investor. For highly conservative investors, hedging may represent 90% or even more of the total mean demand for stocks.

\(^2\)To see this, compare equation (1.33) with the expressions in Proposition 1 of Campbell and Viceira (1999).

\(^3\)Appendix B shows that \(B_1\) and \(C_1\) depend on the loglinearization parameter \(h_1 = E[c_t - w_t]\), which is endogenous. However, one can solve for \(h_1\) using the simple numerical recursive algorithm described in Campbell and Viceira (1999).
1.4 Conclusion

This paper presents a continuous-time version of the model of optimal intertemporal portfolio choice and consumption with time-varying equity premium of Campbell and Viceira (1999). It shows that this model has an exact analytical solution when the investor has unit elasticity of intertemporal substitution in consumption and an approximate analytical solution otherwise. For calibration purposes, we also derive the discrete-time representation of the continuous-time VAR describing the asset return dynamics. This aggregation result is useful to recover the parameters of the model from discrete-time estimates. Our equivalence result shows that intuitive discrete-time representations of univariate continuous-time processes do not translate immediately to multivariate processes which are cross-sectionally correlated.

Our calibration results show that our portfolio choice model exhibits the same properties as its discrete-time counterpart. In particular, given the historical experience in the US stock market, intertemporal hedging motives greatly increase the average demand for stocks by investors who are more risk averse than a logarithmic investor. For highly conservative investors, hedging may represent 90% or even more of the total mean demand for stocks.

1.5 References


1.6 Appendix A

We find $\exp(As)$ by use of an induction proof and some mathematical creativity. We first prove by induction that the matrix $A^n$ is given by

$$A^n = \begin{pmatrix} 0 & (-\kappa)^{n-1} \\ 0 & (-\kappa)^n \end{pmatrix}.$$ \hspace{1cm} (1.35)

To prove this result, assume that $A^n$ is given by (1.35). Then $A^{n+1}$ is given by

$$A^{n+1} = A^n A$$
$$= \begin{pmatrix} 0 & (-\kappa)^{n-1} \\ 0 & (-\kappa)^n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -\kappa \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (-\kappa)^n \\ 0 & (-\kappa)^{n+1} \end{pmatrix},$$

which is the desired result.

The matrix $\exp(As)$ is given by $I + As + \cdots + A^n s^n/n! + \cdots$. Equation (1.35) allows us to write the exponential matrix as
\[
\exp(As) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + \frac{1}{n!} \sum_{n=1}^{\infty} \begin{pmatrix}
0 & (-\kappa)^{n-1} s^n \\
0 & (-\kappa)^n s^n
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & \frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \\
0 & \frac{1}{n!} \sum_{n=0}^{\infty} (-\kappa s)^n
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & \frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \\
0 & \exp(-\kappa s)
\end{pmatrix}
\]

Now, notice that
\[
\frac{d}{ds} \left( \frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \right) = \frac{1}{n!} \sum_{n=0}^{\infty} (-\kappa s)^n = \exp(-\kappa s),
\]
so that
\[
\frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n = \int \exp(-\kappa s) \, ds
\]
\[
= \frac{-1}{\kappa} \exp(-\kappa s) + C.
\]

Since at \( s = 0 \) we have that \( \left( \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n \right) / n! = 0 \), it follows that for the equation to hold at \( s = 0 \) we must have that \( C = \frac{1}{\kappa} \).

Therefore
\[
\frac{1}{n!} \sum_{n=1}^{\infty} (-\kappa)^{n-1} s^n = \frac{1}{\kappa} \left( 1 - \exp(-\kappa s) \right), \tag{1.36}
\]

from which it follows that we can write the matrix \( \exp(As) \) as
\[
\exp(As) = \begin{pmatrix}
1 & \frac{1}{\kappa} (1 - e^{-\kappa s}) \\
0 & e^{-\kappa s}
\end{pmatrix}. \tag{1.37}
\]
1.7 Appendix B

1.7.1 Exact analytical solution when $\psi = 1$

Substitution of (1.28) and (1.29) into the Bellman equation (1.27) leads, after some simplification, to the following equation:

$$0 = -\frac{1}{1-\gamma}\beta \left\{ A_0 + B_0 \mu_t + \frac{C_0}{2} \mu_t^2 \right\} + \beta \log \beta + r - \beta$$

$$+ \frac{\kappa (\theta - \mu_t)}{1-\gamma} (B_0 + C_0 \mu_t) + \frac{\sigma^2_\mu}{2(1-\gamma)} \left\{ C_0 + (B_0 + C_0 \mu_t)^2 \right\}$$

$$+ \frac{1}{2\gamma} \left\{ \frac{(\mu_t - \gamma)^2}{\sigma^2_S} + 2\frac{\rho \sigma_\mu (\mu_t - \gamma)}{\sigma_S} (B_0 + C_0 \mu_t) + \rho^2 \sigma^2_\mu (B_0 + C_0 \mu_t)^2 \right\}.$$

We can now obtain $A_0$, $B_0$, and $C_0$ from the system of recursive equations that results from collecting terms in $\mu_t^2$, $\mu_t$, and constant terms:

$$0 = \frac{\sigma^2_\mu}{2} \left( 1 + \frac{1-\gamma}{\gamma} \rho^2 \right) C_0^2 + \left( -\frac{\beta}{2} - \kappa + \frac{1-\gamma}{\gamma} \frac{\rho \sigma_\mu}{\sigma_S} \right) C_0 + \frac{1}{2\gamma \sigma^2_S}, \quad (1.38)$$

$$0 = \kappa \theta C_0 - \frac{1-\gamma}{\gamma} \frac{r}{\sigma^2_S} - \frac{1-\gamma}{\gamma} \frac{\rho \sigma_\mu}{\sigma_S} C_0$$

$$+ \left( -\kappa - \beta + \frac{\sigma^2_\mu}{2} \left( 1 + \frac{1-\gamma}{\gamma} \rho^2 \right) C_0 + \frac{1-\gamma}{\gamma} \frac{\rho \sigma_\mu}{\sigma_S} \right) B_0, \quad (1.39)$$

$$0 = -\beta A_0 + (1-\gamma) \beta \log \beta + (1-\gamma) (r - \beta) + \frac{\sigma^2_\mu}{2\gamma \sigma^2_S} + \frac{\sigma^2_\mu}{2} C_0$$

$$+ \frac{\sigma^2_\mu}{2} \left( 1 + \frac{1-\gamma}{\gamma} \rho^2 \right) B_0^2 + \left( -\frac{1-\gamma}{\gamma} \frac{\sigma_\mu}{\sigma_S} \rho r + \kappa \theta \right) B. \quad (1.40)$$

We can solve this system by solving equation (1.38) and then using the result to solve (1.39) and finally solve (1.40). Equation (1.38) is a quadratic equation whose only unknown is $C_0$. Thus it has two roots. Campbell and Viceira (1999) show that only one of them maximizes expected utility. This root is the one associated with the negative root of the discriminant of the equation. This is also the only root that ensures that $C_0 = 0$ when $\gamma = 1$, that is, in the log utility case. This is a necessary condition for intertemporal hedging demand to be zero, as we know it must in the log utility case.

We can use these results to obtain the optimal portfolio policy of the investor from the
first order condition (1.26), and the optimal consumption policy from the first order condition (1.25). The optimal portfolio policy is given in equation (1.32) in text. It is easy to see that the optimal consumption policy is $C_t/W_t = \beta$, a constant consumption-wealth ratio equal to the rate of time preference.

### 1.7.2 Approximate analytical solution when $\psi \neq 1$

Substitution of (1.28) and (1.30) into the Bellman equation (1.27) gives, after some simplification, the following ordinary differential equation:

$$
0 = -\beta^\psi H^{-1} + \beta \psi + r (1 - \psi) - \frac{H_\mu}{H} \kappa (\theta - \mu_t) \\
+ \frac{\sigma_\mu^2}{2} \left( -\frac{H_\mu}{H} + \left( 1 + \frac{1 - \gamma}{1 - \psi} \right) \left( \frac{H_\mu}{H} \right)^2 \right) \\
+ \frac{1 - \psi}{2 \gamma} \left( \frac{\mu_t - r}{\sigma_s} \right)^2 - \frac{1 - \gamma}{\gamma} \frac{H_\mu}{H} \rho \sigma_\mu \left( \frac{\mu_t - r}{\sigma_s} \right) + \frac{1}{2} \frac{(1 - \gamma)^2}{\gamma (1 - \psi)} \left( \frac{H_\mu}{H} \right)^2 \rho^2 \sigma_\mu^2.
$$

(1.41)

This ordinary differential equation does not have an exact analytical solution, unless $\psi = 1$.

Though there does not exist an exact analytical solution to (1.41), we can still find an approximate analytical solution following the methods described in Campbell and Viceira (2002) and Chacko and Viceira (1999). First, we note that substitution of the solution guess (1.28)-(1.30) into the first order condition (1.25) gives

$$
\frac{C_t}{W_t} = \beta^\psi H (\mu_t)^{-1}.
$$

We can now use the following approximation for $\beta^\psi H^{-1}$ around the unconditional mean of the log consumption-wealth ratio:

$$
\beta^\psi H (\mu_t)^{-1} = \exp \{c_t - w_t\}
$$

$$
\approx h_0 + h_1 (c_t - w_t)
$$

$$
= h_0 + h_1 (\psi \log \beta - h_t),
$$

(1.42)
where \( c_t = \log C_t \), \( w_t = \log W_t \), \( h_t = \log H (\mu_t) \), and

\[
\begin{align*}
    h_1 & = \exp \{ E [c_t - w_t] \}, \\
    h_0 & = h_1 (1 - \log h_1).
\end{align*}
\]

Substitution of the approximation (1.42) for the first term of (1.41) transforms this ordinary differential equation into another one that has an exact solution, with the following exponential-quadratic form:

\[
H (\mu_t) = \exp \left\{ A_1 + B_1 \mu_t + \frac{C_1}{2} \mu_t^2 \right\}.
\]

The coefficients \( A_1, B_1, \) and \( C_1 \), can be obtained by solving the approximated Bellman equation

\[
0 = -h_0 - h_1 \left\{ \psi \log \beta - \left( A_1 + B_1 \mu_t + \frac{C_1}{2} \mu_t^2 \right) \right\} + \beta \psi + r (1 - \psi) - \kappa (\theta - \mu_t) (B_1 + C_1 \mu_t) + \frac{\sigma_\mu^2}{2} \left[ -((B_1 + C_1 \mu_t)^2 + C_1) + \left( \frac{1 - \gamma}{1 - \psi} + 1 \right) (B_1 + C_1 \mu_t)^2 \right] + \frac{1 - \psi}{2 \gamma} \left( \frac{\mu_t - r}{\sigma_S} \right)^2 - \frac{1 - \gamma}{\gamma} (B_1 + C_1 \mu_t) \rho \sigma_\mu \left( \frac{\mu_t - r}{\sigma_S} \right) + \frac{1 - \gamma}{2 \gamma} \left( \frac{1 - \gamma}{1 - \psi} \right) (B_1 + C_1 \mu_t)^2 \rho^2 \sigma_\mu^2,
\]

which implies the following system of recursive equations:

\[
\begin{align*}
0 & = \frac{\sigma_\mu^2}{2} \frac{1 - \gamma}{1 - \psi} \left( 1 + \frac{1 - \gamma}{\gamma} \rho^2 \right) C_1^2 + \left( \frac{h_1}{2} + \kappa - \frac{1 - \gamma}{\gamma} \rho \sigma_\mu \right) C_1 + \frac{1 - \psi}{2 \gamma \sigma_S^2}, \quad (1.45) \\
0 & = \kappa \theta C_1 + \frac{1 - \psi}{\gamma} \frac{r}{\sigma_S^2} - \frac{1 - \gamma}{\gamma} \rho \sigma_\mu C_1 \quad (1.46) \\
& + \left( \kappa + h_1 + \sigma_\mu^2 \frac{1 - \gamma}{1 - \psi} \left( 1 + \frac{1 - \gamma}{\gamma} \rho^2 \right) C_1 - \frac{1 - \gamma}{\gamma} \rho \sigma_\mu \right) B_1, \\
0 & = h_1 A_1 - h_0 - h_1 \psi \log \beta + \beta \psi + r (1 - \psi) + \frac{1 - \psi}{2 \gamma} \frac{r^2}{\sigma_S^2} - \frac{\sigma_\mu^2}{2} C_1 \quad (1.47) \\
& + \frac{\sigma_\mu^2}{2} \frac{1 - \gamma}{1 - \psi} \left( 1 + \frac{1 - \gamma}{\gamma} \rho^2 \right) B_1^2 + \left( \frac{1 - \gamma}{\gamma} \sigma_\mu \rho r - \kappa \theta \right) B_1.
\end{align*}
\]

We can solve this system by solving equation (1.45) and then using the result to solve (1.46) and finally solve (1.47). Equation (1.45) is a quadratic equation whose only unknown is \( C \). Thus it has two roots. Campbell and Viceira (1999) show that only one of them maximizes expected
utility. This root is the one associated with the positive root of the discriminant of the equation. Note also that this equation implies that \( C/(1 - \psi) \) does not depend on \( \psi \)—except through the loglinearization parameter \( h_1 \)—which in turn implies, through equation (1.46), that \( B/(1 - \psi) \) does not depend on \( \psi \) either.
TABLE 1.1

Continuous-Time Parameter Values
Implied by Campbell-Viceira (2000) VAR Estimates

Model:
\[
\begin{align*}
\frac{dB_t}{B_t} &= r dt \\
\frac{dS_t}{S_t} &= \mu_t dt + \sigma_S dZ_S \\
\mu_t &= \kappa (\theta - \mu_t) dt + \sigma_\mu dZ_\mu \\
\sigma_S \sigma_\mu &= \rho dt
\end{align*}
\]

Parameter Values (at quarterly frequency):
\[
\begin{align*}
r &= 0.0818e - 2 \\
\kappa &= 4.3875e - 2 \\
\theta &= 1.3980e - 2 \\
\sigma_S &= 7.8959e - 2 \\
\sigma_\mu &= 0.5738e - 2 \\
\rho &= -0.9626
\end{align*}
\]
TABLE 1.2

Mean Optimal Percentage Allocation to Stocks and
Percentage Mean Hedging Demand Over Mean Total Demand

<table>
<thead>
<tr>
<th>RRA($\gamma$)</th>
<th>Elasticity of Intertemporal Substitution ($\psi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(A) Mean optimal percentage allocation to stocks:</td>
</tr>
<tr>
<td></td>
<td>1/75  1.00  1/1.5  1/2  1/4  1/10  1/20  1/40</td>
</tr>
<tr>
<td>0.75</td>
<td>180.31 198.02 209.80 214.47 220.42 223.53 224.49 224.97</td>
</tr>
<tr>
<td>1.00</td>
<td>211.12 211.12 211.12 211.12 211.12 211.12 211.12 211.12</td>
</tr>
<tr>
<td>1.50</td>
<td>239.07 223.64 211.64 206.76 200.57 197.39 196.40 195.92</td>
</tr>
<tr>
<td>2.00</td>
<td>248.85 227.85 210.53 203.36 194.22 189.52 188.07 187.36</td>
</tr>
<tr>
<td>4.00</td>
<td>241.99 220.24 200.14 191.27 179.59 173.46 171.56 170.63</td>
</tr>
<tr>
<td>10.00</td>
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</tr>
<tr>
<td>20.00</td>
<td>125.04 123.63 122.17 121.42 120.29 119.61 119.38 119.27</td>
</tr>
<tr>
<td>40.00</td>
<td>75.57  77.74  80.00  81.17  82.98  84.12  84.51  84.71</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(B) Fraction due to hedging demand (percentage)</td>
</tr>
<tr>
<td></td>
<td>1/75  1.00  1/1.5  1/2  1/4  1/10  1/20  1/40</td>
</tr>
<tr>
<td>0.75</td>
<td>56.12 -42.16 -34.17 -31.26 -27.71 -25.94 -25.39 -25.13</td>
</tr>
<tr>
<td>1.00</td>
<td>0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00</td>
</tr>
<tr>
<td>1.50</td>
<td>41.13 37.06 33.49 31.93 29.82 28.69 28.34 28.16</td>
</tr>
<tr>
<td>2.00</td>
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</tr>
<tr>
<td>40.00</td>
<td>93.02 93.21 93.40 93.50 93.64 93.73 93.75 93.77</td>
</tr>
</tbody>
</table>

30
Chapter 2

Hedging Demands, Imperfect Information, and Incomplete Markets

2.1 Introduction

This paper studies consumption and portfolio choice when markets are incomplete and investors cannot observe the current values of the state variable which determine the attractiveness of investment opportunities. I establish conditions under which the investor's optimization problem under incomplete markets can be transformed into a complete markets problem. When prices are the only observable variables, the investor optimally estimates the current expected returns using the realized returns. Because of this, the market is observationally complete for an imperfectly informed investor. In other words, the investors believe changes in the state variable can be perfectly replicated with a portfolio of the available assets. The observational completeness of the market allows me to find analytical, closed-form solutions to the investor's consumption and portfolio choice problem.¹

Motivated by the standard assumption that excess expected returns are a function of volatil-

¹Recent articles in operations research address some of the issues raised in this paper. Lakner (1995,1998), Karatzas and Zhao (2001), and Hershel (1999) study the asset allocation problem under incomplete information. These papers do not consider the consumption aspect of an investor's strategic asset allocation problem.
ity and market price of risk, I apply the model to the case where excess expected returns on the risky asset are time-varying and unobservable. Several empirical studies, particularly Merton (1980), have shown volatility is easily estimated. Under this assumption we should care about parameter uncertainty for the market price of risk, and the problem boils down to uncertainty regarding the current Sharpe ratio. In this case closed-form solutions are obtained when the investors' inference reaches a steady-state, i.e. when additional information does not reduce the variance of estimation error. My results are novel in two dimensions. First, I show how parameter uncertainty, a reasonable assumption given the empirical evidence, can help us simplify the consumption and portfolio choice problem. Second, I can analytically study the link between how the investor perceives predictability in asset returns and their portfolio strategy. Calibrating the model to match the autocorrelation of long-horizon cumulative returns, I find imperfect information reduces both the sensitivity of hedging demand to changes in the state variable, as well as the covariance component of hedging demand. Furthermore, this paper provides another alternative in solving consumption and portfolio choice problems in closed-form.

In the calibration, the variance of the estimation error comes into play in the diffusion of the estimated state variable and is positively correlated with the shock to realized returns while the true correlation between the state variable and the stock price is negative. Therefore, the variance of estimation error has a tempering effect in the hedging demand of the investor. In some cases, the hedging demand of the investor can even be negative. The intuition is as follows: The investor now has to estimate expected returns. When determining how much to change the allocation to the risky asset due to changes in expected returns, the investor must account for the fact that he is using a noisy estimate. Therefore, the investor will be less aggressive in how he changes the composition of the portfolio due to changes in how he perceives the current reward to risk trade-off for the risky asset.

Merton (1971) derives the existence of a hedging portfolio that accounts for changes in variables which determine future investment opportunities. However, the empirical evidence was unable to reject the hypothesis that asset prices followed a random walk. Without time-varying returns, it followed naturally that portfolio choice should be entirely myopic and thus

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their would be no hedging component to the optimal asset allocation policy. Poterba and Summers (1988), Campbell and Shiller (1988) and Fama and French (1989) find evidence of predictability in the time series of asset prices. Recently, Lewellen (2001) shows mean reversion in stock return may be even stronger and quicker than previously perceived. Both papers show the mean reverting component of returns comprises more than 25% of stock return variation, but Lewellen finds mean reversion to be highest in a 12 to 18 month horizon while Fama and French find mean reversion to be highest in the 3 to 5 year horizon. With abundant evidence of time-varying expected returns, Kim and Omberg (1996) study the role of return predictability in the optimal asset allocation problem, finding closed form solutions for the hedging demands.3

Most papers in the portfolio choice literature assume expected returns are observable. Given the amount of evidence regarding predictability in asset prices and the difficulties associated with determining such predictability, a model of portfolio choice should acknowledge a role for parameter uncertainty and imperfect information. Merton (1971), Detemple (1986), Dothan and Feldman (1986), and Gennette (1986) lay the foundation of the portfolio choice problem under imperfect information. They show that the optimization problem where some parameters are unknown can be transformed into an optimization problem using the estimates of the unknown parameters and the price and state variable dynamics obtained by the inference problem. In continuous time, portfolio choice under incomplete information can then be solved in two steps. First, unobservable parameters are estimated by filtering signals from the observable data. Second, the investor chooses optimal consumption and portfolio policies given these estimates. In related work, Barberis (2000) and Xia (2001), consider uncertainty regarding the relation between stock returns and the state variables.4 In contrast, I focus on how uncertainty regarding the current value of the expected returns changes the composition of the investor's consumption and portfolio choice. One interpretation of the model is that business cycles do


4Bawa and Klein (1976) and Bawa, Brown, and Klein (1979) study the role of uncertainty in asset allocation. Kandel and Stambaugh (1996) extend the theory to consider uncertainty about the predictability in asset prices. They find that the predictive relation between returns and the dividend to price ratio, although statistically weak, is economically significant even in the presence of estimation risk. In other words, investors should account for predictability in the portfolio decision, hence it would be suboptimal for the investor to invest under the assumption of a random walk process for asset prices and ignore the role predictability should play in asset allocation even when the evidence of predictability is statistically weak.
occur in the economy, but we are unable to pinpoint where the business cycle currently stands. Furthermore, predictive variable models assume expected returns revert at the same speed as the predictive variable. This assumption can lead to wrong estimates regarding the autocorrelation of returns and lead to possibly suboptimal investment decisions. Finally, the assumption of unobservable mean-reverting expected returns allows me to simplify the model and find closed form solution when the learning process of the investor is considered. Recent work by Brandt and Kang (2002), find little additional explanatory power for predictive variables, once expected returns and volatility are modeled as latent variables when using a vector-autoregression (VAR) approach. The authors interpret this finding as evidence of the inability of the predictive variables to yield a better understanding of the dynamics of the Sharpe ratio, or the return to risk trade-off.

Section 2.2 discusses the structure of the economy and solves the optimization problem of the agent in a partially observable economy in a general setting. I provide a simple application of the separation theorem, the filtering theory of Lipster and Shiryayev (2001), and the martingale methods of Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) as it applies to my model. In Section 2.3, I study stock price predictability under the assumption that the instantaneous Sharpe ratio is not observable and solve for the optimal consumption and portfolio policies. I calibrate the model implied predictable variance to the predictable variance in returns and quantify the effect of imperfect information in the consumption and portfolio policy of the investor. Section 2.4 concludes and offers a variety of extensions for the methodology presented in this paper.

2.2 The Model

I develop a model of consumption and portfolio choice where markets are incomplete and there is uncertainty regarding the current value of the state variables. As shown by Merton (1971), state variables determine the investment opportunity set faced by the investor and the optimal portfolio policy contains a component to hedge the risks associated with those changes. I assume the investor cannot observe those variables. Once the investor estimates the state variables, the market is complete given the information set of the investor. This allows me to obtain
analytical, exact solutions to the consumption and portfolio choice problem.\(^5\)

2.2.1 Market Structure

Assume the existence of a single consumption good which serves as the numeraire. Uncertainty is represented by a probability space \((\Omega, \mathcal{F}, P)\) on which we define a \(d_Z\)-dimensional orthogonal Brownian Motion \(Z_t\) and a \(d_W\)-dimensional orthogonal Brownian Motion \(W_t\). Let \(F\) denote the filtration generated by the Brownian Motions \((Z_t, W_t)\). Assume the filtration is right-continuous and the probability space is complete. The Brownian Motions \(Z_t\) and \(W_t\) are assumed to be orthogonal to each other. For all Itô processes in this paper assume all drift coefficients are defined in \(L^1\) and all diffusion term coefficients are defined in \(L^2\).\(^6\)

Changes in the investment opportunity set of the agent are represented by state variables. The state variables, \(X_t\), are unobservable and not necessarily spanned by the market securities.\(^7\) The state variables follow the process:

\[
dX_t = [a_X(t) + b_X(t) X_t] dt + \sigma_X(t) dZ_t + \sigma_W(t) dW_t
\]

The securities market consists of a riskless asset, the money market account, which pays the locally riskless rates at all times, and \(N\) risky securities which span \(Z\). None of the risky assets are redundant. The money market account grows at the riskless rate of return. The money market account satisfies

\[
 dB_t = B_t r_t dt,
\]

where \(r_t\) is the locally riskless rate of return and observable. The prices for the risky securities

\(^5\)In cases where an analytical solution cannot be obtained, the Monte Carlo methods of Detemple, Garcia, and Rindisbacher (2003) or Cvitanic, Goukasian, and Zapatero (2002) can be used to obtain a numerical solution. Both methods require market completeness which is satisfied given the information set of the investor.

\(^6\)Assume the following definition for the sets described in the paper hold:

\[
 L^1 = \left\{ X \in \mathcal{L} : \int_0^T |X_t| dt < \infty \ a.s. \right\},
\]

\[
 L^2 = \left\{ X \in \mathcal{L} : \int_0^T X_t^2 dt < \infty \ a.s. \right\}.
\]

\(^7\)The extension to both observable and unobservable state variables is straightforward under the assumption observed state variables are spanned by market securities. Otherwise, the market is also incomplete under the information set of the investor.
follow the multidimensional Ito process

\[ dS_t = S_t \left[ \left( a_S(t) + b_S(t) X_t \right) dt + \sigma_S(t) dZ_t \right], \quad (2.3) \]

For simplicity, I assume none of the risky assets are redundant such that \( \sigma_S(t) \) is a square matrix. Equations (2.1), (2.2) and (2.3) represent an economy where expected returns are time-varying and unobservable. Equation (2.3) states there exists a linear relation between expected returns and the unobserved state variables. Since the state variables are not observable, the instantaneous expected return is not observable.

The inference problem is solved with filtering methods covered in Lipster & Shiryayev (2001). I follow their treatment as it applies to (2.1) and (2.3). Assume the investor observes instantaneous returns to the money market account (2.2) and the equity (2.3). Assume the only unknown to the investor is \( X_t \). In this model, prices are the only signals investors have regarding the investment opportunity set. The following lemma states the results from the inference process given the structure of the information set for the investor.

**Lemma 2.1** Define \( \hat{X}_t \), as the investor’s estimate of \( X_t \) under the optimal filter. Given the information set of the investor, the prices of the risky assets and the state variables satisfy the following stochastic differential equations:

\[ dS_t = S_t \left[ \left( a_S(t) + b_S(t) \hat{X}_t \right) dt + \sigma_S(t) d\hat{Z}_t \right], \quad (2.4) \]
\[ d\hat{X}_t = \left[ a_X(t) + b_X(t) \hat{X}_t \right] dt + \left[ \sigma_X(t) \sigma_S'(t) + v(t) b_S(t) \right] \left[ \sigma_S(t) \sigma_S'(t) \right]^{-1} \sigma_S(t) d\hat{Z}_t, \quad (2.5) \]

where \( \hat{X}_t \) is the investor’s estimate of the unobservable state variable, \( v(t) \) represents the variance of the estimation error for the unobservable state variable at time \( t \) and

\[ \sigma_S(t) d\hat{Z}_t = b_S(t) \left( X_t - \hat{X}_t \right) dt + \sigma_S(t) dZ_t \quad (2.6) \]

**Proof.** The proof follows from applying the optimal filter presented in Lipster and Shiryayev (2001, p.22) to the model. ■

Equation (2.4) states the investor decomposes returns to the risky assets into the expected component, under the optimal estimate for the unobservable state variable, and the unexpected
change in returns. Since the investor is using an estimate to gauge expected returns, the unexpected component of stock returns will include the true shock to stock prices as well as the difference between the true expected return and the estimate. This is clearly shown in equation (2.6). Equation (2.5) shows how the optimal filter essentially projects the state variables into the space of market securities. To gain some intuition regarding the inference process, consider the case where no relation exists between the drift of the stock prices and the state variables. Under the previous assumption, equation (2.5) becomes a dynamic version of the projection theorem for Gaussian variables widely used for inference problems in market microstructure models. The projection onto market securities is obvious in this case since there is no estimation error as part of the unexpected component of returns for prices. When the relationship between expected returns and the unobserved state variable holds, the investor projects the shocks of the unobserved state variables to the unexpected component of returns. Since both unobserved components include the estimation error, the projection includes a term related to the variance-covariance matrix of the estimation error. The following lemma states the dynamic behavior of the variance of the estimation error.

**Lemma 2.2** The variance-covariance matrix of the estimation error,

\[ v(t) = E_t' \left( (X_t - \hat{X}_t) (X_t - \hat{X}_t)' \right), \]

satisfy the Riccati equation

\[
\frac{dv(t)}{dt} = b_X(t) v(t) + v(t) b'_X(t) + \sigma_X(t) \sigma'_X(t) + \sigma_W(t) \sigma'_W(t) - \left[ \sigma_X(t) \sigma'_S(t) + v(t) b'_S(t) \right] \left[ \sigma_S(t) \sigma'_S(t) \right]^{-1} \left[ \sigma_X(t) \sigma'_S(t) + v(t) b'_S(t) \right]' \quad (2.7)
\]

**Proof.** The proof follows from applying the optimal filter presented in Lipster and Shiryaev (2001, p.22) to the model. ■

Just as equation (2.5) can be related to the projection theorem, equation (2.7) relates to the improvement in variance due to a informative signal. The first line represents the change in the conditional variance of the estimation error and the second line represents the reduction in variance due to the role of prices as signals.

Equations (2.4) and (2.5) represent the dynamics of the stock prices and the state variables under the information set of the investor. In other words, the investor believes the unexpected component of the state variables is spanned by the shocks to market securities. Hence changes
in the investment opportunity set can be hedged completely with a portfolio of the market securities. The previous statements can be formalized with the following definition.

**Definition 2.1** Markets are complete iff all $F_T$-measurable random variables $V_T$ such that $V_T/B_T \in L^1(Q)$ can be replicated.

In order to prove market completeness under the information set of the investor after the optimal filtering process, I will need to define a state price density for the investor. The following lemma shows the existence of a state price density for the economy implied by the filtered processes for the state variables and the stock prices.

**Lemma 2.3** A state price density (SPD) is a strictly positive Ito process $M_t$ such that for any market security $V_t$, $M_t V_t$ is a martingale, and $M_0 = 1$. Denote $\hat{M}_T$ as a candidate SPD under $F^I$, the investor's information set. The investor specific SPD is given by

$$\hat{M}_T = \exp \left\{ - \int_0^T r_t dt - \int_0^T \hat{\eta}_t d\hat{Z}_t - \frac{1}{2} \int_0^T \hat{\eta}_t^2 \hat{\eta}_t dt \right\}$$

where

$$\hat{\eta}_t = (\sigma_S(t))^{-1} \left( a_S(t) + b_S(t) \hat{X}_t - r_t \right)$$

**Proof.** Provided in the Appendix.

Similar to the incomplete markets consumption and portfolio choice model of He and Pearson (1991), the investor in my model is able to determine a unique stochastic discount factor, but unlike He and Pearson, the stochastic discount factor for the investor is straightforward to obtain and does not require the use of a dual problem. Although the investor has a unique stochastic discount factor, this does not imply it is the unique discount factor for the economy. Basak (2000) studies a dynamic equilibrium model of heterogeneous beliefs and finds individual-specific Arrow-Debreu prices can differ. Similar to Basak, the investor has a uniquely defined stochastic discount factor based on their beliefs on certain parameters in the economy. Therefore even when markets are incomplete, the consumption and portfolio choice problem of the investor can be solved with martingale methods.
If a state price density exists, then an equivalent martingale measure can be defined under the information set of the investor.\textsuperscript{8}

**Definition 2.2** An equivalent martingale measure (EMM) is a probability measure $Q$ such that for any market security $V_t$, $V_t/B_t$ is a martingale under $Q$, and $Q$ is equivalent to $P$.

The equivalent martingale measure is also ignores the stochastic component of $X_t$ not spanned by the market securities. Now I can prove the market is complete under the information set of the investor.

**Proposition 2.1** Let $F^I$ denote the information set of the investor under the optimal filter such that (2.4) and (2.5) represent the dynamics of the stock prices and the state variables under $F^I$. The market is complete under $F^I$.

**Proof.** Provided in the Appendix  

This result is obtained because I assume prices are the only signals investor use to decipher expected returns. A discussion of this possibly controversial assumption is in order. Although Merton (1971) and Brennan (1998) make the same assumption in their models, they only consider the case of constant expected returns. It might seems unnatural to assume away the existence of other predictive variables if expected returns are time-varying. As stated previously, this is not a concern under a straightforward extension of the model to include observable state variables. Also, most predictive variables used in the finance literature include low-frequency signals such as dividends for the dividend to price ratio. If we take the continuous-time model literally, the driver of the dividend to price ratio at high frequencies is just the changes in prices. Therefore, the case for the use of prices as signals can be made due to the need of high-frequency data to follow the implied high-frequency trading strategies of the model.\textsuperscript{9} A more detailed discussion of the issue is given at the end of this section.

\textsuperscript{8}I omit a proof of this assertion. The proof can be found in Duffie (2001).

\textsuperscript{9}I have not considered the issue of transaction costs. Recent work by Lo, Mamaysky, and Wang (2001) finds high-frequency trading needs due to stochastic endowments can lead to high-frequency trading even when there are transaction costs. They show some investors with high-trading needs would rebalance their portfolio at least once every month.
2.2.2 Investor's Optimization Problem

The investor's preferences are assumed to satisfy the standard constant relative risk aversion, power utility function:

\[ u(C_t) = e^{-\phi t \frac{C_t^{1-\gamma}}{1-\gamma}} dt, \]  

(2.8)

where \( C_t \) is the amount of the numeraire good consumed at time \( t \), \( \gamma \) is the coefficient of relative risk aversion and \( \phi \) is the agent's discount rate. Denote \( \alpha_t \) as the vector of portfolio weight for the investor's optimal investment strategy in the risky assets. The investors' budget satisfies:

\[ dW_t = W_t \left\{ \left[ r_t + \alpha'_t (\mu_s - r_t) \right] dt + \alpha'_t \sigma_s(t) dZ_t \right\} - C_t dt \]

(2.9)

and the investor is subject to a non-negative wealth constraint.

Let the superscript \( I \) denote expectation operations taken under \( F^I \), the information set of the investor. Given the process governing the dynamics of the stochastic discount factor, the agent's optimization problem can be solved with martingale methods. The agent's optimization problem is to maximize the expected lifetime utility of consumption \( J_t \) where

\[ J_t = \sup_{\{ \alpha_t, C_t \}} E^I_t \left[ \int_t^T e^{-\phi(s-t) \frac{C_s^{1-\gamma}}{1-\gamma}} ds \right] \]

(2.10)

subject to the dynamic budget constraint under the estimated processes for the securities and a non-negative wealth constraint. The existence of the stochastic discount factor allows us to write the agent's dynamic budget constraint as a static budget constraint given by

\[ E^I_t \left[ \int_t^T \bar{M}_s C_s ds \right] \leq \bar{M}_t W_t. \]

(2.11)

where the expectation is defined under the investor's information set as represented by the results of the inference process described previously. Equation (2.11) states the agent's expected consumption stream in the future appropriately discounted will be less than or equal to his current wealth. Proposition 2.1 implies the investor's optimization problem can be solved via martingale methods as shown by Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987). The following theorem states the optimal consumption and portfolio policies for the
Proposition 2.2 Define the functions $F_{t,s}$ and $G_{t,T}$ as follows

\begin{align}
F_{t,s} &= E_t \left( \frac{\tilde{M}_t}{M_t} \right)^{1-\gamma} + \gamma, \\
G_{t,T} &= \int_t^T e^{-\frac{\gamma}{2} (s-t)} F_{t,s} ds.
\end{align}

The optimal consumption and portfolio policies for an investor maximizing lifetime utility (2.8) subject to the budget constraint (2.11) are

\begin{align}
C_t &= \left( \lambda_0 e^{\delta t} \frac{\tilde{M}_t}{M_t} \right)^{\frac{1}{\gamma}}, \\
\alpha'_t &= \frac{1}{\gamma} \sigma_S(t)^{-1} \tilde{\eta}_t + \frac{\partial \log G_{t,T}}{\partial \tilde{X}_t} \left( \sigma_S(t) \sigma_S(t)^{-1} \right) \left[ \sigma_X(t) \sigma_S(t) + \nu(t) b'_S(t) \right].
\end{align}

respectively.

Proof. Provided in the Appendix.

$F_{t,s}$ shows the relation between the investment opportunity set or the risk-to-reward ratio as it pertains to the investor's hedging needs. In particular, for a myopic investor $F_{t,s} = 1$ always, while an infinitely risk averse investor, $F_{t,s}$ is a linear function of the state price density ratio. Hence, $F_{t,s}$ serves as a measure of hedging aggressiveness due to consumption smoothing.

$G_{t,T}$ weighs the investor reaction to changes in the future investment opportunity set by the investor's impatience and risk aversion. If the investor discounts future utility heavily, then the investor assigns greater weight to the short-term future investment opportunity set. Highly risk averse investors will penalize longer horizon opportunity set changes less since they care about maintaining a low variance in their consumption. Note

\begin{equation}
\frac{\partial G_{t,T}}{\partial \tilde{X}_t} = \int_t^T e^{-\frac{\gamma}{2} (s-t)} \frac{\partial F_{t,s}}{\partial \tilde{X}_t} ds.
\end{equation}

Since $F_{t,s}$ is a function of the ratio of the state price density at time $s$ relative to the state price density at time $t$, equation (2.16) formalizes the relation between the hedging demand and changes in the investment opportunity set. Also, the hedging demand is a weighted function of
the expected changes in the investment opportunity set for all horizons up to retirement. The weighting function is related negatively to the investor’s impatience and positively to its relative risk aversion. Therefore, the more impatient investors care more about hedging demand in a shorter horizon, while more risk averse investor care about longer horizon consumption needs.

The portfolio choice of the agent can be decomposed into its myopic demand, the demand due to the current state of the economy, and the hedging demand, the demand due to expected changes in the investment opportunity set. In the model, the hedging demand is due to the stochastic nature of the estimated state variables. Both the myopic and hedging components are subject to the estimation risk due to the unobserved state variables, while the hedging demand is also subject to the variance of the estimation error. The myopic demand of the agent is a function of the estimated state variables by how those estimates change the investors perception of the current investment opportunity set as proxy by the Sharpe ratio. The hedging demand of the agent is a function of the investor’s perception of the stochastic component for the estimated state variables (how the investment opportunity set changes with time) as well as a function of the current estimate of the unobserved state variables.\textsuperscript{10}

It is well known there exists a relationship between the consumption strategy and the hedging policy of the investor. In particular, the magnitude of the hedging demands is a function of the sensitivity of the consumption profile to changes in the investment opportunity set. The intuition can be formalized by noticing the explicit relation between the consumption to wealth ratio and the function $G_{t,T}$ presented in Theorem 2.2.

**Corollary 2.1** The consumption to wealth ratio of the investor at time $t$ is $G_{t,T}^{-1}$.

**Proof.** Provided in the Appendix 

Corollary 2.1 shows the link between future expected consumption and the hedging strategy of the investor. When markets are complete, the investor essentially can plan the consumption strategy for each possible outcome at each possible horizon, hence hedging demands are linked to

\textsuperscript{10}A similar result for the portfolio strategy could be obtained by assuming the state variable is observable and assigning zero price of risk to the stochastic components of the state variables orthogonal to the stochastic components of the market securities. Yet, this result would miss the effects of estimation risk in the optimal portfolio policy. Similarly, it has been pointed out the results presented can be obtained by assuming the unobserved state variables are spanned by the market securities. It will be apparent in the next section that both market incompleteness and imperfect information contribute to the solution.
the sensitivity of the investor's consumption strategy to changes in the investment opportunity set.

2.2.3 Comment on the Information Structure of the Model

In order to achieve complete markets under the information set of the investor, I assume market securities, or any other random variable spanned by those securities, are the only signals the investor uses in finding estimates for the unobservable state variables. Such an assumption essentially restricts the information set of the investor from utilizing observable predictive variables such as the dividend-to-price ratio, the earnings-to-price ratio, and the book-to-market ratio as long as we believe these ratios are not spanned by the securities in the market. Since most of these ratios are not truly exogenous predictors, the assumption of spanning is defensible. Yet, the literature on stock price predictability has provided us with many more variables which help predict expected returns such as the short rate of interest, the term premium, the default premium, among others.\(^{11}\) I will outline a few reasons why ignoring these variables is not necessarily a weakness for the model.

The literature defending the evidence of predictability in stock returns is as expansive as the literature pointing out the weaknesses of such defense. Stambaugh (1999) highlights the statistical bias of predictive regressions with lagged stochastic regressors. Stambaugh finds the bias can significantly overstate the predictive power of the lagged stochastic variable and offers a correction for the bias. Lewellen (2002), uses the specification of the lagged variable as a stationary variable, and finds previous corrections for the small sample bias might significantly understate forecasting power. Ferson, Sarkissian, and Simin (2002) argue the spurious regression bias is enhanced by the “data mining” issue when looking for a significant predictive variable in a set of candidate variables. Some papers highlight difficulties with the out-of-sample performance and robustness of the predictive power of lagged regressors. Goyal and Welch (2002) find a significant reduction in the predictive power of the dividend to price ratio relative to the unconditional equity premium.

The most encouraging defense to the latent variable approach to the consumption and

\(^{11}\)Ferson and Harvey (1993) contains an extensive list of predictive variables.
portfolio choice problem comes from Brandt and Kang (2002). They study the conditional relationship between the mean and volatility of returns in a vector autoregression model where there are no exogenous sources of information and the mean and volatility of returns are treated as latent variables. The authors do a specification test between a model with only the latent variables and a model that allows for exogenous predictive variables. Their specification test fails to reject the hypothesis of additional predictability due to the exogenous variables. The authors interpret this finding as evidence of the inability of exogenous variables to yield a better understanding of the dynamics of the Sharpe ratio, or the return to risk trade-off.

2.3 Application

I consider the consumption and portfolio choice problem in a market with two assets: a riskless money market account and a risky asset with unobservable, time-varying expected returns. I assume the Sharpe ratio for the risky asset is unobservable, mean-reverting and not perfectly correlated with the stock price. This is the simplest example where the market is naturally incomplete, yet complete under the filtered processes for the stock price and the state variable. I apply the results obtained in the previous section to solve for the optimal consumption and portfolio policies analytically. To quantify the role of imperfect information in the investor's optimal policy, I propose a calibration of the model based on the long-horizon predictability of stock returns. I calibrate the model using the autocorrelation of cumulative returns as studied empirically in Fama and French (1988), Lewellen (2001) among others.

2.3.1 Financial Market

Assume the existence of a money market account where the risk free rate is constant and the existence of one risky securities whose price process satisfies

$$\frac{dS_t}{S_t} = (r + \sigma_S \eta_t) dt + \sigma_S dZ_S,$$

(2.17)

\[A\text{ similar model is used by Wachter (2002) to solve for the optimal consumption and portfolio policies. Wachter assumes market completeness by allowing the shocks to the time-varying component of expected returns and the stock price are perfectly negatively correlated.}\]
such that the Sharpe ratio, \( \eta_t \), is mean reverting, and satisfies

\[
d\eta_t = \kappa (\theta - \eta_t) \, dt + \sigma_\eta dZ_\eta. \tag{2.18}
\]

where \( \kappa \) is the speed of mean reversion and \( \theta \) is the unconditional Sharpe ratio of the risky asset. Assume the shocks to the stock price and shocks to the Sharpe ratio are imperfectly correlated. The correlation coefficient is denoted by \( \rho \). The imperfect correlation between (2.17) and (2.18) implies the market is incomplete. Yet, when the Sharpe ratio is not observable and under the assumptions explained in Section 2.2, the optimization problem under the investor’s information set can be restated in a complete markets framework.

Applying Lemma 2.1 to (2.17) and (2.18) yields the following processes for the stock price and the state variable dynamics respectively:

\[
\frac{dS_t}{S_t} = (r + \sigma_S \tilde{\eta}_t) \, dt + \sigma_S d\tilde{Z}_S, \tag{2.19}
\]

\[
d\tilde{\eta}_t = \kappa (\theta - \tilde{\eta}_t) \, dt + \varepsilon_{\eta t} d\tilde{Z}_S, \tag{2.20}
\]

where

\[
\varepsilon_{\eta t} = \nu_t + \rho \sigma_\eta
\]

and

\[
d\tilde{Z}_S = [(\eta_t - \tilde{\eta}_t) \, dt + dZ_S]. \tag{2.21}
\]

In the previous equations the hat denotes the variables affected by the investor’s estimation process. As shown for the general case in Lemma 2.2, the variance of measurement error for the Sharpe ratio solves the following Riccati Equation

\[
\frac{dv_t}{dt} = -2\kappa v_t + \sigma_\eta^2 - [v_t + \rho \sigma_\eta]^2. \tag{2.22}
\]

Equation (2.22) can be solved following the appendix of Detemple (1986). Under the information set of the investor, shocks to the stock price and the state variable are perfectly correlated. It is straightforward to show Lemma 2.3 and Proposition 2.1 hold.
2.3.2 Optimal Consumption and Portfolio Policy

In order to obtain an analytical solution to the optimal consumption and portfolio choice problem, I assume learning has reached a steady state in which new data and estimation does not reduce the measurement error of the Sharpe ratio. In other words, I assume new return observations contribute to updating the estimated value of the state variable, but do not reduce the variance of the estimation error. Let $\nu_{ss}$ denote the variance of the estimation error under the steady state.\textsuperscript{13} By applying the definition of steady state filtering to (2.22), $\nu_{ss}$ is determined by the quadratic equation

$$0 = -2\kappa \nu_{ss} + \sigma^2 - (\nu_{ss} + \rho \sigma^2)^2.$$  \hspace{1cm} (2.23)

The resulting variance will be the positive root of the quadratic equation obtained from our assumption in (2.22).

Let the superscript $I$ denote expectation operations taken under $F^I$, the information set of the investor. The agent’s optimization problem is to maximize the expected lifetime utility of consumption $J_t$ where

$$J_t = \sup_{\{\alpha_t, C_t\}} E^I_t \left[ \int_t^T e^{-\phi(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \right]$$  \hspace{1cm} (2.24)

subject to the dynamic budget constraint under the estimated processes for the securities and a non-negative wealth constraint. For the remainder of the paper assume $\gamma > 1$, such that the investor has a hedging demand and we can study the range of relative risk aversion which seems to be empirically plausible. The existence of the stochastic discount factor allows us to write the agent’s dynamic budget constraint as a static budget constraint given by

$$E^I_t \left[ \int_t^T \bar{M}_s C_s ds \right] \leq \bar{M}_t W_t.$$  \hspace{1cm} (2.25)

The optimal consumption and portfolio policies are given by the following proposition.

\textsuperscript{13}Barberis (2000) also reduces the state space by not considering variance of estimation error, but he does not assume this occurs due to a steady-state in the equation determining the variance of the estimates. In the Barberis model, steady state learning occurs when parameter uncertainty disappears. My setup allows for the separation of parameter uncertainty and learning about the variance of the estimation error.
Proposition 2.3 Define the functions $H_{t,s}$ and $G_{t,T}$ as follows

$$H_{t,s} = \exp \left[ \frac{1}{\gamma} \left\{ A(s-t) + B(s-t)\tilde{\eta}_t + \frac{1}{2} C(s-t)\tilde{\eta}_t^2 \right\} \right],$$

$$G_{t,T} = \int_t^T H_{t,s} ds$$

The optimal consumption and portfolio policies for an investor maximizing lifetime utility (2.24) subject to the budget constraint (2.25) are

$$C_t = \left( \lambda_0 e^{\theta t} \frac{M_t}{M_0} \right)^{-\frac{1}{\gamma}},$$

$$\alpha_t = \frac{1}{\gamma \sigma_S} \tilde{\eta}_t + \left( \frac{\rho \sigma_\eta + \nu_\sigma}{\sigma_S} \right) \frac{\int_t^T \frac{1}{\gamma} [B(s-t) + C(s-t)\tilde{\eta}_t] H_{t,s} ds}{\int_t^T H_{t,s} ds},$$

respectively.

Proof. Follows directly as an application of Proposition 2.2. Details provided in the Appendix.

Furthermore, due to Corollary 2.1, the consumption to wealth ratio can be stated as

$$\frac{C_t}{W_t} = \left[ \int_t^T H_{t,s} ds \right]^{-1}.$$

The myopic demand of the investor is a function of his estimate for the Sharpe ratio. The hedging demand of the investor has two components: one related to the covariance between the state variable and the stock returns and a second component related to the variance of the estimation error. It is easy to show $\frac{\partial \log C_{t,T}}{\partial \eta_t} \sigma_\eta$ is negative for any $\gamma > 1$ when the market price of risk is positive. Since the variance of the estimation error is positive, accounting for estimation risk lowers the demand for the risky asset. This is the expected result since the variance of the estimation error adds a positively, perfectly correlated component to the diffusion of the state variable. Wachter (2002) shows the hedging demand is positive when the state variable is observable and perfectly negatively correlated with the stock price. In this

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14 The proof follows directly from Wachter (2002).
model, the direction of the hedging demand is unclear even if the state variable is negatively correlated.

2.3.3 Autocorrelation and Predictability

I use the findings on the autocorrelation of returns by Fama and French (1988) and Lewellen (2001) to calibrate for the unobservable Sharpe ratio. The autocorrelation of returns for a given horizon $h$ is estimated by running a regression on the lead $h$ period cumulative returns where the lag $h$ period cumulative return is the dependent variable. Fama and French find negative autocorrelations in return horizons with holding periods beyond a year. Their results suggest the predictable variation of returns can account for 25% of the return variance for holding periods between 3 to 5 years. Using recent data, Lewellen finds a similar relation, but notices reversion is quicker. His evidence suggests a similar magnitude of the return variance is predictable, but mean reversion seems to be strongest for one-year to 18 months holding period. Figure 2-1 shows the return autocorrelations for the raw and excess value-weighted NYSE returns from 1926-1985. This sample is chosen to best match the sample in Fama and French (1988). For short horizon returns, less than one year, the autocorrelation is positive, in line with recent short-horizon momentum evidence. After the one year mark, the autocorrelation curve exhibits a hump-shape pattern with the lowest autocorrelation coefficient achieved for a holding period of 36 months. Figure 2-1 also shows the return autocorrelations for the raw and excess value-weighted NYSE returns from 1932-2001. Following Lewellen (2001), I start the analysis in 1932 so that all returns have at least 5 years of lagged returns available. This curve does not exhibit the positive autocorrelation in short horizon returns seen in the previous figure. It is possible stock return behavior during the Great Depression drives the short-horizon, positive autocorrelation pattern seen in the Fama and French sample. The figure for this sample exhibits a double-hump shape. The first hump occurs in the 15-month holding period, where the autocorrelation of returns is estimated at -0.27. The second hump occurs around the 39 to 42-month holding period. Yet, as in the previous sample autocorrelation move to zero as the holding period increases. This sample implies the temporary component of stock returns

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\(^{15}\)Whether the autocorrelation for long-horizon forecasts implies mean reversion is still an active research question. Richardson and Stock (1989), Kim, Nelson, and Startz (1991), Richardson (1993), and Valkanov (2002) point out reasons why these patterns might not be at odds with the random-walk hypothesis.
exhibits less persistence than it was previously attributed. In terms of portfolio choice, ceteris paribus, lower persistence in returns implies favorable investment opportunities will occur more frequently, but each opportunity will be harder to exploit, because our model of mean reversion in expected returns implies that if investment opportunities are favorable today (expected returns are higher than the long-run value), then investment opportunities should be favorable in the future, but closing in on the long-run value as time passes. Hence, favorable investment opportunities today become much less favorable if mean reversion in returns is greater.

The bottom graph in Figure 2-1 shows the autocorrelations for the raw and excess value-weighted NYSE returns for the period of 1963:05 to 2001:12. This figure implies short-horizon negative autocorrelation in returns followed by long-horizon positive autocorrelation in returns. This evidence is at odds with the common belief returns exhibit short-horizon positive autocorrelations, as shown in various studies on momentum trading strategies, followed by long-horizon negative autocorrelation in returns. In this sample period, the short-horizon negative autocorrelation is strongest at the 18-month holding period. The shift from negative autocorrelation to positive autocorrelation occurs at the 30-month holding period, with positive autocorrelation at its strongest for a 48-month holding period. The estimated autocorrelations during the first thirty month are similar to those for the 1932-2001 holding period, giving further evidence of quick mean reversion in returns. Yet, the positive autocorrelations observed during the next 30 months are in odds with the mean-reversion story. Recent work by Heston and Sadka (2002) suggests stock momentum exhibits periodical patterns. Since the autocorrelation pattern for the last 40 years of data in at odds with a simple mean-reversion or momentum story, a periodicity-based story of returns seems appealing.

I attempt to find a suitable parametrization for (2.17) and (2.18) such that I match the moments of stock returns and volatility as well as match the autocorrelation of return for long horizons observed using only return data. The match to the unconditional moments of stock returns is straightforward. Let \( \phi_h \) denote the autocorrelation of the log cumulative lagged \( h \) month return with the log cumulative lead \( h \) month return. Specifically, let \( \phi_h \) be given by the following regression:

\[
R_S(t, t + h) = \alpha + \phi_h R_S(t - h, t) + \varepsilon
\] (2.30)

where \( R_S(t, t + h) \) is the log cumulative \( h \) month return. In order to match to the autocor-
relation in the data, first I take into account the well-known downward bias in estimating the autocorrelations initially documented in Kendall (1954). Panel A of Table 2.2 shows the autocorrelation estimates obtained from the data as well as the downward bias as shown in the Appendix of Fama and French (1988).

**Proposition 2.4** For the stock price model implied by (2.17) and (2.18), the autocorrelation $\phi_h$, of the log cumulative return from period $t - h$ to $t$ with the log cumulative return from period $t$ to $t + h$ is given by

$$\phi_h = \frac{\left[ \frac{\sigma^2}{2\kappa^3} + \frac{\rho \sigma^2}{\kappa^2} \right] (1 - e^{-\kappa h})^2}{V(h)}.$$  

(2.31)

where

$$V(h) = \frac{\sigma^2}{2\kappa^3} \left(1 - e^{-\kappa h}\right)^2 + \frac{\sigma^2}{\kappa^3} \left[\kappa h - 2 \left(1 - e^{-\kappa h}\right) + \frac{1}{2} \left(1 - e^{-2\kappa h}\right)\right] + 2 \frac{\rho \sigma^2}{\kappa^2} \left[\kappa h + e^{-\kappa h} - 1\right] + h.$$

**Proof.** The Appendix contains a derivation of the result. ■

For the model of stock returns, autocorrelation is a function of the standard deviation of expected returns, the mean reversion in expected returns, and the correlation between the state variable and stock returns. Panel B of Table 2 contains the model-implied autocorrelation obtain by assuming expected returns are a linear function of financial ratios as well as the continuous-time parameters for the mean reversion and standard deviation of the Sharpe ratio as well as the correlation between the state variable and stock returns. The discrete-time estimates are obtained following the VAR methodology of Campbell and Viceira (1999) and presented in Table 2.1. The estimates are converted to their continuous-time equivalent with the procedure outlined in Campbell, Chacko, Rodriguez, and Viceira (2002). The autocorrelations implied by the use of financial ratios as proxies for expected returns do not seem to adequately fit the speed of mean reversion seen in the return data. For example, when using the dividend-yield as a proxy for the state variable, the estimated autocorrelation coefficient are always positive, and the hump in the autocorrelation curve is not observed in less than a five year holding period. Although the use of book-to-marker and earnings-to-price as proxies does seem to capture the negative autocorrelation seen in the data, they also fail to account adequately for the mean reversion speed implied by the return data since the hump in autocorrelation does not occur in
the first five years.

In general, the autocorrelation pattern in the data cannot be fitted by the AR(1) process assumption made for time-varying returns. Under an AR(1) process for expected returns, the autocorrelation pattern can only be positive all the time, positive then negative, or negative all the time. Yet, the data find positive autocorrelation in some cases for longer horizons than those in which negative autocorrelation was observed. Therefore, I cannot exactly use the strategy of Lo and Wang (1995) and pick a few data points in the autocorrelation curve and match to the data via identification. Instead, I resort to minimizing a distance measure between the autocorrelation estimates and the model-implied autocorrelations. Let \( \phi^e_k \) denote the bias-corrected estimates from the return data and \( \phi^M_k \) denote the model implied autocorrelations. In order to obtain \( \kappa, \rho, \) and \( \sigma_\eta \), I use the following distance function to determine the "best-fit" parameters given the bias-corrected autocorrelation estimates:

\[
\{ \kappa, \rho, \sigma_\eta \} \in \arg \min \left[ \sum_{i=1}^{10} \left( \phi^e_{bi} - \phi^M_{bi} \right)^2 \right].
\] (2.32)

Equation (2.32) states the data is fitted up to five year autocorrelations using autocorrelations in six month intervals. The results of the data-fitting process are in Panel C of Table 2.2. For the parameters implied by the autocorrelation pattern in the Fama and French sample we obtain a significantly higher rate of mean reversion than implied by the financial ratios as proxies for expected returns. The results are even more striking when the data is fitted to match the autocorrelation pattern for the data corresponding to the study by Lewellen (2002). In particular, both the mean reversion rate and the standard deviation of expected returns are much higher than those implied by financial ratios. This is not surprising, since persistence in expected returns needs to be smaller so that maximal predictability is achieved in a shorter time horizon. It is possible the reason financial ratios do not seem to fit the autocorrelation well can be attributed to the assumption of an constant relation between expected returns and financial ratios. Recent research points out evidence the relationship between dividend-yield and expected returns is time-varying. Yet, we can also consider expected returns to be a

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16 Please refer to the Appendix for the unconditional and conditional properties of the first and second moments of returns.

17 Xia (2001) entertains the possibility of a time-varying relationship between expected returns and dividend
linear combination of an observable variable and an unobservable variable. In both cases, the
degree of freedom added should be adequate to fit the autocorrelation pattern in the data.
As a final check, I fit the parameters to the autocorrelation curve obtained from the return
data in the sample as the financial ratio data. For the return data from 1963:05-2001:12, the
autocorrelation implies mean reversion and volatility close to the estimates for the financial
ratio models. Furthermore, the correlation between the shock to stock prices and expected
returns is still negative, but considerably higher than the estimate obtained using financial
ratios. 18

In the previous section, I pointed out the difference between the model of returns acting
as the data generating process and the model of return as estimated by the investor. It is
possible differences between the fraction of predictable variation implied by each model can
help us understand the differences in the hedging strategy of the investor conditional on their
information set. Consider the model of stock returns implied by the estimate of the dynamics
of stock price and the Sharpe ratio. Differences in the perception of predictable variation of
returns are to be expected when parameter uncertainty is taken into consideration. Therefore,
there should be a difference in how the investor perceives predictable variation in returns. 19

**Definition 2.3** The model implied \( R^2 \) is the fraction of the log-cumulative variance the model
explains.

\[
R^2 (t, \tau) = \frac{\text{Var} \left( E_t [R_S (t, \tau)] \right)}{\text{Var} \left( R_S (t, \tau) \right)}
\]  \hspace{1cm} (2.33)

where \( R_S (t, \tau) \) is the log return on the asset from time \( t \) to \( t + \tau \).

Campbell (1991) uses this measure to find the implied fraction of variation which can be
explained by predictive variables. Mamaysky (2002) uses the same measure in his research
about predictability of stock prices and the market price of risk when parameters are fully
observable.

yield, but only solves her model under the assumption the relationship is constant and unobservable.

18 The result of finding lower persistence in expected returns by the use of return information only is not
surprising when taking into account the critique in Person, Sarkissian, and Simin (2002). The authors state
data mining as well as an spurious regression bias effect can incorrectly lead to the selection of highly persistent
variable as adequate return predictors.

19 Lewellen and Sbanke (2002) make a similar statement in an equilibrium model of returns. They argue
predictability could be observed by in the data even when the data generating process is a random walk due to
parameter uncertainty.
Proposition 2.5 If stock price dynamics satisfy (2.17) and (2.18), then the $R^2$ for horizon $\tau$ as defined in (2.33) is

$$R^2(t, \tau) = \frac{\sigma^2}{2(1 - e^{-\kappa \tau})^2} \frac{(1 - e^{-\kappa \tau})^2}{\sigma^2 \kappa^2 + \sigma^2 \kappa^2 \left[ \kappa^2 \tau - 2(1 - e^{-\kappa \tau}) + \frac{1}{2}(1 - e^{-2\kappa \tau}) \right] + 2 \lambda \sigma^2 \kappa^2 \left[ \kappa^2 \tau + e^{-\kappa \tau} - 1 \right] + \tau}$$

(2.34)

Proof. Provided in the Appendix. ■

Now consider the case where expected returns are unobservable, then the predictable variance is subject to the inference process of the investor. In particular, we should expect a decrease in predictable variance due to the estimation error since expected returns are not assumed to be precisely estimated, because there is additional noise in the investor's version of the data-generating process.

Corollary 2.2 Denote $\hat{R}^2$ as the fraction of predictable variance implied by the investor's inferred processes for the stock price (2.19) and the Sharpe ratio (2.20). Then the $\hat{R}^2$ for horizon $\tau$ as defined in (2.33) is

$$\hat{R}^2(t, \tau) = \frac{\hat{\sigma}^2}{2(1 - e^{-\hat{\kappa} \tau})^2} \frac{(1 - e^{-\hat{\kappa} \tau})^2}{\hat{\sigma}^2 \hat{\kappa}^2 + \hat{\sigma}^2 \hat{\kappa}^2 \left[ \hat{\kappa}^2 \tau - 2(1 - e^{-\hat{\kappa} \tau}) + \frac{1}{2}(1 - e^{-2\hat{\kappa} \tau}) \right] + 2 \hat{\lambda} \hat{\sigma}^2 \hat{\kappa}^2 \left[ \hat{\kappa}^2 \tau + e^{-\hat{\kappa} \tau} - 1 \right] + \tau}$$

(2.35)

Proof. Apply Proposition 2.5 to the model of stock returns implied by (2.19) and (2.20). ■

Let us consider the implications of (2.35) on the perception of predictability by investors with incomplete information. Figure 2-2 shows the model implied predictable variation in returns under full information for the parameter set implied by the financial ratios and the parameter sets obtained by fitting to the autocorrelation curve. Under the assumption of full information, the financial ratios, when used as proxies for the investment opportunity set, seem to be able to capture most of the variation in long-horizon returns. For all the financial ratios, maximal predictive power seems to occur sometime after a 10-year horizon. Furthermore, the magnitude of predictability is very high, with dividend-yield explaining 70% of return variance at a 10-year horizon. Earnings-to-price and book-to-market explain about 42% and 35% of return variance.
at the same horizon. If this specification is correct, then the optimal portfolio decision under those model will imply high positive hedging demands for long-horizon investors. This is shown more clearly in Table 2.4, Panel B.

The predictable component of variance using only the information in returns for the 1926-1985 and 1932-2001 periods is smaller in magnitude and maximal predictive power is achieved much quicker. For example, in the 1932-2001 sample, maximal predictability occurs at a 12-month horizon, but predictability is stronger by about 12% in the longer data sample. This result agrees with findings that mean reversion in returns is stronger in the pre-World War II era. Yet, in the 1926-1985 sample their seems to be almost no predictability at all. In contrast, the predictability as observed in the 1963-2001 is similar to the predictability observed using financial ratios as proxies for expected returns. In this sample, maximal predictability occurs in a 8-year horizon. Furthermore, about 50% of the variance in predictable under the maximal predictability holding period. Even when mean reversion seems to be stronger for the pre World War II period, predictability in returns seems to have increased considerably after the war. This result suggests return data seems to capture accurately the dynamics of the mean reverting state variable without the need of exogenous information. A similar point is made by Brandt and Kang (2002). The authors show exogenous variable add little explanatory power to a model of time-varying Sharpe ratio, once the latent variable dynamics are considered.

Figure 2-3 shows the graph when the investor cannot observe the instantaneous Sharpe ratio. Is surprising is how strong this effect can be, significantly reducing the maximal predictive power when parameter uncertainty is taken into account. As an example of the effect of imperfect information, let's look at the changes in predictability for the case where the dynamics obtained by using dividend-yield as a proxy for expected returns, but the agent still assumes the state variable is unobservable. In this case, maximal predictability went down from 70% under full information to about 7%. In the analysis of hedging demands, I will show the impact this reduction in predictability has on how much the agent decides to hedge against changes in the investment opportunity set. Although predictability is significantly reduced when parameter unobservability is taken into account, the shape of the model-implied predictability curve remains unchanged. Therefore, while incomplete information significantly reduces how much variance is deemed predictable, it does not significantly change the pattern of predictability.
2.3.4 Hedging Demands

Now I turn my attention to the role of imperfect information in the hedging demand component of the allocation of wealth to the risky asset. Similar to the analysis conducted for model implied forecasting regressions, I study the differences between the hedging demand implied by the data generating process and the hedging demand obtained under the investor's information set. Since the portfolio choice of an investor with perfect information regarding the data generating process cannot be solved in closed-form when utility is over intermediate consumption, I consider the asset allocation strategy of the agent under utility over terminal wealth. This type of utility can also be considered utility over consumption over a specific period in the future.

I solve for the portfolio choice of that agent under perfect information about the process for the Sharpe ratio.

Proposition 2.6 Let the subscript IM denote the policy choices of an investor who observes the data generating process is given by (2.17) and (2.18). The optimal portfolio policy for this investor is

\[
\alpha_t^{IM} = \frac{1}{\gamma \sigma S} \left( \frac{\rho \sigma \eta}{\gamma \sigma S} \right) \left[ B^{IM} (s - t) + C^{IM} (s - t) \eta_t \right],
\]

(2.36)

where \( B^{IM} (\cdot) \) and \( C^{IM} (\cdot) \) are solutions to the following differential equations

\[
\frac{dB^{IM} (\tau)}{d\tau} = B^{IM} (\tau) \left( \frac{(1 - \gamma) \rho \sigma \eta}{\gamma} - \kappa \right) + C^{IM} (\tau) \kappa \theta + B^{IM} (\tau) C^{IM} (\tau) \left( \sigma_\eta^2 + \frac{1 - \gamma}{\gamma} \rho^2 \sigma_\eta^2 \right)
\]

\[
\frac{dC^{IM} (\tau)}{d\tau} = (C^{IM} (\tau))^2 \left( \sigma_\eta^2 + \frac{1 - \gamma}{\gamma} \rho^2 \sigma_\eta^2 \right) + 2C^{IM} (\tau) \left( \frac{(1 - \gamma) \rho \sigma \eta}{\gamma} - \kappa \right) + \frac{1 - \gamma}{\gamma}
\]

with boundary conditions

\[ B (0) = C (0) = 0. \]

Proof. The results follows from Liu (1999). □

Define \( \alpha_t^{PU} \) as the portfolio strategy of the investor following (2.29). I assume the PU-type investor also has utility over terminal wealth. For the rest of this section I assume the investor’s relative risk aversion coefficient is five and show value when the current true value
or estimate of the state variable is equal to its long run mean.\textsuperscript{20} Figures 2-4, 2-5 and 2-6 (upper row) show the hedging demand of the investor as a percentage of the total demand for the risky asset. The hedging demand has two component, the first being related to the covariance of the state variable with the stock price and the second related to sensitivity of the log wealth to consumption ratio to changes in the investment opportunity set. For the set of parameters considered in Table 2.2 Panels B and C, and under the assumption the estimates of the Sharpe ratio for all investors is equal to the long-run value, the hedging demand of the IM-investor is greater than the hedging demand of the PU-investor. For some parameters value, the hedging demand of the investor is not necessarily positive when parameter uncertainty is taken into account. In particular, a reverse horizon effect can occur when the estimation error variance is such that the covariance effect under parameter uncertainty is positive, even if the true covariance is negative. In general, parameter uncertainty does indeed reduce the hedging demand of the investor. Yet, estimation error seems to play a smaller role when the dynamics of expected returns are obtained using the autocorrelation of returns in the longer data samples.

For the dividend-yield model and the model based on the 1963-2001 returns, estimation error plays a significant role in the In all cases, only when imperfect information is accounted for, we see hedging demands correspond to less than 50% of the total demand for risky assets. Under full information, the investor would exhibit an aggressive hedging strategy meant to take advantage of the slow mean reversion and high predictability implied by those models. In contrast, once estimation error is accounted for in such model, hedging demands are negative and strongly so for long-horizon investors. The result follows because the dividend-yield based model and the 1963-2001 returns models are the only case where the steady state variance of estimation error is greater in magnitude than the component of the standard deviation of the Sharpe ratio correlated with the risky asset.

I use the following decomposition to understand the difference in hedging demand between

\textsuperscript{20}These assumption are made to simplify the exposition of the result. Yet, it is clearly possible that differences between the investor's belief regarding the current value of the state variable can drive a further wedge between the decisions of both investors.
investor types:

\[
\alpha_{t}^{\text{hedge,IM}} - \alpha_{t}^{\text{hedge,PU}} = \frac{\rho \sigma_{\eta}}{\sigma_{S}} \left( \frac{1}{\gamma} \frac{\partial \log \phi_{t}^{IM}}{\partial \eta_{t}} - \frac{1}{\gamma} \frac{\partial \log \phi_{t}^{PU}}{\partial \eta_{t}} \right) + \left( \frac{\rho \sigma_{\eta}}{\sigma_{S}} \frac{\rho \sigma_{\eta} + \psi_{SS}}{\sigma_{S}} \right) \frac{1}{\gamma} \frac{\partial \log \phi_{t}^{PU}}{\partial \eta_{t}},
\]

\text{Sensitivity Effect} \quad \text{Covariance Effect} \quad (2.37)

where

\[
\phi_{t}^{IM} = \exp \left[ \left( A_{t}^{IM} (T - t) + B_{t}^{IM} (T - t) \eta_{t} + \frac{1}{2} C_{t}^{IM} (T - t) \eta_{t}^{2} \right) \right],
\]

\[
\phi_{t}^{PU} = \exp \left[ \left( A_{t}^{PU} (T - t) + B_{t}^{PU} (T - t) \tilde{\eta}_{t} + \frac{1}{2} C_{t}^{PU} (T - t) \tilde{\eta}_{t}^{2} \right) \right].
\]

Equation (2.37) shows the difference in the hedging demands of each type of investors can be assigned one of two levers. The first lever, the sensitivity effect, refers to how sensitive the value function is to changes in the state variable. The second lever, the covariance effect, relates to the change in the hedging demand due to difference in the perceived covariance of the state variable to the stock price. Figures 2-4, 2-5 and 2-6 (lower row) show how much of the difference in the hedging demand of the investor with full information, relative to the investor with imperfect information, is due to the role of estimation error in the covariance between the state variable and stock returns. Table 2.3 presents the percentage of the difference due to the covariance effect. Therefore, the remaining percentage must be due to the sensitivity effect. For the financial ratios and the model based on the 1963-2001 return data, the sensitivity effect becomes more important as the investment horizon increases. This result does not apply to the portfolio decisions made under the longer horizon return model. One reason, the sensitivity effect seems to be more important in the financial ratio cases is once again the slow mean reversion implied for expected returns when the dynamics of the financial ratios are assumed to proxy for the dynamics of the investment opportunity set.

Figures 2-7, 2-8, and 2-9 (upper row) show the hedging demand of the investor when intermediate consumption is taken into account and compares to the results when perfect correlation between the state variable and the stock price is assumed.\footnote{Implicitly, the assumption of perfect correlation between the state variable and stock price implies the investor perfectly observes the state variable when learning is in steady state.} The investor under the assump-
tion of market completeness due to perfect correlation is denoted by the superscript $CM$. The decrease in the magnitude of the hedging demand show the importance of accounting for intermediate consumption when one wants to make a quantitative argument regarding hedging demands. Figures 2-7, 2-8, and 2-9 (lower row) show the percentage of wealth each type of investor consumes given the horizon of the investor.

Table 2.3 presents the total demand as a percentage of wealth, hedging demand as percentage of wealth, and hedging demand as percentage of total demand. Panel A describes the portfolio choice of the imperfectly informed investor under utility for intermediate consumption. Panel B describes the portfolio choice of the imperfectly informed investor under utility for terminal wealth. The parameters for the mean reversion and standard deviation of the Sharpe ratio as well as the correlation between the Sharpe ratio and stock returns is taken from Table 2.2. The following parameter values are assumed: $\phi = 0.02/12$, $\gamma = 5$, $\theta = 0.1143$, and $\sigma_S = 0.0426$. When expected returns are not observable, regardless of the estimated dynamics for the Sharpe ratio, the investor would choose fairly similar portfolio policies. In particular, the portfolio decisions derived from the parameters obtain from the return data, are very close to each other. For the dividend-yield based model and the model based on 1963-2001 returns, the allocation to the risky asset involves a negative hedging demand. If investors believe expected returns exhibit the same persistence and volatility as the dividend yield, but assume expected returns are unobservable, then the variance of the estimation error is perceived to be high and dominates the investor's hedging concern.

Table 2.4 shows the total demand as percentage of wealth, hedging demand as percentage of wealth, and hedging demand as percentage of total demand when investors can observe expected returns. Panel A describes the portfolio choice of an investor assuming perfect negative correlation under utility for intermediate consumption. Panel B describes the portfolio choice of the fully investor under utility for terminal wealth. The parameters for the mean reversion and standard deviation of the Sharpe ratio as well as the correlation between the Sharpe ratio and stock returns is taken from Table 2.2. Surprisingly, the asset allocation strategy implied by the financial ratios call for more than 150% of wealth allocated to the risky asset when the investment horizon is long. For the return data, the portfolio does not seem to change radically with investment horizon. The results in Tables 2.3 and 2.4 call into question whether
the horizon effect in asset allocation is really significant. My results seem to point out that when we correctly account for the persistence in expected returns as well as the fact expected returns might not be observable, then the horizon effect is significantly muted.

2.3.5 Remarks on Steady State Learning

A closed-form solution to the investor's consumption and investment problem presented in Section 2.3 is obtained when we assume learning reaches its steady state. In other words, any new observation of the securities will be accounted for by the agent is his new estimates of the unobservable parameters, but the new observation will not contribute in reducing the variance of the estimation error. This assumption begs two questions: (1) How quickly would an agent on average, regardless of prior, reach the steady state in the learning process? (2) Can the estimation risk in the steady state significantly change the investment strategy of the agent? This section provides answers to both questions in the context of the model presented in this section.

To answer the first question, I construct a simulation of how the estimation risk of the agent changes after each observation through time. For the case of stock price predictability, I first obtain the steady state variance of the measurement error and the simulate the learning of the agent under the assumption of priors that are multiples of the steady-state estimation risk. I assume changes in the variance of the estimation error follow (2.22). I assume that new observations are made every 1/10th of a quarter. As expected, from Merton (1980), our results are not sensitive to the assumption of the sampling frequency.

In general, for the parameter sets in Table 2.2, Panels B and C, I find the investor is very close to the steady state solution after observing about 15-25 years of data. Assume the investor has access to the CRSP database, then it is fair to state agents have about 40 years (480 months) of daily data and about 75 years (900 months) of monthly data to earn from before deciding on their consumption and portfolio strategies, thus it is quite believable that a rational agent would achieve a level of learning such that the steady state assumption is innocuous.

In order to understand how the estimation error is reduced with each new observation, I

\footnote{Graphs are omitted in this presentation. They are available from the author by request.}
check the magnitude in which the estimation error variance is reduced with each new observation. I plot the instantaneous reduction in variance for a given point in time. By the time the agent has observed 10 to 20 years of data, the reduction in the variance of the estimates for the unobservable variable is negligible.\textsuperscript{23}

2.4 Conclusion

I study the incomplete markets consumption and portfolio choice optimization problem under partially observable information. I assume state variables are unobservable not spanned by the market securities such that prices serve as signals of the unobservable state variables. Investors infer state variables by looking at current returns and the inferred state variables are spanned by the market securities. Under the investors' information set, the market is complete. This allows me to apply martingale methods to solve for the optimal consumption and portfolio policies.

I consider a example for which the assumption of parameter uncertainty is sensible: unobservable mean reverting expected returns. I apply the theory presented in the paper to the consumption and portfolio choice problem when the current value for the Sharpe ratio, the proxy for the investment opportunity set, is not observable. I calibrate the model to the empirical finding on mean reversion of returns by Fama and French (1988), Lewellen (2001) among others. I find significant quantitative changes in the demand for the risky asset when parameter uncertainty is considered. I complete the analysis with a study of the longevity of learning to validate our assumption regarding steady-state in the learning process and a discussion of the link between hedging demands and the predictability, measured as the predictive component of variance, implied by the model.

In the future, I hope to apply the model presented to this paper to a consumption and portfolio choice variable with multiple predictive variables and considers the role for hedging demands for various linear combinations of such variables. A richer model will be able to better capture the autocorrelation of returns in the data, as well as, give better insight on investment with imperfect information.

\textsuperscript{23}Graphs are omitted in this presentation. They are available from the author by request.
2.5 References


Knox, Thomas, 2002, "Learning How to Invest when Returns are Uncertain," mimeo, Harvard University.


2.6 Appendix

2.6.1 Proof of Lemma 2.3

Let $M_t$ satisfy the Ito process
\[ \frac{dM_t}{M_t} = \mu_{Mt} dt + \sigma_{Mt} dZ_t, \]

By Ito's Lemma
\[ \frac{d(M_tB_t)}{M_tB_t} = (\mu_{Mt} + r_t) dt + \sigma_{Mt} dZ_t. \]

By the Martingale Representation Theorem, for $M_tB_t$ to be a martingale,
\[ \mu_{Mt} = -r_t. \]

Similarly,
\[ \frac{dM_tS_t}{M_tS_t} = \left( a_S(t) + b_S(t) \tilde{X}_t - r_t t + \sigma_S(t) \sigma'_{Mt} \right) dt + (\sigma_S(t) + \sigma_{Mt}) dZ_t, \]

Hence
\[ 0 = a_S(t) + b_S(t) \tilde{X}_t - r_t t + \sigma_S(t) \sigma'_{Mt}, \]
\[ \sigma'_{Mt} = -\sigma_S'(t) (\sigma_S(t) \sigma'_S(t))^{-1} \left( a_S(t) + b_S(t) \tilde{X}_t - r_t \right) = -\tilde{\eta}_t. \]

Taking logs and writing $M_t$ in integral form completes the proof.

2.6.2 Proof of Proposition 2.1

The proof is straightforward because the state variables are spanned by the market securities under $\mathcal{F}^I$ and, by assumption, the market securities span all shocks under $\mathcal{F}^I$. In other words, the rank of the diffusion component of the dynamics of the market securities equals the dimension of the Brownian Motion $\tilde{Z}_t$ and hence $Z_t$ too. A rigorous proof can be obtained by applying the results in Duffie (2001, 6.1).

2.6.3 Proof of Proposition 2.2

Following Lipster and Shirvayev (2001), the optimal filter results in the following dynamics for the stock price
and the state variable:

\[
\begin{align*}
\frac{dS_t}{dt} &= S_t \left[ (a_S(t) + b_S(t) \bar{X}_t) \right] dt + \sigma_S(t) d\tilde{Z}_t, \\
\frac{d\bar{X}_t}{dt} &= \left[ a_X(t) + b_X(t) \bar{X}_t \right] dt + \left[ \sigma_X(t) \sigma_S(t) + \nu_t b'_S(t) \right] \left[ \sigma_S(t) \sigma'_S(t) \right]^{-1} \sigma_S(t) d\tilde{Z}_t,
\end{align*}
\]

where \( \nu_t \), the variance of the estimation error, satisfies the Ricatti Equation

\[
\frac{d\nu_t}{dt} = b_X(t) \nu_t + b'_X(t) \sigma'_S(t) + \sigma_X(t) \sigma'_S(t) + \sigma_W(t) \sigma'_W(t) \\
- \left[ \sigma_X(t) \sigma'_S(t) + \nu_t b'_S(t) \right] \left[ \sigma_S(t) \sigma'_S(t) \right]^{-1} \left[ \sigma_X(t) \sigma'_S(t) + \nu_t b'_S(t) \right].
\]

and

\[
\sigma_S(t) d\tilde{Z}_t = b_S(t) \left( X_t - \bar{X}_t \right) dt + \sigma_S(t) d\tilde{Z}_t
\]

Under the filtered dynamics, the market is complete. I can apply martingale methods to solve for the optimal consumption and portfolio choice.

The investors' value function at time \( t \) is defined as

\[
J \left( W_t, \bar{X}_t, T - t \right) = \sup_{\{a_S, \bar{C}_S\}} E_t^I \left[ \int_t^T e^{-\phi(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \right],
\]

subject to the dynamic budget constraint

\[
dW_t = (r_t W_t - C_t) dt + \alpha'_t W_t \left[ (a_S(t) + b_S(t) \bar{X}_t - r_t) \right] dt + \sigma_S(t) d\tilde{Z}_S.
\]

Define \( \overline{M}_t \) as the state price density at time \( t \), such that

\[
\frac{d\overline{M}_t}{\overline{M}_t} = -r_t dt - \tilde{\eta}_t d\tilde{Z}_S,
\]

where

\[
\sigma_S(t) \sigma'_S(t)\left( a_S(t) + b_S(t) \bar{X}_t - r_t \right) = \tilde{\eta}_t
\]

The consumption strategy is financiable if

\[
E_t^I \left[ \int_t^T \frac{\overline{M}_S}{\overline{M}_t} C_s ds \right] \leq W_t.
\]
The first order condition for consumption is given by

\[ e^{-\phi(s-t)} C_t^{-\gamma} = \lambda_t \frac{M_t}{M_0} \]

\[ C_t = \left( \lambda_t e^{\phi(s-t)} \frac{M_t}{M_0} \right)^{-\frac{1}{\gamma}} \]

(2.38)

The wealth at time \( t \) under the optimal consumption policy can be expressed as

\[ W_t = E_t^I \left[ \int_0^T \left( \frac{M_s}{M_t} \right)^{1-\frac{1}{\gamma}} \left( \lambda_t e^{\phi(s-t)} \frac{M_s}{M_t} \right)^{-\frac{1}{\gamma}} ds \right] \]

Define \( \tilde{M}_u = \exp \left( -\int_0^u r_v dv \right) \tilde{\zeta}_u \). Under the risk-neutral measure of the investor,

\[ \tilde{\xi}_s = \exp \left( \frac{1}{\gamma} \int_s^u \tilde{\eta}_u \| \tilde{\eta}_u \|^2 \, dv - \int_s^u \tilde{\eta}_u \, d\tilde{Z}_v \right) \]

\[ d\tilde{Z}_v = d\tilde{Z}_v + \tilde{\eta}_vdv \]

where \( \tilde{Z}_v \) is the Brownian motion under this measure as defined by Girsanov’s theorem.

Write the wealth process as follows:

\[ W_s = \left( \lambda_t e^{\phi(s-t)} \frac{M_s}{M_t} \right)^{-\frac{1}{\gamma}} \left\{ \int_s^T \exp \left( -\frac{\phi}{\gamma} (u - s) \right) E_t^I \left[ \exp \left( \frac{1-\gamma}{\gamma} \int_s^u r_v dv \right) \left( \frac{\tilde{\xi}_u}{\tilde{\xi}_s} \right)^{-\frac{1}{\gamma}} \right] \right\} \]  

(2.39)

Solving (2.39) requires finding a solution for the conditional expectation. The solution can be found by applying Girsanov’s theorem to find an alternate equivalent martingale measure:

\[ E_t^I \left[ \exp \left( \frac{1-\gamma}{\gamma} \int_s^u r_v dv \right) \left( \frac{\tilde{\xi}_u}{\tilde{\xi}_s} \right)^{-\frac{1}{\gamma}} \right] \]

\[ = E_t^I \left[ \exp \left( \frac{1-\gamma}{\gamma} \int_s^u r_v dv - \frac{1}{2\gamma} \int_s^u \| \tilde{\eta}_u \|^2 \, dv + \frac{1}{\gamma} \int_s^u \tilde{\eta}_u \, d\tilde{Z}_v \right) \right] \]

\[ = E_t^I \left[ \exp \left( \frac{1-\gamma}{\gamma} \int_s^u r_v dv + \frac{1-\gamma}{2\gamma^2} \int_s^u \| \tilde{\eta}_u \|^2 \, dv - \frac{1}{2\gamma^2} \int_s^u \| \tilde{\eta}_u \|^2 \, dv + \frac{1}{\gamma} \int_s^u \tilde{\eta}_u \, d\tilde{Z}_v \right) \right] \]

\[ = E_t^{I_Q} \left[ \exp \left( \frac{1-\gamma}{\gamma} \int_s^u r_v dv + \frac{1-\gamma}{2\gamma^2} \int_s^u \| \tilde{\eta}_u \|^2 \, dv \right) \right] \]

(2.40)

Define the equivalent measure \( Q^\gamma \) such that \( \tilde{Z}_v^Q \), the brownian motion defined under the new measure is given by

\[ d\tilde{Z}_v^Q = d\tilde{Z}_v - \frac{1}{\gamma} \tilde{\eta}_v \, dv. \]
The equality between the second and third equation follow from Bayes' rule.

Let \( D \left( \tilde{X}_s, u - s \right) = E_s^{Q^T} \left[ \exp \left( \frac{1}{\gamma} \int_s^u r_v dv + \frac{1}{2\gamma^2} \int_s^u \| \eta_v \|^2 dv \right) \right] \). The Feynman-Kac formula states equation (2.40) is solved by the partial differential equation

\[
\frac{1}{\gamma} r_s - \frac{1}{2\gamma^2} \eta_s \eta_s = \frac{\partial D}{\partial s} + \frac{\partial D}{\partial \tilde{X}_s} \left( \sigma_X^T(s) + b_X^T(s) \tilde{X}_s \right)
\]

\[
+ \frac{1}{2} \text{tr} \left[ \sigma_X(s) \sigma_S^T(s) + v(s) b_S^T(s) \right] \left[ \sigma_X(s) \sigma_S^T(s) + v(s) b_S^T(s) \right] \frac{\partial^2 D}{\partial \tilde{X}_s^2}
\]

where \( \sigma_X^T, b_X^T \) are the coefficients of the drift for the state variables under the measure \( Q^T \) and the terminal condition is given by

\[
D \left( \tilde{X}_u, 0 \right) = 0
\]

The derivation above shown the expectation is indeed a function of the state variables and time. The wealth process can be stated now as

\[
W_s = \left( \lambda_t \sigma (s - t) \frac{M_t}{M_s} \right)^{-\frac{1}{2}} \int_s^T \left[ \exp \left( -\frac{\phi}{\gamma} (u - s) \right) D \left( \tilde{X}_s, u - s \right) \right] du
\]

(2.41)

From (2.15), the demand for the risky asset is

\[
\alpha'_t = \frac{M_t}{W_t} \frac{\partial W_t}{\partial M_t} \left( \sigma_S(t) \sigma_S'(t) \right)^{-1} \sigma_S(t) \tilde{n}_t + \frac{1}{W_t} \frac{\partial W_t}{\partial \tilde{X}_t} \left( \sigma_S(t) \sigma_S'(t) \right)^{-1} \left[ \sigma_X(t) \sigma_X'(t) + v(t) b_S(t) \right]
\]

Substituting (2.41) in (2.15) gives

\[
\alpha_t = \frac{1}{\gamma} \left( \sigma_S(t) \sigma_S'(t) \right)^{-1} \sigma_S(t) \tilde{n}_t
\]

\[
+ \frac{1}{\gamma} \int_t^T \left[ \exp \left( -\frac{\phi}{\gamma} (u - t) \right) \frac{\partial D \left( \tilde{X}_s, u - t \right)}{\partial \tilde{X}_t} \right] du
\]

\[
+ \frac{1}{\gamma} \int_t^T \left[ \exp \left( -\frac{\phi}{\gamma} (u - t) \right) D \left( \tilde{X}_s, u - t \right) \right] du
\]

\[
\left( \sigma_S(t) \sigma_S'(t) \right)^{-1} \left[ \sigma_X(t) \sigma_X'(t) + v(t) b_S(t) \right]'
\]

\[
= \frac{1}{\gamma} \left( \sigma_S(t) \sigma_S'(t) \right)^{-1} \sigma_S(t) \tilde{n}_t + \frac{\partial \log G_{t,T}}{\partial \tilde{X}_t} \left( \sigma_S(t) \sigma_S'(t) \right)^{-1} \left[ \sigma_X(t) \sigma_X'(t) + v(t) b_S(t) \right]'
\]

The second equality follows from the definition of \( G_{t,T} \).
2.6.4 Proof of Corollary 2.1

Divide (2.39) by (2.38) to obtain the wealth-to-consumption ratio at time $s$

$$
\frac{W_s}{C_s} = \left\{ \int_s^T \exp \left( -\frac{\phi}{\gamma} (u - s) \right) E_s^{Q} \left[ \exp \left( \frac{1 - \gamma}{\gamma} \int_s^u \tau_v dv \right) \left( \frac{\xi_v}{\xi_s} \right)^{-\frac{1}{\gamma}} \right] du \right\} 
= \int_s^T \left[ \exp \left( -\frac{\phi}{\gamma} (u - s) \right) D \left( \tilde{X}_s, u - s \right) \right] du 
= G_{s,T}
$$

The second equality follows from the derivations for Proposition 2.2 and the third equality follows by definition. Assume the initial time to be $t$ to complete the proof.

2.6.5 Proof of Proposition 2.3

The solution to the optimal consumption and portfolio problem under time-varying returns follows closely the derivations by Munk (2002) and Wachter (2002). Following Lipster and Shiryaev (2001), the optimal filter results in the following dynamics for the stock price and the state variable:

$$
\frac{dS_t}{S_t} = (r + \sigma_S \tilde{\eta}_t) dt + \sigma_S d\tilde{Z}_S, \\
\tilde{\eta}_t = \kappa (\theta - \tilde{\eta}_t) dt + (\rho \sigma_\eta + \nu_t) d\tilde{Z}_S,
$$

where $\nu_t$, the variance of the estimation error, satisfies the Ricatti Equation

$$
d\nu_t = \left[ -2\kappa \nu_t + \sigma_\eta^2 - (\rho \sigma_\eta + \nu_t)^2 \right] dt
$$

and

$$
d\tilde{Z}_S = (\eta_t - \tilde{\eta}_t) dt + dZ_S
$$

Under the filtered dynamics, the market is complete. I can apply martingale methods to solve for the optimal consumption and portfolio choice.

The investors' value function at time $t$ is defined as

$$
J(W_t, \tilde{\eta}_t, T - t) = \sup_{\{\phi_s, C_s\}} E_t^I \left[ \int_t^T e^{-\phi(s-t)} C_s^{1-\gamma} \frac{1}{1 - \gamma} ds \right],
$$
subject to the dynamic budget constraint

\[ dW_t = \left( rW_t - C_t \right) dt + \alpha_t W_t \left[ \sigma_S \tilde{\eta}_t dt + \sigma_S d\tilde{Z}_S \right]. \]

Define \( \tilde{M}_t \) as the state price density at time \( t \), such that

\[ \frac{d\tilde{M}_t}{\tilde{M}_t} = -r dt - \tilde{\eta}_t d\tilde{Z}_S, \]

The consumption strategy is financciable if

\[ E_t^f \left[ \int_t^T \frac{\tilde{M}_{s}}{\tilde{M}_t} C_s ds \right] \leq W_t. \]

The first order condition for consumption is given by

\[ e^{-\xi(s-t)} C_s = \lambda_t \frac{\tilde{M}_s}{\tilde{M}_t}, \]

\[ C_s = \left( \lambda_t e^{\xi(s-t)} \frac{\tilde{M}_s}{\tilde{M}_t} \right)^{-\frac{1}{\xi}}. \]

The wealth at time \( t \) under the optimal consumption policy can be expressed as

\[ W_t = E_t^f \left[ \int_t^T \lambda_t^{-\frac{1}{\xi}} e^{-\xi(s-t)} \left( \frac{\tilde{M}_s}{\tilde{M}_t} \right)^{1-\frac{1}{\xi}} ds \right] \]

\[ = \lambda_t^{-\frac{1}{\xi}} \int_t^T e^{-\xi(s-t)} E_t^f \left[ \left( \frac{\tilde{M}_s}{\tilde{M}_t} \right)^{1-\frac{1}{\xi}} \right] ds \]

Let \( \tilde{M}_t = e^{-rt\tilde{\xi}_t} \),

\[ W_t = \lambda_t^{-\frac{1}{\xi}} \int_t^T e^{-\xi(s-t)-r(1-\frac{1}{\xi})(s-t)} E_t^f \left[ \left( \tilde{\xi}_t \right)^{1-\frac{1}{\xi}} \right] ds \]

where

\[ \tilde{\xi}_t = \exp \left( -\frac{1}{2} \int_t^s \tilde{\eta}_u du - \int_t^s \tilde{\eta}_u d\tilde{Z}_S \right). \]

Since

\[ E_t^f \left[ \tilde{\xi}_t \right] = 1 \]

I can apply Bayes' rule to obtain

\[ W_t = \lambda_t^{-\frac{1}{\xi}} \int_t^T e^{-\xi(s-t)-r(1-\frac{1}{\xi})(s-t)} E_t^{fQ} \left[ \left( \tilde{\xi}_t \right)^{-\frac{1}{\xi}} \right] ds \]

\[ \text{71} \]
where $I, Q$ represents an equivalent martingale measure to the investor's original information set. For $I, Q$, the dynamics of the stock price and the state variable are given by

\[
\frac{dS_t}{S_t} = rdt + \sigma_S d\tilde{Z}_S^Q, \\
\eta_t = (\kappa (\theta - \tilde{\eta}_t) - (\rho \sigma_\eta + \nu_t) \tilde{\eta}_t) dt + (\rho \sigma_\eta + \nu_t) d\tilde{Z}_S^Q,
\]

where by Girsanov's Theorem,

\[d\tilde{Z}_S^Q = \tilde{\eta}_t dt + d\tilde{Z}_S.
\]

is a Brownian motion under the measure $I, Q$.

The ratio $\frac{\tilde{\xi}_t}{\xi_t}$ is given by:

\[
\frac{\tilde{\xi}_t}{\xi_t} = \exp \left( -\frac{1}{2} \int_t^s \tilde{\eta}_u^2 du - \int_t^s \tilde{\eta}_u d\tilde{Z}_S \right), \\
= \exp \left( -\frac{1}{2} \int_t^s \tilde{\eta}_u^2 du - \int_t^s \tilde{\eta}_u \left( d\tilde{Z}_S^Q - \tilde{\eta}_u du \right) \right), \\
= \exp \left( \frac{1}{2} \int_t^s \tilde{\eta}_u^2 du - \int_t^s \tilde{\eta}_u d\tilde{Z}_S^Q \right).
\]

Thus

\[E_t^{I, Q} \left[ \left( \frac{\tilde{\xi}_t}{\xi_t} \right)^{-\frac{1}{2}} \right] = E_t^{I, Q} \left[ \exp \left( -\frac{1}{2\gamma} \int_t^s \tilde{\eta}_u^2 du + \frac{1}{\gamma} \int_t^s \tilde{\eta}_u d\tilde{Z}_S^Q \right) \right], \\
= E_t^{I, Q} \left[ \exp \left( \frac{1 - \gamma}{2\gamma^2} \int_t^s \tilde{\eta}_u^2 du - \frac{1}{2\gamma^2} \int_t^s \tilde{\eta}_u^2 du + \frac{1}{\gamma} \int_t^s \tilde{\eta}_u d\tilde{Z}_S^Q \right) \right], \\
= E_t^{I, Q} \left[ \exp \left( -\frac{1}{2\gamma^2} \int_t^s \tilde{\eta}_u^2 du \right) \right],
\]

where

\[d\tilde{Z}_S^Q = d\tilde{Z}_S^Q - \frac{1}{\gamma} \tilde{\eta}_t dt
\]

is a Brownian Motion under the measure $I, Q^\gamma$. The equality between the second and third equation follow from Bayes' rule.

The state variable process is now given by

\[
d\tilde{\eta}_t = (\kappa (\theta - \tilde{\eta}_t) - (\rho \sigma_\eta + \nu_t) \tilde{\eta}_t) dt + (\rho \sigma_\eta + \nu_t) \left( d\tilde{Z}_S^Q - \frac{1}{\gamma} \tilde{\eta}_t dt \right), \\
= (a - b\tilde{\eta}_t) dt + (\rho \sigma_\eta + \nu_t) d\tilde{Z}_S^Q,
\]

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where
\[ a = \kappa \theta, \]
\[ b = \kappa + \left(1 - \frac{1}{\gamma}\right) (\rho \sigma_\eta + v_s). \]

Assume \( v_t = v_{ss} \) such that \( dv_{ss} = 0 \) and \( b = \kappa + \left(1 - \frac{1}{\gamma}\right) (\rho \sigma_\eta + v_{ss}) \). Following Munk (2002) we apply Feynman-Kac formula and define the function \( D(\hat{\eta}_t, s - t) \) as
\[
D(\hat{\eta}_t, s - t) = E_t^{Q_t} \left[ \exp \left( \frac{1 - \gamma}{2\gamma^2} \int_t^s \hat{\eta}_u du \right) \right].
\] (2.42)

Equation (2.42) solves the differential equation
\[
-\frac{1 - \gamma}{2\gamma^2} \eta_t^2 D = \frac{\partial D}{\partial t} + (a - b \eta_t) \frac{\partial D}{\partial \eta_t} + \frac{1}{2} (\rho \sigma_\eta + v_{ss})^2 \frac{\partial^2 D}{\partial \eta_t^2}.
\] (2.43)

under the terminal condition \( D(\hat{\eta}_t, 0) = 1 \). The differential equation (2.43) admits a exponential quadratic solution
\[
D(\hat{\eta}_t, s - t) = \exp \left[ \frac{1}{\gamma} \left\{ A_D (s - t) + B (s - t) \hat{\eta}_t + \frac{1}{2} C (s - t) \hat{\eta}_t^2 \right\} \right]
\]

where
\[
A_D' (\tau) = B (\tau) \kappa \theta + \frac{1}{2} C(\tau) (\rho \sigma_\eta + v_{ss})^2 + \frac{1}{2\gamma} B^2 (\tau) (\rho \sigma_\eta + v_{ss})^2,
\] (2.44)
\[
B' (\tau) = B (\tau) \left[ \frac{1 - \gamma}{\gamma} (\rho \sigma_\eta + v_{ss}) - \kappa \right] + C(\tau) \kappa \theta + \frac{1}{\gamma} B(\tau) C(\tau) (\rho \sigma_\eta + v_{ss})^2,
\] (2.45)
\[
C' (\tau) = 2C(\tau) \left[ \frac{1 - \gamma}{\gamma} (\rho \sigma_\eta + v_{ss}) - \kappa \right] + \frac{1}{\gamma} C^2 (\tau) (\rho \sigma_\eta + v_{ss})^2 + \frac{1 - \gamma}{\gamma}.
\] (2.46)

and the terminal condition for \( D \) can be restated in terms of the terminal conditions for \( A_D, B, \) and \( C \):
\[
A_D (0) = B (0) = C (0) = 0.
\]

Equations (2.44), (2.45), and (2.46) are solved in similar fashion to Kim and Omerg (1996), Wachter (2002),

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and Chacko and Viceira (2001). The solutions to (2.44), (2.45), and (2.46) are

\[
A_D(\tau) = \left[ \frac{1 - \gamma}{\gamma} \left( \frac{2\kappa^2 \theta^2}{\delta^2} + \frac{\gamma^2}{\delta - 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right)} \right) \right]^\tau \\
\times \frac{4^{\frac{1 - \gamma}{\gamma}} \kappa^2 \theta^2 \left( \delta - 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) + \delta}{\delta^3} \left( \delta - 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) + \left( \delta + 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) e^{-\delta \tau}} \\
+ \frac{2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta \right)^2}{\delta^2 - 4 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right)^2} \ln \left( \frac{\delta - 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \left( \delta + 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) e^{-\delta \tau}}{2\delta} \right) \right],
\]

\[
B(\tau) = \frac{4^{\frac{1 - \gamma}{\gamma}} \kappa^2 \theta (1 - e^{-\delta \tau/2})^2}{\delta \left( \delta - 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) + \left( \delta + 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) e^{-\delta \tau}},
\]

\[
C(\tau) = \frac{2^{\frac{1 - \gamma}{\gamma}} (1 - e^{-\delta \tau})}{\left( \delta - 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) + \left( \delta + 2 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right) \right) e^{-\delta \tau}}.
\]

where 

\[
\delta^2 = 4 \left( \frac{1 - \gamma}{\gamma} \epsilon \eta - \kappa \right)^2 - 4 \frac{1 - \gamma}{\gamma^2} \epsilon \eta^2.
\]

The wealth process is now

\[
W_t = \lambda_t^{-\frac{1}{2}} \int_t^T e^{-\frac{\delta}{2} (s-t) - r (1-\frac{1}{2}) (s-t)} G(\tilde{\eta}_t, s - t) \, ds
\]

Define \( H(\tilde{\eta}_t, s - t) \) as

\[
H(\tilde{\eta}_t, s - t) = e^{-\frac{\delta}{2} (s-t) - r (1-\frac{1}{2}) (s-t)} G(\tilde{\eta}_t, s - t).
\]

Then \( H(\tilde{\eta}_t, s - t) \) also has an exponential quadratic solution of the form

\[
H(\tilde{\eta}_t, s - t) = \exp \left\{ \frac{1}{\gamma} \left\{ A(s - t) + B(s - t) \tilde{\eta}_t + \frac{1}{2} C(s - t) \tilde{\eta}_t^2 \right\} \right\},
\]

where

\[
A(\tau) = -\phi \tau + r (1 - \gamma) \tau + A_D(\tau).
\]

I can now apply use the methods in Cox-Huang (1989) to obtain the optimal portfolio policy. Using the
results above, we can write the value function as:

\[
J(W_t, \tilde{\eta}_t, T - t) = E_t \left[ \int_t^T e^{-\phi(s-t)} \frac{\left(\lambda_t e^{\phi(s-t)} \frac{M_s}{M_t}\right)^{1+\frac{\gamma}{1-\gamma}}}{1-\gamma} \, ds \right]
\]

\[
= \frac{\lambda_t}{1-\gamma} \int_t^T e^{-\phi(s-t)} E_t \left[ \left(\frac{M_s}{M_t}\right)^{1+\frac{\gamma}{1-\gamma}} \right] \, ds
\]

\[
= \frac{W_t^{1-\gamma}}{1-\gamma} \left[ \int_t^T e^{-\phi(s-t)} E_t \left[ \left(\frac{M_s}{M_t}\right)^{1+\frac{\gamma}{1-\gamma}} \right] \right]^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma}
\]

\[
= \left[ \int_t^T H (\tilde{\eta}_t, s - t) \, ds \right]^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma}
\]

\[
= G_t^{T, T} \frac{W_t^{1-\gamma}}{1-\gamma}
\]

The optimal portfolio policy for the investor as a function of derivatives of the value function is

\[
\alpha_t = -\frac{JW_t}{W_t J_{Wt} \sigma_s} = -\frac{JW_t}{W_t J_{Wt} \sigma_s} \eta_t
\]

\[
= \frac{\tilde{\eta}_t}{\gamma \sigma_s} + \frac{\epsilon_t}{\gamma \sigma_s} \frac{1}{\partial G_t^{T, T}} \frac{\partial G_t^{T, T}}{\partial \tilde{\eta}_t}
\]

(2.47)

where the derivative of \( G_t^{T, T} \) with respect to the estimate of the Sharpe ratio can be expressed as

\[
\frac{\partial G_t^{T, T}}{\partial \eta_t} = \frac{\partial \left[ \int_t^T H (\tilde{\eta}_t, s - t) \, ds \right]}{\partial \eta_t}
\]

\[
= \int_t^T \frac{\partial H (\tilde{\eta}_t, s - t)}{\partial \tilde{\eta}_t} \, ds,
\]

\[
= \frac{1}{\gamma} \int_t^T \left( B (s - t) + C (s - t) \tilde{\eta}_t \right) H (\tilde{\eta}_t, s - t) \, ds.
\]

(2.48)

Applying (2.48) to (2.47) results in the following expression for the allocation to the risky asset

\[
\alpha_t = \frac{\tilde{\eta}_t}{\gamma \sigma_s} + \frac{\epsilon_t}{\gamma \sigma_s} \frac{1}{\int_t^T H (\tilde{\eta}_t, s - t) \, ds}
\]

2.6.6 Proof of Proposition 2.4

Denote the autocorrelation for the log cumulative return between time \( t - h \) and \( t \) and the log cumulative return
between time $t$ and $t+h$ as $\phi_h$ such that

$$\phi_h = \frac{Cov(RS(t-h), RS(t+h))}{Var(RS(t+h))},$$

by definition. The unconditional variances for the cumulative returns are previously obtained in Appendix C.

For completeness, I restate the result:

$$Var(RS(t,h)) = \sigma_S^2 \left( \frac{\sigma_\eta^2}{2\kappa^3} (1 - e^{-\kappa h})^2 + \frac{\sigma_\eta^2}{\kappa^3} \left[ \kappa h - 2(1 - e^{-\kappa h}) + 1 \right] \left( 1 - e^{2\kappa h} \right) \right) + 2 \frac{\rho \sigma_\eta^2}{\kappa^2} \left[ \kappa h + e^{-\kappa h} - 1 \right] + h.$$  

The unconditional covariance is given by

$$Cov(RS(t-h), RS(t+h)) = Cov \left( \sigma_S \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) e^{-\kappa h} \eta_{t-h} + \sigma_S \sigma_\eta \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) \int_{t-h}^{t} e^{\kappa (s-t)} dZ_\eta, \right),$$

$$\sigma_S \eta_{t-h} \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) + \sigma_S \int_{t-h}^{t} dZ_S + \sigma_S \sigma_\eta \int_{t-h}^{t} \left( 1 - e^{\kappa(s-t)} \right) dZ_\eta,$$

$$= \frac{\sigma_S^2 e^{-\kappa h}}{\kappa} \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) \frac{\sigma_\eta^2}{2\kappa} + \frac{\sigma_S^2}{2\kappa} \sigma_\eta \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) \frac{1}{\kappa} \int_{t-h}^{t} \left( e^{\kappa(s-t)} - e^{2\kappa(s-t)} \right) ds$$

$$+ \rho \sigma_S^2 \sigma_\eta \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) \int_{t-h}^{t} e^{\kappa(s-t)} ds,$$

$$= \frac{\sigma_S^2 \sigma_\eta^2}{\kappa^3} \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) e^{-\kappa h} + \frac{\sigma_\eta^2}{2\kappa} \left( 1 - e^{-\kappa h} \right) \left( 1 - e^{-\kappa h} \right) \frac{1 - e^{-\kappa h}}{2\kappa^2},$$

$$+ \rho \sigma_S^2 \sigma_\eta \left( \frac{1 - e^{-\kappa h}}{\kappa} \right) \left( 1 - e^{-\kappa h} \right),$$

$$= \left[ \frac{\sigma_S^2 \sigma_\eta^2}{2\kappa^3} + \frac{\rho \sigma_S^2 \sigma_\eta^2}{\kappa^2} \right] (1 - e^{-\kappa h})^2.$$

### 2.6.7 Proof of Proposition 2.5

Assume the model for stock prices and mean reverting Sharpe Ratio presented in Section 3. Define $RS(t, \tau)$ as follows

$$RS(t, \tau) = \log \frac{S_{t+\tau}}{S_t} = \int_{0}^{\tau} \left( \frac{r}{2} - \frac{\sigma_S^2}{2} + \sigma_S \eta_{t+v} \right) dv + \int_{0}^{\tau} \sigma_S dZ_{S_{t+v}}$$

76
Then

\[ E_t [R_S (t, \tau)] = \left( r - \frac{\sigma^2 S^2}{2} \right) \tau + \sigma_S \left( \theta \tau + \frac{\eta_t - \theta}{\kappa} (1 - e^{-\kappa \tau}) \right), \]

\[ \text{Var} (E_t [R_S (t, \tau)]) = \frac{\sigma_S^2 \sigma^2_S}{2 \kappa^3} (1 - e^{-\kappa \tau})^2. \]

Let \( \varepsilon (t, \tau) \) be the unexplained component of returns such that \( \varepsilon (t, \tau) = R_t (t, \tau) - E_t [R_t (t, \tau)] \). Then,

\[ \varepsilon (t, \tau) = \sigma_S \left( \int_0^\tau \eta_t + u d\nu_t - \theta \tau \right) - \frac{\eta_t - \theta}{\kappa} (1 - e^{-\kappa \tau}) + \int_0^\tau \sigma_S dZ_{S, t + \nu} \]

\[ = \sigma_S \left( \int_0^\tau \int_0^\nu \sigma_\eta e^{-\kappa (\nu - s)} dZ_{\eta, t + s} d\nu \right) + \int_0^\tau \sigma_S dZ_{S, t + \nu} \]

\[ = \sigma_S \sigma_\eta \int_0^\tau \left( \frac{1 - e^{-\kappa \tau}}{\kappa} \right) dZ_{\eta, t + \nu} + \sigma_S \int_0^\tau dZ_{S, t + \nu} \]

The variance of the unexplained component is given by

\[ \text{Var} (\varepsilon (t, \tau)) = \sigma_S^2 \left[ \frac{\sigma_\eta^2}{\kappa^3} \left( \kappa \tau - 2 (1 - e^{-\kappa \tau}) + \frac{1}{2} (1 - e^{-2\kappa \tau}) \right) + 2 \frac{\rho_\sigma_\eta}{\kappa^2} [\kappa \tau + e^{-\kappa \tau} - 1] + \tau \right] \]

Define \( R^2 (t, \tau) \) as the ratio of variance of returns explained by the model relative to the total variance of returns, then

\[ R^2 (\tau) = \frac{\text{Var} (E_t [R_S (t, \tau)])}{\text{Var} (R_S (t, \tau))} = \frac{\text{Var} (E_t [R_S (t, \tau)]) + \text{Var} (\varepsilon (t, \tau))}{\text{Var} (R_S (t, \tau))} \]

\[ = \frac{\sigma^2_S (1 - e^{-\kappa \tau})^2}{\frac{\sigma_\eta^2}{\kappa^3} (1 - e^{-\kappa \tau})^2 + \frac{\sigma_\eta^2}{\kappa^3} \left( \kappa \tau - 2 (1 - e^{-\kappa \tau}) + \frac{1}{2} (1 - e^{-2\kappa \tau}) \right) + 2 \frac{\rho_\sigma_\eta}{\kappa^2} [\kappa \tau + e^{-\kappa \tau} - 1] + \tau} \]

To obtain the model implied \( R^2 \) for the imperfectly informed investor simply substitute \( \varepsilon_\eta \) for \( \sigma_\eta \) and \( \rho \sigma_\eta \) in the equation presented above.
Figure 2.1: Estimated autocorrelation coefficients obtained using equation (2.30). The top figure shows the autocorrelations using monthly return data from 1926-1985. The middle figure shows autocorrelations using monthly return data from 1932-2001. The bottom figure shows autocorrelations using monthly return data from 1963-2001.
Figure 2-2: Predictable component of variance under full information as defined in Proposition 2.5.
Figure 2.3: Predictable component of variance under incomplete information set as defined in Proposition 2.2.
Figure 2-4: Hedging demand (as % of total demand) and Covariance effect for optimal portfolios obtained for utility over terminal wealth, using the parameters of Table 2 for dividend-yield and book-to-market.
Figure 2-5: Hedging demand (as % of total demand) and Covariance effect for optimal portfolios obtained for utility over terminal wealth, using the parameters of Table 2 for earnings-to-price and returns from 1926 to 1985.
Figure 2-6: Hedging demand (as % of total demand) and Covariance effect for optimal portfolios obtained for utility over terminal wealth, using the parameters of Table 2 for returns from 1932-2001 and 1963-2001.
Figure 2-7: Hedging demand (as % of total demand) and consumption to wealth ratio (%) for optimal portfolios obtained for utility over intermediate consumption, using the parameters of Table 2 for dividend-yield and book-to-market.
Figure 2-8: Hedging demand (as % of total demand) and consumption to wealth ratio (%) for optimal portfolios obtained for utility over intermediate consumption, using the parameters of Table 2 for earnings-to-price and returns from 1926-1985.
Figure 2-9: Hedging demand (as % of total demand) and consumption to wealth ratio (%) for optimal portfolios obtained for utility over intermediate consumption, using the parameters of Table 2 for returns from 1932-2001 and 1963-2001.
Table 2.1
Predictive Regressions for Financial Ratios

This table shows the estimates obtained by running predictive regressions for the log value-weighted returns to the NYSE composite index. The dividend yield, book-to-market, and earnings to price ratios are used as the predictive variables. All data is from 1963:05 to 2001:12. Panel A describes the estimates and standard errors for the regression of log returns on the one month lagged value of the financial ratio. Panel B shows the estimates and standard errors of the coefficients assuming an AR(1) specification for the financial ratio. Panel B also gives the correlation between shocks to the financial ratio and shocks to the stock return.

Panel A: \( r_{S_{t+1}} = \alpha + \beta X_t + \epsilon_{t+1} \)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \sigma_\epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DY</td>
<td>0.04376</td>
<td>0.01018</td>
<td>0.04243</td>
</tr>
<tr>
<td></td>
<td>(0.02016)</td>
<td>(0.00584)</td>
<td></td>
</tr>
<tr>
<td>BM</td>
<td>0.01414</td>
<td>0.00768</td>
<td>0.04248</td>
</tr>
<tr>
<td></td>
<td>(0.00427)</td>
<td>(0.00546)</td>
<td></td>
</tr>
<tr>
<td>EP</td>
<td>0.02857</td>
<td>0.01188</td>
<td>0.04238</td>
</tr>
<tr>
<td></td>
<td>(0.00977)</td>
<td>(0.00575)</td>
<td></td>
</tr>
</tbody>
</table>

Panel B: \( X_{t+1} = \theta + \phi X_t + \eta_{t+1} \)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \theta )</th>
<th>( \phi )</th>
<th>( \sigma_\eta )</th>
<th>( \rho(\epsilon, \eta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DY</td>
<td>-0.01629</td>
<td>0.99564</td>
<td>0.04427</td>
<td>-0.96874</td>
</tr>
<tr>
<td></td>
<td>(0.02104)</td>
<td>(0.0061)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BM</td>
<td>-0.00617</td>
<td>0.99225</td>
<td>0.04925</td>
<td>-0.88928</td>
</tr>
<tr>
<td></td>
<td>(0.00495)</td>
<td>(0.00634)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EP</td>
<td>-0.01910</td>
<td>0.98836</td>
<td>0.05064</td>
<td>-0.85145</td>
</tr>
<tr>
<td></td>
<td>(0.01167)</td>
<td>(0.00688)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 2.2

**Autocorrelations**

This table shows the autocorrelation estimates. Panel A presents the estimates obtained by running long-horizon return regressions. Panel B presents the autocorrelations implied by the estimates of the predictive regressions for κ, σ, and ρ. The estimates for the continuous-time variables are obtained by applying the methods in Campbell, Cao, Rodriguez, and Viceira (2002). Panel C presents the autocorrelations implied by the parameters obtained by solving the minimization problem in equation (32).

#### Panel A: Autocorrelation Estimates using Return Data

<table>
<thead>
<tr>
<th>Period</th>
<th>Return</th>
<th>φ₆</th>
<th>φ₁₂</th>
<th>φ₁₈</th>
<th>φ₂₄</th>
<th>φ₃₀</th>
<th>φ₄₂</th>
<th>φ₴₈</th>
<th>φ₵₄</th>
<th>φ₶₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>1963:05-2001:12</td>
<td>Value-Weighted</td>
<td>-6.28</td>
<td>-15.37</td>
<td>-24.09</td>
<td>-17.47</td>
<td>4.20</td>
<td>18.22</td>
<td>25.43</td>
<td>35.65</td>
<td>32.54</td>
</tr>
<tr>
<td>Bias-Adjustment</td>
<td></td>
<td>1.00</td>
<td>2.00</td>
<td>2.50</td>
<td>3.00</td>
<td>3.50</td>
<td>4.00</td>
<td>5.50</td>
<td>7.00</td>
<td>8.50</td>
</tr>
</tbody>
</table>

#### Panel B: Autocorrelation Estimates using Financial Ratios

<table>
<thead>
<tr>
<th>Ratio</th>
<th>κ</th>
<th>σᵦ</th>
<th>ρ</th>
<th>φ₆</th>
<th>φ₁₂</th>
<th>φ₁₈</th>
<th>φ₂₄</th>
<th>φ₃₀</th>
<th>φ₴₂</th>
<th>φ₴₈</th>
<th>φ₵₄</th>
<th>φ₶₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>DY</td>
<td>0.0044</td>
<td>0.0106</td>
<td>-0.9691</td>
<td>1.44</td>
<td>2.76</td>
<td>3.98</td>
<td>5.10</td>
<td>6.13</td>
<td>7.08</td>
<td>7.95</td>
<td>8.76</td>
<td>9.49</td>
</tr>
<tr>
<td>BM</td>
<td>0.0078</td>
<td>0.0089</td>
<td>-0.8902</td>
<td>-1.66</td>
<td>-3.22</td>
<td>-4.69</td>
<td>-6.07</td>
<td>-7.37</td>
<td>-8.59</td>
<td>-9.74</td>
<td>-10.81</td>
<td>-11.82</td>
</tr>
<tr>
<td>EP</td>
<td>0.0117</td>
<td>0.0143</td>
<td>-0.8534</td>
<td>-1.98</td>
<td>-3.77</td>
<td>-5.38</td>
<td>-6.84</td>
<td>-8.15</td>
<td>-9.32</td>
<td>-10.37</td>
<td>-11.31</td>
<td>-12.14</td>
</tr>
</tbody>
</table>

#### Panel C: Autocorrelation Estimates using estimates from (32)

<table>
<thead>
<tr>
<th>Data</th>
<th>κ</th>
<th>σᵦ</th>
<th>ρ</th>
<th>φ₆</th>
<th>φ₁₂</th>
<th>φ₁₈</th>
<th>φ₂₄</th>
<th>φ₃₀</th>
<th>φ₴₂</th>
<th>φ₴₈</th>
<th>φ₵₄</th>
<th>φ₶₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>1926-1985</td>
<td>0.042</td>
<td>0.011</td>
<td>-0.887</td>
<td>-3.97</td>
<td>-6.54</td>
<td>-8.11</td>
<td>-9.00</td>
<td>-9.42</td>
<td>-9.53</td>
<td>-9.43</td>
<td>-9.21</td>
<td>-8.91</td>
</tr>
<tr>
<td>1932-2001</td>
<td>0.120</td>
<td>0.113</td>
<td>-0.765</td>
<td>-12.05</td>
<td>-15.15</td>
<td>-14.92</td>
<td>-13.65</td>
<td>-12.22</td>
<td>-10.90</td>
<td>-9.77</td>
<td>-8.82</td>
<td>-8.02</td>
</tr>
<tr>
<td>1963-2001</td>
<td>0.010</td>
<td>0.018</td>
<td>-0.646</td>
<td>-0.19</td>
<td>-0.02</td>
<td>-0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Table 2.3
Optimal Portfolio Policies for Imperfectly Informed Investors

This table shows the total demand as % of wealth, hedging demand as % of wealth, and hedging demand as % of total demand. Panel A describes the portfolio choice of the imperfectly informed investor under utility for intermediate consumption. Panel B describes the portfolio choice of the imperfectly informed investor under utility for terminal wealth. The parameters for the mean reversion and standard deviation of the Sharpe ratio as well as the correlation between the Sharpe ratio and stock returns is taken from Table II. The following parameter values are assumed: $\phi = 0.02/12$, $\gamma = 5$, $\theta = 0.1143$, and $\sigma_S = 0.0428$.

<table>
<thead>
<tr>
<th>Data</th>
<th>Total Demand</th>
<th>Hedging Demand</th>
<th>Hedging as % of Total Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.5</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
| **Panel A: Utility over Intermediate Consumption**
| T=              |      |      |     |     |      |      |     |     |      |      |     |     |
| DY              | 52.10| 50.89| 48.12| 46.60| -1.67| -2.95| -5.88| -7.47| -3.00| -5.45| -11.52| -15.16|
| BM              | 57.51| 60.61| 68.15| 72.36| 3.85 | 6.95 | 14.49| 18.70| 6.69 | 11.47| 21.26 | 25.84 |
| EP              | 57.59| 60.48| 66.36| 69.02| 3.93 | 6.80 | 12.70| 15.36| 6.82 | 11.24| 19.14 | 22.25 |
| 1926-1985       | 57.97| 60.14| 62.37| 62.91| 5.54 | 7.71 | 9.94 | 10.48| 9.56 | 12.82| 15.93 | 16.66 |
| 1932-2001       | 68.03| 70.77| 72.70| 73.10| 15.60| 18.34| 20.27| 20.67| 22.93| 25.92| 27.88 | 28.27 |
| **Panel B: Utility over Terminal Wealth**
| T=              |      |      |     |     |      |      |     |     |      |      |     |     |
| DY              | 50.60| 48.36| 43.81| 41.93| -3.06| -5.31| -9.85| -11.73| -6.04| -10.97| -22.48| -27.98|
| BM              | 61.34| 67.49| 82.30| 90.71| 7.67 | 13.83| 28.64| 37.06| 12.51| 20.49| 34.80 | 40.85 |
| EP              | 61.27| 66.46| 75.61| 78.63| 7.61 | 12.80| 21.95| 24.98| 12.42| 19.25| 29.03 | 31.76 |
| 1932-2001       | 75.28| 76.14| 76.20| 76.20| 21.62| 22.48| 22.54| 22.54| 28.72| 29.52| 29.58 | 29.58 |
Table 2.4
Optimal Portfolio Policies under Alternative Specifications

This table shows the total demand as % of wealth, hedging demand as % of wealth, and hedging demand as % of total demand. Panel A describes the portfolio choice of an investor assuming perfect negative correlation under utility for intermediate consumption. Panel B describes the portfolio choice of the fully investor under utility for terminal wealth. The parameters for the mean reversion and standard deviation of the Sharpe ratio as well as the correlation between the Sharpe ratio and stock returns is taken from Table II. The following parameter values are assumed: $\phi = 0.02/12$, $\gamma = 5$, $\theta = 0.1143$, and $\sigma_S = 0.0426$.

<table>
<thead>
<tr>
<th>Data</th>
<th>Total Demand</th>
<th>Hedging Demand</th>
<th>Hedging as % of Total Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T=</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.5  5  15  30</td>
<td>2.5  5  15  30</td>
<td>2.5  5  15  30</td>
</tr>
<tr>
<td>Panel A: Utility under Intermediate Consumption - Perfect Negative Correlation Assumption</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DY</td>
<td>63.63 74.56 116.44 142.02</td>
<td>9.97 20.90 62.78 88.36</td>
<td>15.67 28.03 53.91 62.22</td>
</tr>
<tr>
<td>BM</td>
<td>65.12 75.55 104.78 120.29</td>
<td>11.46 21.88 51.11 66.63</td>
<td>17.60 28.97 48.78 55.39</td>
</tr>
<tr>
<td>EP</td>
<td>71.84 88.51 131.78 149.47</td>
<td>18.18 34.85 78.12 95.81</td>
<td>25.31 39.37 59.28 64.10</td>
</tr>
<tr>
<td>1932-2001</td>
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<td>91.82 118.76 135.25 136.22</td>
<td>63.65 69.37 72.06 72.21</td>
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<tr>
<td>1963-2001</td>
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<td>19.83 41.12 103.49 125.14</td>
<td>26.98 43.38 65.85 69.99</td>
</tr>
<tr>
<td>Panel B: Utility over Terminal Wealth - Full Information</td>
<td></td>
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<tr>
<td>T=</td>
<td>2.5  5  15  30</td>
<td>2.5  5  15  30</td>
<td>2.5  5  15  30</td>
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<tr>
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<td>20.58 46.35 179.25 301.62</td>
<td>27.72 46.34 76.96 84.90</td>
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<tr>
<td>BM</td>
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<td>17.79 34.52 84.57 117.15</td>
<td>24.90 39.15 61.18 68.58</td>
</tr>
<tr>
<td>EP</td>
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<td>25.86 48.70 99.20 116.00</td>
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<tr>
<td>1926-1985</td>
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<td>9.78 12.05 13.00 13.01</td>
<td>15.42 18.34 19.50 19.52</td>
</tr>
<tr>
<td>1932-2001</td>
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<td>59.25 61.70 61.86 61.86</td>
<td>52.47 53.48 53.55 53.55</td>
</tr>
<tr>
<td>1963-2001</td>
<td>73.97 90.40 113.71 116.82</td>
<td>20.31 36.73 60.05 63.16</td>
<td>27.45 40.64 52.80 54.06</td>
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Chapter 3

Strategic Asset Allocation with Labor Income in Stochastic Environments

3.1 Introduction

We study the role of time varying interest rates in the optimal consumption and asset allocation policy of an investor with labor income. We assume a stochastic investment opportunity set in order to consider the asset allocation strategy to both equity and interest-rate sensitive assets such as long-term bonds. It also allows us to consider another way in which human capital is equivalent to holding a risky asset. Even when labor income is riskless, human capital is risky due to its exposure to interest-rate risk. This paper shows how the magnitude of such exposure can determine the allocation policy in both equity and bonds. Furthermore, we find that under reasonable parameter assumptions, the investor's allocation of financial wealth to a long term bond might be hump-shaped. Our results question the monotonic horizon effect for fixed-income assets found in previous work on strategic asset allocation.

The allocation strategies for long-term bonds suggest the sensitivity to interest rate risk in human capital is initially greater than the interest rate risk in total wealth. As the investor nears retirement, the interest rate sensitivity of total wealth decreases at a lower rate than the
interest rate sensitivity of human capital. The shift in the magnitude of the sensitivities explain why the horizon effect might not hold once labor income is considered. This results have not been discussed in the literature previously because most papers study the role of human capital for investment in stocks.\footnote{See Bodie, Merton, and Samuelson (1992), Jaganathan and Kocharlakota (1996), Koo (1998), Viceira (2001)}

We develop a complete markets economy to obtain closed-form solutions for the consumption and portfolio policies. In order to maintain complete markets, we abstract away from idiosyncratic labor income and the moral hazard issue.\footnote{Kimball (1993) shows agents with higher idiosyncratic labor income risk hold a smaller proportion of their wealth in stocks. Viceira (2001) finds employed investor with idiosyncratic labor income still invest more in stocks than employed investors, but increasing idiosyncratic risk reduces their willingness to hold stock. Heaton and Lucas (1997) find idiosyncratic labor income. Koo (1998) shows risk-taking is unambiguously lower when idiosyncratic risk and borrowing constraints are considered.} Although we consider these issue to be important, we believe our results are complementary. Our approach allows us to consider the role of labor supply flexibility, the risk-exposure of human capital, the role of retirement in the investment strategy as well as life-cycle investment in a tractable setting not explored in previous studies.

This paper is an extension of the work on consumption and portfolio choice of Merton (1971), and its extension to labor income presented in Bodie, Merton, and Samuelson (1992). The authors consider consumption and portfolio choice when labor income and human capital are riskless. They find investor will exhibit more conservative investment behavior as they near retirement and riskier human capital should induce investors to hold less of their financial wealth in risky assets. Our paper shows the investor might be willing to invest less conservatively through time initially in order to hedge some of the risk due to human capital.

The structure of the paper is as follows: Section 2 presents the foundations for the model. Section 3 solves the model for the case of inelastic labor supply. Section 4 consider solves the model when labor supply is flexible. Section 5 presents the calibration and results. Section 6 concludes.
3.2 The Model

3.2.1 Stochastic Discount Factor and Interest Rates

We assume a unique stochastic discount factor holds for the economy. The stochastic discount factor allows us to find the unique value for any asset in the market including human capital. Since the focus of this paper is on the role of human capital when the investment opportunity set is stochastic, we assume some of the parameters for the stochastic discount factor are time-varying. Let \( M_t \) denote the stochastic discount factor at time \( t \). Changes in \( M_t \) are given by

\[
\frac{dM_t}{M_t} = -r_t dt - \eta_1 dZ_{1t} - \eta_2 dZ_{2t}.
\]  

(3.1)

The brownian motions \( Z_{1t} \) and \( Z_{2t} \) are assumed to be independent. \( \eta_1 \) and \( \eta_2 \) are the premia associated with each of the independent shocks. We assume the premia to be positive. In this model we assume interest rates are mean reverting. In particular, we use the continuous-time equivalent of an AR(1) process. This process is derived by Vasicek (1977). Let \( r_t \) denote the interest rate at time \( t \). Interest rates movements satisfy

\[
dr_t = \kappa (\theta - r_t) dt + \sigma_1 dZ_{1t} + \sigma_2 dZ_{2t}
\]

(3.2)

where \( \kappa \) is the mean reversion speed for interest rates, \( \theta \) is the long-run interest rate, and \( \sqrt{\sigma_1^2 + \sigma_2^2} \) is the standard deviation of interest rates.

Campbell and Viceira (1999) use a related model to study which investors should allocate funds to long term bonds. We will use the one-factor interest rate model to study how the agent should allocate to bonds. We want to study the allocation strategy, in particular the hedging component, as the agent’s horizon becomes smaller and when the agent’s labor income growth is related to interest rate fluctuations.

We can price long-term bonds via risk-neutral pricing. Let \( P(r_t, m) \) be the price at time \( t \) of a long-term zero coupon bond of unit principal with maturity at time \( t + m \). The price of the bond is given by

\[
P(r_t, m) = E_t^Q \left[ \exp \left( - \int_t^{t+m} r_v dv \right) \right]
\]

(3.3)

where \( E_t^Q [\cdot] \) is the expectation under the risk-neutral measure. From the fixed income pricing
literature, we can use the Feynman-Kac Theorem to obtain the price of the bond in closed form. The price of the bond can be written as an exponential affine function of interest rate:

\[ P(r_t, m) = \exp[A(m) + B(m) r_t] \]  

(3.4)

where \( A(m) \) and \( B(m) \) are solutions to the system of differential equations

\[
\begin{align*}
A'(m) &= B(m) [\kappa \theta - (\sigma_1 r_1 + \sigma_2 r_2)] + \frac{1}{2} B^2 (T - t) \left( \sigma_1^2 + \sigma_2^2 \right), \\
B'(m) &= -1 - B(m) r_t,
\end{align*}
\]

(3.5)

with boundary conditions

\[ A(0) = B(0) = 0 \]  

(3.6)

The boundary conditions (3.6) state the price of the bond at maturity is equal to the cash flow it pays when the bond matures. The solution to (3.5) given (3.6) is

\[
\begin{align*}
A(m) &= -\left[ \frac{\kappa \theta - (\sigma_1 r_1 + \sigma_2 r_2)}{\kappa} \right] \left( m - \frac{1 - e^{-\kappa m}}{\kappa} \right) \\
&\quad + \frac{\sigma_1^2 + \sigma_2^2}{2\kappa^2} \left[ m - 2 \left( \frac{1 - e^{-\kappa m}}{\kappa} \right) + \frac{1 - e^{-2\kappa m}}{2\kappa} \right] \\
B(m) &= -\left( \frac{1 - e^{-\kappa m}}{\kappa} \right)
\end{align*}
\]

We apply Ito's Lemma and obtain the price dynamics for the long-term bond. Changes in the price of the long-term bond are given by

\[
\frac{dP(r_t, m)}{P(r_t, m)} = (r_t + \sigma_r \eta B(m)) dt + \sigma_r B(m) dZ_t
\]

(3.7)

where

\[
\begin{align*}
\sigma_r \eta &= \sigma_1 r_1 + \sigma_2 r_2, \\
\sigma_r dZ_t &= \sigma_1 dZ_{1t} + \sigma_2 dZ_{2t}.
\end{align*}
\]

\(^3\)Duffie and Kan (1996), Dai and Singleton (2003a, 2003b) are excellent references on fixed income pricing theory and evidence.
As expected, the expected premium on the bond depends on the risk through the risk premium assigned to each stochastic shock. The premium for the long-term bond is positive if and only if $\sigma_1 r_1 + \sigma_2 r_2$ is negative. We can think of $B$ as the duration of the bond in the sense that $B$ captures the relationship between volatility in interest rates and volatility for the bond.

### 3.2.2 Labor Income and Human Capital

We consider a model where expected growth in labor income is time-varying. We do this to represent the possibility labor income is tied to the current state of the economy and growth in labor income can change with changes in the business cycle. Let $Y_t$ denote the labor income the investor receives at time $t$. Assume changes in labor income are given by

\[
\frac{dY_t}{Y_t} = (g + g_r r_t) \, dt + \sigma_1 Y \, dZ_{1t} + \sigma_2 Y \, dZ_{2t}. \tag{3.8}
\]

In the model, $g_r$ represents the elasticity of growth in labor income to interest rates. If labor income is non-cyclical, then $g_r = 0$ and we obtain a model of labor income where the growth rate is constant. We assume changes in labor income are correlated with changes in the prices of risky securities. But, because of the multi-shock structure of the model, we do not need to assume perfect correlation between any of the risky assets and labor income. We also assume away any transitory shock component in labor income, and idiosyncratic risk due to labor income. Both these issues are covered by other work on consumption and portfolio decisions and detract from the main purpose of the paper.\footnote{Viceira (1998) shows transitory shocks do not seem to affect the asset allocation decision of the investor. (Mention papers covering idiosyncratic labor income).}

We define human capital as the present value of the labor income stream. Assume the investor always provides one unit of labor until retirement. We will relax this assumption later when we consider the consumption/leisure decision of the investor. Let $R$ denote the time of retirement. Let $H_t$ denote the investor's human capital at time $t$. The investor's human capital is given by

\[
H_t = E_t \left[ \int_t^R \frac{M_s Y_s \, ds}{M_t} \right] \tag{3.9}
\]
The following proposition state human capital as a closed-form solution.

**Proposition 3.1** Assume the investor's labor income at time \( t \) is \( Y_t \) and labor income dynamics are given by (3.8). The investor's human capital is given by

\[
H_t = Y_t \int_{\min(t,R)}^{R} \exp\left( A_H(s-t) + (1-g_r)B(s-t)r_t \right) ds \tag{3.10}
\]

where

\[
A_H(s-t) = (g - \sigma_Y \eta)(s-t) + (g_r - 1) \left[ \frac{\kappa\theta - \sigma_r \eta + \sigma_r \sigma_Y}{\kappa} \right] \left( s - t - \left( 1 - e^{-\kappa(s-t)} \right) \right) \\
+ (g_r - 1)^2 \frac{\sigma_r^2}{2 \kappa^2} \left[ (s-t) - 2 \left( 1 - e^{-\kappa(s-t)} \right) \right] + \left( 1 - e^{-2\kappa(s-t)} \right) \right]
\]

\[
B(s-t) = -\frac{1 - e^{-\kappa(s-t)}}{\kappa}.
\]

**Proof.** The results are obtained through an application of Girsanov’s theorem and Feynman-Kac. The proof is provided in the Appendix. □

We can think of the exponential function inside the integral in (3.10) as the relative value of one unit of labor income at a given time in the future compared to the value of one unit of labor income today. If \( g_r = 1 \), then the present value of human capital is not a function of changes in interest rates. In other words, human capital does not carry interest rate risk when labor income growth is unit elastic to interest rates. Since human capital is an asset for the investor, we would like to understand the dynamics of human capital. In particular, we would like to see if human capital behaves as any known security. If human capital carries interest rate risk, then the amount of interest rate risk carried by a unit of labor income at a future time is proportional to the amount of interest rate a bond with maturity at the time of the future labor income cashflow. This will be the link between human capital as an asset and long-term bonds.

**Proposition 3.2** Assume the value of human capital is given by (3.10). The dynamics of human capital are

\[
\frac{dH_t + Y_t dt}{H_t} = \left( r_t + \sigma_r \eta (1-g_r) \frac{\partial_r H_t}{H_t} + \sigma_Y \eta \right) dt + \left( \sigma_r (1-g_r) \frac{\partial_r H_t}{H_t} + \sigma_Y \right) dZ_t \tag{3.11}
\]
where
\[
\partial_t H_t = Y_t \int_{\min(t,R)}^R B(s-t) \exp(A_H (s-t) + (1 - g_r) B(s-t)r_t) \, ds
\]

**Proof.** Provided in the Appendix. ■

The premium for human capital depends on the exposure to interest rate risk and the stochastic component of labor income. If labor income is locally riskless, then the premium for human capital depends solely on the exposure to interest rate risk. Thus, if labor income is locally riskless, i.e. \( \sigma_1Y = \sigma_2Y = 0 \) human capital is a risky asset similar to a coupon providing long-term bond. If the sensitivity of the growth rate of labor income to interest rate changes is one, i.e. \( g_r = 1 \), then human capital is immunized from interest rate risk. When human capital does not have any interest rate exposure but labor income is stochastic, human capital behaves as a dividend-paying stock. The investor’s asset allocation decision to be sensitive to what type of asset human capital represents. Equation (3.11) shows assumptions on the growth rate of labor income as well as assumption on the interest rate process can significantly change what type of asset human capital represents and therefore should matter in the determination of the investor’s optimal portfolio policy for financial wealth.

### 3.2.3 Assets

We showed in the previous section human capital can be though of a hybrid security with properties similar to those of long-term bonds and stocks. In order to analyze the role of human capital in asset allocation, we assume the investor can allocate her financial wealth to three assets: a money market account, a long-term bond, and equity.

The money market account pays the locally riskless rate continuously. Let \( B_t \) denote the current value of the money market account. Then, changes in the money market account satisfy
\[
\frac{dB_t}{B_t} = r_t dt
\]  

(3.12)

At each point in time, the investor can invest in a zero-coupon long-term bond with time to maturity \( m \). The price of the bond is given in equation (3.3). The investor can also allocate financial wealth to equity. Let \( S_t \) denote the current price of equity. Changes in the price of
equity satisfy the stochastic differential equation

$$\frac{dS_t}{S_t} = (r_t + \sigma S_t \eta) dt + \sigma_1 S_t dZ_{1t} + \sigma_2 S_t dZ_{2t}. \tag{3.13}$$

Although expected returns are time-varying, the excess expected return over the riskless asset is constant. This model for equity has been used in numerous asset allocation studies emphasizing the role of long-term bonds.\(^5\) Since the market has enough tradable securities to span the number of independent stochastic components, we can solve the investor’s consumption and portfolio choice problem via martingale methods as shown in Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987).

### 3.2.4 Investor’s Preferences and Time-Horizon

We assume the investor has utility over consumption and leisure. We normalize the unit of work the investor can provide to one. The utility function of the investor is

$$U(C_t, L_t) = \frac{\left(C_t^\lambda (1 - L_t)^{1-\lambda}\right)^{1-\gamma}}{1 - \gamma} \tag{3.14}$$

where \(C_t\) is consumption at time \(t\), \(L_t\) is the amount of labor done at time \(t\), and \(\lambda\) represents the relative preference for consumption over leisure. As \(\lambda\) approaches 1, equation (3.14) resembles the standard, time-separable, power utility commonly used in the consumption and portfolio choice literature. When the investor’s labor supply is not flexible, we assume \(\lambda = 1\). The investor lives up to time \(T\) and retires at time \(R \leq T\). We can think of retirement as a shift in the preference structure of the investor. When the investor is retired, the investor’s utility function is given by

$$U_R(C_t) = \frac{C_t^{1-\gamma}}{1 - \gamma} \tag{3.15}$$

### 3.2.5 Investor’s Optimization Problem

We now state the investor’s optimization problem subject to her budget constraint.

\(^5\)See Campbell and Viceira (2001), Campbell, Chan, and Viceira (2003), and Brennan and Xia (2000) for examples of work on the role of long-term bond in strategic asset allocation.
Problem 3.1 The investor's consumption and portfolio choice decision is the following.

$$\max_{\{C_t, L_t, a_s, a_P\}} E_0 \left[ \int_0^R e^{-\phi t} U (C_t, L_t) \, dt + \int_R^T e^{-\phi t} U_R (C_t) \, dt \right]$$  \hspace{1cm} (3.16)

subject to the intertemporal budget constraint

$$dF_t = r_t F_t \, dt + \alpha_S F_t \left[ \sigma_S \eta \, dt + \sigma_S \, dZ \right] + \alpha_P F_t \left[ \sigma_P \eta \, dt + \sigma_P \, dZ \right] - (C_t - L_t Y_t) \, dt.$$  \hspace{1cm} (3.17)

where \( \sigma_P \) is the volatility of the long-term bond, and \( F_t \) is the investor's financial wealth.

Let \( W_t \) denote the sum of financial wealth and human capital. We can write the investor budget constraint as a static budget constraint. The dynamic budget constraint presented in equation (3.17) can be restated in static form as:

$$F_t = W_t - H_t$$  \hspace{1cm} (3.18)

$$= E_t \left[ \int_t^T \frac{M_s}{M_t} C_s \, ds \right] - E_t \left[ \int_{\min(t,R)}^T \frac{M_s}{M_t} L_s Y_s \, ds \right]$$

The first order conditions for consumption and labor supply are given by

$$\lambda e^{-\phi t} C_t^{\lambda(1-\gamma)-1} (1 - L_t)^{(1-\lambda)(1-\gamma)} = \varphi_0 \left( \frac{M_t}{M_0} \right)$$  \hspace{1cm} (3.19)

$$(1 - \lambda) e^{-\phi t} C_t^{\lambda(1-\gamma)} (1 - L_t)^{(1-\lambda)(1-\gamma)-1} = \varphi_0 \left( \frac{M_t}{M_0} \right) Y_t$$  \hspace{1cm} (3.20)

When labor supply is not flexible or the investor is retired, the first order condition for consumption collapses to

$$e^{-\phi t} C_t^{-\gamma} = \varphi_0 \left( \frac{M_t}{M_0} \right).$$  \hspace{1cm} (3.21)

Although the first order condition for consumption is the same under retirement and inelastic labor supply, the amount of labor done by the agent is different. If the investor is not retired and her labor supply is inelastic, then \( L_t = 1 \) before retirement and \( L_t = 0 \) after retirement.

The investor's portfolio decision can be obtained by matching the stochastic components of wealth with the stochastic components of the portfolio strategy. The investor's allocation to
equity and the long-term bond must satisfy the following set of equations

\[
\begin{align*}
\alpha S_i F_i \sigma_{1s} + \alpha_P F_i \sigma_{1p} + H_t \sigma_{1H} & = -\frac{\partial W_t}{\partial M_t} M_t \eta_1 + \frac{\partial W_t}{\partial r_t} \sigma_{1r}, \\
\alpha S_i F_i \sigma_{2s} + \alpha_P F_i \sigma_{2p} + H_t \sigma_{2H} & = -\frac{\partial W_t}{\partial M_t} M_t \eta_2 + \frac{\partial W_t}{\partial r_t} \sigma_{2r}.
\end{align*}
\] (3.22)

Since there are two equations and two unknowns, we will find the allocation strategies for the long-term bond and the stock.

### 3.3 Consumption and Portfolio Decisions with Inelastic Labor Supply

First we consider consumption and portfolio decisions when the investor produces one unit of labor until retirement. From proposition 3.1, we know the investor’s human capital is given by

\[
H_t = Y_t \int_{\min(t,R)}^{R} \exp(\log A_H(s-t) + (1-g_r) B(s-t) r_t) ds.
\]

We can use the investor’s first order condition to obtain an expression for total wealth.

**Proposition 3.3** If the investor’s labor supply is inelastic, her total wealth at time \( t \) is given by

\[
W_t = C_t \int_t^T \exp \left[ A_W(s-t) + \left( 1 - \frac{1}{\gamma} \right) B(s-t) r_t \right] ds
\] (3.23)

where \( C_t \) is the investor’s consumption at time \( t \) and \( A_W \) and \( B_W \) are given by

\[
\begin{align*}
A_W(s-t) & = -\frac{\phi}{\gamma} (s-t) + \frac{1-\gamma}{2\gamma^2} \eta^2 - \left( 1 - \frac{1}{\gamma} \right) \left( \frac{\kappa \beta - \left( 1 - \frac{1}{\gamma} \right) \eta \sigma_r}{\kappa} \right) \left[ (s-t) - \frac{1-e^{-\kappa(s-t)}}{\kappa} \right] \\
& \quad + \frac{\sigma_r^2}{2\kappa^2} \left( 1 - \frac{1}{\gamma} \right)^2 \left[ (s-t) - 2 \left( \frac{1-e^{-\kappa(s-t)}}{\kappa} \right) + \left( \frac{1-e^{-2\kappa(s-t)}}{2\kappa} \right) \right] \\
B(s-t) & = -\left( \frac{1-e^{-\kappa(s-t)}}{\kappa} \right)
\end{align*}
\] (3.24)

**Proof.** Provided in the Appendix. ■
Just as the value of human capital takes the form of a dividend bearing asset where the dividend is the investor’s labor income, wealth has a similar form. In this case the investor’s wealth is equivalent to a dividend bearing asset where the dividend is the investor’s consumption stream. It is interesting that wealth does not depend on the growth rate of labor income in direct fashion. This means financial wealth or how the investor’s allocates to financial assets hedges any risks brought on by the inability to tailor labor income through flexibility in providing labor.

Given (3.10) and (3.23) we can find a closed form solution to the asset allocation problem of the investor.

**Proposition 3.4** The allocation of financial wealth to stock and the long-term bond satisfies the following relation.

\[
\begin{align*}
\alpha_{S1}\sigma_{1}\sigma_{1} + \alpha_{P1}\sigma_{1}\sigma_{1} &= \frac{\eta_{1}}{\gamma} \left( 1 + \frac{H_{t}}{F_{t}} \right) + \left( 1 - \frac{1}{\gamma} \right) \left( 1 + \frac{H_{t}}{F_{t}} \right) \left( \frac{\partial_{t} W_{t}}{W_{t}} \right) \sigma_{1r} \quad (3.26) \\
- \frac{H_{t}}{F_{t}} \left( 1 - g_{r} \right) \frac{\partial_{t} H_{t}}{H_{t}} \sigma_{1r} + \sigma_{1Y} \\
\alpha_{S2}\sigma_{2} + \alpha_{P2}\sigma_{2} &= \frac{\eta_{2}}{\gamma} \left( 1 + \frac{H_{t}}{F_{t}} \right) + \left( 1 - \frac{1}{\gamma} \right) \left( 1 + \frac{H_{t}}{F_{t}} \right) \left( \frac{\partial_{t} W_{t}}{W_{t}} \right) \sigma_{2r} \quad (3.27) \\
- \frac{H_{t}}{F_{t}} \left( 1 - g_{r} \right) \frac{\partial_{t} H_{t}}{H_{t}} \sigma_{2r} + \sigma_{2Y} \\
\end{align*}
\]

where

\[
\frac{\partial_{t} W_{t}}{W_{t}} = \frac{\int_{t}^{T} B(s - t) \exp \left[ A_{W} (s - t) + \left( 1 - \frac{1}{\gamma} \right) B(s - t) r_{t} \right] ds}{\int_{t}^{T} \exp \left[ A_{W} (s - t) + \left( 1 - \frac{1}{\gamma} \right) B(s - t) r_{t} \right] ds}
\]

**Proof.** Provided in the Appendix.

Consider the case where stock and interest shocks are independent of each other. Let \( \sigma_{2S} = \sigma_{1r} = 0 \). Equations (3.26) and (3.27) simplify to

\[
\begin{align*}
\alpha_{S1} &= \frac{\eta_{1}}{\gamma \sigma_{1S}} \left( 1 + \frac{H_{t}}{F_{t}} \right) - \frac{H_{t}}{F_{t}} \sigma_{1Y} \quad (3.28) \\
\alpha_{P1} &= \frac{\eta_{2}}{\gamma \sigma_{2r} B(m)} \left( 1 + \frac{H_{t}}{F_{t}} \right) + \left( 1 - \frac{1}{\gamma} \right) \left( 1 + \frac{H_{t}}{F_{t}} \right) \left( \frac{\partial_{t} W_{t}}{W_{t}} \right) \frac{1}{B(m)} \quad (3.29) \\
&\quad - \frac{H_{t}}{F_{t}} \left( 1 - g_{r} \right) \frac{\partial_{t} H_{t}}{H_{t}} \frac{\sigma_{2Y}}{\sigma_{2r} B(m)}
\end{align*}
\]
respectively. When shocks to interest rates and shocks to equity are uncorrelated, the interest rate risk from human capital can only be hedged using long-term bonds. Thus, contrary to popular belief that risk in human capital creates an incentive to change the allocation of wealth in equities, we show human capital can affect allocation to long-term bonds even more. The assumption of uncorrelated shocks between interest rates and stocks in not superfluous. Correlation between the S&P 500 index monthly returns and the 30-day treasury bill is about 0.05 for the sample from 1926 to 2001. Furthermore, the correlation between monthly excess returns in the S&P 500 index and the 30-day treasury bill is -0.05. The data suggest the assumption of no correlation between stocks and interest rates is suitable for the purposes of calibrating asset allocation strategies.

We can rewrite equation (3.28), the allocation of financial wealth to equity, as

$$
\alpha_{St} = \alpha_{St}^{myopic} + \frac{H_t}{F_t} \left( \alpha_{St}^{myopic} \frac{\sigma_{1Y}}{\sigma_{1S}} \right)
$$

(3.30)

where $\alpha_{St}^{myopic}$ is the allocation of financial wealth to stocks by a myopic investor with relative risk aversion $\gamma$. The following proposition states the conditions for the allocation to equity to increase in the presence of human capital:

**Proposition 3.5** Let $\alpha_{St}$ be the optimal allocation of financial wealth to equity as given in (3.28), and define $\alpha_{St}^{myopic}$ as the proportion of financial wealth invested in equity by a myopic investor with relative risk aversion $\gamma$, then

$$
\alpha_{St} > \alpha_{St}^{myopic} \iff \eta_1 > \gamma \rho_S \sigma_Y
$$

(3.31)

where $\rho_S = \frac{\sigma_{1Y}}{\sigma_Y}$ and $\sigma_Y = \sqrt{\sigma_{1Y}^2 + \sigma_{2Y}^2}$.

**Proof.** Follows from equation (3.30). ■

As expected, the correlation between stock volatility and labor income volatility plays a role in determining how much wealth should be allocated to the risky asset. If the correlation between labor income and equity is nonpositive, then the investor will hold more equity than the myopic investor. Version of this result have been obtained previously in the literature. The investor will invest more in equity to hedge its labor income risk and provide the optimal
amount of exposure to that source of uncertainty. Equation (3.31) can be interpreted as a determination of which risk, equity risk or labor income risk, pays better. Consider a model of general equilibrium where the representative agents is endowed with a unit of labor income and a unit of an stochastic endowment such that log consumption follows a random walk with drift. Let financial wealth be represented by the endowment asset and human capital be the present value of labor income. In equilibrium the premia for both asset will be given by

$$E_t \left[ \frac{dP_t}{P_t} - r_t dt \right] = \gamma Cov \left( \frac{dC_t}{C_t}, \frac{dP_t}{P_t} \right).$$

Thus $\gamma \rho \sigma \gamma$ represents the expected premia for holding risk in human capital correlated to the risky asset. If this premia is smaller than the premia offered by holding equity, the investor has an incentive to increase his allocation to the risky asset in order to compensate for the risk being held. Equation (3.31) also sheds light in regards to some arguments on the role of labor income in general equilibrium models with heterogeneous agents. We should expect investors whose labor income is positively correlated with equity to hold less stocks. We should also expect highly risk averse investors to stay away from stocks. Since the holding of equity also depends on the ratio of human capital to financial wealth, in the cross section, the investment in equity of young investors should be more dispersed than that of older investors. To our knowledge, this implication of asset allocation models with labor income has not been tested. Since the myopic investment strategy is equivalent to the strategy of the investor at retirement we have the following corollary to Proposition 3.5.

**Corollary 3.1** Let $\alpha_{St}^R$ denote the proportion of financial wealth allocated to the risky asset at retirement, then

$$\alpha_{St} > \alpha_{St}^R \iff \eta_1 > \gamma \rho \sigma \gamma$$

**Proof.** Follows from equation (3.30) since human capital is zero at retirement. \(\blacksquare\)

We now consider the allocation of financial wealth to the long-term bond. Let $\alpha_{Pt}^{no-labor}$ be the proportion of financial wealth to the long-term bond for an investor with no human capital and facing the same investment opportunity set such that

$$\alpha_{Pt}^{no-labor} = \frac{\eta_2}{\gamma \sigma_2 B(m)} + \left(1 - \frac{1}{\gamma}\right) \frac{\partial_t W_t}{W_t} \frac{1}{B(m)}$$

(3.32)
The following proposition relates the allocation of wealth between both types of investors:

**Proposition 3.6** Let $\alpha_{Pt}$ denote the allocation of financial wealth to the long-term bond as stated in equation (3.29) and $\alpha_{P_t}^{no-labor}$ denote the allocation of financial wealth for an investor with no labor income facing the same investment opportunity set. Then

$$\alpha_{Pt} > \alpha_{P_t}^{no-labor} \iff \frac{\eta_2}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \frac{\partial_r W_i}{W_t} \sigma_{2r} > (1 - q_r) \frac{\partial_r H_t}{H_t} \sigma_{2r} + \rho_B \sigma_Y$$

where $\rho_B = \frac{\sigma_X}{\sigma_Y}$ and $\sigma_Y = \sqrt{\sigma_{1Y}^2 + \sigma_{2Y}^2}$.

**Proof.** Follows from equations (3.29) and (3.32). \qed

For the rest of the section we assume $\sigma_{2r} < 0$ such that the risk premium of the long-term bond is positive. The terms in the left-hand side of the equation is positive by assumption. Thus, if the growth rate of labor income is unit elastic with interest rate changes and the correlation between labor income shocks and interest rate shocks is zero, the investor will allocate more financial wealth to the long term bond when she has positive human capital. If the sensitivity of labor income growth is less than unit elastic, our investor is less compelled to increase its allocation to the long term bond. Positive correlation between the long-term bond and interest rate shocks also decrease the incentive to hold long-term bonds. In the calibration section we find allocation of financial wealth to long-term bonds is reduced in the presence of human capital. Our results suggest human capital is a better substitute for long-term bonds than it is for equity. Although we can find a relationship between the allocation of financial wealth between investor with and without human capital, we cannot make any general assertions in regards to the allocation of financial wealth at retirement relative to the allocation of financial wealth before retirement. The differences between retirement allocation strategies and the strategies while employed are discussed in Section 3.3.

### 3.4 Consumption and Portfolio Decisions with Elastic Labor Supply

In this section we consider how flexible labor supply might affect the asset allocation strategy of the investor. In this case we need to take into account the labor-leisure decision of the investor.
Let $H_t^f$ denote the human capital of the investor accounting for the amount of expected labor done before retirement. Then

$$H_t^f = E_t \left[ \int_{t_{min(t,R)}}^{R} \frac{M_s}{M_t} L_s Y_s ds \right]$$

The following proposition states the relationship between human capital under flexible labor supply and human capital when labor is inelastic.

**Proposition 3.7** Let $H_t^f$ denote the human capital of an investor who can determine how much labor to provide in each period. Let $H_t$ be the investor’s human capital is the unit of labor the investor holds is always supplied. Then

$$H_t^f = H_t + \left( 1 - \frac{1}{\lambda} \right) W_t^E$$

where $W_t^E$ is the amount of wealth expected to be consumed during employment.

**Proof.** Available in the Appendix. ■

Proposition 3.7 states the human capital of an investor with flexible labor supply is equivalent to the human capital of an investor with inelastic labor supply minus a proportion of the present value of the expected consumption stream until retirement. This relationship come from the relationship between the consumption decision and leisure decision of the investor. When the investor’s preference for leisure is greater, the investor’s human capital will be smaller. The presence of the parameter $\lambda$ as the determinant of the difference between the human capital of an investor with utility for leisure and an investor providing his unit of labor inelastically. We can also obtain an expression for total wealth.

**Proposition 3.8** Let $W_t^f$ denote the total wealth of the investor when labor supply is flexible. Denote by $W_t^E$ the present value of the expected consumption stream until retirement and $W_t^R$ the present value of the consumption stream after retirement. The total wealth of the investor is

$$W_t^f = W_t^E + W_t^R$$

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such that

$$W_t^E = \psi \varphi_t^{-\frac{1}{\gamma}} Y_t^{1-\lambda} (1-\lambda)^{-\frac{1}{\gamma}} \int_{\min(t,R)}^R \exp \left[ A_W^f (s-t) + B_W^f (s-t) r_t \right] ds,$$

$$W_t^R = \varphi_t^{-\frac{1}{\gamma}} \int_{\max(t,T)}^T \exp \left[ A_W (s-t) + \left(1-\frac{1}{\gamma}\right) B (s-t) r_t \right] ds.$$

where $\varphi_t$ is the Lagrange multiplier of the static optimization problem and

$$\psi = \left[ \lambda + \gamma - \lambda \gamma \right] \frac{1}{\gamma} \left( 1 - \lambda \right)^{1-\lambda} (1-\gamma)^{\frac{1}{\gamma}}.$$

The function $A_W$ and $B_W$ satisfy (3.24) and (3.25) respectively. The functions $A_W^f$ and $B_W^f$ are given by

$$A_W^f (s-t) = \left[ -\frac{\phi}{\gamma} + \frac{1-\gamma^2}{2\gamma^2} \eta^2 \right] (s-t)$$

$$+ (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \left[ g - \left( 1 - \frac{1}{\gamma} \right) \sigma_y \eta - \left( 1 - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_y^2 \right) \right] (s-t)$$

$$- \left( s-t - \frac{1-e^{-\kappa(s-t)}}{\kappa} \right) \left( 1 - \frac{1}{\gamma} \right) (1-\lambda) g_{tr},$$

$$\times \left[ \frac{\kappa \theta - (1-\frac{1}{\gamma}) \sigma_r \eta - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_r \sigma_y}{\kappa} \right],$$

$$+ \frac{\sigma_r^2}{2\kappa^2} \left[ (s-t) - 2 \left( 1 - \frac{1}{\gamma} \right) \left( 1 - \frac{e^{-\kappa(s-t)}}{\kappa} \right) + \left( 1 - \frac{e^{-2\kappa(s-t)}}{2\kappa} \right) \right] \left( 1 - \frac{1}{\gamma} \right)^2 (1 - (1-\lambda) g_{tr})^2$$

$$B_W^f (s-t) = - \left( 1 - \frac{1}{\gamma} \right) (1-\lambda) g_{tr} \left( 1 - \frac{e^{-\kappa(s-t)}}{\kappa} \right) = \left( 1 - \frac{1}{\gamma} \right) (1 - (1-\lambda) g_{tr}) B (s-t).$$

**Proof.** Provided in the Appendix. □

Wealth needed for retirement consumption acts as wealth needed when the investor does not supply labor income elastically. Yet, wealth needed for consumption during employment does depend on current labor income. We should expect this, since consumption during the employment period is linked to the amount of labor and the current level of labor income.

With the expression we have obtained for human capital and wealth we can proceed to obtain the investors consumption to wealth ratio and asset allocation policies.
Corollary 3.2 Let $C_t^E$ denote consumption while employed and $C_t^R$ denote retirement consumption. The consumption to wealth ratio of an employed investor is

$$\frac{C_t^E}{W_t} = \frac{1}{\int_t^T \exp \left[ A_B(s-t) + (1-\frac{1}{\gamma}) B(s-t) r_t \right] ds + \int_t^T \exp \left[ A_B(s-t) + (1-\frac{1}{\gamma}) B(s-t) r_t \right] ds}.$$ 

The consumption to wealth ratio of a retired investor is

$$\frac{C_t^R}{W_t} = \frac{1}{\int_t^T \exp \left[ A_B(s-t) + (1-\frac{1}{\gamma}) B(s-t) r_t \right] ds}.$$

**Proof.** Follows from first order conditions and Proposition 3.8. ■

The expressions for the consumption to wealth ratio are quite complicated. As expected, the ratio will be a function of how future units of consumption are discounted relative to current consumption. These ratios are finite by construction and imply the proportion of wealth consumed increases with a decrease in the horizon of the investor. If $\gamma = 1$, the consumption to wealth ratio is a function of just time. In other words, the current interest rate and labor income level do not matter in the determination of consumption. This result follows from the fact the substitution and wealth effect for consumption cancel each other out for log utility investors. Everything else equal, an infinitely risk averse investors' consumption to wealth ratio is most sensitive to changes in interest rates. In other words, since the investor want to maintain a constant consumption stream regardless of the shocks to wealth and labor income, the shift in wealth will be fully reflected in the wealth to consumption ratio.

If labor is inelastic and the investor provides her unit of labor at all times before retirement the consumption to wealth ratio during employment can be written as

$$\frac{C_t^E}{W_t} = \frac{1}{\int_t^T \exp \left[ A_B(s-t) + (1-\frac{1}{\gamma}) B(s-t) r_t \right] ds}$$

which the consumption to wealth ratio obtained in Proposition 3.3.

We can now solve for the optimal allocation of financial wealth to equity and the long-term bond.

**Proposition 3.9** The allocation to equity and the long-term bond satisfies the following system
of equations

\[
\alpha_{St}^f \sigma_{1S} + \alpha_{P_t}^f \sigma_{1P} = \frac{\eta_2}{\gamma} \left( 1 + \frac{H_t^f}{F_t} \right) + \left( 1 - \frac{1}{\gamma} \right) \frac{\partial_{\sigma} W_{R}^{E} W_{R}^{E}}{W_{t}^{E}} \frac{F_{t}}{\sigma_{1t}} - (1 - g_r) \frac{\partial_{H} H_{t} H_{t}}{H_{t}} \frac{F_{t}}{\sigma_{1Y}} \\
+ \left( 1 - \frac{1}{\gamma} \right) \frac{\left( 1 - (1 - \lambda) g_r \right)}{\lambda} \frac{\partial_{\sigma} W_{R}^{E} W_{R}^{E}}{W_{t}^{E}} \frac{F_{t}}{\sigma_{1Y}} \\
- \left( 1 - \frac{1}{\lambda} \right) \left( 1 - \frac{1}{\gamma} \right) \frac{W_{E}^{E}}{F_{t}} \sigma_{1Y}
\]

\[
\alpha_{St}^f \sigma_{2S} + \alpha_{P_t}^f \sigma_{2P} = \frac{\eta_2}{\gamma} \left( 1 + \frac{H_t^f}{F_t} \right) + \left( 1 - \frac{1}{\gamma} \right) \frac{\partial_{\sigma} W_{R}^{E} W_{R}^{E}}{W_{t}^{E}} \frac{F_{t}}{\sigma_{2t}} - (1 - g_r) \frac{\partial_{H} H_{t} H_{t}}{H_{t}} \frac{F_{t}}{\sigma_{2Y}} \\
+ \left( 1 - \frac{1}{\gamma} \right) \frac{\left( 1 - (1 - \lambda) g_r \right)}{\lambda} \frac{\partial_{\sigma} W_{R}^{E} W_{R}^{E}}{W_{t}^{E}} \frac{F_{t}}{\sigma_{2Y}} \\
- \left( 1 - \frac{1}{\lambda} \right) \left( 1 - \frac{1}{\gamma} \right) \frac{W_{E}^{E}}{F_{t}} \sigma_{2Y}
\]

where

\[
\frac{\partial_{\sigma} W_{R}^{E}}{W_{t}^{E}} = \frac{\int_{\min(t,R)}^{R} B(s-t) \exp \left[ A_{W}^f (s-t) + \left( 1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) B(s-t) r_t \right] ds}{\int_{\min(t,R)}^{R} \exp \left[ A_{W}^f (s-t) + \left( 1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) B(s-t) r_t \right] ds}
\]

\[
\frac{\partial_{\sigma} W_{R}^{E}}{W_{t}^{R}} = \frac{\int_{\max(t,R)}^{T} B(s-t) \exp \left[ A_{W} (s-t) + \left( 1 - \frac{1}{\gamma} \right) B(s-t) r_t \right] ds}{\int_{\max(t,R)}^{T} \exp \left[ A_{W} (s-t) + \left( 1 - \frac{1}{\gamma} \right) B(s-t) r_t \right] ds}
\]

\[
\frac{\partial_{H} H_{t}}{H_{t}} = \frac{\int_{\min(t,R)}^{R} B(s-t) \exp \left[ A_{H} (s-t) + (1 - g_r) B(s-t) r_t \right] ds}{\int_{\min(t,R)}^{R} \exp \left[ A_{H} (s-t) + (1 - g_r) B(s-t) r_t \right] ds}
\]

Proof. Provided in the Appendix. □
Consider the same assumptions on the variance-covariance matrix of equity volatility made in Section 3.2. Equations (3.33) and (3.34) simplify to

\[
\alpha_{St}^f = \frac{\eta_1}{\gamma \sigma_{1S}} \left(1 + \frac{H_t^f}{F_t}\right) - \frac{H_t}{F_t} \sigma_{1Y} - \left(1 - \frac{1}{\lambda}\right) \frac{W_t^E}{F_t} \sigma_{1Y} \frac{1}{\gamma} \sigma_{1S}
\]

(3.35)

\[
\alpha_{Pt}^f = \frac{\eta_2}{\gamma \sigma_{2rB} (m)} \left(1 + \frac{H_t^f}{F_t}\right) + \left(1 - \frac{1}{\lambda}\right) \frac{1}{\gamma} \frac{\partial_t W_t^R}{W_t^R} \frac{W_t^R}{F_t} - \frac{(1 - \gamma_r) \partial_t H_t}{B (m)} \frac{H_t}{F_t}
\]

(3.36)

Equation (3.35) does not seem very different from the form of equity allocation when labor income is not flexible. This result is not surprising since flexibility in labor income does not vary the per unit risk of labor income. Yet, the investor's ability to change its labor supply given current market conditions seems to play a greater role in the determination of the amount of interest-rate risk per unit of labor income in human capital. The component of the hedging demands due to the sensitivity of employment wealth to interest rates.

**Corollary 3.3** As \( \lambda \to 1 \), the following limits hold:

\[
\lim_{\lambda \to 1} \psi = 1
\]

\[
\lim_{\lambda \to 1} H_t^f = H_t
\]

\[
\lim_{\lambda \to 1} \alpha_{St}^f = \alpha_{St}
\]

\[
\lim_{\lambda \to 1} W_t^f = W_t
\]

\[
\lim_{\lambda \to 1} \frac{\partial_t W_t^R}{F_t} = \frac{1 - (1 - \gamma_r) \gamma_r}{\lambda} \frac{W_t^E}{F_t}
\]

\[
\lim_{\lambda \to 1} \alpha_{Pt}^f = \alpha_{Pt}
\]

**Proof.** The proof are straightforward applications of limit theorems.\(^6\) ■

The results shows that \( \lambda = 1 \) is equivalent to the case of inflexible labor supply studied in the previous section. In this case the investor does not value leisure, but her utility is still

\(^6\) Available upon request.
increasing in consumption, the investor finds it optimal to convert all her human capital into financial wealth since each unit of labor increase income. That income can be invested optimally to manage the risk inherent in holding human capital.

3.5 Calibration and Results

In this section we calibrate the model to parameters for the dynamics of interest rates and stocks implied by securities data. For labor income, we consider various cases for the expected growth, sensitivity of the growth rate to interest rate changes, and labor income volatility. We do not pinpoint one value for each parameter of labor income in order to capture some of the idiosyncracies in optimal investment policies due to investor-specific labor income processes. We also consider various values for the investor's preference and horizon parameters. Table 3.1 provides the values used in the calibrations. We assume shocks to equity and interest rates are independent.

We consider five cases in the calibration of the model to best understand the role of stochastic interest rates and labor supply flexibility. In the first case, we strip the models of all bells and whistles and consider the setup of Bodie, Merton, and Samuelson (1992) where all parameters are constant and the investor's human capital represents a riskless asset. We present both the inelastic and elastic labor supply models under the assumption of a constant investment opportunity set. Case 2, covers the consumption and portfolio strategy of an investor with inelastic labor supply and locally riskless labor income. Case 4 extends Case 2 by considering the consumption and leisure choice of an employed investor. As in Case 2, human capital in Case 4 can be best described as a coupon-bearing long-term bond. Case 3 and Case 5 extend Case 2 and Case 4 respectively by accounting for volatility in labor income. In this cases, human capital acts as a hybrid security with some aspects similar to those of equity as well as exposure to interest rate risk present in long-term bonds.

3.5.1 Case 1: Constant Investment Opportunity Set

We consider the case where labor income is completely riskless and the risk free rate is constant. We assume equity still pays a constant premium relative to the risk free rate. In this setting, the
following equations give the dynamics of the money market account, equity, and labor income respectively:

\[
\frac{dB_t}{B_t} = r dt \\
\frac{dS_t}{S_t} = \mu dt + \sigma dZ, \\
\frac{dY_t}{Y_t} = g dt
\]

The present value of human capital is

\[
H_t = E_t \left[ \int_t^R e^{-r(s-t)} Y_s ds \right] \\
= \frac{Y_t}{r-g} \left[ 1 - e^{-(r-g)(R-t)} \right]
\]

When labor supply is inelastic and the investor must provide one unit of labor at all time before retirement, the allocation of financial wealth to equity is given

\[
\alpha_{St} = \frac{\mu - r}{\gamma \sigma_S^2} \left( 1 + \frac{H_t}{F_t} \right) \tag{3.37}
\]

Since the risk free rate is constant, there is no reinvestment risk in long-term bond strategies. Therefore, the long-term bond is a perfect substitute for a long-term strategy in the money market account. Compared to the asset allocation strategy with no labor income, the investor allocates more financial wealth to the risky asset because human capital is an implicit holding of the riskless asset, therefore an investor who want to optimally allocate total wealth in the risky asset, will allocate more financial wealth to the risky asset in this case. Let \( \alpha_{St}^W \) denote the proportion of total wealth allocated to equity. Equation (3.37) implies

\[
\alpha_{St}^W = \frac{\mu - r}{\gamma \sigma_S^2}
\]

which is the expected demand under a constant investment opportunity set for an investor with risk aversion of \( \gamma \). When labor income is stochastic the investor must consider how the volatility and correlation of labor income can reduce the value of human capital and shift the investment
strategy. Assume the dynamics of labor income satisfy

\[
\frac{dY_t}{Y_t} = g dt + \sigma_Y dZ
\]

When labor income is stochastic, the value of human capital is given by

\[
H_t = \frac{Y_t}{r - g + \sigma_Y \eta} \left[ 1 - e^{-(r-g+\sigma_Y \eta)(R-t)} \right]
\]

The formula for human capital now account for the risk in labor income by assigning a premium to the holding that risk. The optimal allocation to the risky asset is

\[
\alpha_{St} = \frac{\mu - r}{\gamma \sigma_S^2} \left( 1 + \frac{H_t}{F_t} \right) - \frac{\sigma_Y H_t}{\sigma_S F_t}
\]  \hspace{1cm} (3.38)

Table 3.2 shows the optimal allocations for an investor with a relative risk aversion of five. Panel A consider the case where labor income is non-stochastic for various assumption on the retirement horizon and current labor income to financial wealth ratio of the investor. As expected the investor will hold more equity if his current labor income to financial wealth ratio is high and when his retirement horizon is long. We should expect the labor income to financial wealth ratio of the investor to go down as he near retirement because of the human capital the investor has converted to financial wealth during his life. Let assume the diagonal starting from the bottom row and rightmost column towards the top row and the leftmost column roughly represents the conversion of labor income to financial wealth of an investor nearing retirement. It is clear in this case how human capital strengthens the so called horizon effect in equity investment. The horizon effect states younger investor should invest more in equity as reduce their risky portfolio as they near retirement. Panel A shows that even if when the investment opportunity set is non-stochastic, the horizon effect holds because of human capital and the conversion of such capital into financial wealth during the life-cycle. Similarly consider the diagonal starting from the bottom row and leftmost column towards the top row and rightmost column. This shows the strategy of an investor when the market goes down. As long as human capital is available, the investor will allocate about the same amount to equity. This shows labor income can be thought of as a cushion for the investor. In other words, as long as human
capital is a significant component of the investor's total wealth, the investor will risk more of his financial wealth in risky assets.

Panel B considers the case where labor income is negatively correlated with equity. In this case, the horizon effect is even more pronounced than in Panel A. The result is to be expected since equity also serves as a tool to hedge labor income risk. The portfolio of the investor in this case can be seen as a combination of the portfolio obtained in Panel A, plus a positive hedging demand due to labor income risk. Equation (3.38) formalizes the intuition. Panel C considers the case where labor income is positively correlated with equity. Here the horizon effect is muted since the investor already carries a risk similar to that of equity. Had we assumed the investor risk aversion or volatility of labor income to be higher, it is possible the investor would allocate less to equity during employment than during retirement. The intuition behind this result is explained in Proposition 3.5.

For each panel we show the ratio of human capital to financial wealth. The volatility in labor income does not seem to change the ratio much. Therefore, under the parameters we have assumed, the shifts in investment strategies is mostly due to changes in the risk of labor income, not changes in the value of human capital.

Now consider the role of flexibility in labor supply. In this opportunity set the role of human capital is not very different from the previous case. The amount of financial wealth invested in equity is given by

$$
\alpha^f_{St} = \frac{\mu - r}{\gamma \sigma_S^2} \left( 1 + \frac{H^f_t}{F_t} \right) - \frac{H^f_t}{F_t} \frac{\sigma^f_H}{\sigma_S} + (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \frac{W^E_t}{F_t} \frac{\sigma_Y}{\sigma_S} \tag{3.39}
$$

where by Proposition 3.7

$$
H^f_t \sigma^f_H = H_t \sigma_Y + \left( 1 - \frac{1}{\lambda} \right) W^E_t (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_Y
$$

We can rewrite (3.39) as

$$
\alpha^f_{St} = \frac{\mu - r}{\gamma \sigma_S^2} \left( 1 + \frac{H^f_t}{F_t} \right) - \frac{H_t}{F_t} \frac{\sigma_Y}{\sigma_S} - \left( 1 - \frac{1}{\lambda} \right) \left( 1 - \frac{1}{\gamma} \right) \frac{W^E_t}{F_t} \frac{\sigma_Y}{\sigma_S} = \alpha_{St} + \left( 1 - \frac{1}{\lambda} \right) \left( \frac{W^E_t}{F_t} \right) \left( \frac{\mu - r}{\gamma \sigma_S^2} - \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_Y}{\sigma_S} \right). \tag{3.40}
$$
From equation (3.40), the allocation to equity under flexible labor is greater than the allocation to equity under inelastic labor supply when

$$
\eta < (\gamma - 1) \sigma y
$$

Flexibility in labor supply leads to greater equity holdings when leisure and consumption are substitutes and the elasticity of substitution is high. Table 3.3 presents the allocation of financial wealth to equity and the amount of expected human capital when $\lambda = 0.8$. The panels in the table follow the same structure as the panels in Table 3.2. For all panels, the amount of financial wealth invested in equity is less or equal to the amount of financial wealth invested in equity in Table 3.2. The result is not surprising since the condition stated above does not hold for the parameters selected and the investor earn less human capital in this case. As $\lambda \to 1$, the allocation strategies converge.

3.5.2 Case 2: Inelastic Labor Supply, Locally Riskless Labor Income

We extend the previous case and allow for stochastic interest rates. We also allow for labor income growth to change according to the level of the interest rate. But we assume labor income in this case is not stochastic. In other words, labor income is locally riskless. Assume $\sigma y = 0$, then Proposition 3.1, the closed-form solution to human capital in this setting. Proposition 3.2 provides the dynamics for human capital. If human capital is exposed to interest rate risk, then human capital behaves like a long-term bond with coupon payments. The intuition for why labor income would represent a long-term bond in this case is straightforward: The risk in a long-term bond has to do with the exposure to changes in interest rate during the tenure of the bond. Similarly, the risk in human capital has to do with exposure to changes in interest rate before the investor decides to retire. The investor's optimal allocation to equity and the
long-term bond are given by

\[ \alpha_{St} = \frac{\eta_1}{\gamma \sigma_{1S}} \left( 1 + \frac{H_t}{F_t} \right) \]

\[ \alpha_{Pt} = \frac{\eta_2}{\gamma \sigma_{2r} B(m)} \left( 1 + \frac{H_t}{F_t} \right) + \left( 1 - \frac{1}{\gamma} \right) \left( 1 + \frac{H_t}{F_t} \right) \left( \frac{\partial_r W_t}{W_t} \right) \frac{1}{B(m)} 

- \frac{H_t}{F_t} \left( 1 - g_r \frac{\partial_r H_t}{H_t} \right) \]

In Table 3.4, we present the optimal allocations to equity and the long-term bond for an investor with \( \gamma = 5 \). In Panel A, we consider the allocation strategy of an investor whose current labor income to financial wealth ratio is 0.05. Panel B does the same for \( Y_t = 0.10 \), and Panel C shows the strategies when \( Y_t = 0.20 \). Regardless of the assumption on current labor income, the allocation to equity increases with the retirement horizon. In other words, the horizon effect in equity investment still holds when human capital is risky, but labor income is riskless. Since shocks to interest rates are independent of shocks to equity, allocation to equity depends on the risk aversion adjusted premia of the asset and the leverage in wealth due to human capital. Thus, we should expect the horizon effect to hold without question in this case.

The allocation to the long-term bond can be decomposed into a myopic component and two hedging demands. The first hedging demand is related to the exposure of interest rate risk in total wealth. The second hedging demand represents the interest rate risk already held by the investor through her human capital. The difference between these two components determines the effective hedging demand for long-term bonds. Table 3.5 shows the decomposition of the hedging demands into these two components, the hedging demand due to total wealth and the hedging demand due to the investor’s current human capital.

The structure of Table 3.5 follows the structure of Table 3.4. If human capital is riskless, in other words, if \( g_r = 1 \), then human capital serves as a leverage device in the allocation strategy of the investor. In other words, the allocation if measured in terms of total wealth would be the same as that of an investor with no human capital under the same investment opportunity set. As long as \( g_r < 1 \), Table 3.5 shows human capital is equivalent to holding a long-term bond in this setting. Although the result follows from the fact we are considering a one-factor model of interest rates, and thus all bonds are perfectly correlated, we believe the intuition is
robust to the model. What matters for the allocation strategy is the exposure to the risks in the interest rate process. As long as human capital represents an exposure to those factors, the asset allocation strategy of the investor will consider human capital an implicit holding in a bond-like asset. We find it interesting that even with the "leverage" effect of human capital, allocation to the long-term bond does not increase considerably whenever human capital itself represents an exposure to interest rate risk. For example, if labor income grows at a constant rate, the allocation to the long-term bond is not monotonic on the retirement horizon. Figure 3-1 shows the allocation of financial wealth to the long-term bond and equity as a function of the investment horizon of the investor. For $g_r = 0$ and $g_r = -0.5$, the allocation to the long-term bond initially increases with investment horizon and then decreases. This implies the sensitivity to interest rate risk of wealth is initially greater than that of human capital, but at some point in the horizon the magnitudes shift and sensitivity to interest rate in human capital is greater. As expected, the allocation strategy is invariant to labor income once the investor is retired. To the extent of our knowledge, we are the first to document the possibility that normative portfolio choice under linear-stochastic investment opportunity set, need not imply monotonic investment/retirement horizon effects.

In Table 3.6, we consider the role of relative risk aversion in the asset allocation strategy of the investor. In order to study the role of interest rate risk, we assume $g_r = 0$. The results suggest allocation to equity is highly sensitive to the risk aversion parameter, but allocation to the long-term bond is relatively invariant to the risk aversion parameter. This suggests the difference between the interest rate exposure to total wealth the investor desires to hedge and the interest rate exposure due to human capital changes about as much as the myopic demand component for long-term bonds changes when the risk aversion parameter shifts. The hump in the allocation in long-term bonds seems to be more pronounced when the labor income to financial wealth ratio is greater. The results also suggest the horizon at which the allocation to long-term bonds is highest increases with risk aversion and decreases with current labor income.

Similar to Case 1, we consider the possibility of the investor labor income to financial wealth ratio increasing as her horizon shortens. We can roughly approximate the portfolio behavior of such as investor by looking at the allocation strategy of the diagonal beginning at the left-most column and bottom row towards the right-most column and top row. This capture to some
extent the shift in the ratio of financial wealth to human capital as the investor nears retirement. The results in Table 3.6 suggest that once we account for this shift, both the allocation to equity and the long-term bond exhibit a non-monotonic horizon effect. Regardless on the assumption on the coefficient of relative risk aversion, the results suggest the investor first increases the holding of both assets during the life-cycle, and then proceeds to reduce their holdings as they near retirement. Our calibration brings to question the results of most asset allocation papers under stochastic investment opportunities sets, by suggesting the horizon effect disappears once labor income is considered.

3.5.3 Case 3: Inelastic Labor Supply, Stochastic Labor Income

We explore the role of stochastic labor income in the consumption and portfolio choice strategy of an investor with no labor income flexibility. When labor income is stochastic, the allocation of financial wealth to equity and the long-term bond is given by

$$\alpha_{St} = \frac{\eta_1}{\gamma \sigma_{1S}} \left( 1 + \frac{H_t}{F_t} \right) - \frac{H_t \sigma_{1Y}}{F_t \sigma_{1S}}$$

$$\alpha_{Pt} = \frac{\eta_2}{\gamma \sigma_{2r} B(m)} \left( 1 + \frac{H_t}{F_t} \right) + \left( 1 - \frac{1}{\gamma} \right) \left( 1 + \frac{H_t}{F_t} \right) \left( \frac{\partial_t W_t}{W_t} \right) \frac{1}{B(m)}$$

$$\frac{H_t}{F_t} \left( \frac{1 - g_t \partial_t H_t}{H_t} + \frac{\sigma_{2Y}}{\sigma_{2r} B(m)} \right)$$

If all parameters are the same as in Case 2, we can think of the allocations strategies above as linear combinations of the allocation strategies under locally riskless labor income and a hedging demand due to the stochastic component of labor income. In other words, the addition of an stochastic component in labor income means the investor is implicitly holding additional risk and might determine it is optimal to hedge that risk.

In Table 3.7, we assume $Y_t = 0.10, \gamma = 5$ and consider the portfolio allocations of the investor under various assumptions for the investment horizon, the sensitivity of labor income to changes in the interest rate, and some assumptions on the stochastic component of labor income. As in Case 1, the investor allocates more (less) to equity if labor income is negatively (positively) correlated with the stochastic component of equity. Similarly, the investor will allocate more (less) to the long-term bond if labor income is negatively (positively) correlated with stochastic changes.
in interest rates. A decrease in the sensitivity of labor income to interest rates leads to significant reductions in the demand for the long-term bond and negligible changes in the demand for equity. Since the changes in the sensitivity of labor income to interest rate, represent changes in how much human capital serves as a substitute asset to long-term bonds. Our results suggest human capital can be generally considered to be a better substitute to interest-rate sensitive assets.

Figure 3-3 plots the allocation of financial wealth to both assets as a function of investment horizon under various assumption for the shock to labor income correlated with equity. We assume \( g_r = 0 \) and labor income is uncorrelated to interest rate risk. Figure 3-3 shows the allocation to the long-term bond does not change much with changes in the magnitude of labor income volatility. The plot confirms allocation to equity increases (decreases) when labor income volatility correlated to equity is negative (positive).

Figure 3-5 considers allocation strategy when labor income volatility is independent of equity volatility and correlated with interest rate risk. If labor income volatility is positively correlated with interest rate volatility, then the hump-shape allocation to long-term bond is clearer, since human capital represents more exposure to that risk factor, thus the investor optimally wants to invest less in the long-term bond in this case.

3.5.4 Case 4: Elastic Labor Supply, Locally Riskless Labor Income

We allow for flexible labor income when labor income is locally riskless. In this case the investor has a second-tool for interest rate risk management, the amount of labor the investor supplies to the economy at each time before retirement. For the results presented in Table 3.8, we assume \( Y_i = 0.10 \) and \( \gamma = 5 \). We allow for various assumption on \( \lambda \), the relative preference for consumption over leisure and \( g_r \) the sensitivity of labor income to interest rate changes. The
optimal allocation to equity and the long-term bond is given by

\[
\alpha^{f}_{St} = \frac{\eta_1}{\gamma \sigma_{1S}} \left( 1 + \frac{H^f}{F_t} \right)
\]

\[
\alpha^{f}_{Pt} = \frac{\eta_2}{\gamma \sigma_{2R} B(m)} \left( 1 + \frac{H^f}{F_t} \right) + \left( 1 - \frac{1}{\gamma} \right) \frac{\partial_t W^R_t}{W^R_t} \left( \frac{W^R_t}{F_t B(m)} \right) - \left( 1 - g_r \right) \frac{\partial_t H_t}{H_t} \frac{H_t}{F_t B(m)} + \left( 1 - \frac{1}{\gamma} \right) \frac{1 - (1 - \lambda) g_r}{\lambda B(m)} \left( \frac{1 - (1 - \lambda) g_r}{\lambda B(m)} \right) \frac{\partial W^E_t W^E_t}{W^E_t F_t}
\]

A higher preference for leisure seems to reduce the demand for equities. This result follows from the fact the portfolio depend on the amount of human capital expected to be used for labor. Since a higher preference for leisure, implies lower human capital, the asset allocation seems to trend downwards.

Although the result is obvious for equities, it is not quite so for the long-term bond. When \( g_r = -0.5 \), in other words, when human capital is most sensitive to interest rate risk, the allocation to the long-term bond increases with decrease in \( \lambda \). It is in this case where the management in exposure of interest rate risk creates a demand shift greater than the shift created by the reduction of human capital due to higher preference for leisure. Although equity seems to maintain a horizon effect, i.e. investor allocate more financial wealth to equity early in life, we still find the horizon effect need not hold for long-term bonds. Depending on the sensitivity of labor income to interest rate risk we can even obtain the desired negative horizon effect where investor’s allocate more to bonds as they near retirement. After retirement, the investor will proceed to reduce their holding on the bond.

Figure 3-7, shows the allocation to equity and the long-term bond as a function of the investment horizon under various values for \( \lambda \) when labor income growth is riskless and grows at a constant rate, and the investor’s relative risk aversion coefficient is five. The figure shows allocation to both assets is higher when the preference for consumption relative to leisure is greater. In our paper, the effect might be a result of the decrease in human capital due to the preference for leisure. In Figure 3-8, we attempt to control for this possibility by allowing the amount of labor the investor supplies in the inelastic labor supply case vary. In terms of investing in equity, both graphs show similar changes. The allocation decreases as the amount of labor supply decreases. The allocation to the long-term bond under inflexible labor supply
decreases at a much greater rate when less labor is supplied than in the flexible supply case. By controlling for the magnitude of labor output, we can obtain that flexibility in labor supply allows the investor to allocate for risky assets more aggressively. The result is not surprising, since the investor can use labor income now to control how much exposure to interest rate risk she has due to her human capital.

3.5.5 Case 5: Elastic Labor Supply, Stochastic Labor Income

We finish the calibration section of the paper, by allowing for both labor supply flexibility and stochastic labor income. When labor income is stochastic, it serves as a risk-management tool for both equity and the long term bond. The allocation of financial wealth to equity and the long-term bond satisfy equations (3.35) and (3.36). For convenience we provide the allocation results below

\[
\begin{align*}
\alpha^f_{St} &= \frac{\eta_1}{\gamma \sigma_{1S}} \left(1 + \frac{H^f_t}{F_t} \right) - \frac{H_t \sigma_{1Y}}{F_t \sigma_{1S}} - \left(1 - \frac{1}{\lambda} \right) \left(1 - \frac{1}{\gamma} \right) \frac{W^E_t \sigma_{1Y}}{F_t} \\
\alpha^f_{Pt} &= \frac{\eta_2}{\gamma \sigma_{2r} B(m)} \left(1 + \frac{H^f_t}{F_t} \right) + \left(1 - \frac{1}{\gamma} \right) \frac{\partial_t W^R_t}{W^R_t} \frac{W^R_t}{F_t B(m)} - \frac{(1 - g_r)}{B(m)} \frac{\partial_t H_t}{H_t} \frac{H_t}{F_t} \\
&\quad + \left(1 - \frac{1}{\gamma} \right) \frac{(1 - (1 - \lambda) g_r)}{\lambda B(m)} \frac{\partial_t W^E_t W^E_t}{W^E_t F_t} \\
&\quad - \frac{H_t}{F_t} \frac{\sigma_{2Y}}{\sigma_{2r} B(m)} - \left(1 - \frac{1}{\lambda} \right) \left(1 - \frac{1}{\gamma} \right) \frac{W^E_t \sigma_{2Y}}{F_t \sigma_{2r} B(m)}
\end{align*}
\]

We maintain the assumptions on the relative risk aversion of the investor and her current labor income to financial wealth ratio. We consider various parameters for the shocks to labor income and their correlation with equity shocks and interest rate shocks. We also consider various parameters for \( \lambda \), the relative preference for consumption over leisure. Table 3.9, presents the results for \( \lambda = 0.7 \) and Table 3.10 presents the results for \( \lambda = 0.9 \).

Figure 3-2 considers the same situation as Figure 3-1, but assumes \( \lambda = 0.7 \). The allocation to equities is smaller in this setting, since the investor effectively holds less total wealth when a component of his human capital is consumed in terms of leisure. The allocation to long-term bonds seems to be less sensitive to the assumption of the elasticity of the growth rate to changes in interest rates. Figures 3-4 and 3-6 extend Figures 3-3 and 3-5 to allow for labor
supply flexibility and obtain similar results in regards to the quantitative effects of labor supply flexibility. But as shown in Figure 3-8, this results do not control for the fact less labor is supplied when leisure is desired.

Figure 3-9 considers the allocation of financial wealth to the long term bond for two cases: one where 0.8 units of labor are supplied at all times, and another where \( \lambda = 0.8 \). We propose, this cases allow us to control for the magnitude effect in the allocations due to the conversion of human capital into leisure for the investor with preference for leisure. We find the allocation under flexible labor supply to be greater when the amount of human capital is about the same for both investors. The results show flexible labor income does indeed seem to allow for undertaking more risks in the investor’s optimal portfolio.

3.5.6 Labor Income and Investment in the Life Cycle

Our analysis so far has been static in nature. When we studied the asset allocation strategy of the investor, we have assume the labor income to financial wealth of the investor is constant throughout the investment horizon. In this section, we relax the assumption to obtain a more realistic idea of how the asset allocation of the investor changes with investment horizon and the expected labor income to financial wealth ratio at each point during employment. Figure 3-10 and 3-11 show the labor income to financial wealth ratio and the human capital to financial wealth ratio for an investor facing the constant investment opportunity set and riskless human capital presented in Case 1. In Figure 3-10, we assume there are no shocks to equity, while Figure 3-11 allows for the stochastic component. The labor income to financial wealth ratio initially increases and then decreases as the investor nears retirement. What is the role of this hump shape in the scaled labor income profile of the investor in the investor’s portfolio strategy? Figure 3-12 presents the optimal allocation to the long-term bond and equity for an investor whose scaled labor income follows the result in Figure 3-10 under various assumptions for the sensitivity of labor income growth to interest rates. Comparing with Figure 3-1 shows how the curvature of the scaled labor income schedule creates a curvature in the asset allocation strategy of the investor. During the initial employment period, allocation in the long-term bond and equity does not change significantly. As the investor nears retirement, and her scaled labor income decreases, the slope at which the strategies move toward the retirement strategies
increases. Although accounting for changes in scaled labor income does not seem to change our results greatly, it allows for better understanding of how the allocation strategy of the investor should change as she nears retirement.

3.6 Conclusion

We studied the role of human capital in optimal consumption and portfolio strategies when interest rates are stochastic. We found human capital can be risky even when labor income is locally riskless and that it served as a substitute to long-term bonds in the asset allocation of the investor. Aside from the well-known leverage effect of human capital in the allocation strategy of financial wealth, which leads to a horizon effect in equity investment, we found human capital does not necessarily give rise to a horizon effect when allocating financial wealth to long-term bonds. Instead, we find the allocation to long-term bond to depend on the differences in the interest rate risk sensitivity of total wealth and human capital. In our calibration, this can lead to a hump-shape demand for the long-term bond as a function of the investment and retirement horizon.

We extend the model and consider the role of labor income when labor income is stochastic. In this case human capital can act as a hybrid security in the sense that human capital is exposed to both interest rate risk, similar to a long-term bond, and the shocks to equity. Even in this case, we find the sensitivity of human capital to interest-rate risk to be the most important determinant for the asset allocation strategy of the investor.

The results in this paper can be extended by considering a model of time-varying risk premia where labor income is sensitive to changes in risk premia. We believe this model could provide an explanation to the U-shape investment in equities in investor’s portfolios as shown by Zeldes (1994). We also would like to consider the role of idiosyncratic risk to gain a better understanding of both the quantitative and qualitative effects of labor income in strategic asset allocation. Both extensions will be pursued by the author in future papers.
3.7 References


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3.8 Proofs

3.8.1 Proof of Proposition 3.1

Assume \( t < R \). The present value of human capital is given by

\[
H_t = E_t^Q \left[ \int_t^R \exp \left( - \int_t^s r_v \, dv \right) Y_s \, ds \right] \tag{3.41}
\]

where \( r_t \) and \( Y_t \) satisfy the following stochastic differential equation under the \( Q \)-measure

\[
\begin{align*}
\frac{dr_t}{r_t} &= \left[ \kappa (\theta - r_t) - (\sigma_1 \eta_1 + \sigma_2 \eta_2) \right] dt + \sigma_1 dZ_{1t}^Q + \sigma_2 dZ_{2t}^Q, \\
\frac{dY_t}{Y_t} &= \left[ (g + g_r r_t) - (\sigma_1 Y_1 + \sigma_2 Y_2) \right] dt + \sigma_1 dZ_{1t}^Q + \sigma_2 dZ_{2t}^Q. \tag{3.42, 3.43}
\end{align*}
\]

The brownian motions \( Z_{1t}^Q \) and \( Z_{2t}^Q \) are obtained by applying Girsanov’s theorem to \( Z_{1t} \) and \( Z_{2t} \) respectively. The relationship between the \( P \)-measure and \( Q \)-measure brownian motions is given by

\[
dZ_{it}^Q = dZ_{it} + \eta_i dt, \quad i = 1, 2.
\]

Rewrite (3.41) as

\[
H_t = Y_tE_t^Q \left[ \int_t^R \exp \left( - \int_t^s r_v \, dv \right) \exp \left( \ln \frac{Y_s}{Y_t} \right) \, ds \right]
\]

By Ito’s Lemma

\[
\begin{align*}
\frac{d \ln Y_t}{Y_t} &= \left( g + g_r r_t - \sigma_1 Y_1 - \sigma_2 Y_2 \right) dt + \sigma_1 dZ_{1t}^Q \\
\ln Y_s - \ln Y_t &= \int_t^s \left( g + g_r r_v - \sigma_1 Y_1 - \sigma_2 Y_2 \right) dv + \int_t^s \sigma_1 dZ_{1v}^Q
\end{align*}
\]

Substitute the previous equation into (3.41) to obtain

\[
H_t = Y_tE_t^Q \left[ \int_t^R \exp \left( \int_t^s \left( g + (g_r - 1) r_v - \sigma_1 Y_1 - \sigma_2 Y_2 \right) dv + \int_t^s \sigma_1 dZ_{1v}^Q \right) \, ds \right]
\]

We apply another change of measure, let \( Q_Y \) be the measure obtained by applying Girsanov’s theorem as follows

\[
dZ_{it}^{Q_Y} = dZ_{it}^Q - \sigma_Y dt
\]

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such that
\[
\frac{dQ_Y}{dQ} = \exp \left( - \int_t^s \frac{\sigma_s^2}{2} dv + \int_t^s \sigma_y dZ^Q_v \right)
\]

By Bayes' Rule

\[
H_t = Y_t E_t^{Q_Y} \left[ \int_t^R \exp \left\{ \int_t^s \left( g + (g_r - 1) r_v - \sigma_y \eta \right) dv \right\} ds \right] \]
\[
Y_t \left[ \int_t^R X (r_t, s - t) ds \right]
\]

Under the \( Q_Y \) - measure, the interest rate dynamics satisfy the stochastic differential equation
\[
dr_t = [\kappa (\theta - r_t) - \sigma_r \eta + \sigma_r \sigma_Y] dt + \sigma_r dZ^{Q_Y}_t,
\]

We apply the Feynman-Kac theorem to obtain

\[
(1 - g_r) - g + \sigma_y \eta = \frac{\partial X_s}{\partial t} + \frac{\partial X_s}{\partial r_t} \left[ \kappa (\theta - r_t) - \sigma_r \eta + \sigma_r \sigma_Y \right] + \frac{1}{2} \frac{\partial^2 X_s}{\partial r_t^2} \sigma_r^2 
\]

(3.44)

Following the literature of fixed-income pricing, we assume \( X \) is of the exponential-affine form. We guess

\[
X (r_t, s - t) = \exp (A_H (s - t) + B_H (s - t) r_t) 
\]

(3.45)

with boundary conditions

\[
A (0) = B (0) = 0.
\]

Substitute (3.45) into (3.44) to obtain the system of differential equations \( A_H \) and \( B_H \) must satisfy

\[
A'_H (s - t) = g - \sigma_y \eta + B_H (s - t) [\kappa \theta - \sigma_r \eta + \sigma_r \sigma_Y] + \frac{1}{2} B_H^2 (s - t) \sigma_r^2
\]

(3.46)

\[
B'_H (s - t) = (g_r - 1) - B_H (s - t) \kappa
\]

(3.47)

The solutions to (3.46) and (3.47) are

\[
A_H (s - t) = (g - \sigma_y \eta) (s - t) + (g_r - 1) \left[ \frac{\kappa \theta - \sigma_r \eta + \sigma_r \sigma_Y}{\kappa} \left( s - t - \left( \frac{1 - e^{-\kappa (s-t)}}{\kappa} \right) \right) \right] (3.48)
\]

\[
+ (g_r - 1) \frac{\sigma_r^2}{2 \kappa^2} \left( s - t - 2 \left( \frac{1 - e^{-\kappa (s-t)}}{\kappa} \right) + \left( \frac{1 - e^{-2 \kappa (s-t)}}{2 \kappa} \right) \right)
\]

(3.49)

\[
B_H (s - t) = (g_r - 1) \frac{1 - e^{-\kappa (s-t)}}{\kappa} = (1 - g_r) B (s - t)
\]
respectively.

3.8.2 Proof of Proposition 3.2

We apply a combination of Leibniz' Rule and Ito's Lemma to obtain the solution. Assume $t < R$, from Proposition 3.1 we have that human capital is given by

$$H_t = Y_t \int_t^R \exp \left[ A(s-t) + (1-g_r)B(s-t)r_t \right] ds$$

By Ito's Lemma

$$dH_t = \left[ \int_t^R \exp \left[ A(s-t) + (1-g_r)B(s-t)r_t \right] ds \right] dY_t + Y_t D \left[ \int_t^R \exp \left[ A(s-t) + (1-g_r)B(s-t)r_t \right] ds \right]$$

(3.50)

$$+ d\left( Y_t, \int_t^R \exp \left[ A(s-t) + (1-g_r)B(s-t)r_t \right] ds \right)$$

Consider the integral of the exponential function. By Ito's Lemma

$$D \int_t^R \exp \left[ A(s-t) + (1-g_r)B(s-t)r_t \right] ds = D \left[ \int_t^R X(r_t, s-t) ds \right]$$

$$= \frac{\partial}{\partial t} \left[ \int_t^R X(r_t, s-t) ds \right] dt + \frac{\partial}{\partial r_t} \left[ \int_t^R X(r_t, s-t) ds \right] dr_t$$

(3.51)

$$+ \frac{1}{2} \frac{\partial^2}{\partial r_t^2} \left[ \int_t^R X(r_t, s-t) ds \right] (dr_t)^2$$

(3.52)

where by Leibniz Rule,

$$\frac{\partial}{\partial t} \left[ \int_t^R X(r_t, s-t) ds \right] = \int_t^R \left[ \frac{\partial}{\partial t} X(r_t, s-t) \right] ds - X(r_t, 0)$$

$$= \int_t^R \left[ \frac{\partial}{\partial r_t} X(r_t, s-t) \right] ds - 1$$

$$= - \int_t^R \left[ (A'(s-t) + (1-g_r)B'(s-t)r_t) X(r_t, s-t) \right] ds - 1$$

(3.53)
The following equalities follow from Fubini's Theorem:

\[
\frac{\partial}{\partial r_t} \left[ \int_t^R X(r_t, s - t) \, ds \right] = \int_t^R \frac{\partial}{\partial r_t} X(r_t, s - t) \, ds = \int_t^R (1 - g_r) B(s - t) X(r_t, s - t) \, ds \quad (3.54)
\]

\[
\frac{\partial^2}{\partial r_t^2} \left[ \int_t^R X(r_t, s - t) \, ds \right] = \int_t^R \frac{\partial^2}{\partial r_t^2} X(r_t, s - t) \, ds = \int_t^R (1 - g_r)^2 B^2(s - t) X(r_t, s - t) \, ds \quad (3.55)
\]

Substitute (3.53), (3.54), and (3.55) into (3.51) to obtain

\[
d \left[ \int_t^R X(r_t, s - t) \, ds \right]
= (1 - g_r) \tau_t \left[ \int_t^R X(r_t, s - t) \, ds \right] dt - dt
= -\left[ \int_t^R (g - \sigma\gamma\eta) (s - t) + (1 - g_r) B(s - t) [-\sigma_r\eta + \sigma_r\sigma_\gamma] X(r_t, s - t) \, ds \right] dt
+ (1 - g_r) \left[ \int_t^R B(s - t) X(r_t, s - t) \, ds \right] \sigma_r dZ_r. \quad (3.56)
\]

Apply (3.56) to (3.50) to obtain

\[
dH_t + Y_t dt = r_t dt + (1 - g_r) \sigma_r \eta \left[ Y_t \int_t^R B(s - t) X(r_t, s - t) \, ds \right] + \sigma_\gamma \eta H_t dt
+ \left[ (1 - g_r) \sigma_r \left[ Y_t \int_t^R B(s - t) X(r_t, s - t) \, ds \right] + \sigma_\gamma H_t \right] dZ_r. \quad (3.57)
\]

Divide (3.57) by \( H_t \) to obtain the proposed dynamics for human capital.

3.8.3 Proof of Proposition 3.3

Recall wealth is the present value of the consumption stream

\[
W_t = E_t \left[ \int_t^T \frac{M_t}{M_t} C_t \, ds \right] \quad (3.58)
\]
\[ C_t^{-\gamma} = \varphi_t \]

Plug (3.58) into (3.57) to obtain

\[
W_t = \varphi_t^{-\frac{1}{r}} E_t \left[ \int_t^T e^{-\frac{\phi}{\gamma} (s-t)} \left( \frac{M_s}{M_t} \right)^{1-\frac{1}{r}} ds \right]
\]

\[ = C_t E_t \left[ \int_t^T e^{-\frac{\phi}{\gamma} (s-t)} \left( \frac{M_s}{M_t} \right)^{1-\frac{1}{r}} ds \right]. \]

Let \( M_t = \exp \left( -\int_0^t r_s du \right) \xi_t \), where

\[ \xi_t = \exp \left( -\frac{\eta^2}{2} t - \eta Z_t \right) \]

Under the \( Q \)-measure, we can write (3.57) as

\[
W_t = C_t E_t^Q \left[ \int_t^T \exp \left( -\frac{\phi}{\gamma} (s-t) - \left( 1 - \frac{1}{\gamma} \right) \int_t^s r_v du \right) \left( \frac{\xi_s}{\xi_t} \right)^{-\frac{1}{r}} ds \right]
\]

(3.59)

where

\[ \frac{\xi_s}{\xi_t} = \exp \left( \int_t^s \frac{\eta^2}{2} dv - \int_t^s \eta dZ_t^Q \right) \]

We apply Girsanov's Theorem once again to obtain the brownian motions for the measure \( Q_{\gamma} \). Let

\[ dZ_t^{Q_{\gamma}} = dZ_t^Q - \frac{\eta}{\gamma} dt \]

such that

\[ \left( \frac{\xi_s}{\xi_t} \right)^{-\frac{1}{r}} = \exp \left( \int_t^s \frac{1-\gamma}{2\gamma^2} \eta^2 dv + \frac{1}{2} \int_t^s \frac{\eta^2}{\gamma^2} dv - \int_t^s \frac{\eta}{\gamma} dZ_t^{Q_{\gamma}} \right) \]

By Bayes' rule we can write (3.59) as

\[
W_t = C_t E_t^{Q_{\gamma}} \left[ \int_t^T \exp \left( -\int_t^s \frac{\phi}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) r_v - \frac{1-\gamma}{2\gamma^2} \eta^2 \right) ds \right]
\]

\[ = C_t E_t^{Q_{\gamma}} \left[ \int_t^T D(r_t, s-t) ds \right] \]

The interest process under the \( Q_{\gamma} \)-measure is

\[ dr_t = \left[ \kappa (\theta - r_t) - \left( 1 - \frac{1}{\gamma} \right) \sigma_r \eta \right] dt + \sigma_r dZ_t^{Q_{\gamma}} \]

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We apply the Feynman-Kac theorem to obtain the differential equation satisfied by the function $D(r_t, s - t)$

$$\frac{\theta}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \frac{\partial}{\partial t} r_t - \frac{1 - \gamma^2}{2\gamma^2} \eta^2 = \frac{\partial D_s}{\partial t} + \frac{\partial D_s}{\partial r_t} \left[\kappa (\theta - r_t) - \left(1 - \frac{1}{\gamma}\right) \sigma r_t \eta \right] + \frac{1}{2} \frac{\partial^2 D_s}{\partial r_t^2} \sigma^2_t$$  \hspace{1cm} (3.60)

Assume the following functional form for $D(r_t, s - t)$

$$D(r_t, s - t) = \exp \left[ A_W (s - t) + B_W (s - t) r_t \right]$$  \hspace{1cm} (3.61)

with boundary conditions

$$A_W (0) = B_W (0) = 0.$$  

Substitute (3.61) into (3.60) to obtain the system of differential equations $A_W$ and $B_W$ must satisfy

$$A_W (s - t) = -\frac{\theta}{\gamma} + \frac{1 - \gamma^2}{2\gamma^2} \eta^2 + \left[\kappa (\theta - r_t) - \left(1 - \frac{1}{\gamma}\right) \eta \sigma r_t \kappa \right] B_W (s - t) + \frac{1}{2} \sigma^2 \kappa^2 B_W (s - t)$$  \hspace{1cm} (3.62)

$$B_W (s - t) = -\left(1 - \frac{1}{\gamma}\right) B_W (s - t) \kappa$$  \hspace{1cm} (3.63)

The solutions to (3.62) and (3.63) are

$$A_W (s - t) = -\frac{\theta}{\gamma} (s - t) + \frac{1 - \gamma^2}{2\gamma^2} \eta^2 (s - t) - \left(1 - \frac{1}{\gamma}\right) \left(\frac{\kappa \theta - \left(1 - \frac{1}{\gamma}\right) \eta \sigma r_t \kappa}{\kappa}\right) (s - t) - \frac{1 - e^{-\kappa (s - t)}}{\kappa}$$

$$+ \frac{\sigma^2}{2\kappa^2} \left(1 - \frac{1}{\gamma}\right) \left[\frac{(s - t) - 2 \left(1 - e^{-\kappa (s - t)}}{\kappa}\right)}{2\kappa}\right]$$  \hspace{1cm} (3.64)

$$B_W (s - t) = -\left(1 - \frac{1}{\gamma}\right) \left(\frac{1 - e^{-e^{\kappa (s - t)}}}{\kappa}\right) = \left(1 - \frac{1}{\gamma}\right) B(t - s)$$  \hspace{1cm} (3.65)

### 3.8.4 Proof of Proposition 3.4

The portfolio allocation matches the stochastic components of wealth from the allocation perspective to the stochastic component of wealth as a function of the stochastic discount factor and the state variables.

In terms of equation we can express this condition by the following system:

$$\alpha_s T_s \sigma_1 + \alpha_p T_p \sigma_2 + H_i \sigma_{1H} = -\frac{\partial W_i}{\partial M_i} \eta_1 + \frac{\partial W_i}{\partial t} \sigma_{1r}$$  \hspace{1cm} (3.66)

$$\alpha_s T_s \sigma_2 + \alpha_p T_p \sigma_2 + H_i \sigma_{2H} = -\frac{\partial W_i}{\partial M_i} \eta_1 + \frac{\partial W_i}{\partial t} \sigma_{2r}$$  \hspace{1cm} (3.67)
where

\[ \sigma_{iP} = \sigma_m B(m), \]
\[ \sigma_{1H} = \sigma_m (1 - g_r) \frac{\beta_r H_1}{H_t} + \sigma_{iY}, \quad i = 1, 2. \]

From the previous proposition

\[ W_t = \varphi_t^{-\frac{1}{2}} \int_t^T \exp \left[ A_W (s - t) + \left( 1 - \frac{1}{\gamma} \right) B (s - t) r_t \right] ds \quad (3.68) \]

Furthermore, the relationship between the Lagrange Multiplier at time \( t \) and the Lagrange multiplier at time 0 is

\[ \varphi_t = \varphi_0 e^{-\frac{\rho_t M_t}{M_0}} \quad (3.69) \]

where \( \varphi_0 \) and \( M_0 \) are constants. Plugging (3.69) into (3.68) and applying it to equations (3.66) and (3.67), we obtain the following result

\[ \frac{\alpha_{11} F_t \sigma_{1S} + \alpha_{12} F_t \sigma_{1P} + H_t \sigma_{1H}}{\gamma} \]
\[ = \frac{\eta_1 W_t + \left( 1 - \frac{1}{\gamma} \right) \sigma_{1} \varphi_t^{-\frac{1}{2}} \int_t^T B (s - t) \exp \left[ A_W (s - t) + \left( 1 - \frac{1}{\gamma} \right) B (s - t) r_t \right] ds}{\gamma} \quad (3.70) \]
\[ \frac{\alpha_{21} F_t \sigma_{2S} + \alpha_{22} F_t \sigma_{2P} + H_t \sigma_{2H}}{\gamma} \]
\[ = \frac{\eta_2 W_t + \left( 1 - \frac{1}{\gamma} \right) \sigma_{2} \varphi_t^{-\frac{1}{2}} \int_t^T B (s - t) \exp \left[ A_W (s - t) + \left( 1 - \frac{1}{\gamma} \right) B (s - t) r_t \right] ds}{\gamma} \quad (3.71) \]

We divide both sides by \( W_t \), apply the definition of \( \frac{\partial W_t}{W_t} \) and the identity \( W_t = F_t + H_t \) to obtain the result.

### 3.8.5 Proof of Proposition 3.7

We combine the first order conditions (3.19) and (3.20) to obtain the following relation between labor income, consumption, and leisure

\[ \lambda (1 - L_t) Y_t = (1 - \lambda) C_t \quad (3.72) \]

Equation (3.72) can be rewritten as

\[ L_t Y_t = Y_t + \left( 1 - \frac{1}{\lambda} \right) C_t \]
Human capital under flexible labor supply is

\[ H_t^f = E_t \left[ \int_t^R \frac{M_s}{M_t} L_s Y_s ds \right] \]
\[ = E_t \left[ \int_t^R \frac{M_s}{M_t} Y_s ds \right] + \left( 1 - \frac{1}{\lambda} \right) E_t \left[ \int_t^R \frac{M_s}{M_t} C_s ds \right] \]
\[ = H_t + \left( 1 - \frac{1}{\lambda} \right) W_t^E \]

3.8.6 Proof of Proposition 3.8

Total wealth is the sum of the present value of consumption during employment and consumption during retirement. Assume \( t < R \), then total wealth is given by

\[ W_t = E_t \left[ \int_t^R \frac{M_s}{M_t} C_s ds \right] + E_t \left[ \int_R^T \frac{M_s}{M_t} C_s ds \right] \quad (3.73) \]

Combine the first order conditions for leisure and consumption to obtain the following expression for consumption during employment

\[ C_s = \psi \varphi_t^{-\frac{1}{\lambda}} \left( \frac{M_s}{M_t} \right)^{-\frac{1}{\lambda}} e^{-\frac{\xi}{s} (s-t)} Y_t^{(1-\lambda)(1-\frac{1}{\lambda})} \quad (3.74) \]

where

\[ \psi = \left[ \lambda^{\lambda+\gamma-\lambda} (1-\lambda)(1-\gamma) \right]^{-\frac{1}{\lambda}} \]

Substitute (3.74) into equation (3.73) to obtain

\[ W_t^E = \psi \varphi_t^{-\frac{1}{\lambda}} Y_t^{(1-\lambda)(1-\frac{1}{\lambda})} E_t \left[ \int_t^R \left( \frac{M_s}{M_t} \right)^{1-\frac{1}{\lambda}} e^{-\frac{\xi}{s} (s-t)} \left( \frac{Y_s}{Y_t} \right)^{(1-\lambda)(1-\frac{1}{\lambda})} ds \right] \]

We apply Girsanov's theorem and Bayes' Rule as in Proposition 3.3 to obtain the following expression for employment wealth:

\[ W_t^E = \psi \varphi_t^{-\frac{1}{\lambda}} Y_t^{(1-\lambda)(1-\frac{1}{\lambda})} E_t^Q \left[ \int_t^R \exp \left( \int_t^s - \frac{\phi}{\gamma} - \left( 1 - \frac{1}{\gamma} \right) r_s ds \right) \left( \frac{\xi_t}{\xi_s} \right)^{-\frac{1}{\lambda}} \left( \frac{Y_s}{Y_t} \right)^{(1-\lambda)(1-\frac{1}{\lambda})} ds \right] \quad (3.75) \]

where the dynamics for \( r_t \) and \( Y_t \) are given by equations (3.42) and (3.43) respectively. We apply
Girsanov's Theorem again to obtain the brownian motions for the measure $Q_{\gamma}$. Let

$$dZ_t^{Q_{\gamma}} = dZ_t^Q - \frac{\eta}{\gamma} dt$$

such that

$$\left(\frac{\xi_s}{\xi_t}\right)^{-\frac{1}{\lambda}} = \exp \left( \int_t^s \frac{1-\gamma}{2\gamma^2} \eta^2 dv - \frac{1}{2} \int_t^s \frac{\eta^2}{\gamma^2} dv - \int_t^s \frac{\eta}{\gamma} dZ_v^{Q_{\gamma}} \right)$$

By Bayes' rule we can write (3.75) as

$$W_t^E = \psi \varphi_t^{-\frac{1}{\lambda}} \psi_t^{(1-\lambda)(1-\frac{1}{\lambda})} E_t^{Q_{\gamma}} \left[ \int_t^T \exp \left( - \int_t^s \frac{\phi}{\gamma} + \left(1 - \frac{1}{\gamma}\right)r_v - \frac{1-\gamma}{2\gamma^2} \eta^2 dv \right) \left(\frac{Y_s}{Y_t}\right)^{(1-\lambda)(1-\frac{1}{\lambda})} ds \right]$$

(3.76)

The interest process and the labor income process under the $Q_{\gamma}$ measure are

$$dr_t = \left[ \kappa (\theta - r_t) - \left(1 - \frac{1}{\gamma}\right) \sigma_r \eta \right] dt + \sigma_r dZ_t^{Q_{\gamma}}$$

$$\frac{dY_t}{Y_t} = \left[ g + \gamma r_t - \left(1 - \frac{1}{\gamma}\right) \sigma_Y \eta \right] dt + \sigma_Y dZ_t^{Q_{\gamma}}$$

We proceed to express the term $Y_s/Y_t$ in log form by applying Ito's Lemma

$$\ln Y_s - \ln Y_t = \int_t^s \left( g + \gamma r_v - \left(1 - \frac{1}{\gamma}\right) \sigma_Y \eta - \frac{\sigma_Y^2}{2} \right) dv + \int_t^s \sigma_Y dZ_v^{Q_{\gamma}}$$

such that

$$\left(\frac{Y_s}{Y_t}\right)^{(1-\lambda)(1-\frac{1}{\lambda})} = \exp \left( (1-\lambda) \left(1 - \frac{1}{\gamma}\right) \left\{ \int_t^s \left( g + \gamma r_v - \left(1 - \frac{1}{\gamma}\right) \sigma_Y \eta - \frac{\sigma_Y^2}{2} \right) dv + \int_t^s \sigma_Y dZ_v^{Q_{\gamma}} \right\} \right)$$

(3.77)

Define the function $\xi_t^{Q_{\gamma}}$ as follows

$$\xi_t^{Q_{\gamma}} = \exp \left( - \frac{(1-\lambda)^2}{2} \left(1 - \frac{1}{\gamma}\right)^2 \sigma_Y^2 t + (1-\lambda) \left(1 - \frac{1}{\gamma}\right) \sigma_Y Z_t^{Q_{\gamma}} \right)$$

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We can rewrite (3.77) as

\[
\begin{align*}
\left( \frac{Y_s}{Y_t} \right)^{(1-\lambda)(1-\frac{1}{\gamma})} \\
= \exp \left( (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \left\{ \int_t^s \left( g + g_r r_v - \left( 1 - \frac{1}{\gamma} \right) \sigma_Y \eta - \left( 1 - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_Y^2}{2} \right) \right) dv \right\} \left( \frac{\xi_{Q^*}}{\xi_{Q^*}} \right) \right)
\end{align*}
\]

We apply Bayes' rule to obtain

\[
\begin{align*}
E_t^{Q^*} \left[ \left( \frac{Y_s}{Y_t} \right)^{(1-\lambda)(1-\frac{1}{\gamma})} \right] \\
= E_t^{Q^*} \left[ \exp \left( (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \left\{ \int_t^s \left( g + g_r r_v - \left( 1 - \frac{1}{\gamma} \right) \sigma_Y \eta - \left( 1 - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_Y^2}{2} \right) \right) dv \right\} \right) \right]
\end{align*}
\]

where the brownian motions under the new measure satisfy

\[
dZ_t^{Q^*,Y} = dZ_t^{Q^*} - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_Y dt
\]

Under the $Q^*,Y$ measure, the interest rate dynamics are given by

\[
dr_t = dr_t = \left[ \kappa (\theta - r_t) - \left( 1 - \frac{1}{\gamma} \right) \sigma_r \eta - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_r \sigma_Y \right] dt + \sigma_r dZ_t^{Q^*,Y} \tag{3.79}
\]

Plug (3.78) into (3.76) to obtain

\[
W_t^F = \psi_{\theta_r}^{\frac{1}{2}} \left( 1-\lambda \right)^{(1-\frac{1}{\gamma})} E_t^{Q^*,Y} \left[ \int_t^R G(r_t, s - t) ds \right] \tag{3.80}
\]

where

\[
\ln G(r_t, s - t) = - \int_t^s \frac{\phi}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) r_v - \frac{1-\gamma}{2\gamma^2} \eta^2 dv \\
+ (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \left\{ \int_t^s \left( g + g_r r_v - \left( 1 - \frac{1}{\gamma} \right) \sigma_Y \eta - \left( 1 - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_Y^2}{2} \right) \right) dv \right\}
\]

We apply the Feynman-Kac theorem to obtain a partial differential equation that solves $G$:

\[
\frac{\phi}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) r_t - \frac{1-\gamma}{2\gamma^2} \eta^2 - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \left[ g + g_r r_t - \left( 1 - \frac{1}{\gamma} \right) \sigma_Y \eta - \left( 1 - (1-\lambda) \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_Y^2}{2} \right) \right]
= \frac{\partial G}{\partial t} + \frac{\sigma_r^2}{2} \frac{\partial^2 G}{\partial r_t^2} \tag{3.81}
\]

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Assume \( G(r_t, s - t) \) has an exponential-affine functional form:

\[
G(r_t, s - t) = \exp \left( A_W^t (s - t) + B_W^t (s - t) \tau_t \right)
\]

(3.82)

with boundary conditions

\[
A(0) = B(0) = 0
\]

Apply (3.82) to (3.81) and obtain the following system of differential equations

\[
\frac{dA_W^t (s - t)}{dt} = -\frac{\phi}{\gamma} + \frac{1 - \gamma}{2\gamma^2} \eta^2 + B_W^t (s - t) \left[ \kappa \theta - \left( 1 - \frac{1}{\gamma} \right) \sigma_r \eta - (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_r \sigma_Y \right] \\
+ \frac{1}{2} \left( \frac{\sigma_r^2 (s - t)}{\kappa} \right)^2 \\
+ (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \left[ g - \left( 1 - \frac{1}{\gamma} \right) \sigma_Y \eta - \left( 1 - (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \right) \frac{\sigma_Y^2}{2} \right],
\]

(3.83)

\[
\frac{dB_W^t (s - t)}{dt} = - \left( 1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) - B_W^t (s - t) \kappa.
\]

(3.84)

The solutions to (3.83) and (3.84) are

\[
A_W^t (s - t) = \left[ -\frac{\phi}{\gamma} + \frac{1 - \gamma}{2\gamma^2} \eta^2 \right] (s - t) \\
+ (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \left[ g - \left( 1 - \frac{1}{\gamma} \right) \sigma_Y \eta - \left( 1 - (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \right) \frac{\sigma_Y^2}{2} \right] (s - t) \\
- \left[ \left( s - t - \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \right] \left[ \kappa \theta - \left( 1 - \frac{1}{\gamma} \right) \sigma_r \eta - (1 - \lambda) \left( 1 - \frac{1}{\gamma} \right) \sigma_r \sigma_Y \right] \left( 1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) \\
+ \frac{\sigma_r^2}{2\kappa^2} \left[ (s - t) - 2 \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) \right] + \left( \frac{1 - e^{-2\kappa(s-t)}}{2\kappa} \right) \left( 1 - \frac{1}{\gamma} \right)^2 (1 - (1 - \lambda) g_r)^2
\]

\[
B_W^t (s - t) = - \left( 1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) \left( \frac{1 - e^{-\kappa(s-t)}}{\kappa} \right) = \left( 1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) B(s - t)
\]

respectively.

Now we provide an expression for the present value of wealth needed for the expected consumption stream during retirement. Let \( W_t^R \) denote the wealth for retirement consumption. Then

\[
W_t^R = E_t \left[ \int_0^T \frac{M_s}{\bar{M}_t} C_s ds \right]
\]

(3.85)

We plug in the first order condition for consumption at retirement (3.21) to obtain the following expres-
\[
W_t^R = \varphi_t^{-\frac{1}{2}} E_t \left[ \int_{\mathbb{R}} e^{-\frac{\varphi}{2}(s-t)} \left( \frac{M_t}{M_0} \right)^{1-\frac{t}{\lambda}} ds \right]
\]

Following the results of Proposition 3.3 we obtain
\[
W_t^R = \varphi_t^{-\frac{1}{2}} E_t \left[ \int_{\mathbb{R}} \exp (A_W (s-t) + B_W (s-t) r_t) ds \right]
\]

3.8.7 Proof of Proposition 3.9

The portfolio allocation matches the stochastic components of wealth from the allocation perspective to the stochastic component of wealth as a function of the stochastic discount factor and the state variables. In terms of equation we can express this condition by the following system:

\[
\begin{align*}
\alpha^I_{ST} F_0 \sigma_{1S} + \alpha^I_{PT} F_0 \sigma_{1P} + H_t^I \sigma_{1H} &= -\frac{\partial W_t}{\partial M_t} M_t \eta_1 + \frac{\partial W_t}{\partial \sigma_{1s}} \sigma_{1r} + \frac{\partial W_t}{\partial Y_t} Y_t \sigma_{1Y} \\
\alpha^I_{ST} F_0 \sigma_{2S} + \alpha^I_{PT} F_0 \sigma_{2P} + H_t^I \sigma_{2H} &= -\frac{\partial W_t}{\partial M_t} M_t \eta_1 + \frac{\partial W_t}{\partial \sigma_{2s}} \sigma_{2r} + \frac{\partial W_t}{\partial Y_t} Y_t \sigma_{2Y}
\end{align*}
\]  

(3.86)

(3.87)

where

\[
\sigma_{ip} = \sigma_{iQ} B(m), i = 1, 2
\]

From the previous proposition

\[
W_t = W_t^E + W_t^R
\]  

(3.88)

Furthermore, the relationship between the Lagrange Multiplier at time \( t \) and the Lagrange multiplier at time \( 0 \) is

\[
\varphi_t = \varphi_0 e^{\varphi_t M_t}
\]  

(3.89)

where \( \varphi_0 \) and \( M_0 \) are constants. Plugging (3.88) into (3.89) and applying it to equations (3.86) and (3.87), we obtain the following result

\[
\begin{align*}
\alpha^I_{ST} F_0 \sigma_{1S} + \alpha^I_{PT} F_0 \sigma_{1P} + H_t^I \sigma_{1H} &= \frac{\eta_1}{\gamma} W_t + \left( \frac{\partial W_t^E}{\partial \sigma_{1s}} + \frac{\partial W_t^R}{\partial \sigma_{1s}} \right) \sigma_{1r} + \frac{\partial W_t^E}{\partial Y_t} Y_t \sigma_{1Y} \\
\alpha^I_{ST} F_0 \sigma_{2S} + \alpha^I_{PT} F_0 \sigma_{2P} + H_t^I \sigma_{2H} &= \frac{\eta_2}{\gamma} W_t + \left( \frac{\partial W_t^E}{\partial \sigma_{2s}} + \frac{\partial W_t^R}{\partial \sigma_{2s}} \right) \sigma_{2r} + \frac{\partial W_t^E}{\partial Y_t} Y_t \sigma_{2Y}
\end{align*}
\]  

(3.90)

(3.91)

From Proposition 3.7 we have

\[
H_t^I = H_t + \left( 1 - \frac{1}{\lambda} \right) W_t^E
\]

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The stochastic components of $H_t'$ are

$$
\sigma_{1H}' H_t' = \sigma_{1r} (1-g_r) \frac{\partial_r H_t}{H_t} H_t + \sigma_{1Y} H_t + \left(1 - \frac{1}{\lambda} \right) \left(1 - \frac{1}{\gamma} \right) (1 - (1 - \lambda) g_r) \frac{\partial_r W_t^E}{W_t^E} W_t^E \sigma_{1r} \\
+ \left(1 - \frac{1}{\lambda} \right) \left(1 - \frac{1}{\gamma} \right) (1 - \lambda) W_t^E \sigma_{1Y}
$$

(3.92)

where

$$
\partial_r W_t^E = \int_t^R B(s-t) G(r_s, s-t) \, ds
$$

(3.93)

Apply the following identities

$$
\frac{\partial W_t^E}{\partial r_t} = \left(1 - \frac{1}{\gamma} \right) \left(1 - (1 - \lambda) g_r \right) \frac{\partial_r W_t^E}{W_t^E} W_t^E,
$$

(3.94)

$$
\frac{\partial W_t^R}{\partial r_t} = \left(1 - \frac{1}{\gamma} \right) \frac{\partial_r W_t^R}{W_t^R} W_t^R
$$

(3.95)

$$
\frac{\partial W_t^E}{\partial Y_t} = \left(1 - \lambda \right) \left(1 - \frac{1}{\gamma} \right) W_t^E
$$

(3.96)

Substitute (3.92), (3.94), (3.95) and (3.96) into (3.90) and (3.91) to obtain

$$
\alpha_{St} F_t \sigma_{1S} + \alpha_{Fr} F_r \sigma_{1F} = \frac{\eta_t}{\gamma} W_t + \left(1 - \frac{1}{\gamma} \right) \frac{\partial_r W_t^R}{W_t^R} W_t^R \sigma_{1r} \\
- (1 - g_r) \frac{\partial_r H_t}{H_t} H_t \sigma_{1r} - H_t \sigma_{1Y} + \left(1 - \frac{1}{\gamma} \right) \left(1 - \frac{1}{\gamma} \right) \frac{1}{\lambda} \frac{\partial_r W_t^E}{W_t^E} W_t^E \sigma_{1r} \\
- \left(1 - \frac{1}{\gamma} \right) \left(1 - \frac{1}{\gamma} \right) W_t^E \sigma_{1Y}
$$

(3.97)

$$
\alpha_{St} F_t \sigma_{2S} + \alpha_{Fr} F_r \sigma_{2F} = \frac{\eta_t}{\gamma} W_t + \left(1 - \frac{1}{\gamma} \right) \frac{\partial_r W_t^R}{W_t^R} W_t^R \sigma_{2r} \\
- (1 - g_r) \frac{\partial_r H_t}{H_t} H_t \sigma_{2r} - H_t \sigma_{2Y} + \left(1 - \frac{1}{\gamma} \right) \left(1 - \frac{1}{\gamma} \right) \frac{1}{\lambda} \frac{\partial_r W_t^E}{W_t^E} W_t^E \sigma_{2r} \\
- \left(1 - \frac{1}{\gamma} \right) \left(1 - \frac{1}{\gamma} \right) W_t^E \sigma_{2Y}
$$

(3.98)

Divide by $F_t$ and use $W_t = F_t + H_t'$ to complete the proof.
TABLE 3.1
Parameters for the Model

<table>
<thead>
<tr>
<th>Interest Rate Parameters</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>long run mean</td>
<td>(\theta)</td>
<td>0.035</td>
</tr>
<tr>
<td>mean reversion speed</td>
<td>(\kappa)</td>
<td>0.35</td>
</tr>
<tr>
<td>volatility</td>
<td>(\sigma_{2r})</td>
<td>0.04</td>
</tr>
<tr>
<td>risk premia</td>
<td>(\eta_2)</td>
<td>0.1</td>
</tr>
<tr>
<td>bond maturity</td>
<td>(m)</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stock Price Parameters</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>volatility</td>
<td>(\sigma_{1S})</td>
<td>0.2</td>
</tr>
<tr>
<td>risk premia</td>
<td>(\eta_1)</td>
<td>0.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Labor Income Parameters</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>current labor income</td>
<td>(Y_t)</td>
<td>({0.05, 0.10, 0.15, 0.20})</td>
</tr>
<tr>
<td>long run growth</td>
<td>(g + g_r \theta)</td>
<td>0.015</td>
</tr>
<tr>
<td>sensitivity of growth to interest rate</td>
<td>(g_r)</td>
<td>({-0.5, 0, 0.5, 1.0})</td>
</tr>
<tr>
<td>volatility shock correlated to equity</td>
<td>(\sigma_{1Y})</td>
<td>({-0.03, 0, 0.03})</td>
</tr>
<tr>
<td>volatility shock correlated to interest rate</td>
<td>(\sigma_{2Y})</td>
<td>({-0.03, 0, 0.03})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Preference Parameters</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>relative risk aversion</td>
<td>(\gamma)</td>
<td>({1, 2, 5, 10, 20})</td>
</tr>
<tr>
<td>discount rate (impatience)</td>
<td>(\phi)</td>
<td>0.06</td>
</tr>
<tr>
<td>investment horizon</td>
<td>(T)</td>
<td>({10, 20, 30, 40})</td>
</tr>
<tr>
<td>retirement horizon</td>
<td>(R)</td>
<td>(T - 10)</td>
</tr>
<tr>
<td>preference for consumption over leisure</td>
<td>(\lambda)</td>
<td>({0.7, 0.8, 0.9, 1.0})</td>
</tr>
</tbody>
</table>
Table 3.2
Portfolio Choice and Human Capital under Constant Investment Opportunity Set

This table presents the allocation of financial wealth to equity and the human capital to financial wealth ratio of an investor with $\gamma = 5$, inelastic labor supply, and for various assumption on the retirement horizon, labor income volatility and the initial labor income to financial wealth ratio of the investor.

Panel A: Non-Stochastic Labor Income

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Human Capital to Financial Wealth Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Scaled Labor Income Y</td>
<td>Current Scaled Labor Income Y</td>
</tr>
<tr>
<td>R</td>
<td>0.05</td>
</tr>
<tr>
<td>0</td>
<td>20.00</td>
</tr>
<tr>
<td>10</td>
<td>29.06</td>
</tr>
<tr>
<td>20</td>
<td>36.48</td>
</tr>
<tr>
<td>30</td>
<td>42.56</td>
</tr>
</tbody>
</table>

Panel B: Negatively Correlated Labor Income ($\sigma_{1y} = -0.03$)

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Human Capital to Financial Wealth Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Scaled Labor Income Y</td>
<td>Current Scaled Labor Income Y</td>
</tr>
<tr>
<td>R</td>
<td>0.05</td>
</tr>
<tr>
<td>0</td>
<td>20.00</td>
</tr>
<tr>
<td>10</td>
<td>36.33</td>
</tr>
<tr>
<td>20</td>
<td>50.53</td>
</tr>
<tr>
<td>30</td>
<td>62.87</td>
</tr>
</tbody>
</table>

Panel C: Positively Correlated Labor Income ($\sigma_{1y} = 0.03$)

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Human Capital to Financial Wealth Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Scaled Labor Income Y</td>
<td>Current Scaled Labor Income Y</td>
</tr>
<tr>
<td>R</td>
<td>0.05</td>
</tr>
<tr>
<td>0</td>
<td>20.00</td>
</tr>
<tr>
<td>10</td>
<td>22.20</td>
</tr>
<tr>
<td>20</td>
<td>23.90</td>
</tr>
<tr>
<td>30</td>
<td>25.21</td>
</tr>
</tbody>
</table>
Table 3.3  
Portfolio Choice and Human Capital under Constant Investment Opportunity Set: Flexible Labor Supply  
This table presents the allocation of financial wealth to equity and the human capital to financial wealth ratio of an investor with $\gamma = 5$, flexible labor supply ($\lambda = 0.8$), and for various assumption on the retirement horizon, labor income volatility and the initial labor income to financial wealth ratio of the investor.

Panel A: Non-Stochastic Labor Income

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Current Scaled Labor Income Y</th>
<th>Human Capital to Financial Wealth Ratio</th>
<th>Current Scaled Labor Income Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0.05</td>
<td>0.10</td>
<td>0.15</td>
</tr>
<tr>
<td>0</td>
<td>20.00</td>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>10</td>
<td>25.63</td>
<td>33.43</td>
<td>41.23</td>
</tr>
<tr>
<td>20</td>
<td>30.74</td>
<td>44.44</td>
<td>58.13</td>
</tr>
<tr>
<td>30</td>
<td>35.09</td>
<td>53.54</td>
<td>71.97</td>
</tr>
</tbody>
</table>

Panel B: Negatively Correlated Labor Income ($\sigma_{1Y} = -0.03$)

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Current Scaled Labor Income Y</th>
<th>Human Capital to Financial Wealth Ratio</th>
<th>Current Scaled Labor Income Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0.05</td>
<td>0.10</td>
<td>0.15</td>
</tr>
<tr>
<td>0</td>
<td>20.00</td>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>10</td>
<td>30.78</td>
<td>45.02</td>
<td>59.28</td>
</tr>
<tr>
<td>20</td>
<td>41.09</td>
<td>66.90</td>
<td>92.70</td>
</tr>
<tr>
<td>30</td>
<td>50.35</td>
<td>86.08</td>
<td>121.79</td>
</tr>
</tbody>
</table>

Panel C: Positively Correlated Labor Income ($\sigma_{1Y} = 0.03$)

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Current Scaled Labor Income Y</th>
<th>Human Capital to Financial Wealth Ratio</th>
<th>Current Scaled Labor Income Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0.05</td>
<td>0.10</td>
<td>0.15</td>
</tr>
<tr>
<td>0</td>
<td>20.00</td>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>10</td>
<td>20.84</td>
<td>22.55</td>
<td>24.26</td>
</tr>
<tr>
<td>20</td>
<td>21.66</td>
<td>24.51</td>
<td>27.35</td>
</tr>
<tr>
<td>30</td>
<td>22.34</td>
<td>26.03</td>
<td>29.72</td>
</tr>
</tbody>
</table>
Table 3.4

Portfolio Choice with Inelastic Labor Supply and Locally Riskless Labor Income

This table presents the allocation of financial wealth to equity and the human capital to financial wealth ratio of an investor with $\gamma = 5$, inelastic labor supply, and for various assumption on the retirement horizon, sensitivity of labor income growth to interest rate changes and the initial labor income to financial wealth ratio of the investor.

### Panel A: Labor Income to Financial Wealth Ratio = 0.05

<table>
<thead>
<tr>
<th>Labor Income Growth Sensitivity ($g_r$)</th>
<th>Allocation to Equity as % of financial wealth</th>
<th>Allocation to Bond as % of financial wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>-0.5                                 0</td>
<td>0.5                         1.0</td>
</tr>
<tr>
<td>0</td>
<td>20.00                                20.00</td>
<td>20.00                         20.00</td>
</tr>
<tr>
<td>10</td>
<td>28.89                                28.89</td>
<td>28.95                         29.06</td>
</tr>
<tr>
<td>20</td>
<td>35.95                                35.83</td>
<td>36.00                         36.48</td>
</tr>
<tr>
<td>30</td>
<td>41.59                                41.23</td>
<td>41.54                         42.56</td>
</tr>
</tbody>
</table>

### Panel B: Labor Income to Financial Wealth Ratio = 0.10

<table>
<thead>
<tr>
<th>Labor Income Growth Sensitivity ($g_r$)</th>
<th>Allocation to Equity as % of financial wealth</th>
<th>Allocation to Bond as % of financial wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>-0.5                                 0</td>
<td>0.5                         1.0</td>
</tr>
<tr>
<td>0</td>
<td>20.00                                20.00</td>
<td>20.00                         20.00</td>
</tr>
<tr>
<td>10</td>
<td>37.79                                37.79</td>
<td>37.90                         38.13</td>
</tr>
<tr>
<td>20</td>
<td>51.91                                51.65</td>
<td>52.00                         52.97</td>
</tr>
<tr>
<td>30</td>
<td>63.18                                62.46</td>
<td>63.08                         65.12</td>
</tr>
</tbody>
</table>

### Panel C: Labor Income to Financial Wealth Ratio = 0.20

<table>
<thead>
<tr>
<th>Labor Income Growth Sensitivity ($g_r$)</th>
<th>Allocation to Equity as % of financial wealth</th>
<th>Allocation to Bond as % of financial wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>-0.5                                 0</td>
<td>0.5                         1.0</td>
</tr>
<tr>
<td>0</td>
<td>20.00                                20.00</td>
<td>20.00                         20.00</td>
</tr>
<tr>
<td>10</td>
<td>55.58                                55.57</td>
<td>55.79                         56.25</td>
</tr>
<tr>
<td>20</td>
<td>83.81                                83.31</td>
<td>84.00                         85.94</td>
</tr>
<tr>
<td>30</td>
<td>106.36                               104.93</td>
<td>106.15                         110.24</td>
</tr>
</tbody>
</table>
Table 3.5
Hedging Demands due to Risk in Total Wealth and Risk in Human Capital

This table presents the hedging demands due to interest rate risk in the total wealth and human capital processes. We assume the investor has $\gamma = 5$, inelastic labor supply, and show the results for various assumption on the retirement horizon, sensitivity of labor income growth to interest rate changes and the initial labor income to financial wealth ratio of the investor.

Panel A: Labor Income to Financial Wealth Ratio = 0.05

<table>
<thead>
<tr>
<th>R</th>
<th>Labor Income Growth Sensitivity ($g_r$)</th>
<th>Hedging Demand due to Total Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>55.28</td>
<td>55.28</td>
</tr>
<tr>
<td>10</td>
<td>93.21</td>
<td>93.21</td>
</tr>
<tr>
<td>20</td>
<td>121.38</td>
<td>120.96</td>
</tr>
<tr>
<td>30</td>
<td>143.23</td>
<td>141.99</td>
</tr>
</tbody>
</table>

Panel B: Labor Income to Financial Wealth Ratio = 0.10

<table>
<thead>
<tr>
<th>R</th>
<th>Labor Income Growth Sensitivity ($g_r$)</th>
<th>Hedging Demand due to Total Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>55.28</td>
<td>55.28</td>
</tr>
<tr>
<td>10</td>
<td>121.90</td>
<td>121.89</td>
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<tr>
<td>20</td>
<td>175.24</td>
<td>174.39</td>
</tr>
<tr>
<td>30</td>
<td>217.58</td>
<td>215.11</td>
</tr>
</tbody>
</table>

Panel C: Labor Income to Financial Wealth Ratio = 0.20

<table>
<thead>
<tr>
<th>R</th>
<th>Labor Income Growth Sensitivity ($g_r$)</th>
<th>Hedging Demand due to Total Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>55.28</td>
<td>55.28</td>
</tr>
<tr>
<td>10</td>
<td>179.29</td>
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<td>20</td>
<td>282.97</td>
<td>281.26</td>
</tr>
<tr>
<td>30</td>
<td>366.29</td>
<td>361.35</td>
</tr>
<tr>
<td>Allocation to Equity as % of financial wealth</td>
<td>Panel A: Relative Risk Aversion = 1</td>
<td>Panel B: Relative Risk Aversion = 5</td>
</tr>
<tr>
<td>--------------------------------------------</td>
<td>-----------------------------------</td>
<td>-----------------------------------</td>
</tr>
<tr>
<td>Current Labor Income</td>
<td>Allocation to Bond as % of financial wealth</td>
<td>Allocation to Bond as % of financial wealth</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>0.05</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>Allocation to Equity as % of financial wealth</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Current Labor Income</td>
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<td>100.00</td>
</tr>
<tr>
<td>0.10</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Allocation to Equity as % of financial wealth</td>
<td>144.46</td>
<td>144.46</td>
</tr>
<tr>
<td>Current Labor Income</td>
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<td>10</td>
</tr>
<tr>
<td>0.15</td>
<td>233.89</td>
<td>233.89</td>
</tr>
<tr>
<td>Allocation to Equity as % of financial wealth</td>
<td>277.86</td>
<td>277.86</td>
</tr>
<tr>
<td>Current Labor Income</td>
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</tr>
<tr>
<td>0.20</td>
<td>354.53</td>
<td>354.53</td>
</tr>
<tr>
<td>Allocation to Equity as % of financial wealth</td>
<td>418.48</td>
<td>418.48</td>
</tr>
<tr>
<td>Current Labor Income</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Allocation to Equity as % of financial wealth</td>
<td>524.64</td>
<td>524.64</td>
</tr>
<tr>
<td>Current Labor Income</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

This table presents the allocation of financial wealth to equity and long-term bonds for various assumptions on the risk aversion parameter, the current scaled labor income of the investor, and the retirement horizon. We assume the labor income process follows a Geometric Brownian Motion.

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Table 3.7
Portfolio Choice with Stochastic Labor Income and Inelastic Labor Supply

This table presents the allocation of financial wealth to equity and to the long-term bond of an investor with $\gamma = 5$, scaled labor income $Y = 0.10$, and for various assumption on the retirement horizon, labor income volatility and the sensitivity of labor income growth to interest rates.

<table>
<thead>
<tr>
<th>Allocation to Equity as % of financial wealth</th>
<th>Allocation to Bond as % of financial wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>({$\sigma_{1y} = -0.03, \sigma_{2y} = 0$})</td>
<td>({$\sigma_{1y} = 0, \sigma_{2y} = -0.03$})</td>
</tr>
<tr>
<td>Labor Income Growth Sensitivity ($g_r$)</td>
<td>Labor Income Growth Sensitivity ($g_r$)</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>51.13</td>
<td>51.13</td>
</tr>
<tr>
<td>75.84</td>
<td>75.39</td>
</tr>
<tr>
<td>95.57</td>
<td>94.31</td>
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<td>20.00</td>
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<tr>
<td>51.13</td>
<td>51.13</td>
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<td>75.84</td>
<td>75.39</td>
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<td>95.57</td>
<td>94.31</td>
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<tr>
<td>20.00</td>
<td>20.00</td>
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<td>51.13</td>
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<tr>
<td>75.84</td>
<td>75.39</td>
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<tr>
<td>95.57</td>
<td>94.31</td>
</tr>
<tr>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>51.13</td>
<td>51.13</td>
</tr>
<tr>
<td>75.84</td>
<td>75.39</td>
</tr>
<tr>
<td>95.57</td>
<td>94.31</td>
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<tr>
<td>20.00</td>
<td>20.00</td>
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<tr>
<td>51.13</td>
<td>51.13</td>
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<tr>
<td>75.84</td>
<td>75.39</td>
</tr>
<tr>
<td>95.57</td>
<td>94.31</td>
</tr>
<tr>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>51.13</td>
<td>51.13</td>
</tr>
<tr>
<td>75.84</td>
<td>75.39</td>
</tr>
<tr>
<td>95.57</td>
<td>94.31</td>
</tr>
<tr>
<td>20.00</td>
<td>20.00</td>
</tr>
<tr>
<td>51.13</td>
<td>51.13</td>
</tr>
<tr>
<td>75.84</td>
<td>75.39</td>
</tr>
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<td>95.57</td>
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<td>51.13</td>
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<td>75.84</td>
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<td>95.57</td>
<td>94.31</td>
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<td>51.13</td>
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<tr>
<td>95.57</td>
<td>94.31</td>
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<tr>
<td>20.00</td>
<td>20.00</td>
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### Allocation to Equity as % of financial wealth

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Table 3.9
Portfolio Choice with Stochastic Labor Income and Flexible Labor Supply ($\lambda = 0.7$)

This table presents the allocation of financial wealth to equity and to the long-term bond of an investor with $\gamma = 5$, scaled labor income $Y' = 0.10$, and for various assumptions on the retirement horizon, the sensitivity of labor income growth to interest rates, and preferences for consumption over leisure.

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Table 3.10
Portfolio Choice with Stochastic Labor Income and Flexible Labor Supply (λ = 0.9)

This table presents the allocation of financial wealth to equity and to the long-term bond of an investor with γ = 5, scaled labor income Y = 0.10, and for various assumptions on the retirement horizon, the sensitivity of labor income growth to interest rates, and preferences for consumption over leisure.

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Figure 3-1: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for the sensitivity of labor income growth to interest rates. The investor is assumed to have $\gamma = 5$. Labor supply is not flexible.
Figure 3-2: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for the sensitivity of labor income growth to interest rates. The investor is assumed to have $\gamma = 5$. Labor supply is flexible.
Figure 3-3: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for $\sigma_{1Y}$, the volatility of labor income. The investor is assumed to have $\gamma = 5, g_r = 0$. Labor supply is assumed to be inflexible.
Figure 3-4: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for $\sigma_{YY}$, the volatility of labor income. The investor is assumed to have $\gamma = 5$, $g_r = 0$. Labor supply is assumed to be flexible.
Figure 3-5: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for $\sigma_{2y}$, the volatility of labor income. The investor is assumed to have $\gamma = 5$, $g_r = 0$. Labor supply is assumed to be inflexible.
Figure 3-6: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for $\sigma_{1Y}$, the volatility of labor income. The investor is assumed to have $\gamma = 5$, $g_r = 0$. Labor supply is assumed to be flexible.
Figure 3-7: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumption for $\lambda$, the relative preference for consumption over leisure. The investor is assumed to have $\gamma = 5, g_r = 0$. Labor supply is assumed to be inflexible. Labor income is assumed to be locally riskless.
Figure 3.8: Allocation of financial wealth to the long-term bond and equity as a function of the investment horizon under various assumptions for \( L \), the amount of labor supplied. The investor is assumed to have \( \gamma = 5 \), \( g_r = 0 \). Labor supply is assumed to be inflexible.
Figure 3-9: Comparison of the allocation of financial wealth to long-term bonds under flexible and inflexible labor supply for an investor with $\gamma = 5$, and locally riskless labor income. The top graph shows the allocation for $L = 0.8$, $\lambda = 1$. The bottom graph shows the allocation for $L = 1$, $\lambda = 0.8$. 

![Graph 1](image1)

![Graph 2](image2)
Figure 3-10: Scaled labor income and human capital under no stochastic shock, constant growth for labor income and constant investment opportunity set. Scaled relative to financial wealth.
Figure 3-11: Scaled labor income and human capital under stochastic shocks to equity, constant growth for labor income and constant investment opportunity set. Scaled relative to financial wealth.
Figure 3-12: Allocation of financial wealth to long-term bond and equity assuming scaled labor income follows Figure 3-10 for an investor with \( \gamma = 5 \).