Orbifold genera, product formulas and power operations

by

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Abstract

There is a formula by the string theorists Dijkgraaf, Moore, Verlinde and Verlinde, expressing the orbifold elliptic genus of the symmetric powers of an almost complex manifold M in terms of the elliptic genus of M itself. We show that from the point of view of elliptic cohomology an analogous p-typical statement follows as an easy corollary from the fact that the map of spectra corresponding to the genus preserves power operations. We define higher chromatic versions of the notion of orbifold genus, involving h-tuples rather than pairs of commuting elements. Using homotopy theoretic methods we are able to prove an integrality result and show that our definition is independent of the representation of the orbifold. Our setup is so simple, that it allows us to prove DMVV-type product formulas for these higher chromatic orbifold genera in the same way that the product formula for the topological Todd genus is proved. More precisely, we show that any genus induced by an H_∞-map into one of the Morava-Lubin-Tate cohomology theories E_h has such a product formula and that the formula depends only on h and not on the genus. Since the complex H_∞-genera into E_h have been classified in [And95], a large family of genera to which our results apply is completely understood. Loosely speaking, our result says that some Borcherds lifts have a well-known homotopy theoretic refinement, namely total symmetric powers in elliptic cohomology.

Thesis Supervisor: Michael J. Hopkins
Title: Professor

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Contents

1 Introduction

This thesis aims to provide a systematic understanding and homotopy theoretic refinement of the theory of orbifold genera and product formulas as they turn up in the string theory literature (e.g. [DMVV97], [Dij99]).

1.1 Product formulas

The most general and famous of these is probably a formula by Dijkgraaf, Moore, Verlinde and Verlinde expressing the orbifold elliptic genus of the symmetric powers of an almost complex manifold M in terms of the elliptic genus of M itself [DMVV97]:

$$\sum_{n\geqslant 0} \phi_{\mathrm{ell,orb}}(M^n \circlearrowleft \Sigma_n)t^n = \prod_{m,n,l} \left(\frac{1}{1-t^m q^n y^l}\right)^{c(mn,l)}, \tag{1}$$

where

$$\phi_{\mathrm{ell}}(M) = \sum_{n,l} c(n,l) q^n y^l$$

is the two-variable elliptic genus or "equivariant χ_y-genus of the loop space" of M defined in [EOTY89], [Höh91], [HBJ92], and the orbifold version $\phi_{\mathrm{ell,orb}}$ is defined in [BL03]. Algebraic geometers [BL03] as well algebraic topologists [Tam01], [Tam03] have found meanings for this formula in their own worlds and developed the theory further.

1.2 Borcherds lifts

The right hand side of (1) is what people call a *Borcherds lift* of $\phi_{\mathrm{ell}}(M)$, and one can use a calculation of Borcherds to rewrite it as

$$\exp\left[\sum_{m\geqslant 1} V_m(\phi_{\mathrm{ell}}(M))t^m\right] \tag{2}$$

(compare Section 5.3). Here the V_m are a type of Hecke operators acting on q-expansions of Jacobi forms. Concepts like Borcherds lifts and product formulas usually have their home in the Monstrous Moonshine world, and there have been many vague hints that an explanation for the connection between string theory and this story lies hidden in elliptic cohomology. At least in the example of Borcherds lifts and orbifold genera we are able to make this statement precise. Moreover it turns out that the formula has a homotopy theoretic refinement: as stated, formula (1) is a purely algebraic equation, i.e. it lives on the level of coefficients of elliptic cohomology. However, at least in the p-typical case both sides – the orbifold genus as well as (2) – can be given a meaning as operations on elliptic cohomology, and the equation really lives on the level of elliptic cohomology.

1.3 Ando-French definition of orbifold genera

The key to our point of view is in [AF03]. There Ando and French explain how to fit the notion of orbifold (elliptic) genus into the framework of elliptic cohomology. Essential for us is that their definition of orbifold genus is simply an equivariant version of the genus followed by some sort of evaluation maps. It is really expected that the story can be told in terms of equivariant elliptic cohomology, once that is constructed. Since as of now it is not available, Ando and French work with Borel equivariant elliptic cohomology. Thus they have to use an elliptic cohomology theory whose Borel equivariant theory is particularly well understood, namely Morava-Lubin-Tate cohomology $E_2^*(-)$. It is elliptic, because by the Serre-Tate theorem its formal group is isomorphic to the universal deformation of a supersingular elliptic curve, but this fact is not really relevant for us. What is important to us is that in the case of E_2, Hopkins-Kuhn-Ravenel character theory applies, and evaluation at a pair of group elements (of p-power order) is well defined. We give a very short reminder of Hopkins-Kuhn-Ravenel theory in Section 6.2. The drawback of working in this setup as opposed to equivariant elliptic cohomology is that it only reproduces a p-typical analogue of Borisov and Libgober's original definition of orbifold elliptic genus [BL03, 4.1]. However, Ando and French show that their own definition of ϕ_{orb} can be brought into a form that is very similar to that of [BL03].

1.4 Main idea

The left hand side of the DMVV formula involves an object well known to topologists: the assignment

$$M \mapsto \sum_n (M^n \circlearrowleft \Sigma_n) t^n$$

is what is called the total power operation in cobordism (of a point). The right hand side of the DMVV formula is a function in $\phi_{\mathrm{ell}}(M)$ which takes sums into products. Total power operations also have this property. In the context of stable homotopy theory, a genus often is a natural transformation from a cobordism theory to another cohomology theory, applied to a point. If the target of this natural transformation is a form of elliptic cohomology, the genus is called an elliptic genus. This natural transformation is what we referred to above as "homotopy theoretic refinement" of the genus. We explain this point of view for the example of the Todd genus in Section 2. We recall the concept of power operations in a slightly refined setup in Section 4. With these concepts in mind, it is a natural question to ask whether formulas like the DMVV formula simply reflect the fact that the natural transformation in question preserves power operations. A natural transformation that preserves power operations is also called an H_∞-map. We take the opposite approach, and start with a form of elliptic cohomology that has well understood power operations, namely Morava-Lubin-Tate cohomology E_2. We show that any H_∞-map from a cobordism theory into E_2 has a DMVV-type formula for the induced (orbifold) genus.

1.5 Higher chromatic orbifold genera

It is easy to generalize Ando and French's definition of orbifold genus to the higher chromatic case: For each h there is a Morava-Lubin-Tate cohomology theory E_h, and Hopkins-Kuhn-Ravenel character theory defines evaluation at h-tuples of p-power order elements.

Definition 1.1. For a ring map ϕ from MU (or another Thom spectrum) into E_h we define

$$\phi_{\mathrm{orb}}(M \circlearrowleft G) := \frac{1}{|G|} \sum_{\alpha} \mathrm{eval}_{\alpha} \phi_G(M),$$

where the sum runs over all h-tuples of commuting elements of p-power order in G, and ϕ_G is the Borel equivariant version of the genus ϕ, defined by

$$\mathcal{N}_*^{U,G} \to MU_*^G(pt) \to MU_*(BG) \xrightarrow{\phi} E_{h*}(BG) \tag{3}$$

$$(M \circlearrowleft G) \mapsto \phi_G(M),$$

where the first map is the Pontrjagin-Thom map from the equivariant complex bordism ring $\mathcal{N}_*^{U,G}$ of compact, closed, smooth G-manifolds with a complex structure on their stable normal bundle to the coefficients of MU_G.

Remark 1.2. In the case $h = 1$, the cohomology theory E_1 is p-completed K-theory. The standard example of a genus into (non-p-completed) K-theory is the Todd genus. There is an equivariant version of K-theory; here $K_G(pt)$ is the representation ring $R(G)$, and evaluation of an element ρ of $K_G(pt)$ at an element g of G is the trace of $\rho(g)$. Definition 1.1 can then be formulated without the p-power order part, and it becomes the definition of the *topological* Todd genus[1] of the orbifold $M/\!\!/G$,

Definition 1.3.

$$\mathrm{Td}_{\mathrm{top}}(M \circlearrowleft G) := \left(\frac{1}{|G|} \sum_{g \in G} \mathrm{Trace}(g|-) \right) \circ \mathrm{Td}_G(M).$$

In the case $h = 2$ Definition 1.3 is (up to the factor $\frac{1}{|G|}$) Definition [AF03, 6.1]. Ando and French show that this definition is a p-typical analogue of the definitions of orbifold genera discussed in the literature [BL03], [DMVV97]. Thus our point of view fits the *orbifold elliptic genus* (as defined by Ando and French) and its product formula into a common picture with the *topological Todd genus* of an orbifold and its product formula, i.e. the former is exactly the chromatic level two analogue of the latter.

[1] In the literature this turns up as the Euler characteristic of the complex space M/G [AS68a] or the topological Euler characteristic of the orbifold $M/\!\!/G$ [Dij99].

1.6 Statement of results

Remark 1.4. A priori $\mathrm{Td}_{\mathrm{top}}$ appears to take values in $\mathbb{C}$. Note however that

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{Trace}(g|-)$$

equals the inner product with the trivial representation. This shows that $\mathrm{Td}_{\mathrm{top}}$ takes integral values. In a similar way, using an inner product defined by Strickland, we prove in Section 7.3 the following proposition.

Proposition 1.5 (Integrality). *The orbifold genus ϕ_{orb} takes values in E_h^0.*

Definition 1.1 is formulated in terms of the G-space $M \circlearrowleft G$ rather than the orbifold $M/\!/G$. It is a non-trivial fact that ϕ_{orb} depends only on the orbifold, and not on its presentation. of orbifolds. We prove it in Section 8, using a deep homotopy theoretic result of Strickland.

Theorem 1.6 (Well-definedness of ϕ_{orb}). *Let M be a compact manifold acted upon by a finite group G, let N be a compact manifold acted upon by a finite group H, and assume that the orbifold quotients $M/\!/G$ and $N/\!/H$ are isomorphic. Then*

$$\phi_{\mathrm{orb}}(M \circlearrowleft G) = \phi_{\mathrm{orb}}(N \circlearrowleft H).$$

The analogues of the Dijkgraaf-Moore-Verlinde-Verlinde formula for our orbifold genera are given by the following theorem.

Theorem 1.7. *For any H_∞-map ϕ from a Thom spectrum into E_h there is a formula*

$$\sum_{n\geqslant 0}\phi_{\mathrm{orb}}(M^n \circlearrowleft \Sigma_n)t^n = \exp\left[\sum_{k\geqslant 0} T_{p^k}(\phi(M))t^{p^k}\right].$$

Remark 1.8. The right hand side does not look like it depends on h, but it does because T_{p^k} does (cf. Definition 6.13).

There are two side results that are hopefully of independent interest to homotopy theorists. First, as a corollary of the discussion in Section 7.2 we obtain an explicit formula for the Strickland inner product in Morava-Lubin-Tate theory (cf. Corollary 7.14).

Proposition 1.9. *The Strickland inner product in Morava-Lubin-Tate theory is*

$$\begin{aligned} E_h^0(BG)\otimes E_h^0(BG) &\to E_h^0 \\ \chi\otimes\xi &\mapsto \frac{1}{|G|}\sum_{\alpha}\chi(\alpha)\xi(\alpha), \end{aligned}$$

where the sum is over all h-tuples of commuting elements of p-power order.

The main step in proving that ϕ_{orb} is an orbifold invariant is a Theorem about (equivariant) Spanier-Whitehead duals and the Borel construction (cf. Theorem 8.6).

Theorem 1.10. *Let* G *be a finite group. Then there is an ismorphism of functors from the category of finite* G*-CW-spectra to the* $K(h)$*-local category*

$$EG_+ \wedge_G D_G(-) \cong D(EG_+ \wedge_G (-)),$$

where D_G *denotes the* G*-equivariant dual and* D *denotes the dual in the* $K(h)$*-local category.*

1.7 Atiyah-Tall-Grothendieck type definition of Hecke operators

The Hecke operators on the Morava-Lubin-Tate cohomology theories were defined in [And92]; we recall the construction in Section 6.5. At chromatic level one they are the stable Adams operations in $K_p^\wedge(X)$. It was pointed out by Atiyah and Tall [AT69] how Adams' definition of the Adams operations in K-theory fits into the more general theory of λ-rings due to Grothendieck [Gro57]. However, Ando's generalizations are based on a third point of view on the Adams operations, also due to Atiyah, which uses representation theory [Ati66]. One of the many ways to read our result is that we give a formula for the T_{p^k} that translates Ando's definition into an Atiyah-Tall-Grothendieck type definition. This allows us to define Hecke operators in a slightly more general context than Ando does.

1.8 Plan

The thesis falls into two parts. In Part 1, we discuss the chromatic level one case: we explain our point of view in the example of the product formula for the topological Todd genus, and this is done in great detail. The point is that we tell the story in such a way that the exact same discussion goes through in the higher chromatic case. Part 2 is the generalization to the higher chromatic case.

Part 1

The topological Todd genus and its product formula

2 The Todd genus from the point of view of stable homotopy theory

This section is more or less a summary of[2] [CF66, I]. We explain their point of view on the Todd genus and recall how it is related to more classical definitions. Throughout the thesis the Todd genus discussion serves as a model for the discussion of elliptic genera. Indeed, the point of view on the Todd genus we are about to describe was the original motivation to define elliptic cohomology: an elliptic cohomology theory is a cohomology theory realizing an elliptic genus in the same way as K theory realizes the Todd genus.

2.1 The Conner-Floyd map

Atiyah, Bott and Shapiro constructed K-theory Thom classes u_{ABS} for complex vector bundles, which on the universal line bundle L over $\mathbb{C}P^\infty = BU(1)$ give the Euler class

$$x_{ABS} = i^* u_{ABS}(L) = (1 - L)\beta^{-1} \in \widetilde{K}^2(BU(1)) = \widetilde{K}^2(\mathbb{C}P^\infty),$$

where i denotes the zero section. Conner and Floyd [CF66, p.29] show that giving K-theory Thom classes for complex vector bundles is equivalent to giving a map of spectra

$$\mathrm{Td}\colon \mathrm{MU} \to \mathrm{K}$$

(denoted μ_c by Conner and Floyd) which on $\mathrm{MU}(n)$ is given by

$$u_{ABS}(\gamma^n_{\mathrm{univ}}) \in [\mathrm{MU}(n), \mathbb{Z} \times \mathrm{BU}].$$

The map Td is called Conner-Floyd map. The induced map on homotopy groups,

$$\mathrm{Td}_* := \pi_*(\mathrm{Td}),$$

is the Todd genus. This is the definition of Todd genus we are using throughout the thesis. The rest of this section is concerned with with relating this definition to other definitions more common in the non-homotopy theoretic literature. Although the expressions (5) and (6) will seem more explicit, we want to emphasize the fact that we never use them. All our arguments are on a purely conceptual level, no calculations with explicit formulas are made. In the case of the Todd genus this might not seem like a big improvement. The reader familiar with the explicit formulas for more complicated subjects treated in this thesis, e.g. the orbifold elliptic genus, might agree that those formulas soon start to look very complicated and appreciate an approach that does not use them. Proposition 2.2 is going to play a role in Section 8.

[2]Conner and Floyd contribute many of the results mentioned here to Atiyah and Hirzebruch [AH61] and Dold [Aar62].

2.2 The push-forward of 1

Assume we are given a multiplicative cohomology theory $E^*(-)$ with natural Thom classes for complex vector bundles, or equivalently, a map of ring spectra $\phi\colon MU \to E$. Let $[M] \in MU_d$, i.e. let M be a compact closed smooth d-dimensional manifold together with a choice of lift $-[\tau]_K \in \widetilde{K}(M)$ of its stable normal bundle $-[\tau] \in \widetilde{KO}(M)$. Such manifolds are called "manifolds with (normally) almost complex structure". Let

$$\pi\colon M \to pt$$

denote the unique map. The following is a slight reformulation of the definition of "Umkehr" map along π in [Dye69, pp. 40-41], using the language of Thom spaces of virtual bundles set up e.g. in [Rud98, IV].

Definition 2.1. The push-forward along π in $E^*(-)$ is defined by

$$\pi_!^{\phi}\colon E^*(M) \xrightarrow{\cong} \widetilde{E}^{*-d}(M^{-\tau}) \longrightarrow \widetilde{E}^{*-d}(S^0) \cong \pi_{*+d}(E),$$

where the first map is the Thom-Dold isomorphism for $-[\tau]_K$ and the second map is the Pontrjagin-Thom collapse.

Proposition 2.2. *The genus induced by* ϕ,

$$\phi_*\colon MU_* \to E_*$$

sends $[M] \in MU_d$ *to the push-forward of one* $\pi_!^{\phi}(1) \in E_d$.

PROOF : The transformation ϕ maps Thom classes to Thom classes and thus $\pi_!^{\mathrm{id}_{MU}}$ to $\pi_!^{\phi}$. Therefore it is sufficient to consider the universal case $\phi = \mathrm{id}_{MU}$. In this case the statement follows directly from the definition of cobordism Thom class and from the Pontrjagin-Thom construction. □

Definition 2.3. For $a \in K(M)$ we can now define the Todd genus of M with coefficients in a as

$$\mathrm{Td}(M; a) := \pi_!^{\phi}(a).$$

There are two directions to go from this, one using the Riemann-Roch formula and one using the index theorem.

2.3 The Todd genus in terms of characteristic classes

In the case that $E = H$ and that ϕ is given by the standard Euler class $x_H = c_1(L)$, we have

$$\pi_!^H(a) := \pi_!^{\phi}(a) = \int_M a.$$

In the case $E = K$ and $\phi = \mathrm{Td}$, the Riemann-Roch theorem [Aar62, Dyer] relates $\pi_!^{\mathrm{Td}}$ and $\pi_!^{H}$:

$$\mathrm{ch}(\pi_!^{\mathrm{Td}}(a)) = \pi_!^{H}(\mathrm{td}(\nu_M)\,\mathrm{ch}(a)). \tag{4}$$

Here H denotes ordinary cohomology with coefficients in $\mathbb{Q}$, a is an element of $K^0(M)$, and

$$\mathrm{td}(\nu_M) = \prod \frac{1 - e^{x_i}}{x_i},$$

where the x_i are the Chern roots of a normal bundle ν_M. Putting everything together, we have

$$\mathrm{Td}_*(M, a) = \pi_!^{\mathrm{Td}}(a) = \pi_!^{H}(\mathrm{td}(\nu_M)\,\mathrm{ch}(a)) = \int_M \prod \frac{1 - e^{x_i}}{x_i}\,\mathrm{ch}(a). \tag{5}$$

This expression the inverse of the one Conner and Floyd obtain, because they work with the tangent bundle rather than the normal bundle.

2.4 The Todd genus as Euler characteristic

There is yet another point of view on the Todd genus, and in order to show that it is equivalent to the definitions discussed above, one needs to use the index theorem. Let M be a compact complex manifold and V a holomorphic vector bundle over M. Consider the Dolbeault complex

$$\cdots \longrightarrow A^{0,q}(V) \xrightarrow{\bar\partial} A^{0,q+1}(V) \longrightarrow \cdots,$$

where

$$A^{0,q}(V) = \Gamma(\Lambda^0 T^* \otimes \Lambda^q \bar{T}^* \otimes V)$$

denotes the differential forms of type $(0, q)$ with values in V. We have [HBJ92, 5.4]

$$\mathrm{Td}(M; V) = \mathrm{index}(A^{0,\bullet}(V), \bar\partial) = \pi_!^{T^*M}(a(M, V)),$$

where $a(M, V)$ denotes the symbol of $(A^{0,\bullet}(V), \bar\partial)$. This turns out to be

$$\mathrm{Td}(M; V) = \sum (-1)^i \dim(H^i(M, \mathcal{O}(V))) \tag{6}$$

[AS68a, p.542].

3 The equivariant Todd genus

All the constructions of the previous section go through equivariantly.

3.1 Okonek's equivariant Conner-Floyd maps

The following proposition and examples are taken from [Oko82, 1][3].

Proposition 3.1 (Okonek). *If* E^*_G *is a multiplicative,* G*-equivariant cohomology theory with natural Thom classes for complex* G*-bundles, then there is a unique natural, stable transformation*

$$\phi_G\colon MU^*_G(-) \to E^*_G(-)$$

of multiplicative G*-equivariant cohomology theories that takes Thom classes to Thom classes.*

Rather than explaining all the concepts that turn up in the statement of the proposition, we state the two examples that are relevant to us.

Example 3.2. For any complex oriented ring spectrum E, Borel equivariant E-cohomology

$$E(EG \times_G -)$$

has natural Thom classes for complex G-bundles. In this case, ϕ_G factors over Borel equivariant cobordism. If we let ϕ be the orientation of E, then ϕ preserves the equivariant Thom classes, so that we get

$$\phi_G\colon MU_G(-) \to MU(EG \times_G -) \xrightarrow{\phi} E(EG \times_G -).$$

Example 3.3. (Compactly supported) equivariant K-theory has natural Thom classes for complex G-bundles. We denote the resulting equivariant Conner-Floyd maps by

$$Td_G\colon MU_G \to K_G.$$

There is a Pontrjagin-Thom map from the equivariant cobordism ring to the coefficient ring MU^*_G, which in the equivariant case fails to be an isomorphism.

Definition 3.4. The equivariant Todd genus of an almost complex G-manifold is defined to be

$$Td_G(M) := Td_{G*}([M]),$$

where $[M \circlearrowleft G]$ denotes the image of $M \circlearrowleft G$ under the Pontrjagin-Thom map.

3.2 Equivariant push-forward of 1

The Thom spectrum and Thom-Dold isomorphism of a virtual equivariant bundle are defined in [LMSM86, X], and Definition 2.1 goes through for an equivariant theory with Thom classes.

[3]For an English reference see [May96]. There is a difference between the two: Okonek works with tom Dieck's definition of an equivariant cohomology theory [tD71]. In the language of [May96] this is a complex-stable, naive G-equivariant cohomology theory.

In the case of equivariant K-theory this definition is equivalent to the definition[4] in [AS68b]. On the image of the Pontrjagin-Thom map the same argument as in Proposition 2.2 shows

$$\phi_G(M) = \pi_!^{\phi_G}(1) \in E_G(pt),$$

where $\pi\colon M \to pt$ is the unique G-map. The discussion in [AS68a] is of course also equivariant, i.e. the equivariant analogue of (6) holds. The papers [AS68a] and [AS68c] discuss in detail how to express the push-forward in equivariant K-theory using character theory and the Riemann-Roch theorem. The remainder of this section recalls their results. Again this is merely meant to connect our definition to other definitions of topological Todd genus in the literature, which we are not going to use in this thesis.

3.3 The character map in K-theory

The equivariant Todd genus takes values in the representation ring $K_G(pt) = R(G)$, and it is classical that a representation $V \circlearrowleft G$ is determined by its character

$$g \mapsto \mathrm{Trace}(g|V). \tag{7}$$

In the following we recall how to compute

$$\mathrm{Trace}(g|\mathrm{Td}_G(M)).$$

Note that the definition of (7) is extended to $V \in K_G(X)$ as follows.

Definition 3.5 ([AS89]). Let G be a finite group which acts on a compact space X, and let $g \in G$. Evaluation at g is defined to be the map

$$\begin{aligned} \mathrm{Trace}(g|-)\colon K_G(X) &\to K(X^g)^{C_g} \otimes \mathbb{C} \\ V &\mapsto \sum_\zeta (V|_{X^g})_\zeta \otimes \zeta, \end{aligned}$$

where X^g stands for the g-fixed points of X, C_g is the centralizer of g in G, and $(-)_\zeta$ denotes the ζ eigenspace.

3.4 The equivariant Todd genus in terms of characteristic classes

We have by [AS68a, (2.11)]

$$\mathrm{Trace}(g|\pi^M_{!G}(u)) = \pi_!^{M^g}\left(\frac{\mathrm{Trace}(g|i^*u)}{\mathrm{Trace}(g|\Lambda_{-1}(N^g))}\right), \tag{8}$$

[4]More precisely, if one replaces TX by X and assumes that all the Thom classes that are needed exist, Atiyah and Singer's ind_G^X becomes our $\pi^X_{!G}$.

where i is the inclusion of M^g into M, and $\Lambda_{-1}(N^g)$ is the Atiyah-Bott-Shapiro Euler class of its normal bundle N^g. One can express the right hand side of (8) using the Riemann-Roch formula. This has been done by Atiyah and Singer. Their result [AS68c] is summarized by

$$\pi_!^{M^g}\left(\frac{\mathrm{Trace}(g|i^*E)}{\mathrm{Trace}(g|\Lambda_{-1}N^g)}\right) = \int_{M^g} \mathrm{td}(M^g)\,\mathrm{ch}\left(\frac{\sum_\zeta E_\zeta \otimes \zeta}{\sum_\xi (\Lambda_{-1}N^g)_\xi \otimes \xi}\right) \tag{9}$$

$$= \int_{M^g} \frac{\sum \zeta e^{x_i(E_\zeta)}}{\prod_\xi 1 - \xi e^{x_j(N^g_\xi)}},$$

where the first equation is (4), $x_i(E_\zeta)$ and $x_j(N^g_\xi)$ denote the Chern roots of the eigenbundles and the sum and product on the right hand side are taken over all these Chern roots.

4 Power operations

4.1 Power operations and H_∞-ring spectra

Let $\{E_G \mid G \text{ finite}\}$ be a compatible family of equivariant cohomology theories in the sense of [LMSM86, II.8.5], and write $E_G(X)$ for $E^0_G(X)$. "Compatible" means in particular that for a map $\alpha\colon H \to G$ and a G-space X, we have restriction maps

$$\mathrm{res}|_\alpha\colon E_G(X) \to E_H(X),$$

and if α is the inclusion of a subgroup and X a G-space we also have induction maps

$$\mathrm{ind}|_H^G\colon E_H(X) \to E_G(X),$$

such that the axioms of a Mackey structure on E_G spelled out in [tD73] are satisfied.

Remark 4.1. The author could not find the reference for this fact, so here is a short guide through the literature: A compatible family satisfies

$$E_G(G \ltimes_H X) \cong E_H(X) \tag{10}$$

for $H \subseteq G$ and any (pointed) H-space X. If X is already a G-space, one has

$$G \ltimes_H X \cong G/H_+ \wedge X,$$

and the map

$$(p_{G/H})_+\colon G/H_+ \to \mathbb{S}^0$$

sending all of G/H to the non-basepoint induces $\mathrm{res}|_H^G$, while its G-equivariant Spanier-Whitehead dual

$$D_G(p_{G/H})_+\colon \mathbb{S}^0 \to G/H_+$$

induces $\mathrm{ind}|_H^G$. For arbitrary α, the compatibility condition does not provide us with an isomorphism (10), but with a map from the left to the right. Thus if we replace $(p_{G/H})_+$ by the co-unit of the adjunction $(G \ltimes_\alpha -, \text{forget})$ we can still define define $\text{res}|_\alpha$. The Mackey criteria follow from [May96, XIX.3].

Example 4.2. For any E, Borel equivariant E-cohomology $E(EG \times_G -)$ is an example [May96, XXI.1.9]. Here the induction maps equal the transfer maps

$$T_H^G\colon \Sigma_+^\infty(EG \times_G X) \to \Sigma_+^\infty(EH \times_H X).$$

Example 4.3. Equivariant K theory is an example, with the induction maps the induced representation.

We also ask that our family has unitary, commutative and associative *external products*

$$\wedge\colon E_G(X) \otimes E_H(Y) \to E_{G\times H}(X \wedge Y),$$

that are natural in (stable) maps of X and Y. By Remark 4.1 this implies that $\wedge$ also commutes with induction and restriction maps. By *unitary* we mean that for each G, there is an element $1 \in E_G(\mathbb{S}^0)$ with $1 \wedge x = \text{res}|_{\text{pr}_2} x$, where pr_2 is the projection onto the second factor of $G \times H$. We want $\text{res}|_H^G 1 = 1$. Before we recall the Definition of an H_∞-structure on $\{E_G\}$, we need to introduce some notation.

Notation 4.4. Let X be a pointed G-space. We write

$$(X \circlearrowleft G)^n \circlearrowleft \Sigma_n \quad \text{or} \quad X^n \circlearrowleft (G \wr \Sigma_n)$$

for the space $X^{\wedge n}$ acted on by

$$G \wr \Sigma_n = G^n \rtimes \Sigma_n$$

in the following way: G acts on each factor individually, while Σ_n permutes the factors. By abuse of notation, we also write $E_{\Sigma_n}(X^n)$ for $E_{G\wr\Sigma_n}(X^n)$ and in particular $E(X)$ for $E_G(X)$, unless we want to emphasize the equivariant situation.

The following definition is essentially [BMMS86, VIII.1.1].

Definition 4.5. An H_∞-structure on E is given by a collection of natural maps

$$P_n\colon E_G(X) \to E_{G\wr\Sigma_n}(X^n)$$

called *power operations* satisfying the following conditions:

(a) $P_1 = \text{id}$ and $P_0(x) = 1$,

(b) the (external) product of two power operations is

$$P_j(x) \wedge P_k(x) = \text{res}|_{\Sigma_j \times \Sigma_k}^{\Sigma_{j+k}} (P_{j+k}(x)),$$

(c) the composition of two power operations is

$$P_j(P_k(x)) = \operatorname{res}|^{\Sigma_{jk}}_{\Sigma_k \wr \Sigma_j} (P_{jk}(x)),$$

(d) and the P_j's preserve (external) products:

$$P_j(x \wedge y) = \operatorname{res}|^{\Sigma_j \times \Sigma_j}_{\Sigma_j} (P_j(x) \wedge P_j(y)),$$

where the restriction is along the map

$$[((X \backslash\!\!\backslash G)^2)^j \backslash\!\!\backslash \Sigma_j] \longrightarrow [(X \backslash\!\!\backslash G)^{2j} \backslash\!\!\backslash (\Sigma_j \times \Sigma_j)] \cong [((X \backslash\!\!\backslash G)^j \backslash\!\!\backslash \Sigma_j)^2].$$

Remark 4.6. Traditionally people formulated this definition only for Borel equivariant theories. In that case it is a refinement of the notion of ring spectrum up to homotopy, but it is weaker than the notion of E_∞ or A_∞ structure, which would imply the existence of a multiplication on the nose. In the same way our definition is weaker than Greenlees and May's notion of *global $\mathcal{I}_*$ functor with smash product spectrum* in [GM97]. More precisely, using the Yoneda lemma one can reformulate Definitions 4.5 and in terms of maps of $G \wr \Sigma_n$-spectra

$$\xi_n \colon E_G^n \to E_{G \wr \Sigma_n}.$$

The maps $\operatorname{res}|^G_H$ are induced by maps of G-spectra

$$G \ltimes_H E_H \to E_G,$$

and with the Yoneda lemma the external product $\wedge$ becomes a map of $G \times H$-spectra. Thus Conditions (1)-(4) of the definition translate into homotopy commutative diagrams of spectra. A global $\mathcal{I}_*$ functor with smash product spectrum has such ξ_n, and in their case, the diagrams commute strictly.

4.2 Total power operations

Let E be an H_∞-ring spectrum. It is often convenient to consider all power operations at once, i.e. the *total power operation*

$$P \colon E(X) \to \widehat{\bigoplus_{n \geqslant 0}} E_{\Sigma_n}(X^n) t^n$$

which is P_n into each summand. The variable t is a dummy variable, introduced in order to keep track of the summand and also to avoid convergence issues later on.

Remark 4.7. Note that

$$\widehat{\bigoplus_{n \geqslant 0}} E_{\Sigma_n}(X^n) t^n$$

is a graded ring by

$$E_{\Sigma_n}(X^n) \otimes E_{\Sigma_m}(X^m) \xrightarrow{\wedge} E_{\Sigma_n \times \Sigma_m}(X^{n+m}) \xrightarrow{\text{ind}} E_{\Sigma_{n+m}}(X^{n+m}),$$

where

$$\text{ind} = \text{ind}\,|_{\Sigma_n \times \Sigma_m}^{\Sigma_{m+n}}$$

(compare [Seg96]).

Proposition 4.8 (compare [BMMS86, VIII.1.1]). *We have*

(a) the restriction of $P_j(x)$ *to* $E(X^j)$ *is*

$$\text{res}\,|_1^{\Sigma_j} P_j(x) = x^{\wedge j},$$

(b) the operation P_j *applied to* $1 \in E^0(\mathbb{S}^0)$ *is*

$$P_j(1) = 1_{\Sigma_j} := 1 \in E_{\Sigma_j}(\text{pt}),$$

(c) the total power operation takes sums into products:

$$P(x+y) = P(x) \cdot P(y).$$

PROOF : The first two properties are immediate from the definition. The proof of Property (c) in [BMMS86, II.1.6] and [LMSM86, VII.1.10] takes place on the level of equivariant spectra. □

4.3 Power operations in K-theory and cobordism

In [Ati66] Atiyah defines power operations for K-theory. In the case of an (equivariant) vector bundle V over (a G-space) X, they are given by the (external) tensor product

$$P_n([V]) = [V^{\otimes n}] \in K_{\Sigma_n}(X^n).$$

In [tD68] tom Dieck defines power operations for Borel equivariant cobordism and shows that the Conner-Floyd map is an H_∞-map in the classical (i.e. Borel equivariant) sense. We prefer to work on the level of equivariant cobordism $MU^*_G(-)$ (cf. [tD70]). In that case, Greenlees and May show that MU_G is a "global $\mathcal{I}_*$ functor with smash product spectrum" [GM97, 5.8]. Thus, by Remark 4.6 it has power operations. They are made in such a way that the map

$$MU^*_G(-) \to MU^*(EG \times_G -) \tag{11}$$

from Example 3.2 preserves power operations. On coefficients[5] these power operations in

[5]More precisely: on non-equivariant coefficients or on the image of the Pontrjagin-Thom map.

equivariant cobordism are

$$\begin{aligned} P_n\colon MU^*(pt) &\to MU^{n*}_{\Sigma_n}(pt) \\ [M] &\mapsto [M^n \circlearrowleft \Sigma_n]. \end{aligned} \tag{12}$$

Proposition 4.9 (compare [tD68, (A4)]). *The P_n are multiplicative with respect to $\wedge$ and compatible with Thom classes in the following sense.*

$$P_n(u_{MU}(\xi)) = u_{MU}(\xi^{\oplus n}), \tag{13}$$

and $\{MU_G\}$ equipped with the P_n is universal with this property: For any equivariant cohomology theory with multiplicative Thom classes for complex G-bundles and power operations satisfying (13) the maps ϕ_G from Proposition 3.1 preserve power operations.

PROOF : If ξ is the universal complex G-bundle, Equation (13) is immediate from the construction, for other ξ it follows by naturality. Let now E_G be an equivariant family as in the proposition. Then the proof of Proposition 3.1 shows that the ϕ_G preserve power operations. □

Corollary 4.10. *The map (11) is an H_∞-map.*

Corollary 4.11 (compare [tD68]). *The equivariant Conner-Floyd-Okonek maps from Example 3.3 preserve the H_∞-structure.*

Remark 4.12. Let E be an H_∞-spectrum with compatible Thom classes as in the proposition, and let V be a complex d-dimensional G representation. Then E_G comes equipped with natural isomorphisms

$$E^0_G(\mathbb{S}^{2d} \wedge X) \xrightarrow{\cong} E^0(V^c \wedge X),$$

where V^c denotes the one point compactification of V (cf. [GM97, 2.1]). This becomes important when we want to extend our power operations to

$$E^{-2d}(X) = E^0(\mathbb{S}^{2d} \wedge X),$$

because $(\mathbb{S}^{2d})^n \circlearrowleft \Sigma_n$ is an equivariant sphere. In the situation of the proposition we can follow [GM97] to extend the power operations to

$$P_n\colon E^{2d}(X) \to E^{2nd}(X^n).$$

4.4 Internal power operations

We can always compose the power operation P_n with the pullback along the diagonal map of X^n

$$\Delta^*\colon E_{\Sigma_n}(X^n) \to E_{\Sigma_n}(X).$$

Since the action of Σ_n on X is trivial, the target of this map often turns out to be $E_{\Sigma_n}(pt) \otimes E(X)$. We might want to compose further with a map

$$E_{\Sigma_n}(pt) \to E^0$$

in order to obtain operations[6] acting on $E(X)$.

Atiyah's work in K-theory

In the case of K-theory, $E_{\Sigma_n}(pt)$ is the representation ring $R(\Sigma_n)$. Two examples from [Ati66] are important to us.

Example 4.13. Atiyah's definition of the Adams operations is

$$\psi_n(x) = \mathrm{Trace}(c_n | \Delta^* P_n(x)),$$

where c_n is a cycle of length n.

Example 4.14. The operation

$$\begin{aligned} \sigma_n(x) &:= \frac{1}{n!} \sum_{g \in \Sigma_n} \mathrm{Trace}(g | \Delta^* P_n(x)) \\ &= \langle \Delta^* P_n(x), 1 \rangle_{\Sigma_n} \end{aligned}$$

counts the number of times that the trivial representation is a summand in $\Delta^* P_n(x)$. If $x = [V]$ is the class of a vector bundle V, then

$$\sigma_n(x) = [\mathrm{Sym}^n(V)]$$

is represented by the n^{th} symmetric power of V.

Notation 4.15. Let

$$S_t(x) := \sum_{n \geqslant 0} \sigma_n(x) t^n$$

denote the total symmetric power.

Remark 4.16. The total symmetric power S_t is the composite

$$S_t \colon K(X) \xrightarrow{P} \widehat{\bigoplus_{n \geqslant 0}} K_{\Sigma_n}(X^n) t^n \to \widehat{\bigoplus_{n \geqslant 0}} R(\Sigma_n) \otimes K(X) t^n \to K(X)[\![t]\!],$$

where on the n^{th} summand the second map is pullback along the diagonal and the third map is the inner product with 1_{Σ_n}.

[6]In the literature (e.g. [And92]) these compositions are often referred to as power operations and P_n is then called "total power operation". We follow the convention to call them *internal power operations*, since they actually act on $E(X)$.

Remark 4.17. Atiyah's work was generalized to cohomology theories that have Hopkins-Kuhn-Ravenel character theory in [And92]. We recall this in Section 6.5.

5 Product formulas for the Todd genus

5.1 The product formula for the equivariant Todd genus

The following is a reformulation of Corollary 4.11:

Corollary 5.1. *The square*

$$\begin{array}{ccc} MU(X) & \xrightarrow{\mathrm{Td}} & K(X) \\ {\scriptstyle P_{MU}}\downarrow & & \downarrow{\scriptstyle P_K} \\ \widehat{\bigoplus_{n\geqslant 0}} MU_{\Sigma_n}(X^n)t^n & \xrightarrow[\bigoplus_{n\geqslant 0}\mathrm{Td}_{\Sigma_n}]{} & \widehat{\bigoplus_{n\geqslant 0}} K_{\Sigma_n}(X^n)t^n \end{array}$$

commutes.

Remark 5.2. Recall that the key objects in this whole story, the product formulas, express (a variant of) the Todd genus of the $(M^n \circlearrowleft \Sigma_n)$'s in terms of the Todd genus of M itself. In view of Corollary 5.1, this fact is not surprising at all and neither is the fact that the expression should be exponential, i.e. some sort of product formula in $\mathrm{Td}(M)$.

More precisely, for X a point we obtain the following corollary:

Corollary 5.3 (Equivariant product formula). *Let M be an almost complex manifold. Then we have the following equation in $\widehat{\bigoplus_{n\geqslant 0}} R(\Sigma_n)t^n$:*

$$\sum_{n\geqslant 0} \mathrm{Td}_{\Sigma_n}(M^n)t^n = \left(\sum_{n\geqslant 0} 1_{\Sigma_n}t^n\right)^{\mathrm{Td}(M)},$$

where $1_{\Sigma_n} \in R(\Sigma_n)$ denotes the trivial representation.

PROOF : By Corollary 5.1 we have

$$\sum_{n\geqslant 0} \mathrm{Td}_{\Sigma_n}(M^n)t^n = P_K(\mathrm{Td}(M)).$$

By Proposition 4.8 (c), P_K takes sums into products, and we have

$$\mathrm{Td}(M) \in K(\mathrm{pt}) = \mathbb{Z}.$$

Together with Definition 4.5 (a) this proves the claim. □

5.2 Product formula for the topological Todd genus of an orbifold

As explained in the introduction, the correct chromatic level one analogue of the orbifold elliptic genus is the topological Todd genus of an orbifold (cf. Definition 1.3).

Remark 5.4. Note that by Corollary 5.1 and Example 4.14 we have

$$\sum_{n\geqslant 0} \mathrm{Td}_{\mathrm{top}}(M^n \backslash\!\!\backslash \Sigma_n)t^n = S_t(\mathrm{Td}(M)).$$

The product formula for the topological Todd genus is [Dij99]

$$\sum_{n\geqslant 1} \mathrm{Td}_{\mathrm{top}}(M^n \backslash\!\!\backslash \Sigma_n) = \left(\frac{1}{1-t}\right)^{\mathrm{Td}(M)}. \tag{14}$$

We give two proofs for (14).

PROOF NUMBER 1: The formula follows exactly as Corollary 5.3 from the well known fact that S_t is exponential. □
Before we give our second proof, we rewrite the right hand side of the equation in such a way that makes it obvious what the higher chromatic analogues are.

5.3 Borcherds' way to write the right hand side of the DMVV-formula

The following computation is copied from Borcherds' proof of the product formula for the j-function [Bor92]. Let

$$f = \sum_{n,l} c(n,l)q^n y^l$$

be the q-expansion of a (weak) Jacobi form of weight zero. Then

$$\begin{aligned}
\log \prod_{m\geqslant 1, n\geqslant 0, l\in\mathbb{Z}} \left(\frac{1}{1-t^m q^n y^l}\right)^{c(mn,l)} &= \sum_{m,n,l}\sum_{k\geqslant 1} \frac{1}{k} c(mn,l) t^{mk} q^{nk} y^{lk} \\
&= \sum_{m}\sum_{n,l}\sum_{d|(m,n,l)} \frac{1}{d} c\left(\frac{m}{d}\frac{n}{d}, \frac{l}{d}\right) q^n y^l t^m \\
&= \sum_{m\geqslant 1} V_m(f)\, t^m,
\end{aligned}$$

where the V_m are a kind of Hecke operators acting on Jacobi forms and the last equality is [EZ85, I.4.2 (7)].

Remark 5.5. Note the striking similarity of the right hand side of the DMVV-formula with the formal inverse of Rezk's logarithm formula [Rez, p.4]

$$\exp \sum_{k \geqslant 0} T_{p^k}(-). \tag{15}$$

Here the T_{p^k} are as in Definition 6.13.

For the purpose of generalizing to the higher chromatic case, we have to write the right hand side of our product formula as (15), just with the t^{p^k}'s in there (compare Theorem 1.7). Let us first understand how this works for the height one case, i.e. the topological Todd genus. The chromatic level one analogue of the Hecke operators are the stable Adams operations $\frac{\psi_n}{n}$.

5.4 Second proof of the Todd genus product formula

We return to the product formula (14). We thank Charles Rezk for pointing out to us that the following equation follows from Atiyah and Tall's definition of the Adams operations.

$$S_t(x) = \exp\left[\sum_{n \geqslant 1} \frac{\psi_n(x)}{n} t^n\right]. \tag{16}$$

By Remark 5.4, equation (16) provides an alternative proof of (14). More precisely, together with Corollary 5.1 it proves

$$\sum_{n \geqslant 0} \mathrm{Td}_{\mathrm{top}}(M^n \circlearrowleft \Sigma_n) t^n = \exp\left[\sum_{n \geqslant 1} \frac{\psi_n(\mathrm{Td}(M))}{n} t^n\right]. \tag{17}$$

This proof has several advantages over the proof we gave above: it takes place on the level of $K(X)$, rather than just $K(pt)$, it does not use the fact that $K(pt) = \mathbb{Z}$, and it expresses the right hand side in the form (15), the two latter advantages providing the right setup for our generalizations. However, rather than Atiyah and Tall's definition of the ψ_n we need to use Atiyah's definition[7]. Here is the computation with that definition:

$$\exp\left[\sum_{n \geqslant 1} \frac{\psi_n(x)}{n} t^n\right] = \sum_{m \geqslant 0} \frac{1}{m!}\left[\sum_{n \geqslant 1} \frac{\psi_n(x)}{n} t^n\right]^m,$$

and the coefficient of t^l is

$$\sum_{\sum a_n n = l} \frac{(\sum a_n)!}{\prod(a_n!)} \frac{1}{(\sum a_n)!} \prod_n \left(\frac{\psi_n(x)}{n}\right)^{a_n},$$

[7] This is because Atiyah's definition is the one that was generalized to the higher chromatic case in [And95]. One way to say what is going to happen in the following is that we are going to find Atiyah-Tall type definitions of the Hecke operators defined in [And92].

where $\frac{(\sum a_n)!}{\prod(a_n!)}$ counts the number of ways to part a set of $\sum a_n$ (cycles) into subsets of orders a_n, which is the number of times this partition occurs, and $\frac{1}{(\sum a_n)!}$ is $\frac{1}{m!}$. This is

$$\sum_{\sum a_n n = l} \prod_n \frac{1}{n^{a_n} a_n!} \psi_n(x)^{a_n} = \sum_{[g]} \frac{1}{|C_g|} \mathrm{Trace}(g|P_l(x)) = \sigma_l(x),$$

where the last sum is over the conjugacy classes of Σ_l and C_g denotes the centralizer of g in Σ_l.

Remark 5.6. The same calculation makes sense for the a G-space X. Note that in this case e.g. by "pull-back along the diagonal" we mean pull-back along the map

$$\delta\colon X \circlearrowleft G \to (X \circlearrowleft G)^n \circlearrowleft \Sigma_n,$$

which is at the same time restriction along the diagonal

$$G \times \Sigma_n \to G \wr \Sigma_n.$$

In the case $X = \mathrm{pt} \circlearrowleft G$ we obtain a product formula for the $G \wr \Sigma_n$ equivariant Todd genus of $(M \circlearrowleft G)^n$.

Part 2

The orbifold elliptic genus and other higher chromatic relatives of $\mathrm{Td}_{\mathrm{top}}$

The methods of Part 1 appear to be specific to equivariant K-theory: We speak about the inner product of two representations, about symmetric powers of vector bundles, and most importantly, about evaluation maps of characters at group elements. The latter is the key to the first two – one can define symmetric powers and inner products in terms of evaluation maps, cf. Example 4.14. Note that this definition has the disadvantage that it is non-trivial to deduce that the inner product takes values in $\mathbb{Z}$ and thus that the symmetric powers take values in $K(X)$ and the topological Todd genus of orbifolds takes values in $\mathbb{Z}$.

Our discussion in the higher chromatic case relies on the fact that character theory as well as inner products have been defined in much greater generality. Firstly, for E a suitable $K(h)$-local cohomology theory, e.g. $E = E_h$ Morava-Lubin-Tate cohomology, and χ an element of $E^0(BG)$, Hopkins-Kuhn-Ravenel theory defines evaluation of χ at h-tuples of commuting p-power order elements of G. We recall the basic concepts of Hopkins-Kuhn-Ravenel theory in Section 6.2. Secondly, Strickland defines inner products

$$E^0(BG) \otimes E^0(BG) \to E^0$$

for an even larger variety of cohomology theories. Indeed these inner products live on the classifying space BG itself (in the $K(h)$-local stable homotopy category that is). We recall Strickland's results in Section 7.2. In the case that Hopkins-Kuhn-Ravenel theory applies and E^0 is torsion free we identify his bilinear form with

$$b_G(\chi, \xi) = \frac{1}{|G|} \sum_{\alpha} \chi(\alpha)\xi(\alpha) \tag{18}$$

(Corollary 7.14) and conclude that the right hand side takes values in E^0.

The key idea of Ando and French's definition of supersingular orbifold genus is that if we start with a genus into E_2, the appropriate way to reproduce Definition 1.3 is to replace the evaluations at group elements by Hopkins-Kuhn-Ravenel evaluations at pairs of commuting p-power order elements. To justify their definition, Ando and French show that it is equivalent to an expression which is, up to a factor $\frac{1}{|G|}$, very similar to the definition of orbifold genus in [BL03]. Our first definition of higher chromatic orbifold genus (Section 7.4) follows their definition closely, but we choose to put the factor $\frac{1}{|G|}$ back in, using (18) to argue that our orbifold genera still take values in E^0. Following Ando and French one can bring our definition into a form similar to Borisov and Libgober's. Using (18) again we are able to rewrite our definition in a form that makes sense in an even more general situation, namely for any $K(h)$-local cohomology theory E.

More important than the higher generality is that we learned that apart from the Thom-Dold isomorphism, all maps used in the definition of orbifold genus are induced by maps in the $K(h)$-local category. This is the key to Section 8, where we prove that $\phi_{orb}(M^n \circlearrowleft \Sigma_n)$ depends only on the orbifold $M/\!\!/G$.

For $E = E_h$ and ϕ an H_∞ map we prove a DMVV-type product formula (cf. Section 9.2). The right hand side of our product formula also involves objects whose definition requires evaluation maps, namely the Adams operations at chromatic level one getting replaced by

Hecke operators in elliptic cohomology and the other higher chromatic cases. In their present form these Hecke operators were defined by Ando; we recall their definition in Section 6.5. We define higher chromatic analogues of symmetric powers in a similar fashion. Using Strickland inner products it turns out that we can define symmetric powers in the more general situation of a $K(h)$-local cohomology theory with power operations (Section 7.5).

6 Hopkins-Kuhn-Ravenel theory

6.1 Even periodic ring spectra and formal groups

We keep the thesis in the language of even periodic ring spectra, because all our examples are of this kind. This section is a short reminder of their definition and properties. For details see [AHS01].

Definition 6.1. An even periodic ring spectrum is a spectrum E such that the graded coefficient ring E^* is concentrated in even degrees and E^2 contains a unit.

No choice of this unit is specified.

Remark 6.2. In the context of even periodic ring spectra it is often convenient to replace the complex cobordism spectrum MU by its two-periodic version

$$\mathrm{MP} := \bigvee_{j\in\mathbb{Z}} \Sigma^{2j}\mathrm{MU}.$$

Remark 6.3. For even periodic E the Atiyah-Hirzebruch spectral sequence for $E^*(\mathbb{C}P^n)$ collapses, and the system

$$E^*(\mathbb{C}P^n) \leftarrow E^*(\mathbb{C}P^{n+1})$$

is Mittag-Leffler, such that $E^*(\mathbb{C}P^\infty)$ becomes non-canonically isomorphic to $E^*[\![x]\!]$. As usual[8] a good choice of such an x gives rise to E-theory Chern classes, and to a formal group law F over E_* describing the first Chern class of the tensor product of line bundles

$$c_1(L_1 \otimes L_2) = c_1(L_1) +_F c_1(L_2).$$

The advantage of working with even periodic E is that rather than speaking about formal group laws one can use the language of formal groups: For such E the map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

classifying the tensor product of line bundles makes the formal spectrum spf $E^0(\mathbb{C}P^\infty)$ into an (affine one dimensional) formal group scheme, and choosing a coordinate of this formal group is equivalent to specifying a map of ring spectra

$$\mathrm{MP} \to E.$$

[8]Cf. [Ada95], [Rud98]

We do not make much use of these concepts, but we use several results whose proofs rely on a deep understanding of the way these formal groups (group laws) come into the picture. For the moment it is enough to remember that an even periodic ring spectrum E has somehow a formal group attached to it.

6.2 Review of Hopkins-Kuhn-Ravenel theory

This section collects the results from [HKR00] that we are going to use. Rather than good secondary literature on the subject we give merely a summary of the formal properties we need. A nice and short introduction to Hopkins-Kuhn-Ravenel character theory can be found in [AF03, 5], also see [Rez, 8].

Definition 6.4. Let E be an even periodic ring spectrum with associated formal group F. We say that E *has a Hopkins-Kuhn-Ravenel theory* if

(a) E^0 is local with maximal ideal m, and complete in the m-adic topology,

(b) the graded residue field E^0/m has characteristic $p > 0$,

(c) $p^{-1}E^0$ is not zero,

(d) the mod m reduction of F has height $h < \infty$ over E^0/m.

Hopkins Kuhn and Ravenel give a list of examples satisfying the conditions of this definition. One of these examples is in addition an H_∞-spectrum and the interplay between Hopkins-Kuhn-Ravenel theory and the H_∞-structure is well understood. This is the reason why it becomes our favorite example.

Example 6.5 (Lubin-Tate cohomology/Morava E-theory). Consider the graded ring

$$E_* := W\mathbb{F}_{p^h}[\![u_1, \dots, u_{h-1}]\!][u^{\pm 1}],$$

where u_i has degree zero and u has degree 2. There is a cohomology theory called Lubin-Tate cohomology or Morava E-theory, which has E_* as coefficients. On a finite complex X, it is given by

$$E_h^*(X) = MU^*(X) \otimes_{MU^*} E^*,$$

where the map $MU_* \longrightarrow E_*$ classifies the universal deformation of the Honda formal group law. The construction of this cohomology theory can be found in[9] [Rez98].

[9] Rezk omits a subtlety in his exposition: He proves that E_* is Landweber exact over BP, obtaining a homology theory. Via Spanier-Whitehead duality this becomes a cohomology theory on finite complexes as described. The phantom discussion in [HS99] proves that it is (uniquely) represented by a ring spectrum.

6.3 h-tuples of commuting elements

Just as classical characters of G are class functions on G, Hopkins-Kuhn-Ravenel characters are class functions on h-tuples of commuting p-power order elements of G. This section is a short reminder of the basic definitions linked to such h-tuples; we give a more detailed discussion of the case $G = \Sigma_n$ in Section 9.1. Firstly, since G is finite, the set of all h-tuples of commuting elements of p-power order of G can be identified with

$$\mathrm{Hom}(\mathbb{Z}_p^h, G).$$

The group G acts on this set by conjugation:

$$g(g_1, \ldots, g_h)g^{-1} = (gg_1g^{-1}, \ldots, gg_hg^{-1}).$$

Definition 6.6. Let α be an h-tuple of commuting elements (of p-power order) of G. The conjugacy class $[\alpha]_G$ of α is defined to be the orbit of α in $\mathrm{Hom}(\mathbb{Z}^h, G)$ (or $\mathrm{Hom}(\mathbb{Z}_p^h, G)$ respectively) under this G action. The centralizer of α is defined as the stabilizer

$$C_\alpha = C_G(\alpha) := \mathrm{Stab}_G(\alpha) \subseteq G.$$

Definition 6.7. A function on $\mathrm{Hom}(\mathbb{Z}_p^h, G)$ is called a class function if it is invariant under conjugation by elements of G.

6.4 The Hopkins-Kuhn-Ravenel character map

Let E be a spectrum with Hopkins-Kuhn-Ravenel theory, let G be a finite group, let χ be an element of $E^0(BG)$ and let α be an h-tuple of commuting p-power order elements of G, where h is as in Definition 6.4. Then Hopkins, Kuhn and Ravenel define a ring D and an evaluation map

$$\begin{aligned} \mathrm{eval}_\alpha \colon E^0(BG) &\to D \\ \chi &\mapsto \chi(\alpha). \end{aligned}$$

For our purposes it matters neither what the ring D is nor how eval_α is defined. We do need to know that D is independent of the group G and that a fixed $\chi \in E^0(BG)$ defines a class function on the set of h-tuples of commuting p-power order elements of G. This is the sense in which χ is a character. The maps eval_α are analogues of the $\mathrm{Trace}(g|-)$ maps in K-theory, from Section 3.3. The following is a corollary of [HKR00, Thm C].

Theorem 6.8. *Let E be a ring spectrum with Hopkins-Kuhn-Ravenel theory. An element χ of $\frac{1}{p}E^0(BG)$ is uniquely determined by the class function it defines.*

We also need the Hopkins-Kuhn-Ravenel analogue of the formula for the character of an induced representation [Ser77, p.30].

Theorem 6.9 ([HKR00, Thm D]). *Let* $H \subseteq G$ *be a subgroup, and let* α *be an* h*-tuple of commuting* p*-power order elements in* G*. We have*

$$\mathrm{eval}_\alpha(\mathrm{ind}\,|_H^G(-)) = \frac{1}{|H|} \sum_{\substack{g \in G \mid g\alpha g^{-1} \\ \text{maps to } H}} \mathrm{eval}_{g\alpha g^{-1}}(-).$$

6.5 Ando's generalizations of Atiyah's work

We are now ready to carry over the definitions of Section 4.4 and define internal power operations for H_∞ spectra with Hopkins-Kuhn-Ravenel theory. The original reference for this is [And92], also see [And95]. The examples relevant to us are the analogues of the examples in Section 4.4. Let E be an H_∞-ring spectrum with Hopkins-Kuhn-Ravenel theory. For simplicity we assume a Künneth isomorphism for the symmetric groups, i.e. we ask that $E^0(B\Sigma_n)$ be free of finite rank over E^0 and that $E^1(B\Sigma_n) = 0$.

Example 6.10. ([Str98, 3.3],[And95]) The Morava-Lubin-Tate theories E_h satisfy all the above conditions.

Definition 6.11. Let α be an h-tuple of commuting elements of p-power order of Σ_n. Define

$$\psi_\alpha\colon E(X) \to D \otimes E(X)$$

as the composition

$$E(X) \xrightarrow{P_n} E(E\Sigma_n \times_{\Sigma_n} X^n) \xrightarrow{\Delta^*} E(B\Sigma_n \times X) \xleftarrow{\cong} E(B\Sigma_n) \otimes E(X) \xrightarrow{\mathrm{eval}_\alpha} D \otimes E(X),$$

where Δ denotes the diagonal map of X^n.

Notation 6.12. Let α be as above. Then α makes $\{1, \dots, n\}$ into a $\mathbb{Z}_p^h$-set. Conversely, a finite $\mathbb{Z}_p^h$-set A determines an element α of some symmetric group up to conjugacy (cf. Section 9.1). We sometimes write ψ_A for ψ_α.

Definition 6.13. The Hecke operators in Morava-Lubin-Tate theory are defined as

$$T_{p^k}(x) := \frac{1}{p^k} \sum_{\substack{A \in \mathcal{T}_p \\ |A| = p^k}} \psi_A(x),$$

where the sum is over all transitive $\mathbb{Z}_p^h$-sets of order p^k. It is proved in [And92] that these T_{p^k} are additive operations

$$T_{p^k}\colon E_h(X) \to E_h(X).$$

Remark 6.14. Note that on chromatic level one, these Hecke operators are the stable Adams operations:

$$T_{p^k} = \frac{\psi_{p^k}}{p^k}.$$

Definition 6.15. Let E^0 be torsion free. We define the analogues of the symmetric powers as

$$\sigma_n(x) := \frac{1}{n!} \sum_\alpha \mathrm{eval}_\alpha(\Delta^* P_n(x))$$

where this time the sum runs over all h-tuples α of commuting elements of p-power order in Σ_n.

Remark 6.16. It is immediate from [AF03, 5.5] that the operation σ_n takes values in $\frac{1}{|\Sigma_n|}E^0(X)$, but it is a non-trivial fact that it takes values in $E^0(X)$. We postpone the proof to Section 7.3.

Notation 6.17. We write S_t for the "total symmetric power" as above.

Remark 6.18. As in the case of K-theory, S_t turns out to take sums into products. We give a more general definition of the σ_n and prove this exponential property in Section 7.5.

7 Generalized orbifold genera

This section explores several aspects of Definition 1.1. We start off by identifying the target of ϕ_{orb}. A priori ϕ_{orb} takes values in $\mathbb{Q} \otimes E_h^0$. In Section 7.3 we prove that it factors through E_h^0. Our proof uses an inner product on $E_h^0(BG)$ defined by Strickland [Str00]: Analogously to Remark 1.4, we identify the expression

$$\frac{1}{|G|} \sum_\alpha \mathrm{eval}_\alpha(-) \tag{19}$$

as the inner product with one. As a corollary we obtain the required "integrality" statement. We also obtain an explicit formula for the Strickland inner product in terms of Hopkins-Kuhn-Ravenel evaluation maps. This identification motivates a definition of higher orbifold genera in even greater generality than Definition 1.1. We discuss this definition in Section 7.4.

7.1 The K(h)-local categories

Topologists often like to localize the category of spaces or spectra with respect to a generalized homology theory $H_*(-)$ [Bou79]. That is they add more maps to the original category in order to invert all H_*-isomorphisms. More precisely, there is a category $\mathcal{S}_H$, called the H-local (stable homotopy) category, and a functor

$$\gamma \colon \mathcal{S} \to \mathcal{S}_H,$$

which is left-universal with respect to the property that it takes H_*-isomorphisms into isomorphisms. Typically the category $\mathcal{S}_H$ is easier to understand and has nicer properties than $\mathcal{S}$. When it is clear that we are working in $\mathcal{S}_H$, we often omit γ from the notation. As the stable homotopy category $\mathcal{S}$ itself, $\mathcal{S}_H$ is a triangulated category with a compatible closed symmetric monoidal structure. That is it has a symmetric monoidal structure $- \wedge -$ with unit $S = \gamma(\mathbb{S}^0)$ and function objects ("internal hom's") $F(-,-)$, such that

$$\mathrm{Hom}(X \wedge Y, Z) = \mathrm{Hom}(X, F(Y, Z)),$$

and these data are compatible with the triangulated structure in an appropriate sense[10]. The localization functor γ preserves the triangulated structure as well as the monoidal structure and its unit, but not function objects[11]. There is one important exception that is going to play a role for us: Write

$$DX := F(X, S)$$

for the dual of X.

Theorem 7.1 ([LMSM86, III.1.6]). *Let X and Y be objects of a closed symmetric monoidal category, and assume that there are maps*

$$\alpha\colon S \to X \wedge Y \quad \textit{and} \quad \beta\colon Y \wedge X \to S$$

such that the composites

$$X \cong S \wedge X \xrightarrow{\alpha \wedge \mathrm{id}} X \wedge Y \wedge X \xrightarrow{\mathrm{id} \wedge \beta} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{id} \wedge \alpha} Y \wedge X \wedge Y \xrightarrow{\beta \wedge \mathrm{id}} S \wedge Y \cong Y$$

are the respective identity maps. Then the adjoint $\beta^\sharp\colon Y \to DX$ is an isomorphism.

An object X for which such Y, β and α exist is called *strongly dualizable.* It comes with an isomorphism $X \to DDX$. Since γ preserves the monoidal structure, it follows that it also preserves strong dualizability and strong duals.

It turns out that our definition of orbifold genus is naturally at home in such localized categories as we will now explain. We have already seen that we should work with a genus into a cohomology theory E with Hopkins-Kuhn-Ravenel theory. In this case E-cohomology is blind for localization with respect to a certain homology theory $H_*(-) = K(h)_*(-)$, i.e. any map that becomes an isomorphism under $K(h)_*(-)$ also becomes an isomorphism under $E^*(-)$. In other words, E is a $K(h)$-local spectrum[12]. These "Morava K-theory" homology theories $K(h)_*(-)$ were first constructed by Baas and Sullivan and first used by Morava. Today their definition can be found in [Rud98] or [EKMM97]. Since E does not see $K(h)$-localization, we may as well work in the $K(h)$-local category. But this is not the only reason

[10] The details can be found in [HPS97, A.2].

[11] Cf. [HPS97, 3.5.1].

[12] This fact seems to be well known to homotopy theorists, but to the author's knowledge there is no published account of it. In the case that E is Morava-Lubin-Tate cohomology it is proved in [HS99, 5.2], for Noetherian E^0 a written account is available from [Str04], in the generality it is stated here I learned it from Michael Hopkins.

why we do so: in Section 7.3, we describe how the map (19) used to define ϕ_{orb} in Definition 1.1 is induced by a map in the $K(h)$-local category from $\mathbb{S}^0$ to BG_+ that turns up very naturally. This deeper understanding of Definition 1.1 has as corollary the integrality result 7.11 and becomes crucial in Section 8, where we show that Definition 1.1 is independent of the choice of representation of the orbifold $M/\!\!/G$.

Remark 7.2. The functor γ has a fully faithful right-adjoint J, and it is customary among topologists to think of $\mathcal{S}_{K(h)}$ as embedded into $\mathcal{S}$ via J, referring to objects of $\mathcal{S}$ in the image of J as "$K(h)$-local spectra". This point of view is not helpful for our purposes, and we stick to the language of localized categories. The difference is mainly of notational nature: Write $L_{K(h)}$ for the composite $J \circ \gamma$. The functor J does not preserve the monoidal structure. Therefore, for example, where we write

$$\gamma(X) \wedge \gamma(Y) \quad \text{or} \quad X \wedge Y$$

for the smash product in $\mathcal{S}_{K(h)}$, others write

$$L_{K(h)}(L_{K(h)}X \wedge L_{K(h)}Y),$$

and similarly we write $\mathbb{S}^0$ or $\gamma(\mathbb{S}^0)$ for $L_{K(h)}\mathbb{S}^0$.

7.2 Strickland inner products

This section recalls some of the concepts and results in [Str00].

Notation 7.3. Let $\mathcal{C}$ be an additive closed symmetric monoidal category. We use the notation of the previous section and write τ for the twist map $X \wedge Y \to Y \wedge X$. We fix the assumption on $\mathcal{C}$ that every object is strongly dualizable.

Definition 7.4. A *Frobenius object* in $\mathcal{C}$ is an object A equipped with maps $S \xrightarrow{\eta} A$, $A \wedge A \xrightarrow{\mu} A$, $A \xrightarrow{\varepsilon} S$ and $A \xrightarrow{\psi} A \wedge A$ such that

(a) (A, η, μ) is a commutative and associative monoid,

(b) (A, ε, ψ) is a commutative and associative co-monoid,

(c) we have $\psi \circ \mu = (1 \wedge \mu) \circ (\psi \wedge 1)$.

Lemma 7.5 ([Str00, 3.9]). *If $(A, \eta, \mu, \psi, \varepsilon)$ is a Frobenius object in $\mathcal{C}$ then $b := \varepsilon\mu$ defines an inner product on A in the following sense:*

(a) b is symmetric, i.e. $b \circ \tau = b$, and

(b) b is non-degenerate, i.e. the adjoint $b^\sharp \colon A \to DA$ is an isomorphism.

From now on let E be an even periodic $K(h)$-local spectrum.

Theorem 7.6 ([Str00, 9]). *Let* E *be as above, and assume that* $E^0(BG)$ *has finite rank over* E^0*. Then* $\mathbb{Q} \otimes E^0(BG)$ *is a Frobenius object in the category of* E^0*-modules.*

Remark 7.7. Strickland's result is actually much deeper: he shows that BG_+ itself becomes a Frobenius object in the $K(h)$-local category. Note that by applying E^0 we are reversing all the arrows and with them the names for the structure maps. Thus E^0 applied to the map called η in [Str00] is our ε et cetera. Note also that in order to define the augmentation map ε, the multiplication μ and its unit η we do not need to tensor with $\mathbb{Q}$. Thus we have a symmetric bilinear form

$$b_G := \varepsilon \circ \mu$$

on E^0BG, but b_G might not satisfy condition (b) of Lemma 7.5.

Remark 7.8. Over $\mathbb{Q}$, the co-product ψ is defined to be induction along the diagonal map $\delta\colon G \to G \times G$ (cf. [Str00, 8.2])

$$\psi = \operatorname{ind}|_\delta\colon E^0(BG) \to E^0(BG \times BG) \cong_{\mathbb{Q}} E^0(BG) \otimes E^0(BG).$$

By [Str00, 3.11, 8.2, 8.5] the multiplication μ is restriction along δ and the unit η is defined as the pullback along the unique map $BG \to pt$ [Str00, 8.2]. Thus (μ, η) is the classical ring structure on E^0BG. By construction, the augmentation map ε is the same as the inner product with 1,

$$b_G(\chi, 1) = \varepsilon(\chi \cdot 1) = \varepsilon(\chi).$$

The proof of Frobenius reciprocity [Str00, p.25] goes through for arbitrary G:

Proposition 7.9. *Let* $H \subseteq G$ *be finite groups. Then we have*

$$b_G(\operatorname{ind}|_H^G \chi, \xi) = b_H(\chi, \operatorname{res}|_H^G \xi).$$

PROOF : Let $\xi = 1$. We have to show

$$\varepsilon_G \circ \operatorname{ind}|_H^G = \varepsilon_H.$$

This follows from [Str00, 8.5] and the definition of ε [Str00, 8.2,3.11]. Let now ξ be arbitrary. Let $j\colon H \to H \times G$ denote the diagonal inclusion. Note that

$$\operatorname{res}|_j = \operatorname{res}|_{\delta_H} \circ (\operatorname{id} \times \operatorname{res}|_H^G).$$

The proof of [Str00, 8.5] implies

$$\operatorname{ind}|_H^G \circ \operatorname{res}|_j = \operatorname{res}|_{\delta_G} \circ (\operatorname{ind}|_H^G \times \operatorname{id}).$$

Composing both sides with ε_G, we obtain the desired identity. □

7.3 "Integrality" theorem

The goal of this section is to prove the following.

Proposition 7.10. *Over $\mathbb{Q}$ the augmentation map ε is*

$$\varepsilon \otimes \mathbb{Q} = \frac{1}{|G|} \sum_{\alpha} \mathrm{eval}_{\alpha}(-).$$

Corollary 7.11. *If E^0 is torsion free, the right hand side defines a map*

$$E^0(BG) \longrightarrow E^0.$$

Corollary 7.12. *The generalized orbifold genus ϕ_{orb} of Definition 1.1 takes values in E^0_h.*

Corollary 7.13. *The symmetric powers from Definition 6.15 take values in $E^0(X)$.*

PROOF OF PROPOSITION 7.10: Denote the right hand side of the equation in the proposition by a. We need to show that a is a unit of $\psi_{\mathbb{Q}}$. Since units of co-multiplications are uniquely determined, this implies that it is equal to $\varepsilon \otimes \mathbb{Q}$. We first compute ψ in terms of Hopkins-Kuhn-Ravenel character evaluations. Let

$$(\alpha, \beta) := ((a_1, b_1), \ldots, (a_h, b_h))$$

stand for an h-tuple of commuting elements of p-power order in $G \times G$. Then by Theorem 6.9

$$\mathrm{eval}_{\alpha,\beta}(\mathrm{ind}\,|_{\delta}(-)) = \frac{1}{|G|} \sum_{\substack{(s,t) \\ s^{-1}\alpha s = t^{-1}\beta t}} \mathrm{eval}_{s^{-1}\alpha s}(-).$$

Thus, counting the pairs (s, t) and taking into account that $\mathrm{eval}_{s^{-1}\alpha s} = \mathrm{eval}_{\alpha}$, we have

$$\mathrm{eval}_{\alpha,\beta}(\psi(-)) = \frac{1}{|G|} \sum_{s \in G} \sum_{\substack{t \in G \\ s^{-1}\alpha s = t^{-1}\beta t}} \mathrm{eval}_{\alpha}(-) = \begin{cases} |C_{\alpha}| \cdot \mathrm{eval}_{\alpha}(-) & \alpha \sim_G \beta \\ 0 & \text{else.} \end{cases}$$

We are now ready to prove that a is a unit of $\psi \otimes \mathbb{Q}$, i.e. that the equality

$$(\mathrm{id}_{E^0_h(BG)} \otimes a) \circ \psi = \mathrm{id}_{E^0_h(BG)}$$

holds over $\mathbb{Q}$. By Theorem 6.8 it suffices to show that both sides define the same class function. Write

$$\xi := (\mathrm{id} \otimes a) \circ \psi(\chi).$$

We have

$$\begin{aligned}\xi(\alpha) &= \frac{1}{|G|}\sum_{\beta} \mathrm{eval}_{\alpha,\beta}(\psi(\chi)) \\ &= \frac{1}{|G|}\sum_{\beta\in[\alpha]_G} |C_\alpha|\cdot \mathrm{eval}_\alpha(\chi) \\ &= \chi(\alpha).\end{aligned}$$

□

Corollary 7.14. *Let* E *be a cohomology theory to with Hopkins-Kuhn-Ravenel theory, and assume that* E^0 *is torsion free. Then the Strickland inner product on* $E^0(BG)$ *is described by the formula*

$$b_G(\chi,\xi) = \frac{1}{|G|}\sum_{\alpha} \chi(\alpha)\xi(\alpha).$$

7.4 Generalized orbifold genera

The results of the previous section motivate the following definition.

Definition 7.15. Let E be an even periodic $K(h)$-local ring spectrum, and $\phi\colon MU \to E$ a map of ring spectra. Let G be a finite group, and let ϕ_G be the Borel equivariant genus associated to ϕ as in (3). We define the orbifold genus ϕ_{orb} of almost complex G manifolds as the composition

$$\phi_{orb} := \varepsilon \circ \phi_G : \mathcal{N}_*^{U,G} \quad\to\quad E_*,$$

where ε is the augmentation map from Remark 7.7.

Remark 7.16. Instead of MU we could have used any of the classical Thom spectra MSpin, MO, MSp, $MU\langle n\rangle$, $MO\langle n\rangle$ etc.

Remark 7.17. In the case $E = E_h$, Definition 7.15 specializes by Proposition 7.10 to the Ando-French definition 1.1.

Recall from Section 3.2 that ϕ_G is the push-forward of one in Borel equivariant E-theory,

$$\phi_G(M) = \pi_!^{\phi_G}(1),$$

where $\pi\colon M \to pt$ is the unique G-map. The definition of $\pi_!$ in 3.2 involves Borel equivariant E-theory of the G-spectrum $M^{-\tau}$, and the correct generalization of the Borel construction to G-spectra is given by the "twisted half smash product" over G

$$EG \ltimes_G - .$$

These twisted half smash products were introduced and studied extensively in [LMSM86]. A summary of their basic properties can be found in [BMMS86, I.1]. For the suspension spectrum of a pointed G-space X, they specialize to the Borel construction

$$EG \ltimes_G (\Sigma^\infty X) \cong \Sigma^\infty (EG_+ \wedge_G X).$$

In our case of the Thom spectrum $M^{-\tau}$ we have

$$EG \ltimes_G (M^{-\tau}) = (EG \times_G M)^{-EG \times_G \tau} \tag{20}$$

[LMSM86, X.6.3].

Remark 7.18. Ando and French use this identity to avoid speaking about $M^{-\tau}$. The isomorphism (20) identifies the Borel equivariant Thom isomorphism for $-\tau$ with the non-equivariant Thom isomorphism for $-EG \times_G \tau$, and our Borel equivariant push-forward along π becomes their push-forward along $EG \times_G \pi$. We will use (20), but the equivariant picture becomes important in Section 8.

Fixed point descriptions of orbifold genera

The original definition of orbifold genus in [BL03] involves terms looking like the right hand side of (9), whereas Ando and French's definition generalizes the left hand side of (8). The proof of (8) and (9) has two ingredients: character theory and the Riemann-Roch theorem. Character theory is available in the context of E_h, but the Riemann-Roch formula is not. Ando and French explain in detail how to modify the character theoretic discussion in [AS68a] to bring their definition into a form that is modulo a Riemann-Roch theorem very similar to Borisov and Libgober's. Their discussion goes through without changes for our Definition 1.1.

7.5 Generalized symmetric powers

Definition 7.19. Let E be an H_∞ ring spectrum in the classical (i.e. Borel equivariant) sense. Assume moreover that E is even periodic and $K(h)$-local. Recall that $\varepsilon_G = \eta_G^*$ for a map $\eta_G \colon \mathbb{S}^0 \to BG_+$ We define the n^{th} symmetric power in $E(X)$ by

$$\sigma_n := (\eta_{\Sigma_n} \wedge \mathrm{id}_X)^* \circ \Delta \circ P_n$$

and the total symmetric power by

$$S_t := \sum_{n=0}^{\infty} \sigma_n t^n.$$

Remark 7.20. In the situation of Definition 6.15 the two definitions agree by Proposition 7.10.

Lemma 7.21. *The map*

$$\sum_{n\geqslant 1} \varepsilon_{\Sigma_n} \colon \widehat{\bigoplus_{n\geqslant 0}} E^0(B\Sigma_n)t^n \to E^0[\![t]\!]$$

preserves the ring structure of Remark 4.7.

PROOF : Recall from Remark 7.8 that $\varepsilon_G \colon BG_+ \to \mathbb{S}^0$ is $B(-)_+$ applied to the unique map from G to the trivial group and that $\eta_G = D\varepsilon_G$. Thus

$$\varepsilon_{G\times H} = \varepsilon_G \wedge \varepsilon_H \quad \text{and} \quad \eta_{G\times H} = \eta_G \wedge \eta_H.$$

Together with Frobenius reciprocity, this proves the claim. □

Corollary 7.22. *The total symmetric power S_t takes sums into products.*

8 The orbifold genus ϕ_{orb} as orbifold invariant

Up to now we have ignored the question whether our generalized orbifold genus ϕ_{orb} is actually a well defined notion of orbifolds. More precisely, a G-manifold M defines an orbifold $M/\!\!/G$ [Moe02]. But a different group H and H-manifold N can define an isomorphic orbifold. In Definition 1.1 we defined $\phi_{\mathrm{orb}}(M \circlearrowleft G)$, but did not prove that

$$M/\!\!/G \cong N/\!\!/H$$

implies

$$\phi_{\mathrm{orb}}(M \circlearrowleft G) = \phi_{\mathrm{orb}}(N \circlearrowleft H).$$

This section proves this fact. Note however that our definition of ϕ_{orb} only makes sense for finite groups G.

Remark 8.1. For Borisov and Libgober's definition of the orbifold elliptic genus the analogous statement is a consequence of McKay correspondence, proved in [BL02].

8.1 Tangentially almost complex structures

Recall that an almost complex structure on a G-manifold M is a choice of lift $-[\tau]_K \in \widetilde{K}_G(M)$ of the stable normal bundle $-[\tau] \in \widetilde{KO}_G(M)$. The tangent vector bundle is a well defined orbifold notion [Sat57], but

$$\widetilde{K}_G(M) = \operatorname{coker}(K_G(pt) \to K_G(M))$$

is not, since there is no fixed group G. We can however define

$$\check{K}_{\mathrm{orb}}\underline{X} := \operatorname{coker}(K_{\mathrm{orb}}(pt) \to K_{\mathrm{orb}}(\underline{X})),$$

for an arbitrary orbifold $\underline{X}$, and similarly $\check{K}O_{orb}$.

Definition 8.2 (compare [May96, XXVIII.3.1]). A tangentially almost complex structure on an orbifold $\underline{X}$ is a choice of lift $[\tau]_{\check{K}} \in \check{K}_{orb}(\underline{X})$ of $[\tau] \in \check{K}O_{orb}(\underline{X})$.

The input of our orbifold genus are orbifolds that can be represented in the form $M/\!\!/G$ for a finite group G and compact, closed G-manifold M, together with a choice of tangential almost complex structure on $M/\!\!/G$.

Remark 8.3. We could of course do with weaker conditions: By [Moe02, 4.3, 5.4] the Borel construction is a well defined orbifold notion,

$$M/\!\!/G \cong N/\!\!/H \qquad \Rightarrow \qquad EG \times_G M \simeq EH \times_H N,$$

and so is the ring map (in real or complex K-theory)

$$\text{Borel: } K_G(M) = K_{orb}(M/\!\!/G) \longrightarrow K(\text{Borel}(M/\!\!/G)).$$

By (20), all we need to define ϕ_{orb} is a choice of complex lift of

$$-\text{Borel}[\tau] \in KO(\text{Borel}(M/\!\!/G)).$$

8.2 The $K(h)$-local category and ϕ_{orb}

Recall from Section 7.4 that $\phi_{orb}(M \circlearrowleft G) \in E_d$ is the image of one under the composite

$$E^0(EG \times_G M) \to E^{-d}(EG \ltimes_G M^{-\tau}) \xrightarrow{P-T} E^{-d}BG \xrightarrow{\varepsilon} E^{-d}\mathbb{S}^0. \tag{21}$$

Observation 8.4. *It follows from Remark 8.3 and (20) that the spectrum* $EG \ltimes_G M^{-\tau}$ *and its Thom isomorphism can be defined purely in terms of the orbifold. This takes care of the first map in (21).*

The second and third map in (21) clearly depend on the representation of the orbifold, because BG does. Both maps are however induced by maps in the $K(h)$-local category, and the composite of the two maps inducing them

$$\text{Borel}(P - T) \circ \eta : \mathbb{S}^0 \longrightarrow EG \ltimes_G M^{-\tau} \tag{22}$$

has a chance to be independent of the representation $M \circlearrowleft G$, since its source and target are.

A closer look at the definitions of the second and third map in (21) shows that both involve duality: By [May96, XVI.8.1], the Thom spectrum $M^{-\tau}$ is a G-equivariant (strong) dual of M_+, and the G-equivariant Pontrjagin-Thom collapse

$$\mathbb{S}^0 \circlearrowleft G \longrightarrow M^{-\tau} \circlearrowleft G$$

is the dual of the G-map

$$\pi_+ \colon M_+ \to \mathbb{S}^0$$

sending M to the non-basepoint.

The map η is by definition [Str00, 3.11,8.2] the dual in the $K(h)$-local category of the map $(Bp_G)_+$, where

$$p_G\colon \mathrm{pt} \circlearrowleft G \to \mathrm{pt} \circlearrowleft 1$$

is the unique map from G to the trivial group. More precisely, η is the composite

$$\eta\colon \mathbb{S}^0 \xrightarrow{\cong} D\mathbb{S}^0 \longrightarrow D(BG_+) \xleftarrow{\cong} BG_+, \tag{23}$$

where the first and last map are the adjoints to the Strickland inner product, and the second map is $D((Bp_G)_+)$. We need to fit these dualities in two different categories into a common picture.

8.3 Borel construction and duality

Notation 8.5 (Spanier-Whitehead duals). We still stick with the notation of Section 7.1. In this section there are several categories involved, and we write

$$D_G(-) := F_G(-, \mathbb{S}^0)$$

for the G-equivariant dual (cf. [May96, XVI.7]) and

$$D(-) := F_{\mathcal{S}_{K(h)}}(-, \mathbb{S}^0)$$

for the dual in the $K(h)$-local category.

The goal of this section is to prove the following theorem.

Theorem 8.6. *Let G be a finite group and let Y be a finite G-CW spectrum. There is an isomorphism in the $K(h)$-local category*

$$EG \ltimes_G (D_G(Y)) \longrightarrow D(EG \ltimes_G Y), \tag{24}$$

which is natural in Y.

In order to define (24), we construct its adjoint

$$(EG \ltimes_G (D_G(Y))) \wedge (EG \ltimes_G Y) \longrightarrow \mathbb{S}^0. \tag{25}$$

We construct (25) as a map in the (non-localized) stable homotopy category, but the adjointness that defines (24) is in the $K(h)$-local category.

Construction 8.7. *For a finite group G and G-spectra X and Y there is a functor isomorphism*

$$(EG \ltimes_G X) \wedge (EG \ltimes_G Y) \cong (EG \times EG) \ltimes_{G\times G} (X \wedge Y).$$

The map (25) is defined as the composite of several maps. The first is transfer along the diagonal δ of G

$$T\delta\colon (EG \times EG) \ltimes_{G\times G} (D_G(Y) \wedge Y) \longrightarrow (EG \times EG) \ltimes_G (D_G(Y) \wedge Y).$$

The second map is

$$(EG \times EG) \ltimes_G -$$

applied to the G*-map*

$$\beta_G\colon D_G Y \wedge Y \longrightarrow \mathbb{S}^0 \tag{26}$$

from Theorem 7.1. Its target is (the suspension spectrum of)

$$(EG \times EG)_+ \wedge_G \mathbb{S}^0 \simeq BG_+.$$

The last map is

$$(Bp_G)_+ : BG_+ \longrightarrow \mathbb{S}^0,$$

where p_G *denotes the unique map from* G *to the trivial group.*

Observation 8.8. *In the case* $Y = \mathbb{S}^0$*, Construction 8.7 specializes to the definition of Strickland's inner product [Str00, 8.2]*

$$BG_+ \wedge BG_+ \to \mathbb{S}^0.$$

Corollary 8.9. *Theorem 8.6 is true for* $Y = \mathbb{S}^0$*.*

PROOF : This is the fact that the Strickland inner product is non-degenerate [Str00, 8.3]. □
The second easiest special case of Theorem 8.6 is the case that Y is a different zero sphere.

Proposition 8.10. *For* $Y = G/H_+$ *the map (25) is the Strickland inner product on*

$$(EG \times_G G/H)_+,$$

and the theorem holds for $Y = G/H_+$*.*

PROOF : Recall from [May96, p.176] that for finite groups $H \subseteq G$ a G-equivariant (strong) dual of G/H_+ is G/H_+, with the map β_G in (26) given by the composite (of space level maps)

$$(G/H \times G/H)_+ \to G/H_+ \to \mathbb{S}^0, \tag{27}$$

where the first map is the G-equivariant Pontrjagin-Thom collapse along the diagonal inclusion (that is it is the identity on the diagonal and everything else gets mapped to the basepoint) and the second map is p_+, where p is the unique (G-equivariant) map

$$p\colon G/H \to pt.$$

Following Strickland, we write $B\mathcal{G}$ for the Borel construction of the finite groupoid $\mathcal{G}$ defined by $G/H \circlearrowleft G$, remembering that

$$(B\mathcal{G})_+ = EG_+ \wedge_G G/H_+.$$

Strickland defines the inner product on (the suspension spectrum of) $(B\mathcal{G})_+$ as the composite

$$b_{\mathcal{G}}\colon (B\mathcal{G} \times \mathcal{G})_+ \xrightarrow{T\delta} (B\mathcal{G})_+ \xrightarrow{(Bp)_+} \mathbb{S}^0,$$

where

$$\delta = \delta_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{G} \times \mathcal{G}$$

is the diagonal inclusion (of groupoids), and

$$p = p_{\mathcal{G}}\colon \mathcal{G} \to 1$$

is the unique map (of groupoids). Note that $\delta_{\mathcal{G}}$ factors as

$$\delta_{\mathcal{G}}\colon (G/H \circlearrowleft G) \xhookrightarrow{i} (G/H \times G/H) \circlearrowleft G \xhookrightarrow{\delta_G} (G/H \times G/H) \circlearrowleft G \times G,$$

where i is an inclusion of finite G-sets (namely the diagonal inclusion mentioned above) and $T\delta_G$ is the first map of Construction 8.7. We need to identify Ti. However, Bi is a particularly simple example of covering with finite fibers, namely the inclusion of some path components. A look at the construction of Ti in [Ada78, 4.1.1] shows that Ti is given by the (space level) map

$$(EG \times_G (G/H \times G/H))_+ \longrightarrow (EG \times_G G/H)_+$$

that is the identity on $\mathrm{im}(Bi)$ and maps everything else to the basepoint. This is exactly $EG_+ \wedge_G -$ applied to the Pontrjagin-Thom collapse in (27). Also $p_{\mathcal{G}}$ factors as

$$p_{\mathcal{G}}\colon (G/H \circlearrowleft G) \xrightarrow{p} (\mathrm{pt} \circlearrowleft G) \xrightarrow{p_G} (\mathrm{pt} \circlearrowleft 1),$$

where p is as in (27). Together this proves the claim that (25) is the Strickland inner product:

$$b_{\mathcal{G}} = (Bp_{\mathcal{G}})_+ \circ T\delta_{\mathcal{G}} = (Bp_G)_+ \circ (Bp)_+ \circ Ti \circ T\delta_G = (Bp_G)_+ \circ (EG_+ \wedge_G \beta_G) \circ T\delta_G\,.$$

As above, non-degeneracy of the Strickland product in the $K(h)$-local category implies that (24) is an isomorphism for $Y = G/H_+$. ☐

Before we proceed to higher dimensional spheres, we recall that in any closed symmetric monoidal category we have an isomorphism

$$D(X) \wedge D(Y) \xrightarrow{\cong} D(X \wedge Y), \tag{28}$$

which identifies the evaluation map $\beta_{X\wedge Y}$ with

$$(\beta_X \wedge \beta_Y) \circ (\mathrm{id}_{DX} \wedge \tau \wedge \mathrm{id}_Y),$$

where τ switches DY and X.

Lemma 8.11. *Theorem 8.6 holds for spheres*

$$Y = G/H_+ \wedge \mathbb{S}^n.$$

PROOF : The sphere $\mathbb{S}^n$ is strongly dualizable in $\mathcal{S}$ with dual $\mathbb{S}^{-n}$, and both functors $\mathcal{S} \to \mathcal{S}_G$ and $\mathcal{S} \to \mathcal{S}_{K(h)}$ preserve the data of strong dualizability in Theorem 7.1. By (28) we have

$$D_G(G/H_+ \wedge \mathbb{S}^n) \cong D_G(G/H_+) \wedge \mathbb{S}^{-n}.$$

Since $\mathbb{S}^n$ and $\mathbb{S}^{-n}$ have trivial G-action, we have

$$EG \ltimes_G (G/H_+ \wedge \mathbb{S}^{\pm n}) = (EG_+ \wedge_G G/H_+) \wedge \mathbb{S}^{\pm n},$$

and under this identification the map (25) becomes

$$\beta_{EG_+ \wedge_G (G/H_+)} \wedge \beta_{\mathbb{S}^n} : EG_+ \wedge_G (D_G(G/H_+)) \wedge EG_+ \wedge_G (G/H_+) \wedge \mathbb{S}^n \wedge \mathbb{S}^{-n} \to \mathbb{S}^0.$$

Here $\beta_{EG_+ \wedge_G (G/H_+)}$ is the map of the theorem for G/H_+, and by Proposition 8.10, it becomes the evaluation map of a strong duality in $\mathcal{S}_{K(h)}$. The map $\beta_{\mathbb{S}^n}$ is already an evaluation of a strong duality in $\mathcal{S}$ and thus also in $\mathcal{S}_{K(h)}$. Apply (28) again, this time in the $K(h)$-local category, to complete the proof. □

PROOF OF THEOREM 8.6: We prove the theorem by induction over the cells. All our categories have compatible triangulated and closed symmetric monoidal structures. In particular, duals commute with direct sums and take triangles (in the opposite category) into triangles. The Borel construction also preserves the triangulated structure. Since both sides of (24) preserve finite sums, Lemma 8.11 implies that the statement is true for finite bouquets of spheres. Both sides of (24) preserve triangles, thus if the theorem is true for two objects in an exact triangle, it is also true for the third. □

8.4 The orbifold genus as orbifold invariant

We are now ready to prove that ϕ_{orb} is independent of the representation $M \circlearrowleft G$ of the orbifold $M/\!\!/G$.

Proposition 8.12. *Under the isomorphism of Theorem 8.6 the map (22) becomes*

$$D(\mathrm{Borel}(p_{M/\!\!/G})_+) \colon \mathbb{S}^0 \cong D\mathbb{S}^0 \longrightarrow D(\mathrm{Borel}(M/\!\!/G)_+) \cong EG \ltimes_G M^{-\tau},$$

the $K(h)$-local dual of the Borel construction of the unique map of orbifolds

$$p_{M/\!\!/G} \colon M/\!\!/G \to \mathrm{pt}.$$

PROOF : The map $p_{M/\!/G}$ factors as

$$p_{M/\!/G} : M/\!/G \xrightarrow{\pi} pt/\!/G \xrightarrow{p_G} pt.$$

Under the isomorphism of Theorem 8.6 (vertical arrows) the map $EG \ltimes_G (T - P)$ (top row) becomes

$$\begin{array}{ccccc} EG \ltimes_G(D_G M_+) & \longleftarrow & EG_+ \wedge_G (D_G \mathbb{S}^0) & \xleftarrow{\cong} & EG_+ \wedge_G \mathbb{S}^0 \\ \cong\downarrow & & \downarrow\cong & & \\ D(EG_+ \wedge_G M_+) & \longleftarrow & D(EG_+ \wedge_G \mathbb{S}^0), & & \end{array}$$

where the bottom arrow is $D(EG \times_G \pi)_+$; and by Observation 8.8, the composite

$$BG_+ \cong EG_+ \wedge_G \mathbb{S}^0 \to EG_+ \wedge_G (D_G \mathbb{S}^0) \to D(EG_+ \wedge_G \mathbb{S}^0) \cong D(BG_+)$$

is the adjoint of the Strickland inner product $b_G^\sharp$. Together with (23) this proves the claim. □

Combining this with Observation 8.4, we obtain the following corollary.

Corollary 8.13. *The definition of* $\phi_{orb}(M \circlearrowleft G)$ *in Definition 1.1 is independent of the representation of the orbifold* $M/\!/G$.

9 DMVV-type formulas

This section generalizes the discussion of Section 5.4 to the higher chromatic case.

9.1 Conjugacy classes of h-tuples of commuting elements of Σ_l

Just like the conjugacy classes of elements of Σ_l are in one to one correspondence with partitions

$$\sum a_n n = l$$

(i.e. the shape of the Young tableau), one also describes conjugacy classes of h-tuples of commuting elements in terms of the corresponding orbit decomposition of

$$\underline{l} := \{1, \ldots, l\}.$$

More precisely, such an h-tuple $(g_1, \ldots, g_h)$ defines an action of $\mathbb{Z}^h$ on $\underline{l}$, and $\underline{l}$ decomposes into orbits of that action. Two such h-tuples are conjugate by a permutation g of $\underline{l}$ if and only if their orbit decompositions are isomorphic (and an isomorphism is given by g).

Orbits are finite transitive $\mathbb{Z}^h$-sets, and every finite transitive $\mathbb{Z}^h$-set A turns up as a possible orbit for $l \geqslant |A|$.

Let $\mathcal{T} = \{A\}$ contain one representative for each isomorphism class of finite transitive $\mathbb{Z}^h$-sets. The above discussion summarizes as follows. The conjugacy classes of h-tuples of commuting elements in Σ_l are classified by expressions

$$\sum_{A\in\mathcal{T}} a_A A \text{ s.t. } \sum_{A\in\mathcal{T}} a_A|A| = l,$$

where for given $(g_1,\ldots,g_h)$ the expression $\sum_{A\in\mathcal{T}} a_A A$ counts the number of times a_A each isomorphism class of finite transitive $\mathbb{Z}^h$-set A occurs in the decomposition of $\underline{l}$ into $\langle g_1,\ldots,g_h\rangle$-orbits. If $[g_1,\ldots,g_h]$ corresponds to $\sum_{A\in\mathcal{T}} a_A A$, the centralizer of α in G can be described as follows

$$C_{(g_1,\ldots,g_h)} \cong \prod_{A\in\mathcal{T}} \mathrm{Aut}_{\mathbb{Z}^h}(A)^{a_A} \rtimes \Sigma_{a_A}, \tag{29}$$

where Σ_{a_A} permutes the a_A orbits isomorphic to A and $\mathrm{Aut}_{\mathbb{Z}^h}(A)$ acts on each of them individually. Now A is a transitive $\mathbb{Z}^h$-set, and $\mathbb{Z}^h$ is abelian. This means that for any $x,y\in A$ there is an element $z\in\mathbb{Z}^h$ such that $zx=y$, and multiplication with z is the unique automorphism of A mapping x to y. Thus

$$|\mathrm{Aut}_{\mathbb{Z}^h}(A)| = |A|.$$

Since the conjugacy class of $(g_1,\ldots,g_h)$ in $\mathrm{Hom}(\mathbb{Z}^h,\Sigma_l)$ is the orbit of $(g_1,\ldots,g_h)$ under the action of Σ_l by conjugation, we have

$$[g_1,\ldots,g_h]_{\Sigma_l} \cong_{\Sigma_l} \Sigma_l/C_{(g_1,\ldots,g_h)}.$$

Therefore by (29)

$$|[g_1,\ldots,g_h]_{\Sigma_l}| = \frac{l!}{\prod_{A\in\mathcal{T}} |A|^{a_A} a_A}.$$

Assume now that we are only interested in h-tuples of commuting elements of p-power order. Then the same discussion goes through, but we need to replace $\mathcal{T}$ by the set $\mathcal{T}_p$ containing one representative for each isomorphism class of finite transitive $\mathbb{Z}_p^h$-set. Note that elements of $\mathcal{T}_p$ have p-power cardinalities, since each of them can be identified with a quotient of $(\mathbb{Z}/p^j\mathbb{Z})^h$ for some sufficiently large j.

9.2 The computation in the higher chromatic case

Let S_t be as in Notation 6.17 and T_{p^k} as in Definition 6.13.

Proposition 9.1. *We have*

$$S_t(x) = \exp\left[\sum_{k\geqslant 0} T_{p^k}(x)t^{p^k}\right].$$

PROOF : The computation in Section 5.4 generalizes as follows.

$$\exp\left[\sum_{k\geqslant 0} T_{p^k}(x)t^{p^k}\right] = \sum_{m\geqslant 0}\frac{1}{m!}\left[\sum_{k\geqslant 0} T_{p^k}(x)t^{p^k}\right]^m,$$

the coefficient of t^l is

$$\sum_{l=\sum_{A\in\mathcal{T}_p} a_A|A|} \frac{(\sum a_A)!}{\prod(a_A!)}\frac{1}{(\sum a_A)!}\prod_{A\in\mathcal{T}_p}\left(\frac{\psi_A(x)}{|A|}\right)^{a_A},$$

where $\frac{(\sum a_A)!}{\prod(a_A!)}$ counts the number of ways to part a set of $\sum a_A$ (orbits) into subsets of orders a_A (the number of times A occurs as orbit), and $\frac{1}{(\sum a_A)!}$ is $\frac{1}{m!}$. This is

$$\begin{aligned}\sum_{\sum a_A|A|=l}\prod_{\mathcal{T}_p}\frac{1}{|A|^{a_A}(a_A!)}\psi_A(x)^{a_A} &= \sum_{\sum a_A|A|=l}\left(\prod_{\mathcal{T}_p}\frac{1}{|A|^{a_A}(a_A!)}\right)\psi_{(\coprod_{\mathcal{T}_p} a_A A)}(x)\\ &= \sum_{[\alpha]}\frac{1}{|C_\alpha|}\psi_\alpha(x)\\ &= \sigma_l(x).\end{aligned}$$

□

Remark 9.2 (Atitah-Tall-Grothendieck type definition of Hecke operators). The left hand side of the equation in Proposition 9.1 is defined in greater generality than its right hand side, motivating the following definition. Let E be as in Definition 7.19. Then the total symmetric power S_t is defined on elements of $E(X)$ and takes values in

$$1 + tE(X)[\![t]\!].$$

Definition 9.3. In this situation we define additive operators T_n on $E(X)$ by

$$\sum_{n\geqslant 1} T_n t^n := \log S_t.$$

Following Grothendieck [Gro57], or the interpretation for K-theory by Atiyah and Tall [AT69], we note that

$$t\frac{d}{dt}\log S_t(x) = t\frac{\frac{d}{dt}S_t(x)}{S_t(x)} \tag{30}$$

takes values in $E(X)[\![t]\!]$. Thus the Hecke operators are operations

$$T_n\colon E(X) \to \frac{1}{n}E(X).$$

We can make the connection to the Atiyah-Tall-Grothendieck definition of the Adams operations even more precise: Let

$$\Lambda_t := \frac{1}{S_{-t}}$$

denote the "total exterior power" in E-theory. This defines a λ-ring structure on $E(X)$, whose Adams operations are by (30) given by $\psi_n = nT_n$.

9.3 DMVV-type formulas for the higher chromatic case

Using Definition 1.1 Theorem 1.7 follows as a direct corollary of Proposition 9.1.

Corollary 9.4. *Let ϕ be an H_∞-orientation of E_h. Then*

$$\sum_{n\geqslant 0} \phi_{\mathrm{orb}}(M^n \circlearrowleft \Sigma_n) = \exp\left[\sum_{k\geqslant 0} T_{p^k}(\phi(M))t^{p^k}\right].$$

Remark 9.5. As in Remark 5.6, the discussion goes through for almost complex G-manifolds M.

Example 9.6 (σ-orientation). Any elliptic spectrum E has a canonical orientation

$$\sigma_E\colon \mathrm{MU}\langle 6\rangle \to E,$$

and it was shown in [AHS], that in the case $E = E_2$, the map σ is an H_∞-map.

The following result due to Ando classifies the complex genera into MU that can be taken as input for Theorem 1.7:

Theorem 9.7 ([And95]). *The spectrum E_h is an H_∞-spectrum. A map of ring spectra*

$$\phi\colon \mathrm{MU} \to E_h$$

is an H_∞-map if and only if the p-series of its Euler class e_ϕ (of the universal line bundle) satisfies

$$[p]_F(e_\phi) = \prod_{\substack{\nu\in F(D_1)\\ [p]_F(\nu)=0}} (\nu +_F e_\phi),$$

where D_1 denotes the ring extension of E_n^0 obtained by adjoining the roots of the p-series of F, and $F(D_1)$ stands for the maximal ideal of D_1 with the group structure $x +_F y$.

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