Bergman Kernel and Stability of Holomorphic Vector Bundles with Sections

by

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Abstract

In this thesis, we introduce a notion of asymptotic stability for a holomorphic vector bundle with a global holomorphic section on a projective manifold. We prove that the special metric on the bundle studied by Bradlow is the limit of a sequence of balanced metrics that are induced from the asymptotic stability. Conversely, assuming the convergence of a sequence of balanced metrics, we show that the sequence converge to a special metric in the sense of Bradlow. The proof uses the asymptotic expansion of the Bergman kernel for general holomorphic vector bundle and machineries about moment maps involving two group actions developed by Donaldson.

Thesis Supervisor: Gang Tian
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Contents

Introduction

In [29], Yau proposed that there is a connection between Kähler-Einstein metrics and the Chow stability of the underlying algebraic manifolds. In [24], [25], Tian introduced two new stability conditions, K-stability and CM-stability, for algebraic manifolds and proved that the existence of constant scalar curvature metrics implies the two stabilities. In [10], Donaldson proved that the existence of constant scalar curvature metrics implies the Chow stability. The main purpose of this thesis is to extend Donaldson's ideas to those special metrics on bundles with sections in the sense of Bradlow [3].

In [23], Tian proved that a given polarized kähler metric is the limit of Kähler metrics induced from a sequence of projective embeddings using the polarization. Tian established the result by studying the asymptotic expansion of Bergman kernel adapted to the given metric. The asymptotic expansion of Bergman kernel was later studied in greater detail by Ruan [22], Zelditch [30], Catlin [6] and Lu [15]. Notably, Lu evaluated the lower four terms in the asymptotic expansion. The salient one is the second term, which is exactly the scalar curvature. Thus a converging sequence of metrics with constant Bergman kernel converge to a constant scalar curvature metric. Metrics of constant Bergman kernel, so called balanced metrics, has been studied by Luo in [16]. He proved that the existence of balanced metrics imply the Chow stability of the underlying manifold.

Thus to show that a constant scalar curvature metric implies the Chow stability, the main issue is to show the existence of balanced metrics. Donaldson achieved this goal by perturbing a constant scalar curvature metric to metrics which are very close to being balanced. This approximation procedure is based on the asymptotic expansion of Bergman kernel. The metrics close enough to being balanced are then shown to 'flow' to balanced metrics by a collection of estimates. The conceptual motivation for the 'flow' argument is a very interesting 'quantized' moment map interpretation of the balanced condition. Donaldson interpret the condition as the simultaneous vanishing of two moment maps on a certain infinite dimensional manifold. One group is an infinite dimensional group, the other group is a finite dimensional group.

In this thesis, we extend Donaldson's ideas to a special metric for a bundle with a section. We introduce balanced metrics in our context and prove that a special metric is the limit of a sequence of balanced metrics. We basically follow Donaldson's approach. We get balanced metrics in our context using a similar 'quantized' moment map interpretation. Again there are vanishing conditions for two moment maps, with respect to an infinite dimensional group and a finite dimensional group. Our notion of stability also depends on a real parameter, as the case for the stability defined by Bradlow [3]. For the approximation procedure, we use the asymptotic expansion result of Catlin [6]. We work out the explicit expression for the first three terms in the expansion based on the work of Tian [23] and Lu [15]. We prove that the approximation procedure can be carried out to any order by solving a sequence of recursive equations based on the special curvature properties of the special metric. We also work out a similar collection of estimates. The estimates are in the sharp form as suggested by Donaldson in [10].

We also make an observation on the equivalence of two functionals for testing stability in a finite dimensional Kähler geometry set up. The two functionals are introduced from different perspectives, one is from algebraic geometric perspective and the other one is from symplectic geometric perspective. Our observation is based on the geometry linking these two perspectives.

Chapter 1

Preliminaries on Moment Map and Stability

The main purpose of this chapter is to explain the principle that stability conditions are the same as vanishing conditions for moment maps. It's rigorously established for finite dimensional set up ([18]). It serves as a guidance for infinite dimensional set up as initiated by Atiyah and Bott in their study of Yang-Mills equation over Riemann surface ([1]). In section 1.1, we recall the algebraic notion of stability and the theorem of Kempf and Ness which characterizes stability from differential geometry perspective ([12]). In section 1.2, we turn to a standard Kähler geometry set-up. We recall two functionals used to test stability and observe that the two functionals are exactly same. The observation is based on the geometry behind lifting of a group action to a polarized line bundle and moment map for the group action. In section 1.3, we recall a crucial proposition developed by Donaldson on how to find zeros of moment maps inside a complex orbit with suitable assumption ([10]).

1.1 Stability in G.I.T.

The notion of stability is developed as a result of forming a nice moduli space in algebraic geometry ([18]). The simplest situation is the following: suppose we have a connected reductive algebraic group G over complex number, V is a finite dimensional complex vector space, assuming that G acts on V algebraically, the question that we are interested is what structure the orbit space V/G has. It turns out that for the orbit space to have a nice structure in the category of algebraic world, one shouldn't consider all the orbits, instead the orbits with certain stability property.

Definition 1.1 *A nonzero vector $v \in V$ is called*

(1) *semi-stable if zero is not in the closure of the orbit containing v;*

(2) *polystable if the orbit containing v is closed;*

(3) *stable if it is both polystable and the stabilizer is finite.*

Clearly stability property is a property for an orbit. It can be shown that the semi-stable orbits form an algebraic variety ([18]).

Practically to check stability is not quite easy. There are several equivalent criterion from different perspectives available. One is Hilbert numerical criterion ([18]), which tells the stability from the weights of each one algebraic parameter subgroup. Another one is due to Kempf and Ness ([12]), they give a criterion by choosing a Hermitian metric, then examine the length function along the orbit. This approach is very geometric.

Precisely, as G is an reductive algebraic group, it is the complexification of a compact Lie group K (we can take this as the definition of reductiveness). Hence by standard Weyl integration trick, we can choose a Hermitian norm $|| \, ||$ on V so that the action of K preserves this norm.

For each nonzero vector $v \in V$, define a function p_v on G

$$p_v(g) = ||gv||^2. \tag{1.1}$$

Clearly if G_v denotes the stabilizer of v in G, then the function p_v is invariant on the left action by K and on the right action by G_v. Hence p_v is constant on double $K - G_v$ cosets of the form KgG_v.

The theorem of Kempf and Ness is

Theorem 1.1 *([12])*

(1) *Any critical point of p_v is a point where p_v obtains its minimum value.*

(2) *If p_v obtains a minimum value, then the set m where p_v obtains this value consists of a single $K - G_v$ coset and is connected. Moreover, the Hessian of p_v at a point of m in any direction not to tangent to m is positive definite.*

(3) *The function p_v has a critical point if and only if v is polystable.*

(4) *The function p_v is proper if and only if v is stable.*

The above nice theorem tells us that the stability property can be described by their differential geometric property. In the next section, we will see the critical points are exactly zeros of a hidden moment map.

1.2 Symplectic geometry of stability

1.2.1 A standard picture

In this section, we consider the following standard Kähler geometry set up (cf. [8] [26]):

(1) a Kähler manifold $(M,\ J,\ \omega)$, where J is an integrable complex structure, ω is the Kähler form compatible with J, i.e., $\omega(X,\ JX) > 0$, $\forall$ nonzero tangent vector X;

(2) a holomorphic Hermitian line bundle $(L,\ h)$, where h is the Hermitian metric, so that the corresponding curvature form $R^L(h)$ satisfying $\sqrt{-1}R^L(h) = \omega$;

(3) a compact Lie group K and its complexification G act biholomorphically on M. Moreover the action of K preserves the Kähler form ω ;

As before to have a nice structure of the orbit space M/G, we consider only the orbits with stability property. In this case, it turns out that the stable orbits have a nice description via the symplectic geometry, which at the end of day, identifies the stable orbit space with symplectic quotient.

Precisely, we want the following two equivalent assumptions

(4) existence of a linearization of the action (G, K) on (L, h), that is a holomorphic lift of the action of G on M to L, so that the action of K preserves the Hermitian metric h on L;

(5) existence of a K-equivariant moment map $\mu : M \to \mathrm{Lie}(K)^*$, that is

- $d\langle \mu, \xi\rangle + i_{X_\xi}\omega = 0$, for any $\xi \in \mathrm{Lie}K$, where X_ξ is the induced vector field on M, $\langle\ \ \rangle$ is the pairing between vector space and its dual vector space;
- $\langle \mu(gp), \xi\rangle = \langle \mu(p), ad(g^{-1})_*(\xi)\rangle$, where $ad(g)$ is the adjoint action defined by $ad(g)(h) = ghg^{-1}$ for any element $h \in G$.

Convention: During the paper, all the group actions are left actions. And for an element ξ in the Lie algebra of the group, the induced vector field X_ξ is given by $X_\xi(p) = \frac{d}{dt}\exp(-t\xi)p|_{t=0}$ at any point on the manifold. This sign convention is to make sure that the induced map from Lie algebra of the group to the Lie algebra of vector fields on the manifold is a Lie algebra homomorphism. In case the action is on more than one manifolds, we put a superscript on the vector fields to indicate which manifold we are considering.

The equivalence of (4) and (5) is quite interesting. Assumption (4) is usually used by algebraic geometer, while assumption (5) is usually used by differential geometer. Later on we are going to give two criterion for testing stability, one is based on assumption (4), the other is based on assumption (5). We'll show that the two criterion are equivalent using the geometry behind the two perspectives.

The correspondence is as following (cf. [7], [5]). Assuming we have a linearization. For each $\xi \in \mathrm{Lie}(G)$, let R_ξ be the induced action on the space of sections $\Gamma(L)$, that is for any section S of L

$$R_\xi S(p) = \frac{d}{dt}e^{t\xi}S(e^{-t\xi}p)|_{t=0}$$

Clearly, R_ξ is a derivative with respect to X_ξ^M, that is for any smooth function f on M, we have

$$R_\xi f S = X_\xi^M f S + f R_\xi S$$

Hence if D denotes connection on L determined by the metric h and the holomorphic structure, then the difference $R_\xi - D_{X_\xi^M}$ is a homomorphism of the line bundle L, hence there exits a function μ_ξ such that

$$R_\xi S = D_{X_\xi^M} S + \sqrt{-1}\mu_\xi S \tag{1.2}$$

As the action of K preserve the metric h, μ_ξ is real valued for $\xi \in \mathrm{Lie}(K)$. If we define a function $\mu : M \to \mathrm{Lie}(K)^*$ by $\langle \mu, \xi \rangle = \mu_\xi$, then it's straightforward to check that μ is a K-equivariant moment map.

To see the other direction, just note that the identity 1.2 is equivalent to

$$X_\xi^L = \overline{X_\xi^M} - \mu_\xi t_{\mathbf{R}} \tag{1.3}$$

where $\overline{X}$ denotes horizontal lift of a vector field X with respect to the connection D, $t_{\mathbf{R}}$ is the real vector field on L induced by the rotation along the fiber $\xi \to e^{i\theta}\xi$. Thus given a K-equivariant moment map μ, we can use identity 1.3 to define the vector fields X_ξ^L, then the corresponding flow will give a linearization.

Now given a linearization, recall a point p on M is called

Definition 1.2 **(1)** *semi-stable if for all k large enough, there is a G-equivariant holomorphic section of $L^{\otimes k}$ non-vanishing at the point x ;*

(2) *polystable if it's semistable, and the orbit Gp is closed in the set of semistable points ;*

(3) *stable if it's both polystable and the stabilizer G_p is finite.*

To test the stability, there are several equivalent criterion from different perspective. In the following we recall one criterion based on the one given by Kempf and Ness. We'll recall another one from moment map perspective in the next section.

Analogy to the function p_v defined in 1.1, we define, for each point p on M, a function S_p on the group G

$$S_p(g) = \ln \frac{|\xi_p|}{|g\xi_p|} \tag{1.4}$$

where ξ_p is any nonzero element in the fiber L_p.

Clearly, the function S_p is well-defined, it's constant along the double coset $K - G_p$, thus it descends to a function on the homogeneous space G/K. The result of Kempf and Ness holds also in this case ([18]). Particularly, the orbit Gp is polystable if and only if S_p has a critical point. The orbits Gp is stable if and only if S_p is proper. Thus it's desirable to compute the derivative of the function S_p. Note we only need to compute the derivative S_p at the identity $e \in G$, as $S_{gp} = R_g^* S_p$, where R_g is the right multiplication operation.

Lemma 1.1 *(cf. [5]) For any $\xi \in \mathrm{Lie}(K)$, we have*

(1)

$$\frac{d}{dt} S_p(e^{it\xi}) = \langle \mu(e^{it\xi}p),\, \xi \rangle; \tag{1.5}$$

(2)

$$\frac{d^2}{dt^2} S_p(e^{it\xi}) = \omega(X_\xi,\, JX_\xi).$$

The proof uses the geometric identification of linearization and moment map.
For (1), let h denote the norm function on L: $(x, \xi) \mapsto |\xi|^2$, then

$$S_p(g) = \frac{1}{2} \ln h - \frac{1}{2} g^* \ln h.$$

Thus taking derivative, we have

$$\frac{d}{dt} S_p(e^{it\xi}) = \frac{1}{2} \tilde{J} X_\xi^L \ln h,$$

where $\tilde{J}$ is the complex structure on L induced by the complex structure J on M and the connection D on L. Using identity 1.4, we have

$$\tilde{J}X_\xi^L = \overline{JX_\xi^M} - \mu_\xi \tilde{J} t_{\mathbf{R}}.$$

Due to assumption that the action of K preserves the metric h, we have

$$\frac{d}{dt} S_p(e^{it\xi}) = -\frac{1}{2}\mu_\xi \tilde{J} t_{\mathbf{R}} \ln h.$$

To compute the right hand side, we choose a local orthonormal frame for L, which induces a real oriented coordinate (u, v) for the fiber. It's easy to see then

$$h = u^2 + v^2,$$

$$t_{\mathbf{R}} = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u},$$

The proof of (1) is finished by using $\tilde{J}\frac{\partial}{\partial u} = \frac{\partial}{\partial v}$.

The (2) follows from (1) easily by the definition of moment map.

From the above lemma, we see the following important corollary

Corollary 1.1 *([18], [5])(**Moment map interpretation of stability**)*

(1) *A complex orbit Gp is polystable if and only if the moment map μ vanishes along a single $K - G_v$ orbit;*

(2) *A complex orbit Gp is stable if and only the moment map vanishes along a single K orbit with stabilizer G_v finite.*

There are also statements about semistable points ([18]). We omit that as we only discuss the stable orbits in the later discussion.

For later comparison, we list a few properties of function S. They can be checked straightforwardly.

Proposition 1.1 *For any point p on M*

(1) *For any* $g,\, h \in G$, $S_p(g)+S_{gp}(h)=S_p(hg)$;

(2) *For any* $g \in G$, $k \in K$, $S_p(kg) = S_p(g)$ *and* $S_p(e) = 0$;

(3) *For any* $g \in G$, $k \in K$, $S_{kp}(g) = S_p(k^{-1}gk)$;

(4) $g \in G$ *is a critical point of* S_p *if and only if* $\mu(gp) = 0$;

(5) *For any* $\xi \in \mathrm{Lie}(K)$, $\frac{d^2}{dt^2} S_p(e^{it\xi}) \geq 0$ *and the equality holds if and only if* $X_\xi(e^{it\xi}p) = 0$.

1.2.2 Criterion for stability

In last section, we use the function S to test stability. Now we recall another function which I learned in Tian's survey article ([26]).

Now assume we have a moment map μ. For any point p on M, we can define a function T_p on the group G

$$T_p(g) = \int_0^1 \langle \mu(g(t)p),\, \mathrm{Im}(R_{g(t)^{-1}})_*(\dot{g}(t)) \rangle \, dt \tag{1.6}$$

where $g(t)\colon [0,1] \to G$ is any smooth path connecting identity e to g, and Im means the imaginary part under the identification of $\mathrm{Lie}(G) = \mathrm{Lie}(K) \oplus \sqrt{-1}\mathrm{Lie}(K)$.

The well-defineness of the function follows from the following lemma.

Lemma 1.2 *([26]) For any smooth one family path* $g(\cdot, t)\colon [0,1] \to G$ *parameterized by* $s \in [0,1]$, *we have the integration*

$$\int_0^1 \langle \mu(g(s,t)p),\, \mathrm{Im}(R_{g(s,t)^{-1}})_*(g_t(s,t)) \rangle \, dt$$

is independent of s.

The proof uses the definition of moment maps and the following identities in Lie algebra.

Lemma 1.3 *For a family of path* $g\colon (-\varepsilon, \varepsilon) \times (-\eta, \eta) \to G$, *we have*

$$\frac{d}{ds} R_{g^{-1}} g_t - \frac{d}{dt} R_{g^{-1}} g_s = [R_{g^{-1}} g_s, R_{g^{-1}} g_t].$$

The verification for above lemma is quite clear for matrix group, so is for general group.

The function T appears in geometric contexts, mostly in infinite dimensional set up. For instance, Donaldson's functional in studying stability of vector bundles ([7]) and Mabuchi's K-energy functional in studying of metrics with constant scalar curvature on Kähler manifolds ([17]). Here we only discuss the function in the simple finite dimensional context.

From the definition, the two functions S and T are of quite different nature. S is defined using the linearization, while T is define using moment map. And also, in the definition of function S (see 1.4), there is a freedom of choice of a vector ξ, while in the definition of function T(see 1.6), there is a freedom of choice of path. However they do have lots of common properties.

Let's first see the derivative of the function T at identity e.

Lemma 1.4 *For any $\xi \in \mathrm{Lie}(K)$, we have*

(1)

$$\frac{d}{dt}T_p(e^{it\xi}) = \langle \mu(e^{it\xi}p),\, \xi \rangle; \tag{1.7}$$

(2)

$$\frac{d^2}{dt^2}T_p(e^{it\xi}) = \omega(X_\xi,\, JX_\xi).$$

The proof is clear, we simply re-scale the path we are choosing.

Then observe that the function T also have the following properties:

Proposition 1.2 *([10]) For any point p on the manifold M*

(1) *For any $g,\, h\, \in G$, $T_p(g)+T_{gp}(h)=T_p(hg)$;*

(2) *For any $g \in G$, $k \in K$, $T_p(kg) = T_p(g)$ and $T_p(e) = 0$;*

(3) *For any $g \in G$, $k \in K$, $T_{kp}(g) = T_p(k^{-1}gk)$;*

(4) *$g \in G$ is a critical point of T_p if and only if $\mu(gp) = 0$;*

(5) *For any $\xi \in \mathrm{Lie}(K)$, $\frac{d^2}{dt^2}T_p(e^{it\xi}) \geq 0$ and the equality holds if and only if $X_\xi(e^{it\xi}p) = 0$.*

The verification is also clear. They basically follow from the definition.

By now it's reasonable to guess that the two functions coincide. Actually, they are.

Theorem 1.2 *Function S coincide with function T. Explicitly*

$$\ln\frac{|\xi_p|}{|g\xi_p|} = \int_0^1 \langle \mu(g(t)p),\, \mathrm{Im}(R_{g(t)^{-1}})_*(\dot{g}(t))\rangle\, dt$$

The proof by now should be clear. By Lemma 1.1 and lemma 1.4, they have the same derivative along the path $e^{it\xi}$. Clearly they have same initial values. So they coincide along the path. For general point, using proposition 1.1 and Proposition 1.2, we can reduce the proof to the special case.

In the following we give an example in the simplest case.

Example: Let $M = \mathbf{C}$ with the standard Kähler form $\omega = \frac{\sqrt{-1}}{2} dz \wedge d\overline{z}$, L be the trivial complex line bundle with the metric $h(z,u) = |u|^2 e^{-\frac{1}{2}|z|^2}$, $K = S^1$, $G = \mathbf{C}^*$, G acts by multiplication: $z \mapsto zw$, $\forall w \in \mathbf{C}$. For each integer k, there is a linearization $\mathbf{C} \times L \to L$: $(z,u) \mapsto (zw, uw^k)$, $\forall w \in \mathbf{C}$. The corresponding moment map is then

$$\langle \mu(z),\, \xi\rangle = k - \frac{1}{2}|z|^2,$$

where ξ is the standard base for $\mathrm{Lie}(S^1)$, and

$$S_z(e^{it\xi}) = T_z(e^{it\xi}) = \frac{1}{4}(e^{2t} - 1)|z|^2 - ka.$$

1.3 Existence of zeros of moment maps

We have seen that to test stability of a complex orbit is equivalent to find zeros of the moment map along the complex orbit. In his work on the existence of constant scalar curvature, Donaldson([10]) develops a general method to find zeros of a moment map under certain assumption. We discuss this important method in this section.

Suppose we have the standard Kähler geometry set up. Furthermore, we choose an invariant inner product on the Lie algebra Lie(K). Then we use this inner product to identify Lie algebra Lie(K) with its dual Lie$(K)^*$ in the standard way, then we can think the moment map μ as a map $\nu\colon M \to \mathrm{Lie}K$. At each point $p \in M$, we have the infinitesimal action

$$\sigma_p\colon \mathrm{Lie}(K) \to T_pM.$$

Then define an endomorphism of Lie(K) by

$$Q_p = \sigma_p^* \sigma_p,$$

where the adjoint is define using the metrics on Lie(K) and M. By the definition of the moment map, Q_p is also given by

$$Q_p = d\nu_p \circ J \circ \sigma_p.$$

The significance of the endomorphism Q_p lies in the fact that they are very important in the understanding the existence and uniqueness of zeros of the moment map inside a complex orbit. This is well explained in Donaldson's paper [10]. For completeness, we recall his result in the following.

The uniqueness result is

Proposition 1.3 *([10]) **(Uniqueness of zeros of moment map inside a complex orbit)** Suppose for a point $p \in M$, and $g \in G$, such that $\mu(p) = \mu(gp) = 0$, then for some $k \in K$, $gp = kp$, in other words, $gp \in Kp$*

Now we discuss the more interesting and crucial part: the existence of zeros of moment map inside a complex orbit. Suppose the complex orbit we are considering is Gp_0, we assume the stabilizer G_{p_0} is finite, then σ_p is injective and Q_p is invertible for all p in the complex orbit Gp_0. Let Λ_p denote the operator norm of $Q_p^{-1}\colon \mathrm{Lie}(K) \to \mathrm{Lie}(K)$, defined using the invariant Euclidean metric on Lie(K). Precisely

$$\Lambda_p = \max_{\xi \in \mathrm{Lie}(K)\backslash 0} \frac{\|Q_p^{-1}\xi\|}{\|\xi\|}. \tag{1.8}$$

The existence result is

Proposition 1.4 *([10]) **(existence of zeros of moment map inside a complex orbit)** Suppose we have real numbers λ, δ such that $\Lambda_p \leq \lambda$ for all $p = e^{i\xi}p_0$ with $|\xi| \leq \delta$. Furthermore $\lambda|\nu(p_0)| < \delta$. Then there exits a point $p = e^{i\eta}p_0$ with $\nu(p) = 0$, where $|\eta| \leq \lambda|\nu(p_0)|$*

In the assumptions, the main point is certainly the condition $\lambda|\nu(p_0)| < \delta$. It basically means the initial point should be close enough to the zeros of moment map. When we apply this proposition, the main issue is to fulfill this condition. We will see this in our later discussion.

As in [10], to apply this proposition to our later discussion, we need to consider the case $M = W//H$, and the action of K on M is induced by an action of $K \times H$ on the Kähler manifold W. For each point $w \in W$, we have two infinitesimal actions

$$\sigma_{K,w} \colon \mathrm{Lie}(K) \to T_wW, \quad \sigma_{H,w} \colon \mathrm{Lie}(H) \to T_wW.$$

Lemma 1.5 *([10]) Let $p \in W//H$ be represented by a point $w \in W$. Then for $\xi \in \mathrm{Lie}(K)$, the endomorphism Q_p associated to the K action on $W//H$ satisfies*

$$\langle Q_p\xi, \xi\rangle = |\pi(\sigma_{K,w}\xi)|^2$$

where $\pi \colon T_wW \to T_wW$ is the orthogonal projection to $Im(\sigma_{H,w})^{\perp}$. In particular

$$\Lambda_p = \left(\min_{\xi \in \mathrm{Lie}(K)} \frac{|\pi(\sigma_{K,w}\xi)|}{|\xi|}\right)^2.$$

Chapter 2

From Balanced Metrics to Special Metrics

In this chapter we show that the converging balanced metrics converge to a special metric. The proof is based on asymptotic expansion of Bergman kernel of Catlin ([6]). In section 2.1, we recall the special metric studied by Bradlow ([3]) and the vortex equation it governs. We also recall a modified version of the vortex equation by Okonek and Teleman ([19]). In section 2.2.1, we define the asymptotic stability for a bundle with a section in terms of the balanced metrics. In section 2.2.2, we give moment map interpretation of the asymptotic stability. This interpretation fits our problem into the general framework in the previous chapter. In section 2.3, we prove converging balanced metrics converge to special metrics.

2.1 Special metrics and vortex equation

Suppose E is a holomorphic vector bundle of rank $r = r(E)$ over a Kähler manifold (M, ω) of dimension n, $\phi \in \Gamma(E)$ is a global holomorphic section of E, then a Hermitian metric H on E is called special w.r.t. ϕ if it satisfies the following so called vortex equation

$$\sqrt{-1}\Lambda F_H + \phi^* \otimes \phi = cId_E \tag{2.1}$$

where $\Lambda : \Omega^*(M) \longrightarrow \Omega^{*-2}(M)$ is the adjoint operator of L: $\alpha \mapsto \omega \wedge \alpha$, and it extends naturally to bundle valued forms. $F_H \in \Omega^2(M, \operatorname{End} E)$ is the curvature form determined uniquely by the Hermitian metric H and the holomorphic structure on E. $\phi^* \otimes \phi \in \Omega^0(\operatorname{End} E)$ is the endomorphism $\psi \mapsto H(\psi, \phi)\phi$, $\forall \psi \in E$. c is some constant, $Id_E \in \Omega^0(M, \operatorname{End} E)$ is the identity endomorphism.

The equation is introduced by Bradlow ([3]). It's called vortex equation because it generalizes the vortex equation studied in physical Gauge theory (cf. [3]). Note in case ϕ is the zero section, this equation is exactly the Hermitian-Einstein equation. In that case, the existence of the special metric is shown to be equivalent to Mumford stability of the vector bundle ([27], [7]). Analogously, for general section ϕ, the existence of such metric is shown by Bradlow to be equivalent to certain stability defined in a similar way as Mumford stability as well as dependence on a parameter τ. As our main concern is to study the stability from the metric point of view, we are not going into the detail of algebraic notion of τ-stability. However we'd like to explain the dependence of the real parameter τ, which is relevant to our discussion.

In case of Hermitian-Einstein equation, the constant c in the equation 2.1 is a priori determined by the topology of the bundle E. Precisely by taking the trace of both sides of equation 2.1, and integrating over the manifold, we find that c equals to the normalized degree μ_E of E, where

$$\mu_E = \frac{2\pi \int_M c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!}}{rV}$$

which is only a topological invariant. Here $V = Vol(M)$ is the volume of M defined by the Kähler metric ω.

However for the vortex equation, the constant c is not a prior determined. If we set

$$\frac{\tau}{rV} = c - \mu_E.$$

By taking the trace of both sides of vortex equation 2.1, and integrating over the manifold, it's easy to see

$$\|\phi\|^2 = \int_M |\phi|^2 \, d\mu = \tau,$$

where $d\mu$ is the volume form defined by the Kähler metric ω. Hence fixing τ is the same as fixing the L^2 norm of the section ϕ. For fixed τ, we call the metric H to be τ-special.

Convention: To avoid confusion, we use $\langle \quad \rangle$ to denote fiber-wise metric, $| \quad |$ the fiber-wise norm and $(\quad)$ to denote the integrated L^2 metric, $\| \quad \|$ be the corresponding norm.

Note in above discussion, we use the first identity in the following elementary lemma,

Lemma 2.1 **(1)** *$\forall \phi, \psi \in E$, $\operatorname{Tr} \phi^* \otimes \psi = \langle \psi, \phi \rangle$,*

(2) *$\forall T_1, T_2 \in \operatorname{End}E$, $\phi, \psi \in E$, $\langle T_1\phi, T_2\psi \rangle = \langle T_1\psi^* \otimes \phi, T_2 \rangle$,*

Note we are using the induced metric on $\operatorname{End}(E)$, $\langle T_1, T_2 \rangle := \operatorname{Tr} T_1 T_2^*$.

Before we further discuss the vortex equation, we make a digression.

In the later discussion, we'll vary the Hermitian metric on bundle E quite often, we'd like to know the dependence of each term in the vortex equation on the metric.

Recall given a Hermitian metric H on E, it determines a unique connection D_H on E, which in turn induces a connection D_H on $\operatorname{End}E$. We denote the (0,1) part of D_H to be $\overline{\partial}_H$ and the (1,0) part to be ∂_H. Note $\overline{\partial}_H$ is independent of the metric H. We can omit the subscript H. We also omit the subscript in ∂_H when it's clear which metric we are using.

Assume H and K are two Hermitian metrics on E, they are related to each other via

$$K(\varphi, \psi) = H(\eta\varphi, \psi)$$

for some endomorphism $\eta \in \operatorname{End}E$, which is positive definite w.r.t the metric H. Conversely, given such η, the above identity defines a metric. For simplicity we'll write $K = H\eta$. Note the self-adjoint condition is crucial.

Lemma 2.2 *If $K = H\eta$, then*

(1) $F_K = F_H + \overline{\partial}(\eta^{-1}\partial_H \eta)$,

(2) $\phi^{*_K} \otimes \phi = \phi^{*_H} \otimes \phi\, \eta$.

Note $\phi^{*_K} \otimes \phi$ $\in$End$E$ is determined by the metric K. Later if we are clear about which metric we are using, we'll omit the superscript K.

In fact

$$(\phi^{*_K} \otimes \psi)\varphi = K(\varphi, \phi)\psi = H(\eta\,\varphi, \phi)\psi = (\phi^{*_H} \otimes \psi)\,\eta\varphi.$$

Corollary 2.1 *For small enough η $\in$EndE, we have*

$$F_{H(1+\eta)} = F_H + \overline{\partial}\partial\eta + O(\|\eta\|^2).$$

Let's also recall the standard Kähler identity.

Lemma 2.3 *([14])* ***(Kähler identity)***

(1) $\sqrt{-1}\,[\Lambda, \overline{\partial}] = \partial^*$,

(2) $\sqrt{-1}\,[\Lambda, \partial] = -\overline{\partial}^*$

It follows that

$$\sqrt{-1}\,\Lambda\overline{\partial}\partial\,\eta = \partial^*\partial\eta = \Delta_\partial\eta$$

is self adjoint positive operator. We'll omit the subscript in Δ_∂. Keep in mind it's not the usual Laplace. On functions, it's half on the usual Laplace. For general bundle, it's not.

We now continue the discussion of vortex equation. Our purpose is to explain a modified equation, which comes up naturally in our situation.

In the original definition by Bradlow, the right handside term in the vortex equation 2.1 is a constant term. However for our purpose, it is not sufficient to consider only constant. Instead we want a general function. Nonetheless, Okonek and Teleman proved the following theorem

Theorem 2.1 *([19]) The vortex equation*

$$\sqrt{-1}\Lambda F_H + \phi^* \otimes \phi = cId_E$$

is solvable if and only if the modified vortex equation

$$\sqrt{-1}\Lambda F_H + \phi^* \otimes \phi = f Id_E$$

is solvable for any smooth function f on M with

$$\overline{f} : = \frac{\int f \, d\mu}{V} = c.$$

Note in case ϕ is zero section, this theorem is simple. As if $\overline{f} = \mu_E$, then there exists a smooth function g so that

$$f = \mu_E + \Delta g.$$

If the metric H satisfies the Hermitian-Einstein equation

$$\sqrt{-1}\Lambda F_H = \mu_E Id_E.$$

Then for the conformal metric $K = e^g H$, using the identity in Lemma 2.2 and Lemma 2.3,

$$\sqrt{-1}\Lambda F_K = \sqrt{-1}\Lambda F_H + \Delta g = f.$$

However this argument does not work for modified vortex equation. As when we do the conformal change $K = e^g H$, not only

$$\sqrt{-1}\Lambda F_K = \sqrt{-1}\Lambda F_H + \Delta g$$

but also using the identity in Lemma 2.2

$$\phi^{*K} \otimes \phi = e^g \phi^{*H} \otimes \phi.$$

Then altogether we get

$$\begin{aligned}\sqrt{-1}\Lambda F_K + \phi^{*K} \otimes \phi &= \sqrt{-1}\Lambda F_H + \Delta g + e^g \phi^{*H} \otimes \phi \\ &= c + \Delta g + e^g \phi^{*H} \otimes \phi - \phi^{*H} \otimes \phi \\ &= f + e^g \phi^{*H} \otimes \phi - \phi^{*H} \otimes \phi,\end{aligned}$$

which does not equal to f.

This reflects the fact that the term $\phi^* \otimes \phi$ is quite nonlinear. One more evidence about this is that if we consider the simplest case when the bundle E is a line bundle, then the Hermitian-Einstein equation is equivalent to the following one

$$\Delta f = g \tag{2.2}$$

for a smooth function g.

While vortex equation is the following one

$$\Delta f + ae^f = g \tag{2.3}$$

for a smooth function g and some nonnegative function a.

The Laplace equation 2.2 is relatively easy, it's linear and elliptic. The existence and uniqueness of the solution is standard. While the equation 2.3 is harder, it is elliptic, however has a nonlinear term e^f. Equation of this sort has been studied by Kazdan and Warner [13]. This equation raised as a result to prescribe curvature function on a Riemannian manifold. In their excellent written paper [13], Kazdan and Warner proved the existence and uniqueness of the equation 2.3 in case a is nonnegative and positive somewhere and the integration of g is positive. The existence and uniqueness of the equation 2.3 in the general assumption is still largely open. Nonetheless, vortex equation is in the good case.

2.2 Balanced metrics for bundles with sections

2.2.1 Definition of balanced metrics

We assume (M, ω) is polarized, that is, we have a Hermitian holomorphic line bundle (L, h) with the curvature determined by the metric h and holomorphic structure satisfying $\sqrt{-1}R^L(h) = \omega$.

Because of the curvature condition, L is positive. Thus by Kodaira embedding theorem, for integer m large enough, the higher dimensional cohomology group

$$H^i(M, E_m) = 0, \quad \forall i \geq 1 \tag{2.4}$$

where E_m denotes $E \otimes L^m$. Moreover the map

$$\iota_m \colon M \to \mathrm{Gr}(N_m - r, H^0(M, E_m)) \tag{2.5}$$

defined by sending each point $p \in M$ to the sections vanishing at p is an embedding. Where N_m is the dimension of the space of holomorphic sections $\Gamma_{\mathrm{hol}}(E_m)$. Using the Riemann-Roch theorem and vanishing condition 2.4, N_m is given by

$$N_m = \chi(E_m) = \int_M \mathrm{Ch}(E_m)\mathrm{Td}(M) = a_0 m^n + a_1 m^{n-1} + \cdots + a_n.$$

It is easy to see the coefficients a_i's are topological invariants of M and E. The salient ones for the paper are the leading term

$$a_0 = r\int_M \frac{c_1(L)^n}{n!} = \frac{r}{(2\pi)^n}\int_M \frac{\omega^n}{n!}, \tag{2.6}$$

and the second term

$$a_1 = \int_M \left(\frac{r}{2}c_1(X) + c_1(E)\right)\frac{c_1(L)^{n-1}}{(n-1)!} = \frac{r}{(2\pi)^n}\int_M \left(\frac{1}{2}\kappa + \mu_E\right)\frac{\omega^n}{n!}. \tag{2.7}$$

where κ is the complex scalar curvature with respect to the metric ω, which is half of the Riemannian scalar curvature.

Observe that once a basis $\{s_i\}$ for $\Gamma(E_m)$ is chosen, they determine uniquely a Hermitian metric H on E, so that

$$\sum_{i=1}^{N_m} s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi = id_E \tag{2.8}$$

In fact, for a fixed metric H, the endomorphism $\sum_i s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi$ is positive definite, as $\forall\, \psi \in E$, we have

$$H((\sum_i s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi)\psi, \psi) = \sum_i |H(s_i, \psi)|^2 + \frac{1}{m}|H(\phi, \psi)|^2$$

let $\eta^{-1} = \sum_i s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi$, then for the metric $H\eta$, using the first identity in Lemma 2.2, the identity 2.8 holds.

We have used the identification of $\mathrm{End}(E)$ with $\mathrm{End}(E_m)$ in the canonical way.

We call this metric to be canonically determined by the basis $\{s_i\}$. In case $\phi = 0$, geometrically, the metric is the pull-back metric of standard metric of on the tautological bundle on Grassmannian defined by explicit embedding 2.5 using the basis s_i (cf: [28]).

The following lemma is elementary and useful.

Lemma 2.4 *For any $T \in \mathrm{End}(E)$,*

$$|T|_{op}^2 = \sum_i |Ts_i|^2 + \frac{1}{m}|T\phi|^2, \tag{2.9}$$

holds at every point on M. Where the operator norm is defined using metric canonically determined by $\{s_i\}$.

We now define the asymptotic stability in our context. The definition is motivated by metric characterization of Gieseker stability for bundles in X.W.Wang([28]).

Definition 2.1 *Assume E is simple, that is $H^0(\mathrm{End}E) = \mathbf{C}$. $\phi \in \Gamma_{\mathrm{hol}}(E)$. τ is a fixed positive number.*

(1) *A metric H on E is called m-th balanced w.r.t. ϕ if the induced metric on E_m coincides with a metric canonically determined by a basis s_i and for some constant c,*

$$(s_i, s_j) = c\delta_{ij},$$

(2) *A m-th balanced metric H w.r.t. ϕ is called τ-balanced if $||\phi||^2 = \tau$, or equivalently*

$$c = \frac{rV - \frac{\tau}{m}}{N_m},$$

(3) *(E, ϕ) is called asymptotic stable if for any large enough m, E has m-th balanced metrics,*

(4) *(E, ϕ) is called asymptotic τ-stable if for any large enough m, E has m-th τ-balanced metrics.*

Remarks: In case $\phi = 0$, X.W.Wang ([28]) proved the above defined asymptotic stability is exactly the Gieseker stability. A simplified proof is given by Phong and Sturm ([20]) using directly the theorem of Kempf and Ness as explained in the previous Chapter. For general ϕ, the author is not able to find the appropriate algebraic notion yet. It is worth further study to understand the algebraic nature of the above defined stability.

2.2.2 Moment map interpretation

We now start moment map interpretation of balanced metrics according to the principle that stability is equivalent to the vanishing of moment maps. Our equations for balanced condition then correspond to the vanishing condition of moment maps. Actually, this is exactly the reason that we call these metrics to be 'balanced'.

We now fix a background Hermitian metric H on bundle E. We take the compact group $\mathcal{K}$ to be the unitary transformation group of (E, H) and the complex group $\mathcal{G}$ to be the general transformation group of E. Note they are also the corresponding group for bundle E_m.

It's well-known that the space of holomorphic structure on E can be identified with the space $\mathcal{A}^{(1,1)}(E,\, H)$ of unitary connection D on E with $(0,2)$ component of the curvature F_D vanishing (cf. [14]).

The action of $\mathcal{G}$ on the space of smooth sections $\Gamma(E)$ is given by

$$g \cdot \varphi = g\varphi.$$

The action of $\mathcal{G}$ on the space $\mathcal{A}^{(1,1)}(E,\, H)$ is given by

$$\overline{\partial}_{g \cdot D} = g\overline{\partial}_D g^{-1}$$

then extends to the (1,0) part of the connection using the fixed metric H in a standard way.

All actions extend naturally to bundle E_m.

We now give a metric structure and a symplectic structure on $\Gamma(E)$ by

$$(\varphi,\, \psi) = \int_M \langle \varphi,\, \psi \rangle \frac{\omega^n}{n!},$$

$$\Omega(\varphi,\, \psi) = \operatorname{Im} \int_M \langle \varphi,\, \psi \rangle \frac{\omega^n}{n!}.$$

Analogously we have metric structures and symplectic structures on $\Gamma(E_m)$.

The space $\mathcal{A}^{(1,1)}(E,\, H)$ also has a natural symplectic structure defined by Atiyah and Bott ([1])

$$\Omega(a, b) = \int_M \operatorname{Tr}(a \wedge b) \frac{\omega^{n-1}}{(n-1)!}.$$

where $a, b \in \Omega^1(\operatorname{End} E) = T_D \mathcal{A}(E)$.

We now recall the moment map interpretation of the vortex equation ([4]). Consider consider the product action of $\mathcal{K}$ on $\mathcal{A}^{(1,1)}(E,\, H) \times \Gamma(E)$.

Lemma 2.5 *([4]) The moment map μ is given by*

$$(\sqrt{-1}\Lambda F_H + \phi^* \otimes \phi - f) \frac{\omega^n}{n!}$$

for some smooth function f on M.

So the modified vortex equation corresponds exactly to the vanishing condition of moment map μ. Hence the theorem Bradlow proved confirms the principle that stability condition is the condition for the vanishing of moment map in the above infinite dimensional context ([3]).

To explain the balanced condition, we consider the action on another interesting space with an interesting symplectic structure with the help of polarize bundle L. This sort of construction is first given by Donaldson ([10]).

First of all, the group $\mathcal{G}$ has a natural action on

$$\Gamma(E_m) \times \cdots \Gamma(E_m) \times \Gamma(E) \times \mathcal{A}^{(1,1)}(E,\, H),$$

taking N_m copies of $\Gamma(E_m)$

We then let $\mathcal{H}_m$ be the subset of this product consisting of those $(s_1, \cdots, s_{N_m}; \phi; \overline{\partial}_D)$ such that s_i's are holomorphic with respect to the holomorphic structure on E_m determined by $\overline{\partial}_D$ and the fixed holomorphic structure on L and are linearly independent, as elements of $\Gamma(E_m)$, and ϕ is a holomorphic section of E with respect to the holomorphic structure $\overline{\partial}_D$.

Note the space $\mathcal{H}_m$ is invariant under the action of $\mathcal{G}$. Each complex orbit can be think as an equivalent class of holomorphic structures coupled with a set of holomorphic basis and a holomorphic section.

The interesting part is the symplectic structure on $\mathcal{H}_m$. Consider the projection π from $\mathcal{H}_m$ to $\Gamma(E_m)^{N_m} \times \Gamma(E)$, we have

Proposition 2.1 *(cf. [10]) The projection π is an injective immersion of $\mathcal{H}_m$ into $\Gamma(E_m)^{N_m} \times \Gamma(E)$*

In fact it's just another way of expressing that the map ι_m defined in 2.5 is an embedding. Take a variation vector $(\delta s_i; \delta\phi; \delta\overline{\partial}_D)$ which vanishes under projection π, that is

$$\delta s_i = \delta\phi = 0.$$

On the other hand, by the definition of $\mathcal{H}_m$, we have constraints

$$\overline{\partial}_D \, \delta s_i + \delta \overline{\partial}_D (s_i) = 0$$

which is a result of differentiating the equation $\overline{\partial}_D s = 0$. Thus we get

$$\delta \overline{\partial}_D (s_i) = 0.$$

Recall the map ι_m is an embedding map, we thus have $\delta \overline{\partial}_D = 0$. This proves that the map π is immersion. Injective is also clear.

We can use the above proposition to define a symplectic structure on $\mathcal{H}_m$. We start with the standard form Ω_{E_m} on $\Gamma(E_m)$ and the standard form Ω_E on $\Gamma(E)$, take the sum of Ω_{E_m} over N copies of $\Gamma(E_m)$ and Ω_E over $\Gamma(E)$ with weight $\frac{1}{m}$, then lift this form to $\mathcal{H}_m$ using the projection π. We write Ω for the resulting form on $\mathcal{H}_m$. Be aware of the weight $\frac{1}{m}$ we have.

Proposition 2.2 *(cf. [28]) The group $\mathcal{K}$ acts on $\mathcal{H}_m$ preserving the symplectic structure Ω. The moment map for the action is given by*

$$\mu_{\mathcal{K}}(s_1, \cdots, s_{N_m}; \phi; \overline{\partial}_D) = \frac{\sqrt{-1}}{2} \left(\sum_{i=1}^{N_m} s_i^* \otimes s_i + \frac{1}{m} \phi^* \otimes \phi \right)$$

There is another natural symmetry group acting on $\mathcal{H}_m$. This is the finite-dimensional unitary group $U(N_m)$ acting on the basis $\{s_1, \cdots, s_{N_m}\}$. Each of the group $\mathcal{G}$ and $U(N_m)$ has nontrivial center containing trivial multiplication. To avoid this duplication, we restrict to the subgroup $SU(N_m)$ of $U(N_m)$.

Proposition 2.3 *(cf. [10]) The moment map μ_{SU} for the action $SU(N_m)$ is a map from $\mathcal{H}_m$ to trace-free matrices $su(N_m)$*

$$\mu_{SU}(s_1, \cdots, s_{N_m}; \phi; \overline{\partial}_D) = \sqrt{-1}((s_i, s_j) - c\delta_{ij}) \tag{2.10}$$

where $c = \frac{1}{N_m}(\sum_i \|s_i\|^2)$

Observe that the action of $\mathcal{G}$ and $GL(N_m, \mathbf{C})$ on $\mathcal{H}_m$ commute, so we have an action of $\mathcal{G} \times GL(N_m)$ on $\mathcal{H}_m$.

Now for a fixed $a > 0$ we may consider the symplectic quotient

$$\mathcal{H}_m//(\mathcal{K} \times SU(N_m)) = \frac{\mu_{\mathcal{K}}^{-1}(\sqrt{-1}a) \cap \mu_{SU}^{-1}(0)}{\mathcal{K} \times SU(N_m)}. \tag{2.11}$$

The following two propositions fit our stability into the general framework of stability we discussed in the first chapter. The proof simply use definition.

Proposition 2.4 *(cf. [10])*

(1) *Any $\mathcal{G}$-orbit in $\mathcal{H}_m$ contains a point in $\mu_{\mathcal{K}}^{-1}(\sqrt{-1}a)$, unique up to the action of $\mathcal{K}$*

(2) *Any $SL(N_m, \mathbf{C})$-orbit in $\mathcal{H}_m$ contains a point in $\mu_{SU}^{-1}(0)$, unique up to the action of $SU(N_m)$.*

Proposition 2.5 *(cf. [10]) Assume $(E, \overline{\partial}_D)$ is simple. A triple $(E, \phi, \overline{\partial}_D)$ is m-th stable if and only if, for any $a > 0$, the corresponding complex orbit $o \subset \mathcal{H}_m$ contains a point in $\mu_{\mathcal{K}}^{-1}(\sqrt{-1}a) \cap \mu_{SU}^{-1}(0)$. In other words the complex orbit is represented by a point in the symplectic quotient.*

2.2.3 Explicit formulae

We have now fit the problem of constructing balanced metrics into the standard set up we have in the first chapter. Hence the key point is to estimate the operator norm Λ of Q^{-1} defined in 1.8 along a complex orbit. In the following, we formulate this norm in terms of the data we have. Also note we are in the situation of having two group actions.

We consider the action of the group $SU(N_m)$ on the symplectic quotient

$$\mathcal{Z} = \mathcal{H}_m//\mathcal{K}$$

We fix attention to a single orbit of the complex group $SL(N_m, \mathbf{C})$, say $z \in \mathcal{Z}$. Keep in mind, an element of the orbit is represented by a tuple $(s_1, \cdots, s_{N_m}; \phi; \overline{\partial}_D)$ which determine a unique Hermitian metric. Note we omit the component $\overline{\partial}_D$ in the following as it is not relevant in the metric structure and symplectic structure as we defined.

Given a matrix $\sqrt{-1}\,A = \sqrt{-1}(a_{ij})$ of $su(N_m)$, we write

$$\sigma_{A,i} = \sum_j a_{ij} s_j.$$

It is exactly the infinitesimal of action $\sqrt{-1}\,A$ at point $z = (s_1, \cdots, s_{N_m}; \phi)$.

To apply lemma(1.5), we need to find the orthogonal projection in the Hilbert space $\Gamma(E_k)^{N_m} \times \Gamma(E)$ of

$$\sigma_A = (\sigma_{A,1}, \cdots, \sigma_{A,N_m}; 0)$$

to the orthogonal complement of the subspace

$$P = \{(Ts_1, \cdots, Ts_{N_m}; T\phi) \colon T \in \mathrm{End}(E)\}.$$

which is exactly the image of infinitesimal action of $\mathcal{K}$ at z.

Proposition 2.6 *Given s_i, ϕ and $A = (a_{ij})$ as above, define an endomorphism $T_A \in \mathrm{End}(E)$ by*

$$T_A = \sum_{ij} a_{ij} s_i^* \otimes s_j.$$

Then the orthogonal projection of σ_A to the subspace P is

$$p_A = (T_A s_1, \cdots, T_A s_n; T_A \phi).$$

(Note we are using the metric canonically determined by the tuple $(s_1, \cdots, s_{N_m}; \phi)$).

Actually, we want to find $T_A \in \mathrm{End}(E)$ so that

$$\psi_A = \sigma_A - (T_A s_1, \cdots, T_A s_n; T_A \phi), \tag{2.12}$$

is orthogonal to $(Ts_1, \cdots, Ts_{N_m}; T\phi)$, for any $T \in \mathrm{End}(E)$.

Written out, we want for any $T \in \mathrm{End}(E)$

$$a_{ij}(s_j, Ts_i) - (T_A s_i, Ts_i) - \frac{1}{m}(T_A\phi, T\phi) = 0$$

Applying the identity $(T_1\phi, T_2\psi) = (T_1\psi^* \otimes \phi, T_2)$, we have

$$a_{ij}(s_i^* \otimes s_j, T) - (T_A s_i^* \otimes s_i, T) - \frac{1}{m}(T_A\phi^* \otimes \phi, T) = 0$$

for any $T \in \mathrm{End}(E)$.

Hence

$$a_{ij}s_i^* \otimes s_j - T_A s_i^* \otimes s_i - \frac{1}{m}T_A\phi^* \otimes \phi = 0$$

Using the metric is determined canonically by $(s_1, \cdots, s_{N_m}; \phi)$. We get

$$T_A = \sum_{ij} a_{ij}s_i^* \otimes s_j.$$

We can now apply Lemma 1.5 to see that the quantity Λ_z associated to our problem with respect to the Hilbert-Schmidt norm on Hermitian matrices is given by

$$\Lambda_z^{-1} = \min \sum ||\psi_A||^2 \tag{2.13}$$

where the minimum runs over the trace-free Hermitian matrices $A = (a_{ij})$ with $||A||^2 = \sum_{ij} |a_{ij}|^2 = 1$.

While the quantity $\Lambda_{op,z}$ associated to operator norm on Hermitian matrices is given by the same expression

$$\Lambda_{op,\,z}^{-1} = \min \sum ||\psi_A||^2 \tag{2.14}$$

however the minimum runs over the trace-free Hermitian matrices $A = (a_{ij})$ with $||A||_{op}^2 = 1$.

Recall the two standard norms on Hermitian matrices are defined as:

the Hilbert-Schmidt norm

$$\|A\|^2 = \sum_{i,j} |A_{ij}|^2$$

and the operator norm

$$\|A\|_{op} = \max \frac{|A\varphi|}{|\varphi|}$$

Our problem is to find a lower bound for the quantity Λ_z^{-1} and $\Lambda_{op,\,z}$.

Proposition 2.7 *We have identity*

$$|\overline{\partial}\psi_A|^2 = |\overline{\partial}T_A|^2_{op}$$

holds point wisely on M

In fact

$$|\overline{\partial}\psi_A|^2 = \sum |\overline{\partial}(a_{ij}s_j - T_A s_i)|^2 + \frac{1}{m}|\overline{\partial}(T_A\phi)|^2.$$

Notice s_i and ϕ are holomorphic sections, hence we get

$$|\overline{\partial}\psi_A|^2 = \sum |\overline{\partial}(T_A)s_i|^2 + \frac{1}{m}|\overline{\partial}(T_A)\phi|^2.$$

Using the identity: for any $T \in \mathrm{End}(E)$,

$$|T|^2_{op} = \sum |Ts_i|^2 + \frac{1}{m}|T\phi|^2,$$

we have $|\overline{\partial}\psi_A|^2 = |\overline{\partial}T_A|^2_{op}$.

Corollary 2.2 *We have identity*

$$\|\overline{\partial}\psi_A\|^2 = \|\overline{\partial}T_A\|^2_{op}$$

Simply integrating.

2.3 From balanced metrics to special metrics

We now prove converging balanced metrics for converge to a special metric.

Theorem 2.2 *Assume for each m large enough, there exist m-th τ-balanced metric H_m. Furthermore, assuming H_m converge to a smooth metric H_∞ in at least C^{l_0} norm. Then H_∞ satisfies the following modified vortex equation*

$$\sqrt{-1}\Lambda F_{H_\infty} + \phi^* \otimes \phi + \frac{\kappa}{2} = \frac{\bar{\kappa}}{2} + \frac{\tau}{rV} + \mu_E.$$

Hence H_∞ is also τ-special.

Here l_0 is given in Theorem 4.1 which tells us that the expansion of Bergman kernel is uniform in C^0 to the second term as we vary metrics in bounded family in C^{l_0} norm. We take $l_0 \geq 2$.

To prove the theorem, we first give a characterization of balanced metrics in terms of Bergman kernel. Recall the m-th Bergman Kernel $B_m(H) \in \mathrm{End}(E)$ is

$$B_m(H) = \sum_{i=1}^{N_m} s_i^* \otimes s_i,$$

where $\{s_i\}$ is an orthonormal basis for $\Gamma_{\mathrm{hol}}(E_m)$ w.r.t. the L^2-metric.

Proposition 2.8 *A Hermitian metric H on the bundle E is m-th balanced for (E, ϕ) if and only if for some constant $c > 0$*

$$B_m(H) = \frac{1}{c}(id_E - \frac{1}{m}\phi^* \otimes \phi).$$

It's τ-special if moreover the constant c equals $\frac{rV - \frac{\tau}{m}}{N_m}$.

In fact if

$$B_m(H) = \frac{1}{c}(id_E - \frac{1}{m}\phi^* \otimes \phi),$$

then by definition of Bergman kernel, we can choose an orthonormal basis $\{t_i\}$ so that

$$\sum t_i^* \otimes t_i = \frac{1}{c}(Id_E - \frac{1}{m}\phi^* \otimes \phi).$$

Then the basis $\{s_i = \sqrt{c} t_i\}$ will satisfy

$$\sum_{i=1}^{N_m} s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi = Id,$$

and

$$(s_i, s_j) = c\delta_{i,j},$$

which exactly means H is m-th balanced.

The other direction can be proved analogously. The statement about τ-stability is just the definition.

With this characterization and the asymptotic expansion Theorem 4.1 for Bergman kernel by Catlin ([6]), we give the proof the theorem.

Proof of the theorem:

Because H_m is m-th τ-special, using the above characterization by Bergman kernel, we have

$$B_m(H_m) = \frac{N_m}{rV - \frac{\tau}{m}}(id_E - \frac{1}{m}\phi^{*H_m} \otimes \phi).$$

The dimension N_m is a polynomial of m by Riemann-Roch formula of degree n, say

$$N_m = a_0 m^n + a_1 m^{n-1} + \cdots + a_n$$

By the identity 2.6 and 2.7, we have

$$a_0 = \frac{rV}{(2\pi)^n}, \quad a_1 = \frac{rV}{(2\pi)^n}(\frac{1}{2}\overline{\kappa} + \mu_E).$$

Thus the expansion of $B_m(H_m)$ for large m is

$$B_m(H_m) = \frac{m^n}{(2\pi)^n}(1 + \frac{1}{m}(\frac{1}{2}\overline{\kappa} + \mu_E - \phi^{*m} \otimes \phi + \frac{\tau}{rV}) + O(\frac{1}{m^2})),$$

note $\phi^{*m} \otimes \phi$ is determined by metric H_m.

We also have the expansion for $B_m(H_\infty)$ for large m

$$B_m(H_\infty) = \frac{m^n}{(2\pi)^n}(1 + \frac{1}{m}(\sqrt{-1}\Lambda F_{H_\infty} + \frac{1}{2}\kappa) + O(\frac{1}{m^2})),$$

Precisely,

$$\|B_m(H_\infty) - \frac{m^n}{(2\pi)^n}(1 + \frac{1}{m}(\sqrt{-1}\Lambda F_{H_\infty} + \frac{1}{2}\kappa)\|_{C^0} \leq Cm^{n-2}$$

And the constant C can be chosen to be uniform for the metrics varying in a bounded family of metrics in C^{l_0} norm.

As we assume H_m converge to H_∞ in C^{l_0}, we can have

$$\|\sqrt{-1}\Lambda F_{H_m} + \frac{1}{2}\kappa - \frac{1}{2}\overline{\kappa} + \mu_E - \phi^{*m} \otimes \phi + \frac{\tau}{rV}\|_{C^0} \leq Cm^{-1}$$

Thus, using $l_0 \geq 2$, we get

$$\sqrt{-1}\Lambda F_H + \frac{1}{2}\kappa = \frac{1}{2}\overline{\kappa} + \mu_E - \phi^* \otimes \phi + \frac{\tau}{rV}.$$

This is exactly what we want.

Chapter 3

From Special Metrics to Balanced Metrics

In this chapter we prove that a τ-special metric is the limit of a sequence of τ-balanced metrics. The proof is basically based on the idea of Donaldson ([10]). In section 3.1, we work out the required estimates in order to construct balanced metrics. Our estimates are in the shape form as suggested in the last part of Donaldson's paper ([10]). This estimates is enough to derive the existence of balanced metrics if we only use the approximation to the third term. In section 3.2, we work out the approximation procedure. We show that the approximation procedure can be carried out to any order by solving a sequence of recursive equations. In the last section 3.3, we prove the existence of balanced metrics and their convergence to the given special metric.

3.1 Analytical estimates

In last Chapter, we have fit the problem of finding balanced metrics into the general moment map framework. To construct balanced metrics is then equivalent to find zeros of moment map.

For that purpose, we want to apply the general Proposition 1.4. The most important issue is thus to estimate the quantity Λ and Λ_{op}. In this section we obtain explicit estimate on the quantity Λ based on analytic arguments in Donaldson ([10]).

Fix any reference metric H_0 on bundle E and an integer l. For $R > 0$, we say that another metric H on E is $R-$ *bounded* if $H > R^{-1}H_0$ and

$$\|H - H_0\|_{C^l(H_0)} < R,$$

where $\| \ \|_{C^l(H_0)}$ is the norm determined by the fixed referenced metric H_0. Clearly, at the cost of changing R, this notion is independent of the choice of H_0. Now, as in 2.8, we consider a basis s_i for $H^0(E_m)$ which together with ϕ determines a unique metric on E such that

$$\sum_{i=1}^{N_m} s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi = Id$$

at each point. We say that the basis $\{s_i\}$ has R-bounded geometry if the Hermitian metric they determined is R-bounded. Notice that, working with this metric,

$$\sum_i \|s_i\|^2 + \frac{1}{m}\|\phi\|^2 = rV$$

This is a crucial observation. It tells the sum of L^2 norm of s_i are bounded by universal constant.

We write

$$(s_i, s_j) = \frac{rV - \frac{\|\phi\|^2}{m}}{N_m}\delta_{ij} + \Upsilon_{ij}.$$

Then the matrix $\Upsilon = (\Upsilon_{ij})$ is a trace-free Hermitian matrix. Υ=0 if and only if the metric is m-th balanced.

We continue to use the notation in subsection 2.2.3, so for any matrix $A = (a_{ij}) \in \sqrt{-1}\, su(N_m)$ we define a $T_A \in$End(E) and sections ψ_A by (2.12).

We make use of two standard norms on Hermitian matrices:

the Hilbert-Schmidt norm

$$||A||^2 = \sum_{i,j} |A_{ij}|^2$$

and the operator norm

$$||A||_{op} = \max \frac{|A\varphi|}{|\varphi|}$$

We have the following elementary inequalities, for $N \times N$ Hermitian metrics A, B

$$|\mathrm{Tr}(ABA)| \leq ||A||^2 ||B||_{op}, \tag{3.1}$$

$$|\mathrm{Tr}(AB)| \leq \sqrt{N} ||A|| \, ||B||_{op}. \tag{3.2}$$

The goal of this section is to prove:

Theorem 3.1 *Suppose E is simple. For any R and $\varepsilon > 0$, arbitrary small, there is a constant $C: = C(R, H_0, \varepsilon)$ and $\epsilon: = \epsilon(R, H_0, \varepsilon) < \frac{1}{10}$ such that, for any m large enough, if the basis $\{s_i\}$ for $H^0(E_m)$ has R-bounded geometry and with error term $||\Upsilon|| < \frac{\epsilon}{m^n}$. Then, for any traceless Hermitian matrix A, we have*

$$||A|| \leq C m^{\frac{n}{2}+\frac{1}{2}+\varepsilon} ||\psi_A||.$$

By identity 2.13 and $||A||_{op} \leq ||A||$, this yields

Corollary 3.1 *If z is the point in $\mathcal{Z}$ determined by a basis $\{s_i\}$ which satisfies the condition in the Theorem 3.1 we have*

$$\Lambda_z \leq C^2 m^{n+1+2\varepsilon},$$

and

$$\Lambda_{op,\,z} \leq C^2 m^{n+1+2\varepsilon}.$$

The analytical estimate required to prove Theorem 3.1 is summarized in the following:

Proposition 3.1 *Suppose E is simple. If the basis $\{s_i\}$ for $H^0(E_m)$ has R-bounded geometry and $||\Upsilon||_{op} \leq \frac{1}{10m^n}$ then there are constants C_1, C_2 and C_p, depending only on R, ω and H_0,* *such that for m sufficiently large and any traceless Hermitian matrix A, we have:*

(1) $\|\bar{\partial} T_A\|_{op}^2 \leq C_1 m^{1+\frac{n}{2}} \|\psi_A\| \, \|A\|$,

(2) $\|T_A\|^2 \leq C_2 \|\bar{\partial} T_A\|^2 + \frac{N_m}{rV} \|\Upsilon\|_{op}^2 \|A\|^2$,

(3) $\|A\|^2 \leq \frac{5}{4} m^n (\|T_A\|_{op}^2 + \|\psi_A\|^2)$,

(4) *Moreover for large integer p, we have a sharper version for (1),*

$$\|\bar{\partial} T_A\|_{op}^2 \leq C_p m^{1+\frac{n}{2p}} \|\psi_A\|^{2-\frac{1}{p}} \, \|A\|^{\frac{1}{p}}.$$

Here all the norms are defined by the metric determined by the sections $\{s_i\}$ together with the given section ϕ.

We now give the proof of Theorem 3.1 given the above proposition.

Proof of Theorem 3.1:

If we have, say $\|A\|^2 \leq \frac{5}{2} m^n \|\psi_A\|^2$, then Theorem 3.1 holds trivially. If not, then

$$\frac{5}{2} m^n \|\psi_A\|^2 \leq \|A\|^2.$$

Using inequality (3), we then have

$$\|A\|^2 \leq \frac{5}{4} m^n \|T_A\|_{op}^2 + \frac{1}{2} \|A\|^2.$$

It follows then

$$\|A\|^2 \leq \frac{5}{2} m^n \|T_A\|_{op}^2.$$

By inequality (2) and $\|T_A\|_{op}^2 \leq \|T_A\|^2$, we have

$$\|A\|^2 \leq \frac{5C_2}{2} m^n \|\bar{\partial} T_A\|^2 + \frac{5m^n}{2} \frac{N_m}{rV} \|\Upsilon\|_{op}^2 \|A\|^2.$$

Using the assumption $||\Upsilon||_{op} \leq \frac{1}{10m^n}$ and $N_m \approx rVm^n$, we have, say

$$||A||^2 \leq \frac{5C_2}{2} m^n ||\overline{\partial} T_A||^2 + \frac{1}{50}||A||^2.$$

Thus we have, for some constant C

$$||A||^2 \leq Cm^n ||\overline{\partial} T_A||^2.$$

We now apply stronger inequality (4) and $||\overline{\partial} T_A||^2 \leq r^2 ||\overline{\partial} T_A||^2_{op}$, we have

$$||A||^2 \leq C_p C r^2 m^{n+1+\frac{n}{2p}} ||\psi_A||^{2-\frac{1}{p}} ||A||^{\frac{1}{p}},$$

or equivalently

$$||A|| \leq (C_p C r^2)^{\frac{p}{2p-1}} m^{\frac{n+1+\frac{n}{2p}}{2-\frac{1}{p}}} ||\psi_A||.$$

Notice $\frac{n+1+\frac{n}{2p}}{2-\frac{1}{p}} \approx \frac{n+1}{2}$ for large p, hence for any ε small enough, we can choose p large enough, so that for some constant, say C, we have

$$||A|| \leq Cm^{\frac{n}{2}+\frac{1}{2}+\varepsilon} ||\psi_A||.$$

This finish the proof of the main Theorem 3.1.

We now begin the proof of Proposition 3.1. The most important inequality is (1), so we start with that. We begin with a Lemma which expresses the face that we can control the size of the derivatives of the holomorphic sections s_i and ϕ.

Lemma 3.1 *Under the assumption of Theorem 3.1, For each fixed integer K, there is a constant $C = C(K)$ such that for any integer $j \leq K$*

(1) $\sum_i |\nabla^j s_i|^2 + \frac{1}{m}|\nabla^j \phi|^2 \leq Cm^{j+n}$,

at each point of M;

(2) $||\nabla^j T_A||^2 \leq Cm^j ||A||^2$, We emphasis again that in this Lemma we use the metric canonically determined by the sections $\{s_i\}$ and the given connection and metric on the line bundle L^m.

Proof of (1) in the Lemma 3.1 uses the following inequality in Donaldson ([10], cf. [28]). For a fixed metric H on bundle E, there is a constant C, depending only on ω and H, so that for each point p on M and any holomorphic section s of E_m

$$|(\nabla^j s)_p|^2 \leq Cm^{j+n}||s||^2.$$

It is clear then, under the assumption of R-bounded geometry, we can choose a fixed constant C, depending only on R and H_0.
The the inequality can be derived by simply summing up combining the fact

$$\sum_{i=1}^{N_m} ||s_i||^2 + \frac{1}{m}||\phi||^2 = rV.$$

To prove (2) in the Lemma 3.1 uses same trick in Donaldson [10]. Notice T_A $\in$End(E) is not a holomorphic section, however it is a holomorphic section of the bundle $F_m = \pi_1^* \overline{E_m}^\vee \otimes \pi_2^* E_m$ over the manifold $X = \overline{M} \times M$. Here $\overline{M}$ ($\overline{E}$) is M (E) with opposite complex structure,π_1 is the projection to the first factor and π_2 is the projection to the second factor. Then for any Hermitian matrix $A = (a_{ij})$ we get a holomorphic section

$$\tilde{T_A} = \sum a_{ij} \tilde{s_i} \otimes s_j$$

of F over X. Because it's holomorphic, we can use the same inequality as in the proof of (1). To apply the inequality, we compute the L^2 norm of $\tilde{T_A}$ which is

$$||\tilde{T_A}||^2 = \sum_{ii'jj'} a_{ij}\overline{a_{i'j'}}(s_i, s_j)(s'_j, s'_i)$$

or in matrix notation

$$||\tilde{T}_A||^2 = \mathrm{Tr}(A(\frac{rV - \frac{||\phi||^2}{m}}{N_m} Id + \Upsilon)(\frac{rV - \frac{||\phi||^2}{m}}{N_m} Id + \Upsilon^*)A^*)$$

Since $||\Upsilon|| \leq \frac{1}{10m^n}$ we deduce, using 3.1, $N_m \approx rVm^n$ and $||\phi||$ is bounded due to R-boundedness that

$$||\tilde{T}_A|| \leq Cm^{-n}||A||.$$

The proof of (2) of Lemma 3.1 is done by noticing that T_A is the restriction of $\tilde{T}_A$ to the diagonal of $\tilde{M} \times M$ and applying the same sort inequality in the proof of (1).

Now we can give the proof of first inequality (1) in Lemma 3.1. Recall from the Corollary Proposition 2.7 that

$$||\bar{\partial} T_A||_{op}^2 = ||\bar{\partial}\psi_A||^2 = \sum_i ||\bar{\partial}\psi_{A,i}||^2.$$

We use $i = 0$ to denote the component of $\Gamma(E)$.

We also have identity

$$||\bar{\partial}\varphi||^2 = (\varphi, \Delta\varphi),$$

and the inequality

$$(\varphi, \Delta\varphi) \leq ||\varphi|| \, ||\Delta\varphi||,$$

and we get

$$||\bar{\partial} T_A||_{op}^2 \leq (\sum ||\Delta\psi_{A,i}||^2)^{1/2} (\sum ||\psi_{A,i}||^2)^{1/2} = (\sum ||\Delta\psi_{A,i}||^2)^{1/2} \, ||\psi_A||,$$

Now for each $i > 0$, we have

$$\Delta\psi_{A,i} = \Delta(a_{ij}s_j - T_A s_i) = -\Delta(T_A s_i),$$

and for $i = 0$, we have

$$\Delta\psi_{A,0} = -\Delta(T_A\phi).$$

Now use the identity

$$\Delta(T\varphi) = \Delta(T)\varphi + 2\nabla T \cdot \nabla\varphi$$

and Lemma 3.1, we have

$$\begin{aligned} \|\bar{\partial} T_A\|_{op}^2 &\leq C(\sum \|\Delta T_A s_i\|^2 + \frac{1}{m}\|\Delta T_A \phi\|^2)^{1/2} + (\sum \|\nabla T_A \cdot \nabla s_i\|^2 + \frac{1}{m}\|\nabla T_A \cdot \nabla \phi\|^2)^{1/2}) \\ &\leq Cm^{n+1}\|A\| \, \|\psi\| \end{aligned}$$

We now give the last inequality (4) in Proposition 3.1. Basis idea is the same, however we use the following inequality

$$(\varphi, \Delta\varphi) \leq \|\varphi\|^{2-\frac{1}{p}} \, \|\Delta^p \varphi\|^{\frac{1}{p}}.$$

It can be proved using Hölder inequality and eigenspace decomposition for positive elliptic operator Δ.

Applying Hölder inequality, we get then

$$\|\bar{\partial} T_A\|_{op}^2 \leq \|\psi_A\|^{2-\frac{1}{p}} \, \|\Delta^p \psi_A\|^{\frac{1}{p}}.$$

Then apply Lemma 3.1, we get, for some constant C_p

$$\|\bar{\partial} T_A\|_{op}^2 \leq C_p m^{1+\frac{n}{2p}} \|\psi_A\|^{2-\frac{1}{p}} \, \|A\|^{\frac{1}{p}}.$$

We now proceed to prove the second inequality (2) in Proposition 3.1.

Recall the following inequality,

Lemma 3.2 *Suppose E is simple. Then for any $L > 1$, there is a constant C, depending only on H_0 and L, such that if H is any Hermitian metric on E with $LH_0 > H > L^{-1}H_0$ and if Φ is any endomorphism of E, we have*

$$\|\Phi\|^2 \leq C\|\bar{\partial}\Phi\|^2 + \frac{\left|\int_M \mathrm{Tr}\Phi \, d\mu\right|^2}{rV}, \tag{3.3}$$

The norms in above inequality are defined in terms of metric H. In case Φ is Hermitian, we can omit the absolute value, because trace is then a real number.

In fact, as we have seen in the proof of Lemma 3.3, the kernel of the elliptic operator consists of constant multiple of identity, because of the simpleness assumption. And the deduction of above inequality for a fixed metric is standard and the constant C can be taken as the first positive eigenvalue of the positive elliptic operator $\overline{\partial}^*\overline{\partial}$. For varying metric in a bounded family, as operator $\overline{\partial}$ doesn't depend on the metric, we can choose C uniformly. Note this is different from Donaldson's case ([10]). The operator there depends on the metric. The proof of corresponding lemma is harder.

We apply the above lemma to prove (2) in Lemma 3.1. In fact using

$$T_A = a_{ij} s_i^* \otimes s_j,$$

we have

$$\begin{aligned}
\int_M \mathrm{Tr}\, T_A \, d\mu &= \int_M \mathrm{Tr}\, a_{ij} s_i^* \otimes s_j \, d\mu \\
&= \int_M a_{ij} \langle s_j, s_i \rangle \, d\mu \\
&= a_{ij}(s_j, s_i) \\
&= \mathrm{Tr}(A\Upsilon) \\
&\leq \sqrt{N_m} ||\Upsilon||_{op} ||A||
\end{aligned}$$

We used the fact that A is trace-free and the inequality 3.2 in the above deduction.

We now prove (3) in Lemma 3.1.

Recall we have an orthogonal decomposition in $\Gamma(E_m)^{N_m} \times \Gamma(E)$

$$\sigma_A = \psi_A + p_A,$$

so we have

$$||\sigma_A||^2 = ||\psi_A||^2 + ||p_A||^2.$$

Using identity 2.9, we also have

$$\|p_A\|^2 = \|T_A\|_{op}^2.$$

For the rest σ_A, we have

$$\|\sigma_A\|^2 = \sum_{ijk} a_{ij}\overline{a_{ik}}(s_j, s_k) = \sum_{ij} \frac{rV - \frac{\|\phi\|^2}{m}}{N_m}|a_{ij}|^2 + \mathrm{Tr}A\Upsilon A^*.$$

Then (3) is proved by the fact $N_m \approx rVm^n$, boundedness of $\|\phi\|^2$ and inequality 3.1.

3.2 Constructions of approximate balanced metrics

In this section, we want to construct metric which are very close to being balanced assuming the existence of a special metric.

Suppose H is an Hermitian metric on E. Fix an integer q, we call H is called approximatible to be balanced to the order q if there are smooth self-adjoint endomorphisms η_1, η_2, $\cdots$, η_q $\in$End(E) such that if H_m is of the form

$$H_m = H(1 + \sum_{j=1}^{q} \eta_j m^{-j})$$

which is a Hermitian metric for large enough m, then

$$B_m(H_m) = \frac{N_m}{rV - \frac{\tau}{m}}(id_E - \frac{1}{m}\phi^{*H_m} \otimes \phi) + \sigma_q(m)$$

where $\sigma_q(m) = O(m^{n-q-2})$. Precisely, for any integer $l \geq 0$, we have

$$\|\sigma_q(m)\|_{C^l} \leq C_{l,q} m^{n-q-2}.$$

where $C_{l,q}$ is some constant depending on l, q, metric H and ω.

Note, when $q = 0$, by our discussion in Section 2.3, we see the obstruction for the metric H to be approximatible to be balanced to order 0 is that it satisfies the following modified vortex equation

$$\sqrt{-1}\Lambda F_H + \phi^* \otimes \phi + \frac{\kappa}{2} = \frac{\bar{\kappa}}{2} + \frac{\tau}{rV} + \mu_E.$$

for some $\tau \geq 0$.

Also recall H_m is m-th balanced if and only if $\sigma_q(m)$ vanish by the characterization of balanced metric in terms of Bergman kernel.

The above definition can be used in Hermitian-Einstein case simply by setting $\phi = 0$ and $\tau = 0$.

Our main result in this section is

Theorem 3.2 *Suppose E is simple, (E, ϕ) has a τ-special metric H_∞ such that*

$$\sqrt{-1}\Lambda F_{H_\infty} + \phi^* \otimes \phi + \frac{\kappa}{2} = \frac{\bar{\kappa}}{2} + \frac{\tau}{rV} + \mu_E.$$

Then it is approximatible to any order. Moreover the endomorphisms $\eta_1, \eta_2, \cdots$ are uniquely determined by H_∞ in the form of a sequence of recursive equations.
Furthermore, the L^2 norm of the section ϕ is fixed during the approximation procedure.

The following lemma is crucial for the proof of the theorem. Basically we are studying the linearized equation. The first is standard. The second one is not hard, however it uses the special curvature property of special metrics.

Lemma 3.3 *Assume E is simple, $\phi \in \Gamma_{\mathrm{hol}}(E)$ is nontrivial.*

(1) *Let H be any Hermitian metric on E, then the linearized equation*

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi + \phi^* \otimes \phi\,\Phi = \Psi,$$

is always solvable and has a unique solution for any smooth endomorphism Ψ of E. Moreover, we have

$$(\Phi\,\phi, \phi) = (\Psi, id) = \int_M \mathrm{Tr}\Psi\,\frac{\omega^n}{n!}.$$

(2) *Furthermore if H satisfies the following modified vortex equation*

$$\sqrt{-1}\,\Lambda F_H + \phi^* \otimes \phi = f,$$

for some smooth function f. Then the adjoint of the unique solution Φ^* satisfying the following equation

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi^* + \phi^* \otimes \phi\,\Phi^* = \Psi^*.$$

Thus by the unique of solution, Φ is self-adjoint if and only if Ψ is self-adjoint.

Proof of the lemma:

(1): We use the standard technique: Fredholm alternative for elliptic equation.

On one hand using the identity in Lemma 2.3

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi = \partial^*\partial\Phi = \Delta'\Phi$$

is an self-adjoint elliptic operator.

On the other hand, $\forall\,\Phi, \Psi \in \mathrm{End}(E)$

$$\begin{aligned}
\langle \phi^* \otimes \phi\,\Phi, \Psi\rangle &= \mathrm{Tr}(\phi^* \otimes \phi\,\Phi\,\Psi^*) \\
&= \mathrm{Tr}(\Phi\,\Psi^*\,\phi^* \otimes \phi) \\
&= \mathrm{Tr}(\Phi\,(\phi^* \otimes \phi\,\Psi)^*) \\
&= \langle \Phi, \phi^* \otimes \phi\,\Psi\rangle.
\end{aligned}$$

Altogether we see the operator

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial + \phi^* \otimes \phi$$

is self-adjoint and elliptic.

Thus to prove the lemma, we need only to show it has trivial kernel.

Suppose $\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi + \phi^* \otimes \phi\,\Phi = 0$ for some $\Phi \in \mathrm{End}(E)$.

Taking the inner product with Φ itself, using the identities in Lemma 2.2 and 2.3, we get

$$(\partial\Phi, \partial\Phi) + (\Phi^*\phi, \Phi^*\phi) = 0.$$

Both term on the left hand side are nonnegative, so we must have

$$(\partial\Phi, \partial\Phi) = 0, \quad (\Phi^*\phi, \Phi^*\phi) = 0.$$

Hence point wisely

$$\partial\Phi = 0 \quad \Phi^*\phi = 0$$

By the assumption E is simple and identity

$$(\partial\Phi)^* = \overline{\partial}\Phi^*,$$

we have $\Phi = c\, id_E$.

However ϕ is a nontrivial section, hence $\Phi^*\phi = 0$ forces $c = 0$, that is $\Phi = 0$.

For the other statement in (1) about the trace, we take the inner product with id, using the identity in Lemma 2.3

$$(\sqrt{-1}\,\Lambda\overline{\partial}\partial\Phi, id) = (\partial\Phi, \partial id) = 0$$

and the lemma 2.2

$$(\phi^* \otimes \phi\,\Phi, id) = (\Phi\,\phi, \phi).$$

(2) For this statement, we use the special curvature property of the metric H and Kähler

identity. In fact

$$\begin{aligned}
\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi^* &= \sqrt{-1}\,\Lambda\bar{\partial}(\overline{\partial\Phi})^* \\
&= \sqrt{-1}\,\Lambda(\partial\bar{\partial}\Phi)^* \\
&= -(\sqrt{-1}\,\Lambda\partial\bar{\partial}\Phi)^* \\
&= (\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi - [\sqrt{-1}\,\Lambda F_H, \Phi])^*
\end{aligned}$$

Now using the vortex equation and the fact $f\,id$ is in the center of Lie algebra $\mathrm{End}(E)$

$$[\sqrt{-1}\,\Lambda F_H, \Phi] = -[\phi^* \otimes \phi, \Phi]$$

Then we have

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi^* + \phi^* \otimes \phi\,\Phi^* = (\sqrt{-1}\,\Lambda\bar{\partial}\partial\Phi + \phi^* \otimes \phi\,\Phi)^* = \Psi^*$$

This finishes the proof of the lemma. Now we discuss the approximation procedure.

Proof of the theorem:

Recall we have expansion for the Bergman kernel

$$B_m(H) = \frac{m^n}{(2\pi)^n}(id + A_1(H)m^{-1} + \cdots + A_{q+1}(H)m^{-q-1} + O(m^{-q-2}))$$

where A_p's polynomials in the curvature of H and ω and its covariant derivatives, and the error term is uniformly bounded in C^l for all metrics H in a bound family in $C^{l'}$ (l' is suitably related to l).

Recall the integration of the trace of $A_p(H)$ are topological invariants

$$\int_M \mathrm{Tr} A_p(H)\frac{\omega^n}{n!} = a_p,$$

where a_p are p-th coefficients in N_m. Also $A_p(H)$ are self-adjoint (cf. Remark 4.1).

We can make a Taylor expansion of the coefficients

$$A_p(H(1+\eta)) = A_p(H) + \sum_{k=1}^{q} A_{p,k}(\eta) + O(\|\eta\|_{C^{l'}}^{q+1})$$

where $A_{p,k}(\eta)$ is a homogeneous polynomial of degree k, depending on H, ω, in η and its covariant derivatives and l' is sufficient large (depending on l and q).
Thus for any $\eta_1, \eta_2, \cdots, \eta_q \in \mathrm{End}(E)$, we can write

$$A_p(H(1+\sum_{j=1}^{q} \eta_j m^{-j})) = A_p(H) + \sum_{k=1}^{q} b_{p,k}(\eta) m^{-k} + O(m^{-q-1}),$$

where the $b_{p,k}$ are certain multi-linear expressions in the η_j, and their covariant derivatives, beginning with

$$b_{p,1} = A_{p,1}(\eta_1).$$

Thus we get

$$B_k(H_m) = \frac{m^n}{(2\pi)^n}(\sum_{p=1}^{q+1} A_p(H_\infty) m^{-p} + \sum_{p,k=1}^{p+k=q+1} b_{p,k} m^{-p-k} + O(m^{-q-2})). \tag{3.4}$$

where $H_m = H_\infty(1 + \sum_{j=1}^{q} \eta_j m^{-j})$.
We now should choose the η_j so that the terms in the right handside of above expression are the same as

$$\frac{N_m}{rV - \frac{\tau}{m}}(id_E - \frac{1}{m}\phi^{*H_m} \otimes \phi) \tag{3.5}$$

up to m^{n-q-1}.
We then try inductively. Suppose, inductively, that we have chosen the $j \leq p$ so that the coefficient of m^{n-j} coincide for $j \leq p$. The new term η_{p+1} appears only once in the coefficient of both 3.4 and 3.5, in the form $A_{1,1}(\eta_{p+1})$ in 3.4 and in the form $-\phi^* \otimes \phi\, \eta_{p+1}$ in 3.5.
Now by the expression for $A_2(H)$ and Corollary 2.1, we have

$$A_{1,1}(\eta) = \sqrt{-1}\,\Lambda\bar{\partial}\partial\,\eta. \tag{3.6}$$

Thus to construct η_{p+1}, all we have to do is to solve the following linear equation

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\,\eta_{p+1} + \phi^* \otimes \phi\,\eta_{p+1} = \zeta_{p+1}, \tag{3.7}$$

for some $\zeta_{p+1} \in \mathrm{End}(E)$

For instance, the equation for η_1 is

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\,\eta_1 + \phi^* \otimes \phi\,\eta_1 = a_2 - A_2(H_\infty) + a_1\frac{\tau}{rV} - a_1\phi^* \otimes \phi + \frac{\tau^2}{(rV)^2} - \frac{\tau}{rV}\phi^* \otimes \phi. \tag{3.8}$$

The solvability of the equation 3.7 has been established in Lemma 3.3. It is always solvable. However to establish that H_m is Hermitian metric, we have to show the unique solution η_{p+1} for 3.7 is self-adjoint. Then by the Lemma 3.3, ζ_{p+1} has to be self-adjoint.

For ζ_1, it is true by the expression in 3.8. For higher order terms, let us take a look at ζ_2. By straightforward computation, we get

$$\begin{aligned}
\zeta_2 &= a_3 - A_3(H_\infty) + \sqrt{-1}\,\Lambda\bar{\partial}(\eta_1\partial\,\eta_1) - b_{21}(\eta_1) \\
&- (\frac{\tau}{rV} + a_1)\phi^* \otimes \phi\,\eta_1 \\
&- (\frac{\tau^2}{(rV)^2} + a_1\frac{\tau}{rV} + a_2)\phi^* \otimes \phi \\
&+ \frac{\tau^3}{(rV)^3} + a_1\frac{\tau^2}{(rV)^2} + a_2\frac{\tau}{rV}.
\end{aligned} \tag{3.9}$$

The terms on the last two rows are self-adjoint. However the terms on the first two rows are really involved. For example, the term $b_{21}(\eta_1)$, which is the first variation of $A_2(H)$ with respect to η_1, which we compute in the last chapter, is really involved. And also the term $\sqrt{-1}\,\Lambda\bar{\partial}(\eta_1\partial\,\eta_1)$ is also complicated. Now it seems that there is no way to establish the self-adjointness in general. However, it turns out that to show the self-adjointness, we do not really have to know the term ζ_{p+1} explicitly.

In fact, let

$$H_m = H_\infty(1 + \eta_1 m^{-1} + \cdots + \eta_p m^{-p}).$$

We write

$$B_m(H_m) = \frac{m^n}{(2\pi)^n}(1 + B_1 m^{-1} + \cdots + B_{p+1} m^{-p-1} + B_{p+2} m^{-p-2} + O(m^{-p-3})),$$

And

$$\frac{N_m}{rV - \frac{\tau}{m}}(id - \frac{1}{m}\phi^{*_{H_m}} \otimes \phi) = \frac{m^n}{(2\pi)^n}(1 + D_1 m^{-1} + \cdots + D_{p+1} m^{-p-1} + D_{p+2} m^{-p-2} + O(m^{-p-3})).$$

Clearly the endomorphisms C's and D's are determined by $\eta_1, \cdots, \eta_p$. Then the approximation to order p just means that

$$B_i = D_i, i = 1, \cdots, p+1$$

Clearly the equation 3.7 is nothing but

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\,\eta_{p+1} + \phi^* \otimes \phi\,\eta_{p+1} = D_{p+2} - B_{p+2}.$$

Our goal is thus to show

$$B^*_{p+2} - B_{p+2} = D^*_{p+2} - D_{p+2}.$$

Claim: We have the following recursive formulae

$$B^*_{p+2} - B_{p+2} = \eta_1 B_{p+1} - B^*_{p+1}\eta_1 + \eta_2 B_p - B^*_p \eta_2 + \cdots + \eta_p B_2 - B^*_2 \eta_p.$$

and

$$D^*_{p+2} - D_{p+2} = \eta_1 D_{p+1} - D^*_{p+1}\eta_1 + \eta_2 D_p - D^*_p \eta_2 + \cdots + \eta_p D_2 - D^*_2 \eta_p.$$

In fact, as we have seen that Bergman kernel $B_m(H_m)$ is self-adjoint w.r.t. the metric H_m. In terms of the metric H_∞, that reads

$$(1 + \eta_1 m^{-1} + \cdots + \eta_p m^{-p}) B_m(H_m) = B^*_m(H_m)(1 + \eta_1 m^{-1} + \cdots + \eta_p m^{-p}).$$

Thus the claim follows by comparing the coefficient of m^{n-p-2}.

The other one is also clear by the fact that $\phi^{*_{H_m}} \otimes \phi$ is also self-adjoint w.r.t. the metric H_m. By the same reasoning as above, we confirm our claim.

Thus using the inductive assumption, we conclude that ζ_{p+1} is self-adjoint, so is η_{p+1}.

For the invariant of L^2 norm of ϕ, we prove inductively that

$$\int_M \operatorname{Tr} \zeta_{p+1} \, d\mu = 0.$$

In fact, using the Corollary 4.1 that the integration of the trace of the terms in Bergman kernel is of topological nature, we have

$$\int_M \operatorname{Tr} B_{p+2} \, d\mu = a_{p+2}.$$

On the other hand, by the inductive assumption we have

$$\int_M \operatorname{Tr} \phi^{*_{H_m}} \otimes \phi \, d\mu = \tau.$$

Thus we also have

$$\int_M \operatorname{Tr} D_{p+2} \, d\mu = a_{p+2}.$$

Together applying the result in Lemma 3.3, we have

$$(\eta_{p+1} \phi, \phi) = 0.$$

which exactly means the invariant of L^2 norm of ϕ under the metric H_m.

This completes our proof.

Remark. In [28], X.W.Wang worked the analogous problem for a Hermitian-Einstein metric. However, in the paper, he wrote

$$A_{1,1}(\eta) = \sqrt{-1}\Lambda\bar{\partial}(\eta^{-1}\partial\eta)$$

As we have seen, it should be $\sqrt{-1}\,\Lambda\bar{\partial}\partial\,\eta$.

As we have seen, the Hermitian-Einstein equation is a special case of vortex equation. However, the corresponding linearized equations are slightly different. In the following we study the Hermitian-Einstein case.

For the linearized problem, we use the following analogous Lemma to Lemma 3.3. Also the first one is standard. The second one use the special curvature property of Hermitian-Einstein metric.

Lemma 3.4 *Assume E is a simple bundle,*

(1) *For any Hermitian metric H, equation*

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta = \zeta. \tag{3.10}$$

is always solvable for any smooth endomorphism ζ of E with $\int_M \mathrm{Tr}\zeta\,\frac{\omega^n}{n!} = 0$. The solution is unique if we fix $\int_M \mathrm{Tr}\eta\,\frac{\omega^n}{n!} = 0$.

(2) *Moreover, if H is a Hermitian-Einstein metric on E satisfying the following equation*

$$\sqrt{-1}\,\Lambda F_H = f$$

for some smooth function. Then the adjoint of the unique solution η^* satisfying the following equation

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta^* = \zeta^*.$$

and $\int_M \mathrm{Tr}\eta^*\,\frac{\omega^n}{n!} = 0$.

Thus by the uniqueness of the solution, η is self-adjoint if and only if ζ is self-adjoint.

The proof use the similar strategy in the proof of Lemma 3.3. Using the Fredholm alternative, we only need to compute the kernel of operator $\sqrt{-1}\,\Lambda\bar{\partial}\partial$. As we have already seen the kernel consist endomorphism of form $c\,id$, for some constant c, due to the assumption

E is simple. Thus by Fredholm alternative, equation 3.10 has a solution if and only if

$$(\zeta, id) = 0,$$

which written out is the same as $\int_M \mathrm{Tr}\zeta \frac{\omega^n}{n!} = 0$.
For the second statement, we use the Hermitian-Einstein equation and Kähler identity. In fact, we've already seen

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta^* = (\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta - [\sqrt{-1}\,\Lambda F_H, \eta])^*.$$

Using the equation, this implies

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta^* = (\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta)^* = \zeta^*$$

For the approximation part, it is also true that the approximation procedure can be carried out to any order. In fact, the derivation is simpler then those for vortex equation. the reason is that the corresponding $B_1, B_2, \cdots, B_{p+1}$ are constants. It thus follows that B_{p+2} are self-adjoint. Then use the fact that integration of trace of Bergman kernel is of topological nature, we complete the inductive process. For comparison, we write the first two equations below:

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta_1 = a_2 - A_2(H_\infty),$$

$$\sqrt{-1}\,\Lambda\bar{\partial}\partial\eta_2 = a_3 - A_3(H_\infty) + \sqrt{-1}\,\Lambda\bar{\partial}(\eta_1\partial\,\eta_1) - b_{21}(\eta_1).$$

Remark: The author originally thought that the approximation was unlikely to be carried out to any order. As we have seen, the second equation 3.9 in the approximation procedure is very involved. To show that the solution of the second equation is self-adjoint directly using the explicit formula of each term seems unpromising. Later Tian suggested me to find a recursive formula to see the adjointness. It turns out working. The author is thankful for his suggestion. The author is also thankful for Richard Melrose for questioning the author on the approximation procedure.

3.3 Convergence to balanced metrics

In this section, we prove the existence of balanced metrics given a special metric and the convergence of balanced metrics to the given metric.

Theorem 3.3 *Suppose E is simple, (E, ϕ) has a τ-special metric H_∞ such that*

$$\sqrt{-1}\Lambda F_{H_\infty} + \phi^* \otimes \phi + \frac{\kappa}{2} = \frac{\bar{\kappa}}{2} + \tau + \mu_E. \tag{3.11}$$

Then for each integer m large enough, there exit m-th τ-balanced metric H_m on E. Moreover, we have convergence

$$H_m \to H_\infty$$

in any C^l norm.

The plan for the proof is, starting with a special metric, first use the approximation Theorem 3.2 to get metric which is close to being balanced for large m, then shift to the context of moment map set-up, using the estimate in Theorem 3.1, we then can apply Proposition 1.4 to prove the existence of balanced metrics.

Assume H_∞ can be approximated to order q. We wrote

$$B_m(H_m) = \frac{N_m}{rV - \frac{\tau}{m}}((id_E + \epsilon_m)^{-1} - \frac{1}{m}\phi^{*H_m} \otimes \phi)$$

where $\epsilon_m = O(m^{-q-2})$. Then we pick up any orthonormal basis $\{\sqrt{\frac{N_m}{rV-\frac{\tau}{m}}}s_i\}$ for $H^0(E_m)$ with respect to metric H_m, then we have

$$\sum_i s_i^* \otimes s_i + \frac{1}{m}\phi^* \otimes \phi = (id_E + \epsilon_m)^{-1}.$$

We then define $H'_m = H_m(id_E + \epsilon_m)$. Because ϵ_m is self-adjoint w.r.t. H_m, H'_m define an Hermitian metric for large m. Under this metric

$$\sum_i s_i^{*'} \otimes s_i + \frac{1}{m}\phi^{*'} \otimes \phi = id_E.$$

Thus we are in the situation considered in Section 1.3, with a point, say z, in the symplectic quotient $\mathcal{Z} = \mathcal{H}_m//\mathcal{K}$. For any trace-free Hermitian matrix $B \in \sqrt{-1}su(N_m)$ we can use the action of $SL(N_m, \mathbf{C})$ to get another point $e^B z$ of the symplectic quotient, which will give another Hermitian metric H_B on E. We take the reference metric H_0 to be H_∞.

Proposition 3.2 *If $\|B\|_{op} \leq \frac{1}{10}$ and integer $l \geq 0$, then:*

(1) *There is a constant c such that if*

$$\|B\|_{op} + \|\epsilon\|_{C^l} + m^{-1} \leq cR,$$

then the metric H_B are R-bound.

(2) *There is a constant c' such that*

$$\|\Upsilon_B\|_{op} \leq c' m^{-n}(\|B\|_{op} + \|\epsilon\|_{C^0}).$$

The proof is straightforward. We simply write down explicitly the metric H_B in terms of H_m, which in turn relates to H_∞ explicitly. Because all the consideration is invariant under the action of $SU(N_m)$, we can assume that B is in the diagonal form $B = \mathrm{diag}(\lambda_i)$.
Let $H_B = H_m(1 + \eta_B)$, then by the definition of metric H_B, we have

$$\sum_i e^{2\lambda_i} s_i^* \otimes s_i(1 + \eta_B) + \frac{1}{m}\phi^* \otimes \phi(1 + \eta_B) = id,$$

which is also equivalent to

$$(id + \sum_i (e^{2\lambda_i} - 1)s_i^* \otimes s_i)(1 + \eta_B) = id.$$

The above explicit expression plus the inequality in Lemma 3.1 prove (1) in the Proposition 3.2.

For the second one, notice the L^2 inner product of $e^{\lambda_i}s_i$ is now given by

$$(e^{\lambda_i}s_i, e^{\lambda_j}s_j)_{H_B} = \int_M \langle (1+\eta_B)e^{\lambda_i}s_i, e^{\lambda_j}s_j\rangle \, d\mu,$$

the above metric is in H_m. (2) is proved by using the fact

$$(s_i, s_j) = \frac{N_m}{rV - \frac{\tau}{m}},$$

and the explicit formula for η_B, and the following elementary lemma in [10]

Lemma 3.5 *Let $V \to Y$ be a Hermitian vector bundle over a measurable space Y and let $s_i, i = 1, \cdots, N$ be an orthonormal set of sections of V with respect to the usual L^2 inner product. If F is a bounded function on Y and we define a matrix Υ by*

$$\Upsilon_{ij} = \int_Y F \langle s_i, s_j \rangle \, d\mu,$$

then $||\Upsilon||_{op} \leq ||F||_{L^\infty}$

We now continue the proof of Theorem 3.3. Let l be a fixed integer-we want to show that there is a sequence of balanced metrics converging in C^l. Fix an number R. Let H_∞ can be approximated to order q, then we can choose metric H'_m with error term $\epsilon_m = O(m^{-q-2})$. We choose m large enough that $||\epsilon_l||_{C^l} + m^{-1} \leq cR/2$, where c is the constant in Proposition 3.2. Then Proposition 3.2 tells us that if $||B||_{op} \leq \min(cR/2, 1/10)$, then the metric H_m is R-bounded. Under this assumption we can apply Theorem 3.1, which tells us that $\Lambda_{op,B} \leq \lambda$ where we take $\lambda = C^2 m^{n+1+2\varepsilon}$ for the constant C in Theorem 3.1. Now we apply Proposition 1.4. We take the constant δ of Proposition 1.4 to be $\min(cR/2, 1/10)$. We know that $||B||_{op} \leq \delta$ implies $\Lambda_{e^{B}z} \leq \lambda$. Now the other ingredient entering into Proposition 1.4, which in our context is $||\Upsilon||$. Applying

$$||\Upsilon|| \leq \sqrt{N_m}||\Upsilon||_{op}$$

which is less than $C'm^{-n/2-q-2}$. Thus we can apply Proposition 1.4 so long as

$$\lambda C'm^{-n/2-q-2} = C^2C'm^{n/2+2\varepsilon-q-1} \leq \delta.$$

Take $q > n/2+2\varepsilon+1$, this holds for large enough m and we obtain a solution to our problem with

$$\|B\|_{op} \leq C^2C'm^{n/2+2\varepsilon-q-1}.$$

This inequality then show the metric H_B which corresponds to the solution differs from H_∞ in C^l norm by $O(m^{l+n/2+2\varepsilon-q-1})$. Thus for $q > n/2+2\varepsilon+l+1$, we can derive the existence of balanced metrics and convergence of the balanced metrics to special metrics in C^1 norm. The τ-property is also clear as τ is preserved during approximation procedure as in Theorem 3.2.

Chapter 4

Asymptotic Expansion of Bergman Kernel for Bundles

In this chapter, we derive the lower three terms of Bergman kernel for bundles. One motivation is to know explicitly each term in the second equation 3.9 coming up during the approximation procedure. In section 4.1, we discuss the background for the problem and state the result. In section 4.2, we recall the important gadget, peak section, as developed by Tian ([23]). In section 4.3, we do the expansion to the third term based on the work of Lu ([15]).

4.1 Introduction

Assume we have a Kähler manifold (M, g, ω), g is a Kähler metric, ω is a Kähler form. Suppose M has a polarization, that is a Hermitian line bundle (L, h^L) with $\sqrt{-1}R^L(h) = \omega$, where $R^L(h)$ is the curvature of L determined by the holomorphic structure on L and the metric h. Let (E, H) is a Hermitian holomorphic bundle of rank $r = r(E)$.

The metrics induce metrics on the bundles $E_m = E \otimes L^m$, for each integer m, correspondingly the L^2 metrics on the space of sections $\Gamma(E_m)$. We choose m large enough, so that the map ι_m define in 2.5 is an embedding map. N_m denote the dimension of space of holomorphic sections $\Gamma_{\text{hol}}(E_m)$.

The m-th Bergman Kernel $B_m(H) \in \text{End}(E)$ associated to these data is

$$B_m(H) = \sum_{i=1}^{N_m} S_i^* \otimes S_i, \tag{4.1}$$

where $\{S_i\}$ is an orthonormal basis for $\Gamma_{\text{hol}}(E_m)$ w.r.t. the L^2-metric. Clearly, $B_m(H)$ is well-defined. It's the kernel function for the L^2 orthonormal projection from L^2 section space $L^2(E_m)$ to holomorphic section space $\Gamma_{\text{hol}}(E_m)$.

Notice that the Bergman kernel $B_m(H)$ does not really depend on the metric h on the polarized bundle L, even thought in the definition we use this metric h. The reason is that the metric h is determined by the Kähler metric ω up to a constant factor.

Our purpose is to derive the lower order terms in the expansion of $B_m(H)$ for large m. We write these terms in terms of various curvatures of M, E and their covariant derivatives.

In case the bundle E is a trivial line bundle, the problem has been extensively studied. It's initiated by Tian [23]. Where Tian developed the idea of peak global sections as well as fundamental properties of peak sections. Tian also got the first term in the expansion of Bergman kernel and proved convergence in C^2 norm. Ruan [22] improved the result by introducing K-coordinates. Ruan showed the general convergence result of expansion of Bergman kernel. Zelditch [30] and Catlin [6] reproved the result based on the general theory of Szegö kernel. Catlin [6] actually proved the existence of expansion for general bundle. We'll use his result in this paper.

The explicit evaluation of coefficients in the expansion is hard work in general. Lu in [15] got the first four terms in case of E is trivial line bundle. X.W.Wang ([28]) got the first two terms for general bundle E. We derive the third term in this chapter. Our one motivation is to know explicitly each term in the second equation 3.9 coming up from the approximation procedure.

Recall the result of Catlin([6]) about the existence of expansion for general bundle E.

Theorem 4.1 *For a fixed metric H, there is an asymptotic expansion as $m \to \infty$*

$$(2\pi)^n B_m(H) = A_0(H)m^n + A_1(H)m^{n-1} + \cdots ,$$

where $A_i(H) \in \mathrm{End}(E)$ are determined by the geometry of ω and H. Precisely, for any integer $l, R \geq 0$

$$\|(2\pi)^n B_m(H) - \sum_{j<R} A_j(H)m^{n-j}\|_{C^l} \leq C_{l,R,H}m^{n-R},$$

where $C_{l,R,H}$ depends on l, R, ω and H .

Note in [10], Donaldson remarked that the constants $C_{l,R,H}$ can be chosen to be uniform if we vary the metric H in a bounded set of metrics in the $C^{l'}$ norm. Here l' depends on l and R. In case $R = 2$ and $l = 0$, we denote the l' by l_0. This integer is important for the derivation of special metrics, see Section 2.3. It is interesting to know this number explicitly.

By the result of Catlin, we see right away

Corollary 4.1 **(1)** *The integration of $\mathrm{Tr}A_j(H)$ is of topological nature. In fact*

$$\int_M \mathrm{Tr}A_j(H)\frac{\omega^n}{n!} = ((2\pi)^n a_j$$

where a_j's are $(n-j)$-th coefficient of m in N_m.

(2) *Each A_j is self-adjoint with respect to the fixed metric H.*

The (2) is simply because that Bergman kernel $B_m(H)$'s are self-adjoint.

Our expansion result is

Theorem 4.2 **(0)** *$A_0(H) = Id$;*

(1) *$A_1(H) = \sqrt{-1}\Lambda F_H + \frac{1}{2}\kappa Id$;*

(2)
$$A_2(H) = \tfrac{1}{3}\Delta\kappa Id + \tfrac{1}{24}(|R|^2 - 4|Ric|^2 + 3\kappa^2)Id + \tfrac{1}{2}(\Delta'' Ric^E + \kappa Ric^E + Ric^E Ric^E - R^E R^E - \langle R^E, Ric\rangle).$$

Here R, Ric and κ represent the curvature tensor, the Ricci curvature and the scalar curvature of g, and Δ represents the Laplace operator of M, $Ric^E = \sqrt{-1}\Lambda F_H$ and F_H represent the curvature of (E, h^E) and $\Delta'' = \sqrt{-1}\,\Lambda\partial\bar{\partial}$.

4.2 Constructing peak section

The main gadget to derive the coefficients in the Bergman kernel expansion is the peak global section introduced by Tian ([23]). It provides a way to construct global holomorphic sections basically concentrating on a prescribed point.

Fix a point x_0 on M. To simplify the computation, we choose K-coordinates introduced by Ruan in [22]. Precisely we choose local coordinates $(z_1, \cdots, z_n)$ centered at the given point x_0 such that the Kähler metric $g_{i\bar{j}}$ satisfying

$$g_{i\bar{j}}(x_0) = \delta_{ij},$$

$$\frac{\partial^{p_1+\cdots+p_n} g_{i\bar{j}}}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(x_0) = 0$$

for any nonzero n-tuple $P = (p_1, \cdots, p_n) \in \mathbf{Z}_+^n$.
and we choose a local holomorphic frame e^L of L centered at x_0 so that the local representative function a of the Hermitian metric h^L, i.e., $a = h^L(e^L, e^L)$ satisfying

$$a(x_0) = 1,$$

$$\frac{\partial^{p_1+\cdots+p_n} a}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(x_0) = 0$$

for any nonzero n-tuple $P = (p_1, \cdots, p_n) \in \mathbf{Z}_+^n$.
and we choose a local holomorphic frame $\{e_1^E, \cdots, e_r^E\}$ of E centered at x_0, so that the local representative function $b_{\alpha\bar{\beta}}$ of the Hermitian metric h^E, i.e., $b_{\alpha\bar{\beta}} = h^E(e_\alpha^E, e_\beta^E)$ satisfying

$$b_{\alpha\bar{\beta}}(x_0) = \delta_{\alpha\beta},$$

$$\frac{\partial^{p_1+\cdots+p_n} b_{\alpha\bar{\beta}}}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(x_0)=0$$

for any nonzero n-tuple $P=(p_1,\cdots,p_n)\in \mathbf{Z}_+^n$.

We recall the result on the peak global section is

Theorem 4.3 *([23]) For any n-tuple $P=(p_1,\cdots,p_n)\in\mathbf{Z}_+^n$, and an integer $p'>p_1+\cdots+p_n$, there is an integer m_0, such that for any $m>m_0$, there is a global holomorphic section $S_{P,\alpha,m}^{p'}$ of E_m so that*

$$\|S_{P,\alpha,m}^{p'}\|=1$$

$$\int_{M\setminus\{|z|<\frac{\log m}{\sqrt{m}}\}}|S_{P,\alpha,m}^{p'}|^2\,dV_g=O(\frac{1}{m^{2p'}})$$

and $S_{P,\alpha,m}^{p'}$ can be decomposed as

$$S_{P,\alpha,m}^{p'}=\tilde{S}_{P,\alpha,m}^{p'}+u_{P,\alpha,m}^{p'},$$

so that

$$\tilde{S}_{P,\alpha,m}^{p'}=\begin{cases}\lambda_{P,\alpha}z_1^{p_1}\cdots z_n^{p_n}e_\alpha^E\otimes(e^L)^{\otimes m}(1+O(\frac{1}{m^{2p'}})) & |z|\leq\frac{\log m}{\sqrt{m}}\\ 0 & \text{otherwise}\end{cases}$$

$$u_{P,\alpha,m}^{p'}(z)=O(|z|^{2p'})$$

and

$$\|u_{P,\alpha,m}^{p'}\|^2=O(\frac{1}{m^{2p'}})$$

and

$$\lambda_{P,\alpha}^{-2}=\int_{|z|<\frac{\log m}{\sqrt{m}}}|z_1^{p_1}\cdots z_n^{p_n}|^2a^m b_{\alpha\bar{\alpha}}\,dV_g.$$

where $dV_g=\frac{1}{(2\pi)^n}d\mu$ is the re-scaled metric for convenience.

Note the original statement is for bundle E trivial. The extension to general bundle E is exactly the same, see X.W.Wang [28]

The basic idea is that we first construct a local holomorphic section of E_m, say S, with the prescribed local form. Then we extend this section to a global smooth section using cut-off function. We still denote it by S. This section is of course not holomorphic, to get holomorphic section, the idea is to solve the equation

$$\overline{\partial}T = \overline{\partial}S. \tag{4.2}$$

If we can solve the equation, then the difference $S-T$ is holomorphic. That's not enough, we also want section $S-T$ has prescribed local form. So we want to choose solution T so that it won't perturb the local form too much. This requires a delicate study for the equation 4.2 using the positiveness of line bundle L and taking care of the integer m. These are all done in Tian [23].

We also recall the following lemma in Lu ([15]) which is useful in the following computation.

Lemma 4.1 *Let A be a function on $\{1, \cdots, n\}^p \times \{1, \cdots, n\}^p$. Then for any $p' \geq 0$,*

$$\sum_{I,J} \int_{|z|<\frac{\log m}{\sqrt{m}}} A_{I,\overline{J}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0$$
$$= \Big(\frac{1}{p!} \sum_I \sum_{\sigma \in \Sigma_n} A_{I,\overline{\sigma(I)}}\Big) \frac{p!(n+p+q-1)!}{(p+n-1)! m^{n+p+q}} + O\Big(\frac{1}{m^{p'}}\Big)$$

where $dV_0 = (\frac{\sqrt{-1}}{2\pi})^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$ is the normalized volume form for $\mathbf{C}^n$. Note the factor $(2\pi)^n$.

4.3 Calculation of the first three terms

4.3.1 Proof of the result

We start with the following simple and useful observation(cf. [15]).

Lemma 4.2 *Let $S_1, S_2, \cdots, S_{N_m}$ be any basis of $\Gamma_{\mathrm{hol}}(E_m)$ so that at a given point x_0 on M, $S_1(x_0), S_2(x_0), \cdots, S_r(x_0)$ form a basis of the fiber E_m at x_0, while $S_k(x_0) = 0$ for $k > r$. If* *we set* $F_{i\bar{j}} = (S_i, S_j), i, j = 1, \cdots, d_m$, *then obviously* $(F_{i\bar{j}})$ *is a positive definite matrix, hence it can be written as the product of a matrix* G *with it's complex transposition* $\bar{G}^t$, *explicitly,*

$$F_{i\bar{j}} = \sum_{k=1}^{d_m} G_{i\bar{k}} G_{k\bar{j}}$$

Let $(H_{i\bar{j}})$ *be the inverse matrix of* $(G_{i\bar{j}})$, $(I_{i\bar{j}})$ *be the inverse matrix of* $(F_{i\bar{j}})$,

Then $\{\sum_{j=1}^{d_m} H_{i\bar{j}} S_j\}$ *form an orthonormal basis of* $\Gamma_{\mathrm{hol}}(E_m)$ *and at the point* x_0, *Bergman kernel is of the form*

$$B_m(H)(x_0) = \sum_{\alpha\beta=1}^{r} I_{\alpha\bar{\beta}} S_\alpha^*(x_0) \otimes S_\beta(x_0).$$

In fact

$$\begin{aligned}
(H_{i\bar{k}} S_k, H_{j\bar{l}} S_l) &= H_{i\bar{k}} \overline{H_{j\bar{l}}} F_{k\bar{l}} \\
&= H_{i\bar{k}} \overline{H_{j\bar{l}}} G_{k\bar{p}} G_{p\bar{l}} \\
&= \delta_{i\bar{p}} \delta_{p\bar{j}} \\
&= \delta_{i\bar{j}}.
\end{aligned}$$

Our plan to compute the lower term of the Bergman kernel expansion is then, we choose a special set of basis $\{S_i\}$ satisfying the assumption of above lemma, then to compute the lower term of the expansion of the terms on the right hand side.

First we take $P = (0, \cdots, 0)$ in Theorem 4.3, and get r sections, which for simplicity, we'll denote as S_α, $\alpha = 1, \cdots, r$. These sections play crucial roles in the calculation of lower order terms in the asymptotic expansion.

Next we complete the sections $\{S_\alpha, \quad \alpha = 1, \cdots, r\}$ to a basis $\{S_\alpha, T_i, \quad \alpha = 1, \cdots, r, \quad i = 1, \cdots, d_m - r\}$ of $H^0(M, E \otimes L^{\otimes m})$, such that

$$\{T_i, \quad i = 1, \cdots, d_m - r\} \quad \text{are orthonormal and vanishing at } x_0$$

and

$$(S_\alpha, T_i) = 0, \quad \text{whenever} \qquad \alpha = 1, \cdots, r, \quad i > r$$

We can choose such basis T_i simply for dimension reason. If $\mathcal{V}$ is the linear space of sections vanishing at x_0, $\mathcal{W}$ is the orthogonal space to the span of S_α, then clearly

$$\dim \mathcal{V} = \dim \mathcal{W} = N_m - r$$

Hence the dimension of their intersection

$$\dim(\mathcal{V} \cap \mathcal{W}) = \dim \mathcal{V} + \dim \mathcal{W} - \dim(\mathcal{V} \cup \mathcal{W})$$

is no less than then $N_m - r + N_m - r - N_m = N_m - 2r$.

With respect to this basis, the matrix $(F_{i\bar{j}})$ has the form

$$(F_{i,\bar{j}}) = \begin{pmatrix} (S_\alpha, S_\beta) & (S_\alpha, T_i) & 0 \\ (T_i, S_\alpha) & I_r & 0 \\ 0 & 0 & I_{d_m - 2r} \end{pmatrix}$$

If we set $M = (\,(S_\alpha, S_\beta)\,)$ and $N = (\,(S_\alpha, T_i)\,)$, then it's easy to see the first block of the inverse matrix of F is

$$I = (M - N\bar{N}^t)^{-1}. \tag{4.3}$$

Clearly, the basis $\{S_\alpha, T_i\}$ satisfying the conditions in the lemma 4.2, hence by the local form of S_α in Theorem 4.3, our main task is now to compute the lower terms for

$$\sum_{\alpha\beta=1}^{r} I_{\alpha\bar{\beta}} S_\alpha^*(x_0) \otimes S_\beta(x_0) = \sum_{\alpha\beta=1}^{r} \lambda_\alpha \lambda_\beta I_{\alpha\bar{\beta}} (e_\alpha^E)^*(x_0) \otimes e_\beta^E(x_0) \tag{4.4}$$

We can further simplify the computation by recalling the following Lemma in Lu [15] (cf. [28]).

Lemma 4.3 *For any holomorphic section T of E_m that vanishing at x_0, we have*

$$(S_\alpha, T) = O(\frac{1}{m^{\frac{3}{2}}})\|T\|$$

Thus we have the matrix N is $O(\frac{1}{m^{\frac{3}{2}}})$ and I is essentially M^{-1} modulo higher order terms. So all we have to do is to compute the lower three terms in λ_α's and (S_α, S_β)'s. In the following we first state the result for these terms and use them to give the expansion for Bergman kernel. Then give the computation for λ_α's and (S_α, S_β)'s separately in next section.

We fix the notation first. For the tangent bundle TM, Let R be the curvature determined by the metric g. Locally, we write out R

$$R\frac{\partial}{\partial z_i} = R_{i\bar{j}}\frac{\partial}{\partial z_j} = R_{i\bar{j}k\bar{l}}dz_k \wedge d\bar{z}_l \otimes \frac{\partial}{\partial z_j}$$

and the Ricci curvature Ric

$$Ric_{i\bar{j}} = R_{i\bar{j}k\bar{l}}g^{k\bar{l}}$$

and the scalar curvature κ

$$\kappa = Ric_{i\bar{j}}g^{i\bar{j}}.$$

For the Hermitian bundle E, let R^E be curvature determined by the metric H. Locally we write out R^E

$$R^E e^E_\alpha = R^E_{\alpha\bar{\beta}}e^E_\beta = R^E_{\alpha\bar{\beta}i\bar{j}}dz_i \wedge d\bar{z}_j \otimes e^E_\beta$$

and $Ric^E = \sqrt{-1}\Lambda R^E$

$$Ric^E_{\alpha\bar{\beta}} = R^E_{\alpha\bar{\beta}i\bar{j}}g^{i\bar{j}}$$

Proposition 4.1 *For the above chosen basis $\{S_\alpha, T_i\}$, we have*

(1)

$$\lambda^2_\alpha = m^n(1 + \frac{A_\alpha}{m} + \frac{B_\alpha}{m^2} + O(\frac{1}{m^3}))$$

where

$$
\begin{aligned}
A_\alpha =& \frac{1}{2}\kappa + Ric^E_{\alpha\bar\alpha}, \\
B_\alpha =& \frac{1}{3}\Delta\kappa + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\kappa^2) + \\
& \frac{1}{2}(\Delta'' Ric^E_{\alpha\bar\alpha} - Ric^E_{\alpha\bar\beta}Ric^E_{\beta\bar\alpha} - |R^E_{\alpha\bar\beta}|^2 - \langle R^E_{\alpha\bar\alpha}, Ric\rangle + \kappa Ric^E_{\alpha\bar\alpha} + 2Ric^E_{\alpha\bar\alpha}Ric^E_{\alpha\bar\alpha})
\end{aligned}
$$

summing over β

(2) *For* $\alpha \neq \beta$,

$$
(S_\alpha, S_\beta) = \frac{C_{\alpha\bar\beta}}{m} + \frac{D_{\alpha\bar\beta}}{m^2} + O(\frac{1}{m^2}),
$$

where

$$
\begin{aligned}
C_{\alpha\bar\beta} =& - Ric^E_{\alpha\bar\beta}, \\
D_{\alpha\bar\beta} =& - \frac{1}{2}Ric^E_{\alpha\bar\beta}(Ric^E_{\alpha\bar\alpha} + Ric^E_{\beta\bar\beta}) - \\
& \frac{1}{2}(\Delta'' Ric^E_{\alpha\bar\beta} - Ric^E_{\alpha\bar\gamma}Ric^E_{\gamma\bar\beta} - \langle R^E_{\alpha\bar\gamma}, R^E_{\beta\bar\gamma}\rangle - \langle R^E_{\alpha\bar\beta}, Ric\rangle)
\end{aligned}
$$

summing over γ.

We now calculate the first three terms in the expansion for Bergman kernel.

Proof of the main result

By the identity 4.4, we only need to compute the coefficients of λ_α's and $I_{\alpha\overline{\beta}}$'s. The λ_α's are clear by the Proposition 4.1. For $I_{\alpha\overline{\beta}}$'s, using the identity 4.3 and $N = O(\frac{1}{m^{\frac{2}{3}}})$, we get

$$
I = M^{-1} + O(\frac{1}{m^3}).
$$

If we set $C_{\alpha\bar\alpha} = D_{\alpha\bar\alpha} = 0$, using the Proposition4.1, and

$$
(I + A)^{-1} = 1 - A + A^2 + \cdots
$$

for small matrix A. We have

$$I_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}} - \frac{C_{\alpha\bar{\beta}}}{m} - \frac{D_{\alpha\bar{\beta}}}{m^2} + \frac{C_{\alpha\bar{\gamma}}C_{\gamma\bar{\beta}}}{m^2} + O(\frac{1}{m^3}).$$

For $A_0(H)$, it's simply

$$m^{-n} m^{\frac{n}{2}} m^{\frac{n}{2}} \delta_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}.$$

For $A_1(H)$, using $(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \cdots$,we get

$$(\frac{A_\alpha}{2} + \frac{A_\beta}{2})\delta_{\alpha\beta} - C_{\alpha\bar{\beta}}$$

which is $\frac{\kappa}{2} + Ric^E$.

For $A_3(H)$, using $(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$, we get

$$(\frac{A_\alpha}{2} + \frac{A_\beta}{2})C_{\alpha\bar{\beta}} + C_{\alpha\bar{\gamma}}C_{\gamma\bar{\beta}} - D_{\alpha\bar{\beta}}$$

for $A_3(H)_{\alpha\bar{\beta}}$ with $\alpha \neq \beta$ and

$$B_\alpha + C_{\alpha\bar{\gamma}}C_{\gamma\bar{\alpha}}$$

for $A_3(H)_{\alpha\bar{\alpha}}$.

Then you put the exact expressions in, you get the result.

4.3.2 Expansion for λ_α's

We do the expansion for λ_α in this section. The main gadget is Lemma 4.1.

Recall the λ_α is

$$\lambda_\alpha^{-2} = \int_{|z|\leq\frac{\log m}{\sqrt{m}}} b_{\alpha\bar{\alpha}} a^m \det(g_{i\bar{j}})\, dV_0$$

To apply the Lemma 4.1, we expand each term in the above integrand to certain order. This is the same strategy in Lu [15].

Recall in the K-coordinates ([15]), around x_0, we have following Taylor expansions:

$$\log a = -|z|^2 + a_4 + a_5 + a_6 + O(|z|^7)$$
$$\log\det(g_{i\bar{j}}) = g_2 + g_3 + g_4 + O(|z|^5)$$
$$b_{\alpha\bar{\beta}} = \delta_{\alpha\beta} + b_{\alpha\bar{\beta}2} + b_{\alpha\bar{\beta}3} + b_{\alpha\bar{\beta}4} + O(|z|^5)$$

where a's are

$$\begin{cases} a_4 = \frac{1}{4}R_{i\bar{j}k\bar{l}}z_i\bar{z}_jz_k\bar{z}_l \\ a_5 = \frac{1}{12}(R_{i\bar{j}k\bar{l},p}z_i\bar{z}_jz_k\bar{z}_lz_p + R_{i\bar{j}k\bar{l},\bar{p}}z_i\bar{z}_jz_k\bar{z}_l\bar{z}_p) \\ \tilde{a}_6 = \frac{1}{36}(R_{i\bar{j}k\bar{l},p\bar{q}} - R_{i\bar{s}k\bar{q}}R_{s\bar{j}p\bar{l}} - R_{i\bar{j}s\bar{l}}R_{k\bar{s}p\bar{q}} - R_{i\bar{s}p\bar{q}}R_{s\bar{j}k\bar{l}})z_i\bar{z}_jz_k\bar{z}_lz_p\bar{z}_q \end{cases}$$

where $\tilde{a}_6$ is the (3,3) part of a_6 and $R_{i\bar{j}k\bar{l},p}$, $R_{i\bar{j}k\bar{l},\bar{p}}$ and $R_{i\bar{j}k\bar{l},p\bar{q}}$ are the coefficients of the covariant derivatives.

And g's are

$$\begin{cases} g_2 = -Ric_{i\bar{j}}z_i\bar{z}_j \\ g_3 = -\frac{1}{2}R_{i\bar{j},k}z_i\bar{z}_jz_k - \frac{1}{2}R_{i\bar{j},\bar{k}}z_i\bar{z}_j\bar{z}_k \\ \tilde{g}_4 = \frac{1}{4}(-R_{i\bar{j},k\bar{l}} + R_{i\bar{s}k\bar{l}}R_{s\bar{j}})z_i\bar{z}_jz_k\bar{z}_l \end{cases}$$

where $\tilde{g}_2$ is the (2,2) part of g_2.

And b's are

$$\begin{cases} b_{\alpha\bar{\beta}2} = -R^E_{\alpha\bar{\beta}i\bar{j}}z_i\bar{z}_j \\ b_{\alpha\bar{\beta}3} = -\frac{1}{2}R^E_{\alpha\bar{\beta}i\bar{j},k}z_i\bar{z}_jz_k - \frac{1}{2}R^E_{\alpha\bar{\beta}i\bar{j},\bar{k}}z_i\bar{z}_j\bar{z}_k \\ \tilde{b}_{\alpha\bar{\beta}4} = -\frac{1}{4}(R^E_{\alpha\bar{\beta}i\bar{j},k\bar{l}} - R^E_{\alpha\bar{\gamma}i\bar{l}}R^E_{\gamma\bar{\beta}k\bar{j}} - R^E_{\alpha\bar{\gamma}k\bar{l}}R^E_{\gamma\bar{\beta}i\bar{j}} - R^E_{\alpha\bar{\beta}p\bar{j}}R_{i\bar{p}k\bar{l}})z_i\bar{z}_jz_k\bar{z}_l \end{cases}$$

where $\tilde{b}_{\alpha\bar{\beta}4}$'s are (2,2) part of $b_{\alpha\bar{\beta}4}$.

Also if $b_{\alpha\bar{\beta}i\bar{j}k\bar{l}}$ denotes the coefficient of $z_i\bar{z}_j z_k\bar{z}_l$ in $b_{\alpha\bar{\beta}}$, then

$$\sum_{ij} b_{\alpha\bar{\beta}i\bar{i}j\bar{j}} = \langle R^E_{\alpha\bar{\gamma}}, R^E_{\beta\bar{\gamma}}\rangle + Ric^E_{\alpha\bar{\gamma}} Ric^E_{\gamma\bar{\beta}} + \langle R^E_{\alpha\bar{\beta}}, Ric\rangle - \Delta'' Ric^E_{\alpha\bar{\beta}} \tag{4.5}$$

The computation for these terms are straightforward. We simply use the local expression for the curvatures in terms of metrics.

Now for each function ϕ, we define

$$K(\phi) = \int_{|z|\le\frac{\log m}{\sqrt{m}}} \phi e^{-m|z|^2}\, dV_0.$$

In terms of function K, λ_α's can be rewritten as

$$\begin{aligned}
\lambda_\alpha^{-2} &= \int_{|z|\le\frac{\log m}{\sqrt{m}}} b_{\alpha\bar{\alpha}} a^m\, dV_g \\
&= \int_{|z|\le\frac{\log m}{\sqrt{m}}} b_{\alpha\bar{\alpha}} e^{m\log a} \det(g_{i\bar{j}})\, dV_0 \\
&= \int_{|z|\le\frac{\log m}{\sqrt{m}}} b_{\alpha\bar{\alpha}} e^{m(\log a+|z|^2)} e^{\log\det(g_{i\bar{j}})} e^{-m|z|^2}\, dV_0 \\
&= K(b_{\alpha\bar{\alpha}} e^{m(\log a+|z|^2)} e^{\log\det(g_{i\bar{j}})})
\end{aligned}$$

Using the Taylor expansions for a's, b's and g's and Lemma 4.1, we have

$$\begin{aligned}
\lambda_\alpha^{-2} =& K((1 + b_{\alpha\bar{\alpha}2} + b_{\alpha\bar{\alpha}3} + b_{\alpha\bar{\alpha}4}) \\
& (1 + m(a_4 + a_5 + a_6) + \frac{1}{2}m^2 a_2^2) \\
& (1 + g_2 + g_3 + g_4 + \frac{1}{2}g_2^2)) + O(\frac{1}{m^{n+3}}) \\
=& K(1) + K(g_2) + K(b_{\alpha\bar{\alpha}2}) + mK(a_4) + \\
& K(g_4) + K(b_{\alpha\bar{\alpha}2}g_2) + K(b_{\alpha\bar{\alpha}4}) + \frac{1}{2}K(g_2^2) + mK(a_6) + \\
& mK(g_2a_4) + mK(b_{\alpha\bar{\alpha}2}a_4) + \frac{1}{2}m^2K(a_2^2) + O(\frac{1}{m^{n+3}})
\end{aligned}$$

We collect the expansion for the these K's in the following lemma

Lemma 4.4

$$
\begin{aligned}
K(1) &= \frac{1}{m^n} + O(\frac{1}{m^{n+3}}) \\
K(g_2) &= -\kappa\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}}) \\
K(b_{\alpha\bar{\alpha}2}) &= -Ric^E_{\alpha\bar{\alpha}}\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}}) \\
mK(a_4) &= \frac{1}{2}\kappa\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}}) \\
K(g_4) &= -\frac{1}{2}(\Delta\kappa - |Ric|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
K(b_{\alpha\bar{\alpha}2}g_2) &= (\kappa Ric^E_{\alpha\bar{\alpha}} + \langle R^E_{\alpha\bar{\alpha}}, Ric\rangle)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
K(b_{\alpha\bar{\alpha}4}) &= -\frac{1}{2}(\Delta'' Ric^E_{\alpha\bar{\alpha}} - Ric^E_{\alpha\bar{\beta}}Ric^E_{\beta\bar{\alpha}} - \langle R^E_{\alpha\bar{\alpha}}, Ric\rangle - |R^E_{\alpha\bar{\beta}}|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
\frac{1}{2}K(g_2^2) &= \frac{1}{2}(\kappa^2 + |Ric|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
mK(a_6) &= \frac{1}{6}(\Delta\kappa - 2|Ric|^2 - |R|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
mK(g_2a_4) &= -\frac{1}{2}(\kappa^2 + 2|Ric|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
mK(b_{\alpha\bar{\alpha}2}a_4) &= -\frac{1}{2}(\kappa Ric^E_{\alpha\bar{\alpha}} + 2\langle R^E_{\alpha\bar{\alpha}}, Ric\rangle)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
\frac{1}{2}m^2K(a_2^2) &= \frac{1}{8}(\kappa^2 + 4|Ric|^2 + |R|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})
\end{aligned}
$$

Here $\Delta'' Ric^E := \sqrt{-1}\Lambda\partial\bar{\partial}Ric^E$

The derivation is straightforward using Lemma 4.1. We omit the proof here. However, in the next section, we write all the analogous K's out. The K's here can be done analogously.

The expansion for λ_α's is then clear.

4.3.3 Expansion for (S_α, S_β)'s

The expansion for (S_α, S_β)'s uses the same strategy as for λ_α's.

Recall, using the local forms for S_α's in Theorem 4.3

$$
\begin{aligned}
(S_\alpha, S_\beta) &= \int_M \langle S_\alpha, S_\beta\rangle \, dV_g \\
&= \lambda_\alpha \lambda_\beta \int_{|z|\le \frac{\log m}{\sqrt{m}}} b_{\alpha\bar\beta} a^m \, dV_g + O(\frac{1}{m^{2p'}})
\end{aligned}
$$

Rewrite it in terms of function K

$$
\begin{aligned}
\int_{|z|\le \frac{\log m}{\sqrt{m}}} b_{\alpha\bar\beta} a^m \, dV_g &= \int_{|z|\le \frac{\log m}{\sqrt{m}}} b_{\alpha\bar\beta} e^{m\log a} \det(g_{i\bar j}) \, dV_0 \\
&= \int_{|z|\le \frac{\log m}{\sqrt{m}}} b_{\alpha\bar\beta} e^{m(\log a + |z|^2)} e^{\log\det(g_{i\bar j})} e^{-m|z|^2} \, dV_0 \\
&= K(b_{\alpha\bar\beta} e^{m(\log a+|z|^2)} e^{\log\det(g_{i\bar j})})
\end{aligned}
$$

Recall in case $\alpha \neq \beta$, we have following expansion for b's

$$
b_{\alpha\bar\beta} = b_{\alpha\bar\beta 2} + b_{\alpha\bar\beta 3} + b_{\alpha\bar\beta 4} + O(|z|^5)
$$

Hence using the Lemma 4.1, we have

$$
\begin{aligned}
\int_{|z|\le \frac{\log m}{\sqrt{m}}} b_{\alpha\bar\beta} a^m \, dV_g \;&=\; K((b_{\alpha\bar\beta 2} + b_{\alpha\bar\beta 3} + b_{\alpha\bar\beta 4}) \\
&\quad (1 + m(a_4 + a_5 + a_6) + \frac{1}{2} m^2 a_2^2) \\
&\quad (1 + g_2 + g_3 + g_4 + \frac{1}{2} g_2^2)) + O(\frac{1}{m^{n+3}}) \\
&= K(b_{\alpha\bar\beta 2}) + K(b_{\alpha\bar\beta 4}) + mK(b_{\alpha\bar\beta 2} a_4) + \\
&\quad K(b_{\alpha\bar\beta 2} g_2) + O(\frac{1}{m^{m+3}})
\end{aligned}
$$

We collect the K's in the following lemma

Lemma 4.5

$$
\begin{aligned}
K(b_{\alpha\bar\beta 2}) &= -Ric^E_{\alpha\bar\beta}\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}})\\
mK(b_{\alpha\bar\beta 2}a_4) &= -(\frac{1}{2}\kappa Ric^E_{\alpha\bar\beta} + \langle R^E_{\alpha\bar\beta}, Ric\rangle)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
K(b_{\alpha\bar\beta 2}g_2) &= (\kappa Ric^E_{\alpha\bar\beta} + \langle R^E_{\alpha\bar\beta}, Ric\rangle)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
K(b_{\alpha\bar\beta 4}) &= \frac{1}{2}(\langle R^E_{\alpha\bar\gamma}, R^E_{\beta\bar\gamma}\rangle + Ric^E_{\alpha\bar\gamma}Ric^E_{\gamma\bar\beta} + \langle R^E_{\alpha\bar\beta}, Ric\rangle - \Delta'' Ric^E_{\alpha\bar\beta})\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})
\end{aligned}
$$

The derivation is straightforward using Lemma 4.1. In fact

$$
\begin{aligned}
K(b_{\alpha\bar\beta 2}) &= -\int_{|z|\le\frac{\log m}{\sqrt{m}}} R^E_{\alpha\bar\beta i\bar j} z_i \bar z_j e^{-m|z|^2}\, dV_0\\
&= -Ric^E_{\alpha\bar\beta}\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}})\\
mK(b_{\alpha\bar\beta 2}a_4) &= -\frac{m}{4}\int_{|z|\le\frac{\log m}{\sqrt{m}}} R^E_{\alpha\bar\beta i\bar j} R_{k\bar l p\bar q} z_i \bar z_j z_k \bar z_l z_p \bar z_q e^{-m|z|^2}\, dV_0\\
&= -\frac{1}{4}(R^E_{\alpha\bar\beta i\bar i}(R_{j\bar j k\bar k} + R_{j\bar k k\bar j}) + 2R^E_{\alpha\bar\beta i\bar j}(R_{j\bar i k\bar k} + R_{j\bar k k\bar i}))\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
&= -(\frac{1}{2}\kappa Ric^E_{\alpha\bar\beta} + \langle R^E_{\alpha\bar\beta}, Ric\rangle)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
K(b_{\alpha\bar\beta 2}g_2) &= \int_{|z|\le\frac{\log m}{\sqrt{m}}} R^E_{\alpha\bar\beta i\bar j} Ric_{k\bar l} z_i \bar z_j z_k \bar z_l e^{-m|z|^2}\, dV_0\\
&= (R^E_{\alpha\bar\beta i\bar i} Ric_{j\bar j} + R^E_{\alpha\bar\beta i\bar j} Ric_{j\bar i})\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
&= (\kappa Ric^E_{\alpha\bar\beta} + \langle R^E_{\alpha\bar\beta}, Ric\rangle)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
K(b_{\alpha\bar\beta 4}) &= \frac{1}{4}\int_{|z|\le\frac{\log m}{\sqrt{m}}} b_{\alpha\bar\beta i\bar j k\bar l} z_i \bar z_j z_k \bar z_l e^{-m|z|^2}\, dV_0\\
&= \frac{1}{2} b_{\alpha\bar\beta i\bar i j\bar j}\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})\\
&= \frac{1}{2}(\langle R^E_{\alpha\bar\gamma}, R^E_{\beta\bar\gamma}\rangle + Ric^E_{\alpha\bar\gamma}Ric^E_{\gamma\bar\beta} + \langle R^E_{\alpha\bar\beta}, Ric\rangle - \Delta'' Ric^E_{\alpha\bar\beta})\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}})
\end{aligned}
$$

For the last one, we use the identity 4.5. The expansion for (S_α, S_β)'s is then clear.

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