Tearing-Mode Transport Model in the Reversed Field Pinch Concept

by

Antonio Bruno

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Abstract 

In this thesis, a self-consistent model for analyzing the transport performance of a Reversed Field Pinch (RFP)-type of thermonuclear fusion reactor has been developed. The study has been focused on determining equilibrium configurations which describe a plasma evolution dominated by particular instabilities originated by plasma electrical resistivity (tearing-, or resistive interchange, modes). The ultimate goal is to provide a model of transport scaling in an RFP which can evaluate the global parameters describing the plasma confinement performance. Starting from a magnetic field configuration essentially given by Taylor’s relaxation model, the self-consistent pressure profile is determined by assuming that the ohmic heating source raises the plasma pressure until the profile is locally marginally stable to tearing modes. A critical point here is the long held belief that an RFP, because of its bad curvature, would always be unstable to tearing or resistive interchange modes; that is, no marginally stable state exists. This belief turns out to be untrue. The basis for this statement is a careful ordering of the resistive layer dynamics, showing that thermal conductivity dominates over convection and compressibility. Thus, the use of the adiabatic equation of state in earlier work is not accurate for an RFP. As a result, tearing and interchange modes can indeed be stabilized in an RFP. In this model, a proper, self-consistent definition of tearing-mode marginality has been used as a prescription for building the pressure profile. The actual numerical determination of the marginally stable profiles can be solved by using state-of-the-art personal computers. It is worth emphasizing that there are no free parameters in the model. Point checks indicate reasonable agreement with typical experimental data. Parametric numerical studies are also shown, spanning the operational space of RFP experiments, and finally providing the tearing mode transport scaling relations for the global confinement parameters. Comparisons with experiments as well as other transport models are shown.

Thesis Supervisor: Jeffrey P. Freidberg 
Title: Professor of Nuclear Engineering/Department Head
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A special thank goes to Jim Hastie, for his valuable help and for very instructive discussions on the subject of tearing modes.

Alla mia famiglia, e a Rania, con tanto ammore.
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Chapter 1

Introduction to RFP

1.1 Concept and Main Features of an RFP

The Reversed Field Pinch is a member of the toroidal pinch class of systems in which the plasma is confined by an externally generated magnetic field, following the basic principle for confinement known as the Pinch Effect (see Ref. [4]). According to this confinement scheme, magnetic tension can balance the particle pressure on a time scale shorter than the resistive-diffusion time scale of the magnetic field in the plasma, reaching an equilibrium which can be described by the steady-state one-fluid MHD equations. The poloidal component of the magnetic field \( B_\theta \) is due to a toroidal current induced in the plasma, using the transformer principle in which the plasma column plays the role of the secondary circuit, while the toroidal component \( B_\phi \) is directly generated by external magnets. There are essentially two main features that characterize the RFP (see schematic and comparison with the tokamak configuration in Figs. 1-1, 1-2). One is that the poloidal and toroidal magnetic field components are of the same magnitude, i.e. \( B_\theta \sim B_\phi \); this implies that the safety factor \( q \equiv \frac{rB_\phi}{RB_\theta} \ll 1 \), since \( q \) scales like the inverse aspect ratio of the torus (where \( r \) and \( R \) are respectively its minor and major radius). The second main feature is that the toroidal magnetic field changes its sign (reverses) on the outside of the plasma, with respect to its value in the core; hence the name of the fusion concept. This peculiar magnetic configuration turns out to be self-generated by the plasma, as observed
experimentally, whenever operating in certain parameter ranges. The RFP ordering ($B_\theta \sim B_\phi$) indicates that an accurate MHD stability analysis can be obtained by the one-dimensional treatment of the geometry, that is the straight cylindrical model for an RFP; in other words, toroidal corrections are not crucial to study the global behavior of an RFP, and cylindrical theory is adequate for most cases. Moreover, the MHD ordering for an RFP shows the possibility of obtaining equilibria with stable high-$\beta$ plasmas (as was shown by early ideal MHD stability studies (see Ref. [5])), which makes the RFP concept quite appealing as a future commercial reactor for energy production. Unfortunately, plasma resistivity plays an important role, limiting overall RFP confinement performances to lower values of $\beta$. In fact, it has been shown (Ref. [5]) that axisymmetric 1-D steady state profiles that naturally arise from resistive MHD (without dynamo action) cannot have a toroidal field reversal, for a typical plasma resistivity profile; this causes the safety factor to have a minimum on the outside of a plasma with vanishing current density at the edge (as $j_\theta \sim B'_\perp \sim 0$ and $B_\theta \sim 1/\tau$). This turns out to be a configuration that is very unstable to ideal internal pressure- or current-driven modes (see Ref. [6]). Brian Taylor, in his well-known relaxation theory, gives an explanation of the mechanism by which resistive instabilities make the system relax to a minimum energy state, showing a reversal in the toroidal field and overall improved stability; naturally the loss of initially stored energy in the system is the price to pay in order to gain in stability, thus forcing the RFP to operate at significantly lower $\beta$ conditions than the ideal predictions. Taylor’s relaxed state essentially minimizes the two sources of free energy: the gradients of $j_\parallel/B$ (that drive current-driven kink instabilities), and the pressure gradient (that drives pressure-driven kinks). It has also been shown (see [6]), that an RFP is very unstable to current-driven and pressure-driven external modes, as well as the internal modes, the $m = 1$, low $n$ modes being the most dangerous instabilities. This indicates the need for a “perfectly” conducting wall to surround the RFP discharge in order to fully suppress such modes; in reality, the presence of an electric resistivity in the wall still makes external modes unstable, although with a much reduced growth rate, and the impact of these resistive wall modes on the RFP confinement has recently been
Figure 1-1: Schematic of Typical Magnetic Field Profiles in an RFP

an important topic for theoretical research (see Refs. [7], [8]).

Another very interesting feature of an RFP is the presence of a “motor-dynamo” or “dynamo” effect, analogous to the generation of magnetic field by MHD effects in planetary and stellar cores. Very schematically, the toroidal plasma current driven by the externally applied toroidal loop voltage, generates flows in the plasma such that magnetic flux is spontaneously and dynamically transferred between poloidal and toroidal directions in different plasma locations: poloidal flux is converted into toroidal flux by differential rotation, but contrarily to the classical astrophysical or terrestrial dynamo, where toroidal flux is cyclically reconverted into poloidal flux, in the RFP dynamo, the poloidal flux is continually supplied by the external circuit. In particular, a poloidal current is driven in the outer region of the plasma, generating the reversal in the toroidal field. The dynamo effect then in a sense breaks down axisymmetry, with small scale fluctuations sustaining steady field reversed states against resistive diffusion. The inherent nonlinearity of the RFP dynamo has made the de-
1.2 Past Experiments

The originating idea behind the RFP concept dates back to Bennett’s work in 1934 (Ref. [4]) on the Pinch Effect. The first attempt to magnetically confine plasma using a small toroidal device without toroidal magnetic field was done in 1951 by Cousins and Ware in UK. The main result from their pinch experiments was that the plasma column was highly unstable to MHD instabilities. Research then concentrated on the “stabilized pinch”, in which a toroidal component of the magnetic field was applied, following the theoretical work of Rosenbluth et al. (see for example Ref. [9])
that would predict better stability. Despite reaching some gross stability for the
pinch, energy losses still were very large, and thus the confinement was very poor.
The development of ideal MHD stability theory, with in particular Suydam’s stability
criterion and Newcomb’s analysis in the 1960s helped determining two possible means
of stabilizing axisymmetric pinches: the tokamak approach, with a very large toroidal
field to suppress the \( m = 1 \) kink instability, followed in USSR, and the RFP approach
with its high magnetic shear to stabilize pressure- and current-driven modes. The
first observation of the importance of field reversal on plasma stability was seen in
the ZETA experiment in the UK, showing inconsistency with theoretical predictions
of that time. The RFP program continued by exploring higher sizes and currents,
showing significant confinement improvement even though not as encouraging as the
tokamak program that was running in parallel. In the 1970s, Taylor’s theory of
relaxation was probably the major theoretical breakthrough in understanding RFP
physics (see Refs. [10],[11]); at the same time highly improved computer capabilities
allowed important numerical work to be done (see for example Ref. [5]). From the
experimental point of view, fast programming opened up a new way to operate and
set up the plasma discharge. An excellent paper written by Bodin and Newton in
1980 (Ref. [12]) contains a much more detailed review, as well as a complete list of
references for all the past RFP experiments. In the 1980s, upgraded RFP experiments
continued in Culham-UK, Los Alamos-USA and Padova-Italy. A database began to
be built, collecting data from those experiments. However, the data turned out to
be noisy and quite limited information on RFP in general could be extracted. In
1996, Werley et al. (Refs. [3],[13]) derived an experimental scaling for the energy
confinement time \( \tau_E \) from the past RFP experiments: they found that a special
subset of this database, including only the best confinement shot of each device (one
operating point per machine, that is), was following (in a log-log plot) a scaling law
that was found by Taylor and Connor from dimensional analysis (see Refs. [14],[15])

\[
\tau_E \sim \frac{I^3 \cdot a^2}{N^{3/2}}
\]  

(1.1)
where \( I \) is the total toroidal plasma current, \( a \) is the plasma minor radius, and \( N \) is the area integrated plasma density defined as follows

\[
N = 2\pi \int_0^a nr \, dr
\]  

(1.2)

This scaling law shows a strong dependency of the energy confinement time on the plasma current \( I \), which indicates significantly improved confinement for high currents. However, this scaling law was not found to agree with confinement performance of individual machines as its parameters were varied. Still the scaling of the best shots was so appealing that it was decided to build larger devices, which represent the current status of RFP research. One last characterization found in RFP plasmas worth mentioning is an approximate empirical limit involving the parameter \( \frac{I}{N} \), the ratio of plasma current to line density. Even though this parameter is not tied to any obvious physical phenomenon, it can be thought as analogous to the tokamak Greenwald parameter \( \frac{na^2}{I} \), thus giving an RFP version of the tokamak density limit (see Ref. [16])

\[
\frac{N}{I} < \left( \frac{N}{I} \right)_{\text{crit}}
\]  

(1.3)

Experimental observations in the RFP devices have indicated an operational window in terms of \( \frac{I}{N} \), whose boundaries describe both a high density limit, which can be interpreted in terms of a balance between ohmic heating and impurity radiation losses, and a low density limit probably related to an excessive amount of runaway electrons (high streaming parameter). In general, although many pinches can easily operate at low density, it is clear that when operating close to the high density limit, lower impurities, higher \( \beta \) and longer confinement times are obtained. For this reason, RFP's operate at an optimized value of \( \frac{I}{N} \), which is typically \( 2 \sim 4 \cdot 10^{-14} \text{A} \cdot \text{m} \), above the critical value. This yields the RFP version of the tokamak Greenwald density limit

\[
\frac{I}{N} > \left( \frac{I}{N} \right)_{\text{crit}} \sim 2 \cdot 10^{-14} \text{A} \cdot \text{m}
\]  

(1.4)
This empirical relation is satisfied in present RFP devices as well, like MST and RFX.

1.3 Current Experiments

1.3.1 MST

The Madison Symmetric Torus is a toroidal RFP research device situated in the Department of Physics at the University of Wisconsin-Madison. Its construction started in 1985, and its first plasma operation began in August 1988. The machine has a minor radius of 0.52m, and a major radius of 1.5m; the vacuum vessel, also acting as first wall, closely surrounding the plasma (i.e. close-fitting conducting shell), is made of aluminum, and it has a thickness of 5cm. A maximum current of 0.75MA can be induced in the plasma, by using a 2Wb iron core transformer. The specially designed toroidal field winding directly generates the initial bias field in the vacuum chamber, which together with the presence of only a few large port holes for diagnostics guarantees excellent field uniformity and a high degree of toroidal symmetry. The role of MST in the worldwide program during its first five years of operation (1988-1993) has been to investigate RFP confinement at large size, to determine the extent of optimization due to field error reductions and minimizing the vacuum region between the plasma and the wall, and to measure plasma parameters and fluctuations for comparison with theoretical predictions (MHD, turbulent transport, anomalous ion heating, etc.). The main results of MST during its first five years of operation are well summarized in Ref. [17]. Standard ohmic RFP confinement has been characterized by analyzing a large database of shots, recently enriched by new improved diagnostics, in which the plasma current \( I \) and poloidal area integrated density \( N \) were varied independently, keeping the magnetic geometry unchanged (contrarily to many other RFP devices). Direct measurements were done of core electron temperature (by Thomson scattering) and ion temperature (by a central-chord charge-exchange analyzer), showing roughly the same values. The radiated power fraction is constant (around 30%) for \( \frac{I}{N} > 3 \cdot 10^{-14}Am \), below which it rises sharply; impurity content
<table>
<thead>
<tr>
<th>$I[MA]$</th>
<th>$\Phi_t[Wb]$</th>
<th>$F$</th>
<th>$\Theta$</th>
<th>$r_0[m]$</th>
<th>$n[10^{19}m^{-3}]$</th>
<th>$\beta_p[%]$</th>
<th>$\tau_E[ms]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.376</td>
<td>0.0696</td>
<td>-0.22</td>
<td>1.77</td>
<td>0.418</td>
<td>1.4</td>
<td>6.29</td>
<td>1.38</td>
</tr>
</tbody>
</table>

Table 1.1: Plasma Parameters in a Standard MST Shot

<table>
<thead>
<tr>
<th>$I[MA]$</th>
<th>$\Phi_t[Wb]$</th>
<th>$F$</th>
<th>$\Theta$</th>
<th>$r_0[m]$</th>
<th>$n[10^{19}m^{-3}]$</th>
<th>$\beta_p[%]$</th>
<th>$\tau_E[ms]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.381</td>
<td>0.0531</td>
<td>-0.97</td>
<td>2.35</td>
<td>0.41</td>
<td>1.4</td>
<td>10.83</td>
<td>6.47</td>
</tr>
</tbody>
</table>

Table 1.2: Plasma Parameters in a Typical MST Profile Controlled Shot

(mostly aluminum, carbon from the limiter, and oxygen) is of course another important factor in MST. Ensemble averaging of this large database has given the following values for what can be called the MST Standard Shot, as reported in Table 1.1. Here, together with the physical quantities which is easy to recognize, like toroidal current ($I$), toroidal magnetic flux ($\Phi_t$), location of field reversal ($r_0$), and plasma density ($n$), some other plasma parameters are introduced, namely the reversal parameter ($F$) and the pinch parameter ($\Theta$), defined in Equation (2.41), and the confinement parameters beta poloidal ($\beta_p$) and energy confinement time ($\tau_E$), respectively defined in Eqs. (6.1),(6.6). It is important to note that it is currently not possible to measure the energy confinement in a single MST shot, because the Thomson scattering diagnostic requires many similar shots to map out the electron temperature profile. This implies that all the confinement values quoted in MST experimental reports are shot averages. Another interesting consideration is that standard MST plasmas are punctuated by sawtooth crashes, unlike other RFP machines. Thus, the reported standard parameters are for the most part measurements between sawteeth; for instance, in order to give a quantitative idea of what this suggests, consider the following: if the time-averaged energy confinement time were estimated including the sawtooth crashes, a value of 1 ms or less would be found, instead of the value of 1.38 ms as reported in Table 1.1. From the MHD studies, resonant tearing modes with poloidal
number $m = 1$ and toroidal mode numbers $n \gtrsim 6$ are always found in the plasma: they are detected by using bispectral analysis, and they show nonlinarily coupling to higher modes during sudden MHD sawtooth relaxation events (similar to those seen in tokamaks). MST sawteeth are actually discrete dynamo events, as they produce deepened reversal and larger toroidal flux. Soft x-ray core emission tomography has been used to analyze the radial structure of the modes, showing consistency with linear MHD eigenfunctions for $m = 1$. While in the plasma core the resonant modes show global structure and particular coherence, at the edge the high-$n$ resonant modes indicate different mode structure and coherence.

After 1993, with the advent of the 2MA RFP experiment in Padova-Italy (RFX), the MST effort has changed its focus to the problem of improving confinement at low plasma current by means of external profile control, such as Pulsed Poloidal Current Drive (PPCD), electrostatic or oscillating current injection at the edge, and radiofrequency techniques. It is worth briefly describing the innovative and interesting concept of Oscillating Field Current Drive (OFCD), as recently implemented in MST. An AC toroidal voltage is applied to the plasma 90 degrees out of time phase with the applied AC poloidal voltage; this results in net magnetic helicity injection (quadratic effect), adding flux and current via a mechanism called $F-\Theta$ pumping. In practice, electro-magnetic fluctuations are driven at the plasma edge, causing extra velocity fluctuations (from the $\mathbf{E} \times \mathbf{B}$ drift) which, if properly phased, will act as the RFP dynamo and consequently will drive a poloidal current at the edge. The frequency of the oscillating fields is a key issue, as it has to be faster than resistive diffusion time scales ($\sim 1s$) in order to fully develop at the edge, and slower than tearing-mode time scales ($\sim 1ms$) in order to propagate in the core. The appeal of OFCD technique versus the operationally simpler and currently more developed PPCD mode (which is a transient method, where 2 or 3 pulses properly applied generate extra poloidal voltage in the discharge by using additional capacitors), lies in the possibility of reaching steady state operation, for no transformer drive is needed. Apparently, successful OFCD has already been reached in a past device, ZT40, even though under significantly different conditions (see Ref. [18]). Significant improve-
ment in confinement, due to suppression of the fluctuation levels has been observed during pulsed poloidal current drive experiments in MST, increasing the energy confinement time by about one order of magnitude (see Table 1.2). The underlying idea behind profile control is to flatten the parallel (to the magnetic field) current density profile $j_\parallel$ in the edge region by externally driving a poloidal current at the edge (remember that at the plasma edge the magnetic field is mostly in the poloidal direction). The goal is to reduce the mode activity in the edge by artificially lowering the local gradients of $j_\parallel$, broadly speaking trying to push the equilibrium towards a Taylor relaxed state (see Chapter 2). Different ways of injecting poloidal current at the edge have been explored (see Refs. [19]-[20]), but the full understanding of the mechanism is still far from complete. Actually one of the latest (numerical) results (see Ref. [21]) shows that poloidal current injected in the edge of a cylindrical RFP globally readjusts the current density profile, causing a flattening in the core rather than in the edge, so that the improvement in the confinement is due more to global reduction of fluctuations in the core. This is still a work in progress at MST.

1.3.2 RFX

The RFX machine is situated in Padova-Italy and it was constructed from 1985 to 1991, entering into operation in 1992. The machine has a minor radius of 0.48m, and a major radius of 2.0m. The vacuum vessel is made of Inconel-625, and its overall thickness is 3cm; the innerwall (or first wall) which faces the plasma, is covered by 18mm thick graphite tiles. The particular choice for this design was based on the consideration that a metallic first wall would have produced high-Z impurity contamination in the plasma, and could have been seriously damaged by instability driven thermal flux concentrations; furthermore, its small thickness was meant to keep the plasma edge as near as possible to the stabilizing shell. A considerable number of ports and gaps in the shell (96) contributes to enhanced field errors. RFX is designed to reach maximum plasma current of 2MA; unfortunately, the vessel design did not behave as expected, and several problems arose, significantly limiting the machine performance, and producing different physical phenomena than the ones observed on
<table>
<thead>
<tr>
<th>$I[MA]$</th>
<th>$\Phi_t[Wb]$</th>
<th>$F$</th>
<th>$\Theta$</th>
<th>$r_0[m]$</th>
<th>$n[10^{19}m^{-3}]$</th>
<th>$\beta_p[%]$</th>
<th>$\tau_E[ms]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5/1</td>
<td>0.12/0.16</td>
<td>0/-0.4</td>
<td>1.5/1.7</td>
<td>0.38</td>
<td>2/5</td>
<td>5</td>
<td>0.5/1.5</td>
</tr>
</tbody>
</table>

Table 1.3: Range of Plasma Parameters in a Typical RFX Shot

MST. In particular, in RFX the tearing modes that are responsible for the dynamo process tend to lock in phase, due to their non-linear interaction, forming a localized magnetic perturbation that locks onto the enhanced magnetic field errors at the wall, and remains stationary throughout the discharge; this phenomenon is called wall mode locking. A consequence of this mode locking is that a portion of the graphite first wall gets exposed to severe heat loads ($\sim 100 MW/m^2$), which can cause carbon blooms; for this reason, plasma currents have mostly been kept limited to $800kA$. A solution to this impasse has been found by inducing a rotation of the locked modes, thus uniformly distributing the heat load over the entire first wall, allowing RFX to run currents up to $1.1MA$. Another phenomenon observed on RFX experiments is the Quasi Single Helicity (QSH) state (see Ref. [22]), where the $m = 1$ resistive tearing mode spectrum, which usually includes a broad range of toroidal numbers $n$, shrinks around a dominant $m = 1, n = n_0$ mode. In this situation, a coherent helical structure appears in the plasma core, and a hot $m = 1$ island is formed at the corresponding resonant radius; QSH states are obtained in RFX either spontaneously or as a result of Pulsed Poloidal Current Drive (PPCD), and typically tend to show a better confinement compared to normal RFX operation. The main original goal for RFX was to explore confinement at higher current densities, and especially to test the Taylor-Connor confinement scaling law for an RFP. The machine performance, though, has shown very poor confinement during its operation, being mostly limited by wall mode locking. In Table 1.3, a typical range for plasma parameters observed in RFX shots is reported. Improvements were achieved with poloidal current drive experiments (PPCD), which confirmed the suppression of magnetic fluctuations as seen in MST. Interestingly, the electron temperature profile was seen to peak in the
plasma core during PPCD (see Ref. [23]), consistently with reduction of thermal conductivity. Unfortunately, in December 1999 the machine experienced a severe fire accident, and it has been under repair since then.

1.4 Future Designs

A detailed analysis of future RFP designs was done by Werley at al. in 1996 (Ref. [3]), using the Taylor-Connor scaling for the energy confinement time in order to extrapolate parameters for an ignition experiment. Based on the criterion for ignition requiring the plasma alpha power to equal the ohmic power input, and taking into account impurity effects from past experiments, they were able to show that ignition in a purely ohmically heated RFP could be reached with a minimum current of $12.1\, MA$. This very favourable result strongly depends on the particular property of the Taylor-Connor scaling that shows the energy confinement time increasing with the plasma current to the third power. Two experiments were suggested, based on this calculation: the NEPI experiment - a machine designed to operate just with deuterium, and thus create deuterium-deuterium fusion reactions, with the goal of demonstrating deuterium-tritium effective ignition conditions - and the TITAN RFP reactor - designed to reach ignition and a desired beta poloidal of 20% by using an $18\, MA$ plasma current. Both of these designs are compact size, and aside from the validity of Taylor-Connor scaling, they also rely on the possibility to ohmically heat the RFP to ignition. Recent results, both experimental and theoretical (including the present thesis), show that Taylor-Connor scaling is actually too optimistic in predicting confinement for individual machines, so that the validity of those future designs is disputable.

1.5 Thesis Outline

In this thesis, a self-consistent model for analyzing the performance of a Reversed Field Pinch (RFP) type of thermonuclear fusion reactor is formulated and developed.
The RFP reactor concept belongs to the family of toroidal devices which aim at confining an hot ionized gas (plasma) by externally generated magnetic fields. The main features of an RFP device include the reversal of the toroidal component of the magnetic field with respect to its value on axis, and the fact that both the components (poloidal and toroidal) of the equilibrium magnetic field configuration are of the same order of magnitude. This means that, in contrast to the tokamak concept, the RFP has high magnetic shear, and a safety factor considerably less than unity overall the plasma region. Consequently, the RFP is conceptually appealing for its capability of confining the plasma with relatively low (compared to a tokamak) externally imposed magnetic fields (high-β MHD ordering), and for the possibility of reaching ignition by pure ohmic heating, without any auxiliary heating. Its main disadvantage, though, is related to its stability, as it has been shown that the high-β ordering in an RFP is unstable to resistive modes, and the plasma loses most of its initially stored energy through a relaxation process, which sets it into a relatively steady state compared to resistive time scales. Brian Taylor in 1974 started developing his well-known relaxation theory: taking the system to be isolated both electrically and thermally, he assumed that small scale turbulence generated by plasma resistivity dissipates energy under the constraints that total toroidal magnetic flux and total plasma helicity are invariant during the relaxation process. Taylor found that the system reaches a minimum energy state (relaxed state) which is analytically described by the Bessel Function Model (BFM), where the poloidal and toroidal component of the magnetic field are given by Bessel functions, while no pressure gradient is allowed in the entire plasma region. His result, applied to the RFP concept, describes experimental observations very well, not only from a qualitative point of view but also from a quantitative point of view in a first order approximation. However, since it does not account for a driven system, as an experimental plasma in fact is, due to ohmic effect, his theory cannot predict the amount of plasma pressure that can be confined by the magnetic configuration. Moreover, having neglected plasma-wall interaction effects and edge plasma physics, Taylor's magnetic fields are not accurate in the plasma edge region; in particular, the more realistic boundary condition of vanishing current
densities at the wall is not satisfied by BFM. In short, Taylor's theory provides a
good description of the magnetic field configuration in the plasma core, but it does
not account for a driven pressure gradient, and for a vanishing edge current density.
The strong instabilities that largely limit RFP confinement are due to the small
safety factor, which allows a virtually infinite number of resonant surfaces inside the
plasma region, making transport a crucial issue for confinement. Characterization
of plasma transport in an RFP is still not fully understood. One conjecture is that
RFP transport is determined by saturated, internal resonant tearing-mode fluctua-
tions (originated by a non-zero plasma electrical resistivity); experimental evidence is
still not sufficient to strongly support this transport scenario. In this work, the basic
idea is to determine equilibrium configurations which describe a plasma evolution
dominated by tearing modes; specifically, a transport model has been formulated to
provide RFP equilibrium profiles which correspond to a driven-relaxed state that is
determined by stability to tearing modes. The final goal is to provide a model of
transport scaling in an RFP which can evaluate the global parameters describing the
plasma confinement performance. The theory is based on two underlying assump-
tions:

(1) The self consistent equilibrium magnetic field is basically given by Taylor's relax-
ation model, with appropriate modifications to take into account the presence of a
cold wall. As mentioned before, Taylor's theory captures the essential structure of
the magnetic field but predicts a zero pressure gradient as it is a relaxation theory
that does not attempt to take into account heating sources which drive a non-zero
steady-state pressure.

(2) The self-consistent pressure profile is determined by assuming that the ohmic heat-
ing source raises the plasma pressure up until the profile becomes locally marginally
stable to tearing modes.

The intuitive picture of this transport scenario is the following: imagine starting from
a Taylor-like relaxed equilibrium state, which has no pressure and is stable to tearing
modes everywhere in the plasma region. Now the presence of ohmic heating raises the
plasma pressure. It eventually becomes so high that tearing modes are excited. The
induced transport is so strong that the plasma is driven back to the marginal state to tearing modes. A critical point here is the long held belief that an RFP, because of its bad curvature, would always be unstable to tearing or resistive interchange modes (also known as “g-modes”); that is, no marginally stable state exists. This belief turns out to be untrue. The basis for this statement is some recent work by Lutjens et al who considered the effect of thermal conductivity on tearing and resistive interchange modes in a tokamak. When these results are applied to an RFP the conclusion is that tearing and interchange modes can indeed be stabilized, and a prescription for building a pressure profile is given. From a physics point of view a careful ordering of the resistive layer dynamics shows that thermal conductivity dominates over convection and compressibility. Thus, the use of the adiabatic equation of state in earlier work is not accurate for an RFP. The balancing of the thermal conductivity terms (respectively parallel and perpendicular) dominating the perturbed energy equation introduce a new scale length which is essentially given by the ratio of perpendicular thermal diffusivity to parallel thermal diffusivity, to the 1/4 power; this new scale length turns out to be much larger than the tearing mode scale length found in the previous works. Thermal conduction effects then tend to flatten the pressure profile, thereby stabilizing the “g-mode”. To the author’s knowledge, this is the first time a proper, self-consistent definition of tearing-mode marginality has been used to describe an RFP equilibrium.

The derivation of this marginality condition (whose details can be found in Chapter 3) is performed under the assumption of constant plasma density (which is in good agreement with experimental observations). Essentially the form of the tearing mode marginality criterion is one which equates the plasma parameter $\Delta'$ to a normalized pressure gradient divided by the new tearing-mode scale length. This suggests a simple self-consistent way to determine the equilibrium profiles: starting from a Taylor-like profile for the axial magnetic field, properly modified at the edge to account for vanishing current density, the remaining poloidal field and pressure gradient are found in terms of the thermal conductivities by imposing MHD pressure balance and the tearing-mode marginality criterion. Under the assumption of clas-
sical parallel thermal diffusivity and electrical resistivity, the perpendicular thermal diffusivity can be determined self-consistently with the plasma equilibrium profiles by solving the power balance relation between ohmic heating and thermal losses. A further constraint is imposed on the pressure profile, and that is that it should never violate Suydam's marginal criterion, which is a local necessary condition for stability against ideal localized interchange modes. The actual numerical determination of the marginally stable profiles is somewhat challenging, but it can be solved by using state-of-art personal computers. It is worth emphasizing that there are no free parameters in the model. Point checks indicate reasonable agreement with typical experimental data from the two presently existing main RFP devices, MST and RFX. Calculated $\beta_p$ and energy confinement time fall in the ballpark of the observed values. The $m = 1$ mode turns out to be the most constraining mode in terms of pressure confinement overall the plasma region. The resulting, self consistent thermal diffusivity turns out to be about $20 - 60 m^2/s$ in the plasma core, about two orders of magnitude larger than the classical value of $\chi_\perp$. Plasma pressure turns out to be constrained by Suydam's criterion only on axis, and in a very narrow region around the field reversal layer, where very large values of $\Delta'$ show strong tearing-mode stability, allowing the build up of a very steep pressure gradient. Results show that plasma transport is very strong on axis, flattening the plasma pressure, while it is quite weak around the reversal layer, which turns out to be the region of best confinement. It is worth noting that in the linear tearing-mode theory, the $m = 0$ mode is only active right at the reversal layer, so it does not interfere with the very stable $m = 1$ modes which resonate in that area; in a nonlinear analysis, mode coupling and overlapping of the island widths would change this aspect, allowing the $m = 0$ mode to have a bigger effect at the reversal layer (from an experimental point of view, $m = 0$ activity seems to be observed in RFPs around the field reversal region, even though it is not clear how it interacts and compares with the $m = 1$ mode activity). Parametric numerical studies are also shown, spanning the operational space of RFP experiments, finally providing the tearing mode transport scaling relations for the global confinement parameters $\tau_E$ and $\beta_p$. Comparison with experiments as well as other transport models
confirms the important result that the long believed Taylor-Connor scaling for an RFP \( \tau_E \sim \frac{I^3a^2}{N^{3/2}} \) is too optimistic in describing the confinement performance in individual machines.

As an outline of this thesis, in the next chapter a review of Taylor's relaxation theory is provided, including the derivation of the BFM and its properties that will be used in the formulation of the model. In Chapter 3, the new tearing-mode dispersion relation is derived in detail, after reviewing the earlier analysis based on the adiabatic form of the energy equation. Included is the effect of a resistive diffusive flow in the resonant layer, which turns out to be too small to stabilize the "g-mode" in an RFP. It is interesting to note that the new tearing-mode derivation also eliminates the "rippling-mode", an instability arising in the early analysis from the presence of a gradient in the plasma electrical resistivity. Also the gyro-thermal diffusivity is considered, showing that it is negligible in the tearing-mode ordering. The new derivation is linked to an early work by J. Hastie (whose help has been very valuable in this thesis) et al., in which a local flattening in the plasma pressure around the resonant layer was analytically shown to stabilize the "g-mode", although no physical meaning nor scaling was given to the flattening width. Chapter 4 describes the numerical evaluation of the plasma parameter \( \Delta' \), which directly enters in the determination of the self-consistent pressure gradient, as mentioned before. Finally, the complete description of the model, together with the numerical implementation of the solving algorithm, is shown in Chapter 5, before presenting the results and comparisons in Chapter 6.
Chapter 2

Review of Plasma Relaxation in an RFP

2.1 Definition of Plasma Relaxation

Extensive study in laboratory plasma experiments within the international fusion reactor program has shown that the magnetic field configurations in a variety of devices tend to naturally evolve toward a small number of preferred, reproducible states, whose particular structure depends only upon a few global parameters (like total current, magnetic flux, applied voltage, geometry...). These states are almost independent of the details of the initial conditions of the system, that is the way the system was prepared. This phenomenon is called “plasma relaxation” (Refs. [24],[25]), and it belongs to the more general phenomenon of self-organization that can also be found in many physical systems outside plasma physics. Typically, self-organized systems are described by nonlinear partial differential equations with some type of dissipation, which plays a key role in the relaxation process. In fact, dissipation will eventually cause all the physical global quantities that would otherwise be conserved (i.e. invariants) to decay during the evolution; the decaying process for those invariants, though, will happen at different rates, such that certain quantities will experience much more dissipation than others (selective decay of the invariants). A variational theory can then be developed, by considering the minimum dissipated invariants as
constant (on the opportune scale lengths), while a minimization with respect to the maximum dissipated quantity is performed: the correspondent final state describes the relaxed system. Use of a variational theory, however, provides no insight into the dynamics of relaxation. Experiments have shown that the Reversed Field Pinch closely resembles a relaxed configuration, as defined in Ref. [12]. In 1974, J. B. Taylor first illustrated the theory explaining this relaxation process (Refs. [10], [11]). Taylor's Relaxation Theory is remarkable in that it is able to predict the global state of many experimental plasma configurations - not only RFPs - with at least qualitative, and in many cases quantitative, accuracy. In the RFP case, this theory turns out to be particularly successful. In the following section, Taylor's theory will be illustrated in detail. A discussion of the resulting relaxed state for an RFP will follow, together with comparison with experimental observations.

2.2 Taylor's Relaxation Theory

In order to formulate a variational theory of relaxation, the underlying physics has to be provided. The plasma discharge system can be described by an ionized gas permeated by a magnetic field and surrounded by a fixed solid boundary that, with a good approximation, is a perfect electrical conductor. This means that both the tangential electric field and the normal magnetic field must vanish at the system boundary. Consequently there can be no inward or outward Poynting flux \( \vec{S}_v = \vec{E} \times \vec{B} \) (no electromagnetic energy can enter or leave the system). Also thermal isolation from outside environment is assumed. The dissipation process in a plasma discharge is given by plasma resistivity; Furth et al. (Ref. [26]) showed that a small but non-zero plasma resistivity significantly changes the topology of the field lines. This comes from the resistive Ohm's Law for a conducting fluid moving with macroscopic velocity \( \vec{U} \)

\[
\vec{E} + \vec{U} \times \vec{B} = \eta \vec{j} \tag{2.1}
\]

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where \( \vec{E} \) and \( \vec{B} \) are the electric and magnetic field vectors, respectively, \( \eta \) is the electrical resistivity of the fluid and \( \vec{j} \) the current density: the presence of a non-zero resistivity invalidates the 'frozen-in law' of ideal MHD, Ref. [6] (where the magnetic field lines evolve within the fluid elements, and as such they cannot reconnect, keeping the magnetic topology unchanged). The plasma system is described by the following set of equations from resistive MHD:

\[
\begin{align*}
\vec{E} + \vec{v} \times \vec{B} &= \eta \vec{j} \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\
\vec{j} \times \vec{B} &= \nabla p + \rho \frac{d\vec{v}}{dt} \\
\frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) &= 0 \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{B} &= \mu_0 \vec{j} \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{A} &= \vec{B}
\end{align*}
\]

where \( p \) and \( \rho \) are the plasma pressure and mass density, respectively, while

\[
\frac{d}{dt} \triangleq \frac{\partial}{\partial t} + \vec{v} \cdot \nabla
\]

is the convective time derivative, and \( \vec{A} \) is the vector potential, such that from Equation (2.8),(2.9)

\[
\frac{\partial \vec{A}}{\partial t} = -\vec{E} - \nabla \phi
\]
Here \( \phi \) is the scalar potential to be determined by the gauge condition. The following boundary conditions complete the formulation:

\[
\begin{align*}
\vec{B} \cdot \hat{n} \bigg|_{S_0} &= 0 \\
\vec{E} \times \hat{n} \bigg|_{S_0} &= 0 \\
\vec{v} \cdot \hat{n} \bigg|_{S_0} &= 0 \\
\nabla p \cdot \hat{n} \bigg|_{S_0} &= 0
\end{align*}
\] (2.11-2.14)

where \( \hat{n} \) is the unit vector normal to \( S_0 \), the boundary surface of the plasma system, occupying the volume \( V_0 \). Since the system is isolated from the outside environment, it is reasonable to assume that dissipation will minimize the total energy \( W \), which is the sum of the magnetic energy and thermal energy stored in the volume \( V_0 \)

\[
W = \int_{V_0} \left( \frac{B^2}{2\mu_0} + \frac{3}{2}p \right) dV
\] (2.15)

The direct minimization of Equation (2.15) with respect to arbitrary variations in \( \vec{B} \) and \( p \) leads to the trivial uninteresting result \( W = 0 \). Hence, physical invariants of the system have to be found and used as constraints in the variational principal. The first invariant is the total toroidal flux \( \Phi_t \)

\[
\Phi_t = \int_{S_p} \vec{B} \cdot \hat{n}_p dS_p
\] (2.16)

where \( \hat{n}_p \) is the unit vector normal to the poloidal surface \( S_p \). It is easy to see that \( \Phi_t \) is indeed an exact invariant, even with plasma dissipation, due to the presence of the fixed, perfectly conducting wall:

\[
\frac{d\Phi_t}{dt} = \int_{S_p} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}_p dS_p = -\int_{S_p} \nabla \times \vec{E} \cdot \hat{n}_p dS_p = -\oint \vec{E} \cdot d\vec{l}_p = 0
\] (2.17)
The energy minimization then can be performed with the method of Lagrange multipliers, which consists in minimizing the functional $I_1$ such defined

$$I_1 = W - \lambda \Phi_t, \quad \lambda \text{ constant} \quad (2.18)$$

by finding the condition that will lead to $\delta I_1 = \delta W - \lambda \delta \Phi_t = 0$ for a generic plasma variation; thus, the next step is to calculate

$$\delta W = \int_{V_0} \left( \frac{\vec{B}}{\mu_0} \cdot \delta \vec{B} + \frac{3}{2} \delta p \right) dV = \int_{V_0} \left[ \frac{\vec{B}}{\mu_0} \cdot \left( \nabla \times \delta \vec{A} \right) + \frac{3}{2} \delta p \right] dV$$

$$= \int_{V_0} \left\{ \nabla \cdot (\delta \vec{A} \times \vec{B}) + \delta \vec{A} \cdot \nabla \times \vec{B} \right\} dV$$

$$= \int_{V_0} \delta \vec{A} \cdot (\nabla \times \vec{B}) dV + \int_{S_0} (\delta \vec{A} \times \vec{B}) \cdot \hat{n} dS + \int_{V_0} \frac{3}{2} \delta p dV \quad (2.19)$$

$$\delta \Phi_t = \int_{S_p} \delta \vec{B} \cdot \hat{n}_p dS_p = \int_{S_p} (\nabla \times \delta \vec{A}) \cdot \hat{n}_p dS_p = \int_{L_p} \delta \vec{A} \cdot \vec{dl} \quad (2.20)$$

whence

$$\delta I_1 = \int_{V_0} \left[ \delta \vec{A} \cdot (\nabla \times \vec{B}) + \frac{3}{2} \delta p \right] dV + \int_{S_0} (\delta \vec{A} \times \vec{B}) \cdot \hat{n} dS - \lambda \int_{L_p} \delta \vec{A} \cdot \vec{dl} \quad (2.21)$$

By determining the gauge condition (see Equation (2.10)) such that

$$\delta \vec{A} \times \hat{n} \bigg|_{S_0} = 0 \quad (2.22)$$

it follows that both the surface and the line integral vanish, and the minimum energy state with constant toroidal flux is given by

$$\nabla \times \vec{B} = 0, \quad p = 0 \quad (2.23)$$

which is also an uninteresting configuration (no current and no pressure). Taylor realized that there is another physical quantity which is invariant during the relaxation.
process, the total plasma helicity $K_h$ defined here as

$$K_h = \int_{V_0} \vec{A} \cdot \vec{B} dV$$

(2.24)

The concept of helicity in a plasma has to do with the topology of the field lines, intuitively giving a quantitative idea of the linkage of the magnetic flux tubes in the system. Taylor's insight was that only the total helicity in the system would be conserved in the presence of dissipation, and not the local helicity within any given flux surface which is conserved in a dissipation free system. It is instructive to prove that $K_h$ also gets dissipated during the plasma evolution, but that happens on a much longer time scale than for the energy, so that it can be considered constant as explained in Par. 2.1. Taylor proved the invariance of $K_h$ via his conjecture that plasma relaxation is produced by small scale turbulence. In fact, computing directly the time derivatives of $W$ and $K_h$

$$\frac{dW}{dt} = \int_{V_0} \left( \frac{\vec{B}}{\mu_0} \cdot \frac{\partial \vec{B}}{\partial t} + \frac{3}{2} \frac{\partial p}{\partial t} \right) dV = \int_{V_0} \left[ -\frac{\vec{B}}{\mu_0} \cdot \nabla \times \vec{E} + \frac{3}{2} \frac{\partial p}{\partial t} \right] dV$$

$$= -\frac{1}{\mu_0} \int_{V_0} \left[ \nabla \cdot \left( \eta \vec{j} \times \vec{B} \right) + \eta \vec{j} \cdot \nabla \times \vec{B} \right] dV +$$

$$+ \int_{V_0} \left[ \nabla \cdot \left( \vec{v} \times \vec{B} \right) \times \frac{\vec{B}}{\mu_0} \right] + \vec{v} \times \vec{B} \cdot \vec{j} + \frac{3}{2} \frac{\partial p}{\partial t} \right] dV$$

$$= -\int_{V_0} \left[ \eta j^2 + \vec{v} \cdot \nabla p - \frac{3}{2} \frac{\partial p}{\partial t} \right] dV$$

$$= -\int_{V_0} \left[ \eta j^2 - \frac{3}{2} \frac{dt}{dt} + \frac{5}{2} \left( \nabla \cdot (p\vec{v}) - p\nabla \cdot \vec{v} \right) \right] dV$$

$$= -\int_{V_0} \left[ \eta j^2 + \frac{5}{2} \nabla \cdot (p\vec{v}) \right] dV = -\int_{V_0} \eta j^2 dV$$

(2.25)
\[
\frac{dK_h}{dt} = \int_{V_0} \frac{\partial \tilde{A}}{\partial t} \cdot \tilde{B} dV + \int_{V_0} \tilde{A} \cdot \frac{\partial \tilde{B}}{\partial t} dV = -\int_{V_0} \left[ \tilde{E} \cdot \tilde{B} - \nabla \phi \cdot \tilde{B} + \tilde{A} \cdot \nabla \times \tilde{E} \right] dV
\]

\[
= -\int_{V_0} \left[ 2\tilde{E} \cdot \tilde{B} - \nabla \cdot \left( \tilde{A} \times \tilde{B} + \phi \tilde{B} \right) \right] dV = -2\int_{V_0} \tilde{E} \cdot \tilde{B} dV + \int_{S_0} \left( \tilde{A} \times \tilde{E} \right) \cdot \hat{n} dS
\]

\[
= -2\int_{V_0} \eta \tilde{j} \cdot \tilde{B} dV + \int_{S_0} \tilde{A} \times \left( \eta \tilde{j} - \tilde{v} \times \tilde{B} \right) \cdot \hat{n} dS
\]

\[
= -2\int_{V_0} \eta \tilde{j} \cdot \tilde{B} dV + \int_{S_0} \eta \mu_0 \tilde{A} \times \left( \nabla \times \tilde{B} \right) \cdot \hat{n} dS + \int_{S_0} \left[ \left( \tilde{A} \cdot \tilde{v} \right) \tilde{B} - \left( \tilde{A} \cdot \tilde{B} \right) \tilde{v} \right] \cdot \hat{n} dS
\]

\[
= -2\int_{V_0} \eta \tilde{j} \cdot \tilde{B} dV + \int_{S_0} \eta \mu_0 \tilde{A} \times \left( \nabla \times \tilde{B} \right) \cdot \hat{n} dS
\]

\[
\text{having used Gauss' Theorem together with Eqs. (2.2)-(2.15). As expected, both quantities } W \text{ and } K_h \text{ decay due to non-zero plasma resistivity; Taylor's conjecture assumes that for small } \eta, \text{ the relaxation process is driven by small scale turbulence.}
\]

In fact, by Fourier decomposing the magnetic field

\[
\tilde{B} = \sum_k \tilde{b}_k \exp \left( ik \cdot r \right)
\]

and by substituting into Eqs. (2.25)-(2.26), it is easy to show that

\[
\frac{dW}{dt} \sim -\eta \sum_k k^2 b_k^2, \quad \frac{dK_h}{dt} \sim -\eta \sum_k k b_k^2
\]

which shows that for small scale spatial turbulence (equivalently large \(k\)),

\[
\frac{dK_h}{dW} \sim k^{-1} << 1
\]

This implies that the total plasma helicity can be considered an invariant while the energy is dissipated in the relaxation process. Defining the functional

\[
I_2 = W - \mu \frac{K_h}{2}, \quad \mu \text{ constant}
\]
the energy minimization under the constraint of constant $K_h$ can now be carried out by the method of Lagrange multipliers. The variation of $I_2$ yields

$$\delta I_2 = \delta W - \frac{\mu}{2} \delta K_h = 0$$ (2.31)

for a general plasma displacement. The expression for $\delta W$ is given by Equation (2.19), while

$$\delta K_h = \int_{V_0} \left[ \delta \vec{A} \cdot \vec{B} + \vec{A} \cdot \nabla \times \delta \vec{A} \right] dV = 2 \int_{V_0} \delta \vec{A} \cdot \vec{B} dV + \int_{S_0} \left( \delta \vec{A} \times \vec{A} \right) \cdot \hat{n} dS$$ (2.32)

Thus,

$$\delta I_2 = \int_{V_0} \delta \vec{A} \cdot \left[ \nabla \times \vec{B} - \mu \vec{B} \right] dV + \int_{V_0} \frac{3}{2} \delta p \ dV + \int_{S_0} \left( \delta \vec{A} \times \vec{n} \right) \cdot \left[ \frac{\vec{A}^\mu}{2} - \vec{B} \right] dS$$

$$= \int_{V_0} \delta \vec{A} \cdot \left[ \nabla \times \vec{B} - \mu \vec{B} \right] dV + \int_{V_0} \frac{3}{2} \delta p \ dV$$ (2.33)

where the gauge condition in Equation (2.22) has been used. The minimum energy state then corresponds to

$$\left\{ \nabla \times \vec{B} = \mu \vec{B} , \quad \mu \ constant \quad p = 0 \right\}$$ (2.34)

which describes Taylor's relaxed state. Note that the variations in $\delta \vec{A}$ and $\delta p$ are considered independent in the presence of resistive turbulence in which tearing and reconnection of magnetic field lines can occur. They are no longer connected by the ideal MHD relations

$$\delta \vec{B} = \nabla \times \left( \vec{\xi} \times \vec{B} \right) ; \quad \delta p = -\vec{\xi} \cdot \nabla p - \gamma p \nabla \cdot \vec{\xi}$$ (2.35)

The following section discusses in detail the properties of such a state in a cylindrical geometry, showing comparisons with RFP experimentally observed relaxed states.
2.3 Taylor’s Relaxed State: The Bessel Function Model

In this section, Taylor’s relaxed state described by Equation (2.34) is discussed in a cylindrical geometry with axial periodicity (topologically equivalent to a torus with major radius \( R \) and minor radius \( a \): \( 0 \leq r \leq a \), \( 0 \leq \theta \leq 2\pi \), \( 0 \leq z \leq 2\pi R \)). In cylindrical geometry, the general solution to Equation (2.34) with boundary condition (2.11) is easily found in terms of the Bessel functions \( \hat{J}_m \) (Ref. [27])

\[
B_z = \sum_{m,k} b_{m,k} \hat{J}_m(\alpha r) \exp[i(m\theta + kz)]
\]

(2.36)

where \( \alpha^2 = \mu^2 - k^2 \). It can be shown (Refs. [28]-[29] and [6]) that only two of the solutions given by Equation (2.36) can have absolute minimum energy, and those are a cylindrically symmetric solution \( \{ m = 0, \ k = 0 \} \), also known as the Bessel Function Model (BFM)

\[
\begin{align*}
B_z &= B_0 \hat{J}_0(\mu r) \\
B_\theta &= B_0 \hat{J}_1(\mu r) \\
B_r &= 0
\end{align*}
\]

(2.37)

and a helically symmetric state \( \{ m = 1, \ k \text{ arbitrary} \} \)

\[
\begin{align*}
B_z &= B_0 \left[ \hat{J}_0(\mu r) + b_{1,k} \hat{J}_1(\alpha r)\cos(\theta + kz) \right] \\
B_\theta &= B_0 \left[ \hat{J}_1(\mu r) + \frac{b_{1,k}}{\alpha} \left( \mu \hat{J}_1(\alpha r) + \frac{k}{\alpha r} \hat{J}_1(\alpha r) \right) \cos(\theta + kz) \right] \\
B_r &= -B_0 \frac{b_{1,k}}{\alpha} \left[ k \hat{J}_1(\alpha r) + \frac{\mu}{\alpha r} \hat{J}_1(\alpha r) \right] \sin(\theta + kz)
\end{align*}
\]

(2.38)

where \( B_0 \) gives the value of the axial magnetic field on axis \( (r = 0) \), and

\[
\mu = \mu_0 \frac{\vec{j} \cdot \vec{B}}{B^2}
\]

(2.39)
The states given by BFM are completely determined by the two constants, $B_0$ and $\mu$, or equivalently by $\Phi_t$ and $K_h$ by using the relation

$$\frac{K_h}{\Phi_t^2} = \frac{R}{a} \left\{ \frac{\mu a \left[ \hat{J}_0^2(\mu a) + \hat{J}_1^2(\mu a) \right] - 2 \hat{J}_0(\mu a) \hat{J}_1(\mu a)}{\hat{J}_1^2(\mu a)} \right\}$$

(2.40)

together with Eqs. (2.16) or (2.24). In the RFP experiments it is usual to interpret the plasma state in terms of the field reversal parameter $F$ and the pinch parameter $\Theta$ defined as

$$F = \frac{B_z(a) \pi a^2}{\Phi_t}, \quad \Theta = \frac{B_\theta(a) \pi a^2}{\Phi_t} = \frac{\mu_0 I a}{2\Phi_t}$$

(2.41)
where $I$ is the total axial current. These two parameters can be easily expressed in terms of $\mu$

$$F = \frac{\hat{J}_0(\mu a)}{\hat{J}_1(\mu a)} \frac{\mu a}{2}, \quad \Theta = \frac{\mu a}{2}$$ (2.42)

The main feature of BFM state to be noticed is that, for $\mu a > \hat{J}_{0,1} \approx 2.405...$ (that is the first zero of the Bessel Function $\hat{J}_0$), or equivalently for $\Theta > 1.202...$, $F$ becomes negative; that is, the axial magnetic field reverses its sign in the proximity of the wall (plasma edge) with respect to its value on axis. This is in substantial agreement with experiments. The locus of minimum energy states can be represented in an $F - \Theta$ diagram, which shows the condition for reversal (see Figure 2-1). As $\Theta$ is increased (from an experimental point of view, more volt-seconds are supplied to the plasma for a given toroidal flux), a stronger reversal is observed. At $\Theta \geq 1.56...$ ($\mu a \geq 3.11...$) the helical configuration becomes the minimum energy state, and the system undergoes an $m = 1$ instability due to helical distortion: in other words, the $m = 1$ tearing mode becomes unstable for $\Theta \approx 1.56...$ and higher (see Chapter 4).

### 2.4 Discussion

In this chapter, Taylor's theory has been reviewed, together with its predictions for an RFP configuration. Since most of the features that agree with experimental observations have been pointed out (field reversal, $F - \Theta$ diagram...), in this final section the main differences between Taylor's theory and RFP experiments will be discussed. The first aspect is related to plasma pressure: in Taylor's theory, dissipation and reconnection of field lines due to resistivity allows the pressure to equilibrate over the entire plasma volume, so that the pressure gradient sustained by Equation (2.4) vanishes. In terms of the variational principle, pressure variations become independent on $\delta \tilde{A}$, so the minimization of the functional gives $\delta p = 0$. This of course is not in good agreement with observations, in which small but finite pressure is confined, mainly due to the fact that experimental plasmas are not isolated systems, as
Taylor's theory assumes, but are instead ohmically driven: for this reason, Taylor's theory cannot predict how much plasma pressure can be confined in an RFP. The second aspect is related to the description of the plasma edge: Taylor's theory does not include either plasma-wall interaction, or plasma edge physics (colder plasmas) which are actually quite important and indeed very complicated to model. This can be seen in the theory prediction that the plasma current density $\vec{j}$ at the wall is in general non zero:

$$\vec{j}|_a = \frac{\mu}{\mu_0} \vec{B}|_a$$  \hspace{1cm} (2.43)

and of the order of the current density on axis:

$$\vec{j}|_a \sim \vec{j}|_0$$  \hspace{1cm} (2.44)

This is not in agreement with experiments as well, since the plasma current density drops drastically at the edge. To obtain a quantitative idea about how Taylor's relaxed states compare to the experimentally observed ones in an RFP, it is interesting to examine them in the aforementioned $F - \Theta$ diagram. The main difference is that in experimental plasmas, values of $\Theta$ significantly higher than 1.56 (which corresponds to Taylor's stability threshold to helical instability) can be achieved. For example, measured values of $\Theta$ up to 2 have been found in Standard MST shots, reaching values as high as 3.5 during PPCD operation; similar behavior is observed in RFX, even though in this device plasmas tend to get closer to a Taylor's relaxed state.

Several attempts have been made to improve or extend Taylor's Relaxation theory: mainly a theoretical explanation for the existence of relaxed states that could support an equilibrium pressure gradient and still show field reversal has been sought. The basic idea has been to consider the plasma system as not only supported by a dc magnetic field, but also driven by an applied electric field, which more closely describes the situation in an experimental plasma; this means that the system is not isolated anymore, and consequently other invariants could decay faster than the energy, due to the presence of an input energy source now in the system. Relaxed states of a
magnetized plasma with minimum dissipation, or dissipation rate, have been explored (see Refs. [30], [31]); also energy minimization under the constraint of constant Ohmic energy dissipation rate has been investigated (Ref. [32]). Even though some of those theories indeed show a better agreement with experiments in terms of $F - \Theta$ diagram (i.e. allowing profiles with higher values of $\Theta$ and non-zero pressure gradients), none has in reality a strong physical explanation to justify their description of a driven-relaxation theory, in the author’s opinion. For this reason, in this work Taylor’s theory will be used as the starting point for describing RFP relaxed states in the plasma core.

A final comment concerning Taylor’s conjecture: as mentioned in Section 2.2, Taylor assumed that the relaxation process is dominated by small scale turbulence. To the author’s knowledge, to this day in the RFP program there is no systematic evidence to validate this conjecture: in fact, in both experiments and numerical simulations the dominant magnetic fluctuations associated with plasma relaxation appear to have more global, long wavelength structure. Nonetheless, simulations display the same relative invariance of total helicity with respect to plasma energy. Overall, experiments show that Taylor’s theory gives the relaxed magnetic field configuration in the core of an RFP with quite good accuracy, from a qualitative as well as quantitative point of view; for this reason, it will be used as a starting point of the transport model developed in this thesis.
Chapter 3

Classical Tearing- Modes in a Cylindrical Plasma

3.1 Historical Review

The influence of a small but non-zero resistivity on plasma stability was first pointed out by Furth et al. [26] in 1963, for a plane slab geometry. Later (1966), Coppi et al. [33] extended the analysis to a circular cylinder, giving a better hint of what to expect in the more realistic toroidal geometry (see Ref.[34]). Both of these works investigated low-\( \beta \) plasmas in stationary equilibrium. About a decade later, in 1977, Dobrott et al. [35] showed that the equilibrium diffusion velocity of a plasma was indeed a significant and general phenomenon whose contribution to the analysis of tearing modes had been mistakenly left out in all the previous works. This important remark was followed, in 1979, by an extensive reformulation of the tearing mode problem for a cylindrical, low-\( \beta \) plasma (Pollard et al. [36]). The main weakness of all these important works lies in the energy equation used in the resistive MHD model; in fact, an adiabatic law is assumed, neglecting terms such as thermal conduction, which can add important physics to the problem. Only very recently, a tearing mode analysis which takes into account a more complete form of the energy equation was carried out by Lutjens et al. [37], illustrating an important new contribution. In this chapter, all the forementioned works will be illustrated and discussed in detail. In order to
better understand all the different tearing mode derivations, and consequently give a proper interpretation of their results, it is appropriate to start by looking at the general resistive MHD model which contains all effects, and then derive them from there. A description of this general model follows in Section 3.2. The method of solving this model is common to all these works, and consists in carrying out an analytic expansion in terms of a small plasma resistivity; this approach shows that in this limit, resistivity is important only in a thin layer \( \delta \) (called resistive layer, or resonant layer) outside of which the plasma follows the ideal MHD equations with the inertial term playing a negligible role. As will be shown in this chapter, this layer scales like \( \delta \sim \eta^{\frac{3}{2}} \) for tearing modes, whose growth rate \( q \) scales like \( q \sim \eta^{\frac{3}{2}} \). It is then straightforward to see that the plasma diffusion (whose velocity \( v_0 \) scales like \( v_0 \sim \eta \) across the resistive layer is a physically relevant process, and as such, it cannot be neglected a posteriori [35]. By virtue of this consideration, the following section 3.3 will show in detail the full tearing mode derivation by Pollard and Taylor [36] and its results for a cylindrical geometry, recovering the previous treatments in [26] and [33] as a special case \( (v_0 = 0) \). However, it is shown that corrections due to resistive diffusive flow are negligible in an RFP. The implication of this result is that RFP configurations are always unstable to resistive interchange modes ('g-modes') for an adiabatic energy equation. A detailed description of the new tearing-mode analysis as suggested by [37] is then presented in Section 3.5, properly adapted and extended to the RFP concept. It is shown that thermal conductivity is more important than adiabatic thermal effects and indeed leads to stabilization of the 'g-mode'. As stated, for many years researchers did not believe the 'g-mode' could be stabilized.

### 3.2 General Resistive MHD Model

The general form of resistive MHD which will be considered in the present work is the following:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{3.1}
\]
\[
\rho \frac{d\vec{v}}{dt} = \vec{j} \times \vec{B} - \nabla p \tag{3.2}
\]
\[
\vec{E} + \vec{v} \times \vec{B} = \eta_\perp \vec{j}_\perp + \eta_\parallel \vec{j}_\parallel \tag{3.3}
\]
\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.4}
\]
\[
\nabla \times \vec{B} = \mu_0 \vec{j} \tag{3.5}
\]
\[
\nabla \cdot \vec{B} = 0 \tag{3.6}
\]
\[
p = nT \tag{3.7}
\]
\[
\frac{3}{2} \frac{dp}{dt} + \frac{5}{2} p \nabla \cdot \vec{v} = \eta_\perp \vec{j}_\perp^2 + \eta_\parallel \vec{j}_\parallel^2 + \nabla \cdot \left[ n \chi_\parallel \nabla \| T + n \chi_\perp \nabla \perp T - n \chi_\Lambda \frac{\vec{B}}{B} \times \nabla \perp T \right] \tag{3.8}
\]

In this model, the mass (3.1) and momentum (3.2) equations can be easily recognized, being the same as for ideal MHD [6], together with the low-frequency version of Maxwell’s equations, (3.4), (3.5), and (3.6), and the relation (3.7) linking plasma pressure \( p \) and temperature \( T \) through particle density \( n \). Here \( \vec{E} \) and \( \vec{B} \) are respectively the electric and magnetic fields, \( \vec{j} \) is the plasma current density, \( \vec{v} \) the plasma velocity, and \( \rho \) the plasma mass density. The convective derivative \( \frac{d}{dt} \) is related to the time derivative by the following expression:

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla
\]

The remaining equations represent Ohm’s law (3.3), and the energy conservation equation (3.8). Plasma thermal conductivity \( \chi \) and electrical resistivity \( \eta \) are here distinguished in their orthonormal components: parallel to the magnetic field \( \vec{B} \) (\( \| \)), perpendicular to \( \vec{B} \) and in the plane containing the vector (\( \perp \)), and perpendicular to this plane (\( \Lambda \)) in such a way as to give the following orthonormal set of unit vectors: \( \hat{e}_\parallel \times \hat{e}_\perp = \hat{e}_\Lambda \).
3.3 Tearing-Mode Derivation with Diffusion for Cylindrical Plasma

In this paragraph a detailed illustration of Pollard-Taylor tearing-mode derivation [36] will be found for a circular cylinder. The usual set of cylindrical coordinates \((r, \theta, z)\) will therefore be adopted. By imposing periodicity along the \(z\)-axis, the cylindrical system will correspond to its topologically equivalent torus with major radius \(R = \frac{L}{2\pi}\). The main assumptions in this treatment are:

1. Consider only one scalar value for the plasma resistivity \(\eta_\perp = \eta_\parallel = \eta = \text{const}\).

2. Neglect the ohmic and thermal conduction terms in the energy equation (3.8) (adiabatic approximation).

While the first assumption is mostly a simplification for the sake of solving the problem analytically, the second one neglects important physical phenomena due to plasma thermal conduction. This last statement will be extensively discussed in the next sections. The general model described in section 3.2 can then be reduced to:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\
\frac{d\vec{\jmath}}{dt} &= \frac{(\nabla \times \vec{B})}{\mu_0} \times \vec{B} - \nabla p \\
\frac{\partial \vec{B}}{\partial t} &= \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \vec{B} \right) - \nabla \times (\vec{\jmath} \times \vec{B}) \\
\frac{dp}{dt} + \frac{5}{3} p \nabla \cdot \vec{v} &= 0 \\
\nabla \cdot \vec{B} &= 0,
\end{align*}
\]

where the variables \(\vec{E}\) and \(\vec{j}\) have been eliminated through Eqs. (3.4) and (3.5). Equation (3.12) can be rewritten in the more common form of the adiabatic law by eliminating \(\nabla \cdot \vec{v}\) via (3.9):

50
\[
\frac{dp}{dt} - \frac{5}{3} \frac{d\rho}{dt} = 0 \implies \frac{d(p\rho^{-\frac{3}{2}})}{dt} = 0 \tag{3.14}
\]

This set of equations will now be solved analytically for a cylindrical geometry by expanding in terms of plasma resistivity, here considered as a small parameter.

### 3.3.1 Equilibrium

For a steady state equilibrium \(\left( \frac{\partial}{\partial t} = 0 \right)\), it follows from Equation (3.11) that \(\vec{v}_0\) satisfies the equation

\[
\nabla \times (\vec{v}_0 \times \vec{B}_0) = -\frac{\eta}{\mu_0} \nabla^2 \vec{B}_0, \tag{3.15}
\]

whence \(\vec{v}_0 = v_0(r) \hat{e}_r \propto \eta\), so that to the zero-th order in the expansion in \(\eta\) the system of equations becomes:

\[
\hat{j}_0 \times \vec{B}_0 = \nabla p_0 \tag{3.16}
\]
\[
\nabla \times \vec{B}_0 = \mu_0 \hat{j}_0 \tag{3.17}
\]
\[
\nabla \cdot \vec{B}_0 = 0 \tag{3.18}
\]
\[
\rho_0 = \rho_0(r) = m_i n_0(r) \text{ arbitrary,} \tag{3.19}
\]

\((m_i = \text{ion mass, } n_0 = \text{plasma particle density})\)

The following static equilibrium fields are given by:
\[ B_0 = B_{0\theta}(r)\hat{e}_\theta + B_{0z}(r)\hat{e}_z \]  
(3.20)

\[ j_0 = j_{0\theta}(r)\hat{e}_\theta + j_{0z}(r)\hat{e}_z, \]  
(3.21)

where from Equation (3.17): 
\[
\begin{align*}
\mu_0 j_{0\theta} &= -\frac{B_{0z}'}{r} \\
\mu_0 j_{0z} &= \frac{(r B_{0\theta})'}{r}
\end{align*}
\]  
(3.22)

and from Equation (3.16): 
\[ \mu_0 p_0' + B_{0z}B_{0z}' + \frac{B_{0\theta}}{r} (r B_{0\theta})' = 0 \]  
(3.23)

Here, the symbol \('\) represents the derivative with respect to the radial coordinate \(r\).

### 3.3.2 Evolution of the equilibrium state

A perturbative system analysis, carried out by linearizing Equations (3.9)-(3.13), describes the evolution of the steady state given in Section 3.3.1. Due to the periodicity of the system along its poloidal \((\theta)\) and axial \((z)\) components, the linearization process is performed by applying to any equilibrium field \(f_0(r)\), a perturbation \(f_1\) such that

\[ f(r, \theta, z) = f_0(r) + f_1(r) \exp[i(m\theta - k z) + q t], \]  
(3.24)

where \(m\) and \(k = \frac{n}{R}\) are respectively the poloidal and axial wave number \((m, n \in \mathbb{N})\), and \(q\) is the growth rate. In other words, the plasma evolution will be Fourier-Laplace decomposed in its spatial modes and characteristic frequencies; the linearization will of course neglect mode coupling, interference, and other non-linear phenomena.
Equations (3.9)-(3.13) then become as follows:

\[
\begin{align*}
\frac{\partial p_1}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 + \rho_1 \nabla \cdot \vec{v}_0 + \vec{v}_0 \cdot \nabla \rho_1 + \vec{v}_1 \cdot \nabla \rho_0 &= 0 \quad \Rightarrow \\
q p_1 + \rho_0 q \nabla \cdot \vec{v}_1 + \rho_1 \frac{(rv_0)^r}{r} + v_0 \rho'_1 + q \xi_r \rho'_0 &= 0 \\
\rho_0 \left[ \frac{\partial \vec{v}_1}{\partial t} + \vec{v}_0 \cdot \nabla \vec{v}_1 + \vec{v}_1 \cdot \nabla \vec{v}_0 \right] + \rho_1 \vec{v}_0 \cdot \nabla \vec{v}_0 &= -\nabla p_1 + \vec{j}_0 \times \vec{B}_1 + \vec{j}_1 \times \vec{B}_0 \quad \Rightarrow \\
q \rho_0 \left[ q \vec{v}_1 + v_0 (\xi_r \vec{e}_r + \xi_{\theta} \vec{e}_{\theta} + \xi_z \vec{e}_z) + \vec{v}_1 \cdot \nabla \vec{v}_0 \right] + \rho_1 v_0 \vec{v}'_0 \hat{e}_r &= -\nabla p_1 + \vec{j}_0 \times \vec{B}_1 + \vec{j}_1 \times \vec{B}_0 \\
- \frac{\partial \vec{B}_1}{\partial t} &= \frac{n}{\mu_0} \nabla \times (\nabla \times \vec{B}_1) - \nabla \times (\vec{v}_0 \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_0) \quad \Rightarrow \\
q \vec{B}_1 &= \frac{n}{\mu_0} \nabla^2 \vec{B}_1 + iv_0 \left( \frac{m}{r} B_{1\theta} - k B_{1z} \right) \hat{e}_r - (v_0 B_{1\theta})' \hat{e}_{\theta} - \left( \frac{rv_{0} B_{1z}}{r} \right)' \hat{e}_z + \\
&\quad + q \left\{ i \xi_r \left( \frac{m}{r} B_{0\theta} - k B_{0z} \right) \hat{e}_r - \left[ ik (\xi_{\theta} B_{0z} - \xi_z B_{0\theta}) + (\xi_{\theta} B_{0\theta})' \right] \hat{e}_{\theta} \right. \\
&\quad \left. - \left[ \frac{r \xi_r B_{0z}}{r} + \frac{im}{r} (\xi_{\theta} B_{0z} - \xi_z B_{0\theta}) \right] \hat{e}_z \right\} \\
\frac{\partial}{\partial t} \left( p_1 \rho_0 - \frac{5}{3} \rho_0 - \frac{\rho_1}{\rho_0^{\frac{5}{3} - 1}} \right) + \vec{v}_0 \cdot \nabla \left[ p_1 \rho_0 - \frac{1}{3} - \frac{5}{3} \rho_0 - \frac{\rho_1}{\rho_0^{\frac{5}{3} - 1}} \right] + \vec{v}_1 \cdot \nabla \left( p_0 \rho_0^{-\frac{5}{3}} \right) &= 0 \quad \Rightarrow \\
q \left( p_1 \rho_0 - \frac{5}{3} \rho_0 - \frac{\rho_1}{\rho_0^{\frac{5}{3} - 1}} \right) + v_0 \left[ p_1 \rho_0 - \frac{1}{3} - \frac{5}{3} \rho_0 - \frac{\rho_1}{\rho_0^{\frac{5}{3} - 1}} \right]' + q \xi_r \left( p_0 \rho_0^{-\frac{5}{3}} \right)' &= 0 \quad \Rightarrow \\
\nabla \cdot \vec{B}_1 &= 0 \quad \Rightarrow \\
\left( \frac{r B_{1z}}{r} \right)' + \frac{im B_{1\theta}}{r} - ik B_{1z} &= 0 \\
\end{align*}
\]

\(3.25\)
\(3.26\)
\(3.27\)
\(3.28\)
\(3.29\)
where
\[ \mu_0 \vec{j}_1 = \nabla \times \vec{B}_1, \] (3.30)

and the plasma displacement \( \vec{\xi} \) has been introduced, such that \( \vec{u}_1 = \frac{\partial \vec{\xi}}{\partial \tau} \).

### 3.3.3 Solution in the limit \( \eta \to 0 \)

Before solving Equations (3.25)-(3.29) for a finite \( \eta \), it is important to study what happens to the solution in the case of a perfectly conducting plasma column. In this limit \( \eta \to 0 \), and Equations (3.25)-(3.29) reduce to:

\[
\begin{align*}
\rho_1 + \rho_0 \nabla \cdot \vec{\xi} + \vec{\xi} \times \nabla \rho_0 &= 0 \quad (3.31) \\
\nabla p_1 &= \vec{j}_0 \times \vec{B}_1 + \vec{j}_1 \times \vec{B}_0 \quad (3.32) \\
\vec{B}_1 &= \nabla \times (\vec{\xi} \times \vec{B}_0) \quad (3.33) \\
p_1 - \frac{5}{3} \frac{p_0}{\rho_0} \rho_1 + \vec{\xi} \cdot \nabla p_0 - \frac{5}{3} \vec{\xi} \cdot \nabla \rho_0 \frac{p_0}{\rho_0} &= 0 \quad (3.34) \\
\nabla \cdot \vec{B}_1 &= 0 \quad (3.35)
\end{align*}
\]

Here all the terms containing \( v_0 \propto \eta \) have been neglected, together with the small (assuming ideal MHD stability and considering only resistive instabilities) inertia term \( q^2 \rho_0 \vec{\xi} \) in Equation (3.32). By algebraically eliminating \( \rho_1 \) in Equation (3.34) from Equation (3.31) and then \( p_1 \) in Equation (3.32) from Equation (3.34), the system reduces to:

\[ \nabla \left[ \frac{5}{3} p_0 \nabla \cdot \vec{\xi} + \vec{\xi} \cdot \nabla p_0 \right] = -\vec{j}_0 \times \vec{B}_1 - \vec{j}_1 \times \vec{B}_0 \] (3.36)

where \( \vec{B}_1 \) and \( \vec{j}_1 \) are given by Eqs. (3.33) and (3.30), respectively.

After some cumbersome vector algebra, whose details can be found in Appendix A (see also [6]), a second order differential equation is obtained for the normalized radial
The perturbed magnetic field $\Psi$:

$$
\Psi'' + \frac{\Psi}{r^2} \left\{ -\frac{2k^2r^3}{F^2} \mu_0 p_0 + \frac{2k^2r^2}{F} m B_{\theta 0} + k r B_{0z} - \frac{F''}{F} r^2 + \frac{F'}{F} r \frac{m^2 - k^2 r^2}{D} + 1 - \frac{D}{D} + \frac{m^4 - 3k^4 r^4 + 10m^2 k^2 r^2}{4D^2} \right\} = 0 \quad (3.37)
$$

where

$$
\Psi \equiv B_{1r} \frac{r^3}{\sqrt{D}}, \quad F \equiv m B_{\theta 0} - k r B_{0z}, \quad \text{and} \quad D \equiv m^2 + k^2 r^2. \quad (3.38)
$$

Note that the radial plasma displacement $\xi_r$ is directly related to $B_{1r}$ via the radial component of Equation (3.33),

$$
B_{1r} = \frac{i \xi_r}{r} F \quad (3.39)
$$

Once the equilibrium profiles are given, the solution of Equation (3.36) can be obtained in a straightforward manner over the entire plasma region. This solution will have a singularity at the radius $r_s$ where the function $F(r_s) \equiv [m B_{\theta 0} - k r B_{0z}]$, is equal to zero, or equivalently where the safety factor $q \equiv \frac{r B_{0z}}{R B_{\theta 0}}$ is a rational number: $q(r_s) = \frac{m}{n}$.

It is easy to show that at this point the radial perturbed magnetic field is finite, while the radial displacement becomes infinite. It is in the vicinity of the points $r_s$, which are also called resonant points, that the plasma resistivity plays a role: it removes the singularity, and therefore cannot be neglected. It is important to look at the solution of Equation ((3.37)) around the singular layer $r_s$, where the differential equation has the form:

$$
\Psi'' + \left[ \frac{a_1}{x^2} + \frac{a_2}{x} + a_3(x) \right] \Psi = 0 \quad x \equiv r - r_s, \quad \left\{ \begin{array}{l} a_1, a_2 \text{ constants} \\ a_3(x) \text{ regular function of } x \end{array} \right. \quad (3.40)
$$
where \( a_1 \equiv -\frac{2k^2 r_s}{|F'(r_s)|^2} \mu_0 p_0(r_s) \equiv D \), \( F' = mB'_0 - k(B_2 + rB'_s) \), and

\[
a_2 = a_1 \frac{p''(r_s)}{F'(r_s)} + \frac{a_1}{r_s} - \frac{F''(r_s)}{F'(r_s)} (a_1 + 1) + \frac{4k^2 mB_0(r_s)}{(m^2 + k^2 r_s^2) F'(r_s)} - \frac{m^2 - k^2 r_s^2}{r_s (m^2 + k^2 r_s^2)} \tag{3.41}
\]

By asymptotic expansion, the solution of Equation (3.40) is easily found (see [38]):

\[
\Psi(x) = |x|^{\nu_1} \left[ 1 - \frac{a_0 x}{2 \nu_1} + x^2 \frac{a_0^2 - 2 \nu_1 a_0(0)}{4 \nu_1 (1 + 2 \nu_1)} + O(x^3) \right] \tag{3.42}
\]

\[
+ |x|^{\nu_2} \left[ 1 - \frac{a_0 x}{2 \nu_2} + O(x^2) \right] \left( \Delta + \frac{a_0 x}{2 \nu_1 |x|} \right)
\]

where \( \nu_1 = \frac{1}{2} (1 - \sqrt{1 - 4D}) \), \( \nu_2 = \frac{1}{2} (1 + \sqrt{1 - 4D}) \). \tag{3.43}

Notice that \( a_1 \) is exactly the parameter \( D \) (see [6]) appearing in Suydam’s stability criterion for localized ideal interchange modes. Furthermore, the normalization constant here has been fixed, so there is only one arbitrary constant, namely \( \Delta \), that has to be determined from boundary conditions: \( \Psi(r = 0) = 0 \) and \( \Psi(r = 0) = a \) (perfectly conducting wall); as a consequence of the singularity, the solution will have a jump in the derivative at \( r_s \), such that its analytic expression will differ from one side of the resonance to the other, and as such it will later be distinguished between an inner solution \( \Psi_L \) for \( x < 0 \), and an outer solution \( \Psi_R \) for \( x > 0 \). It will also be important later to work with the asymptotic solution of Equation (3.40) in the limit of small pressure (\( a_1 \to 0 \Rightarrow \nu_1 \to 0 \), \( \nu_2 \to 1 - a_1 \)).

\[
\Psi_{L,R} = 1 - a_0 x \ln |x| + O(x^2 \ln |x|) + \Delta_{L,R}|x| + O(x^2) + a_1 \ln |x|. \tag{3.44}
\]

### 3.3.4 Tearing Mode Ordering in the Resonant Layer

The solution of Equations (3.25)-(3.29) in the limit \( \eta \to 0 \) has shown the presence of a singularity, or resonance, at the radial point \( r_s \) : \( F(r_s) = 0 \), where the function \( F \) is given in the Equation (3.38). In order to remove this singularity, the full system

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of Equations (3.25)-(3.29) has to be solved for a small but non-zero resistivity in the proximity of \( r_s \), in a region that is known in the literature as the resonant, or the resistive, layer. It is then more appropriate to use as radial variable the distance \( x \) from the resonance, such that \( x \approx r - r_s \). To serve the purpose, the limit of small resistivity is taken by applying the following Tearing-Mode Ordering procedure (see Refs. [33], [36]):

\[
\eta \sim \mathcal{E}^5, \quad v_0 \sim \mathcal{E}^5, \quad m \sim 1, \quad q \sim \mathcal{E}^3, \quad x \sim \mathcal{E}^2, \quad \beta \sim \mathcal{E}^2,
\]

where for ease of notation, \( \mathcal{E} \) is introduced and used as the small parameter of the ordering. Notice that \( \frac{d}{dr} = \frac{d}{dx} = (') \); in the resonant layer the equilibrium quantities \( (\rho_0, v_0, B_0, p_0, \ldots) \) can be considered constant with \( x \), while the perturbed quantities \( (\rho_1, \xi_r, B_1, p_1, \ldots) \) are allowed to vary with \( x \). As a matter of fact they can be expanded in powers of \( x \), or - consistently with the previous choice of ordering parameter - equivalently in powers of \( \mathcal{E} \). For a generic perturbed quantity \( f \), the expansion is written in the following notation:

\[
f = \sum_{n=0}^{\infty} \mathcal{E}^{2n} f^{(2n)}.
\]

The next step is to set the ordering for the perturbed quantities. By balancing the second and the last term of the lefthandside of Equation (3.25),

\[
\nabla \cdot \vec{\xi} \sim \xi_r,
\]

where

\[
\nabla \cdot \vec{\xi} = \frac{(r \xi_r)'}{r} + \frac{im}{r} \xi_\theta - ik \xi_z \sim \frac{\xi_r}{\mathcal{E}^2} + \{\xi_\theta, \xi_z\}
\]

whence

\[
\xi_r \sim \mathcal{E}^2 \sim \nabla \cdot \vec{\xi}, \quad \{\xi_\theta, \xi_z\} \sim \mathcal{E}^0.
\]

In particular, \( \xi_r^{(2)'} + \frac{im}{r} \xi_\theta^{(0)} - ik \xi_z^{(0)} = 0 \). Then, from the radial component of Equation (3.33),

\[
B_{1r} \sim \xi_r F \sim \xi_r \mathcal{E}^2 \sim \mathcal{E}^4
\]
Balancing \( \mathbf{j}_0 \times \mathbf{B}_1 \) and \( \mathbf{j}_1 \times \mathbf{B}_0 \) in Equation (3.26) implies \( B_{1r} \sim \{B_{1\theta}, B_{1z}\} \sim \mathcal{E}^4 \). Note that in this ordering, Equation (3.29) gives 
\[
\frac{d B_{1r}^{(4)}}{d r} = 0 \quad \text{and} \quad \frac{d B_{1r}^{(6)}}{d r} + \frac{im}{r} B_{1r}^{(4)} - i k B_{1z}^{(4)} = 0,
\]
which is also known as the constant-\( \psi \) approximation, as is explained later. From Equation (3.25),
\[
\rho_1 \sim \xi_r \sim \mathcal{E}^2
\]
Finally, from the parallel (to \( \mathbf{B}_0 \)) component of Equation (3.26)
\[
\mathbf{B}_0 \cdot \nabla p_1 \sim q^2 \rho_0 B_0 \xi || \sim B_{1r} p_0',
\]
whence \( p_1 \sim \mathcal{E}^4 \), \( \xi || \sim \mathcal{E}^4 \) and \( p_0' \sim \beta \sim \mathcal{E}^2 \).

Consequently, the expansions for the perturbed quantities in terms of \( \mathcal{E} \) are:
\[
\hat{\xi} = \{\mathcal{E}^2 \xi_r^{(2)} + O(\mathcal{E}^4)\} \hat{\varepsilon}_r + \{\xi_\theta^{(0)} + O(\mathcal{E}^2)\} \hat{\varepsilon}_\theta + \{\xi_z^{(0)} + O(\mathcal{E}^2)\} \hat{\varepsilon}_z
\]
\[
\mathbf{B}_1 = \{\mathcal{E}^4 B_{1r}^{(4)} + \mathcal{E}^6 B_{1r}^{(6)} + O(\mathcal{E}^8)\} \hat{\varepsilon}_r + \{\mathcal{E}^4 B_{1\theta}^{(4)} + O(\mathcal{E}^6)\} \hat{\varepsilon}_\theta + \{\mathcal{E}^4 B_{1z}^{(4)} + O(\mathcal{E}^6)\} \hat{\varepsilon}_z
\]
\[
p_1 = \mathcal{E}^4 p_1^{(4)} + O(\mathcal{E}^6)
\]
\[
\rho_1 = \mathcal{E}^2 \rho_1^{(2)} + O(\mathcal{E}^4)
\]

In order to decompose the vector Equations (3.25)-(3.29) into scalar equations, it is convenient to examine the components with respect to a more natural orthonormal set of unit vectors, given by the equilibrium magnetic field geometry:
\[
\{\hat{\varepsilon}_r, \hat{\varepsilon}_\theta, \hat{\varepsilon}_z\} \rightarrow \{\hat{\varepsilon}_r, \hat{\varepsilon}_{||}, \hat{\varepsilon}_\perp\}:
\]
\[
\begin{align*}
\hat{\varepsilon}_{||} &= \frac{\mathbf{B}_0}{B_0} = \frac{B_{0\theta}}{B_0} \hat{\varepsilon}_\theta + \frac{B_{0z}}{B_0} \hat{\varepsilon}_z; \\
\hat{\varepsilon}_\perp &= \hat{\varepsilon}_r \times \hat{\varepsilon}_{||} = -\frac{B_{0z}}{B_0} \hat{\varepsilon}_\theta + \frac{B_{0\theta}}{B_0} \hat{\varepsilon}_z; \\
\hat{\varepsilon}_\theta &= \frac{B_{0z}}{B_0} \hat{\varepsilon}_{||} = \frac{B_{0\theta}}{B_0} \hat{\varepsilon}_\perp;
\end{align*}
\]

Note that despite the similar notation, kept here for the sake of tradition, this coordinate system is in general different from the one already introduced in Equation (3.8). They coincide only for a vector having no radial component, in which case \( \hat{\varepsilon}_r \equiv \hat{\varepsilon}_\Lambda \).

Before rewriting the Equations (3.25)-(3.29) in this new coordinate system, it is
straightforward to identify the terms which are small for all the components, and as such can immediately be neglected:

From Equation (3.25) : \[ \frac{\rho_1 \nabla \cdot \mathbf{v}_0}{q \rho_1} \sim \mathcal{E}^2 ; \]

From Equation (3.26) : \[ \frac{\mathbf{v}_1 \cdot \nabla \mathbf{v}_0}{\mathbf{v}_0 \cdot \nabla \mathbf{v}_1} \sim \mathcal{E}^2 ; \]
\[ : \rho_1 \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 \sim \mathbf{v}_0 \rho_1 \nabla \cdot \mathbf{v}_0 \sim \mathbf{v}_0 \mathcal{E}^2 q \rho_1 \sim \mathbf{v}_0 \mathcal{E}^2 \rho_0 \nabla \cdot \mathbf{v}_1 \lesssim \mathcal{E}^2 \rho_0 \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 ; \]

From Equation (3.27) : \[ \nabla \times (\mathbf{v}_0 \times \mathbf{B}_1) \lesssim \mathbf{v}_0 \mathbf{B}_1' \sim \mathcal{E}^3 \mathbf{B}_1 \lesssim \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}_1 \sim \mathcal{E} \mathbf{B}_1 \]

Finally, keeping in mind the following relations:

\[ \bullet \ (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 = - (B_{0z} B_{1x} + \frac{(r B_{0 \theta} v_x)}{r} B_{1 \theta}) \hat{e}_r + \frac{(r B_{0 \theta} v_x)}{r} B_{1x} \hat{e}_\theta + B_{0z} B_{1r} \hat{e}_z \quad (3.45) \]

\[ \bullet \ \mathbf{B}_0 \cdot \left[ (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 \right] = \frac{(r B_{0 \theta} v_x)}{r} B_{0 \theta} B_{1r} + B_{0z} B_{0z} B_{1r} \]
\[ = - B_{1r} \mu_0 \rho_0 \quad \text{from Equation (3.23)} \quad (3.46) \]

\[ \bullet \ (\hat{e}_r \times \hat{e}_{ll}) \cdot \left[ (\mathbf{j}_0 \times \mathbf{B}_1) + (\mathbf{j}_1 \times \mathbf{B}_0) \right] = \left[ \mathbf{j}_0 \times (\hat{e}_{ll} \times \hat{e}_r) \right] \cdot \mathbf{B}_1 + \left[ (\hat{e}_{ll} \times \hat{e}_r) \times \hat{e}_{ll} \right] \cdot \mathbf{j}_1 \mathbf{B}_0 \]
\[ = - (\mathbf{j}_0 \cdot \mathbf{B}_0) \frac{B_{0z}}{B_0} B_{1r} + B_0 j_{1r} \quad (3.47) \]
\[ = - (\mathbf{j}_0 \cdot \mathbf{B}_0) \frac{B_{0z}}{B_0} + \frac{\rho_0}{\mu_0} (m B_{1z} + k B_{1 \theta}) \]

\[ \bullet \ \nabla \times (\mathbf{\tilde{f}} \times \mathbf{\tilde{B}}_0) = \mathbf{\tilde{f}} (\nabla \cdot \mathbf{\tilde{B}}_0) - \mathbf{\tilde{B}}_0 (\nabla \cdot \mathbf{\tilde{f}}) + \mathbf{\tilde{B}}_0 \cdot \nabla \mathbf{\tilde{f}} - \mathbf{\tilde{f}} \cdot \nabla \mathbf{\tilde{B}}_0 \quad (3.48) \]

\[ \bullet \ \mathbf{\tilde{B}}_0 \cdot \nabla \mathbf{\tilde{f}}_1 = \left[ B_{0 \theta} \frac{im}{r} f_{1r} - B_{0z} k f_{1r} - \frac{B_{0 \theta}}{r} f_{1 \theta} \right] \hat{e}_r + \left[ B_{0 \theta} \frac{im}{r} f_{1z} - B_{0z} k f_{1z} \right] \hat{e}_z + \left[ B_{0 \theta} \frac{im}{r} f_{1 \theta} - ik B_{0z} f_{1 \theta} + \frac{B_{0 \theta}}{r} f_{1r} \right] \hat{e}_\theta \quad (3.49) \]
\[ = i (\frac{m}{r} B_{0 \theta} - k B_{0z}) \mathbf{\tilde{f}}_1 - B_{0 \theta} f_{1 \theta} \hat{e}_r + \frac{B_{0 \theta}}{r} f_{1r} \hat{e}_\theta = i F_{1r} \mathbf{\tilde{f}}_1 + \frac{B_{0 \theta}}{r} \hat{e}_z \times \mathbf{\tilde{f}}_1 \]

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\[ \xi \cdot \nabla \vec{B}_0 = -\xi_\theta \frac{B_{0\theta}}{r} \hat{e}_r + \xi_r B_{0\theta} \hat{e}_z + \xi_r B_{0z} \hat{e}_z \]  
(3.50)

\[ \nabla^2 \vec{B}_1 = \left[ \nabla^2 B_{1r} - \frac{2im}{r^2} B_{1\theta} - \frac{B_{1r}}{r^2} \right] \hat{e}_r + \nabla^2 B_{1z} \hat{e}_z \]  
(3.51)

\[ \hat{e}_r \cdot \left[ j_0 \times \vec{B}_1 + j_1 \times \vec{B}_0 \right] = - \left[ B'_{0z} B_{1z} + \frac{(r B_{0\theta})'}{r} B_{1\theta} \right] \frac{1}{\mu_0} + j_{1\theta} B_{0z} - j_{1z} B_{0\theta} \] 
(3.52)

\[ = - \left[ B'_{0z} B_{1z} + \frac{(r B_{0\theta})'}{r} B_{1\theta} \right] \frac{1}{\mu_0} - B_{0z} B'_{1z} - B_{0\theta} \frac{(r B_{0\theta})'}{r} \]

\[ \vec{j}_0 \times \vec{B}_1 + \vec{j}_1 \times \vec{B}_0 = \vec{B}_1 \cdot \nabla \vec{B}_0 + \vec{B}_0 \cdot \nabla \vec{B}_1 - \nabla (\vec{B}_1 \cdot \vec{B}_0), \]  
(3.53)

The remaining terms from Equations (3.25)-(3.29) can be decomposed as follows:

\[ Eq. \ (3.25) \quad \Rightarrow \quad q \rho_1 + \rho_0 q \nabla \cdot \vec{\xi} + v_0 \rho'_1 + q \xi_\rho \rho'_0 = 0; \]

\[ Eq. \ (3.26) \cdot \vec{B}_0 \quad \Rightarrow \quad q^2 \rho_0 B_0 \xi_\parallel + q \rho_0 v_0 \left( \xi_\theta B_{0\theta} + \xi_z B_{0z} \right) = -\vec{B}_0 \cdot \nabla p_1 - B_{1r} p'_0 \]

where

\[ \vec{B}_0 \cdot \left[ (\nabla \times \vec{B}_0) \times \vec{B}_1 \right] = \frac{(r B_{0\theta})'}{r} B_{0\theta} B_{1r} + B'_{0z} B_{0z} B_{1r} = -B_{1r} \mu_0 \rho'_0 \]

From Equations (3.40) and (3.23)

\[ Eq. \ (3.26) \cdot \hat{e}_\perp \quad \Rightarrow \quad q^2 \rho_0 \xi_\perp + q \rho_0 \frac{v_0}{B_0} \left( -B_{0z} \xi_\theta + B_{0\theta} \xi_z' \right) = \]

\[ \frac{ip_1}{\hat{B}_0} \left( \frac{m}{r} B_{0z} + kB_{0\theta} \right) - B_{1r} \frac{B_{0z}}{B_0} \left( \hat{j}_0 \cdot \hat{B}_0 \right) + \frac{ip_0}{\mu_0} \left( \frac{m}{r} B_{1z} + kB_{1\theta} \right) \]

The terms on the lefthandside scale like \( \mathcal{E}^6 \), while the terms on the righthandside
scale like $\mathcal{E}^4$.

\begin{equation}
(3.27) \cdot \dot{B}_0 \implies qB_0B_{1\parallel} = \eta \frac{\mu_0}{\mu_0} \left[ B_{0\theta} \left( \nabla^2 B_{1\theta} + \frac{2im}{r^2} B_{1r} - \frac{B_{1\theta}}{r^2} \right) + B_{0z} \nabla^2 B_{1z} \right] + qB_0^2 \nabla \cdot \hat{\xi} + qF B_0\xi_{||} - q\xi_r \left( B_{0\theta}B'_{0\theta} + B_{0z}B'_{0z} - \frac{B_{0\theta}^2}{r} \right)
\end{equation}

The largest terms in $\eta \nabla^2 \vec{B}_1$ are the ones containing the second radial derivatives, i.e. $\eta(B_{0\theta}B''_{1\theta} + B_{0z}B''_{1z}) \sim \mathcal{E}^5$, which are larger than $qB_{1\parallel} \sim \mathcal{E}^7$, whence

\begin{equation}
\frac{\eta}{\mu_0 q} \left( B_{0\theta}B''_{1\theta} + B_{0z}B''_{1z} \right) - B_0^2 \nabla \cdot \hat{\xi} + \frac{iF}{r} B_0\xi_{||} + \xi_r \left( \mu_0 p' + \frac{2B_{0\theta}}{r} \right)
\end{equation}

\begin{equation}
(3.27) \cdot \dot{e}_r \implies qB_{1r} = \eta \frac{\mu_0}{\mu_0} \left[ \nabla^2 B_{1r} - \frac{2im}{r^2} B_{1\theta} - \frac{B_{1r}}{r^2} \right] + q\frac{F\xi_r}{r} - q\frac{B_{0\theta}}{r} \xi_0 + q\xi_0 B_{0\theta}
\end{equation}

In this equation it is possible to directly observe the impact of the forementioned constant-$\Psi$ approximation: in fact, the largest term in $\eta \nabla^2 \vec{B}_1$ is once again $\eta B''_{1r}$, and in order to balance the other terms, it has to involve the component that varies with $x$, i.e. $B^{(6)}_{1r}$. Thus,

\begin{equation}
B^{(4)}_{1r} = \eta \frac{\mu_0}{\mu_0} B^{(6)}_{1r} + \frac{iF\xi_r}{r}
\end{equation}

Next, note that from

\begin{equation}
(3.28) \implies q \left( p_1 - \frac{5}{3} \rho_0 \rho_1 \right) + \rho_0 \frac{5}{3} v_0 \left( p_1' - \frac{5}{3} \rho_0 \rho_1' \right) + q\xi_r \left( p_0' - \frac{5}{3} \rho_0 \rho_0' \right) = 0
\end{equation}

The term containing the diffusion velocity $v_0$ will compete with the others to first order, when the derivative is applied to the perturbed quantities:

\begin{equation}
q \left( p_1 - \frac{5}{3} \rho_0 \rho_1 \right) + v_0 \left( p_1' - \frac{5}{3} \rho_0 \rho_1' \right) + q\xi_r p_0 \left( p_0' - \frac{5}{3} \rho_0 \rho_0' \right) = 0
\end{equation}

Continuing,

\begin{equation}
(3.26) \cdot \dot{e}_r \implies \mu_0 \left( q^2 \rho_0 \xi_r + q \rho_0 v_0 \xi_r \right) = -\mu_0 p'_1 - \left( B_{0z}^r B_{1z} + \frac{(rB_{0\theta})'}{r} B_{1\theta} \right) - B_{0z} B_{1z}' - \frac{(rB_{1\theta})'}{r} B_{0\theta}
\end{equation}
which at the highest order shows that \( \mu_0 p'_1 + \vec{B}_0 \cdot \vec{B}'_1 = 0 \), or equivalently,

\[
(\mu_0 p_1 + B_0 B_{1||})' = 0,
\]

whence \( \mu_0 p_1 + B_0 B_{1||} = \text{const.} \)

Note that no extra information is given by this component, since this relation is already contained in Eq. (3.26) \( \cdot \vec{e}_\perp \) to the leading order, keeping in mind that \( \frac{dB^{(4)}_r}{dr} = 0 \). In order to extract another relevant relation from Equation (3.26), it is necessary to go to higher order, by operating on it via an appropriate annihilator operator: \( \vec{B}_0 \cdot \nabla \times \):

\[
\vec{B}_0 \cdot \nabla \times \left( q^2 \rho_0 \vec{\xi} \right) \simeq q^2 \rho_0 \vec{B}_0 \cdot \nabla \times \vec{\xi} \simeq q^2 \rho_0 \left[ -B_{0\theta} \xi'_r + B_{0z} \xi'_\theta \right] ;
\]

\[
\vec{B}_0 \cdot \nabla \times \left[ q \rho_0 v_0 \left( \xi'_r \vec{e}_r + \xi'_\theta \vec{e}_\theta + \xi'_z \vec{e}_z \right) \right] \simeq q \rho_0 v_0 \left[ -B_{0\theta} \xi'_z + B_{0z} \xi''_\theta \right] ;
\]

\[
\vec{B}_0 \cdot \nabla \times [\nabla p_1] = 0 ;
\]

\[
\vec{B}_0 \cdot \nabla \times [\vec{j}_0 \times \vec{B} + \vec{j}_1 \times \vec{B}_0]
\]

\[
= \nabla \cdot \left[ (\vec{j}_0 \times \vec{B}_1) \times \vec{B}_0 + (\vec{j}_1 \vec{B}_0) \times \vec{B}_0 \right] + \left[ \vec{j}_0 \times \vec{B}_1 + \vec{j}_1 \times \vec{B}_0 \right] \cdot \nabla \times \vec{B}_0
\]

\[
= \nabla \left[ \vec{B}_0 j_{0||} \vec{B}_1 - B_0 B_{1||} \vec{j}_0 + B_0 j_{1||} \vec{B}_0 - B_0^2 \vec{j}_1 \right] - \mu_0 \vec{j}_0 \times \vec{B}_0 \cdot \vec{j}_1
\]

\[
= \vec{B}_1 \cdot \nabla (\vec{j}_0 \cdot \vec{B}_0) - \vec{j}_0 \cdot \nabla (B_0 B_{1||}) + \vec{B}_0 \cdot \nabla (B_0 j_{1||}) - j_{1r} (B_0^2)' - \mu_0 p_{0j_{1r}} ;
\]

\[
\vec{j}_0 \cdot \nabla (B_0 B_{1||}) \simeq B_0 \vec{j}_0 \cdot \nabla B_{1||} = -\frac{B_0}{\mu_0} \left[ \frac{im}{r} B'_0 + i k \left( \frac{r B_{0\theta}}{r} \right)' \right] B_{1||} ;
\]

\[
\vec{B}_0 \cdot \nabla (B_0 \vec{j}_{1||}) \simeq i B_0 \frac{F}{r} j_{1||} = \frac{i B_0 F}{r} \left( \nabla \times \vec{B}_1 \right)_{1||} \simeq \frac{i F}{\tau \mu_0} \left( -B_{0\theta} B'_{1z} + B_{0z} B'_{1\theta} \right) ;
\]

\[
\mu_0 j_{1r} = (\nabla \times \vec{B}_1)_{1r} = \frac{im}{r} B_{1z} + i k B_{1\theta} .
\]

Finally, the annihilated form of Equation (3.26) is:

\[
\vec{B}_0 \cdot \nabla \times (3.26) \implies q^2 \rho_0 (\xi'_r B_{0z} - \xi'_z B_{0\theta}) + q \rho_0 v_0 (B_{0z} \xi''_\theta - B_{0\theta} \xi''_r)
\]

\[
= B_{1r} (\vec{j}_0 \cdot \vec{B}_0)' + \frac{B_0}{\mu_0} \left[ \frac{im}{r} B'_0 + i k \left( \frac{r B_{0\theta}}{r} \right)' \right] B_{1||} + \frac{i F}{\tau \mu_0} (B_{0z} B'_{1\theta} - B_{0\theta} B'_{1z}) - \left( \frac{im}{r} B_{1z} + i k B_{1\theta} \right) \left( p_0 + \frac{B_0^2}{\mu_0} \right)' .
\]

Now it is important to rewrite all these equations in terms of the radial, parallel,
and perpendicular quantities, consistently with the choice of the coordinate system:

\[ \tilde{f}_1 = f_{1r} \hat{e}_r + f_{1\theta} \hat{e}_\theta + f_{1z} \hat{e}_z = f_{1r} \hat{e}_r + f_{1\theta} \hat{e}_\theta + f_{1z} \hat{e}_z \]

where

\[
\begin{align*}
& f_{1||} = \frac{f_{1\theta} B_{0\theta} + f_{1z} B_{0z}}{B_0} ; \\
& f_{1\perp} = \frac{f_{1z} B_{0\theta} - f_{1\theta} B_{0z}}{B_0} ; \\
& f_{1\theta} = \frac{f_{1||} B_{0\theta} - f_{1\perp} B_{0z}}{B_0} ; \\
& f_{1z} = \frac{f_{1||} B_{0\theta} + f_{1\perp} B_{0z}}{B_0} ;
\end{align*}
\]

(3.26) \( \cdot \tilde{B}_0 \, \Rightarrow \, \xi'_\theta B_{0\theta} + \xi'_z B_{0z} \simeq \left[ \xi'_{||} B_{0\theta}^2 - \xi'_{\perp} B_{0\theta} B_{0z} + \xi'_{\perp} B_{0\theta} B_{0z} + \xi'_{||} B_{0z}^2 \right] \frac{1}{B_0} = \xi'_{||} B_0 ;

(3.26) \cdot \hat{e}_\perp \, \Rightarrow \, \frac{m}{r} B_{1z} + k B_{1\theta} = \frac{m}{r} B_{1||} B_{0\theta} + B_{1\perp} B_{0z} + k \frac{B_{1||} B_{0\theta} - B_{1\perp} B_{0z}}{B_0} =
\left( \frac{m}{r} B_{0\theta} - k B_{0z} \right) \frac{B_{1||}}{B_0} + \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \frac{B_{1\perp}}{B_0} \simeq \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \frac{B_{1||}}{B_0} ;

(3.27) \cdot \tilde{B}_0 \, \Rightarrow \, B_{0\theta} B_{1\theta}'' + B_{0z} B_{1z}'' \simeq \left[ B_{0\theta}^2 B_{1||}'' - B_{0z} B_{1\perp}'' B_{0\theta} + B_{1\perp}'' B_{0\theta} B_{0z} + B_{1||}'' B_{0z}^2 \right] \frac{1}{B_0} = B_0 B_{1||}'' ;

\tilde{B}_0 \cdot \nabla \times (3.26) \Rightarrow \xi'_{\theta} B_{0z} - \xi'_z B_{0\theta} \simeq \left[ \xi'_{||} B_{0\theta} B_{0z} - \xi'_{\perp} B_{0\theta}^2 - \xi'_{\perp} B_{0z}^2 - \xi'_{||} B_{0\theta} B_{0z} \right] \frac{1}{B_0} = -\xi'_{\perp} B_0 ;

\frac{m}{r} B_{0z}' + k \left( \frac{r B_{0\theta}}{r} \right)' \simeq \frac{k}{B_{0\theta}} \left[ B_{0z} B_{0z}' + \left( \frac{r B_{0\theta}}{r} \right)' \right] = -\mu_0 k \frac{\rho'}{B_{0\theta}} ;

B_{0z} B_{1\theta}' - B_{0\theta} B_{1z}' \simeq \left[ B_{0z} B_{0\theta} B_{1||}' - B_{0z}^2 B_{1\perp}' - B_{0\theta} B_{1\perp}' - B_{0\theta} B_{0z} B_{1||}' \right] \frac{1}{B_0}
\simeq -B_{1\perp}' B_0 ;

(3.29) \Rightarrow \frac{m}{r} B_{1\theta} - k B_{1z} = \frac{m}{r} B_{1||} B_{0\theta} - B_{1\perp} B_{0z} - k \frac{B_{1\perp} B_{0\theta} + B_{1||} B_{0z}}{B_0} =
\left( \frac{m}{r} B_{0\theta} - k B_{0z} \right) \frac{B_{1||}}{B_0} - \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \frac{B_{1\perp}}{B_0} \simeq \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \frac{B_{1||}}{B_0} ;

(\nabla \cdot \xi = 0) \Rightarrow \frac{m}{r} \xi_{\theta} - k \xi_z = -\left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \frac{\xi_z}{B_0} ;

Finally, the full set of equations for the unknowns \( B_{1r}^{(4)}, B_{1r}^{(6)}, B_{1\perp}^{(4)}, B_{1\perp}^{(6)}, \xi_{r}^{(2)}, \xi_{r}^{(4)}, \xi_{\perp}^{(2)}, \xi_{\perp}^{(4)}, \xi_{||}^{(0)}, \)
\[ \nabla \cdot \hat{\xi}^{(2)} + \rho_{1}^{(2)} + p_{1}^{(4)}, \text{ becomes:} \]

\[ B_{1r}^{(4)'} = 0 \quad (3.54) \]
\[ B_{1r}^{(6)'} = \frac{i B_{1}^{(4)}_{\perp}}{B_0} \left( \frac{m}{r} B_{0x} + kB_{0\theta} \right) \quad (3.55) \]
\[ \xi_{r}^{(2)'} = \frac{i \xi_{r}^{(4)}}{B_0} \left( \frac{m}{r} B_{0z} + kB_{0\theta} \right) \quad (3.56) \]
\[ q \rho_{1}^{(2)} + q \rho_{0} \nabla \cdot \xi^{(2)} + v_{0} \rho_{1}^{(2)'} + q \xi_{r}^{(2)'} \rho_{0}' = 0 \quad (3.57) \]
\[ q^2 \rho_{0} B_{0} \xi_{r}^{(0)} + q \rho_{0} v_{0} B_{0} \xi_{r}^{(0)'} + \frac{i F}{r} p_{1}^{(4)} + B_{1r}^{(4)} p_{0}' = 0 \quad (3.58) \]
\[ p_{1}^{(4)} + \frac{B_{0} B_{1}^{(4)}_{\parallel}}{\mu_{0}} + \frac{i r (j_{0} \cdot \vec{B}_{0})}{m B_{0x} + kB_{0\theta}} B_{1r}^{(4)} = 0 \quad (3.59) \]
\[ \frac{\eta}{\mu_{0} q} B_{1}^{(4)n} + B_{0} \nabla \cdot \xi^{(2)} + \frac{i F}{r} \xi_{r}^{(0)} - \frac{\xi_{r}^{(2)}}{B_0} (\mu_{0} p_{0} + B_{0}^{2})' = 0 \quad (3.60) \]
\[ B_{1r}^{(4)} = \frac{\eta}{\mu_{0} q} B_{1}^{(6)n} + \frac{i F}{r} \xi_{r}^{(2)} \]  

\[ q \left( p_{1}^{(4)} - \frac{5 p_{0}}{3 \rho_{0}} \rho_{1}^{(2)} \right) + v_{0} \left( p_{1}^{(4)'} - \frac{5 p_{0}}{3 \rho_{0}} \rho_{1}^{(2)'} \right) + q \xi_{r}^{(2)} p_{0} \left( \frac{p_{1}^{(4)}}{p_{0} \rho_{0}} - \frac{5}{3} \right) \rho_{0}' = 0 \quad (3.62) \]

\[ -q^2 \rho_{0} \xi_{r}^{(4)'} B_{0} - q \rho_{0} v_{0} \xi_{r}^{(4)'} B_{0} = B_{1r}^{(4)} (j_{0} \cdot \vec{B}_{0})' - \frac{i q}{\mu_{0}} k B_{0} B_{1}^{(4)} + \]
\[ - \frac{i F}{r \rho_{0} q} B_{0} B_{1}^{(4)'} - \frac{i k}{\rho_{0} q} B_{0} B_{1}^{(4)} \left( p_{0} + \frac{B_{0}^{2}}{\mu_{0}} \right)' \quad (3.63) \]

Eliminating \( B_{1\perp} \) and \( \xi_{\perp} \) by using, respectively, Equations (3.55) and (3.56), it follows that Equation (3.63) becomes:

\[ v_{0} \xi_{r}^{(2)'''} + q \xi_{r}^{(2)''} = \frac{i F}{r \rho_{0} q \mu_{0}} B_{1r}^{(6)''} - \frac{i k}{\rho_{0} q B_{0\theta}} (j_{0} \cdot \vec{B}_{0})' B_{1r}^{(4)} - \frac{k^2 B_{0} (B_{0}^{2})'}{\rho_{0} q B_{0\theta} \mu_{0}} B_{1}^{(4)} \quad (3.64) \]

Eliminating \( v_{0} \rho_{1} + q \rho_{1} \) by using Equation (3.57), it follows that Equation (3.62) be-
comes:

\[ q p_1^{(4)} + v p_1^{(4)}' + \frac{5}{3} p_0 q \nabla \cdot \xi^{(2)} + q \rho_0 \xi^{(2)}_r = 0 \]  

(3.65)

Eliminating \( p_1^{(4)} \) and \( \nabla \cdot \xi^{(2)} \) by using respectively Equations (3.59) and (3.60), the set of equations is finally reduced to:

\[ B_{1r}^{(4)'} = 0 \]  

(3.66)

\[ \frac{\eta}{\rho_0 q} B_{1r}^{(4)''} = B_{1r}^{(4)} - \frac{i F}{r} \xi^{(2)}_r \]  

(3.67)

\[ v_0 \xi^{(2)''}_r + q \xi^{(2)''}_r = \frac{i F}{r \rho_0 q \mu_0} B_{1r}^{(4)''} - \frac{i k}{\rho_0 q \mu_0} (\vec{J}_0 \cdot \vec{B}_0) B_{1r}^{(4)} - \frac{k^2 B_0 (B_0^2)'}{\rho_0 q B_0^2 \mu_0} B_{1||} \]  

(3.68)

\[ 5 \frac{p_0}{3 B_0} \left[ \frac{\eta}{\mu_0 q} B_{1||}'' + \frac{i F}{r} \xi_{||} - \frac{\xi_{||}}{B_0^2} (B_0^2)'' \right] + \frac{p_0}{q \rho_0 B_0} \xi_{||} - \frac{i B_0}{k B_0} (\vec{J}_0 \cdot \vec{B}_0) B_{1r}^{(4)} = \left[ \frac{v_0}{q} B_{1||} + B_{1||} \right] \frac{B_0}{\mu_0} \]  

(3.69)

\[ v_0 \xi^{(2)'}_{||} + q \xi_{||} = - \frac{p_0}{q \rho_0 B_0} B_{1r}^{(4)} + \frac{i F}{q \rho_0 B_0 r} \left[ \frac{B_0}{\mu_0} B_{1||} + \frac{i B_0}{k B_0} (\vec{J}_0 \cdot \vec{B}_0) B_{1r}^{(4)} \right] \]  

(3.70)

which is a system of five equations in the five unknowns \( \xi^{(2)}_r, B_{1||}^{(4)}, B_{1r}^{(4)}, B_{1r}^{(6)}, \xi_{||}^{(0)} \).

The next step is to simplify the equations by introducing standard normalization and expanding \( F = F' x \):

\[ q \equiv \left[ \frac{\eta^3 F'}{F' F' F'} \right]^{\frac{1}{2}} Q, \quad x \equiv \left[ \frac{\rho_0 q}{F' F'} \right]^{\frac{1}{2}} x, \quad v_0 \equiv \frac{\eta}{\rho_0} C, \]  

(3.71)

\[ B_{1r}^{(4)} \equiv -i \left( \frac{F' F'}{\mu_0} \right)^{\frac{1}{2}} \Psi_0, \quad B_{1r}^{(6)} \equiv -i \left[ \frac{F' F'}{r^3 F'} \right]^{\frac{1}{2}} \Psi_2, \quad B_{1||} \equiv - \left[ \frac{F' F'}{r^4 F'} \right]^{\frac{1}{2}} \Gamma, \]  

\[ \xi_r \equiv \xi, \quad \xi_5 \equiv \frac{i}{r^3} \left[ \frac{F'}{r^3 F'} \right]^{\frac{1}{2}}, \quad J_r \equiv J + J, \quad D \equiv -2 \mu_0 \rho_0 \frac{k_{||}}{r^2}, \]  

\[ F' \equiv k_{||} B_{||z} \left( B_{||z} - \frac{1}{r} - \frac{B_{||z}}{B_{||z}} \right), \quad \xi_{||} \equiv i \left[ \frac{F' F'}{r^4 F'} \right]^{\frac{1}{2}} \Lambda, \quad \beta \equiv 3 \frac{\mu_0}{B_0}, \]  

\[ S \equiv \left( \frac{2k B_0}{F'} \right)^2 \]  

(3.71)
In terms of the new dimensionless quantities, the system of equations becomes:

\[ \psi' = 0 \]  
\[ \psi'' = q_0 + zq_0 \xi \]  
\[ Cq_0'' + q^2q_0'' = -\Gamma + z^2q_0\xi + qzq_0 - Jp_q - J\psi_0 \]  
\[ \Gamma'' = q_0 \left[ s - \frac{D}{\beta} \right] + \frac{e^6q}{\beta} \left( \Gamma + \frac{C}{Q}\Gamma' \right) - zq_0\Lambda - \frac{\ell_6q}{\beta} J\psi_0 \]  
\[ q^2\Lambda + QC\Lambda' = -z\Gamma - \frac{D}{\ell_6}\psi_0 + z\psi_0J \]  

(3.72)  
(3.73)  
(3.74)  
(3.75)  
(3.76)

where \( q_2 \) has been eliminated from Equation (3.74) via Equation (3.73). Notice that all the terms are of the same order: \( \beta \sim p_0 \sim D \sim p_0 \sim \epsilon^2 \sim \eta^2 \sim l^6 \). Eqs. (3.74)-(3.76) form a closed system of three equations in the three unknowns \( \xi, \Gamma, \Lambda \), where the quantities \( C, J_p, J, D, \beta, S \) are constants known from the equilibrium, \( \psi_0 \) is an arbitrary constant (see Equation (3.72)), \( z \) is the variable of differentiation, and \( Q \) is the eigenvalue. Once the solution of Eqs. (3.72)-(3.76) is found, it still has to match with the solution outside the resonant layer (see Section (3.3.3)). In particular, the solution for \( \psi \) inside the resistive layer for large \( z \) will have to match the corresponding solution for \( \psi \) outside the resistive layer for small \( x \): this matching condition will give the tearing-mode Dispersion Relation. In this connection, after introducing \( \hat{\Gamma} = \Gamma - J\psi_0 \), Eqs. (3.72)-(3.76) can be solved for large \( z \):

\[
\begin{align*}
-\hat{\Gamma} + z^2q_0\xi + qzq_0 - \ell T\psi_0 &= 0 \\
q_0\left[ s - \frac{D}{\beta} \right] + \frac{e^6q}{\beta} \hat{\Gamma} - zq_0\Lambda &= 0 \\
q^2\Lambda &= -z\hat{\Gamma} - \frac{D}{\ell_6}\psi_0
\end{align*}
\]  

(3.77)  
(3.78)  
(3.79)
where eliminating $\Lambda$ gives

\[
\begin{align*}
-\hat{\Gamma} + z^2Q\xi + Qz\Psi_0 - j_T\Psi_0 &= 0 & (3.80) \\
Q^2\xi \left[ S - \frac{D}{\beta} \right] + \hat{\Gamma} \left[ \frac{\ell^6}{\beta} Q^2 + z^2 \right] + z\frac{D}{\ell^6}\Psi_0 &= 0 & (3.81)
\end{align*}
\]

and eliminating $\hat{\Gamma}$

\[
z^2Q\Psi_0 - z^2j_T\Psi_0 + z\frac{D}{\ell^6}\Psi_0 + Q\xi \left\{ z^4 + Q \left[ S - \frac{D}{\beta} \right] \right\} = 0 & (3.82)
\]

The asymptotic form of the solution for large $z$ is easily obtained:

\[
\begin{align*}
\xi &= \Psi_0 \left[ -\frac{1}{z} + \frac{j_T}{Qz^2} - \frac{D}{\ell^6Qz^3} + O(z^{-4}) \right] \\
\hat{\Gamma} &= \Psi_0 \left[ -\frac{D}{\ell^6z} - Q^2 \left[ S - \frac{D}{\beta} \right] \left( -\frac{1}{z^3} + \frac{j_T}{Qz^4} + O(z^{-5}) \right) \right]
\end{align*}
\]

Substituting into Equation (3.73), the asymptotic equation for $\Psi$ for large $z$ is:

\[
\Psi''_z = \Psi_0 \left[ \frac{j_T}{z} - \frac{D}{\ell^6z^2} + O(z^{-3}) \right]
\]

which leads to

\[
\Psi_z = \Psi_0 \left[ j_T\ln |z| + \ln |z|\frac{D}{\ell^6} \right]
\]

By recalling that $x \equiv r - r_s \equiv \ell^6rz$ and $\frac{B_{\ell r}}{\Psi_0} = \frac{B_{\ell r}^{(6)}}{\ell^6\Psi_z}$, and by using Equation (3.72), the asymptotic solution for $\Psi$ can be written as follows (normalized to 1 at resonance):

\[
\Psi = 1 + Ax + j_T\frac{x}{r_s}\ln |x| + D\ln |x| & (3.83)
\]

Even though this expression is valid for both positive and negative $x$, a distinction has to be made between inner ($x < 0$) and outer ($x > 0$) solutions. The arbitrary integration constant $A$ is determined by matching Equation (3.83) with Equation (3.44)
on both sides of the resonance. (Note that $A$ is actually also a function of $Q$, the eigenvalue of the problem). Recall that in the T-M ordering, the pressure is small $\beta \sim D \sim \eta^2$, which is why the matching has to be done with the small pressure limit expression of Equation (3.42). Note that

$$x \equiv r - r_s, \quad a_1 \equiv D, \quad \frac{B_{1r}^{(4)}}{\Psi_0} = \frac{B_{1r}^{(6)}}{\ell^6 \Psi_2},$$

and

$$a_2 = \frac{-F''(r_s) \cdot r_s (m^2 + k^2 r_s^2) + 4k^2 m B_\theta(r_s) \cdot r_s - F'(r_s) (m^2 - k^2 r_s^2)}{F'(r_s) r_s (m^2 + k^2 r_s^2)}$$

$$= \frac{-F''(r_s) B^2 r_s + 4k B_z B_\theta^2 - F'(r_s) (B_z^2 - B_\theta^2)}{F'(r_s) r_s B^2}$$

$$= \frac{k}{B_\theta F'(r_s) r_s B^2} \left[ 4B_\theta^3 B_z - \mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0) r_s^2 B^2 - \mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0) [3B_z^2 - B_\theta^2] r_s + 2B_\theta B_z B^2 - \mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0) [B_z^2 - B_\theta^2] r_s + 2B_z B_\theta (B_z^2 - B_\theta^2) \right]$$

$$= \frac{-\mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0) k r_s}{F'(r_s) B_\theta} - \frac{2B_\theta \mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0) k}{F'(r_s) B^2}$$

$$= \frac{-J_p}{r} - \frac{J}{r} = -\frac{j_T}{r}$$

where the following relations have been used

$$F'(r_s) = kr B_z \left( \frac{B_\theta'}{B_\theta} - \frac{B_z'}{B_z} - \frac{1}{r} \right) \bigg|_{r_s} = kr B_z \left( \frac{\mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0)}{B_\theta B_z} - \frac{2}{r} \right).$$

$$F''(r_s) = kr B_z \left( \frac{B_\theta''}{B_\theta} - \frac{2B_z'}{r B_z} - \frac{B_z''}{B_z} \right) = kr B_z \left[ \frac{\mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0)'}{B_\theta B_z} - \frac{3B_z'}{r B_z} - \frac{B_\theta'}{r B_\theta} + \frac{1}{r^2} \right]$$

$$= \frac{B_z B_\theta'}{r} - B_z' B_\theta + \frac{B_\theta^2}{r} = \mu_0 \mathbf{p}' \quad \Rightarrow \quad \begin{cases} \frac{B_z B_\theta'}{r} - B_z' B_\theta = \mu_0 (\vec{\mathbf{j}}_0 \cdot \vec{\mathbf{E}}_0) \\ B_z B_\theta' + B_\theta B_z' + \frac{B_\theta^2}{r} = \mu_0 \mathbf{p}' \end{cases} \Rightarrow \begin{cases} \frac{B_z'}{B_\theta} = \frac{B_z}{B_\theta} \frac{\mu_0 \mathbf{j} \cdot \vec{\mathbf{B}}}{B_\theta^2} - \frac{1}{r} - \frac{\mu_0 \mathbf{p}'}{B_\theta^2} \\ \frac{B_\theta'}{B_z} = -\frac{B_z}{B_\theta} \frac{\mu_0 \mathbf{j} \cdot \vec{\mathbf{B}}}{B_\theta^2} - \frac{\mu_0 \mathbf{p}'}{B_\theta^2} \end{cases}$$

The matching can now easily be carried out: because of the integration constant

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$A$, it is always possible to match the solution on one side of the resonance, say $x < 0$, so that

$$\Psi_L = 1 - \Delta_L x + j_T \frac{x}{T_s} \ln |x| + D \ln |x| \quad x \to -\infty,$$

Next, rewriting the solution for $x > 0$ as

$$\Psi_R = 1 - \Delta_L x + \Delta x + j_T \frac{x}{T_s} \ln |x| + D \ln |x| \quad x \to +\infty,$$

where

$$\Delta \equiv \lim_{x \to \infty} \frac{\Psi'|_x - \Psi'|_{-x}}{\Psi|_0}$$

is the jump in the logarithmic derivative of $\Psi$ at the resonance, it is straightforward to show that

$$\Delta(Q) \equiv \lim_{x \to \infty} \frac{B_{1r}'|_x - B_{1r}'|_{-x}}{B_{1r}|_0} = \lim_{x \to \infty} \frac{\Psi'|_x - \Psi'|_{-x}}{\Psi|_0}$$

$$= \lim_{x \to \infty} \frac{(\Psi_0 + \ell^6 \Psi_2)'|_x - (\Psi_0 + \ell^6 \Psi_2)'|_{-x}}{\Psi_0}$$

$$= \lim_{x \to \infty} \ell^6 \frac{\Psi_2'|_x - \Psi_2'|_{-x}}{\Psi_0} \quad \text{(3.86)}$$

$$= \lim_{x \to \infty} \ell^6 \int_{-x}^{x} \frac{\Psi''(\tilde{x})}{\Psi_0} d\tilde{x} = \lim_{x \to \infty} \int_{-x}^{x} Q \left[ 1 + \frac{\tilde{z}\xi}{\Psi_0} \right] d\tilde{z}$$

In order to have matching on the outer side as well ($x > 0$), the following condition then has to be satisfied, involving the parameter $\Delta'$:

$$\Delta(Q) = \Delta_R + \Delta_L \equiv \Delta' \quad \text{(3.87)}$$

where Equation (3.73) has been used. This leads to the dispersion relation for T-M:

$$\Delta' = \frac{Q}{r_s} \lim_{z \to \infty} \int_{-z}^{z} \left[ 1 + \frac{\tilde{z}\xi}{\Psi_0} \right] d\tilde{z}.$$  \quad \text{(3.88)}$$

Note that $\Delta'$ is a parameter that arises from the solution outside the resonant layer, and as such it is a known quantity once the plasma equilibrium profiles are given. The dispersion relation (3.88) will then give information about the eigenvalue.
Q, the normalized growth rate, and consequently on the stability of the equilibrium profiles. In the following section, the tearing-mode dispersion relation will be solved in the presence of non-zero pressure and resistive diffusive flow, as was done in Ref. [36].

### 3.4 The Equilibrium Diffusion Velocity in an RFP

The equilibrium resistive radial diffusion velocity \(v_0\), introduced in Section 3.3.1, satisfies the equation

\[
\nabla \times (\vec{v}_0 \times \vec{B}_0) = -\frac{\eta}{\mu_0} \nabla^2 \vec{B}_0 \quad , \quad \vec{v}_0 = v_0 \hat{e}_r
\]

(3.89)

In this paragraph, \(v_0\) will be determined from the equilibrium fields, and related to the plasma parameters in order to better quantify its importance in an RFP. Equation (3.89) leads to the following two scalar relations for \(v_0\) and \(v'_0\)

\[
\begin{align*}
\left( v_0 B_{0\theta} \right)' &= \frac{\eta}{\mu_0} \left[ \frac{(r B'_{0\theta})'}{r} - \frac{B_{0\theta}}{r^2} \right] \\
\left( \frac{r v_0 B_{0z}}{r} \right)' &= \frac{\eta}{\mu_0} \left( \frac{r B'_{0z}}{r} \right)
\end{align*}
\]

(3.90)

Eliminating \(v'_0\) yields

\[
v_0 = \frac{\eta}{\mu_0} \frac{B''_{0\theta} B_{0z} + B_{0z} B'_{0\theta} - B_{0\theta} B'_{0z} - B_{0\theta} B''_{0z} - B'_{0\theta} B_{0z}}{B_{0z} B'_{0\theta} - B'_{0z} B_{0\theta} - B_{0\theta} B''_{0z}}
\]

\[
= \frac{\eta}{\mu_0} \left[ \left( j_0 \cdot \vec{B}_0 \right)' + \left( j_0 \cdot \vec{B}_0 \right) \frac{2 B_{0\theta}^2}{r B_0^2} + \frac{2 B_{0\theta} B_{0z} p'_0}{r B_0^2} \right]
\]

(3.91)

The normalized \(v_0\) (i.e. \(C\)) can then be rewritten in terms of the dimensionless parameters \(J_p, J, D\) introduced in Equation (3.71)

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\[ C = J_\rho + J - D \frac{4B_{0z}^2B_{0\theta}^2}{B_0^4 \left[ J - 2 \frac{B_{0\theta}^2}{B_0^2} \right]} \]  

(3.92)

where the following relation has been used

\[ F'' = \frac{kr}{B_{0\theta}} \left( \mu_0 \bar{\gamma}_0 \cdot \bar{B}_0 - 2 \frac{B_{0\theta}B_{0z}}{r} \right) = \frac{4kB_{0z}^2B_{0\theta}^2}{B_0^4} \frac{J - 2 \frac{B_{0\theta}^2}{B_0^2}}{J - 2 \frac{B_{0\theta}^2}{B_0^2}} \]  

(3.93)

In order to quantify how influential the resistive diffusive flow is in an RFP, it is helpful to rewrite \( C \) in terms of plasma parameters which are more characteristic of such a configuration:

\[ \mu = \frac{\mu_0 (\bar{\gamma}_0 \cdot \bar{B}_0)}{B_0^2}, \quad \text{the } \mu \text{ parameter from Taylor's relaxation theory (3.94)} \]

\[ p_N = \frac{\mu_0 p_0'}{B_0^2}, \quad \text{related to plasma } \beta. \]

By using

\[ \mu' = \mu_0 \frac{(\bar{j}_0 \cdot \bar{B}_0)'}{B_0^2} + \mu_0 \frac{\bar{j}_0 \cdot \bar{B}_0}{B_0^2} \frac{2}{B_0^2} \left( \mu_0 p_0' + \frac{B_{0\theta}^2}{r} \right) \]  

(3.95)

then Equation (3.91) can be rewritten as follows:

\[ C = r \left[ \frac{\mu' - 2\mu p_N + 2 \frac{B_{0\theta}B_{0z}}{r B_0^2} p_N}{\mu - 2 \frac{B_{0\theta}B_{0z}}{r B_0^2}} \right] = r \left[ \frac{\mu' - \mu p_N}{\mu - 2 \frac{B_{0\theta}B_{0z}}{r B_0^2}} - p_N \right] \]  

(3.96)

The normalized diffusive flow parameter \( C \) is basically given by two terms, one proportional to the plasma pressure gradient, and the other proportional to \( \mu' \). Note that in an RFP experiment, \( \mu \) is essentially constant over all the plasma core region, \( (\mu \text{ is exactly constant in Taylor's Relaxation Theory - the Bessel Function Model}) \), implying that \( \mu' \) is very small everywhere but near the edge, where the current drops
drastically. Also the other term, which is related to the plasma $\beta$ parameter, is in general very small. It is also interesting to note that even though $\mu'$ is very small in the core region, it is not zero on axis: in fact, the term $\frac{\mu'}{\mu - 2 \frac{B_{0a} B_{0x}}{r B_0^2}}$ is finite for $r \to 0$. A typical curve of $C$ versus the plasma radius for a typical RFP equilibrium is plotted in Fig. 3-1, showing that it is indeed very small in the core region, and large (of the order of unity or larger) only at the edge. Overall, small values of $C$ will strongly limit the impact of the diffusive flow on the tearing mode dispersion relation in an RFP (as is shown later). This limitation does not apply for example to a tokamak-type of reactor, where $\mu'$ is non-zero, and thus the diffusive flow has a non-negligible effect through the entire plasma region (see [35] and [36]).

It is possible to solve Equations (3.74)-(3.76) analytically for a small $C$ and a non-zero plasma pressure. In fact, it is easy to show that Equations (3.74)-(3.76) reduce to the system analytically solved by Coppi et al. in [33], in the limit of $C \to 0$,
as expected. After rescaling the variables for the sake of simplicity:

\[
\Gamma - J\Psi_0 \equiv \hat{\Gamma} = Q^{\frac{3}{4}} M, \quad \xi \equiv Q^{-\frac{1}{4}} R, \quad C \equiv Q^{\frac{1}{4}} \lambda, \quad z \equiv Q^{\frac{1}{4}} y, \quad Q^{\frac{3}{4}} j^T, \quad \beta \equiv Q^{\frac{1}{4}} Y, \quad D \equiv Q^{\frac{3}{4}} \delta, \quad \Lambda = Q^{-\frac{1}{4}} N
\]  

(3.97)

the system becomes

\[
\begin{aligned}
\lambda R'''' + R'' & = -M + y^2 R + y\Psi_0 - j^T\Psi_0 \\
M'' &= R \left[ S - \frac{\delta}{Y} \right] + \frac{M + \lambda M''}{Y} - yN \\
N + \lambda N' &= -yM - \delta\Psi_0
\end{aligned}
\]

(3.98)

where \( \Psi_0 \) is the normalization constant (from Equation (3.72)), providing the magnitude of the unknowns \( R, M, \) and \( N \). Thus, Eqs. (3.74)-(3.76) can be solved by perturbation analysis, \( \lambda \) being the small parameter \( f = f_0 + \lambda f_1 + O(\lambda^2) \).

To the 0-th order:

\[
\begin{aligned}
R_0'' - y^2 R_0 + M_0 - y\Psi_0 + j^T\Psi_0 &= 0 \\
M_0'' - \frac{M_0}{Y} + yN_0 - R_0 \left[ S - \frac{\delta}{Y} \right] &= 0 \\
N_0 &= -yM_0 - \delta\Psi_0 \quad \implies \quad (3.99)
\end{aligned}
\]

\[
\begin{aligned}
R_0'' - y^2 R_0 + M_0 - y\Psi_0 &= -j^T\Psi_0 \\
M_0'' - \frac{M_0}{Y} - y^2 M_0 - R_0 \left[ S - \frac{\delta}{Y} \right] &= \delta\Psi_0 y \quad \implies \quad (3.100)
\end{aligned}
\]

The solution is found analytically in terms of Hermite Polynomials \( \hat{H}_n(y) \) (see [33]).

\[
\begin{aligned}
R_0 &= \exp \left( -\frac{y^2}{2} \right) \sum_n a_n \hat{H}_n(y) \\
M_0 &= \exp \left( -\frac{y^2}{2} \right) \sum_n b_n \hat{H}_n(y)
\end{aligned}
\]

(3.101)
By using properties of Hermite Polynomials ([27], [39]):

\[
R_0' = -y R_0 + \exp \left( -\frac{y^2}{2} \right) \sum_n a_n \hat{H}_n'(y),
\]

\[
R_0'' = -R_0 + y^2 R_0 - 2y \exp \left( -\frac{y^2}{2} \right) \sum_n a_n \hat{H}_n'(y) + \exp \left( -\frac{y^2}{2} \right) \sum_n a_n \hat{H}_n''(y)
\]

(3.102)

with analogous expressions for \( M_0 \). Equation (3.100) becomes

\[
\sum_n \left[ a_n (1 + 2n) - b_n \right] \hat{H}_n = \frac{\sqrt{2} \Psi_0}{4^n \Gamma(n + 1)} \left( \hat{H}_{2n+1}^{*} - \hat{H}_{2n} \right)
\]

\[
\sum_n \left[ b_n \left( 1 + 2n + \frac{1}{Y} \right) + a_n \left( S - \frac{\delta}{Y} \right) \right] \hat{H}_n = -\frac{\sqrt{2} \Psi_0 \delta}{4^n \Gamma(n + 1)} \hat{H}_{2n+1}
\]

(3.103)

where the following identities have been used:

\[
\hat{H}_n'' - 2y \hat{H}_n' + 2n \hat{H}_n = 0, \quad 1 = \sqrt{2} \sum_n \frac{\hat{H}_{2n}(y)}{4^n \Gamma(n + 1)}, \quad y = \sqrt{2} \sum_n \frac{\hat{H}_{2n+1}(y)}{4^n \Gamma(n + 1)}
\]

(3.104)

Due to the orthogonality of \( \{ \hat{H}_n \} \), it is possible to solve for the coefficients \( a_n, b_n \).

If \( n \) is even \( (\Rightarrow n = 2t) \)

\[
a_{2t} (1 + 4t) - b_{2t} = \frac{\sqrt{2} \Psi_0}{4^t \Gamma(t + 1)} j_T^*
\]

\[
b_{2t} \left( 1 + 4t + \frac{1}{Y} \right) + a_{2t} \left[ S - \frac{\delta}{Y} \right] = 0
\]

whence

(3.105)

\[
a_{2t} = \frac{\sqrt{2} \Psi_0 j_T^* \left( 1 + 4t + \frac{1}{Y} \right)}{4^t \Gamma(t + 1) \left[ (1 + 4t + \frac{1}{Y}) (1 + 4t) + \left( S - \frac{\delta}{Y} \right) \right]}
\]

\[
b_{2t} = -\frac{\sqrt{2} \Psi_0 j_T^* \left( S - \frac{\delta}{Y} \right)}{4^t \Gamma(t + 1) \left[ (1 + 4t + \frac{1}{Y}) (1 + 4t) + \left( S - \frac{\delta}{Y} \right) \right]}
\]

(3.106)
If \( n \) is odd \((\Rightarrow n = 2t + 1)\)

\[
\begin{align*}
  a_{2t+1}(3 + 4t) - b_{2t+1} &= -\frac{\sqrt{2}\psi_0}{4t\Gamma(t+1)} \\
  b_{2t+1}\left(3 + 4t + \frac{1}{Y}\right) + a_{2t+1}\left[S - \frac{\delta}{Y}\right] &= -\frac{\sqrt{2}\psi_0\delta}{4t\Gamma(t+1)} \\
  a_{2t+1} &= -\frac{\sqrt{2}\psi_0}{4t\Gamma(t+1)} \left[\left(3 + 4t + \frac{1}{Y}\right) - \left(S - \frac{\delta}{Y}\right)\right] \\
  b_{2t+1} &= -\frac{\sqrt{2}\psi_0}{4t\Gamma(t+1)} \left[\left(3 + 4t + \frac{1}{Y}\right) - \left(S - \frac{\delta}{Y}\right)\right]
\end{align*}
\]

whence (3.107)

The general solution of Equation (3.100) can then be written as

\[
\begin{align*}
  R_0 &= R_{0,\text{even}} + R_{0,\text{odd}} = \exp\left(-\frac{y^2}{2}\right) \sum_t \left[a_{2t}\hat{H}_{2t} + a_{2t+1}\hat{H}_{2t+1}\right] \\
  M_0 &= M_{0,\text{even}} + M_{0,\text{odd}} = \exp\left(-\frac{y^2}{2}\right) \sum_t \left[b_{2t}\hat{H}_{2t} + b_{2t+1}\hat{H}_{2t+1}\right]
\end{align*}
\]

(3.109)

To the 1-st order:

\[
\begin{align*}
  R''_1 - y^2 R_1 + M_1 &= -R''_0 \\
  M''_1 - \frac{M'_1}{Y} + yN_1 - R_1\left[S - \frac{\delta}{Y}\right] &= \frac{M'_0}{Y} \\
  N_1 + yM_1 &= -N'_0 = M_0 + yM'_0
\end{align*}
\]

(3.110)

\[
\begin{align*}
  R''_1 - y^2 R_1 + M_1 &= M'_0 - 2yR_0 - y^2 R'_0 - \psi_0 \\
  M''_1 - \frac{M'_1}{Y} - y^2 M_1 - R_1\left[S - \frac{\delta}{Y}\right] &= M'_0 \left(\frac{1}{Y} - y^2\right) - yM_0
\end{align*}
\]

(3.111)

A similar treatment will give the analytic solution in terms of \(\hat{H}_n(y)\)

\[
\begin{align*}
  R_1 &= \exp\left(-\frac{y^2}{2}\right) \sum_n c_n \hat{H}_n(y) \\
  M_1 &= \exp\left(-\frac{y^2}{2}\right) \sum_n d_n \hat{H}_n(y)
\end{align*}
\]

(3.112)
Using the following expressions:

\[
R'_0 = \exp \left( -\frac{y^2}{2} \right) \sum_t \left\{ \hat{H}_{2t} \left[ (2t + 1) a_{2t+1} - \frac{a_{2t-1}}{2} \right] + \hat{H}_{2t+1} \left[ 2(t+1)a_{2t+2} - \frac{a_{2t}}{2} \right] \right\}
\]

\[
yR_0 = \exp \left( -\frac{y^2}{2} \right) \sum_t \left\{ \hat{H}_{2t} \left[ (2t + 1) a_{2t+1} + \frac{a_{2t-1}}{2} \right] + \hat{H}_{2t+1} \left[ 2(t+1)a_{2t+2} + \frac{a_{2t}}{2} \right] \right\}
\]

\[
yR'_0 = \exp \left( -\frac{y^2}{2} \right) \sum_t \left\{ \hat{H}_{2t} \left[ 2(t+1)(2t+1)a_{2t+2} - \frac{a_{2t}}{2} - \frac{a_{2t-2}}{4} \right] + \right.
\]
\[
+ \hat{H}_{2t+1} \left[ (2t+3)2(t+1)a_{2t+3} - \frac{a_{2t+1}}{2} - \frac{a_{2t-1}}{4} \right] \right\} \quad (3.113)
\]

\[
y^2 R'_0 = \exp \left( -\frac{y^2}{2} \right) \sum_t \left\{ \hat{H}_{2t} \left[ (2t + 3)2(t+1)(2t+1)a_{2t+3} + (2t+1)(2t-1)\frac{a_{2t+1}}{2} + ight. \right.
\]
\[
- 2(t+1)\frac{a_{2t-1}}{4} - \frac{a_{2t-2}}{8} \right\] + \hat{H}_{2t+1} \left[ 4(t+2)(2t+3)(t+1)a_{2t+4} + \right. \]
\[
+ 2t(t+1) a_{2t+2} - (2t+3)\frac{a_{2t+3}}{4} - \frac{a_{2t-2}}{8} \right\}
\]

then the coefficients \( c_n, d_n \) can be determined as before:

\[
\begin{align*}
\left\{ \begin{array}{l}
 c_{2t}(1+4t) - d_{2t} = -(2t+1)b_{2t+1} + \frac{b_{2t-1}}{2} + a_{2t+3}(2t+3)(2t+2)(2t+1) + + a_{2t+1}(2t+1)(2t+3) - a_{2t-1}(t-1)^2 - \frac{a_{2t-3}}{8} + \frac{\sqrt{2\Psi_0}}{4\Gamma(t+1)} \\
 d_{2t} \left( 1 + 4t + \frac{1}{Y} \right) + c_{2t} \left( S - \frac{\delta}{Y} \right) = b_{2t+3}(2t+3)(2t+2)(2t+1) + + b_{2t+1}(t+\frac{1}{2} - \frac{1}{Y}) + + \frac{b_{2t-1}}{2} \left( \frac{1}{Y} - t \right) - \frac{b_{2t-3}}{8}
\end{array} \right.
\end{align*}
\]

\[
(3.114)
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
 c_{2t+1}(3+4t) - d_{2t+1} = -2(t+1)b_{2t+2} + \frac{b_{2t}}{2} + a_{2t+4}(2t+2)(2t+3)(t+1) + + a_{2t+2}(2t^2 + 6t + 4) - a_{2t}(\frac{2t-1}{4}) - \frac{a_{2t-2}}{8} \equiv F_{1t}(\{a_n, b_n\}) \\
 d_{2t+1} \left( 3 + 4t + \frac{1}{Y} \right) + c_{2t+1} \left( S - \frac{\delta}{Y} \right) = b_{2t+4}(t+2)(2t+3)(t+1) + + b_{2t+2}(t+1)(t+\frac{1}{2} - \frac{1}{Y}) + + \frac{b_{2t}}{2} \left( \frac{1}{Y} - t - \frac{1}{2} \right) - \frac{b_{2t-2}}{8} \equiv F_{2t}(\{a_n, b_n\})
\end{array} \right.
\end{align*}
\]

\[
(3.115)
\]

Focus is put on the odd solution, because it is the only one that enters in the calcu-
lation of $\Delta'$. It is easy to show that

$$c_{2t+1} = \frac{(4t + 3 + \frac{1}{Y}) F_{1t}(\{a_n, b_n\}) F_{2t}(\{a_n, b_n\})}{(4t + 3 + \frac{1}{Y})(4t + 3) + \left[S - \frac{\delta}{Y}\right]} ,$$

$$d_{2t+1} = \frac{F_{2t}(\{a_n, b_n\}) - F_{1t}(\{a_n, b_n\}) \cdot \left[S - \frac{\delta}{Y}\right]}{(4t + 3 + \frac{1}{Y})(4t + 3) + \left[S - \frac{\delta}{Y}\right]}$$

(3.116)

where

$$F_{1t}(\{a_n, b_n\}) \equiv \frac{\sqrt{2} \Psi_0 j_T^2}{4 \Gamma(t + 1)} \left\{ \frac{-\frac{t}{2} (4t - 3 + \frac{1}{Y})}{(4t - 3 + \frac{1}{Y})(4t - 3) + \left[S - \frac{\delta}{Y}\right]} + \frac{3(t + \frac{1}{2})(4t + 1 + \frac{1}{Y})}{(4t + 1 + \frac{1}{Y})(4t + 1) + \left[S - \frac{\delta}{Y}\right]} + \frac{3(t + 1)(4t + 5 + \frac{1}{Y})}{(4t + 5 + \frac{1}{Y})(4t + 5) + \left[S - \frac{\delta}{Y}\right]} - \frac{(\frac{t}{2} + \frac{3}{4})(4t + 9 + \frac{1}{Y})}{(4t + 9 + \frac{1}{Y})(4t + 9) + \left[S - \frac{\delta}{Y}\right]} \right\}$$

(3.117)

$$F_{2t}(\{a_n, b_n\}) \equiv \frac{\sqrt{2} \Psi_0 j_T^2 \left[S - \frac{\delta}{Y}\right]}{4 \Gamma(t + 1)} \left\{ \frac{\frac{t}{2}}{(4t - 3 + \frac{1}{Y})(4t - 3) + \left[S - \frac{\delta}{Y}\right]} + \frac{\frac{t}{2} + \frac{1}{4} - \frac{1}{2Y}}{(4t + 1 + \frac{1}{Y})(4t + 1) + \left[S - \frac{\delta}{Y}\right]} + \frac{\frac{t}{2} + \frac{1}{4} - \frac{1}{2Y}}{(4t + 5 + \frac{1}{Y})(4t + 5) + \left[S - \frac{\delta}{Y}\right]} - \frac{(\frac{t}{2} + \frac{3}{4})}{(4t + 9 + \frac{1}{Y})(4t + 9) + \left[S - \frac{\delta}{Y}\right]} \right\}$$
In particular, after some algebra from Equation (3.116)

\[
c_{2t+1} = \frac{\sqrt{2} \Psi_{0,j}^*}{4 T (t + 1)} \left\{ -\frac{3}{4} - \frac{t (3 + \frac{1}{2} \lambda)}{(4t - 3 + \frac{1}{2}) (4t - 3) + [S - \frac{\delta}{\lambda}]} + \frac{3 \left( \frac{1}{\lambda} - 2 \right) (t + 1) (4t + 5 + \frac{1}{\lambda}) - \left( \frac{3}{2} (t + 1) + \frac{1}{2} \lambda \right) [S - \frac{\delta}{\lambda}]}{(4t + 5 + \frac{1}{\lambda}) (4t + 5) + [S - \frac{\delta}{\lambda}]} + \left( \frac{t}{2} + \frac{3}{4} \right) \left( \frac{1}{\lambda} - 6 \right) (4t + 9 + \frac{1}{\lambda}) - 2 \left[ S - \frac{\delta}{\lambda} \right] \right\}
\]

(3.118)

At this point, it is possible to derive the dispersion relation for T-M, by rewriting

Equation (3.88) in terms of the newly rescaled variable in Equation (3.97):

\[
\Delta' = \frac{Q^3}{r_s} \lim_{y \to \infty} \int_{-y}^{y} \left( 1 + \frac{y[R_0 + \lambda R_1]}{\Psi_0} \right) dy \\
= \frac{Q^3}{r_s} \int_{-\infty}^{\infty} \left( 1 + \frac{y[R_{0,odd} + \lambda R_{1,odd}]}{\Psi_0} \right) dy \\
= \Delta_0' + \Delta_1'
\]

(3.119)

where \( \Delta_0' \) and \( \Delta_1' \) are respectively the 0-th order and the 1-st order contribution of the diffusive flow to \( \Delta' \). Carrying out the algebra,

\[
\Delta_0' = \frac{Q^3}{r_s} \int_{-\infty}^{\infty} \left[ 1 + \frac{y R_{0,odd}}{\Psi_0} \right] \frac{\Psi_0 H_{2t+1}(\hat{y})}{2 \hat{y}} \left( 1 - \frac{2(2t + 1)(\frac{\delta}{\lambda} + 4t + 3 + \frac{1}{\lambda})}{(4t + 3 + \frac{1}{\lambda}) (4t + 3) + [S - \frac{\delta}{\lambda}]} \right) dy \\
= \frac{2 Q^3}{r_s} \sum_{t=0}^{\infty} \frac{4t + 3 + \frac{1}{\lambda} + S - \frac{\delta}{\lambda} - \frac{1}{4} (4t + 2) \delta}{4 \Gamma(t + 1)} \left( \frac{\Gamma(t + \frac{1}{2})}{\Gamma(t + 1)} \right) \\
= \frac{Q^3}{r_s} \sum_{t=0}^{\infty} \frac{\Gamma(t + \frac{1}{2})}{\Gamma(t + 1)} \left\{ \frac{s_1 - s_2}{s_3} + \frac{s_1 + s_2}{s_3} \right\}
\]

(3.120)
where the following relations have been used

\[
\int_{-\infty}^{+\infty} \hat{y}\hat{H}_n(\hat{y}) \exp \left(-\frac{\hat{y}^2}{2}\right) d\hat{y} = 2n \int_{-\infty}^{+\infty} \hat{H}_{n-1}(\hat{y}) \exp \left(-\frac{\hat{y}^2}{2}\right) d\hat{y} ;
\]

\[
1 = \sqrt{2} \exp \left(-\frac{\hat{y}^2}{2}\right) \sum_n \frac{\hat{H}_{2n}(\hat{y})}{4^n n! (n+1)} ;
\]

\[
\int_{-\infty}^{+\infty} \hat{H}_{2n}(\hat{y}) \exp \left(-\frac{\hat{y}^2}{2}\right) d\hat{y} = \sqrt{2\pi} \frac{(2n)!}{n!} , \quad \text{and}
\]

\[
\frac{\sqrt{\pi} (2n)!}{4^n n!} = \frac{\sqrt{\pi}}{2^n} [1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)] = \Gamma \left(n + \frac{1}{2}\right) ;
\]

and the following parameters have been defined

\[
s_1 = 1 - \delta , \quad s_2 = \frac{1 - \delta}{2Y} + \delta + S , \quad s_3 = \sqrt{\frac{1}{4Y^2} + \frac{\delta}{Y} - S} .
\]

The series can now be evaluated (see Refs. [26] and [40]) to obtain:

\[
\Delta'_0 = \frac{Qs}{4r_s} \left\{ \begin{array}{c}
1 - \delta + \frac{1 - \delta}{2Y} + \delta + S \\
\sqrt{\frac{1}{4Y^2} + \frac{\delta}{Y} - S} \\
\Gamma \left\{ \begin{array}{c}
\frac{1}{4} \left( 3 + \frac{1}{2Y} - \sqrt{\frac{1}{4Y^2} + \frac{\delta}{Y} - S} \right) \\
\frac{1}{4} \left( 5 + \frac{1}{2Y} - \sqrt{\frac{1}{4Y^2} + \frac{\delta}{Y} - S} \right)
\end{array} \right\} \\
+ \left( 1 - \delta - \frac{1 - \delta}{2Y} + \delta + S \right) \frac{1}{4} \left( 3 + \frac{1}{2Y} + \sqrt{\frac{1}{4Y^2} + \frac{\delta}{Y} - S} \right) \\
\Gamma \left\{ \begin{array}{c}
\frac{1}{4} \left( 5 + \frac{1}{2Y} + \sqrt{\frac{1}{4Y^2} + \frac{\delta}{Y} - S} \right)
\end{array} \right\}
\end{array} \right\}
\]

(3.122)

Notice that \(\Delta'_0\) from Equation (3.122) coincides with the expression evaluated in [33], when the resistive diffusive flow was omitted from the derivation. The correction term
arising from a non-zero diffusive flow is

\[
\Delta_1^* = \frac{\lambda Q_i^4}{r_s} \int_{-\infty}^{\infty} \tilde{y} \exp \left( -\frac{\tilde{y}^2}{2} \right) \sum_t \frac{C_{2t+1}}{\psi_0} \hat{H}_{2t+1}(\tilde{y}) d\tilde{y} =
\]

\[
= \frac{C \sqrt{2}}{r_s} \sum_t \frac{C_{2t+1}}{\psi_0^3} (2t+1) 2^{2t+1} t \left( t + \frac{1}{2} \right) =
\]

\[
= \frac{4C_{JT}^2}{Q_i^4 r_s} \sum_t \frac{\Gamma \left( t + \frac{1}{2} \right)}{\Gamma(t+1)} \left( \frac{2t+1}{4t+3 + \frac{1}{2}} \right) \left( \frac{4t+3 + \frac{1}{2}}{4t+3} \right) \left( S - \frac{8}{3} \right) \left( \frac{3}{4} \right)
\]

\[
- \frac{t \left( 3 + \frac{1}{2} \right) - t \left[ S - \frac{8}{3} \right]}{\left( 4t - 3 + \frac{1}{2} \right) \left( 4t - 3 \right) + \left[ S - \frac{8}{3} \right]} +
\]

\[
- \frac{3 \left( 2 + \frac{1}{2} \right) \left( t + \frac{1}{2} \right) \left( 4t + 1 + \frac{1}{2} \right) - \left( \frac{5}{2} \left( t + \frac{1}{2} \right) + \frac{1}{2} \right) \left[ S - \frac{8}{3} \right]}{\left( 4t + 1 + \frac{1}{2} \right) \left( 4t + 1 \right) + \left[ S - \frac{8}{3} \right]} +
\]

\[
- \frac{3 \left( \frac{1}{2} - 2 \right) \left( t + 1 \right) \left( 4t + 5 + \frac{1}{2} \right) - \left( \frac{3}{2} \left( t + 1 \right) + \frac{1}{2} \right) \left[ S - \frac{8}{3} \right]}{\left( 4t + 5 + \frac{1}{2} \right) \left( 4t + 5 \right) + \left[ S - \frac{8}{3} \right]} +
\]

\[
+ \left( \frac{t}{2} + \frac{3}{4} \right) \left( \frac{1}{2} - 6 \right) \left( 4t + 9 + \frac{1}{2} \right) - 2 \left[ S - \frac{8}{3} \right] \right) \right) \right) \right) \right) \right)
\]

Given the cumbersome algebra generated by this analytical approach, it was also decided to implement a numerical code that directly solves Equations (3.72)-(3.76), thus providing a numerical support and double-checking the analytical framework. Direct testing was done with the well-known analytic solution (see Ref. [33]) in the absence of flow \((C = 0)\), to compare with the simplified form of Equation (3.122) for the relevant range of plasma parameters \((D \lesssim 1)\)

\[
r_s \Delta_0' = 2\pi \frac{\Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} Q_i^4 \left( 1 - \frac{\pi D}{4Q_i^4} \right)
\]

(3.123)

In the same way, a simplified form for Equation (3.123) is found:

\[
r_s \Delta_1' = 0.78 \frac{C_{JT}}{Q_i^4}
\]

(3.124)

Finally, the tearing-mode dispersion relation (see Fig. 3-2) with first order corrections due to small resistive diffusive flow \(\left( \frac{C}{Q_i^{5/4}} << 1 \right)\) is
Figure 3-2: Dispersion Relation for the Strongly Unstable Case $\frac{C}{Q^{5/4}} << 1$

$$r_s \Delta' = 2\pi \Gamma \left(\frac{3}{4}\right) Q^{3/4} \left(1 - \frac{\pi D}{4Q^{3/2}}\right) + 0.78 \frac{C j_T}{Q^{3/4}} + O(C^2) \quad (3.125)$$

The appearance of the flow coefficient $C$ divided by $Q^{5/4}$ could possibly balance the destabilizing term $\frac{D}{Q^{1/4}}$ if Equation (3.125) would remain valid for larger values of $C$, or equivalently in the limit of small growth rate $Q \to 0$ when a possible condition for marginal stability is explored. Unfortunately, in that limit Equation (3.125) changes form completely, as it was investigated numerically. The dispersion relation for $\frac{C}{Q^{5/4}} >> 1$ and negligible pressure is found numerically to satisfy the formula

$$r_s \Delta' = \text{sgn}(C j_T) |j_T| \pi \tan \left(\frac{\pi}{10}\right) + 0.735 \frac{Q}{|C|^{2/5}} [3|C| - \text{sgn}(C j_T)|j_T|] \quad (3.126)$$

in agreement with Ref. [36] (see Fig. 3-3). This shows (as predicted), that the correction to the tearing-mode dispersion relation due to resistive flow, while important in a tokamak, is negligible for an RFP. The reason is that the flow term contribution,
Figure 3-3: Dispersion Relation for the Weakly Unstable Case $\frac{C}{Q^{5/4}} >> 1$

valid for both small and large $\frac{Q^{5/4}}{C}$, scales schematically as $\frac{A_1}{1 + \frac{Q^{5/4}}{C}}$, $A_1$ being a constant. Thus, for large negative $\Delta'$ and small $Q$ the flow term approaches a constant. The interchange term $\frac{D}{Q^{1/4}}$ dominates and the system remains unstable, the flow only making a change in the interchange growth rate.

In conclusion, the tearing-mode dispersion relation as found in the literature shows the cylindrical RFP to always be unstable to the resistive interchange mode ('g-modes'). Therefore no marginally stable state can be found in an RFP. Even adding the resistive diffusive flow, which has been shown by Taylor and Pollard to be stabilizing in a tokamak (see Ref. [36]) does not eliminate this instability, due to the numerically small flow parameter $C$ in an RFP. This is the impasse that faced all computational works in the past, trying to model an equilibrium state of marginally stable tearing modes in an RFP. In the next section, a new tearing-mode analysis is derived, based on the physical argument that thermal conduction cannot be neglected in the physics.
of the resistive layer; this treatment, suggested by Lutjens et al for a tokamak (see Ref. [37]), will be applied to the RFP configuration, resulting in the stabilization of the ‘g-mode’ and consequently, the possibility of finally being able to properly define a marginally stable state to tearing-modes in an RFP.

3.5 Lutjens’ Derivation

3.5.1 Introduction

As shown in the previous chapter, the tearing-mode dispersion relation for a cylindrical RFP as derived by CGJ ([33]) always gives an instability in the presence of a finite negative pressure gradient, regardless of its magnitude. From a mathematical point of view, this instability, known in the literature as the 'g-mode', arises from the pressure term which in practical cases has been shown to be equal to $-\frac{\pi}{4} \frac{D}{Q^{\frac{3}{2}}}$ (see Equation (3.123)). Since $D$ is always positive (see Equation (3.71)) in the RFP configurations of interest for plasma confinement, the CGJ tearing-mode dispersion relation predicts instability even as $\Delta' \to -\infty$. In other words, one cannot construct RFP equilibrium profiles which are stable (not even marginally) against tearing modes. This impasse has been observed in several RFP simulations carried out in the past (see Ref. ([41])), forcing one for instance to introduce small but non-zero thresholds for growth rates in order to find a “stable” RFP profiles in the presence of pressure. To the author’s knowledge, this impasse has for the very first time been overcome in the present work, thanks to the use of a recent paper (see Ref. [37]) in which the authors, Lutjens, Luciani and Garbet, derived the tearing-mode dispersion relation for a tokamak configuration from MHD resistive equations in which the full energy equation (3.8) was used instead of the usual, poorly justified adiabatic approximation (3.14). As already pointed out in Section 3.3, all previous tearing-mode analysis neglected the ohmic and the conduction terms in the energy equation (3.8), reducing it to its adiabatic form (3.14). Lutjens et al. in their work showed that these terms not only cannot be neglected, but also turn out to be dominating in the tearing-
mode ordering, setting up a new scale length in the resistive layer that ultimately
resolves the impasse. From the physical point of view, the thermal conductivities
along the parallel (to the magnetic field) and perpendicular directions cause a diffu-
sive flattening of the plasma pressure in the proximity of the resonance surface on
the tearing-mode scale length $\delta \sim \eta^{1/3}$. As a consequence, this local flattening in the
pressure eliminates the 'g-mode'. This physical but still intuitive consideration, is
confirmed on more mathematical grounds by a previous work by Hastie et al. ([1]),
in which the pressure profile was artificially flattened around the resonance over an
arbitrary scale length $\delta_f$ which had no link to physical quantities, aside from being
larger than the resistive tearing mode layer width $\delta$, and smaller than the plasma
size $a$. The flattening function investigated by Hastie et al. is such that the pressure
gradient near the resonant surface is modelled as

$$p_f'(x) = p_i'(x) \frac{x^2}{x^2 + \delta_f^2}$$

(3.127)

where $x = r - r_s$, $p_f'$ is the flattened pressure gradient and $p_i'$ is the initial pressure
gradient. This set up, schematically shown in Fig. 3-4, defines three separate regions
in the derivation of the tearing-mode dispersion relation: in regions $I_{R,L}$ and $II_{R,L}$,
the equation (3.37) for the radial perturbed magnetic field $\Psi$ near the resonant surface
but outside the resistive layer becomes

$$\Psi'' + \Psi \frac{D_f}{x^2} = 0$$

(3.128)

where the pressure gradient term dominates over the current gradient term, with

$$D_f(x) = -\frac{2k^2(x + r_s)}{[F'(r)]^2} \mu_0 p_f'(x)$$

(3.129)

By introducing the flattening function, the differential equation takes the form

$$(x^2 + \delta_f^2)\Psi'' + \Psi D_i = 0$$

(3.130)
where

\[ D_i = - \frac{2k^2r_s}{|F'_\nu|_{x=0}|^2} \mu_0 \mu'_0 |x=0 |_{\nu} \]  \hspace{1cm} (3.131)

The general solution to within the usual multiplicative constant is found in terms of the Legendre functions \( P_\nu, Q_\nu \), for both positive and negative \( x \) (see Ref.[27])

\[ \Psi_{R,L} \left( \frac{x}{\delta_f} \right) = \left( 1 + \frac{x^2}{\delta_f^2} \right) \left[ P_\nu \left( \frac{ix}{\delta_f} \right) + \lambda_{R,L} Q'_\nu \left( \frac{ix}{\delta_f} \right) \right] \hspace{1cm} (3.132) \]

where \( \nu = -\frac{1}{2} + \sqrt{\frac{1}{4} - D_i} \), and the prime symbol ('') denotes differentiation with respect to the argument, and \( \lambda_{R,L} \) are arbitrary constants. Now in region \( I_{R,L} \), in the limit \( x \gg \delta_f \) where the pressure is unchanged, the asymptotic solution for \( \Psi \) is
known to be (see Equation (3.42))

$$\Psi_{I_{R,L}} \simeq |x|^{-\nu} + \Delta_{I_{R,L}} |x|^{1+\nu}$$  \hspace{1cm} (3.133)

while in the region $I_{II_{R,L}}$, at the limit $x \sim \delta << \delta_f$ where the pressure profile is fully flattened ($p' = 0$), the asymptotic solution for $\Psi$ is known to be (see Equation (3.44))

$$\Psi_{I_{II_{R,L}}} \simeq 1 + \Delta_{I_{II_{R,L}}} |x|$$  \hspace{1cm} (3.134)

Using the appropriate asymptotic forms (see Ref. [42]) of the Legendre functions at large and small arguments, it is straightforward to relate $\Delta_{I_{II}}$ to $\Delta_{I_R}$ and $\Delta_{I_{II}}$ to $\Delta_{I_L}$, obtaining the following result:

$$\Psi_{I_{II_{R,L}}} = -\frac{F_2(\nu)}{\delta_f} \tanh \left( \frac{\pi \nu}{2} \right) + \frac{F_2(\nu)}{F_1(\nu)} \left[ 1 - \tan^2 \left( \frac{\pi \nu}{2} \right) \right] \Delta_{I_{R,L}} \delta_f^{2\nu}$$  \hspace{1cm} (3.135)

where

$$F_1(\nu) = \frac{2^{2\nu+1} \Gamma(1-\nu) \Gamma(\nu+\frac{1}{2}) \Gamma(\nu+\frac{3}{2}) \cos(\pi \nu)}{\pi \Gamma(\nu+2)}$$  \hspace{1cm} (3.136)

$$F_2(\nu) = \frac{(1+\nu) \Gamma(1-\nu/2) \Gamma(\nu/2)}{\Gamma(1+\nu/2) \Gamma(\nu/2)}$$  \hspace{1cm} (3.137)

Thus

$$\Delta'_{I_{II}} = -2 \frac{F_2(\nu)}{\delta_f} \tanh \left( \frac{\pi \nu}{2} \right) + 2 \delta_f^{2\nu} \Delta'_{I} \frac{F_2(\nu)}{F_1(\nu)} \left[ 1 - \tan^2 \left( \frac{\pi \nu}{2} \right) \right]$$  \hspace{1cm} (3.138)

where $\Delta'_{I_{II}} = \Delta_{I_{II}} + \Delta_{I_{R}}$ and $\Delta'_{I} = \Delta_{I_{L}} + \Delta_{I_{R}}$ as seen in Equation (3.86). In the low-$\beta$ limit, when $D_1 << 1$ and $\nu \sim -D_1$, Equation (3.138) becomes

$$\Delta'_{II} = 2 \delta_f^{2\nu} \Delta'_{I} + \frac{\pi D_1}{\delta_f}$$  \hspace{1cm} (3.139)

Finally the matching of the solution outside the resistive layer has to be carried
out with Equation (3.123), namely the solution inside the resistive layer (with \( p' = 0 \)) in order to get the tearing-mode dispersion relation for a pressure profile flattened near the resonant surface. As already see in Section 3.3, this matching condition is

\[
\Delta'_{H} = \frac{2\pi}{r_s} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} Q^{\frac{5}{4}}
\]  

(3.140)

This can be written as a condition for \( \Delta'_{I} \) by using Equation (3.139)

\[
\Delta'_{I} = \frac{\pi}{r_s \delta_f^{2\nu}} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} Q^{\frac{5}{4}} \left( 1 - \frac{\pi D_i}{2\delta_f^{1+2\nu}} \right)
\]  

(3.141)

It is straightforward to see that this dispersion relation allows one to find stable and marginally stable configurations having eliminated the 'g-mode'. In fact, the pressure gradient now enters in the term \(-\frac{\pi D_i}{2\delta_f^{1+2\nu}}\), with no inverse dependence on the growth rate \( Q^{1/4} \). This result, then, should give the reader an idea of how a local pressure flattening mathematically modifies the tearing-mode description when applied near the resonant surface on a scale length \( \delta_f \) larger than the tearing-mode width \( \delta \). Still no physical meaning is attached to \( \delta_f \), and that is the goal of the remaining of this chapter, where the derivation by Lutjens et al., properly adapted to an RFP, will be illustrated in detail; furthermore, it will be extended to include the gyrokinetic terms in the energy equation, as well as to solve another impasse existing in the literature, namely the presence of an always unstable rippling mode (see Refs. [26],[43],[44]), due to the presence of a gradient in the plasma resistivity. Even though the rippling mode has never really been believed to be physically important by the scientific community, it is reassuring that there is a formal and rigorous way to prove it.

### 3.5.2 New Tearing-Mode Analysis

In this paragraph, the appropriate tearing-mode dispersion relation is derived as suggested in Ref. [37] for a tokamak, extending it to an RFP configuration. The analysis includes spatial dependence of the transport parameters \( (\eta, \chi) \) allowing for
a clarifying discussion on the rippling mode as well. The gyrokinetic term is also included and discussed. In order to properly perform the analysis, it is important first to have a feeling of the numerical values of the relevant plasma parameters, transport coefficients, time scales and characteristic spatial lengths in a typical RFP

\[
T_e = T_i = 200\text{eV} ; \quad B = 0.5T ; \quad n = 3 \cdot 10^{19}m^{-3} ; \quad a = 0.5m ;
\]

(3.142)

for a Deuterium plasma (ion mass \(m_i = 3.34 \cdot 10^{-27}Kg\)). In particular, the thermal diffusivity and plasma resistivity can be estimated by using the classical formulation by Braginskii (see Ref. [45])

\[
\chi_{||} \sim \chi_{||e} \sim 1.8 \cdot 10^{8}m^2/s ; \quad \chi_{\perp} \sim \chi_{\perp i} \sim 2.4 \cdot 10^{-1}m^2/s ;
\]

\[
\chi_{\Lambda} \sim 9.9 \cdot 10^{2}m^2/s ; \quad \eta_{\perp} = 1.96\eta_{||} \sim 2.6 \cdot 10^{-7}\Omega m .
\]

(3.143)

where for simplicity, values of \(Z = 1\) and \(ln\Lambda = 20\) for the Coulomb logarithm have been used. Furthermore, an estimate of the typical tearing-mode growth rate and layer width from the early derivation leads to

\[
\delta \sim 7.4 \cdot 10^{-4}m ; \quad q \sim 300s^{-1}
\]

(3.144)

Those estimates will turn useful during the new tearing-mode analysis. The starting point of the derivation is still the general form of resistive MHD as shown in Eqs. (3.1)-(3.8). It is important to highlight here that in this derivation a constant plasma density \(\rho\) is assumed. Experimental observations show evidence that this assumption is quite accurate for describing an RFP equilibrium density. The impact of density perturbations in this treatment, though, is left as a future work. Furthermore, the diffusive velocity \(v_0\) from Section 3.4 is now neglected, having proved that it is very small in an RFP. With this setup, the general model described in Section 3.2 reduces to:
\[ \rho \text{ arbitrary constant} \]
\[ \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla p \quad (3.145) \]
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) - \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (3.147) \]
\[ \nabla \cdot \mathbf{B} = 0 \quad (3.148) \]

\[ \frac{3}{2} \frac{dp}{dt} + \frac{5}{2} \rho \nabla \cdot \mathbf{v} = \eta_\perp j_{\perp}^2 + \eta_\parallel j_{\parallel}^2 + \nabla \cdot \left[ \chi_\parallel \nabla_\parallel p + \chi_\perp \nabla_\perp p - \chi_\Lambda \frac{\mathbf{B}}{B} \times \nabla_\perp p \right] \quad (3.149) \]

The distinction between the parallel and perpendicular components of plasma transport is maintained; this will be crucial for the thermal conductivities \( \chi \), while not critical for the plasma resistivity. In fact, the large difference between \( \chi_\parallel \) and \( \chi_\perp \) (for classical transport \( \chi_\perp / \chi_\parallel \sim 10^{-9} \)) will set the new scale length previously mentioned; for classical transport, \( \eta_\perp \sim \eta_\parallel \), so that maintaining their distinction will only result in a small numerical coefficient, without adding any extra physical insight to the problem.

**Equilibrium**

Since \( \chi_\parallel \gg \chi_\perp \), the steady state equilibrium solution to Eqs. (3.145)-(3.148) is exactly as in Par. 3.3.1, with the following additional relations

\[ \nabla_\parallel p_0 = 0 \quad ; \quad \nabla \cdot (\chi_\parallel \nabla_\parallel p_0) = - \left( \eta_\perp j_{\perp,0}^2 + \eta_\parallel j_{\parallel,0}^2 \right) \quad (3.150) \]

a consequence of the full energy equation (3.149). Notice that at equilibrium the ohmic power balances the power losses of the system due to thermal conduction, and \( \chi_\parallel = \chi_\parallel(r) \), \( \chi_\perp = \chi_\perp(r) \), and \( \chi_\Lambda = \chi_\Lambda(r) \). The gyroterm \( \chi_\Lambda \) does not contribute to
the equilibrium, for

$$\nabla \cdot \left[ \chi_\lambda p_0' \left( \frac{\tilde{B}_0 \times \hat{e}_r}{B_0} \right) \right] = \nabla \cdot \left[ \chi_\lambda p_0' \frac{B_{0\theta} \hat{e}_\theta - B_{0\psi} \hat{e}_z}{B_0} \right] = 0$$ \hspace{1cm} (3.151)

In other words, the contributions from $\chi_{\parallel}$ and the gyroterm $\chi_\lambda$ do not enter equilibrium because of $(\theta, z)$ symmetry.

**Evolution of the Equilibrium State**

The study of the evolution of the steady state described in Section 3.5.2, is performed via a perturbative analysis by linearization as was carried in Section 3.3.2:

$$f(r, \theta, z) = f_0(r) + f_1(r) \exp[(i(m\theta - kz) + qt)]$$ \hspace{1cm} (3.152)

While obtaining exactly the same expressions from Eqs. (3.146), (3.147), (3.148), a more careful perturbation analysis of the energy equation (3.149) points out the physical and mathematical differences between this derivation and the traditional one as found in the literature (Refs. [26], [33]-[36], [40]). In order to properly linearize Equation (3.149), it is worth noting that to evaluate the thermal conduction terms, the following definitions are used: $\nabla_{\parallel} \equiv \frac{\tilde{B}}{B^2} \cdot \nabla$ and $\nabla_{\perp} \equiv \nabla - \nabla_{\parallel}$. The thermal conduction terms can then be linearized as follows

$$\nabla \cdot (\chi_{\parallel} \nabla_{\parallel} p) = \nabla \cdot \left( \chi_{\parallel} \frac{\tilde{B} \tilde{B}}{B^2} \cdot \nabla p \right) = \chi_{\parallel} \nabla \cdot \left( \frac{\tilde{B} \tilde{B}}{B^2} \cdot \nabla p \right) + \frac{(\tilde{B} \cdot \nabla p)}{B^2} (\tilde{B} \cdot \nabla \chi_{\parallel})$$

$$\approx \chi_{\parallel} \nabla \cdot \left[ \frac{\tilde{B}_0 \tilde{B}_1 \cdot \nabla p_1}{B^2_0} + \frac{\tilde{B}_0 \tilde{B}_1 \cdot \nabla p_0}{B^2_0} \right] + \frac{(\tilde{B}_1 \cdot \nabla \chi_{\parallel})}{B^2_0} \left[ \tilde{B}_0 \cdot \nabla p_1 + \tilde{B}_1 \cdot \nabla p_0 \right]$$

$$\approx \chi_{\parallel} \nabla \cdot \left[ \frac{\tilde{B}_0 \tilde{B}_1 \cdot \nabla p_1}{B^2_0} + \frac{\tilde{B}_0 B_{1r} \cdot \nabla p'_0}{B^2_0} \right]$$

$$= \chi_{\parallel} \tilde{B}_0 \cdot \nabla \left[ \frac{\tilde{B}_0 \cdot \nabla p_1 + B_{1r} p'_0}{B^2_0} \right]$$

$$= \frac{\chi_{\parallel}}{B^2_0} \left[ - \left( \frac{m B_{0\theta}}{r} - k B_{0z} \right)^2 p_1 + i \left( \frac{m B_{0\theta}}{r} - k B_{0z} \right) B_{1r} p'_0 \right]$$ \hspace{1cm} (3.153)
\[ \nabla \cdot (\chi \nabla \nabla p) = \chi \nabla \cdot \left[ \left( \nabla - \frac{\vec{B} \vec{B}}{B^2} \nabla \right) p \right] + \left( \nabla - \frac{\vec{B} \vec{B}}{B^2} \nabla \right) p \cdot \nabla \chi \]

\[ \approx \chi \nabla \cdot \left[ \left( \nabla - \frac{\vec{B}_0 \vec{B}_0}{B_0^2} \nabla \right) p_1 - \frac{\vec{B}_0 \vec{B}_1 \cdot \nabla p_0}{B_0^2} \right] + \chi' \nabla \cdot p_1' + p_0' \chi' \]

\[ = \chi \left[ \nabla^2 p_1 - \vec{B}_0 \cdot \nabla \left( \frac{\vec{B}_0 \cdot \nabla p_1 + B_{1r}p_0'}{B_0^2} \right) \right] + \chi' \nabla \cdot p_1' + p_0' \chi' \]

\[ \approx \chi \left[ \frac{(r p_1')'}{r} - \left( \frac{m^2}{r^2} + k^2 \right) p_1 + \left( \frac{m B_{0\theta}}{r} - k B_{0z} \right)^2 \frac{p_1}{B_0^2} + \right. \]

\[ \left. - i \left( \frac{m B_{0\theta}}{r} - k B_{0z} \right) \frac{B_{1r}p_0'}{B_0^2} \right] + \chi' \nabla \cdot p_1' + p_0' \chi' \] \hspace{1cm} (3.154)

\[ \nabla \cdot \left( \frac{\chi \vec{B} \times \nabla p}{B} \right) = \nabla \cdot \left[ \frac{\chi}{B_0} \left( \vec{B}_0 \times \nabla p_1 + \vec{B}_1 \times \nabla p_0 - B_{1||} \frac{\vec{B}_0 \times \nabla p_0}{B_0} \right) + \frac{\chi}{B_0} p_0' \vec{B}_0 \times \hat{e}_r \right] \]

\[ = \nabla \cdot \left[ \frac{\chi}{B_0} \left\{ - \hat{r} \cdot \left( \frac{m}{r} B_{0\theta} + \frac{m B_{0\theta}}{r} \right) + \hat{e}_z \left( B_{0\theta}p_1' + p_0' \left( B_{1z} - B_{1||} \frac{B_{0z}}{B_0} \right) \right) \right\} \right] \]

\[ \approx \frac{i m}{r} \frac{\chi}{B_0} p_1' \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) + \frac{i m \chi}{r} \frac{\chi}{B_0} \left[ B_{0\theta}p_1' + p_0' \left( B_{1\theta} - B_{1||} \frac{B_{0\theta}}{B_0} \right) \right] + \frac{i m}{r} \frac{\chi}{B_0} p_0' \frac{\chi}{B_0} \left( B_{0\theta}p_1' + p_0' \left( B_{1\theta} - B_{1||} \frac{B_{0\theta}}{B_0} \right) \right) + \frac{i m}{r} \frac{\chi}{B_0} p_0' \frac{\chi}{B_0} \]

\[ = \frac{i m}{r} \frac{\chi}{B_0} p_1' \left( \frac{m}{r} B_{0\theta} - k B_{0z} \right) \left( B_{0\theta}B_{1z} - B_{0z}B_{1\theta} \right) + \frac{i m}{r} \frac{\chi}{B_0} p_0' \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \] \hspace{1cm} (3.155)

The linearized form of Equation (3.149) then becomes

\[ \frac{3}{2} q(p_1 + \xi p_0') + \frac{5}{2} q p_0 \nabla \cdot \vec{F} = 2(\eta_{||}j_{||0j_{||1}} + \eta_{||}j_{||0j_{||1}}) + \frac{\chi}{B_0} \left[ \frac{F^2}{r^2} p_1 + i B_{1r} p_0' \frac{F}{r} \right] + \chi' \nabla \cdot p_1' + \]

\[ + \left( p_0' \chi'_{||1} + \chi_{||0} \left( \frac{(r p_1')'}{r} - \left( \frac{m^2}{r^2} + k^2 \right) p_1 + \frac{F^2}{r^2} \frac{p_1}{B_0^2} - i \frac{F}{r} \frac{B_{1r} p_0'}{B_0^2} \right) \right) + \]

\[ - i \frac{\chi}{B_0} p_0' \frac{F}{r} \left( B_{0\theta}B_{1z} - B_{0z}B_{1\theta} \right) - i \frac{\chi}{B_0} p_0' \left( \frac{m}{r} B_{0z} + k B_{0\theta} \right) \] \hspace{1cm} (3.156)
where $F$ is defined in Equation (3.38). Notice that no ordering has been used so far. By applying the tearing-mode scaling as seen in Section 3.3.4, Equation (3.156) becomes

$$
\frac{3}{2} q (p_1 + \xi_r p_0') + \frac{5}{2} q p_0 \nabla \cdot \vec{\xi} = 2 (\eta_1 j_{\perp 0} j_{\perp 1} + \eta_{||} j_0 j_{||}) + \frac{\chi_{||}}{B_0^2} \left[ -\frac{F'^2}{r^2} x^2 p_1 + i B_{1^r} p_0' \frac{F'}{r} x \right] + \\
+ \chi_{\perp 0} p_0'' + p_0' \chi_{\perp 1} - i \frac{\chi_{\perp 0}}{B_0^2} p_0 \frac{F'}{r} (B_{0\theta} B_{1z} - B_{0z} B_{1\theta}) + \\
- i \frac{\chi_{\perp 1}}{B_0} p_0' \left[ \frac{r}{r} B_{0z} + k B_{0\theta} \right] \tag{3.157}
$$

where the following ordering has been used $[x \sim \epsilon^2, (') \sim \epsilon^{-2}, \chi_{\perp 0} \sim \chi_{\perp 0}, \chi_{||} \gg \chi_{\perp 0}]$, and recalling that spatial derivatives of equilibrium quantities scale like 1, while spatial derivatives of perturbed quantities scale like $\epsilon^{-2}$. Now, since

$$
j_{\perp 0} \sim p_0' \sim \epsilon^2 \ , \quad j_{||} \sim \epsilon^0 \ , \quad \mu_0 j_{||} \sim (\nabla \times \vec{B}_1) \cdot \vec{B}_0 \sim -B'_{1z} B_0 \tag{3.158}
$$

then

$$
\eta_1 j_{\perp 0} j_{\perp 1} + \eta_{||} j_0 j_{||} \lesssim q p_1 \sim q \xi_r p_0' \sim +q p_0 \nabla \cdot \vec{\xi} \tag{3.159}
$$

All these terms turn out to be small compared to $\chi_{\perp 0} p_1''$ because

$$
\chi_{\perp 0} \gg q \xi^2 \tag{3.160}
$$

Equation (3.160) holds even for classical transport as it can be easily seen by substituting typical RFP parameters (see Eqs. (3.143), (3.144)). Next step involves estimating the terms coming from the gyro-diffusivity $\chi_A$; comparison with the parallel thermal diffusivity term, yields

$$
\frac{\chi_{||}}{B_0^2} \frac{F'^2}{r^2} x^2 p_1 \cdot \left[ \frac{B_0^3 r}{\chi_{\perp 0} p_0' F' x B_1} \right] \sim \frac{\chi_{||}}{B_0^2} \frac{x p_1}{\chi_{\perp 0} B_1 p_0'} \gg 1 ; \\
\frac{\chi_{||}}{B_0^2} \frac{F'^2}{r^2} x^2 p_1 \cdot \left[ \frac{1}{\chi_{\perp 1} p_0'} \right] \sim \frac{\chi_{||}}{B_0^2} \frac{x^2 p_1}{\chi_{\perp 1} p_0'} \gg 1 \tag{3.161}
$$

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using that \( f \alpha_0 x \alpha_1 x \alpha_0 \lesssim \epsilon^2 \). To conclude the scaling, it remains to evaluate the last term containing gradients of \( \chi \): that is \( \chi' \ll p_1 \). This term also turns out to be small; assuming a general dependence for \( \chi \sim p^\alpha \) (notice that \( \alpha = -\frac{1}{2} \) for classical transport, see Ref. [45]), it follows that

\[
\chi' \ll p_1^{-1} p_1 \sim \chi_\perp \frac{p_1}{p_0}
\]

Thus

\[
\chi' p_0 \sim \chi_\perp p_1 \ll \chi_\parallel p_1''
\]

Finally, the linearized form of Equation (3.149) in a tearing-mode scaling is:

\[
\chi_\parallel p_1^{(4)''} - \chi_\parallel p_1^{(4)} \frac{(F')^2}{\tilde{B}_0^2} x^2 + \frac{p_0'}{B_0^2} \frac{F'}{r} x B^{(4)}_{1r} \chi_\parallel = 0
\]

which replaces Equation (3.62) in the full set of layer equations, as derived in detail in Section 3.3.4. All the other equations are left unchanged, and are here rewritten for the present case of no diffusion velocity \( (v_0 = 0) \) and constant plasma density (see Equation (3.145))

\[
B_{1r}^{(4)}' = 0
\]

\[
\frac{\eta}{\mu_0 q} B_{1r}^{(6)''} = B_{1r}^{(4)} - \frac{iF}{r} \xi_r^{(2)}
\]

\[
\frac{i}{\mu_0} + \frac{B_0 B_{1r}^{(4)}}{\mu_0} + \frac{i \rho_0 B_0 B_{1r}^{(4)}}{m B_{0z} + k_r B_{0\theta}} B_{1r}^{(4)} = 0
\]

\[
q \xi_r^{(4)''} = \frac{iF}{r \rho_0 q \mu_0} B_{1r}^{(6)''} - \frac{i k}{\rho_0 q B_0 B_{0\theta}} (j_0 \cdot \tilde{B}_0)' B_{1r}^{(4)} - \frac{k^2 B_0 (B_0^2)' B_{1r}^{(4)}}{\rho_0 q B_{0\theta}^2 \mu_0}
\]

\[
\frac{\eta}{\mu_0 q} B_{1r}^{(4)''} + \frac{iF}{r} \xi_r^{(2)} = \frac{i}{B_0} \left( \mu_0 p_0 + B_0' \right) = 0
\]

\[
q^2 \rho_0 B_0 \xi_r^{(4)} + \frac{iF}{r} p_1^{(4)} + B_{1r}^{(4)} p_0'' = 0
\]
By virtue of Equation (3.165), i.e. the constant-$\Psi$ approximation discussed in Section 3.3.4, it is possible to directly solve Equation (3.164) and derive the expression for $p_1$. It is straightforward to see that a new scale length $\delta_x$ appears in Equation (3.164), describing the spatial variation length for $p_1$: in fact, it can be rescaled as follows

$$\bar{p}_1 - p_1z_x^2 - i\frac{D}{\delta_x} \frac{F'}{2\mu_0 k^2} z_x B_{1r}^{(4)} = 0$$

(3.171)

where

$$z_x = \frac{x}{\delta_x}, \quad \delta_x = \left[\frac{\chi_{\perp 0}}{\chi_{\parallel 0}} \left(\frac{r B_0}{F'}\right)^2\right]^{\frac{1}{4}} \quad \left(\bar{p} = \frac{dp}{dz_x}\right)$$

(3.172)

Here $D$ is the same as defined in Equation (3.71). This new scale length $\delta_x$ is determined by the ratio of perpendicular and parallel thermal coefficients. For typical RFP parameters and classical transport (see Eqs. (3.142), (3.143)) it is easy to see that $\delta_x \sim 3 \cdot 10^{-3} m$, whence $\frac{\delta_x}{\delta} \gg 1$, where $\delta$ is the tearing-mode scale length as defined in Equation (3.71) and estimated in Equation (3.144). This consideration demonstrates the physical explanation for the flattening of pressure over a layer larger than the tearing-mode layer, as mentioned earlier. The solution for Equation (3.171) can be found in terms of Hermite Polynomials

$$p_1 = \exp \left(-\frac{z_x^2}{2}\right) \sum_{n=0}^{\infty} f_n \hat{H}_n(z_x)$$

(3.173)

which leads to

$$-\sum_{n=0}^{\infty} f_n (1 + 2n) \hat{H}_n(z_x) = i\frac{D}{\delta_x} \frac{F' D_{1r}^{(4)}}{2\mu_0 k^2} \sqrt{2} \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z_x)}{4^n \Gamma(n + 1)}$$

(3.174)

whence

$$f_{2n+1} = -i\frac{D}{\delta_x} \frac{F' B_{1r}^{(4)}}{2\mu_0 k^2} \sqrt{2} \frac{\sqrt{2}}{4^n (4n + 3) \Gamma(n + 1)}$$

(3.175)
The full solution for $p_1$ is

$$p_1 = \exp \left( -\frac{z_x^2}{2} \right) \frac{D}{\delta_x} \frac{F'B^{(4)}_{1r}}{2\mu_0 k^2} \sqrt{2} \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z_x)}{4^n(4n+3)\Gamma(n+1)}$$  \hspace{1cm} (3.176)

Substituting Equation (3.176) into Equation (3.167), it is straightforward to determine $B^{(4)}_{11r}$ and then use it into Equation (3.168), which together with Equation (3.166) will give one differential equation for $\xi^{(2)}_r$, the quantity needed to determine the new dispersion relation (via Equation (3.88)). Moreover, as already seen in Section 3.3.4, only the odd part of $\xi^{(2)}_r$ is needed in order to calculate $\Delta(Q)$, which means that only the odd part of $B^{(4)}_{11r}$ is needed in Equation (3.168); the odd part of $B^{(4)}_{11r}$ is given by $p_1$ in Equation (3.167), since the term proportional to $B^{(4)}_{11r}$ is even:

$$B^{(4)}_{11r,odd} = -\mu_0 \frac{p^{(4)}_1}{B_0}$$  \hspace{1cm} (3.177)

Finally, Equation (3.168) becomes

$$\xi^{\prime\prime}_{odd} - z^2 \xi_{odd} = \Psi_0 \left[ z + \frac{D}{\delta_x} \frac{\eta \sqrt{2}}{\mu_0 g \delta} \exp \left( -\frac{\delta^2 z^2}{2\delta_x^2} \right) \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(\delta z)}{4^n(4n+3)\Gamma(n+1)} \right]$$  \hspace{1cm} (3.178)

where the tearing-mode notation already introduced in Section 3.3.4 has been used:

$$\xi^{(2)}_r = r \xi \ , \ \ \ z = \frac{x}{\delta} \ , \ \ \ \delta = \left[ \eta \rho_0 \frac{r^2}{(F')^2} \right]^{\frac{1}{4}}, \ \ \ \Psi_0 = \frac{i}{2} \frac{B^{(4)}_{11r}}{F' \delta}$$  \hspace{1cm} (3.179)

Again, Equation (3.178) holds for the odd part of $\xi$, so that the even term proportional to $(\vec{j}_0 \cdot \vec{B}_0)'B^{(4)}_{11r}$ in Equation (3.168) is omitted. Solution of Equation (3.178) can be found by separating the contributions to $\xi_{odd}$ coming from the two driving terms on the RHS: the first of those terms, $z\Psi_0$, is the old tearing-mode term that gives the $Q^{\frac{3}{2}}$ part of the dispersion relation, as seen in Equation 3.123, and acts on the tearing-mode scale length $\delta$. The second term, is the contribution from the plasma pressure, and in contrast to the adiabatic analysis, it acts on the larger scale length...
\( \delta_x \) determined by plasma thermal conductivity in the resonant layer. It is then simple to write

\[
\xi_{\text{odd}} = \xi_{T-M} + \xi_x
\]  

(3.180)

where

\[
\xi_{T-M}(z) = -e\exp\left(-\frac{z^2}{2}\right) \Psi_0 \sqrt{2} \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z)}{4^n(4n + 3)\Gamma(n + 1)}
\]  

(3.181)

is the solution to

\[
\xi''_{T-M} - z^2 \xi_{T-M} = z\Psi_0
\]  

(3.182)

Similarly, (since \( \delta << \delta_x \))

\[
\xi_x \approx -\Psi_0 \frac{D}{\delta_x^2} \frac{\eta \sqrt{2}}{\mu_0 q \delta} \frac{1}{z^2} e\exp\left(-\frac{\delta^2 z^2}{2\delta_x^2}\right) \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}\left(\frac{\delta z}{\delta_x}\right)}{4^n(4n + 3)\Gamma(n + 1)}
\]  

(3.183)

is a good approximate solution to

\[
\xi''_x - z^2 \xi_x = \Psi_0 \frac{D}{\delta_x^2} \frac{\eta \sqrt{2}}{\mu_0 q \delta} e\exp\left(-\frac{\delta^2 z^2}{2\delta_x^2}\right) \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}\left(\frac{\delta z}{\delta_x}\right)}{4^n(4n + 3)\Gamma(n + 1)}
\]  

(3.184)

as shown later. The tearing-mode dispersion relation can now be derived by substituting Eqs. (3.180), (3.181), and (3.183) into Equation (3.88):

\[
\Delta' = \Delta(Q) = \lim_{z \to \infty} \int_{-\infty}^{z} \frac{B''_{1r}}{B_{1r}} \, dx = \frac{q\mu_0 \delta}{\eta} \int_{-\infty}^{+\infty} \left( 1 + z \frac{(\xi_{T-M} + \xi_x)}{\Psi_0} \right) \, dz = \Delta'_{T-M} + \Delta'_x
\]  

(3.185)
where (using the Equations in (3.121))

\[
\Delta'_{T-M} = \frac{q\mu_0\delta}{\eta} \int_{-\infty}^{+\infty} \left(1 + z \frac{\xi_{T-M}}{\Psi_0}\right) dz
\]

\[
= \frac{Q_\delta^5}{r_s} \int_{-\infty}^{+\infty} \left(1 - z \exp\left(-\frac{z^2}{2}\right) \right) \sqrt{2} \sum_{n=0}^{\infty} \frac{\tilde{H}_{2n+1}(z)}{4^n(4n+3)\Gamma(n+1)} dz
\]

\[
= \frac{Q_\delta^5}{r_s} \sqrt{2} \int_{-\infty}^{+\infty} \exp\left(-\frac{z^2}{2}\right) \sum_{n=0}^{\infty} \frac{\tilde{H}_{2n}(z)}{4^n\Gamma(n+1)} \left[1 - \frac{4n + 2}{4n + 3}\right] dz
\]

\[
= 2\frac{Q_\delta^5}{r_s} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} \frac{1}{4n + 3} = \frac{Q_\delta^5}{r_s} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}
\]

(3.186)

is the part of the dispersion relation coinciding with the one found in Equation (3.123).

The term arising from the thermal conductivities is

\[
\Delta'_{\chi} = \frac{q\mu_0\delta}{\eta} \int_{-\infty}^{+\infty} z \frac{\xi_{\chi}}{\Psi_0} dz = -\frac{D}{\delta_{\chi}} \sqrt{2} \int_{-\infty}^{+\infty} \frac{1}{z_{\chi}} \exp\left(-\frac{z_{\chi}^2}{2}\right) \sum_{n=0}^{\infty} \frac{\tilde{H}_{2n+1}(z_{\chi})}{4^n(4n+3)\Gamma(n+1)} dz_{\chi}
\]

\[
= -2\sqrt{2} \frac{D}{\delta_{\chi}} \sum_{n=0}^{\infty} \frac{1}{4n+3} \int_{-\infty}^{+\infty} \exp\left(-\frac{z_{\chi}^2}{2}\right) \sum_{i=0}^{n} (-1)^{i+n} \frac{\tilde{H}_{2i}(z_{\chi})}{4^i i!} dz_{\chi}
\]

\[
= -4 \frac{D}{\delta_{\chi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} \sum_{i=0}^{n} (-1)^{i+n} \frac{\Gamma(i + \frac{1}{2})}{\Gamma(i + 1)}
\]

\[
= -4 \frac{D}{\delta_{\chi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+3} \left[\sqrt{\frac{\pi}{2}} + (-1)^n \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n + 2)} \right] _2F_1 \left[1, n + \frac{3}{2}, n + 2, -1\right]
\]

\[
= -\frac{\pi^{\frac{3}{2}} D}{2 \delta_{\chi}}
\]

(3.187)

where the following relations have been used (Ref. [27], [39])

\[
\frac{\tilde{H}_{2n+1}(z_{\chi})}{2n! z_{\chi}} = \sum_{i=0}^{n} (-1)^{i+n} \frac{\tilde{H}_{2i}(z_{\chi})}{i!}
\]

\[
\sum_{i=0}^{n} (-1)^{i} \frac{\Gamma(i + \frac{1}{2})}{\Gamma(i + 1)} = \sqrt{\frac{\pi}{2}} + (-1)^n \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n + 2)} \right] _2F_1 \left[1, n + \frac{3}{2}, n + 2, -1\right]
\]
Here $\mathbf{2F}_{1}$ is the hypergeometric function which satisfies

$$
\sum_{n=0}^{\infty} \mathbf{2F}_{1} \left[ 1, n + \frac{3}{2}, n + 2, -1 \right] \frac{4 \Gamma(n + \frac{3}{2})}{(4n + 3) \Gamma(n + 2)} =
\frac{4}{\Gamma(\frac{1}{2})} \sum_{n=0}^{\infty} \frac{1}{4n + 3} \int_{0}^{1} t^{n + \frac{1}{2}} (1 - t)^{-\frac{1}{2}} (1 + t)^{-1} dt =
\frac{4}{\Gamma(\frac{1}{2})} \int_{0}^{1} \sum_{n=0}^{\infty} \frac{t^{n + \frac{1}{2}}}{4n + 3} (1 - t)^{-\frac{1}{2}} (1 + t)^{-1} dt =
\frac{4}{3 \sqrt{\pi}} \int_{0}^{1} \sqrt{t} \mathbf{2F}_{1} \left[ 1, \frac{3}{4}, \frac{7}{4}, t \right] (1 - t)^{-\frac{1}{2}} (1 + t)^{-1} dt
= \sqrt{\pi} \left\{ \ln \left[ \sin \left( \frac{3\pi}{8} \right) \right] - \ln \left[ \sin \left( \frac{\pi}{8} \right) \right] \right\}
$$

(3.189)

and

$$
4 \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{4n + 3} = \frac{\pi^{\frac{3}{2}}}{2} + \sqrt{\pi} \ln \left[ \sin \left( \frac{\pi}{8} \right) \right] - \sqrt{\pi} \ln \left[ \sin \left( \frac{3\pi}{8} \right) \right]
$$

(3.190)

Finally, the tearing-mode dispersion relation becomes:

$$
\Delta' = \frac{Q_{s}^{\frac{3}{2}}}{r_{s}^{\frac{3}{2}}} \pi \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} - \frac{\pi^{\frac{3}{2}} D}{2 \delta_{x}}
$$

(3.191)

As expected, the pressure flattening in the resistive layer due to thermal conduction has eliminated the 'g-mode' for sufficiently negative $\Delta'$: that is the term related to the pressure $\frac{D}{Q^{1/4}}$ in Equation (3.123) no longer depends on the (normalized) growth rate $Q$. In agreement with Hastie et al. (Ref. [1]), the pressure term has the form $-\frac{D}{\delta_{f}}$, with $\delta_{f}$ being the local pressure flattening width; thanks to the derivation pointed out by Lutjens et al. (Ref. [37]) now there is a physical meaning attached to this flattening width, contained in the definition of $\delta_{x}$ as given in Equation (3.172):

$$
\delta_{x} = \left[ \frac{\chi_{L0} B_{0}^{2}}{\chi_{||0} B_{0}^{2}} \left( \frac{q_{s}}{k q_{s}} \right) \right]^{\frac{1}{4}}
$$

(3.192)
where
\[
q_s = \left. \frac{RB_{0z}}{RB_{0\theta}} \right|_{r_s}
\]  \hspace{1cm} (3.193)

is the safety factor evaluated at the resonance layer.

To complete this derivation, it is required to rigorously prove that the expression given for \( \xi_z \) in Equation (3.183) is a good approximation for the evaluation of \( \Delta' \). In order to show this, an estimate of the error will be provided. It is obvious to observe that Equation (3.183) is the solution of Equation (3.184) for large \( z \). For \( z < 1 \), the solution to Equation (3.184) is
\[
\hat{\xi} = -\exp \left( -\frac{z^2}{2} \right) \frac{D}{\delta x} \frac{\eta}{q \mu_0 \delta x} \frac{[\Gamma(\frac{3}{4})]^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z)}{4^n(4n+3)\Gamma(n+1)}
\]  \hspace{1cm} (3.194)
since \( \delta \ll \delta \), and
\[
\exp \left( -\frac{\delta^2 z^2}{2\delta^2} \right) \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(\frac{\delta z}{\delta x})}{4^n(4n+3)\Gamma(n+1)} \approx \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)!!}{2^{n-1}(4n+3)\Gamma(n+1)} \cdot \frac{\delta z}{\delta x} \cdot \frac{[\Gamma(\frac{3}{4})]^2}{2\sqrt{\pi}} \cdot \frac{\delta z}{\delta x}
\]  \hspace{1cm} (3.195)

Thus, the error in calculating \( \Delta' \) in Equation (3.187) lies in using Equation (3.183) also for \( z < 1 \) in the integral. This error can be quantified as follows: for \( \alpha \sim 1 \)
\[
\text{err} = \frac{q \mu_0 \delta}{\eta} \int_{-\alpha}^{\alpha} \frac{\xi_z - \hat{\xi}_z}{\Psi_0} dz = -\frac{D}{\delta x} \sqrt{2} \int_{-\alpha}^{\alpha} \frac{1}{z} \exp \left( -\frac{\delta^2 z^2}{2} \right) \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(\frac{\delta z}{\delta x})}{4^n(4n+3)\Gamma(n+1)} dz + \frac{D}{\delta x} \frac{[\Gamma(\frac{3}{4})]^2}{\sqrt{\pi}} \int_{-\alpha}^{\alpha} z \exp \left( -\frac{z^2}{2} \right) \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z)}{4^n(4n+3)\Gamma(n+1)} dz \approx -\frac{D}{\delta x} \frac{[\Gamma(\frac{3}{4})]^2}{\sqrt{2\pi}} \left\{ 2\alpha - \sqrt{2} \int_{-\alpha}^{\alpha} \exp \left( -\frac{z^2}{2} \right) \sum_{n=0}^{\infty} \frac{z\hat{H}_{2n+1}(z)}{4^n(4n+3)\Gamma(n+1)} dz \right\} \sim -\frac{D}{\delta x} \frac{\delta}{\delta x}
\]  \hspace{1cm} (3.196)

Equation (3.196) provides an estimate of the error in Equation (3.187), which can
be written more rigorously as

$$\Delta'_x = -\frac{D \pi^{\frac{3}{2}}}{\delta_x} \left[ 1 + O\left( \frac{\delta}{\delta_x} \right) \right]$$  \hspace{1cm} (3.197)

### 3.5.3 The Rippling Mode

As mentioned in Section 3.5.1, another unstable mode appears in parallel with the 'g-mode' in the old tearing-mode derivation; that is the rippling mode, related to the presence of a spatial gradient in the plasma resistivity. In Section 3.5.2 it was shown that the gradients of the plasma thermal conductivity scale out in the tearing-mode ordering; the plasma resistivity \( \eta \), though, was still considered to be constant. In this paragraph, the impact of a gradient of \( \eta \) in the new tearing-mode derivation just presented will be discussed, and as a result, it will be shown that the rippling mode will disappear. The mode turns out to be a direct consequence of the adiabatic condition, and more generally the neglecting of thermal conductivity. Starting from the original set of Equations (3.145)-(3.148), it is easy to see that a gradient of resistivity will add extra terms only in Equation (3.147), repeated here for convenience

$$- \frac{\partial \vec{B}}{\partial t} = \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \vec{B} \right) - \nabla \times (\vec{v} \times \vec{B})$$  \hspace{1cm} (3.198)

Assuming \( \eta = \eta_0(r) + \eta_1(r) e^{im\theta - ikz + qt} \), consistent with the perturbation analysis used before, one finds that the perturbed form of Equation (3.198) gives (for \( \vec{v}_0 = 0 \))

$$- \eta_0 \nabla^2 \vec{B}_1 - \eta_1 \left\{ \left[ \frac{(rB_{0g})'}{r} \right]' \hat{\theta} + \frac{(rB'_{0z})}{r} \hat{z} \right\} + \eta_1 \hat{r} \left[ \frac{im}{r} \frac{(rB_{0g})'}{r} - ikB'_{0z} \right] +$$

$$+ \eta_0' \left[ \left( \frac{im}{r} B_{1r} - \frac{(rB_{1g})'}{r} \right) \hat{\theta} - (ikB_{1r} + B'_{1z}) \hat{z} \right]$$

$$= -q\mu_0 \vec{B}_1 + q\mu_0 \nabla \times (\xi \times \vec{B}_0)$$  \hspace{1cm} (3.199)
As seen in Section 3.3.4, only the radial and parallel components of Equation (3.199) will enter in the final set of equations

\[
\mu_0 q B_{1r} = q \mu_0 \frac{e}{r} \frac{F'}{r} + \eta_0 B_{1r}' - \eta_1 \left[ \frac{im}{r} \frac{(r B_{0\theta})'}{r} - ik B_{0z}' \right] \\
\mu_0 q B_{1\parallel} B_0 = \frac{\eta_0 (B_{1\theta \parallel} B_{0\theta} + B_{1z} B_{0z}) + \eta_1 \left\{ B_{0\theta} \left[ \frac{(r B_{0\theta})'}{r} \right]' + \frac{B_{0z}}{r} (r B_{0\parallel}') \right\} + \\
\frac{\eta_0}{r} - \frac{F'}{r} x B_{1r} + \eta_0' (B_{1\theta \parallel} B_{0\theta} + B_{1z} B_{0z}) - \eta_1' \mu_0 p_0' + \\
+ \mu_0 q [B_0 \vec{B}_0 \cdot \nabla \zeta_{\parallel} - B_0^2 \nabla \cdot \zeta - \zeta_{r} (\mu_0 p_0 + B_0^2)']
\]  

(3.201)

where the tearing-mode ordering has already been used. One more relation is needed to link \( \eta \) to plasma quantities; the equilibrium form of Equation (3.198) does not provide the missing relation because the growth rate of the instability is faster than \( \eta_0 \). Thus, the classical formula for plasma resistivity is used (see Ref. [46]), whence (for constant density)

\[
\frac{\eta_1}{\eta_0} \propto \frac{p_1}{p_0}
\]  

(3.202)

The other non trivial equations needed to describe the plasma evolution in the resonant layer are the same as given by Equations (3.54)- (3.56), (3.58), (3.59), (3.63), together with the new linearized form of the energy equation (3.164). For the purpose of looking for changes in the dispersion relation, all that is needed is a subset of these equations that allows one to solve for the radial displacement \( \xi_r \), and consequently for \( \Delta(Q) \). First of all, \( B_{1\parallel} \) and \( \xi_\perp \) can still be eliminated from Equation (3.63) by using respectively Eqs. (3.55) and (3.56), as shown in Section 3.3.4, obtaining Equation (3.64) here rewritten as:

\[
q \xi_r^{(2)} = \frac{i F_x}{r \rho_0 q \mu_0} B_{1r}^{(6)} - i k \frac{(\vec{\gamma}_0 \cdot \vec{B}_0)'}{\rho_0 q B_{0\theta}} B_{1r}^{(4)} - \frac{k^2 B_0 (B_0^2)'}{\rho_0 q \mu_0 B_{0\theta}^2} B_{1\parallel}^{(4)}
\]  

(3.203)

Then, Equation (3.164) can be solved for \( p_1 \), exactly as in the derivation of Equation (3.176); \( p_1 \) turns out to be an odd function, and so also are \( B_{1\parallel}^{(4)} \) and the new term \( \eta_1 \), from Eqs. (3.59) and (3.202), respectively. It is then evident that the cor-
resection term related to \( \eta_l \) in Equation (3.200) and subsequently introduced in Equation (3.203) has the wrong parity. Hence it does not alter the expression for \( \Delta(Q) \), and the dispersion relation shown in Equation (3.191) remains unchanged. Finally, it can be asserted that the tearing-mode dispersion relation found in Equation (3.191) does not change if a spatial dependence in the thermal diffusivities \( (\chi_\perp, \chi_\parallel) \) and plasma resistivity \( (\eta) \) is taken into account. This implies that there is no rippling mode. Thus, the interesting question is now understanding how the rippling mode ever came up in the earlier literature (Refs. [26],[43],[44]). Hereafter, it will be shown how the rippling mode can be recovered from the tearing-mode derivation illustrated in Section 3.5.2. Once again, the crucial point involves the thermal conductivity; as explained in great detail in Ref. [26], the assumption in the past has been to simply neglect thermal conductivities in the energy equation. This corresponds to the following relation for the perturbed resistivity

\[
\eta_1 = -\xi_r \eta'_0 \tag{3.204}
\]

which is no longer a purely odd function. Thus it will contribute to the evaluation of \( \Delta(Q) \) through its even part. By simply using Equation (3.204) in the tearing-mode derivation, it is possible to find the rippling mode as originally obtained in Ref. [26]; by substituting Equation (3.204) in Equation (3.200)

\[
B''_{1r} = \frac{\mu_0 q}{\eta_0} B_{1r} \frac{\xi_r}{\eta_0} i \xi_r \frac{F'x}{r} - \xi_r \frac{\eta'_0}{\eta_0} \left[ \frac{im (r B_{0\theta})'}{r} - ik B_{0z}' \right] \tag{3.205}
\]

and then by substituting Eqs. (3.205) and (3.59) into Equation (3.203), the equation for the radial displacement becomes

\[
\rho_0 q \xi_r^{(2)''} = i \frac{F'x}{r \eta_0} B_{1r} + \frac{(F'x)^2}{r^2 \eta_0} \frac{\xi_r}{r q \mu_0 \eta_0} \left[ \frac{m (r B_{0\theta})'}{r} - k B_{0z}' \right] + \frac{2k^2}{q r} \left[ p_1 + \frac{ir (\vec{j}_0 \cdot \vec{B}_0) B_{1r}}{m B_{0z} + kr B_{0\theta}} \right] - \frac{ik (\vec{j}_0 \cdot \vec{B}_0)' B_{1r}}{q B_{0\theta}} \tag{3.206}
\]
Note that the even terms cannot be neglected, as they now contribute to \( \Delta(Q) \). By scaling all the variables as in Eqs. (3.172) and (3.179)

\[
\xi^{(2)} = r \xi , \quad z = \frac{z}{\delta} , \quad \delta = \left[ \frac{\eta_0 q_0 r^2}{(F')^2} \right]^{1/4} , \quad \Psi = \frac{iB_{tr}^{(4)}}{F' \delta} , \quad \chi = \frac{x}{\delta_x}
\]

\[
\delta_x = \left[ \frac{\chi \chi_0 B_0^2}{\chi \chi_0 B_0^2} \left( \frac{q_s}{k q_s} \right)^2 \right]^{1/4} , \quad \hat{h} = \frac{\eta_0 r}{F' \delta \mu_0} \left[ \frac{m (r B_0)}{r} - k B_0' \right]
\]

\[
\hat{j}_t^* = \frac{k \eta_0}{\delta q F'} \left[ 2 (\hat{z}_0 \cdot \hat{B}_0) \frac{B_0}{B_0^2} + \frac{r (\hat{z}_0 \cdot \hat{B}_0)}{B_0} \right]
\]

then Equation (3.206) becomes

\[
\xi'' - z^2 \xi - \hat{h} z \xi - z \Psi_0 + \Psi_0 \hat{j}_t^* - \exp \left( -\frac{z^2}{2} \right) \frac{\sqrt{2 D \eta_0}}{\delta_\chi \mu_0 q} \Psi_0 \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z_x)}{4^n (4n + 3) \Gamma(n+1)} = 0
\]

(3.208)

Similarly as done in Section 3.5.3, the solution is split in two parts, one arising from the large scale length \( \delta_x \), related to thermal transport, and the other having the contribution from \( \eta_0 \), that is the rippling mode:

\[
\xi = \xi_{ripp} + \xi_x
\]

(3.209)

The solution for \( \xi_x \) is easily found. To a good approximation

\[
\xi x \cdot (z + \hat{h}) = -\exp \left( -\frac{\delta^2 z^2}{2 \delta_x^2} \right) \frac{\sqrt{2 D \eta_0}}{\delta_\chi \mu_0 q} \Psi_0 \sum_{n=0}^{\infty} \frac{\hat{H}_{2n+1}(z_x)}{4^n (4n + 3) \Gamma(n+1)}
\]

(3.210)

The equation for \( \xi_{ripp} \), after changing variables to \( y = z + \frac{\hat{h}}{2} \), is given by

\[
\xi_{ripp}'' - y^2 \xi_{ripp} + \frac{\hat{h}^2}{4} \xi_{ripp} = \Psi_0 \left( y - \frac{\hat{h}}{2} - \hat{j}_t^* \right)
\]

(3.211)
Its solution can be found in terms of Hermite Polynomials:

\[ \xi_{ripp} = \exp \left( -\frac{y^2}{2} \right) \sum_{n=0}^{\infty} a_n \hat{H}_n(y) \]  

leading to the following relation

\[ -\sum_{n=0}^{\infty} a_n \left( 1 + 2n - \frac{\hat{h}^2}{4} \right) \hat{H}_n = \Psi_0 \sqrt{2} \sum_{m=0}^{\infty} \frac{\hat{H}_{2m+1}(y)}{4^m \Gamma(m + 1)} - \left( \frac{\hat{h}}{2} + j^* \right) \sum_{m=0}^{\infty} \frac{\hat{H}_{2m}(y)}{4^m \Gamma(m + 1)} \]  

(3.213)

The coefficients \( \{a_n\} \) are then determined as follows:

\[ a_{2l} = \Psi_0 \frac{\sqrt{2}}{4^l \Gamma(l + 1)} \left( \frac{\hat{h}}{2} + j^* \right) \frac{1}{1 + 4l - \frac{\hat{h}^2}{4}} \]  

(3.214)

\[ a_{2l+1} = \Psi_0 \frac{\sqrt{2}}{4^l \Gamma(l + 1)} \frac{1}{\frac{\hat{h}^2}{4} - 4l - 3} \]  

(3.215)

It is now possible to compute \( \Delta'(Q) \) again from its definition (Equation (3.88)), and substitute it into the dispersion relation, Equation (3.185). The result is

\[ \Delta' = \Delta'(Q) = \lim_{x \to \infty} \int_{-x}^{x} \frac{B_{1r}''}{B_{1r}} \, dx = \frac{q \mu_0 \delta}{\eta} \int_{-\infty}^{+\infty} \left( 1 + (z + \hat{h}) \frac{\xi_{ripp} + \xi_{x}}{\Psi_0} \right) \, dz = \Delta'_{ripp} + \Delta'_{x} \]  

(3.216)

It is easy to show that \( \Delta'_{x} = -\frac{\pi \frac{3}{2}}{2} D_{\delta x} \), as previously found in Equation (3.187). Now

\[ \Delta'_{ripp} = \frac{q \mu_0 \delta}{\eta} \int_{-\infty}^{+\infty} \left( 1 + (z + \hat{h}) \frac{\xi_{ripp}}{\Psi_0} \right) \, dz = \frac{Q_{\delta}}{r_s} \int_{-\infty}^{+\infty} \left[ 1 + \frac{y \xi_{ripp, odd}}{\Psi_0} \frac{\hat{h}}{2} \xi_{ripp, even} \right] \, dy = \]  

\[ = \frac{Q_{\delta}}{r_s} \sqrt{2} \sum_{l=0}^{\infty} \frac{1}{4^l \Gamma(l + 1)} \int_{-\infty}^{+\infty} \left[ \frac{\left( 1 + 4l + \frac{\hat{h}^2}{2} j^* \right) \hat{H}_{2l}(y)}{1 + 4l - \frac{\hat{h}^2}{4}} - \frac{y \hat{H}_{2l+1}(y)}{3 + 4l - \frac{\hat{h}^2}{4}} \right] \, \exp \left( -\frac{y^2}{2} \right) \, dy = \]  

\[ = \frac{Q_{\delta}}{r_s} \sqrt{2} \sum_{l=0}^{\infty} \frac{\sqrt{2\pi}}{4^l \Gamma(l + 1)} \left\{ \frac{1 + 4l + \frac{\hat{h}^2}{2} j^*}{1 + 4l - \frac{\hat{h}^2}{4}} \hat{H}_{2l}(2l)! - \frac{2^{2l+1}(2l + 1)!}{(4l + 3 - \frac{\hat{h}^2}{4}) 4^l \Gamma(l + 1)} \right\} = \]  

\[ = 2 \frac{Q_{\delta}}{r_s} \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{1}{2})}{\Gamma(l + 1)} \left\{ \frac{\frac{\hat{h}}{2} j^*}{1 + 4l - \frac{\hat{h}^2}{4}} - \frac{\frac{\hat{h}^2}{4} - 1}{4l + 3 - \frac{\hat{h}^2}{4}} \right\} \]  

(3.217)
which corresponds to the same expression found in Ref. [26] [pág. 465, Eq. (47)]. In conclusion, it has been proven that the rippling mode is a mathematical consequence that arises due to neglecting thermal conductivity in the plasma evolution.
Chapter 4

Evaluation of $\Delta'$

4.1 Introduction

In the previous chapter, the tearing-mode dispersion relation as found by Lutjens et al. in Ref. 37 was derived for an RFP with constant plasma density (see Equation (3.191)). The interesting feature of this dispersion relation is that it allows one to easily define a marginal stability condition with respect to tearing modes for an RFP equilibrium, by simply looking for the condition with zero growth rate ($Q = 0$)

$$\Delta' = -\frac{\pi^3 D}{2 \delta_x}$$

Equation (4.1) is the relationship that will provide a prescription for building a plasma pressure gradient profile together with a consistent description of the plasma transport coefficients ($\chi$). Before going into the details describing the full consistent model, which is the main contribution of this thesis work (see Chapter 5), it is worthwhile it to spend some effort discussing the actual computation of $\Delta'$, as it presents some interesting features that will characterize the results shown later on. The formula defining $\Delta'$ was given in Section 3.3, and is rewritten here as follows

$$\Delta' = \Delta_R + \Delta_L,$$
where $\Delta_R$ and $\Delta_L$ are the coefficients of the asymptotic expansion of the solution $\Psi$ of the differential equation (3.37) around the resonant surface $r = r_s$, respectively for the region at its right ($r > r_s$) and the region at its left ($r < r_s$) (as shown in Eqs. (3.42), (3.44)). The function $\Psi$ is physically linked to the radial perturbed magnetic field from its definition (see Equation (3.38))

$$
\Psi = B_{1r} \frac{r^2}{\sqrt{m^2 + k^2 r^2}}
$$

(4.3)

By examining the differential equation (3.37), it is easy to see that $\Psi$ (and consequently $\Delta'$) depends on the plasma equilibrium profiles $B_\theta$, $B_z$, $p'$, as well as on the poloidal mode number $m$ and the location of the resonant layer $r_s$ (or equivalently the axial wavenumber $k$). Notice that $\Delta'$ depends only on the shape of the profiles, not their magnitude. Two boundary conditions are needed. One is given by assuming a perfectly conducting wall at the plasma boundary $r = a$; the justification is that only modes resonating inside the plasma region (internal modes) are being considered. Also, with typical thick walls, the wall diffusion time is long. This assumption corresponds to having no component of the magnetic field perpendicular to the wall ($B_{1r}|_a = 0$). The other boundary condition is given by expanding Equation (3.37) around the axis ($r = 0$); assuming $B_{1r} = r^\alpha$ as well as typical behavior for RFP profiles,

$$
B_z|_{r \to 0} = B_{z0}(1 - d_1 r^2 + O(r^4)) , \quad B_\theta|_{r \to 0} = B_{\theta 0}(b_1 r + b_2 r^3 + O(r^5)) , \quad p'|_{r \to 0} = g_1 r + O(r^3)
$$

(4.4)

it is easy to show that to the lowest order in $r$

$$
\left(\alpha + \frac{3}{2}\right) \left(\alpha + \frac{1}{2}\right) + \frac{1}{4} - m^2 = 0 \Rightarrow \alpha = m - 1 \quad \text{for } m \geq 1
$$

(4.5)
Thus

\[ \Psi|_{r=0} \sim r^{m+\frac{1}{2}} + O(r^{m+\frac{5}{2}}) \quad \text{for} \quad m \geq 1 \]  

(4.6)

Special attention has to be given to the \( m = 0 \) case. In this case Equation (3.37) becomes

\[ \Psi'' + \frac{\Psi}{r^2} \left\{ -\frac{2rp'}{B_z^2} - k^2 r^2 - \frac{3}{4} \frac{B_z' + rB_z''}{B_z} r \right\} = 0 \]  

(4.7)

and the resonance is exactly at the reversal point \( r_s = r_0 \), such that \( F(r_s) = -kr_s B_z|_{r_s} = 0 \) for any wavenumber \( k \). Similarly as for \( m \geq 1 \), it is easy to show that

\[ \Psi|_{r=0} \sim r^{\frac{3}{2}} \quad \text{for} \quad m = 0 \]  

(4.8)

leading to the boundary condition \( \Psi|_{r=0} = 0 \). Since the boundary conditions require \( \Psi \) to vanish both at the wall (assumed perfectly conducting) and on axis, it is easy to recognize that \( \Delta' \) is really an integrated quantity, which in general depends upon the profiles integrated over the entire plasma region (for a given \( m \) and \( r_s \)). Furthermore, the presence of the singularity at \( r = r_s \) makes the computation of the coefficients \( \Delta_{L,R} \) difficult from a numerical point of view. It is also important to notice that the type of singularity depends on the pressure profile: if \( p'|_{r_s} \neq 0 \), then Equation (3.37) will have a \( \frac{1}{r^2} \) type of singularity; otherwise, if there is no pressure gradient at resonance, the singularity will be of the \( \frac{1}{r^2} \) type (for special force-free configurations like BFM, Equation (3.37) will have no singularity, as will be shown later in this chapter).

Two codes have been developed to efficiently compute \( \Delta' \), respectively for the cases with and without pressure. In the following sections, the corresponding numerical algorithms will be illustrated. Finally, tests and benchmarks of the two codes will be performed, showing good numerical accuracy. Both the codes are extensively used to calculate \( \Delta' \) in the model.
4.2 Numerical Evaluation of $\Delta'$:

Case with no pressure

It has been shown in Section 3.3 that the asymptotic expansion for $\Psi$ around the resonant layer in the limit of zero pressure gradient is (see Equation (3.44))

$$\Psi_{L,R} = 1 - a_2 x \ln |x| - \left( \frac{3}{4} a_2^2 + \left. a_3 \right|_{x=0} \right) x^2 + \frac{a_2^2}{2} x^2 \ln |x| + \Delta_{L,R} |x| \left( 1 - \frac{a_2}{2} x \right) \quad (4.9)$$

where $x = r - r_s$ and $a_3|_{x=0}$ is the regular function introduced in Equation (3.40) and evaluated at $x = 0$. The key point of calculating $\Delta'$ is to accurately extract the coefficients $\Delta_L$ and $\Delta_R$ from the numerical solution of $\Psi$. As easily seen in Equation (4.9), $\Psi$ is constant (here normalized to 1) at resonance, but the actual quantity that has to be carefully evaluated is its derivative; in fact, another expression for $\Delta'$ (valid only in the case with no pressure) is

$$\Delta' = \lim_{\epsilon \to 0} \frac{\Psi'|_{x=\epsilon} - \Psi'|_{x=-\epsilon}}{\Psi|_{x=0}} \quad (4.10)$$

That is $\Delta'$ represents the jump in the logarithmic derivative of $\Psi$ for a case of negligible pressure gradient. It is easy to realize that the logarithmic terms in Equation (4.9) exactly cancel out in the evaluation of $\Delta'$, even though they appear in the numerical solution for $\Psi$. The numerical method adopted to accurately extract $\Delta'$, is a type of shooting technique: instead of integrating Equation (3.37) from the axis and from the edge toward the resonant layer, where the logarithmic derivative $\Psi$ then has to be evaluated (and where an 'exact' cancellation of the logarithmic terms has to be numerically guaranteed), the approach is to start by assigning the solution of $\Psi$ at the resonant surface (via Equation (4.9)), guessing the initial values of $\Delta_L$ and $\Delta_R$, and then to integrate Equation (3.37) respectively toward the axis and toward the wall. The actual values of $\Delta_L$ and $\Delta_R$ are the ones which make the solution satisfy the boundary conditions. This method, even though requiring an iteration process, is still quite fast (few seconds on a Sun Ultra 5 workstation) thanks to an optimization
of the initial guesses for $\Delta_L$ and $\Delta_R$; more importantly, it gives very good accuracy in $\Delta'$, as it is shown next in the testing and benchmarking session.

4.3 Numerical Evaluation of $\Delta'$: Case with pressure

A similar shooting technique as the one described for the case without pressure is also adopted for evaluating $\Delta'$ in the presence of a pressure gradient. The main difference is in the nature of the singularity, which now is of type $\frac{1}{x^2}$. This considerably changes the properties of the solution $\Psi$: in particular, the asymptotic expansion around the resonance now becomes (see Equation (3.42))

$$
\Psi_{L,R} = |x|^\nu_1 \left[ 1 - \frac{a_2 x}{2
u_1} + \frac{a_2^2 - 2
u_1}{4\nu_1(1 + 2\nu_1)} x^2 \right] + |x|^\nu_2 \left[ 1 - \frac{a_2 x}{2\nu_2} \right] \left( \Delta_{L,R} + \frac{a_2 x}{2\nu_1|x|} \right)
$$

(4.11)

where

$$
\nu_1 = \frac{1}{2} \left( 1 - \sqrt{1 - 4D} \right), \quad \nu_2 = \frac{1}{2} \left( 1 + \sqrt{1 - 4D} \right)
$$

(4.12)

In the limit of small pressure, Equation (4.11) transforms into Equation (4.9). Thus, $\Psi$ now vanishes at resonance, and extracting the coefficients $\Delta_L$ and $\Delta_R$ is numerically more difficult. Specifically, $\Delta'$ cannot be calculated by Equation (4.10) any longer, and its original definition has to be used, given by Equation (4.2). Aside from these differences, the solving method is the same as illustrated for the case without pressure, where an optimized iterative guessing process has been developed, which converges when $\Psi$ vanishes both at the wall and on axis. As expected, the values of $\Delta'$ in the presence of a pressure gradient show higher sensitivity to the numerical parameters (like grid size, convergency thresholds...), especially near the axis and the reversal layer, regions in which Equation (3.37) assumes peculiar limits. The method was designed such that it allows a benchmark with the code without pressure.
Unfortunately, no analytic expressions or asymptotic formulas for \( \Delta' \) are known for testing the code in cases where the pressure gradient is large (in the sense of Suydam's marginal criterion for ideal MHD modes, for instance).

### 4.4 Testing and Benchmarking the \( \Delta' \) codes

#### 4.4.1 The Bessel Function Model

A good test for the \( \Delta' \) code is given by the force-free configuration known as BFM, and described in Chapter 2. This configuration was proven to correspond to the minimum energy state under the constraints of constant toroidal flux and total helicity, and was able to quantitatively describe the main features of an RFP equilibrium in the core. The main characteristics of the BFM can be summarized by the following relations

\[
\begin{align*}
\mu & = \frac{\mu_0 \bar{j} \cdot \bar{B}}{B^2} \quad \text{constant} \\
p' & = 0 \Leftrightarrow \bar{j} \times \bar{B} = 0
\end{align*}
\] (4.13)

When extended to the entire plasma region, this model allows a complete analytical calculation of the solution for the radial perturbed magnetic field, and consequently \( \Delta' \), providing a very reliable test for the \( \Delta' \) code without pressure. The differential equation (3.37) for the special case of BFM, presents no singularity: in fact as was already pointed out in Appendix A, the term containing the \( \frac{1}{x^2} \) type of singularity is proportional to \( p' \), while the term coming from the \( \frac{1}{x} \) type of singularity is proportional to \( \mu' \) (which is also physically related to the gradient of the current density parallel to the magnetic field), and both those quantities are exactly zero in the BFM.

The differential equation (3.37) can then be rewritten for the BFM as follows:

\[
\Psi'' + \Psi \left[ \frac{\mu^2}{r^2} - \frac{\dot{D}}{D} - \frac{2\mu m k}{D} + \frac{m^4 + 10m^2k^2r^2 - 3k^4r^4}{4r^2D^2} \right] = 0
\] (4.14)

The analytic solution of Equation (4.14) was first given by Voslamber and Calle-
baut in 1962 (see Ref. [47]). By changing variable

\[ y_{km} = \Psi \sqrt{\frac{\hat{D}}{r}} = rB_{1r} \quad (4.15) \]

Equation (4.14) becomes

\[ y''_{km} + y_{km} \frac{1}{r} \left( 1 - \frac{2k^2r^2}{\hat{D}} \right) + y_{km} \left( \mu^2 - k^2 - \frac{m^2}{r^2} - \frac{2\mu mk}{\hat{D}} \right) = 0 \quad (4.16) \]

whose general solution is

\[ y_{km} = -\frac{k\tilde{r}}{\mu + k} \hat{Z}_{m-1}(\tilde{r}) + m\hat{Z}_m(\tilde{r}) \quad (4.17) \]

where \( \tilde{r} \equiv r\sqrt{\mu^2 - k^2} \), and \( \hat{Z}_i \) is a linear combination of the Bessel \( \{\hat{J}_i\} \) and Neumann \( \{\hat{Y}_i\} \) functions

\[ \hat{Z}_i = c_1\{\hat{J}_i\} + c_2\{\hat{Y}_i\} \quad (4.18) \]

The integration constants \( c_1 \) and \( c_2 \) have to be determined by applying the following boundary conditions (together with a normalization condition, since Equation (4.16) is homogeneous):

\[
\begin{align*}
    y_{km}|_{r=0} &= 0 \\
    y_{km}|_{r=a} &= 0 \\
    y_{km}|_{r=r_s} &= y_{km}|_{r=r_s^+}
\end{align*}
\quad (4.19)
\]

Here \( r_s \) is always the radial point at which \( F \) vanishes. That is for a BFM

\[ m\hat{J}_1(\mu r_s) = kr_s\hat{J}_0(\mu r_s) \quad (4.20) \]

Notice that the resonant location \( r_s \) is not a singular point for \( B_{1r} \), anymore, but it still remains singular for the plasma displacement (see Equation (3.39)). Once \( y_{km} \)

is found on both sides of \( r = r_s \), it is straightforward to calculate \( \Delta' \) by rewriting
Equation (4.10) in terms of $y_{km}$

$$\Delta' = \lim_{\epsilon \to 0} \frac{y'_{km} |_{r_s + \epsilon} - y'_{km} |_{r_s - \epsilon}}{y_{km} |_{r_s}}$$  \hspace{1cm} (4.21)$$

A complete analysis of $\Delta'$ for different values of the poloidal mode number $m$ has been carried by Gibson and Whiteman (see Ref. [48]), and the main results are presented below.

Case $m = 0$

The simplest case to start with is $m = 0$, where the singular point coincides with the first zero of the Bessel function $\hat{J}_0$ (that is the field reversal location)

$$r_s \equiv \hat{\lambda}_{0,1} = 2.405...$$  \hspace{1cm} (4.22)$$

The differential equation (4.16) becomes

$$y''_{k0} - \frac{y_{k0}}{r} + y_{k0} \left( \mu^2 - k^2 \right) = 0$$  \hspace{1cm} (4.23)$$

whose general solution is

$$y_{k0} = c_1 r \hat{J}_1(\hat{r}) + c_2 r \hat{Y}_1(\hat{r})$$  \hspace{1cm} (4.24)$$

The constants $c_1$ and $c_2$ are determined by the boundary conditions:

$$\begin{align*}
\begin{cases}
y_{k0} |_{r=0} = 0 & \Rightarrow & c_{2,L} = 0 \\
y_{k0} |_{r=a} = 0 & \Rightarrow & c_{2,R} = -c_{1,R} \frac{\hat{J}_1(\hat{a})}{\hat{Y}_1(\hat{a})} \\
y_{k0} |_{r=r_s^-} = y_{k0} |_{r=r_s^+} & \Rightarrow & c_{1,L} \hat{J}_1(\hat{r}_s) = c_{1,R} \left\{ \hat{J}_1(\hat{r}_s) - \frac{\hat{J}_1(\hat{a})}{\hat{Y}_1(\hat{a})} \hat{Y}_1(\hat{r}_s) \right\}
\end{cases}
\end{align*}$$  \hspace{1cm} (4.25)$$
where \( \tilde{a} \equiv a\sqrt{\mu^2 - k^2} \) and \( \tilde{r}_s \equiv r_s\sqrt{\mu^2 - k^2} \). Thus the solution for \( y_{k0} \) on both sides of the resonance can be normalized as follows

\[
y_{k0} = \tilde{A} \cdot \begin{cases} 
0 \leq r \leq r_s \quad r \tilde{J}_1(\tilde{r}) \\
0 \leq r \leq a \quad r \tilde{J}_1(\tilde{r}) \tilde{Y}_1(\tilde{a}) - \tilde{J}_1(\tilde{a}) \tilde{Y}_1(\tilde{r}) \\
\tilde{J}_1(\tilde{r}) \tilde{Y}_1(\tilde{a}) - \tilde{J}_1(\tilde{a}) \tilde{Y}_1(\tilde{r}) \\
\tilde{J}_1(\tilde{a}) \tilde{Y}_1(\tilde{r}) - \tilde{J}_1(\tilde{r}) \tilde{Y}_1(\tilde{a}) \end{cases}
\]

(4.26)

with \( \tilde{A} \) being the arbitrary multiplicative constant. It follows from Equation (4.26) that

\[
\left| \begin{array}{l}
\frac{y'_{k0}}{y_{k0}} \\
\frac{y'_{k0}}{y_{k0}} \\
\frac{y'_{k0}}{y_{k0}}
\end{array} \right|_{r_s} = \sqrt{\mu^2 - k^2} \frac{\tilde{J}_0(\tilde{r}_s)}{\tilde{J}_1(\tilde{r}_s)}
\]

\[
\left| \begin{array}{l}
\frac{y'_{k0}}{y_{k0}} \\
\frac{y'_{k0}}{y_{k0}} \\
\frac{y'_{k0}}{y_{k0}}
\end{array} \right|_{r_s} = \sqrt{\mu^2 - k^2} \frac{\tilde{Y}_1(\tilde{a}) - \tilde{J}_1(\tilde{r}_s) \tilde{Y}_1(\tilde{a})}{\tilde{J}_1(\tilde{r}_s) \tilde{Y}_1(\tilde{a}) - \tilde{J}_1(\tilde{a}) \tilde{Y}_1(\tilde{r}_s)}
\]

(4.27)

Thus the analytical expression for \( \Delta' \) for \( m = 0 \) is given by

\[
r_s \Delta' = \frac{2}{\pi} \frac{\tilde{J}_1(\tilde{a})}{\tilde{J}_1(\tilde{r}_s)} \cdot \frac{1}{\tilde{J}_1(\tilde{r}_s) \tilde{Y}_1(\tilde{a}) - \tilde{J}_1(\tilde{a}) \tilde{Y}_1(\tilde{r}_s)}
\]

(4.28)

**Case \( m \geq 1 \)**

Setting the special \( m = 0 \) case aside, the results for \( m \geq 1 \) are illustrated here. The resonant point is now defined by Equation (4.20), and the integration constants are given by the boundary conditions in Equation (4.19):

\[
y_{km} |_{r=0} = 0 \quad \Rightarrow \quad c_{2,L} = 0
\]

\[
y_{km} |_{r=a} = 0 \quad \Rightarrow \quad c_{2,R} = -c_{1,R} \frac{m \tilde{J}_m(\tilde{a}) - \frac{k\tilde{a}}{\mu + k} \tilde{J}_{m-1}(\tilde{a})}{m \tilde{Y}_m(\tilde{a}) - \frac{k\tilde{a}}{\mu + k} \tilde{Y}_{m-1}(\tilde{a})}
\]

(4.29)

\[
y_{km} |_{r=r_s} = y_{km} |_{r=r_s^+} \quad \Rightarrow \quad c_{1,L} = c_{1,R} + c_{2,R} \frac{m \tilde{Y}_m(\tilde{r}_s) - \frac{k\tilde{r}_s}{\mu + k} \tilde{Y}_{m-1}(\tilde{r}_s)}{m \tilde{J}_m(\tilde{r}_s) - \frac{k\tilde{r}_s}{\mu + k} \tilde{J}_{m-1}(\tilde{r}_s)}
\]

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Introducing the following simplifying notation:

\[
\mathcal{J}_m(w) = m\hat{J}_m(w) - \frac{kw}{\mu + k}\hat{J}_{m-1}(w) \\
\mathcal{Y}_m(w) = m\hat{Y}_m(w) - \frac{kw}{\mu + k}\hat{Y}_{m-1}(w)
\] (4.30)

the solution for \( y_{km} \) on both sides of the resonance can then be written as follows:

\[
y_{km} = c_{1,L} \begin{cases}
\mathcal{J}_m(\bar{r}) & 0 \leq r \leq r_s \\
[\mathcal{J}_m(\bar{r}_s) \cdot \mathcal{Y}_m(\bar{a}) - \mathcal{J}_m(\bar{a}) \cdot \mathcal{Y}_m(\bar{r})] / [\mathcal{J}_m(\bar{a}) \cdot \mathcal{J}_m(\bar{r}_s) - \mathcal{J}_m(\bar{a}) \cdot \mathcal{Y}_m(\bar{r}_s)] & r_s \leq r \leq a
\end{cases}
\] (4.31)

Finally, the analytical expression for \( \Delta' \) for \( m \geq 1 \) is

\[
r_s \Delta' = \frac{\mathcal{J}_m(\bar{a}) \cdot \frac{2\mu-k}{\mu+k} \left( m^2 + k^2 r_s^2 \right)}{\mathcal{J}_m(\bar{a}) \cdot \mathcal{J}_m(\bar{r}_s) - \mathcal{J}_m(\bar{a}) \cdot \mathcal{Y}_m(\bar{r}_s)}
\] (4.32)

In Fig. 4-1, numerical values from the \( \Delta' \) code without pressure are compared with the analytical curves of \( \Delta' \) as a function of the resonant location \( \frac{r_s}{a} \) described by Equation (4.32), for different values of \( m \). The results show very good agreement. It is easy to see that \( m = 1 \) is the most stringent mode for the BFM in terms of stability (\( \Delta' = 0 \)). For \( m = 1 \) the marginal stability condition, \( \Delta' = 0 \), is described by the relation

\[
\mathcal{J}_1(\bar{a}) \equiv \hat{J}_1(\bar{a}) - \frac{k\bar{a}}{\mu + k} \hat{J}_0(\bar{a}) = 0
\] (4.33)

leading to the threshold \( \mu a \approx 3.11 \) above which a resistive \( m = 1 \) tearing-mode instability appears in the plasma, as already mentioned in Chapter 2. Further analysis of Equation (4.32) shows also that \( m = 1 \) is the most stringent mode for BFM in terms of stability (\( \Delta' = 0 \)). Moreover, the value of \( \mu a \approx 3.18 \) is the limit above which a zero in \( y_{k1} \) appears inside the plasma region (\( r < a \)): this corresponds to the marginal con-
Figure 4-1: Code Testing: $\Delta'$ vs. Resonant Surface $r_s$ for BFM ($\mu a = 2.9$)

Condition for stability to ideal MHD modes, according to Newcomb's stability analysis (see Refs. [49],[6]). In Fig. 4-2, these stability results for $m = 1$ are illustrated with the $\Delta'$ code: the tearing-mode stability threshold $\Delta' = 0$ is crossed for $\mu a = 3.113$, while the ideal MHD instability condition $\Delta' \to +\infty$ is reached at $\mu a = 3.176$.

Case $m \geq 1$, $r_s \to \hat{j}_{0,1} \approx 2.405$

Finally, special consideration has to be given when the resonance coincides with the field reversal layer for $m \geq 1$. In this case, Equation (4.20) shows that $k \to +\infty$. The differential equation (4.14) becomes

$$\Psi'' - k^2 \Psi = 0$$

(4.34)

with general solution ($\kappa \equiv |k|$)

$$\Psi = c_1 e^{\kappa r} + c_2 e^{-\kappa r}$$

(4.35)
Figure 4-2: Code Testing: Stability Analysis for BFM

The boundary conditions now give

\[
\begin{align*}
\Psi_{r=0} &= 0 \quad \Rightarrow \quad c_{1,L} + c_{2,L} = 0 \\
\Psi_{r=a} &= 0 \quad \Rightarrow \quad c_{1,R} e^{\kappa a} + c_{2,R} e^{-\kappa a} = 0 \\
\Psi_{r=r_s^-} &= \Psi_{r=r_s^+} \quad \Rightarrow \quad c_{1,R} [e^{\kappa r_s} - e^{-\kappa r_s}] = c_{1,R} [e^{\kappa r_s} - e^{2\kappa a - \kappa r_s}]
\end{align*}
\]  

(4.36)

The solution on both sides of the resonance is

\[
\Psi = c_{1,R} e^{\kappa a} \cdot \begin{cases} 
\frac{e^{\kappa(r_s-a)} - e^{-\kappa(r_s-a)}}{e^{\kappa r_s} - e^{-\kappa r_s}} \cdot (e^{\kappa r} - e^{-\kappa r}) & 0 \leq r \leq r_s \\
(e^{\kappa(r-a)} - e^{-\kappa(r-a)}) & r_s \leq r \leq a
\end{cases}
\]  

(4.37)
Finally, $\Delta'$ in this case is

$$
\Delta'|_{r_s-\hat{j}_{0,1}} = 2\kappa \frac{e^{\kappa a} - e^{-\kappa a}}{(e^{\kappa r_s} - e^{-\kappa r_s})(e^{\kappa (r_s-a)} - e^{-\kappa (r_s-a)})}
$$

(4.38)

As $\kappa \to +\infty$

$$
\Delta'|_{r_s-\hat{j}_{0,1}} \approx -2\kappa = -\left. \frac{2m\hat{J}_1(\mu r_s)}{r_s\hat{j}_0(\mu r_s)} \right|_{r_s-\hat{j}_{0,1}} \to -\infty
$$

(4.39)

Even in this limit, $m = 1$ turns out to be the most restrictive case for tearing-mode stability.

### 4.4.2 Solution for $\Delta'$ on axis when $p' = 0$

Another analytic expression for $\Delta'$ is known for the case without pressure and for $m \geq 1$. This expression, found by Papaloizou (see Refs. [2], [50]), gives $\Delta'$ when the resonance $r_s$ is close to the axis ($r = 0$), where Equation (3.37) shows a $\frac{1}{r^2}$ type of singularity, in addition to the usual $\frac{1}{r-r_s}$ singularity due to $\mu' \neq 0$. The derivation is based on the assumption that the eigenfunction $\Psi$ is very localized around the resonant surface $r_s$ when it approaches zero: in this way, $\Delta'$ will mostly depend on the plasma profile around $r = r_s$, justifying the solution of Equation (3.37) for $r_s \approx 0$ via the expansion around the axis. In this paragraph, a brief illustration of Papaloizou's derivation is given (details to be found in Ref. [2]). First, Equation (3.37) (or its equivalent form given in Equation (A.15)) is written in the limit of small $r$:

$$
\Psi'' + \Psi \left[ \frac{1}{4r^2} - \frac{m^2}{r^2} - \mu' \frac{m B_z + k r B_\theta}{m B_\theta - k r B_z} \right] = 0
$$

(4.40)

By changing variable $\Phi = \frac{\Psi}{\sqrt{r}}$, Equation (4.40) becomes

$$
\Phi'' + \frac{\Phi'}{r} + \Phi \left[ \frac{m^2}{r^2} - \mu' \frac{m B_z + k r B_\theta}{m B_\theta - k r B_z} \right] = 0
$$

(4.41)
Moreover, an expansion for \( B_\theta \) around the axis is assumed to be valid through the entire plasma region:

\[
B_\theta = b_1 r + b_2 r^3
\]

(4.42)

The resonant layer \( r_s \) is defined by

\[
F(r_s) = mr_s(b_1 + b_2 r_s^2) - kr_sB_z = 0 \Rightarrow r_s^2 = \frac{kB_z - mb_1}{mb_2}
\]

(4.43)

Finally, by making the transformation

\[
z = \left(\frac{r}{a}\right)^2; \quad z_s = \left(\frac{r_s}{a}\right)^2; \quad \Phi = z^\frac{3}{2} y_{km}(z)
\]

(4.44)

Equation (4.41) becomes

\[
z \frac{d^2 y_{km}}{dz^2} + \frac{dy_{km}}{dz} (m + 1) + \frac{2 y_{km}}{z_s - z} = 0
\]

(4.45)

whose solution can be found in terms of hypergeometric functions for \( 0 < z < z_s \), and imposing the proper boundary condition on axis. The solution for \( r > r_s \) is also found from Equation (4.45) as a combination of hypergeometric functions by defining the new variable \( u = z - z_s \) and thus obtaining the equation

\[
u(u + z_s) \frac{d^2 y_{km}}{du^2} + \frac{dy_{km}}{du} u(m + 1) - 2 y_{km} = 0
\]

(4.46)

together with the matching condition at resonance, and the boundary condition

\[y_{km}|_{1-z_s} = 0\]

(4.47)

As assumed at the beginning, the eigenfunction \( y_{km} \) is indeed very localized on axis, justifying a posteriori its analytic derivation. Once the function \( y_{km} \) is determined,
\( \Delta' \) can be evaluated from its definition, and the resulting analytic expression is

\[
  r_s \Delta'|_{r_s \rightarrow 0} = -2\pi \Lambda \cot |\chi| + 2\Lambda \left( \frac{1}{\chi} + \frac{1}{|\chi|} \right)
\]  

(4.48)

where

\[
  \Lambda = \frac{1 + 2\gamma}{\gamma}, \quad \gamma = \frac{r B_z}{2 B_\theta} \frac{d^2}{dr^2} \left[ \frac{r B_z}{B_\theta} \right]_{r=0}, \quad \chi = \frac{m}{2} - \sqrt{\frac{m^2}{4} + \Lambda}
\]  

(4.49)

The value of \( \Delta' \) on axis will then depend only upon \( \gamma \) (related to the safety factor on axis) and \( m \). It is easy to observe that the condition for real \( \chi \) corresponds to the criterion of ideal hydromagnetic stability

\[
  \Lambda > -\frac{m^2}{4} \Rightarrow \left\{ \begin{array}{l} \gamma > 0, \quad \gamma < -\frac{4}{m^2 + 8} \end{array} \right.
\]  

(4.50)

The on-axis stability condition to tearing modes (\( \Delta' = 0 \)) can also be written in terms of \( \gamma \)

\[
  \frac{4}{\gamma} < 2m - 7
\]  

(4.51)

In Fig 4-3, \( \Delta' \) is plotted versus \( \gamma \) for a few values of \( m \). It is important to note that once again \( m = 1 \) sets up the most stringent constraint for stability versus tearing modes. Finally, it is worthwhile to check this formula by applying it to the BFM. For this configuration, it is easy to verify that on axis

\[
  2\gamma = -1 - \frac{\mu^2 r^2}{8} + O(\mu^4 r^4), \quad \Lambda = \frac{\mu^2 r^2}{4} + O(\mu^4 r^4), \quad \chi = -\frac{\mu^2 r^2}{4m} + O(\mu^4 r^4)
\]  

(4.52)

Thus

\[
  r_s \Delta'|_{r_s \rightarrow 0} = -2m + O(\mu^2 r^2) \quad m \geq 1
\]  

(4.53)

in agreement with Equation (4.32), using the following expansion for the Bessel func-
tions for $z \cong 0$

$$
\begin{align*}
\hat{J}_m(z) &= \left( \frac{\hat{x}}{\hat{2}} \right)^m \frac{1}{m!}, \\
\hat{Y}_0(z) &= \frac{2}{\pi} \ln \left( \frac{\hat{x}}{\hat{2}} \right) \hat{J}_0(z) \\
\hat{Y}_m(z) &= -\left( \frac{\hat{x}}{\hat{2}} \right)^{-m} \frac{(m-1)!}{\pi}, & m \geq 1
\end{align*}
$$

In Fig. 4-4, the $\Delta'$ code without pressure is tested around the axis with the analytic expression (4.48) for some typical RFP profiles, again showing good agreement.

4.4.3 Analytic Solution at Reversal Layer for $m \geq 1$ when $p' = 0$

When the resonance $r_s$ approaches the reversal layer $r_0$, another particular case for the Equation (3.37) is met. In fact, since $B_z$ is very small at the resonance, the
wavenumber $k$ becomes very large and dominating:

$$ k = \left. \frac{mB_\theta}{rB_z} \right|_{r=r_s} \simeq \left. \frac{mB_\theta}{rB_z} \right|_{r=r_s} \frac{1}{\mathcal{E}}, \quad \mathcal{E} \equiv r_s - r_0 \ll r_s \quad (4.55) $$

The differential equation for $r_s \to r_0$ becomes:

$$ \Psi'' - \Psi \left[ k^2 + \frac{b}{x} \right] = 0 \quad (4.56) $$

where $b \equiv \left[ \frac{1}{r} + \frac{B''_z}{B_z^2} + \frac{4\mathcal{E}}{r^2} \right]_{r_s}$ is now a constant, which stays finite even for $\mathcal{E} \to 0$; also $x \equiv r - r_s$.

The general solution of Equation (4.56) is given in terms of the confluent hyper-
geometric functions

\[
\begin{align*}
U(a, b, z) &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}t^{a-1}(1 + t)^{b-a-1} dt \\
_1F_1(a, b, z) &= \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}
\end{align*}
\] (4.57)

with

\[
(a)_0 = 1; \quad (a)_k = a(a + 1)(a + 2) \cdots (a + k - 1)
\]

\[
(b)_0 = 1; \quad (b)_k = b(b + 1)(b + 2) \cdots (b + k - 1)
\]

It is given here in its form valid on both sides of the resonance:

\[
\begin{align*}
\Psi_L &= -e^{kx} \cdot x \left\{ _1F_1 \left[ 1 - \frac{b}{2k}, 2, -2kx \right] c_L + U \left[ 1 - \frac{b}{2k}, 2, -2kx \right] d_L \right\} \quad x < 0 \quad (4.58)
\Psi_R &= e^{-kx} \cdot x \left\{ _1F_1 \left[ 1 + \frac{b}{2k}, 2, 2kx \right] c_R + U \left[ 1 + \frac{b}{2k}, 2, 2kx \right] d_R \right\} \quad x > 0 \quad (4.59)
\end{align*}
\]

The boundary conditions of \( \Psi \) vanishing on axis and at the wall give \( c_L = c_R = 0 \),

since

\[
\begin{align*}
\lim_{kx \to +\infty} xe^{-kx} _1F_1 \left[ 1 + \frac{b}{2k}, 2, 2kx \right] &= e^{kx} x^{\frac{b}{2k}} \to \infty \\
\lim_{kx \to -\infty} xe^{kx} _1F_1 \left[ 1 - \frac{b}{2k}, 2, -2kx \right] &= e^{-kx} x^{-\frac{b}{2k}} \to \infty \\
\lim_{kx \to +\infty} xe^{-kx} U \left[ 1 + \frac{b}{2k}, 2, 2kx \right] &= e^{-kx} x^{\frac{b}{2k}} \to 0 \\
\lim_{kx \to -\infty} xe^{kx} U \left[ 1 - \frac{b}{2k}, 2, -2kx \right] &= e^{kx} x^{\frac{b}{2k}} \to 0
\end{align*}
\] (4.60)

After matching the condition at \( r = r_s \), the solution becomes

\[
\begin{align*}
\Psi_L &= -2e^{kx} kx U \left[ 1 - \frac{b}{2k}, 2, -2kx \right] \cdot \Gamma (1 - \frac{b}{2k}) \quad x < 0 \\
\Psi_R &= 2e^{-kx} kx U \left[ 1 + \frac{b}{2k}, 2, 2kx \right] \cdot \Gamma (1 + \frac{b}{2k}) \quad x > 0
\end{align*}
\] (4.61)
Moreover, using the expansion for \( kx \to 0 \),

\[
U \left[ 1 \pm \frac{b}{2k}, 2, \pm 2kx \right] \cdot \Gamma \left( 1 \pm \frac{b}{2k} \right) \simeq \pm \frac{1}{2kx} + O(\ln x) \tag{4.62}
\]

Equation (4.61) can be written as

\[
\begin{cases}
\Psi_L &= e^{kx} [1 + O(x \ln |x|)] \\
\Psi_R &= e^{-kx} [1 + O(x \ln |x|)]
\end{cases} \tag{4.63}
\]

Thus

\[
E_0 \Delta'_{|r_s = r_0} = -2 \left| \frac{m B_\theta}{B_\| (r_s - r_0)} \right| \tag{4.64}
\]

This analytic expression extends Equation (4.39), found for BFM, to general configurations (as long as there is no pressure). This means that the term containing \( k^2 \) in Equation (4.56) dominates over the singular term \( \frac{b}{x} \) in the evaluation of \( \Delta' \). Similarly as found in Equation (4.39), \( \Delta'_{|r_s = r_0} \to -\infty \), and it is easy to show once again that \( m = 1 \) is the most restrictive mode in terms of stability. In Fig. 4-5, the \( \Delta' \) code without pressure is tested around the reversal layer with the analytic expression (4.64) for some typical RFP profiles, again showing good agreement. Note that in the outward region \( (r_s > r_0) \) the actual \( \Delta' \) deviates from the analytical expression sooner than in the inward region \( (r_s < r_0) \) due to the presence of the perfectly conducting wall.

### 4.4.4 The Case with Pressure

Unfortunately, in the presence of a pressure gradient, Equation (3.37) becomes quite a bit more difficult to solve analytically, mainly due to the presence of the \( \frac{1}{x^2} \) singularity that adds to the ones existing in the case with no pressure.

To the author’s knowledge, there is no analytical formula for \( \Delta' \) when \( \rho' \neq 0 \). This significantly reduced the testing performed on the \( \Delta' \) code with pressure (as it will
be used later on) to just benchmarking it with the $\Delta'$ code without pressure in the limit $p' \to 0$, together with some numerical convergence analysis.

It has to be pointed out, though, that the $\Delta'$ code with pressure will be used later on to determine marginally stable RFP configurations which have small plasma pressure, so the code will be operated mainly at the low pressure limit, where it has been successfully benchmarked to the $\Delta'$ code without pressure, which also passed the analytical tests shown before. As will be seen in Chapter 5, only two regions in the plasma will turn out to have a pressure gradient which is large (of the order of Suydam’s marginal pressure gradient); mainly the axis and the reversal layer. Overall, these regions will give a small contribution to the final results presented in this thesis, such that their validity should not be affected.

Now, a few comments will be said in order to show the degree of difficulty in finding an analytic expression for $\Delta'$ in those regions in the presence of a finite $p'$. A possible analytic solution in order to get a better testing of the $\Delta'$ code with pressure
is left for future work.

**Case of** \( r_s \approx 0 \) **for** \( p' \neq 0 \) **and** \( m \geq 1 \)

When the resonant surface approaches the axis \( r_s \approx 0 \) in the presence of a pressure gradient, Equation (3.37) around the resonance becomes (by Taylor expansion)

\[
\Psi'' + \Psi \left[ \frac{\frac{2k^2r^3}{F^2} \mu_0 p'_0 - \frac{F'}{F} r + \frac{5}{4} - m^2}{r^2} \right] = 0
\]  

(4.65)

Following Papaloizou's analytic approach as shown for the case with no pressure, that is, using Eqs. (4.42), (4.43) together with MHD pressure balance, Equation (4.65) can be written as follows

\[
\Psi'' + \Psi \left[ \frac{2b_1^2}{B_z^2 \beta_0 (r^2 - r_s^2)^2} \left( \frac{2b_1^2 + B_z B'_z}{r} \right) - \frac{2}{r^2 - r_s^2} + \left( \frac{1}{4} - m^2 \right) \frac{1}{r^2} \right] = 0
\]  

(4.66)

This case, differently from the case without pressure, is more complicated to solve analytically, because now the singular term has a \( \frac{1}{x^2} \) type of component that competes with the stabilizing contribution due to the line bending term \( \left( \frac{1}{4} - m^2 \right) \frac{1}{r^2} \). It is important to notice that in this limit the coefficient of the \( \frac{1}{x^2} \) term is not necessarily small; in fact, remember that \( D = \frac{1}{4} \) for Suydam's marginally stable pressure gradients. An analytic derivation, as the one carried out by Papaloizou for the case with no pressure, would certainly provide a valid test for the \( \Delta' \) code with pressure.

**Case of** \( r_s \approx r_0 \) **for** \( p' \neq 0 \) **and** \( m \geq 1 \)

In the limit of the resonant layer \( r_s \) approaching the reversal surface \( r_0 \) in the case of non-zero pressure gradient, Equation (3.37) for \( r \approx r_s \) becomes (by Taylor expansion)

\[
\Psi'' + \Psi \left\{ \frac{D}{(r - r_s)^2} - \frac{1}{r - r_s} \left[ (D + 1) \left( \frac{1}{r_s} + \frac{B''_z}{B'_z r_s} \right) - D \left. \frac{p''_0}{p'_0} \right|_{r_s} \right] - k^2 \right\} = 0
\]  

(4.67)

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where \( k \equiv \frac{mB^2(r_s)}{r_s B^2|_{r_s} (r_s - r_0)} \) is a very large number, going to infinity in the limit of \( r_s \) approaching \( r_0 \).

Similarly as seen for the limit \( r_s \approx 0 \), the case with pressure is more complicated to solve analytically because now the singular term is of the \( \frac{1}{x^2} \) type and it is destabilizing, since \( D \) is positive, so that it competes with the stabilizing term \( -k^2 \) more than the \( \frac{1}{x^2} \) type of singularity would do. Overall, this should still give \( \Delta'|_{r_s \rightarrow r_0} \rightarrow -\infty \), even though the trend with the distance from the reversal \( (r_s - r_0) \) is weaker than the one found in Equation (4.64). These intuitive considerations and expectations are qualitatively confirmed by the \( \Delta' \) code with pressure; nonetheless a formal analytic solution of Equation (4.67) (written in the Wittaker standard form) in this limit would provide a solid test for the \( \Delta' \) code with pressure.
Chapter 5

Description of the Model

5.1 Introduction

Once the criterion for tearing-mode marginal stability is found, it is possible to formulate a consistent model that describes a general RFP equilibrium, and consequently determine the confinement performance of a standard RFP-type of fusion reactor. The main picture that motivates the model is that the RFP equilibrium relaxes to a minimum energy state which is marginally stable versus tearing modes. As illustrated in Chapter 2, Taylor’s Relaxation Theory provides the absolute minimum energy state, that corresponds to the Bessel Function Model (see Equation (2.37)) with zero pressure gradient. Thus, intuitively, one can see the idea behind the model that leads to a reasonable way to include a finite pressure gradient (corresponding to a confined plasma) into Taylor’s minimum energy magnetic configuration: imagine starting from one particular Taylor state which is stable versus tearing modes, and then add as much pressure as needed to reach tearing-mode marginal stability at every point in the plasma region. As described in Ref. [41] this idea has been supported in the past by some experimental work, but it is not conclusive. That is, it is not clear from an experimental point of view if an RFP equilibrium is actually tearing-mode marginally stable. Still, it is very interesting to develop a fully consistent model which finds such equilibria for an RFP configuration, and then compare them with the experimental results. It is important to recognize that some attempts to create such a
model have been made in the past (see Ref. [41]), but to the author's knowledge this is the first time that a proper criterion to define tearing-mode marginal stability is used. In this chapter, a detailed description of the model is given, together with its computational implementation.

5.2 The Model

5.2.1 The Geometry

First, the geometry will be defined. The RFP is a toroidal device, with a major radius $R$ and a minor radius $a$. A good approximation is given by considering the topologically equivalent cylinder, having radius $a$, and axial length $L = 2\pi R$, and imposing axial periodicity to simulate toroidicity. In the model here presented, a cylindrical geometry with axial periodicity is adopted. In fact, toroidicity effects can be considered secondary in a global analysis of an RFP, as many calculations and theories have proved; the main features of an RFP equilibrium can be quantitatively well described even in a cylindrical geometry (see Ref. [12]).

5.2.2 The Equations

In this paragraph, a set of differential equations defining the model in cylindrical geometry is derived and discussed. The first step is to determine the minimum energy relaxed state. As discussed in Chapter 2, Taylor's theory provides a good description of the core magnetic configuration of an RFP. However, it fails in describing the profiles at the edge of the plasma, because it does not take into account the plasma-wall interaction. It should be recognized that including edge physics is not a trivial task. Lacking a more general theory, for the moment it is assumed that the axial component of the equilibrium magnetic field $B_z(r)$ is a known function, properly chosen in such a way that it not only guarantees that the configuration is a minimum energy state, but also makes the plasma profiles at the edge behave in accordance with experimental evidence. Due to the complexity of the edge, the basic philosophy
of the model on how to treat this region, is to minimize the edge plasma contribution, especially in terms of the global confinement parameters. For instance the formation of artificial pedestals in the profiles is avoided. This is what might be called the "unobtrusive model for the edge". It is critical to treat the edge, at least quasi-realistically. The pure Taylor model which predicts a finite current density at the wall, coupled with the experimental condition of temperature vanishing at the wall, leads to an infinite ohmic power, and consequently the energy confinement time would be zero. This is clearly not physical. Thus, granted that one component of the equilibrium magnetic field (say $B_z$) is known, and characterized by several parameters, then two profiles remain to completely describe the equilibrium configuration in a cylindrical geometry. These profiles are $B_\theta(r)$ and $p(r)$. MHD force balance provides one obvious relationship between these quantities:

$$B_z B'_z + \frac{B_\theta}{r} (r B_\theta)' + \mu_0 p' = 0$$

(5.1)

The second relation is given by the marginal stability condition, as found in Section 3.5: at marginality ($Q = 0$) Equation (3.191) becomes

$$\Delta' = -\frac{\pi^{3/2}}{2} \frac{D}{\delta_x}$$

(5.2)

This condition holds only at the resonant surfaces on which $\Delta'$ is evaluated. In the adopted geometry (cylinder with axial periodicity), the resonant surfaces are discrete for a given aspect ratio $\frac{R}{a}$, that is for a particular device. In order to keep the model as general as possible, valid for every machine, it is important to keep the major radius unspecified: the results have to be independent from the aspect ratio. Moreover, the tearing-mode eigenfunctions are global, practically extending overall the entire plasma region. This justifies using Equation (5.2) as a continuum rather than discrete constraint. In short, in the model the marginality condition will be imposed continuously at every radial location in the plasma region. In this sense, Equation (5.2) is a differential equation relating several plasma quantities; in particular, it is a relationship involving not only $B_\theta(r)$ and $p(r)$, but also the thermal
conductivities in the parallel and perpendicular direction. Hence, further physics needs to be added in order to close the system. Determining the plasma transport coefficients is also another large area of plasma physics. It is important to observe, though, that the perpendicular transport has to be consistent with the resulting pressure gradient. In fact, the assumption that tearing-mode dominated turbulence generates and is consistent with the plasma equilibrium implies certain information about the (perpendicular) transport in the discharge. The formula that states what was just said in words is the equilibrium power balance, as found in Equation (3.150) and rewritten here in cylindrical geometry (for constant $n$):

$$\frac{1}{r}(r \chi_{\perp} p')' = -\left(\eta_{\perp} j_{\perp}^2 + \eta_{\parallel} j_{\parallel}^2\right)$$  \hspace{1cm} (5.3)

In order to close the system, parallel thermal diffusivity and plasma resistivity should finally be assigned. It is reasonable to assume the following classical values of these transport coefficients as given in Ref. [45]:

$$\chi_{\parallel}[m^2/s] = \tilde{\chi}_{\parallel} \frac{T_{eV}^{5/2}}{n_{[m^{-3}]}}$$  \hspace{1cm} (5.4)

$$\eta_{\parallel}[\Omega m] = \frac{\eta_{\perp}}{1.96} = \tilde{\eta} T_{eV}^{-3/2}$$  \hspace{1cm} (5.5)

where

$$\tilde{\chi}_{\parallel} = 1.72 \times 10^{21}$$  \hspace{1cm} (5.6)

$$\tilde{\eta} = 1.05 \times 10^{-3}$$  \hspace{1cm} (5.7)

Keep in mind that the transport model deriving from this choice is not going to be classical, as the self-consistently determined $\chi_{\perp}$ will turn out to be much higher (about two orders of magnitude in the plasma core) than its classical value as found in Ref. [45]; despite its simplicity, one of the attractive features of this model lies in the derivation of the radial profile for $\chi_{\perp}(r)$ consistent with the marginally stable equilibrium, and this can also be looked at as an interesting radial estimate of
perpendicular anomalous transport in an RFP. Finally, the closed set of differential
equations, whose unknowns are \( B_\theta(r) \), \( p(r) \), and \( \chi_\perp(r) \) can be rewritten as

\[
\begin{align*}
B_z B'_z + \frac{B_\theta}{r}(r B_\theta)' + \mu_0 p' &= 0 \\
\frac{1}{r}(r \chi_\perp p')' &= -\frac{n_0}{T_{\text{[eV]}}} (1.96 j_\perp^2 + j_\parallel^2) \\
\frac{\mu_0 p' T^\frac{5}{3}_{\text{[eV]}}}{\chi_\perp^{\frac{2}{3}}} &= \frac{\Delta' n_0^\frac{1}{3} B_z^2}{\frac{B}{m B_\theta}} \left( \left| 1 + \frac{r B'_z}{B_z} - \frac{r B'_\theta}{B_\theta} \right| \right)^{\frac{3}{2}}
\end{align*}
(5.8)
\]

Plasma density \( n \) is assumed constant, in good agreement with experimental observa-
tions, and its value is also assigned in the model. Temperature and current densities
are linked to the three unknowns via the MHD relations

\[
\nabla \times \vec{B} = \mu_0 \vec{j} \\
p = n T_{\text{[eV]}} e
\]
(5.9)

Here \( e \) is the proton charge expressed in Coulombs, and \( m \) is the poloidal mode
number. Aside from the temperature, expressed in energy units \([eV]\), as traditionally
done in plasma physics, all other quantities are given in MKS units. The system of
equations (5.8) is highly nonlinear, but the main difficulty arises from \( \Delta' \), which is
a global quantity. It has to be determined via Equation (3.86) after integrating the
differential equation (3.37) through the entire plasma region. Equation (3.37) depends
upon the entire equilibrium magnetic field and pressure profiles, and this makes the
numerical solution of Equation (5.8) extremely challenging. In the present work,
an iterative approach has been used to solve Equation (5.8): the basic assumption
is that the plasma pressure will turn out to be small \((\beta_p \ll 1)\), as suggested by the
experiments. This approach dramatically simplifies the solving procedure, as it will
be shown next; moreover, the convergence of the method will turn out to be quite
fast, and indeed the resulting equilibrium pressure will be small as assumed.
5.2.3 The Solving Procedure

If the plasma had zero pressure gradient, \( B_\theta(r) \) could be evaluated immediately by numerically integrating the MHD force balance, since \( B_z \) is an assigned function in the model. Furthermore, \( \Delta' \) could also be determined numerically by integrating (3.37) with no pressure gradient, obtaining the dependency for \( \Delta' \) on the (continuous) radial location of the resonant surface, that is

\[
\Delta' = \Delta'_0(r)
\]

(5.10)

Once \( \Delta'_0(r) \) is known, the other two unknowns \( p(r) \) and \( \chi_\perp(r) \) can be found by numerically solving the remaining equations in (5.8), which now can be written as

\[
\begin{align*}
T_{\nu[eV]}^3 \cdot (r \chi_{\perp} T_{\nu[eV]}^\nu)' &= -\frac{\vec{\chi}_3 \cdot \vec{r}}{eH} \\
\frac{T_{\nu[eV]}^\nu T_{\xi[eV]}^{\xi\nu}}{\chi_{\perp}^{1/4}} &= g(r) = \frac{\Delta'_0 B_z^2 n}{e\pi^{1/2} \mu_0 \chi_{\parallel}} \frac{3}{2} \sqrt{\frac{B}{m B_\parallel}} \left( \left| 1 + \frac{r B_z^2}{B_z^2 - \frac{r B_\parallel^2}{B_\parallel}} \right| \right)^{3/2}
\end{align*}
\]

(5.11)

Since in the RHS of Equation (5.11) is now known, the following numerical algorithm is adopted for the solution:

\[
\begin{align*}
z_1 = T_{\nu[eV]}; & \quad z_2 = r \chi_{\perp} T_{\nu[eV]}^\nu \\
\begin{align*}
z'_1 &= \frac{z_2}{r \chi_{\perp}} \\
\chi_{\perp} &= \left( \frac{z_2}{r g(r)} \right)^{4/3} \sqrt{z_1}
\end{align*}
\end{align*}
\]

(5.12)

Notice that in order to solve Equation (5.12), the value of temperature (or pressure) on axis \( T_0 \) (or \( p_0 \)) should be given; since this value will be determined by the boundary condition at the wall \( p|_a = 0 \) (see Section 5.2.4), the solving procedure then involves an initial guess for \( T_0 \). The entire algorithm described in this section is carried and then iteratively repeated by adjusting the initial guess of \( T_0 \) until the condition \( p|_a = 0 \) is satisfied.
Analytic Solution on Axis

By expansion around \( r = 0 \), it is possible to find the analytic solution of Equation (5.12) on axis. Assigning

\[
B_z = B_{z0}(1 - d_1 r^2 + d_2 r^4 + O(r^6))
\]

then \( B_\theta \) is given by MHD force balance with no pressure

\[
B_\theta = B_{z0}(b_1 r + b_2 r^3 + O(r^5))
\]

where \( b_1 = \sqrt{d_1} \) and \( b_2 = -\frac{d_1^2 + 2d_2}{3\sqrt{d_1}} \). Furthermore,

\[
\mu_0 j_\parallel = 2B_{z0}b_1 + O(r^2))
\]

\[
\Delta' = -\frac{c_1}{r}
\]

as shown in Chapter 4 (see [2] and [50]), whence \( g(r) \cong g_0 r^{\frac{3}{2}} \). The leading order radial dependence for \( p(r) \) and \( \chi_\perp(r) \) on axis can then be evaluated from Equation (5.12): using the boundary condition that \( (r\chi_\perp T')_0 = 0 \) (no heat source on axis), and taking \( T = T_0 + T_1 r^\alpha \), it is easy to show that

\[
\begin{align*}
    r\chi_\perp T'_0 & \sim r^2 \Rightarrow \chi_\perp \sim r^{2-\alpha} \\
    \frac{r^{\alpha-1}}{r^{2-\alpha}} & \sim r^{\frac{3}{2}} \Rightarrow \alpha = \frac{12}{7}
\end{align*}
\]

so that the equilibrium temperature (or equivalently pressure) profile on axis generated by Equation (5.12) will be

\[
T = T_0 + T_1 r^{\frac{12}{7}}
\]

This profile, however, turns out to be unstable to ideal MHD modes; in particular,
it violates Suydam's criterion on axis (see [6]):

\[
\mu_0 p' = -\frac{B_z^2}{8r} \left[ 1 + \frac{rB_z'}{B_z} - \frac{rB_\theta'}{B_\theta} \right]^2 \Rightarrow \mu_0 p'|_0 = -\frac{B_z^2}{2} \left[ d_1 + \frac{b_2}{b_1} \right]^2 r^3
\]  

(5.19)

As Equation (5.19) shows, Suydam’s criterion for marginal stability to ideal MHD (localized) modes forces the pressure to be very flat on axis, much flatter than Equation (5.12) would yield; this suggests that the equilibrium profiles for temperature (or pressure) and \( \chi_\perp \) on axis will be given by Suydam’s condition, and not by Equation (5.18). As the evaluation of the profiles moves away from the axis, \( \Delta_0 \) becomes finite and smaller, and that lessens the pressure gradient generated by the tearing-mode marginality condition, until eventually it becomes smaller than the one generated by Suydam’s condition. At the radial location where tearing-mode marginality becomes more restrictive than Suydam’s, a transition will occur, in such a way that the pressure and the correspondent \( \chi_\perp \) will always be determined by the most restrictive criterion everywhere in the plasma region. From a numerical point of view, it is easier to localize this transition by looking at the thermal diffusivity rather than the pressure gradients. Having established that \( p' \) on axis is given by Suydam’s criterion, it can be immediately evaluated via Equation (5.19); \( \chi_\perp \) is then found directly from integrating the power balance. The transition can easily be seen by comparing \( \chi_\perp \) with the correspondent function \( \chi_{\perp,T-M} \) that would result from the tearing-mode marginal condition for the same profile of Suydam-marginal pressure gradient:

\[
\chi_{\perp,T-M} = \left[ \frac{T' T^{\frac{4}{3}}}{g(r)} \right]^4
\]  

(5.20)

It is easy to show that on axis \( \chi_\perp \sim r^{-2} \), while \( \chi_{\perp,T-M} \sim r^6 \), so that it is straightforward to distinguish among the two quantities. The radial location where the two functions eventually intersect, defines the transition point \( r_1 \) (see Fig. 5-1); for \( r > r_1 \), \( p' \) and \( \chi_\perp \) will be determined by the tearing-mode marginality condition (which will now be the most restrictive one), always coupled together with the power balance.
Analytic Solution at the Reversal Point

Similarly as was carried out on axis, expansion around the point of axial field reversal \( r_0 \) will also lead to an analytic solution for Equation (5.12).

Now \( B_z = B'_z|_{r_0} (r - r_0) + O(r - r_0)^2 \), while \( B_\theta = B_{\theta|_{r_0}} \) and \( \mu_0 j_\parallel = - (B_\theta B'_z)|_{r_0} \) are finite quantities. Moreover, as shown in Chapter 4, \( \Delta_0' \sim -|r - r_0|^{-1} \). Thus \( g \sim |r - r_0|^{-1/2} \). By taking \( T = T_0 + T_1 |r - r_0|^\alpha \), Equation (5.11) then gives

\[
\begin{align*}
\chi_\perp T' &\sim |r - r_0|^0 \quad \Rightarrow \quad \chi_\perp \sim |r - r_0|^{2/5} \\
\chi_\perp \sim T' \sim |r - r_0|^2 
\end{align*}
\]

As the reversal point is approached, the temperature (equivalently pressure) gradient given by Equation (5.11) becomes steeper and steeper, and eventually it approaches infinity; this means that there will be a thin region around \( r = r_0 \) in which again the tearing-mode marginal pressure gradient profile will exceed Suydam’s limit,
as happens on axis. Similarly, the model will go through a transition in which the pressure gradient in this layer around $r_0$ is determined by Suydam's marginal condition (5.19). As is shown later, from a numerical point of view it turns out that this region is very localized around $r = r_0$, due to the weak dependency of $p'$ versus $r - r_0$, and also due to the large value of Suydam's limit with respect to tearing-mode marginal pressure gradient in the plasma core. Thus in practical simulations, for the used radial grids (typically five hundred radial grid points), this transition never occurs.

**Considerations about the solution at the Edge**

As the evaluation of the profiles moves past the reversal point and the thin layer around it where the pressure gradient is limited by Suydam’s condition, $p'$ and $\chi_\perp$ are again determined by tearing-mode marginality until the edge region is approached. As discussed in Chapter 4, $\Delta'_0 |_{r_r} \rightarrow -\infty$, due to the presence of the ideal wall which provides a strong artificial stabilizing effect at the plasma boundary ($r \sim a$). An accurate treatment of the edge is quite complicated. For the present purpose, it is sufficient to consistently and uniquely determine the radial location $r = r_2$ at which the adopted model loses validity due to the artificially stabilizing effect of the ideal wall; this location will set the final transition point, between the tearing-mode marginally stable region and the plasma edge region which is then defined by $r_2 < r < a$. As mentioned earlier, in the plasma edge region additional physics needs to be taken into account; for simplicity’s sake, though, in the present model an analytical continuation of the profiles is used for $r_2 < r < a$, by imposing matching conditions at $r = r_2$ and appropriate boundary conditions at $r = a$. In the next paragraph it will be shown in detail that this continuation of the profiles is suggested by experimental observations, together with the idea of “unobtrusive edge”; that is mainly avoiding the formation of artificial pedestals in the edge profiles. But before discussing these details, the continuation of the iterative process will now be illustrated.
**Iteration Process**

After having found $B_\theta(r)$, $p(r)$, and $\chi_\perp(r)$ for $0 < r < a$ in the first iteration with zero pressure on the RHS of Equation (5.8), it is straightforward to repeat the procedure for the second iteration by simply replacing the pressure profile just found from completing the first iteration into the RHS of Equation (5.8). This includes recalculating $\Delta'$ with a non-zero pressure (see Chapter 4). Then, the second iteration is performed by following exactly the same steps of the first one, leading to new profiles for $p(r)$ and $\chi_\perp$. Convergence, basically measured by the change in the global parameters of plasma confinement, is quickly reached at the end of the second iteration, as is shown later.

### 5.2.4 The Edge

**Determination of the Edge Profiles**

A dedicated section has to be reserved for the edge modelling. As mentioned in Section 5.2.2, the model assumes as an initial input the axial magnetic field profile $B_z(r)$, properly chosen so that it describes a minimum energy state as given by Taylor's theory, and at the same time taking into account the presence of the wall at $r = a$. Experiments indicate that at the wall the plasma current density, pressure and pressure gradient drop dramatically from their respective values in the plasma core region. The presence of a wall then mainly cools the plasma at the edge; this feature is not contained into Taylor's theory, for the BFM allows finite current densities at $r = a$. It has to be remembered that edge measurements typically record a substantial drop in those plasma quantities, although they don't completely vanish at the wall. For the purpose of this model, the following boundary conditions at the wall are imposed

$$
\tilde{J}|_a = 0 \ ; \quad p'|_a = 0 \ ; \quad p|_a = 0 \ ; \quad T|_a = 0 \ ; \quad (5.22)
$$

By forcing all those quantities to be exactly zero at $r = a$, the model does not
depend upon free parameters which are impossible to determine without invoking edge physics. This is also part of making the edge profiles relatively unimportant in the evaluation of the RFP global parameters. The appropriate choice for $B_z(r)$ has to take into account (5.22). In the present work, the following profile for the radial derivative of $B_z(r)$ is used:

$$B'_z(r) = -B_{z0} \mu j_1(\mu r) \left[ \frac{\tanh \left( f \left( 1 - \frac{r^2}{a^2} \right) \right)}{\tanh(f)} \right]^{\nu}$$

(5.23)

This function is the product between the BFM solution (see Equation 2.37) and an edge term, described by the hyperbolic tangent (see Fig. 5-2), such that the resulting profile will coincide with Taylor’s relaxed state in the plasma core, while it will be modified only at the edge in order to automatically satisfy the condition of vanishing current density at the wall (see Equation (5.22)). Two free parameters, $f$ and $\nu$, appear in the edge term: basically, $f$ defines the radial location of the boundary
between core and edge region, while $\nu$ determines the behavior of the profiles right at the wall. From physical considerations, a prescription will now be provided in order to uniquely determine $f$ and $\nu$. The first step is related to finding the radial location $r = r_2$ at which the edge region begins. As mentioned in the previous section, in this model the plasma edge is the region in which the stabilizing effect due to the perfectly conducting wall would become significant in the determination of the plasma profiles. This stabilising effect is 'artificial': it appears as a large and negative (i.e. very stable) $\Delta'$ (see Chapter 4) at $r \approx a$, which would generate a large pressure gradient (and corresponding pedestal) near the wall, were the tearing-mode marginality condition applied. Due to these considerations, a reasonable way to determine $r_2$ is found by examining the typical $\Delta'$ profile (see Fig. 5-3): $r_2$ is defined as the radial location between the reversal point $r = r_0$ and the wall $r = a$, at which $\Delta'_0$ (that is the $\Delta'$ calculated in the first iteration, with zero pressure) has a maximum. In this way, the tearing-mode marginality condition is not used to determine the plasma profiles in the region near the wall where $\Delta'$ becomes large and negative. This procedure avoids the 'artificially' stabilizing wall effect. Once $r_2$ is defined, a natural prescription for determining the free parameter $f$ can be devised: $f$ is such that the edge term introduced in Equation (5.23), evaluated at $r = r_2$, is equal to half its value on axis

$$\frac{\tanh \left[ f \left( 1 - \frac{r_2^2}{a^2} \right) \right]}{\tanh(f)} = \frac{1}{2}$$

(5.24)

Since $\Delta'_0$ is dependent on $f$ (through $B'_z$), some iteration is initially required at the beginning of each simulation in order to find $f$. This iteration is straightforward. Now what remains is to give a prescription for $\nu$, to completely define the model. The free parameter $\nu$ basically determines the plasma profiles right at the wall $r \approx a$, since

$$B'_z(r \approx a) \sim \left( 1 - \frac{r}{a} \right)^\nu$$

(5.25)

Aside from the boundary conditions (5.22), an additional constraint at the wall
arises from Ohm’s Law:

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$$ \hfill (5.26)

The electric field at $r = a$ has to stay finite, which means that

$$T^3 \sim j \sim \left(1 - \frac{r}{a}\right)\nu$$ \hfill (5.27)

based on Spitzer’s formula for plasma resistivity (see Equation (5.5)). Furthermore, MHD force balance requires that

$$p' B'_z \sim \left(1 - \frac{r}{a}\right)\nu \Rightarrow p \sim \left(1 - \frac{r}{a}\right)^{\nu + 1} \hfill (5.28)$$

Thus the plasma density cannot be held constant anymore, but it has to vanish at
the wall

\[ n \sim \left(1 - \frac{r}{a}\right)^{\frac{5}{2} + 1} \]  \hspace{1cm} (5.29)

Several attempts have been made to extend the plasma profiles in the edge region, satisfying smooth matching at \( r = r_2 \) and the boundary conditions at \( r = a \); in most cases, not all the profiles could be smoothly matched up to the first derivative, or else the resulting edge profiles would show a pedestal. Hence, only the best solution obtained is shown, and then used in the remaining part of the work. This edge solution gives the analytic extension of the plasma profiles in the edge region as follows

\[
p' = \left[p' \big|_{r_2}(1 + \nu) + p'' \big|_{r_2}(a - r_2)\right] \left(\frac{a - r}{a - r_2}\right)^\nu + \hspace{1cm} (5.30)
\]

\[
- \left[\nu p' \big|_{r_2} + p'' \big|_{r_2}(a - r_2)\right] \left(\frac{a - r}{a - r_2}\right)^{\nu + 1}
\]

\[
T = \left[\left(\frac{2}{3} \nu + 1\right) T \big|_{r_2} + T' \big|_{r_2}(a - r_2)\right] \left(\frac{a - r}{a - r_2}\right)^{\frac{2}{3} \nu} + \hspace{1cm} (5.31)
\]

\[
- \left[\frac{2}{3} \nu T \big|_{r_2} + T'' \big|_{r_2}(a - r_2)\right] \left(\frac{a - r}{a - r_2}\right)^{\frac{2}{3} \nu + 1}
\]

\[
j_\parallel = \frac{j_\theta B_\theta + j_z B_z}{B} \hspace{1cm} (5.32)
\]

In words, the smooth, edge modified BFM solution for \( j_\parallel \) is simply extended into the edge region, while the pressure is matched at \( r = r_2 \) up to its second derivative, and the temperature is continuous up to its first derivative. As far as the determination of the free parameter \( \nu \) is concerned, only a narrow range of values for \( \nu \) is satisfactory: typically for \( \nu < 1 \), pedestals in the current density and temperature appear at the edge, while for \( \nu > \frac{3}{2} \) the plasma pressure profiles at the wall become too flat (see Equation (5.28)). In the present model, results with \( \nu = 1 \) are shown. It is interesting to note that for \( 1 < \nu < \frac{3}{2} \), the global plasma parameters do not change very much, and thus are insensitive to the particular choice for \( \nu \).
Chapter 6

Results and Comparisons

In this chapter, results from the model just described, and comparisons with existing experimental, theoretical, and computational works are presented. As mentioned earlier, the goal of the model is to provide an insight to the global features of the RFP, mainly in terms of confinement performance. Two types of results are shown: a comparison with standard operating shots in the main modern devices, MST and RFX (see description in Chapter 1), and the extraction of a scaling law for plasma confinement parameters by building a large database of simulations. Before starting, it is helpful to introduce and define the two main confinement parameters that will be evaluated by the model: the plasma beta parameter $\beta_p$, and the energy confinement time $\tau_E$. The plasma beta is a dimensionless parameter, defined as the ratio of the plasma pressure to the magnetic pressure; thus, it gives a measure of confinement efficiency, the externally generated magnetic pressure representing the cost to confine the plasma pressure. The formula for $\beta_p$ used in this work is the following (see Ref. [6]):

$$\beta_p = \frac{4\mu_0}{a^2 B_0^2} \int_0^a pr \, dr$$

(6.1)

where the volume-averaged pressure is taken in the numerator, while the (poloidal) magnetic pressure at the wall is in the denominator. This $\beta_p$ then is directly related
to the plasma axial current $I$ via the relation

$$B_{\theta a} = \frac{\mu_0 I}{2\pi a} \quad (6.2)$$

The energy confinement time is a measure of the typical time scale in which the plasma energy remains confined without external supply. In other words, it is the ratio of the plasma thermal energy $W_p = \int_V \frac{3}{2} p \, dV$ (see Equation (2.15)) to the power loss, so it has the dimensions of a time. The expression for $\tau_E$ can be evaluated from energy balance; for instance, dot-multiplying Faraday’s law (Equation (3.4)) by $\vec{B}$

$$\frac{\partial \left( \frac{B^2}{2} \right)}{\partial t} = -\nabla \cdot (\vec{E} \times \vec{B}) - \vec{E} \cdot \vec{j}_{\mu_0} \quad (6.3)$$

It is easy to recognize the Poynting vector $\vec{S}_y$ in the first term on the RHS of Equation (6.3), which when integrated over the plasma volume gives the total power externally supplied to the discharge ($P_{in}$). In steady state operation $\left( \frac{\partial}{\partial t} = 0 \right)$, and this power input equals the power lost by the system ($P_{out}$), so that Equation (6.3) gives

$$P_{out} = P_{in} = -\int_V \nabla \cdot \left( \frac{\vec{E} \times \vec{B}}{\mu_0} \right) \, dV = \int_V \vec{E} \cdot \vec{j} \, dV \quad (6.4)$$

Using Ohm’s law (see Equation (3.3)), $P_{out}$ can be written as

$$P_{out} = \int_V \eta j^2 \, dV + \int_V \vec{j} \times \vec{B} \cdot \vec{v} \, dV =$$

$$= \int_V \eta j^2 \, dV + \int_V \nabla p \cdot \vec{v} \, dV =$$

$$= \int_V \eta j^2 \, dV - \int_V p \nabla \cdot \vec{v} \, dV \quad (6.5)$$

having used the boundary condition that $p\vec{v} \cdot \hat{n}$ exactly vanishes at the plasma wall ($\hat{n}$ being the unit vector normal to the surface containing the plasma volume $V$).

Equation (6.5) puts in evidence the ohmic power term and the convective term,
related to the dynamo activity. Thus, the energy confinement time can be written as

$$\tau_E \equiv \frac{3}{2} \frac{\int_0^a pr \, dr}{\int_0^a (\eta j^2 - p \nabla \cdot \vec{v}) r \, dr}$$ (6.6)

While the ohmic term is easily found in terms of plasma equilibrium profiles (assuming for instance Spitzer’s formula for plasma resistivity), the proper evaluation of the dynamo term is quite cumbersome, and it requires a much more complicated model. It is interesting to note that under the assumption of plasma incompressibility ($\nabla \cdot \vec{v} = 0$), the dynamo term exactly vanishes; it is easy to show that this term would also be zero if the adiabatic form of the energy equation (3.14) were used. Moreover, even without invoking incompressibility, or adiabaticity, there is some belief based on experimental data that the dynamo term might average out to zero when integrated over the plasma volume. That is, physically the dynamo transfers energy from one region of the plasma to another, but does not directly generate a large net energy loss. In any event, an accurate treatment of the dynamo term goes beyond the purpose of this thesis, and so for this reason it is neglected in the calculation of $\tau_E$. Finally, the expression for the energy confinement time used in this work is

$$\tau_E \equiv \frac{3}{2} \frac{\int_0^a pr \, dr}{\int_0^a \eta j^2 r \, dr}$$ (6.7)

In the following section, results from the model simulating typical standard shots at MST and RFX will be shown; these special runs will also be used to give the reader a deeper insight into the solution procedure and iterative convergence as described in Chapter 5.

### 6.1 Comparison with Standard RFP Experiments in MST and RFX

In this section, the standard RFP shots, given in Table 1.1 and Table 1.3 for MST and RFX are simulated by the model. Note that MST data are ensemble averages of
similar shots, while RFX data corresponds more to the observed range in standard RFP operation (i.e. no profile control, no PPCD, etc.) The first step is defining the plasma parameters to be assigned such that a fair comparison with the experiments can be performed. It is important to notice that the model has only three parameters that can be adjusted in order to match experimental data: two of them correspond to the pair of plasma parameters which define Taylor’s relaxed state, as explained in Chapter 2 (i.e. $\mu$ and $B_0$, or $\Phi_t$ and $K_h$, or $\Theta$ and $I$, and so on), while the third one is the core plasma density (which is constant for $0 < r < r_2$). All these parameters are known physical quantities; it is important to stress again that the model has no free fitting parameters to adjust. Having direct data for plasma density from Table 1.1 and Table 1.3, one can choose the remaining two parameters to use in the model based on the total magnetic flux $\Phi_t$ and plasma current $I$. These are the ones which hopefully have better accuracy in the measurements. In Table 6.1, the known fixed plasma parameters used for the simulated standard shots in MST and RFX
<table>
<thead>
<tr>
<th>Device</th>
<th>$I_{[MA]}$</th>
<th>$\Phi_{[Wb]}$</th>
<th>$n_{[10^{19}m^{-3}]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MST</td>
<td>0.376</td>
<td>0.0696</td>
<td>1.4</td>
</tr>
<tr>
<td>RFX</td>
<td>0.750</td>
<td>0.141</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Table 6.1: Plasma Parameters Held Fixed in the Simulation of RFPs Standard Shots

are recorded. By running the model, the first interesting general result is that the poloidal number $m = 1$ is the most stringent one in terms of the marginal stability condition (see Fig. 6-1), as expected from the discussion in Chapter 4. Fig. 6-1 also shows the curves for $\Delta'_{m=1}$ from Iteration I (zero pressure gradient) and Iteration II (containing the plasma pressure from Iteration I). It is important to observe that no significant changes occur by introducing the pressure, aside from small differences in the regions around the axis, and much less evidently around the reversal. Both of those regions have a small impact on the pressure profile generated from Iteration I, because on axis $p'$ is indeed very flat, due to Suydam's marginality condition, while around the reversal the change in $\Delta'$ is quite small: overall, the change in plasma pressure from Iteration I to Iteration II is less than 10%, so that Iteration II will already show convergence. In Fig. 6-2 and Fig. 6-3, the temperature profiles for MST and RFX are shown, together with the radial locations of the transition points in the model, as defined them in Chapter 5. In Fig. 6-4 and Fig. 6-5, the profiles for the self-consistent determined $\chi_\perp$ for MST and RFX are shown. It is interesting to observe the area of low thermal transport around the reversal layer, corresponding to the best confinement region (steepest pressure gradient). It is also interesting to point out that the $\chi_\perp$ corresponding to tearing-mode marginal stability transport, as evaluated from the model, is about $20 - 60m^2/s$ through most of the plasma core; that is about two orders of magnitude larger than the classical value (see Equation (3.143)). Finally, in Table 6.2 and Table 6.3, the experimental confinement
parameters as given in Table 1.1 and Table 1.3 are compared with the results from
the model. These comparisons show a fair agreement (certainly much better than
other transport scenarios, as it will be discussed in the next section).

6.2 Scaling Laws for Confinement Parameters

The second type of result obtained with the model is the extraction of scaling laws
for confinement parameters. A large ($\sim 100$) database of simulations has been built,
each taking about two hours of computing time on a Sun Untra5 workstation; most
of this time is spent calculating a detailed radial profile for $\Delta'$ on a fine grid ($\sim 500$
grid points). Due to the structure of the model, the dependence of $b_p$ and $\tau_E$ on the
Figure 6-3: Tearing-Mode Marginally Stable Temperature Profile for RFX Standard Shot

<table>
<thead>
<tr>
<th></th>
<th>MST Standard Shot</th>
<th>T-M Marginality</th>
<th>Suydam Marginality</th>
<th>Classical Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_p[%]$</td>
<td>6.3</td>
<td>5.3</td>
<td>19.3</td>
<td>21.3</td>
</tr>
<tr>
<td>$\tau_E[ms]$</td>
<td>1.4</td>
<td>2.0</td>
<td>41.0</td>
<td>37.4</td>
</tr>
</tbody>
</table>

Table 6.2: Plasma Confinement Parameters For the MST Standard Shot
<table>
<thead>
<tr>
<th></th>
<th>RFX Standard Shot</th>
<th>T-M Marginality</th>
<th>Suydam Marginality</th>
<th>Classical Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_p[%]$</td>
<td>5</td>
<td>4.0</td>
<td>17.4</td>
<td>19.8</td>
</tr>
<tr>
<td>$\tau_E[ms]$</td>
<td>0.5/1.5</td>
<td>2.4</td>
<td>43.4</td>
<td>57.1</td>
</tr>
</tbody>
</table>

Table 6.3: Plasma Confinement Parameters For the RFX Standard Shot

The following plasma parameters can be extracted:

\[
I = 2\pi \int_0^a j_\varphi r \, dr = \frac{2\pi a}{\mu_0} B_{\theta a},
\]

\[
N = 2\pi \int_0^a n r \, dr
\]

\[a \to \text{minor radius}\]

\[\Theta = \frac{2\pi a B_{\phi n}}{\Phi_t}\]

(6.8)

It is possible to extract the scaling laws for $\beta_p$ and $\tau_E$ analytically for the Iteration I, when $a\Delta_0'$ is only a function of $\mu a$ (for a given poloidal mode $m$, of course), or equivalently $\Theta$. In this case, using the following scaling relations (together with Spitzer resistivity)

\[
B_0 \sim \frac{\mu_0 I}{a}, \quad j \sim \frac{I}{a^2}, \quad n \sim \frac{N}{a^2}, \quad \tau_E \sim \frac{p^2 a^7}{N \beta I^2}, \quad \beta_p \sim \frac{p a^2}{\mu_0 I^2},
\]

(6.9)

it is possible to find the scaling for $p$ from the tearing-mode marginality condition (see Equation (4.1))

\[
p \sim (a\Delta_0') \left( \frac{\chi_\perp}{\chi_\parallel} \right)^{\frac{1}{2}} B_0^2 \sim (a\Delta_0') \frac{1}{p^{7/8}} \frac{I^2 N^{7/8}}{a^{15/4}}
\]

(6.10)
Hence

$$\tau_e \sim \frac{a^2}{\chi_\perp} \sim (a\Delta_c')^4 \frac{I^8 N^{7/2}}{p^{13/2} a^{13}}$$  \hspace{1cm} (6.11)$$

which together with Equation (6.9) gives

$$p \sim \frac{I^{10/9} N^{5/9}}{a^{20/9}} (a\Delta_c')^{4/9}$$  \hspace{1cm} (6.12)$$

This gives the following analytical scaling laws from Iteration I

$$\tau_E = \frac{I^{7/5} \cdot a^{13/9}}{N^{1/5}} \cdot G_I(\Theta), \quad \beta_p = \frac{N^{5/8}}{I^{3/8} \cdot a^{2/5}} \cdot K_I(\Theta)$$  \hspace{1cm} (6.13)$$

These formulas are in good agreement with the numerical scaling laws found using only the Iteration I in the database. Iteration I also allows one to determine the functions $G_I(\Theta)$ and $K_I(\Theta)$ (shown in Fig. 6-6 and Fig. 6-7, respectively, for the
parameters corresponding to the MST Standard Shot as in Table 6.1); these functions show the dependence of plasma confinement upon the pinch parameter $\Theta$. Note that $G_\ell(\Theta)$ and $K_\ell(\Theta)$ are completely determined. There are no free parameters. This leads to a magnitude as well as scaling for $\tau_E$ and $\beta_p$. It is interesting to note that varying $\Theta$ throughout its range, that is for $\Theta$ between $\Theta_{\text{min}}$ (given by the reversal condition) and $\Theta_{\text{max}}$ (given by the tearing-mode instability condition $\Delta_0' > 0$ occurring somewhere in the plasma region), only a relatively small change in $\beta_p$ and $\tau_E$ is found, showing a broad peak around typical RFP operational values, and giving somewhat poorer confinement (but still finite) in the proximity of $\Theta_{\text{min}}$ and $\Theta_{\text{max}}$. This is reasonable agreement with experimental results (see Ref. [51]), where no strong effect of confinement performance versus $\Theta$ is observed. It is much more difficult to extract the scaling laws from Iteration II analytically, as now $\Delta'$ is not just a function of $\Theta$, but it depends upon $B_0$ and $n$ as well, in a very complicated way. Even the numerical extraction from the database is less straightforward, as all
the global plasma parameters change a little from the ones assigned in Iteration I, so a multiple linear regression has to be performed in order to extract all the coefficients. The following scaling laws are extracted from Iteration II:

\[
\tau_E = I^{1.1} \cdot N^{-0.25} \cdot a^{1.6} \cdot G_{II}(\Theta) \quad , \quad \beta_p \sim I^{-0.75} \cdot N^{0.49} \cdot a^{-0.16} \cdot K_{II}(\Theta) \quad (6.14)
\]

The dependence of plasma confinement upon the pinch parameter \( \Theta \) is shown in Fig. 6-8 and Fig. 6-9, for the parameters corresponding to the MST Standard Shot (see Table 6.1). Like for Iteration I, the dependency on \( \Theta \) is completely determined by the model, and it shows similar features as discussed above.
6.3 Comparison with Other Transport Models

It is interesting to compare the results found by using the tearing-mode marginal stability transport model, with the predictions and the scalings given by other transport scenarios. In particular, it is instructive to observe the main quantitative difference in predicting the confinement performance in an RFP. In this section, three different transport scenarios are discussed. The first one assumes plasma turbulence to be dominated by ideal interchange modes; this corresponds to replacing the tearing-mode marginal condition with its analogue for ideal interchange modes in order to find the equilibrium profiles in an RFP. The second scenario assumes the transport to be completely classical, as described by Braginskii in Ref. [45]. The final type of transport to be investigated is the one described by Bohm diffusion (see Ref. [52]). A brief physical description of all these scenarios is now presented, before showing the respective predictions and comparing them with the results obtained by the tearing-mode marginal stability transport model.

6.3.1 Ideal Interchange-Mode Dominated Transport

This scenario is obtained by replacing the tearing-mode marginal stability condition with its analogue for ideal interchange modes. Suydam’s criterion provides a local necessary (but not sufficient) condition for marginality (see Ref. [6])

$$\mu_0 p' + \frac{r}{\delta} \left[ \frac{B_z}{r} + B'_z - \frac{B_z B'_\theta}{B_\theta} \right]^2 = 0$$

(6.15)

derived by examining localized trial functions in the MHD energy principle. This condition sets a relation for the three profiles describing the cylindrical plasma equilibrium $B_\theta(r), B_z(r),$ and $p(r)$. In analogy with the transport model described in this thesis work, the plasma equilibrium in the ideal interchange mode dominated scenario is simply determined as follows: assuming the same analytical expression for
\( B_z(r) \) as given in Equation (5.23)

\[
B'_z = -B_{z0} \mu H(r) \cdot H(\mu r) , \quad H \equiv \frac{\tanh \left[ j \left( 1 - \frac{r^2}{a^2} \right) \right]}{\tanh (j)}
\tag{6.16}
\]

solve the MHD force balance (Equation (5.1)) and the marginal stability condition (Equation (6.15)) for the remaining profiles \( B_\theta(r) \) and \( p'(r) \), with the same boundary conditions as used in Equation (5.22)

\[
p'|_a = 0 \quad p|_a = 0 \quad j'|_a = 0
\tag{6.17}
\]

It is easy to observe that Equation (6.15) does not allow one to satisfy Equation (6.17) and simultaneously have a reversal in the axial magnetic field \( (B_z)|_a < 0 \). This requires a modification of Equation (6.15) at the edge; for the sake of consistency with the adopted edge approach, the edge-modified Suydam's criterion is taken to be

\[
\mu_0 p' + \frac{r}{8} \left[ \frac{B_z}{r} + B'_z - \frac{B_\theta B'_\theta}{B_\theta} \right]^2 \cdot H(r) = 0
\tag{6.18}
\]

Eliminating the pressure gradient from MHD pressure balance, Equation (6.18) becomes

\[
\frac{rH}{8} \left\{ \left[ \frac{rB_z}{r} \right]^2 + \frac{B'_z}{r} \left( \frac{B'_\theta}{B_\theta} \right)^2 - 2B_z \left( \frac{rB_z}{r} \right)' \frac{B'_\theta}{B_\theta} \right\} - B_z B'_z - \frac{B^2_\theta}{r} - B_\theta B'_\theta = 0
\tag{6.19}
\]

Solving for \( B'_\theta \) yields

\[
\left[ \frac{B'_\theta}{B_\theta} \right]_\pm = 1 + \frac{B'_z}{B_z} + \frac{4B^2_\theta}{HBz} \pm \sqrt{\frac{16B^4_\theta}{H^2B^2_z} + \frac{16B^2_\theta}{HBz} + \frac{8}{H} \frac{rB'_z}{B_\theta} \frac{B^2_\theta}{B^2_z}}
\tag{6.20}
\]

which can be numerically solved for \( B_\theta(r) \). The root with the \((-\)\) sign has to be chosen in order to satisfy the boundary condition

\[
j_z|_a = 0 \implies B'_\theta|_a \approx -\frac{B_\theta|_a}{a}
\tag{6.21}
\]
Figure 6-7: $\Theta$ Dependence for $\beta_p$ in the Tearing-Mode Marginality Scaling Law from Iteration I

Furthermore, as already seen in the previous chapter, Suydam's criterion forces the pressure gradient to be quite flat on axis, mainly $p'_{r=0} \sim r^3$. Hence the boundary condition on the axis for $B_\theta$ is given by:

$$B_\theta|_{r=0} = \sqrt{-\frac{B_{z0}}{2} \frac{B''_{z0}}{2}} \ r = B_{z0} \frac{\mu}{2} \ r$$

Once $B_\theta$ is found by integrating Equation (6.20), $p'$ and $p$ can be evaluated via MHD pressure balance, using the boundary condition $p|_{r=a} = 0$, This completes the RFP configuration of equilibrium. So far, nothing has been said about the plasma density profile in this scenario; it only comes into play in the evaluation of the energy confinement time, as defined by Equation (6.7). It is straightforward to see that the last boundary condition to still be used, the one that guarantees a finite electric field
Figure 6-8: $\Theta$ Dependence of $\tau_E$ in the Tearing-Mode Marginality Scaling Law from Iteration II

at the plasma wall (see Equation (5.27))

$$\eta j|_a \text{ finite } \implies j|_a \sim T|_a^{\frac{3}{2}}$$  \hspace{1cm} (6.23)

can be satisfied by a plasma density that vanishes like $n|_a \sim (1 - \frac{r}{a})^{\frac{4}{3}}$ as shown in Section (5.2.4) for the tearing-mode transport model. In this case, not having a prescription for the plasma temperature $T$, the density profile at the edge is directly chosen accordingly

$$n = n_0 \cdot [H(r)]^{\frac{4}{3}}$$  \hspace{1cm} (6.24)

Once the equilibrium profiles are found, they undergo a global ideal interchange-mode stability test, because Suydam’s criterion only guarantees stability to local modes. In Fig. 6-10, the temperature profile satisfying Suydam marginal stability
condition is numerically evaluated from Eqs. (6.16), (6.20), and (6.18) for a typical MST Standard Shot. Compared with the tearing-mode marginal stability case, much steeper gradients are allowed by this transport scenario, resulting in high values of temperature on axis $T_0 \sim 1.7keV$. Consequently, significantly higher values of $\beta_p$ and $\tau_E$ are found, compared with the MST and RFX experiments (see Table 6.2 and Table 6.3).

Finally, the scaling law for the ideal interchange-mode dominated transport can be extracted for the confinement parameters: note that from Equation (6.18) and the relation $I = \frac{2\pi a}{\mu_0} B_\theta(a)$

$$
\mu_0 p \sim B_{z0}^2; \quad B_{z0} \sim \mu_0 \frac{I}{a}; \quad j \sim \frac{I}{a^2}; \quad n \sim \frac{N}{a^2} \quad (6.25)
$$

where $N$ is the plasma density integrated over the poloidal cross section and $I$ is the total axial current, as defined in Equation (6.8). The energy confinement time scales as follows

$$
\tau_E = \frac{I^3_{[MA]} \rho_{[m]}^2}{N^{\frac{3}{2}}_{10^{20} m^{-1}}} \cdot G_{Suydam}(\Theta) \quad (6.26)
$$

while the beta poloidal scales as

$$
\beta_p = K_{Suydam}(\Theta). \quad (6.27)
$$

In Figure (6-11), a typical plot of $G_{Suydam}(\Theta)$ and $K_{Suydam}(\Theta)$ is shown, for $\Theta_{min} < \Theta < \Theta_{max}$, where $\Theta_{min} = 1.39$ is still given by the condition of field reversal, while $\Theta_{max} = 3.03$ is given by the instability condition to ideal MHD internal modes.

### 6.3.2 Classical Transport

This scenario is obtained by assuming that the plasma transport is described by the classical theory of transport (see Ref. [45]). In practice, the transport coefficients are assumed to be known, and to be given by Braginskii's expressions [45] in terms of
plasma equilibrium profiles. Again, as performed for other transport scenarios, $B_\theta(r)$ and $p'(r)$ are determined by integration of the following set of differential equations containing the usual MHD pressure balance and the power balance:

$$\left\{ \begin{array}{l}
\mu_0 p' + B_z B'_z + \frac{B_\theta}{r} (r B_\theta)' = 0 \\
\frac{e}{r} (r n \chi_{\perp} T_{[eV]}')' = -\eta J^2
\end{array} \right.$$  \hspace{1cm} (6.28)

where

$$\chi_{\perp [m^2/s]} = \hat{\chi}_{\perp} \cdot \frac{n}{B^2 T_{[eV]}^2} , \quad \hat{\chi}_{\perp} = 3.92 \cdot 10^{-20}$$  \hspace{1cm} (6.30)

$$\eta_{[eV]} = \hat{\eta} \cdot \frac{1}{T_{[eV]}}, \quad \hat{\eta} = 1.05 \cdot 10^{-3}$$
and $B_z$ is always assumed given by Equation (6.16). All the quantities are given in MKS units, except the temperature, which is given in energy units. Before numerically solving Eqs. (6.28) and (6.29), it is interesting to consider a few points about the resulting equilibrium profiles. By analytical expansion around the axis $r = 0$, it is easy to determine the behavior of $B_\theta$ and $p'$:

\[
B_\theta = B_{z0} \left[ b_1 r + O(r^3) \right] ; \\
B_z = B_{z0} \left[ 1 - \frac{\mu^2 r^2}{4} + O(r^4) \right] \\
p = p|_{r=0} - \frac{\hat{\eta} b_0^2 B_{z0}^4 e^2}{\hat{\chi}_\perp P_{r=0} \mu_0^2} r^2 + O(r^4) ; \\
b_1 = \frac{\mu}{2} \left[ 1 - \frac{\hat{\eta} B_{z0}^2 e^2}{\hat{\chi}_\perp P_{r=0} \mu_0} \right]^{-\frac{1}{2}}
\]

(6.31)

In Fig. 6-12, the temperature profile for classical transport scenario is found by numerically solving the system of equations (6.28). Notice that classical transport allows a much more peaked temperature profile on axis ($T_0 \sim 2.7keV$) than Suydam's.
criterion does, so that even larger pressure gradients can be built in the core (see Fig. 6-13). Classical transport at the edge, though, becomes stronger than in Suydam’s marginal stability configuration. The global confinement parameters $\beta_p$ and $\tau_E$ are shown in Table 6.2 and Table 6.3. The numerical solution of Equations (6.28) and (6.29) is found using the following algorithm:

\[
\begin{align*}
z_1 &= T_{[n\phi]} ; \\
z_2 &= \frac{r n^2 T_{[e\phi]}^2}{B^2 T_{[e\phi]}^2} ; \\
\zeta_1' &= \frac{z_2^{1/2}}{\mu_0} \left( B^2_z + \frac{z_3^2}{r^2} \right) ; \\
\zeta_2' &= - \frac{\hat{\eta} r \left( B^2_z'^{1/2} + \left( \frac{z_3'}{r} \right)^2 \right)}{\bar{\chi}_1 \zeta_1^2 \mu_0^2} ; \\
\zeta_3' &= - \left[ \mu_0 (n'z_1 + nz_1') \cdot e + B_z B_z' \right] \frac{r^2}{z_3}
\end{align*}
\]

Finally, in order to extract the scaling laws for the confinement parameters, the usual relations have to be used

\[
B_0 \sim \frac{\mu_0 I}{a} , \quad j \sim \frac{I}{a^2} , \quad n \sim \frac{N}{a^2} , \quad \chi_1 \sim \frac{n}{D_0^2 T_1^2} , \quad \tau_E \sim \frac{p a^2}{N^2 I^2} , \quad \beta_p \sim \frac{p a^2}{\mu_0 I^2} ,
\]

(6.32)

together with the scaling for the plasma pressure. For the specific case of classical transport the following relation has to be used

\[
\tau_E \sim \frac{a^2}{\chi_1} \sim \frac{a^3 p_0^2 \mu_0^3 I^2}{N^3}.
\]

(6.33)
whence \( p \sim \mu_0 \frac{I^2}{\alpha^2} \), thus the scaling laws for \( \tau_E \) and \( \beta_p \) in the classical transport scenario are

\[
\tau_E = \frac{I^3_{[MA]} \cdot \alpha_m^2}{N_{[10^{20}m^{-1}]}^{3/2}} G_{\text{classic}}(\Theta) \\
\beta_p = K_{\text{classic}}(\Theta)
\] (6.34)

exactly coinciding with the ones found for the ideal interchange-mode dominated transport scenario (Equations (6.26),(6.27)), although the coefficients \( G_{\text{classic}} \) and \( K_{\text{classic}} \) are different.

### 6.3.3 Transport described by Bohm Diffusion

Another transport scenario was suggested by David Bohm in 1949 (see Ref. [52]) in an empirical way. Bohm proposed the following scaling for the particle diffusion
Figure 6-12: Temperature Profile from Classical Transport Model for a Standard MST Shot

The coefficient $D$

$$D \sim \frac{T}{B}$$  \hspace{1cm} (6.35)

Starting from the classical particle diffusion coefficient for a gas of neutral particles

$$D \sim \frac{v_{th}^2}{\nu_c} \sim \frac{T}{m \nu_c}$$  \hspace{1cm} (6.36)

where $v_{th}$ is the thermal velocity of the gas, and $\nu_c$ is the collision frequency, Bohm conjectured that in a plasma (ionized gas), the effective collision frequency would be set by plasma instabilities related to the cyclotron frequency $\Omega_c = \frac{eB}{m}$. This assumption leads to Equation (6.35). Moreover, Bohm also indicated a coefficient for
Figure 6-13: Comparison of Pressure Gradients given by Classical Transport and Suydam Marginality for a Standard MST Shot

his scaling relation

\[ D = \frac{T_{|eV|}}{16B}, \tag{6.37} \]

probably suggested by empirical analysis. Note that, to this day, no rigorous derivation of Equation (6.37) is known, in terms of a specific physical mechanism. In the literature, Bohm diffusion is usually used as a scaling, while the numerical coefficient is extracted by data fitting. For the purpose of describing the Bohm-diffusive transport scenario, the same dependence as given in Equation (6.35) is assumed for the thermal diffusivity:

\[ \chi_t = \frac{T}{B}, \tag{6.38} \]

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but no physical argument is available to justify the choice of a particular numerical coefficient. Thus, only the scaling for \( \tau_E \) and \( \beta_p \) will be evaluated and compared with the other scenarios. The same relations as in Equation (6.32) are used, with the only exception made for the scaling of \( \chi_\perp \) with plasma parameters:

\[
B_0 \sim \frac{\mu_0 I}{a}, \quad j \sim \frac{I}{a^2}, \quad n \sim \frac{N}{a^2}, \quad \chi_\perp \sim \frac{T}{B}, \quad \tau_E \sim \frac{p\frac{5}{2}a^2}{N^\frac{5}{2}I^2}, \quad \beta_p \sim \frac{pa^2}{\mu_0 I^2} \tag{6.39}
\]

The scaling for plasma pressure is found by using the relation

\[
\tau_E \sim \frac{a^2}{\chi_\perp} \sim \frac{\mu_0 IN}{pa} \tag{6.40}
\]

This yields

\[
\tau_E \sim I^{\frac{1}{7}} N^\frac{5}{7} a^2, \quad \beta_p \sim \frac{N^5}{I^{8}a^2} \tag{6.41}
\]

The scaling law derived from Bohm diffusion essentially shows a very week dependence of \( \tau_E \) upon \( I \) and \( N \).

### 6.4 Comparison with Recent Experimental and Computational Work

A final comparison worth noting involves two recent papers whose goal was also to provide a scaling law in an RFP.

#### 6.4.1 3D Model by Scheffel and Schnack

The first work (see Ref. [53]) is a computational analysis carried out by Scheffel and Schnack in 2000. They implemented a 3D, resistive MHD code, using classical transport coefficients (including thermal conduction and viscosity). No radiation losses, and no resistive wall effect was taken into account. Even though classical values for \( \chi_\perp \) and \( \chi_\parallel \) were used, an anomalous global transport (\( \chi_\perp \)) is generated by
parallel transport due to field stochasticity in the core region. This code is certainly more complex than the one developed in this thesis, not only because of the geometry and the presence of the plasma viscosity, but also because of its nonlinear treatment of instabilities. Even so, a few important issues remain in Scheffel and Schnack's model: first, only the dependence versus plasma current $I$ and area-integrated density $N$ was extracted, mostly due to the long computing time required by each simulation, and second, the actual regression was performed on $\frac{I^2}{N}$, rather than on $I$ and $N$ separately. In fact, the scaling law found in that work is

$$
\tau_E \sim I^{0.68} \cdot N^{-0.34}, \quad \beta_p \sim I^{-0.80} \cdot N^{0.40}
$$

(6.42)

This originates from the fact that the parameter used in the linear regression was a form of local plasma beta and local Lunquist number (which is defined as the ratio of the resistive diffusion time to the Alfvén transit time): $\beta_0$ and $S_0$ in their notation. It is easy to show that both of these parameters scale with $\frac{I^2}{N}$.

### 6.4.2 MST Experimental Scaling

The second work (see Ref. [54]) is an experimental fit of confinement shots, carried out by the MST group in 1998. By using a large database of shots, they generated a direct fit of velocity and magnetic field fluctuations experimentally measured in the MST device. Here again, the fit was carried out with the Lunquist number $S$, with the purpose of comparing experiments with theoretical predictions for scaling of fluctuations (see Refs. [55],[56]); the following dependence was found

$$
B_{1r} \sim S^{-0.2}
$$

(6.43)

In order to extract a scaling law for the confinement parameters from this experimental fit, the theoretical assumption of dominant stochastic magnetic field diffusivity
was made, allowing the use of the following relation between $\tau_E$ and $\beta_p$ (see Ref. [54])

$$
\tau_E \sim \beta_p^{0.1} \cdot I^{0.6} \cdot N^{-0.3}
$$

(6.44)

for the specific power law scaling given in Equation (6.43). Equation (6.44), together with the other scaling relation arising from the definitions of $\tau_E$ and $\beta_p$ (see Eqs. (6.1), (6.7))

$$
\tau_E \sim \beta_p^{0.5} \cdot \frac{I^3 \cdot a^2}{N^3}
$$

(6.45)

leads once again to an actual scaling versus $\frac{I^2}{N}$. Specifically, the scaling law resulting from this analysis is

$$
\tau_E \sim I^{0.5} \cdot N^{-0.25}, \quad \beta_p \sim I^{-1.0} \cdot N^{0.5}
$$

(6.46)

### 6.4.3 Comparison

In order to compare the different scalings just presented, in Fig. 6-14 and Fig. 6-15 experimental confinement data from three MST standard shots (see Ref. [57]) are plotted versus the parameter $\frac{I^2}{N}$ as it appears in the MST experimental scaling laws (see Equation (6.46)); for comparison, the scaling law found in this thesis (see Equation (6.14)) and the one given by Scheffel and Schnack's 3D Model (see Equation (6.42)) are shown respectively by the dashed line and solid line. The data relative to the three shots plotted in these figures are listed in Table 6.4. Shot 1 is the MST standard shot already used in this chapter for earlier comparisons; the corresponding data are also listed in Table 6.1 and Table 6.2. The other two shots (Shot 2 and Shot 3) were recently reported by the MST group in Ref. [57]: they essentially have the same plasma density, while they differ in the plasma current by a factor of two. All of these shots have a value of $\Theta$ of about 1.7 – 1.8; moreover, all of these parameters were measured between sawtooth crashes, which occur regularly throughout standard MST plasmas, and which temporarily degrade the confinement. It is
Figure 6-14: Comparison of Scaling Laws for $\beta_p$ for MST Standard Shots

Figure 6-15: Comparison of Scaling Laws for $\tau_E$ for MST Standard Shots
<table>
<thead>
<tr>
<th>Shot 1</th>
<th>0.376</th>
<th>1.2</th>
<th>6.3</th>
<th>1.4</th>
</tr>
</thead>
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<td>Shot 2</td>
<td>0.210</td>
<td>0.68</td>
<td>9.0</td>
<td>1.4</td>
</tr>
<tr>
<td>Shot 3</td>
<td>0.430</td>
<td>0.85</td>
<td>4.8</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Table 6.4: Plasma Parameters for the MST Standard Shots used in Fig. 6-14 and Fig. 6-15

important to remember that the scaling laws predicted by the tearing-mode transport model determine not only the dependence on $I$, $N$, and $a$, but also the magnitude, given as a function of $\Theta$. Both the experimental fit and the 3D model only give a prescription for the dependence on $I$ and $N$. Hence, in order to be represented in Fig. 6-14 and Fig. 6-15 the magnitude of the MST and 3D scaling relations was determined by matching with the first experimental experimental value. The tearing-mode scaling relation uses the self consistent magnitude factor predicted by the theory.

One further comment should be noted on the scaling laws found by the 3D model: this scaling does not satisfy the relation between $\tau_E$ and $\beta_p$ that comes from the definitions (see Equation (6.45)). Notice that this relation is indeed closely followed by the scaling laws found in this thesis, despite the fact that it was not automatically given as a constraint in the determination of $\tau_E$ and $\beta_p$; the small deviation from fully satisfying Equation (6.45) gives an idea of the error bars related to the power laws for current, density and minor radius. Finally, judging from these figures, it is interesting to point out that for the theories $\beta_p$ seems to follow quite well with the experimental fit, while $\tau_E$ appears to have a stronger dependence with the plasma current than the experimental fit.
6.5 Discussion

Having presented several scaling laws related to different transport scenarios, it is now time to discuss the results in more detail. As a first interesting observation, Equation (6.26) for $\tau_E$ is the same as the Taylor-Connor scaling, which has long been used in past RFP confinement studies (see [3],[13]). This scaling was found by Taylor and Connor in 1977 (see [14],[15]) in a dimensional analysis applied to several fusion reactor concepts; the treatment was based on the concept of invariance for the scaling laws under any transformation that leaves the equations regulating plasma evolution themselves invariant. This idea was then applied to specific models for the magnetic configuration (typically Tokamak and RFP), and for the physical mechanism of transport, thus providing extra conditions on the nature of the turbulence. Specifically, for an RFP, transport was assumed to be determined by resistive g-mode activity, leading the scaling law for $\tau_E$ as seen in Equation (6.26). Furthermore, the same approach predicts a scaling for $\beta_p$ (in an ohmically heated RFP) that is independent of the machine parameters, so long as radiation is not important. This is in agreement with Equation (6.27). The derivation of Taylor-Connor scaling, then, relies on the previous tearing-mode dispersion relation (see Ref. [33]), which neglects heat conductivity and thus observes the presence of "g-modes," which are then assumed to dominate plasma turbulence. In terms of plasma turbulence, this scaling corresponds to the following dependence of magnetic fluctuations with the Lundquist number

$$B_{1r} \sim S^{-0.5} \quad (6.47)$$

to be compared with the experimental relation given in Equation (6.43). This leads to a much more optimistic scaling law for confinement in an RFP, because of its strong dependence on the current $I$. Recent targeted experimental campaigns, performed mostly on MST (see Ref. [54]), and less on RFX (see Ref. [58]), demonstrated a consistent deviation from Taylor-Connor scaling, mostly a much less favorable dependence of $\tau_E$ versus the plasma current, as seen in the previous section.

The scaling laws found with the simple, 1D transport model presented in this thesis
confirm reasonably well the recent experimental and computational results, showing a much less optimistic scaling law for confinement in an ohmic RFP. In particular, a weaker dependence of $\tau_E$ upon the plasma current is found, which is a crucial issue for the future large RFP fusion experiments. In addition to previous work, the results found with the present model provide considerably more information on the scaling laws. First, the fit was done by independently varying both the plasma current $I$ and the area-integrated density $N$, which had not been done in the existing scaling laws. It is important to stress that the MST device can operate with independent values of $I$ and $N$, while not all the past devices were able to do so. Furthermore, the present model gives the scaling of plasma confinement with two other important parameters: the minor radius and the pinch parameter $\Theta$. While very weak dependence on $\Theta$ is found, in agreement with experimental observations, a quite significant dependence of $\tau_E$ versus the minor radius was found (see Eqs. (6.13),(6.14)). The tearing-mode model also allows a comparison of multiple shots from a single machine, showing reasonably good agreement in terms of global parameters. The tearing-mode model also explicitly predicts an anomalous transport coefficient for perpendicular thermal diffusivity in a self-consistent manner by imposing the marginal stability condition on a given ohmically driven Taylor's relaxed state; in a sense, this effectively proceeds inversely from traditional transport models, where the profiles are evaluated from the transport coefficients. Here $\chi_{\perp}$ is evaluated from the temperature profile. Classical values for parallel thermal diffusivity and plasma resistivity were also chosen in the model. Anomalous parallel transport may be important in an RFP, but from a physical point of view even the classical value of $\chi_{\parallel}$ is enormously large compared to the perpendicular one, moreover, in the tearing-mode marginal stability condition, only the ratio of parallel to perpendicular diffusivity to the one quarter power enters, so the sensitivity of the model to those parameters is relatively low. Typical values of $20 - 60 m^2/s$ for the anomalous $\chi_{\perp}$ in the core region were found from simulating standard RFP shots; these values exceed the classical value of $\chi_{\perp}$ by about two orders of magnitude. It would be interesting to have a more targeted experimental campaign trying to verify this result, as well.
<table>
<thead>
<tr>
<th>///</th>
<th>$I[MA]$</th>
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<th>$a[m]$</th>
<th>$\tau_E[ms]$</th>
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<tr>
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<td>0.20</td>
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<td>0.537</td>
<td>0.26</td>
<td>0.50</td>
</tr>
<tr>
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<td>1.458</td>
<td>0.50</td>
<td>1.60</td>
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<tr>
<td>RFX</td>
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<td>3.125</td>
<td>0.457</td>
<td>1.30</td>
</tr>
</tbody>
</table>

Table 6.5: RFP International Database, as reported in Ref. [3]

An interesting consideration is worthy noting regarding the Taylor-Connor scaling. As mentioned before, this scaling has been used until fairly recently as a reference to fit experimental values of the energy confinement time in an RFP. However, even though the Taylor-Connor scaling law was not in good agreement with the data for different operating points in an individual machine, it has been shown by Werley et al in 1996 (see Ref. [3]) that a special restricted database made of only the “best” confinement shots provided by each individual RFP device (i.e. the best operating point per machine) would, broadly speaking, fit the Taylor-Connor scaling (in a log – log plot). This result is potentially very important for the RFP community, since it predicts that when operating at its best performance, an RFP device drastically improves its confinement as the plasma current increases. Given the importance and the validity
Figure 6-16: Scaling Law for $\tau_E$ extracted from RFP International Database of Best Confinement Shots

of this result by Werley et al, it is interesting to try to see if it can be recovered in the context of the present analysis and the very recent results on RFP confinement. The key issue is the definition of "best" performance. In Table 6.5, confinement data from the RFP International Database are listed, exactly as reported in Ref. [3]. A multiple linear regression allows one to confirm that this special database indeed follows Taylor-Connor scaling (see Fig. 6-16)

$$\tau_E[ms] \simeq 243 \cdot \frac{J_{[MA]}^{3.05} \cdot a_{[m]}^{1.72}}{N_{[10^{19} m^{-1}]}^{1.62}} \quad (6.48)$$

The point which is addressed here is to investigate how this result fits with the scaling law found in this thesis. For this purpose, the key issue consists in the proper characterization of the best confinement shot for each individual device. A plausible interpretation comes from the RFP "Greenwald limit", given in Equation (1.4). This relation suggests that there is a maximum density (whose value depends upon the
plasma current) which the plasma can reach before its confinement is degraded, for example due to enhanced radiation effects. Defining this maximum density as $N_{\text{best}}$, it is then reasonable to postulate that the best confinement shot corresponds to operating the machine at the best density $N_{\text{best}}$ for a given current. A multiple linear fit of the density in Table 6.5 with current and minor radius leads to the following general scaling (see Fig. 6-17)

$$N_{\text{best}[10^{19} m^{-1}]} \approx 7.13 \cdot I_{[MA]}^{1.50} \cdot a_{[m]}^{-0.063} \quad (6.49)$$

It is interesting to note that the dependence of $N_{\text{best}}$ with the minor radius $a$ is indeed very weak; nonetheless, the dependence with the current is not linear, as the tokamak Greenwald limit would suggest, but a bit stronger. Substituting Equation (6.49) into
Figure 6-18: Scaling Law for $\tau_E$ at the highest possible density $N_{\text{best}}$, from the RFP International Database, compared with the tearing-mode transport model

Equation (6.48), leads to the following scaling relation for best confinement shots

$$\tau_E[\text{ms}] \simeq 9.85 \cdot I_{[\text{MA}]}^{0.62} \cdot a_{[\text{m}]}^{1.82}$$  \hspace{1cm} (6.50)

Now it is possible to directly compare this scaling law with the one found in this thesis (see Fig. 6-18); in particular, substituting in Equation (6.49) for the density in Equation (6.14), yields the scaling for best confinement shots, as predicted by the tearing-mode transport model

$$\tau_E[\text{ms}] \simeq 10.0 \cdot I_{[\text{MA}]}^{0.72} \cdot a_{[\text{m}]}^{1.62}$$  \hspace{1cm} (6.51)

showing reasonably good agreement with Equation (6.50). It is important to notice that the numerical coefficient in Equation (6.51) is given by the model as a function of the pinch parameter $\Theta$. Since the experimental values for $\Theta$ are not reported in
the RFP International Database for best confinement shots, the coefficient in Equation (6.51) is calculated for $\Theta \sim 1.7$; the weak dependence of $\tau_E$ upon the pinch parameter ensures a good accuracy in the estimation of the numerical coefficient.

A few comments are in order concerning the data in Table 6.5. First, it has to be kept in mind that only seven data points were used to extract the scaling laws in Eqs. (6.48), (6.49). Moreover, the accuracy of these data, especially for the early experiments, is characterized by large error bars. This for instance prevents a similar analysis on $\beta_\rho$ to be carried out, due to high uncertainty in the measured values of plasma beta in the early experiments. Values from the new devices, namely MST and RFX, are of course the most reliable in the database; it has to be stated that, even though the shots used in Table 6.5 for MST and RFX describe good confinement in standard operated RFP plasmas, it still has to be verified that they really represent the best confinement found in these machines. Anyway, the deviation in energy confinement time so far observed in both of these currently operating RFPs is quite small. A final comment concerns the two best shots that were originally included in the RFP International Database, both belonging to an early Japanese device, TPE-1RM; these shots were omitted from this analysis, because they are the only ones which clearly violate the (empirical) rule that optimum scaling is obtained around the Greenwald density limit (see Ref. [3]): that is, their best performance does not occur at the RFP Greenwald limit.

In conclusion, the scaling laws found with the tearing-mode transport model agree reasonably well with experimental and computational analysis. This implies that the designed future devices (NEPI and TITAN) would not reach their design goals (breakeven and ignition) if operated as a purely ohmic RFP. The suggestion is to keep exploring improved confinement by new techniques aimed at reducing the turbulence in an RFP; a few techniques are already being used in modern devices (edge profiles control, PPCD), others are still being developed (Oscillating Current Drive), and indeed some significant improvement in confinement has already been observed (in MST, profile control increases $\tau_E$ by almost one order of magnitude, as shown in Table 1.1 and Table 1.2). The present theory implies that one or more of these
techniques must be successfully developed if the RFP is to become competitive with the tokamak as a fusion reactor.
Appendix A

Derivation of Equation (3.37)

In this appendix, a detailed derivation of Equation (3.37) is given, starting from the two vector equations (3.33) and (3.36). The procedure simply consists in reducing the system to one scalar equation for the radial component of the perturbed magnetic field \( B_{1r} \), by eliminating all the other components and by using the relations in Eqs. (3.24), (3.23) and (3.30).

The first step is to split Equation (3.33) into its cylindrical components

\[
\begin{align*}
B_{1r} &= \frac{im}{r} \xi_r B_{0\theta} - ik \xi_r B_{0z} \\
B_{1\theta} &= -ik (\xi_\theta B_{0z} - \xi_z B_{0\theta}) - (\xi_r B_{0\theta})' \\
B_{1z} &= -\frac{1}{r} (r \xi_z B_{0z})' - \frac{im}{r} (\xi_\theta B_{0z} - \xi_z B_{0\theta})
\end{align*}
\]  

(A.1) (A.2) (A.3)
Similarly for Equation (3.36)

\[
\begin{align*}
\mu_0 \frac{5}{3} \left[ p_0 \left( \frac{1}{r} (r \xi_r)' + \frac{im}{r} \xi_\theta - ik \xi_z \right) \right]' + \mu_0 (\xi_r p_0)' - B''_0 + B_z B_{1z} - \frac{B_{0\theta}}{r} (r B_{0\theta})' + \\
- B_{0z} (ik B_{1r} + B_{1z}') - \frac{B_{0\theta}}{r} (r B_{1\theta})' + \frac{im}{r} B_{1r} B_{0\theta} = 0 \\
\mu_0 \frac{5}{3} \frac{im}{r} p_0 \left( \frac{1}{r} (r \xi_r)' + \frac{im}{r} \xi_\theta - ik \xi_z \right) + \mu_0 \frac{im}{r} \xi_r p_0' + \frac{B_{1r}}{r} (r B_{0\theta})' - B_z \left( \frac{im}{r} B_{1z} + ik B_{1\theta} \right) = 0 \\
- \mu_0 \frac{5}{3} ik p_0 \left( \frac{1}{r} (r \xi_r)' + \frac{im}{r} \xi_\theta - ik \xi_z \right) - \mu_0 ik \xi_r p_0' + B_{1r} B_{0z} + B_{0\theta} \left( \frac{im}{r} B_{1z} + ik B_{1\theta} \right) = 0
\end{align*}
\] (A.4) (A.5) (A.6)

Multiplying Equation (A.5) by \( k \) and Equation (A.6) by \( \frac{m}{r} \), and then adding them together, yields

\[
(k B_{0z} - \frac{m}{r} B_{0\theta}) \left[ \left( \frac{m^2}{r^2} + k^2 \right) (\xi_\theta B_{0z} - \xi_z B_{0\theta}) - \frac{im}{r^2} (r \xi_r B_{0z})' - ik (\xi_r B_{0\theta})' \right] = \frac{B_{1r}}{r} (r B_{0\theta})' + m B_{0z}'
\] (A.7)

where Equations (A.2) and (A.3) have been used to eliminate \( B_{1\theta} \) and \( B_{1z} \). Equation (A.7) can be solved for \( \xi_\theta B_{0z} - \xi_z B_{0\theta} \), giving:

\[
\xi_\theta B_{0z} - \xi_z B_{0\theta} = \frac{im}{m^2 + k^2 r^2} \left[ \frac{m}{r} (r \xi_r B_{0z})' + kr (\xi_r B_{0\theta})' - \xi_r (k (r B_{0\theta})' + m B_{0z}') \right]
\] (A.8)

Another useful relation is found by multiplying Eqs. (A.5) and (A.6) respectively by \( B_{0\theta} \) and \( B_{0z} \), and then adding them together:

\[
\mu_0 \frac{5}{3} p_0 i \left( \frac{m B_{0\theta}}{r} - k B_{0z} \right) \left[ \frac{1}{r} (r \xi_r)' + \frac{im}{r} \xi_\theta - ik \xi_z \right] + \\
+ \xi_r i \left( \frac{m}{r} B_{0\theta} - k B_{0z} \right) \left[ \mu_0 p_0' + \frac{B_{0\theta}}{r} (r B_{0\theta})' + B_{0z} B_{0z}' \right] = 0
\] (A.9)
Using Equation (3.23) leads to

\[
\frac{5}{3} p_0 i \left[ \frac{1}{r} (r \xi_r)' + \frac{im}{r} \xi_\theta - ik \xi_z \right] = 0 \tag{A.10}
\]

Substituting in Eqs. (A.8) and (A.10) into Equation (A.4), after using Eqs. (A.1) - (A.3), again to eliminate \( B_{1\theta} \) and \( B_{1z} \), yields the following differential equation for \( \xi_r \):

\[
\left\{ \frac{\xi_r F^2}{\bar{D}} - \frac{\xi_r}{\bar{D} r} \left[ m^2 B_{0\theta} - k^2 r^2 B_{0z}^2 \right] \right\}' + \xi_r \frac{2 B_{0\theta}}{\bar{D} r} (m^2 B_{0\theta} - mkr B_{0z}) + \\
+ \xi_r \left\{ \frac{2 B_{0\theta}}{r} \left[ B'_{0\theta} - k_r \left( \frac{m B_{0z}}{r} - k B_{0\theta} \right) \right] - \frac{F^2}{r^2} \right\} = 0 \tag{A.11}
\]

where \( F = m B_{0\theta} - k_r B_{0z} \) and \( \bar{D} = m^2 + k^2 r^2 \). Introducing the function

\[
\Psi \equiv i F \xi_r \frac{r^3}{\bar{D}^2} = B_{1r} \frac{r^2}{\sqrt{\bar{D}}} \tag{A.12}
\]

one obtains the corresponding differential equation for the radial perturbed magnetic field as given in Equation (3.37):

\[
\Psi'' + \frac{\Psi}{r^2} \left\{ - \frac{2k^2 r^3}{F^2} \mu_0 p'_0 + \frac{2k^2 r^2}{F} \frac{m B_{0\theta}}{\bar{D}} + \frac{kr B_{0z}}{\bar{D}} - \frac{F''}{F} r^2 + \\
- \frac{F'}{F} \left( \frac{m^2 - k^2 r^2}{\bar{D}} + 1 \right) + \frac{m^4 - 3k^4 r^4 + 10m^2 k^2 r^2}{4 \bar{D}^2} \right\} = 0, \tag{A.13}
\]

Equation (A.13) can be cast in a slightly different form (see Ref. [50]), less practical and compact, but clearer in terms of putting in evidence the physical meaning of the terms appearing in it: by introducing Taylor’s parameter, as defined in Chapter 2,

\[
\mu \equiv \frac{\mu_0 \tilde{J}_0 \cdot \tilde{B}_0}{B_0^2}, \tag{A.14}
\]
Equation (A.13) can be rewritten as follows:

\[ \Psi'' + \Psi \left\{ -\frac{2k^2r}{F^2} \mu_0 p_0' + \frac{2\mu_0 p_0'}{B_3^2 F} \left[ \mu \left( mB_{0z} + krB_{0r} \right) - \frac{mB_{0x}}{r} \right] + \right. \]

\[ -\mu \frac{mB_{0x}}{F} + \frac{krB_{0r}}{F} + \frac{m^2 - k^2r^2}{rB_3^2} \left( \frac{\mu_0 p_0'}{B_3^2} \right)^2 + \left( \frac{\mu_0 p_0'}{B_3^2} \right)' + \mu^2 \left( \frac{2\mu m k}{D} - \frac{\hat{D}}{r^2} + \frac{m^4 - 3k^4r^4 + 10m^2k^2r^2}{4r^2\hat{D}^2} \right) \right\} = 0 \]

\[ (A.15) \]

The first observation is that a singularity appears in the differential equation when \( F = 0 \), for terms proportional to \( \frac{1}{F^2} \) and \( \frac{1}{F} \) are present; letting \( r_s \) be the singular point, then \( F(r_s) = 0 \), and assuming (as is usually the case) that \( F'(r_s) \neq 0 \), the singularity is of the \( \frac{1}{x^2} \) type only in the presence of a pressure gradient. The coefficients of the \( \frac{1}{F} \) term depend on the pressure gradient as well as on \( \mu' \), which is related to the gradient of the current density parallel to the magnetic field \( (j_{||}) \). This implies that a force-free configuration (as for example BFM) will show no singularity, while a pressure-free model will still have a singular layer due to gradients in \( j_{||} \). Tearing-mode stability considerations can be carried out by simply solving Eqs. (A.13) or (A.15), as shown in Chapter 3; it is then obvious that the term \( -\frac{2k^2r}{F^2} \mu_0 p'_0 \) for a confined plasma \( (p'_0 < 0) \) is always destabilizing, while the line-bending term \( -\frac{\hat{D}}{r^2} \) (important especially for high \( k \)-numbers) is always stabilizing. The \( \frac{1}{F} \) terms change sign across the singularity; thus they will be predominantly more stabilizing or destabilizing based on the specific plasma profiles.
Bibliography


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