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STOCHASTIC QUANTIZATION OF FIELD THEORY IN FINITE AND INFINITE VOLUME

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ABSTRACT

Using the theory of Dirichlet forms a distribution-valued symmetric Markov process is constructed so that it has the $(\phi^4)_2$ measure as its invariant probability measure. Several properties of this process are proved and an infinite volume limit theorem is established.

KEY WORDS

$(\phi^4)_2$ measure, Distribution-valued stochastic differential equations, Euclidean quantum field theory, Dirichlet forms, Ergodic Markov processes.

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I. INTRODUCTION

Recently, Jona-Lasinio and Mitter [7], following up on a program proposed by Parisi and Wu [15], constructed a nonlinear stochastic differential equation taking values in the space of distributions on a finite rectangle in R^2 so that the resultant process is an ergodic Markov process whose unique invariant measure is coincident with the finite volume Euclidean $(\phi^4)_2$ measure. (See also [13], [14] for related expositions.) They called this procedure the stochastic quantization of field theory. This process, which we shall refer to as the $(\phi^4)_2$ process, is the continuum analog of the statistical mechanical models known as the interacting particle systems [10]. The motivation for the latter was to introduce dynamics in the study of equilibrium statistical mechanics so as to shed light on critical phenomena such as phase transitions. This was achieved by constructing a stochastic process which has the equilibrium (Gibbs') distribution as its invariant measure. Similar considerations motivate the study of the $(\phi^4)_2$ process.

In the existing work on distribution-valued s.d.e.s. (see, e.g., [6], [8]) the drift term of the s.d.e. in question is explicity specified as a suitable map of the state process. In contrast to this, the drift term of the $(\phi^4)_2$ process involves Wick ordering and is thus specified only as the L^p-limit of a suitable approximating sequence with respect to a specific probability measure. This presents certain technical problems. The approach of [7] was to generalize the concept of a weak solution for a finite dimensional s.d.e. to this infinite dimensional situation. This solution is defined by a change of measure argument based on the Cameron-Martin-Girsanov theorem [11] as in the finite dimensional case, but a rigorous justification of this procedure presents considerable technical difficulties. The key step is to show that the so-called Girsanov density (see (4.4)) is an exponential martingale. In [7] this is proved using moment estimates for the Girsanov density based on Feynman graph calculations and a variant of Nelson's estimates for exponents of integrals of semibounded Wick polynomials ([17], pp. 148-154).

This paper extends the results of [7] in several ways. The restriction in [7] that the parameter ε appearing in the s.d.e under study (see (3.7)) should belong to (0,1/10) is dropped. The process is constructed in both finite and infinite volume with general boundary conditions in the former case and the relation between them is clarified. The proof avoid several complicated computations needed in [7] .

The organization of the paper is as follows: Section II establishes the notation that will be followed throughout and briefly reviews the essentials of the infinite dimensional Ornstein-Uhlenbeck process. Section III gives a construction of the

$(\phi^4)_2$ process in finite volume using the theory of Dirichlet forms [1]. A stochastic differential equation satisfied by the process is also derived. Section IV establishes a suitable uniqueness in law result for this process for the finite volume case, in addition to establishing its path continuity, absolute continuity in law with respect to the Ornstein-Uhlenbeck process on finite time intervals and ergodicity. Section V proves a finite to infinite volume limit theorem under half-Dirichlet conditions at the process level, i.e., the corresponding $(\phi^4)_2$ processes are shown to converge in an appropriate sense.

REMARKS

The methods of this paper will also work for general $P(\phi)_2$ processes for semibounded P.

II. NOTATION AND PRELIMINARIES

Let $\Lambda \subset \Re^2$ be a finite open rectangle in $\Re^2$. Let

$S_1 = \{f:\Lambda \to \Re | f \varepsilon C^\infty(\bar{\Lambda})$ with $f = 0$ on the boundary$\}$,

$S_2 = C^\infty(\bar{\Lambda})$,

S' will denote $D'(\Lambda)$, the space of distributions on Λ and $\mathbf{S}'$, the space of tempered distributions in $\Re^2$.

Let $C_i = (-\Delta+I)^{-1}$, i = 1,2 with Dirichlet (resp. free) boundary conditions for Λ finite and $C_1 = C_2 = (-\Delta+I)^{-1}$ for $\Lambda = \Re^2$. Let $\{\lambda^{-1}_{n=1}, e_n\}$ and $\{\xi^{-1}_{n=1}, g_n\}$ denote the corresponding normalized eigenbases on $L^2(\Lambda)$, so that $\{e_n\}^\infty_{n=1} \subset S_1$ and $\{g_n\}^\infty_{n=1} \subset S_2$

For $\alpha \in \Re$, i = 1,2, let $H_i^\alpha(\Lambda)$ denote the Hilbert space obtained by completing S_i with respect to the inner product

$$\langle f,g\rangle_\alpha = \sum_n \lambda_n^\alpha \langle f,e_n\rangle \langle g,e_n\rangle \qquad \text{for } i = 1$$

and

$$\langle f,g\rangle_\alpha = \sum_n \xi_n^\alpha \langle f,g_n\rangle \langle g,g_n\rangle \qquad \text{for } i = 2$$

where $\langle\cdot,\cdot\rangle$ denotes L^2 scalar product. Clearly $\left\{\lambda_n^{\frac{-\alpha}{2}}, e_n\right\}$, $\left\{\xi_n^{\frac{-\alpha}{2}}, g_n\right\}$ will be orthonormal bases for $H_i^\alpha(\Lambda)$, i =1,2. In much of this paper, we shall prove results simultaneously for i = 1,2 and all finite Λ. Hence for simplicity, we may often delete the sub-index i, leaving the reader to infer the specific i from the context. The following is immediate:

(i) $H_i^\alpha \subset H_i^\beta$ for $\alpha \geq \beta$, i = 1,2,

(ii) $H_1^0 = L^2(\Lambda) = H_2^0$

$$\text{(iii)}\quad S_i \subset \cap_\alpha H_i^\alpha \qquad \cup_\alpha H_i^\alpha \subset S' \tag{2.1}$$

where $\cap_\alpha H_i^\alpha$ is topologized by countably many seminorms $\|\cdot\|_n = \sqrt{\langle\cdot,\cdot\rangle_n}$ and $\cup_\alpha H^{-\alpha}$ is then topologized via duality.

For a covariance operator C, $C(\cdot,\cdot)$ will denote its integral kernel, C^α, its α-th operator power and μ_C the centered Gaussian measure on S' with covariance operator C. Let $:\,:_j$, j = 1,2 denote the Wick ordering with respect to μ_{C_j}, j = 1,2 respectively (for details of Wick ordering, see [4], chapter 8). The $(\phi^4)_2$ measures $\mu_\Lambda(i,j)$, $i,j \in \{1,2\}$, for finite Λ are defined by

$$\frac{d\mu_\Lambda(i,j)}{d\mu_{C_i}} = \exp\Big(-\frac{1}{4}\int_\Lambda :\phi^4:_j\, dx\Big)/Z_\Lambda(i,j) \tag{2.2}$$

where

$$Z_\Lambda(i,j) = \int_\Lambda \exp\Big(-\frac{1}{4}\int_\Lambda :\phi^4:_j\, dx\Big)d\mu_{C_i}(\phi) \tag{2.3}$$

For finite Λ let Σ_Λ denote the sub-σ-field of the Borel σ-field of S' generated by the maps $\varphi \to \varphi(f)$ for smooth f supported in Λ. The infinite volume $(\phi^4)_2$ measure μ is defined by

$$\bar{\mu}\,|_{\Sigma_\Lambda} = \underset{\Lambda\subset\Lambda'\uparrow\Re^2}{\text{weak limit}}\ \mu_{\Lambda'}(1,2)|_{\Sigma_\Lambda} \tag{2.4}$$

See [4], Ch. 11, for details. $\bar{\mu}$ is supported on $\mathbf{S}'$ (Theorem VIII.26, pp.294, [17]). μ_{C_i} and hence $\mu_\Lambda(i,j)$ is supported on $H_i^{-1}(\Lambda)$, i=1,2 [7].

Let $0 < \varepsilon < 1$ and $\beta_i(\cdot)$, $i \geq 1$, a collection of independent standard Brownian motions. Define

$$W(t) = \sum \lambda_i^{-(1-\varepsilon)/2}\beta_i(t)e_i, \quad t \geq 0\ . \tag{2.5}$$

Since we are in the two dimensional case,

$$\sum \lambda_n^{-1-\delta} < \infty, \quad \sum \xi_n^{-1-\delta} < \infty \text{ for } \delta > 0 \ . \tag{2.6}$$

Thus the right hand side of (2.5) converges in the norm $E[\|\cdot\|^2_{H_1^{-1}(\Lambda)}]^{1/2}$ for each t.

Furthermore, for t>s, a straightforward computation using (2.6) shows that for any $\beta>\varepsilon$,

$$E\left[\|W(t) - W(s)\|^4_{H_1^{-\beta}(\Lambda)}\right] \le k|t - s|^2$$

for a suitable constant k. By Kolmogorov's test for sample path continuity, we can realize $W(\cdot)$ as an $H_1^{-\beta}(\Lambda)$–valued process with continuous sample paths. Using its independent increment property and centered Gaussian law at each time (both immediate from the definition), one can easily see that it is in fact an H_1^{-1}-valued Wiener process with covariance $C_1^{1-\varepsilon}$.

The $\mathbf{S}'$-valued Ornstein-Uhlenbeck process of [7] is described by the s.d.e.

$$d\phi(t) = -\frac{1}{2} C_1^{-\varepsilon}\phi(t)dt + dW(t), \ \phi(0) = \varphi\varepsilon\mathbf{S}' \tag{2.7}$$

Letting $X_n(t) = \phi(t)(e_n)$, $n \ge 1$, (2.7) decouples into an infinite system of noninteracting one dimensional O.-U. processes $X_n(\cdot)$ described by

$$dX_n(t) = -\frac{\lambda_n^\varepsilon}{2} X_n(t)dt + \lambda_n^{-(1-\varepsilon)/2} d\beta_n(t), \ n \ge 1 \ , \tag{2.8}$$

which can be explicitly solved as

$$X_n(t) = \exp(-\lambda_n^\varepsilon t/2)X_n(0) + \int_0^t \exp(-\lambda_n^\varepsilon (t-s)/2)\lambda_n^{-(1-\varepsilon)/2} d\beta_n(s) \tag{2.9}$$

Let $A_n(t)$, $B_n(t)$ denote respectively the first and the second term on the right hand side of (2.9). By (2.6), we have

$$\Sigma\lambda_n^{\alpha}\exp(-\lambda_n^{\varepsilon}t) < \infty, \quad \alpha\varepsilon\Re, t > 0$$

implying that

$$\Sigma A_n(t) \in H^{\alpha} \ \forall\alpha \in \Re, t > 0 .$$

Since

$$E[B_n(t)^2] = \lambda_n^{-1}(1-\exp(-\lambda_n^{\varepsilon}t)) \le \lambda_n^{-1} ,$$

(2.6) implies that $\Sigma B_n(t) \varepsilon H^{-\alpha}$ a.s. for $\alpha > 0$, $t \geq 0$.

In particular, we conclude that (2.7) defines an $H^{-\alpha}$-valued process on $(0, \infty)$ for each $\alpha > 0$ (on $[0, \infty)$ if $\varphi\varepsilon H^{-\alpha}$). Similar calculations show that for $0 < t < T < \infty$, $t_1 < t_2$ in $[t,T]$, we have

$$|A_n(t_2) - A_n(t_1)| \le K|t_2 - t_1|\lambda_n^{\varepsilon} \exp(-\lambda_n^{\varepsilon}\theta)\varphi(e_n) \tag{2.10}$$

$$E[(B_n(t_2) - B_n(t_1))^2] \le K|t_2 - t_1|\lambda_n^{-1-\varepsilon}\exp(-\lambda_n^{\varepsilon}\theta) \tag{2.11}$$

for suitable constant K, θ. Thus for $\alpha > 0$,

$$E\left[\|\phi(t_2) - \phi(t_1)\|^4_{H^{-\alpha}}\right] \le K'\left[\left(\Sigma\lambda_n^{-\alpha}(A_n(t_2) - A_n(t_1))^2\right)^2\right]$$

$$+ E\left[\left(\Sigma\lambda_n^{-\alpha}(B_n(t_2) - B_n(t_1))^2\right)^2\right]$$

$$= K'\left[\left(\Sigma\lambda_n^{-\alpha}(A_n(t_2) - A_n(t_1))^2\right)^2\right]$$

$$+ E\left[\Sigma\lambda_n^{-\alpha}(B_n(t_2) - B_n(t_1))^2\right]^2$$

$$+ 2\Sigma\lambda_n^{-2\alpha}E\left[(B_n(t_2) - B_n(t_1))^2\right]^2$$

(by properties of Gaussian measures)

$$\leq K''|t_2 - t_1|^2$$

for suitable constants K', K" by virtue of (2.6), (2.10), (2.11). By Kolmogorov's test and the arbitrariness in the choice of t, T, it follows that $\phi(\cdot)$ has a path-continuous version as an $H^{-\alpha}$-valued process on $(0,\infty)$. If $\varphi \varepsilon H^{-\alpha}$, a simple additional computation allows us to extend this claim to $[0,\infty)$.

For each n, (2.8) has $N(0, \lambda_n^{-1})$ as its unique invariant measure. Thus $\phi(\cdot)$ has μ_{C_1} as its invariant measure. In fact $\phi(\cdot)$ is ergodic with μ_{C_1} as its unique invariant measure ([7], pp. 418).

The above goes through in toto if C_1, μ_{C_1}, $\{\lambda_n\}$, $\{e_n\}$ is replaced by C_2, μ_{C_2}, $\{\xi_n\}$, $\{g_n\}$ respectively.

The following lemma will be of crucial importance in Section IV.

Lemma 2.1: For $\alpha>0$, $\|:\phi^3:_j\|_{H^{-\alpha}} \varepsilon L^p(\mu_{C_i})$, i, j ε {1,2), $p\geq 2$

Proof:

To begin with, let p = 2, and write i=j=1. Then

$$\int \|:\phi^3:_1\|^2_{H^{-\alpha}}\, d\mu_C(\phi) = \lim_{n\to\infty} \sum_{m=1}^{n} \lambda_m^{-\alpha} \int (:\phi^3:_1(e_m))^2 d\mu_C(\phi)$$

$$= \lim_{n\to\infty} 3! \int_\Lambda \Big(\sum_{m=1}^{n} \lambda_m^{-\alpha} e_m(x) e_m(y)\Big)(C_1(x,y))^3 dxdy$$

(see [17], p. 137)

$$\leq 3! \int_{\Re^2} C^{\alpha}(x,y)(C(x,y))^3 dxdy \, .$$

(An explicit expression for $C^{\alpha}(x,y)$ can be recovered from (5.1) in Section V).

$$= 3! \int_{\Re^2} \frac{1}{(|k|^2+1)^{\alpha}} \left(\frac{1}{|k|^2+1} * \frac{1}{|k|^2+1} * \frac{1}{|k|^2+1} \right) dk$$

(by Parseval's equation)

$$\leq K < \infty$$

Similar arguments prove the claim for $p = 2$, Λ finite when $i = j = 2$. For $i \neq j$, the claim follows from the additional fact that $:\phi^3:_1$, $:\phi^3:_2$ differ by a constant multiple of ϕ and $\|\phi\|_{H^{-\alpha}} \varepsilon \, L^2(\mu_{C_i})$ for $i = 1,2$ as the following computation (for $i=1$) shows:

$$\int \|\phi\|^2_{H^{-\alpha}} \, d\mu_{C_1} = \Sigma \lambda_n^{-\alpha} \int \phi(e_n)^2 d\mu_{C_1} = \Sigma \lambda_n^{-1-\alpha} < \infty \quad .$$

For general $p \geq 2$, it suffices to consider $p > 4$. By Theorem 1.22, pp. 38, [17], we have

$$\int \left(\sum_{m=1}^{n} \lambda_m^{-\alpha} (:\phi^3:(e_n))^2 \right)^{p/2} d\mu_c \leq \left(\frac{p}{2} - 1 \right)^{3p/2} K^{p/2} ,$$

where K is as in (2.12). Taking $n \to \infty$, the claim follows. q.e.d.

Corollary 2.1: For $g \varepsilon S$ with supp. $g \subset \Lambda$, $:\phi^3:(g) \varepsilon L^P(\mu_{C_i})$ for $i = 1,2$.

Proof:

Note that $|:\phi^3:(g)| \leq \|:\phi^3:\|_{H^{-\alpha}} \|g\|_{H^{\alpha}}$ for $\alpha > 0$. q.e.d.

III. CONSTRUCTION OF THE $(\phi)_2^4$ PROCESS IN FINITE VOLUME

In this section we construct the $(\phi)_2^4$ process by using the theory of Dirichlet forms.

Let $\Lambda \subset \Re^2$ be finite and let C, C' denote resp. C_i, C_j for some i,j in {1,2}. Let μ denote the corresponding measure $\mu_\Lambda(i,j)$ (see Eq. (2.2)) and : : denote Wick ordering with respect to C'. Let $\{\alpha_n^{-1}, h_n\}$ denote either $\{\lambda_n^{-1}, e_n\}$ or $\{\xi_n^{-1}, g_n\}$ depending on whether i = 1 or 2.

Let $P = \{h:H^{-1}(\Lambda) \to R \mid h(\phi) = p(\phi(h_{i_1}),...,\phi(h_{i_n}))$ for some $\{i_1,...,i_n\} \subset N$ and some smooth functions p s.t. p and its derivatives have polynomial growth $p:R^n \to R\}$. Define $\nabla:P \to K = L^2(\Lambda \times H^{-1}(\Lambda), d^2\chi \otimes d\mu)$ by

$$\nabla h(\phi) = \sum_{j=1}^{n} \alpha_{i_j}^{\frac{-1+\varepsilon}{2}} h_{i_j} \partial_j p(\phi(h_{i_1}),...,\phi(h_{i_n})) \tag{3.1}$$

where ε is a prescribed number in (0,1), $h(\phi) = p(\phi(h_{i_1}),...,\phi(h_{i_n}))$ and ∂_{ip} denotes $\frac{\partial p(x_1,...,x_n)}{\partial x_i}$. Let

$P^* = \{F(\cdot,\cdot) \mid F:\Lambda \times H^{-1}(\Lambda) \to \Re$ s.t. $F(x,\phi) = \Sigma_{j=1}^{n} p_j(\phi(h_{i_1}),...,\phi(h_{i_{k_j}}))f_j(x)$.
for some n, $k_j \in N$, $p_j \in P$, $f_j \in \mathbf{S}$, $\{i_1,...,i_{k_j}\} \subset N\}$. It is not difficult to check that P^* is dense in K. Define $\nabla^*:P^* \to L^2(\mu)$ by

$$\nabla^*(F(\phi)) = -\sum_{j=1}^{n} \alpha_{i_j}^{\frac{-1+\varepsilon}{2}} \partial_j p(\phi(h_{i_1}),...,\phi(h_{i_n}))\langle e,h_{i_j}\rangle$$

$$+\sum_{j=1}^{n} \alpha_{i_j}^{\frac{-1+\varepsilon}{2}} p(\phi(h_{i_1}),...,\phi(h_{i_n})):\phi^3:(h_{i_j})\langle e,h_{i_j}\rangle$$

$$+\sum_{j=1}^{n} \alpha_{i_j}^{\frac{1+\varepsilon}{2}} \phi(h_{i_j})(p(\phi(h_{i_1}),...,\phi(h_{i_n}))\langle e,h_{i_j}\rangle\ )$$

for $F(\phi) = p(\phi(h_{i_1}),...,\phi(h_{i_n}))e$, extended by linearity to all of P*. Here and in the sequel $\langle , \rangle$ denotes the scalar product on $L^2(\Lambda)$.

Lemma 3.1:

$$\int\langle q(\phi(h_{i_1}, ...,\phi(h_{i_n}))e, \nabla p(\phi(h_{j_1}),...,\phi(h_{j_m}))\rangle d\mu$$

$$= \int\nabla^*(q(\phi(h_{i_1}),...,\phi(h_{i_n}))e)p(\phi(h_{j_i}),...,\phi(h_{j_m}))d\mu \tag{3.2}$$

for q, p$\in$ P, $\{i_1,...,i_n\} \subset$ N, $\{j_1,...,j_m\} \subset$ N, e $\in$ S.

Proof:

Note that

$$\langle qe, \nabla p\rangle = \sum_{i=1}^{n} \alpha_{j_i}^{\frac{-1+\varepsilon}{2}} (\partial_i p)q\langle e,h_{j_i}\rangle \tag{3.3}$$

and

$$\begin{aligned}(\nabla*(qe))p = &- \sum_{j=1}^{n} \alpha_{i_j}^{\frac{-1+\varepsilon}{2}} (\partial_j q)p\langle e,h_{i_j}\rangle \\ &+ \sum_{j=1}^{n} \alpha_{i_j}^{\frac{1+\varepsilon}{2}} \phi(h_{i_j})qp\langle e,h_{i_j}\rangle \\ &+ \sum_{j=1}^{n} \alpha_{i_j}^{\frac{-1+\varepsilon}{2}} qp{:}\phi^3{:}(h_{i_j})\langle e,h_{i_j}\rangle\end{aligned} \tag{3.4}$$

Using the integration by parts formula for the $P(\phi)_2$ measure (cf. Eq. (9..1.32) in [4]) we get

$$\int\phi(h_{i_j})pqd\mu = \int\alpha_{i_j}^{-1}(\partial_j p)qd\mu$$

$$+ \int\alpha_{i_j}^{-1}(\partial_j q)pd\mu$$

$$- \int \alpha_{i_j}^{-1} pq{:}\phi^3{:}(h_{i_j})d\mu \tag{3.5}$$

Rearranging these terms, multiplying both sides by $\alpha_{i_j}^{1+\varepsilon/2}\langle e,h_{i_j}\rangle$, summing over j, we get an expression for the R.H.S. of (3.4) which when substituted into (3.4) yields (3.2) by virtue of (3.3). q.e.d.

Corollary 3.1: ∇ is closable.

Proof:

The above theorem shows that ∇^* is the adjoint of ∇. Since ∇^* is densely defined, the claim follows from a standard fact of functional analysis ([9] Theorem III.5.28). q.e.d

The closure of ∇ will be denoted by $\bar{\nabla}$. Set $L = -\frac{1}{2}\bar{\nabla}^* \cdot \bar{\nabla}$ on $D(L) = D(\bar{\nabla}) \subset L^2(\mu)$.

Corollary 3.2: -L is a densely defined positive definite self-adjoint operator on a dense domain D(-L) in $L^2(\mu)$ and $T_t = e^{-tL}$, $t \geq 0$ defines a self-adjoint strongly continuous contraction semigroup on $L^2(\mu)$.

Proof:

Immediate from the previous results. q.e.d

Direct computation show that for $h \in P$ of the form $h(\phi) = p(\phi(h_{i_j}),\ldots,\phi(h_{i_n}))$,

$$Lh = \frac{1}{2}\sum_{j=1}^{n} \{\alpha_{ij}^{-1+\varepsilon}\partial_j^2 p \ - \alpha_{ij}^{\varepsilon}\phi(h_{ij})\partial_j p \ - \alpha_{ij}^{-1+\varepsilon}\partial_j p{:}\phi^3{:}(h_{ij})\}. \tag{3.6}$$

Thus L coincides with the desired generator of the $(\phi^4)_2$ process [7] on P.

Remark:

In what follows, we use several results from the theory of Dirichlet forms as in [3]. The Dirichlet form theory is developed in [3] under a local compactness hypothesis on the state space of the symmetric Markov process under consideration. The state space of our process is not locally compact. However, the specific implications we have here do not require this restriction, as a careful look at the proofs of [3] shows.

Theorem 3.1: T_t, $t \geq 0$ is the transition semigroup of a Markov process.

Proof:

Note that L 1 = 0 implying that $T_t 1 = 1$ on $L^2(\mu)$ where 1 denotes the constant function identical to 1 on $H^{-1}(\Lambda)$. We need to show that T_t is positivity preserving i.e., is Markovian. By Theorem 1.4.1 [3] it suffices to show that the associated Dirichlet form $\varepsilon:\bar{D}(\bar{\nabla}) \otimes D(\bar{\nabla}) \to R$ defined by $\varepsilon(F,G) = \int\langle\bar{\nabla}F(\phi), \bar{\nabla}G(\phi)\rangle d\mu(\phi)$,

is Markovian in the sense of [3], p. 5. The properties of $\bar{\nabla}$ established above imply that ε is a closed symmetric form ([9], example VI.1.23). Letting ψ_ε denote the infinitely differentiable function ϕ_ε in Problem 1.2.1 of [3], p. 7 we have

$$\varepsilon(\psi_\varepsilon \circ F, \psi_\varepsilon \circ F) = \int(\psi'_\varepsilon(F(\phi)))^2 \|\bar{\nabla}F(\phi)\|^2 d\mu(\phi)$$

$$\leq \int\|\bar{\nabla}F(\phi)\|^2 d\mu(\phi)$$

$$= \varepsilon(F,F) \ .$$

Thus ε is Markovian.

By Theorem 1.4.1 of [3] it follows that T_t, $t \geq 0$ is the transition semigroup of a Markov process. Since -L is self-adjoint on D(-L) $L^2(\mu)$, T_t is self-adjoint on $L^2(\mu)$, $\forall t \geq 0$. Thus $\langle T_t f,g\rangle = \langle f,T_t g\rangle$, $\forall f, g \in L^2(\mu)$ implying (taking f=1) $\int g d\mu = \int T_t g d\mu \ \forall g \in L^2(\mu)$, that is μ is an invariant probability measure for this process. The symmetricity of the process follows from the self-adjointness of (T_t).

q.e.d.

We shall refer to the process constructed above as the $(\phi^4)_2$-process. It will be denoted by $\phi(\cdot)$.

Theorem 3.2: There exists an $H^{-1}(\Lambda)$-valued Wiener process $\overline{W}(t)$ with covariance $C^{1-\varepsilon}$ such that the following s.d.e. holds

$$d\phi(t) = -1/2(C^{-\varepsilon}\phi(t) + C^{1-\varepsilon}\!: \phi(t)^3\!:)dt + d\overline{W}(t) \tag{3.7}$$

with $\phi(0)$ having the law μ.

Proof:

(3.7) is equivalent to

$$\phi(t)(f) - \phi(0)(f) = -\int_0^t 1/2(\phi(s)(C^{-\varepsilon}f) + :\phi(s)^3:(C^{1-\varepsilon}f))ds + \overline{W}(t)(f) \tag{3.8}$$

for $f \in S$. Using (3.6) and mimicking the arguments of [1], pp. 19-21 one has

$$\phi(t)(h_i) - \phi(0)(h_i) = -1/2 \int_0^t (\alpha_i^{\varepsilon}\phi(s)(h_i) + (\alpha_i^{1-\varepsilon}:\phi(s)^3:(h_i))ds + \alpha_i^{\frac{-1+\varepsilon}{2}} \bar{\beta}_i(t) , \qquad (3.9)$$

$i = 1,2,\ldots$, where $\beta_i(.)$ are independent standard Brownian motions. Define

$$\bar{W}(t) = \sum_{i=1}^{\infty} (\alpha_i^{\frac{-1+\varepsilon}{2}})\bar{\beta}_i(t)h_i$$

show that $\bar{W}(\cdot)$ is an H-b-valued Wiener process for b > e with covariance C^{1-e}. Using arguments identical to those used for the process $W(\cdot)$ in Section II, we can show that $W(\cdot)$ is an $H^{-\beta}$-valued Wiener process for $\beta > \varepsilon$ with covariance $C^{1-\varepsilon}$. Then (3.9) becomes a special case of (3.8). The general case follows by linearity and limiting arguments, since μ_C and hence μ is supported on $H^{-1}(\Lambda)$. q.e.d.

IV. THE $(\phi^4)_2$ PROCESS IN FINITE VOLUME

Let Λ be finite and $\phi(\cdot)$ the $(\phi^4)_2$ process constructed in the preceding section. Let $\phi_0(\cdot)$ denote the Ornstein-Uhlenbeck process as in (2.7) with Λ as above with C replacing C_1 in (2.7), the initial law being the invariant law μ_C. Let $\alpha = (1-\varepsilon)/2$. From Lemma 2.1, we have

$$E\left[\int_0^T \langle :\phi_0^3(t):,\ C^{1-\varepsilon}:\phi_0^3(t):\rangle_{H^{-\alpha},H^{\alpha}}\,dt\right]$$

$$= TE\left[\|:\phi_0^3(0):\|^2_{H^{-\alpha}}\right] \quad \text{(by stationarity)}$$

$$< \infty \tag{4.1}$$

for all T>0. Let Λ be finite. By Lemma 2.1 and the Schwartz inequality, it follows that

$$E\left[\int_0^T \langle :\phi^3(t):,\ C^{1-\varepsilon}:\phi^3(t):\rangle_{H^{-\alpha},\,H^{\alpha}}\,dt\right]$$

$$= TE\left[\|:\phi^3(0):\|^2_{H^{-\alpha}}\right]$$

$$\leq T\left(E\left[\|:\phi_0^3(0):\|^4_{H_{-\alpha}}\right]\right)^{1/2}\left(E\left[\left(\frac{d\mu}{d\mu_C}\right)^2\right]\right)^{1/2} < \infty. \tag{4.2}$$

This allows us to define as in [7] the stochastic integral

$$\int_0^t \langle :\phi^3(s):,\ d\bar{W}(s)\rangle,\ \ t \geq 0 \tag{4.3}$$

by means of finite dimensional approximations. The reader is referred to [7] for details of the actual construction. Note that (4.3) is a zero mean square-integrable martingale whose quadratic variation process is

$$\int_0^t \left\langle :\phi^3(s):,\ C^{1-\varepsilon}:\phi^3(s):\right\rangle_{H^{-\alpha},H^{\alpha}} ds, \quad t \geq 0,$$

with α as above, and whose quadratic covariation process with $\beta_i(\cdot)$, $i \geq 1$, is

$$\alpha_i^{-(1-\varepsilon)/2} \int_0^t :\phi^3(s):(h_i)ds, \quad i \geq 1,$$

respectively. Either observation is immediate from the finite dimensional approximations used to construct (4.3).

Let $(\Omega, \mathcal{F}, P)$ be the probability space underlying (4.1). Define a new measure P_0 on $(\Omega, \mathcal{F})$ by

$$\Gamma \stackrel{\text{def}}{=} \frac{dP_0}{dP} = \exp\left(\frac{1}{2.}\int_0^T \left\langle :\phi^3(s):, d\tilde{W}(s)\right\rangle - \frac{1}{8}\int_0^T \left\langle :\phi^3(s):, C^{1-\varepsilon}:\phi^3(s):\right\rangle_{H^{-\alpha},H^{\alpha}} ds + \frac{1}{4}\int_\Lambda :\phi^4(0):dx\right)/Z_\Lambda(i,j) \qquad (4.4)$$

Lemma 4.1: P_0 is a probability measure under which the law of $\phi(\cdot)$ is the law of $\phi_0(\cdot)$ described above.

Proof:

For $n \geq 1$, define

$$\tau_{0n} = \inf\left\{t \geq 0 \,\middle|\, \int_0^t \left\langle :\phi_0^3(s):, C^{1-\varepsilon}:\phi_0^3(s):\right\rangle_{H^{-\alpha},H^{\alpha}} ds > n\right\}$$

$$\tau_n = \inf\left\{t \geq 0 \,\middle|\, \int_0^t \left\langle :\phi^3(s):, C^{1-\varepsilon}:\phi^3(s):\right\rangle_{H^{-\alpha},H^{\alpha}} ds > n\right\}$$

By (4.1), (4.2), it follows that $\tau_{0n} \uparrow \infty$ a.s., $\tau_n \uparrow \infty$ a.s. For each $n \geq 1$, define Γ_n as in (4.4) with T replaced by $T \wedge \tau_n$ on the right hand side. By Novikov's criterion, ([5], pp. 142), it follows that $\int \Gamma_n dP_0 = 1$. Since $\Gamma n \to \Gamma$ a.s., it suffices to prove that $\{\Gamma_n\}$ are uniformly integrable. A sufficient condition for this is

$$E\{\Gamma_n \log \Gamma_n] \le K \ \forall n, \tag{4.5}$$

for some K independent of n. However,

$$E[\Gamma_n \log \Gamma_n] = E\Big[\Big(\frac{1}{2}\int_0^{T\wedge\tau_n} \langle :\phi^3(s):, d\bar{W}(s)\rangle$$

$$\frac{1}{8}\int_0^{T\wedge\tau_n} \langle :\phi^3(s):, C^{1-\varepsilon}: \phi^3(s):\rangle_{H^{-\alpha}, H^{\alpha}} ds + \frac{1}{4}\int_\Lambda :\phi^4(0):dx\Big)\Gamma_n\Big] - \log Z_\Lambda(i,j) \cdot$$

By the Cameron-Martin-Girsanov theorem [11] and the observations immediately preceding the statement of the Lemma, it follows that under probability measure P_n defined by

$$\frac{dP_n}{dP} = \Gamma_n ,$$

the process $f(\cdot\wedge\tau_n)$ satisfies on $[0, \tau_n]$ the equation (2.7) with $\overline{W}(\cdot)$ replaced by

$$W(t) = \bar{W}(t) - \frac{1}{2}\int_0^t C^{1-\varepsilon}:\phi^3(s):ds, \qquad t \ge 0,$$

which is a Wiener process on $[0, \tau_n)$. Furthermore, the initial law of $\phi(\cdot)$ under P_n is μ_C. By uniqueness of a strong solution to (2.7), it follows that the law of $\phi(\cdot)$ on $[0,\tau_n)$ coincides with the law of $\phi_0(\cdot)$ on $[0,\tau_{0n})$ under P_0. Thus (4.6) equals

$$E\Bigg[\Bigg(\frac{1}{2}\int_0^{T\wedge\tau_{0n}} \langle :\phi_0^3(s):, dW(s)\rangle$$

$$+\frac{1}{8}\int_0^{T\wedge\tau_n} \langle :\phi_0^3(s):, C^{1-\varepsilon}:\phi_0^3(s):\rangle_{H^{-\alpha}, H^{\alpha}} ds$$

$$+\frac{1}{4}\int_\Lambda :\phi_0^4(0):dx\Big) - \log Z_\Lambda(i,j)\Big]$$

$$\le \frac{1}{8} E\left[\int_0^T \langle :\phi_0^3(s):, C^{1-\varepsilon}:\phi_0^3(s):\rangle_{H^{-\alpha}, H^{\alpha}} ds - \log Z_\Lambda(i,j)\right] < \infty$$

by (4.2). (4.5) follows, implying $E[\Gamma] = 1$. Thus P_0 is a probability measure. The rest follows from an application of the Cameron-Martin-Girsanov theorem along the lines already indicated above.

q.e.d.

Corollary 4.1: The solution of (3.7) with initial law μ is unique in law for finite Λ and yields an $H^{-\beta}$-valued process with continuous paths for each $\beta > 0$.

Proof:

This is immediate from the foregoing and the sample path properties of the O.-U. process. q.e.d.

Theorem 4.1: For finite Λ, $\phi(\cdot)$ as in (3.7) with initial law μ is an ergodic process.

Proof:

Suppose not. Then there exists a Borel set $A \subset H^{-1}(\Lambda)$ such that $\mu(A) > 0$, $\mu(A^c) > 0$ and for some $t > 0$,

$$E[I\{\phi(t)\varepsilon A^c\}I\{\phi(0)\varepsilon A\}] = 0 ,$$

Without any loss of generality, we let $T > t$ and obtain

$$E[I\{\phi_0(t)\varepsilon A^c\}I\{\phi_0(0)\varepsilon A\}\Gamma_0)] = 0 ,$$

where

$$\Gamma_0 = \exp\Bigg(-\frac{1}{2}\int_0^T \langle :\phi_0^3(s):, dW(s)\rangle$$

$$\frac{1}{8}\int_0^T \langle :\phi_0^3(s):, C^{1-\varepsilon}:\phi_0^3(s):\rangle_{H^{-\alpha},H^{\alpha}} ds - \frac{1}{4}\int_\Lambda :\phi_0^4(0):dx\Bigg)/Z_\Lambda(i,j) ,$$

for $\alpha = (1-\varepsilon)/2$, where the construction of the stochastic integral is analogous to that of (4.3). Since $\Gamma_0 > 0$ a.s.,

$$E[I\{\phi_0(t)\ \varepsilon\ A^c\}I\{\phi_0(0)\varepsilon A\}] = 0 .$$

By ergodicity of $\phi_0(\cdot)$, either $\mu_C(A) = 0$ or $\mu_C(A^c) = 0$. But μ, μ_C are mutually absolutely continuous. Thus either $\mu(A) = 0$ or $\mu(A^c) = 0$, a contradiction. The claim follows. q.e.d.

COROLLARY 4.2: If $\tilde{\phi}(\cdot)$ is a stationary solution of (4.1) with the law of $\tilde{\phi}(0) = \tilde{\mu}$ satisfying $\tilde{\mu} << \mu$, then $\tilde{\mu} = \mu$ and $\tilde{\phi}(\cdot)$, $\phi(\cdot)$ agree in law.

The proof follows easily from Theorem 4.1 and the ergodic theorem. We shall strengthen these results in what follows.

Let $\varphi \varepsilon H^{-\alpha}$ from some $0 < \alpha \leq 1$. Let $\phi_0(\cdot)$ denote the O.-U. process with the same dynamics as that of $\phi_0(\cdot)$ but with initial condition φ. Let $p(t,\phi,\phi')$ denote the transition probabilities for $\phi_0(\cdot)$, $t > 0$. From [7], pp. 417, we know that they are mutually absolutely continuous with respect to μ_C with the Radon-Nikodym derivative $\rho_{t,\phi}(\phi')$ being square integrable with respect to μ_C. Furthermore (from Eq. 2.28, p. 419 [7]),

$$\int |\rho_{t,\phi}(\phi')|^2 d\mu_C(\phi') = \left[\det\left(I - e^{-2tC^{-\varepsilon}}\right)\right]^{-1/2} \exp\left(\left\langle \phi, C^{-1} e^{-tC^{-\varepsilon}}\left(1 + e^{-tC^{-\varepsilon}}\right)^{-1} \phi\right\rangle\right) .$$

Let $F(\varphi,t)$ denote the R.H.S. with φ replacing ϕ. Then for $\beta > 0$, $T > 0$, $p \geq 2$,

$$E\left[\int_0^T ||\tilde{\phi}_0(t)||^p_{H^{-\beta}} dt\right] \leq \int_0^T E\left[||\phi_0(t)||^{2p}_{H^{-\beta}}\right]^{1/2} F(\varphi,t)^{1/2} dt$$

$$= E\left[||\phi_0(0)||^{2p}_{H^{-\beta}}\right]^{1/2} \int_0^T F(\varphi,t)^{1/2} dt$$

Note that

$$\int_0^T F(\varphi,t)dt \leq K \int_0^T \frac{dt}{(\det(I - \exp(-2tC^{-\varepsilon})))^{1/2}}$$

for some constant K. Since $\det(I - \exp(-2t\,C^{-\varepsilon})) = \Pi_n(1 - \exp(-2t\lambda_n^{\varepsilon}))$ and $\lambda_n \to \infty$, it follows that for some constant K',

$$\det(I - \exp(-2t(C^{-\varepsilon}))) \leq K'^{-1} \prod_n (1 - \exp(-2t\lambda_n^{-1}))$$

$$= \det(I - \exp(-2tC))/K'$$

Now

$$I - \exp(-2t(C)) = (2tC)D(t)$$

where

$$D(t) = 1 - tC + \frac{(2t)^2}{3!} C^2 - \dots ,$$

the series being absolutely convergent. Thus det(I - exp(-2t(C)) = 2t det C det D(t). Since det D(t) $\to$ 1 as t $\to$ 0, $(\det(I - \exp(-2tC))^{1/2} \sim t^{1/2}$ for small t. Thus

$$\int_0^T F(\varphi, t)dt < \infty$$

implying

$$E\left[\int_0^T \|\bar{\phi}_0(t)\|^p_{H^{-\beta}}\, dt\right] < \infty \tag{4.7}$$

Suppose (3.7) has a solution $\bar{\phi}(\cdot)$ with initial condition φ. (This is certainly true for μ_C-a.s. φ by virtue of the foregoing.)

Lemma 4.2. The law $\bar{P}$ of $\bar{\phi}(\cdot)$ restricted to [0,T] is mutually absolutely continuous with respect to the law $\bar{P}_0$ of $\bar{\phi}_0(\cdot)$ restricted to [0,T] with.

$$\frac{d\bar{P}_0}{d\bar{P}} = \exp\left(\frac{1}{2}\int_0^T \langle :\bar{\phi}^3(s):, d\bar{W}(s)\rangle - \frac{1}{8}\langle :\bar{\phi}^3(s):, C^{1-\varepsilon}:\bar{\phi}^3(s):\rangle_{H^{-\alpha}, H^{\alpha}}\, ds\right)$$

Proof:

Define τ_n, τ_{0n} the same way as τ_n, τ_{0n} with $\phi(\cdot)$ replacing $\phi(\cdot)$, $\phi(\cdot)$ respectively. As in (4.3), we can define $\int_0^t \langle :\phi^3(s):, dW(s)\rangle$ for $t\varepsilon[0,\lim_n\tau_n)$. Let $\tau = \lim_n\tau_n$. Mimicking the arguments of Lemma 4.1, one reaches the desired conclusions with $\phi(\cdot)$ replaced by $\phi(\cdot\wedge\tau)$. (Here we make a crucial use of (4.7) in the obvious fashion.) Let $\tau_0 = \lim_n \tau_{0n}$. Since $\tau_0 = \infty$ a.s. in view of (4.7), $\tau_0 \wedge T = T$ a.s. By mutual absolute continuity of the probability measures under consideration, $\tau \wedge T = T$ a.s. and we are done. q.e.d.

Theorem 4.2: $\phi(\cdot)$ has a unique invariant probability measure as an H^{-1}-valued process.

Proof:

Suppose $\mu' \neq \mu$ is an invariant probability measure for $\phi(\cdot)$ as an H^{-1}-valued process. Let $t > 0$. From Lemma 4.2 and the fact that the transition probabilities of $\phi_0(\cdot)$ for $t > 0$ are mutually absolutely continuous with respect to μ_C and hence with respect to μ, it follows that the law of $\phi(t)$ above is mutually absolutely continuous with respect to μ. Since this is true for μ'-a.s. φ, μ' itself must be mutally absolutely continuous with respect to μ. The claim is now immediate from Corollary 4.2. q.e.d.

Theorem 4.3: If (3.7) has a solution on [0,T] for some initial law η (supported on H^{-1}) then it is unique in law and yields an $H^{-\beta}$-valued process with continuous paths on $(0,\infty)$ for each $\beta>0$. This can be strengthened to $[0,\infty]$ if η is supported on $H^{-\beta}$.

This is immediate from Lemma 4.2.

V. THE INFINITE VOLUME LIMIT AT THE PROCESS LEVEL

Recall that $\overline{\mu}$ was constructed by taking the limit of finite volume $(\phi^4)_2$ measures under half-Dirichlet conditions, i.e., i = 1, j = 2. We prove here an analogous convergence theorem at the process level.

Let $\{\Lambda_n\}$ denote a sequence of open rectangles in R^2 increasing to R^2. Let C_{1n}, C_{2n} denote the operator $(-\Delta+1)^{-1}$ on Λ_n with Dirichlet and free boundary conditions respectively, $C = (-\Delta+1)^{-1}$ on R^2 and $::_n$, $::$ the Wick ordering with respect to C_{2n}, C respectively.

Lemma 5.1: For $\alpha \in (0,1]$; $n \in N$ and $f \in C(R^2)$ with $f \geq 0$ and supp. $f \subset \Lambda_n$, $0 \leq C^{\alpha}_{1m} f \uparrow C^{\alpha} f$ on Λ_n as $n \leq m \to \infty$ and $C^{\alpha} f \in L_p(R^2)$ $\forall p \in [1,\infty)$ (and hence for all $f \in C(R^2)$ with supp. $f \subset \Lambda_n$, $C^{\alpha} f \in L^p(R^2)$ $\forall p \in [1,\infty)$ and $C^{\alpha}_{1m} f \to C^{\alpha} f$ on Λ_n as $n \leq m \to \infty$ pointwise and in $L^p(R^2)$ if we extend $C^{\alpha}_{1m} f$ to R^2 by setting it equal to zero outside Λ_m.)

Proof:

From the expression for fractional powers of operators in [16], p. 70 and the probabilistic representation of semigroups generated by C^{-1}_{1n} and C^{-1} [18], it follows that

$$C^{\alpha}_{1n} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} E(t^{\alpha-1} e^{-t} f(x+B(t)) I[\tau_n \geq t])dt$$

$$\begin{aligned} C^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} E(t^{\alpha-1} e^{-t} f(x+B(t)))dt \\ &= \frac{1}{2\pi\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-2} e^{-t} \int_{R^2} f(y) \exp\left(\frac{-\|x-y\|^2}{2t}\right) dydt \end{aligned} \tag{5.1}$$

where $B(\cdot)$ is the two-dimensional standard Brownian motion and $\tau_n = \inf \{t \geq 0 | x+B(t) \notin \Lambda_n\}$. The claim follows immediately. q.e.d.

Lemma 5.2: There exists a constant $c < \infty$ such that for any g in $D(\Lambda)$ and $p = 4/(4-j)$, $j<4$ we have

$$\int \exp(:\phi^j:(g))d\mu \le \exp(c(\|g\|_{L_1} + \|g\|_{L_p}^p)) \tag{5.2}$$

where c does not depend on Λ.

Proof:

The case for finite Λ is proved in Theorem 12.2.2[4]. Let g be compactly supported such that supp. $g \subset \Lambda_i$ with $\Lambda_i \uparrow R^2$ as in Theorem 1.2.1 [4]. By this theorem, the laws of $\exp(:\phi^j:(g))$ under μ_{Λ_i} converges weakly to that under μ for $\Lambda = R^2$ as $i \to \infty$. The claim now follows by Fatou's Lemma and Skorohod's theorem that allows us to pass from convergence in law to almost sure convergence ([5], p.9). q.e.d.

Remark

By a simple approximation argument, this extends to g in $\bigcap_\alpha H_1^\alpha(\Lambda)$ for finite Λ. This, in fact, is the case we use later.

Corollary 5.1: For c,j,p,g as above and n = 1,2,..., there exists a constant M depending on n such that,

$$\int \left| :\phi^3(g): \right|^n d\mu \le M(n) \exp\left(c\left(\|g\|_{L_1} + \|g\|_{L_p}^p \right) \right) \tag{5.3}$$

For $n \ge 1$, let $\{W^n(t)\}$ be an $H^{-1}(\Lambda_n)$-valued process with covariance operator $C_{1n}^{1-\varepsilon}$ and $\{\phi^n(t)\}$ the $H^{-1}(\Lambda_n)$-valued process satisfying

$$d\phi^n(t) = -1/2\left(C_{1n}^{-\varepsilon}\phi^n(t) + C_{1n}^{1-\varepsilon} :\left(\phi^n(t)\right)^3 :_n \right)dt + dW^n(t) \tag{5.4}$$

with initial law μ_n = its invariant measure. For $n \ge m$, $\phi^{n,m}(t)$, $W^{n,m}(t)$ will denote the restriction of $\phi^n(t)$, $W^n(t)$ respectively to Λ_m. Fix $m \ge 1$ and $f \in \bigcap_\alpha H^\alpha(\Lambda)$. Let $h = (-\Delta+1)f$. Since the restriction of an element of $H_1^{-1}(\Lambda_n)$ to Λ_m lies in $H_1^{-1}(\Lambda_m)$, $\phi^{n,m}(\cdot)$, $W^{n,m}(\cdot)$ are $H_1^{-1}(\Lambda_m)$-valued processes. We can similarly define H^- 1,1)(Λ_m)-valued processes $\phi_i^{n,m}(\cdot)$, i = 1,2,3, by

$$\phi_1^{n,m}(t)(f) = -\frac{1}{2}\int_0^t \phi^n(s)(C_{1n}^{1-\varepsilon}f)ds, \qquad f \varepsilon \cap_\alpha H^\alpha(\Lambda_m)$$

$$\phi_2^{n,m}(t)(f) = -\frac{1}{2}\int_0^t :\phi^n(s))^3:_n(C_{1n}^{1-\varepsilon}f)ds), \qquad f \varepsilon \cap_\alpha H^\alpha(\Lambda_m)$$

$$\phi_3^{n,m}(t) = W^{n,m}(t)$$

Then

$$\phi^{n,m}(t)(f) - \phi^{n,m}(0)(f) = \phi_1^{n,m}(t)(h) + \phi_2^{n,m}(t)(f) + \phi_3^{n,m}(t)(f),\ f\varepsilon D(\Lambda_m). \quad (5.5)$$

Lemma 5.3. The laws of $(\phi^{n,m}(\cdot)(f), \phi_1^{n,m}(\cdot)(h), \phi_2^{n,m}(f), \phi_3^{n,m}(\cdot)(f), n \geq m$ are tight as probability measures on $(C[0, \infty))^4$.

Proof.

Let $T \geq t_2 \geq t_1 \geq 0$. In what follows, M denotes a constant, not always the same. Note that

$$E\left[\left|\phi^{n,m}(t_2)(f) - \phi^{n,m}(t_1)(f)\right|^4\right] \leq M\sum_{i=1}^{3} E\left[\left|\phi_i^{n,m}(t_2)(f) - \phi_i^{n,m}(t_1)(f)\right|^4\right] \quad (5.6)$$

Now

$$\begin{aligned} E\left[\left|\phi_3^{n,m}(t_2)(f) - \phi_3^{n,m}(t_1)(f)\right|^4\right] &\leq M(t_2 - t_1)^2(C_{1n}^{1-\varepsilon}(f,f))^2 \\ &= M(t_2 - t_1)^2 \|C_{1n}^{(1-\varepsilon)/2}f\|^2_{L^2(\Lambda_n)} \\ &\leq M(t_2 - t_1)^2 \|C_{1n}^{(1-\varepsilon)/2}|f|\|^2_{L^2(R^2)} \end{aligned} \quad (5.7)$$

where the last step follows from Lemma 5.1 and the observation that C_{1n}^{α}, $\alpha\varepsilon(0,1)$, as a map from $L^2(\Lambda_n)$ to $L^2(\Lambda_n)$ preserves positivity (see (5.1).) By the stationarity of $\{\phi^{n,m}(t)\}$, Corollary 5.1 and Lemma 5.1, we have

$$E\left[\|\phi_1^{n,m}(t_2)(h) - \phi_1^{n,m}(t_1)(h)\|^4\right] \le M(t_2 - t_1)^2 E\left[\|\phi^n(0))C_{1n}^{1-\varepsilon}h)\|^4\right]$$

$$\le M(t_2 - t_1)^2 \exp\left(c\left(\|C_{1n}^{1-\varepsilon}h\|_{L^1(\Lambda_n)}^{1} + \|C_{1n}^{1-\varepsilon}h\|_{L^{4/3}(\Lambda_n)}^{4/3}\right)\right)$$

$$\le M(t_2 - t_1)^2 \exp\left(2c\left(\|C^{1-\varepsilon}|h|\,\|_{L^1(\Re^2)}^{1} + \|C^{1-\varepsilon}|h|\,\|_{L^{4/3}}^{4/3}(\Re^2)\right)\right)$$

A similar argument shows that

$$E\left[\left|\phi_2^{n,m}(t_2)(f) - \phi_2^{n,m}(t_1)(f)\right|^4\right] \le M(t_2 - t_1)^2$$
$$\exp\left(2c\left(\|C^{1-\varepsilon}|f|\|_{L^1(\Re^2)}^{1} + \|C^{1-\varepsilon}|f|\|_{L^4(\Re^2)}^{4}\right)\right) \qquad (5.9)$$

From Theorem 11.2.1, [4], it follows that the laws of $\phi^{n,m}(0)(f)$, $n \ge m$, converges weakly and hence are tight. The claim follows from (5.6)-(5.9) and the tightness criterion of [2], p. 95. q.e.d.

Consider the processes $\phi^{n,m}(\cdot)$, $n \ge m$. Fix m for the time being. By Theorem 3.1, [12], the above implies that these are tight as $C([0,\infty); \cup_\alpha H^{-\alpha}(\Lambda_m))$-valued random variables. Let $\{e_j\}$ denote the normalized eigenvectors of $(-\Delta+I)$ on Λ_m with Dirichlet boundary conditions. By a diagonal argument, we can pick a subsequence of $n \ge m$, denoted $\{n\}$ again, along which $\{\phi^{n,m}(\cdot)(e_1), \phi^{n,m}(\cdot)(e_2)\ldots\}$ converge in law as $(C([0,\infty); R))^\infty$-valued random variables as $n \to \infty$. In particular, for any finite subset $\{t_1, t_2,\ldots,t_k\}$ of $[0,\infty)$ and a collection $\{g_1,\ldots,g_k\}$ of finite linear combinations of $\{e_i\}$, the joint laws of $\{\phi^{n,m}(t_1)(g_1),\ldots,\phi^{n,m}(t_k)(g_k)\}$ converge. Consider $f_1,\ldots,f_h \in \cap_\alpha H^\alpha(\Lambda_m)$. Argue as in the proof of Corollary 12.2.4, pp. 222-3, [4], to conclude that for $1 \le i \le k$,

$$E[|\phi^{n,m}(t_i)(f_i) - \phi^{n,m}(t_i)(g_i)|^2] \le M\|f_i - g_i\|_{L^{4/3}(\Lambda_m)} \qquad (5.10)$$

for a suitable constant M depending on m. It is easy to see that for given $\{f_i\}$, the right hand side of (5.10) can be made smaller than any prescribed $\delta>0$ for each i by a suitable choice of $\{g_i\}$.

Let $\{h_i\}$ denote an enumeration of finite linear combinations of $\{e_j\}$ with rational coefficients. By Skorohod's theorem ([5], pp.9), we can construct on some probability space random variables X_{nil}, Y_{il} such that $\{X_{nil}\}$, $\{\phi^{n,m}(t_i)(h_l)$ agree in law for each n, i and $X_{nil} \to Y_{il}$ a.s. as $n \to \infty$. By augmenting this probability space if necessary, construct on it random variables Z_{ni} such that the joint law of $[Z_{ni}, X_{nil}, X_{ni2},...]$ coincides with that of $[\phi^{n,m}(t_i)(f_i), \phi^{n,m}(t_i)(h_i), \phi^{n,m}(t_i)(h_2),...]$ for each n,i. Let $\delta > 0$. Now

$$E[|Z_{ni} - X_{nil}|^2] = E[|\phi^{n,m}(t_i)(f_i - h_l)|^2],$$

which, in view of (5.10), can be made smaller than δ uniformly in n by a suitable choice of h_l, l depending on i. Thus for $n' \geq n \geq m$,

$$\lim_{n',n \to \infty} E[|Z_{n'i} - Z_{ni}|^2] \leq 2\delta + \lim_{n',n \to \infty} E[|X_{n'il} - X_{nil}|^2] = 2\delta$$

where the passage from a.s. convergence to zero of $X_{n'il} - X_{nil}$ to mean square convergence is obtained by moment bounds derivable from (5.2). Thus $[Z_{nl},....,Z_{nk}]$ converges in mean square as $n \to \infty$. Hence $[\phi^{n,m}(t_1)(f_1),...,\phi^{n,m}(t_k)(f_k)]$ converges in law as $n \to \infty$. Theorem 5.3, [12], can now be invoked to claim that $\phi^{n,m}(\cdot)$, $n \geq m$, (converge in law as $C([0, \infty)$, $\cup_\alpha H^{-\alpha}(\Lambda_m))$-valued random) variables. By a diagonal argument, we can find a further subsequence of $\{n\}$, again denoted by $\{n\}$, such that this is true for each $m \geq 1$. This allows us to consistently define a probability measure on $C([0,\infty); D'(R^2))$, endowed with its Borel σ-field, according to the following recipe: Let $\phi(w,t)$, $w \varepsilon C([0,\infty), D'(R^2))$, $t \geq 0$, be defined by $\phi(w,t) = w(t)$. Put on $C([0,\infty); D'(R^2))$ the probability measure under which the restriction of $\phi(w, \cdot)$ to Λ_m coincides in law with the limit of $\phi^{n,m}(\cdot)$ above as $m \leq n \to \infty$. We suppress the w-dependence and write $\phi(\cdot)$ for $\phi(w,\cdot)$. By (2.4), it follows that the law of $\phi(t)$ for each t is $\overline{\mu}$.

This procedure can be carried out for the quadruplet $(\phi^{n,m}(\cdot), \phi_1^{n,m}(\cdot), \phi_2^{n,m}(\cdot), \phi_3^{n,m}(\cdot))$ instead of just $\phi^{n,m}(\cdot)$ in exactly the same manner. The anlogs of (5.10) needed are:

$$E[|\phi_1^{n,m}(t_i)(f_i) - \phi_1^{n,m}(t_i)(g_i)|^2] \le M\|C_{1m}^{-\varepsilon}(f_i - g_i)\|^2_{L^{4/3}(\Lambda_m)}$$

$$\le M\|f_i - g_i\|_{H_{1m}^{-2\varepsilon}}(\Lambda_m),$$

$$E[|\phi_2^{n,m}(t_i)(f_i) - \phi_2^{n,m}(t_i)(g_i)|^2] \le M\|C_{1m}^{-\varepsilon}(f_i - g_i)\|^2_{L^4(\Lambda_m)}$$

$$E[|\phi_3^{n,m}(t_i)(f_i) - \phi_3^{n,m}(t_i)(g_i)|^2] \le M\|C_{1m}^{(1-\varepsilon)/2}(f_i - g_i)\|^2_{L^2(\Lambda_m)}$$

$$= M\|f_i - g_i\|^2_{H_1^{1-\varepsilon}(\Lambda_m)}$$

where M denotes a constant depending on m and $\max(t_1,\ldots,t_k)$ not always the same. The first two are proved by using stationarity and arguments similar to those of Corollary 12.2.4, pp. 222-3, [4]. (Analogous arguments are used in the proof of Theorem 5.1 below.) The third is easy. Clearly, $\{g_i\}$ can also approximate $\{f_i\}$ arbitrarily closely in $H_1^{-2\varepsilon}(\Lambda_m)$ and $H_1^{1-\varepsilon}(\Lambda_m)$. As for the second case, let $\tilde{f}_i = C_{1m}^{1-\varepsilon} f_i$, $\tilde{g}_i = C_{1m}^{1-\varepsilon} g_i$. Then $\tilde{g}$ is also a finite linear combinations of $\{e_i\}$. A suitable g_i can be obtained simply by truncating the Fourier series for $\tilde{f}_i \in \cap_\alpha H_1^\alpha(\Lambda_m)$ so as to approximate the latter arbitrarily closely in $H_1^\alpha(\Lambda_{m,})$ for any prescribed α. By Sobolev embedding, we see g_i can approximate f_i arbitrarily closely in $C(\Lambda_m)$ and hence in $L_4(\Lambda_m)$.

Theorem 5.1: $\phi(\cdot)$ is a stationary solution of (3.7).

Proof:

Let $f \in D(\Lambda_m)$, $h = (-\Delta+1)f$. As noted above, we can use exactly the same arguments as above to draw the following stronger conclusion: (By dropping to an appropriate subsequence of $\{n\}$ if necessary) the $(\cup_\alpha H^{-\alpha}(\Lambda_m))^4$-valued processes $(\phi^{n,m}(\cdot), \phi_1^{n,m}(\cdot), \phi_2^{n,m}(\cdot), \phi_3^{n,m}(\cdot))$ converge in law to a limit process $(\tilde{\phi}^m(\cdot), \tilde{\phi}_1^m(\cdot), \tilde{\phi}_2^m(\cdot), \tilde{\phi}_3^m(\cdot))$ which agrees in law with the restriction to Λ_m of a $D(\Re^2)^4$–valued process $(\phi(\cdot), \phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot))$. Let $\{h_n\} \in C_0^\infty$ with $0 \le h_n(\cdot) \le 1$. supp. $h_n \subset \Lambda_n$, $h_n \uparrow 1$. Let $n \ge m$. Argue as in the proof of Corollary 12.2.4, pp. 222-3, [4], to conclude that for some constant M independent of n,

$$E\left[|\phi^n(t)(C_{1n}^{1-\varepsilon}h) - \phi^n(t)(h_m C^{1-\varepsilon}h)|^2\right]$$

$$\leq M\left[\|C_{1n}^{1-\varepsilon}h - h_m C^{1-\varepsilon}h\|^2_{L^{4/3}(\Lambda_n)} + \|C_{1n}^{1-\varepsilon}h - h_m C^{1-\varepsilon}h\|^2_{L^1(\Lambda_n)}\right]$$

$$\leq M\left[\|C_{1n}^{1-\varepsilon}h - h_m C^{1-\varepsilon}h\|^2_{L^{4/3}(\Re^2)} + \|C_{1n}^{1-\varepsilon}h - h_m C^{1-\varepsilon}h\|^2_{L'(\Re^2)}\right]$$

where we extend h, $C_{1n}^{1-\varepsilon}h$ to $\Re^2$ by setting it equal to zero outside Λ_n. Thus

$$E\left[\left(\int_0^t \phi^n(s)(C_{1n}^{1-\varepsilon}h)ds - \int_0^t \phi^n(s)(h_m C^{1-\varepsilon}h)ds\right)^2\right]$$

$$\leq t^3 M\left[\|C_{1n}^{1-\varepsilon}h - h_m C^{1-\varepsilon}h\|^2_{L^{4/3}(\Re^2)} + \|C_{1n}^{1-\varepsilon}h - h_m C^{1-\varepsilon}h\|^2_{L^1(\Re^2)}\right]$$

From this, Corollary 5.1, and Lemma 5.1,

$$\lim_{m\to\infty} \lim_{n\to\infty} E\left[\left(\int_0^t \phi^n(s)(C_{1n}^{1-\varepsilon}h)ds - \int_0^t \phi^n(s)((h_m C^{1-\varepsilon}h)ds\right)^2\right] = 0$$

i.e.

$$\lim_{m\to\infty} \lim_{n\to\infty} E\left[\left(-\frac{1}{2}\int_0^t \phi^n(s)(h_m C^{1-\varepsilon}h)ds - \phi_1^{n,m}(t)(h)\right)^2\right] = 0$$

i.e.

$$\lim_{m\to\infty} E\left[\left(-\frac{1}{2}\int_0^t \phi(s)(h_m C^{1-\varepsilon}h)ds - \phi_1(t)(h)\right)^2\right] = 0$$

i.e.

$$E\left[\left(\frac{1}{2}\int_0^t \phi(s)(C^{1-\varepsilon}h)ds - \phi_1(t)(h)\right)^2\right] = 0$$

Thus

$$\phi_1(t)(h) = \frac{1}{2}\int_0^t \phi(s)(C^{-\varepsilon}f)ds \qquad \text{a.s.}$$

Similarly, we have

$$E\left[\left(:(\phi^n(t))^3:_n (C_{1n}^{1-\varepsilon}f) -: (\phi^n(t))^3:_n (h_m C_1^{1-\varepsilon}f)\right)^2\right]$$

$$\leq M\left[\|C_{1n}^{1-\varepsilon}f - h_m C^{1-\varepsilon}f\|^2_{L^4(\Re^2)} + \|C_{1n}^{1-\varepsilon}f - h_m C^{1-\varepsilon}f\|^2_{L^1(\Re^2)}\right],$$

leading as above to,

$$\lim_{m\to\infty}\ \lim_{n\to\infty}\left[\left(\int_0^t :(\phi^n(s))^3:_n (C_{1n}^{1-\varepsilon}f)ds - \int_0^t :(\phi^n(s))^3:_n (h_m C^{1-\varepsilon}f)ds\right)^2\right] = 0$$

i.e.

$$\lim_{m\to\infty}\ \lim_{n\to\infty}\ \lim_{k\to\infty}\ E\left[\left(\int_0^t :(\phi^n(s))^3:_n (C_{1n}^{1-\varepsilon}f)ds - \int_0^t :(\phi^n(s))_k^3:_n (h_m C^{1-\varepsilon}f)ds\right)^2\right] = 0$$

where the subscript k denotes the ultraviolet cutoff as in [4], p. 221. By the results of [4], p.221, we can interchange the order of the first and the second limit to obtain

$$\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} E\Big[\Big(\int_0^t :(\phi^n(s))^3 :_n (C_{1n}^{1-\varepsilon}f)\, ds$$

$$- \int_0^t :(\phi^n(s))_k^3 :_n h_m C^{1-\varepsilon} f) ds\Big)^2\Big] = 0$$

i.e.

$$\lim_{m\to\infty}\lim_{k\to\infty} E\left[\left(\frac{1}{2}\int_0^t : (\phi(s)_k^3) : (h_m C^{1-\varepsilon}f)ds - \phi_2(t)(f)\right)^2\right] = 0 ,$$

i.e.,

$$E\left[\left(\frac{1}{2}\int_0^t :\phi(s)^3: (C^{1-\varepsilon}f)ds - \phi_2(t)\right)^2\right] = 0$$

Thus

$$\phi_2(t)(f) = \frac{1}{2}\int_0^t :\phi(s)^3: C^{1-\varepsilon}f)ds \qquad \text{a.s.}$$

Finally, $\phi_3^n(\cdot)$ is a Wiener process with covariance $(t\wedge s)C_{1n}^{1-\varepsilon}$. It is not hard to see that $\phi_3^n(\cdot)$ is a Wiener process with covariance $(t\wedge s)C_{1n}^{1-\varepsilon}$. It is not hard to see that $\phi_3(\cdot)$ will have to be a Wiener process with covariance $(t\wedge s)C^{1-\varepsilon}$. From (5.5), it follows that

$$\phi(t)(f) - \phi(0)(f) = \phi_1(t)(f) + \phi_2(t)(f) + \phi_3(t)(f) \quad \text{a.s} \tag{5.11}$$

for $f\varepsilon C_0^\infty$. Thus $\phi(.)$ satisfies (3.7) as an $D'(\Re^2)$-valued process. That $\phi(\cdot)$ is in fact an S'-valued process follows from the fact that μ is supported on S'. One can then extend (5.11) to $f\varepsilon S$ by an approximation argument. q.e.d.

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