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I. Introduction

In this paper we present results related to the smoothing problem and related generalized Riccati equations for the two-point boundary value descriptor system (TPBVDS)

\[ \begin{align*}
E_x(k+1) &= Ax(k) + Bu(k) \\
V_i x(0) + V_f x(N) &= v \\
y(k) &= Cx(k)
\end{align*} \] (1)

where \( E, A, V_i, \) and \( V_f \) are possibly singular \( n \times n \) matrices, and \( B \) and \( C \) are \( n \times m \) and \( pxn \) matrices respectively.

II. System Theory for TPBVDSs

In [1-2] we develop a basic theory for (1)-(3).

Many of the aspects of this theory have a similar flavor to that in [4-5], although the possible singularity of \( E \) and \( A \) create some significant differences. As discussed in [1,2], when (1)-(2) is well-posed, we can assume that it is in standard form, i.e. for some constants \( \alpha \) and \( \beta \)

\[ \begin{align*}
&\alpha E' + A' = 1 \\
&V_i E' + V_f A' = 1
\end{align*} \] (4)

As in [4-5], \( x(k) \) can be decomposed into an outward process \( z_o \) and an inward process \( z_i \). The outward process \( z_o \) is defined as

\[ z_o(k,\ell) = E^{-k} x(\ell) - A^{-k} x(k) \quad k \leq \ell. \] (6)

By eliminating \( x \)'s in (6), we find that \( z_o(k,\ell) \) is only a function of the inputs inside the interval \([k,\ell]\). Also note that \( z_o \) does not depend in any way on the boundary matrices \( V_i \) and \( V_f \). The expression for the inward process \( z_i \) is in general very complex, although in the so-called stationary case there is a simple expression for \( z_i \) [1].

The system (1)-(2) is strongly reachable on \([k,\ell]\) if the map \( \{u(m): m \in [k,\ell]\} \rightarrow z_o(k,\ell) \) is onto. System (2.1) is called strongly reachable if it is reachable on some \([k,\ell]\).

Theorem 1:
The following statements are equivalent
a) System (1)-(2) is strongly reachable.

b) The strong reachability matrix

\[ R = \begin{bmatrix} A^{n-1}E & A^{n-2}E & \ldots & AE & E \end{bmatrix} \] (7)

has full rank.

c) The matrix \([\alpha E - \alpha A]B\) has full rank for all \((s,t) \in \{(0,0)\}.

d) The state \( x(1) \) where \( i \in [n,N-n] \) can be made arbitrary by proper choice of the inputs \( u(\cdot): j \in [n-1,n+1] \) with all other inputs and the boundary value \( v \) set to zero, and for all pair of matrices \( V_i \) and \( V_f \) in standard form.

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In the case of the optimal smoother, it is shown in [3] that if the following generalized Riccati equations

\[ \Theta = A'(E \Theta^{-1} E' + BQB')^{-1} A + C'R^{-1} C \]  
(14)

\[ \Psi = A(E \Psi^{-1} E + C'R^{-1} C)A' + BQB' \]  
(15)

have positive definite solutions \( \Psi \) and \( \Theta \) then there exist invertible matrices \( M \) and \( N \) such that

\[ MN^{-1} = I \]
(16)

\[ MN^{-1} = \begin{bmatrix} I & 0 \\ 0 & A'S^{-1}E\Theta^{-1} \end{bmatrix} \]  
(17)

Moreover, the eigenvalues of \( AT^{-1}E'\Psi^{-1} \) and \( A'S^{-1}E\Theta^{-1} \) are inside or on the unit circle. Equation (3.5) is called the descriptor Hamiltonian equation and the above decomposition is the descriptor Hamiltonian diagonalization. Of course, we would like \( AT^{-1}E'\Psi^{-1} \) and \( A'S^{-1}E\Theta^{-1} \) to be strictly stable. This occurs only when the descriptor Hamiltonian has no eigenmodes on the unit circle i.e. it is forward-backward stable.

**Theorem 3:**

If the system is forward-backward detectable and stabilizable (i.e. the modes on the unit circle are strongly reachable and strongly observable) then the corresponding smoother is forward-backward stable.

**IV. Generalized Riccati Equations**

In this section we study the generalized algebraic Riccati equation.

\[ \Psi = A(E'\Psi^{-1} E + C'R^{-1} C)^{-1} A' + BQB' \]  
(18)

**Theorem 4:**

If \( (E,A,B) \) and \( (C,E,A) \) are strongly reachable and observable respectively then (18) has a unique positive definite solution.

The approach used to prove this theorem is similar to that in [6] for the standard Riccati equation. Details will be presented in a future paper. Existence proceeds as follows. From Theorem 3 and the fact that eigenmodes of the smoother occur in reciprocal pairs, we know that we can write

\[ \begin{bmatrix} E & -BQB' \\ 0 & A' \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} \]  
(19)

The proof then proceeds by first showing that \( F \) is invertible, then that \( E'GF + C'R^{-1}C > 0 \) and finally that

\[ \Psi = (A(E'GF^{-1} + C'R^{-1}C)^{-1} A' + BQB') \]  
(20)

satisfies (18). To prove uniqueness, let \( \phi_1 \) and \( \phi_2 \) be two positive definite solutions of (18), let \( \Delta \phi = \phi_1 - \phi_2 \), and

\[ T_i = E'\phi_i^{-1} E + C'R^{-1} C \text{ for } i=1,2. \]  
(21)

Some algebra then yields

\[ \Delta \phi = AT^{-1}E'\phi_1^{-1} A\phi_2 - ET^{-1}A'. \]  
(22)

But \( AT^{-1}E'\phi_1^{-1} \) and \( ET^{-1}A' \) are strictly stable (see [3]); thus \( \Delta \phi = 0 \).

**References**