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ON THE ONSAGER-MACHLUP FUNCTIONAL OF DIFFUSION PROCESSES AROUND NON C^2 CURVES

by

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ABSTRACT

The Onsager Machlup function, namely the fictitious density of diffusions paths in function space is considered, where the density is evaluated around non C^2 curves, thus extending earlier results [3]-[6]. The extension holds also for the case of diffusions evolving on a manifold.[1]

Key Words and Phases: Diffusion Processes, Onsager-Machlup Functional, Fundamental solution.

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I. Introduction

Let w. be a standard n-dimensional Brownian motion, and let x. be an n-dimensional diffusion which is the solution to the following stochastic differential equation:

$$dx_t = f(x_t)dt + dw_t \tag{1.1}$$

where $f_i \in C^1{}_b(R^n)$, i=1,...,n. We are interested in computing the asymptotic behavior of

$$\frac{P(\|\phi - x\| < \varepsilon)}{P(\|w\| < \varepsilon)} = J(\phi,\varepsilon); \tag{1.2}$$

as $\varepsilon \to 0$, where ϕ is a deterministic n dimensional continuous function on [0,T] and, for any $\psi \in C([0,T] \to R^n)$,

$$\|\psi\| \triangleq \max_{t\varepsilon[0,T]} |\psi(t)| \tag{1.3}$$

and $|\ |$ denotes the Euclidean norm in R^n.

This problem was investigated by physicists in the context of statistical mechanics and quantum theory cf. [1], [2]. A rigorous mathematical treatment was initiated by Stratonovich and carried out by Ikeda-Watanabe, Takahashi-Watanabe and Fujita-Kotani, in various degrees of generality, cf. [3], [4], [5], [6]. In particular, the two last references treat the case where (1.1) is a general s.d.e (i.e., with state-dependent diffusion coefficients), and the diffusion evolves on a manifold.

The analysis above was restricted to the case of $\phi \in C^2$ (C^∞ in [5], [6]; however it seems that their technique can be pushed through up to cover C^2). In that case, it was shown that

$$J(\phi)\exp((-\|\dot{\phi}\|-1)K(\varepsilon)) < J(\phi,\varepsilon) < \exp((\|\dot{\phi}\|+1)K(\varepsilon) + \|\ddot{\phi}\|\varepsilon)J(\phi) \tag{1.4}$$

where $K(\varepsilon) \to 0$ and

$$J(\phi) = \lim_{\varepsilon\to 0} J(\phi,\varepsilon) = \exp -[\frac{1}{2}\int_0^T (|\dot{\phi}(s)-f(\phi(s)|^2)ds + \frac{1}{2}\int_0^T \nabla f(\phi(s))ds] \qquad (1.5)$$

In the context of the estimation of trajectories of diffusions, there was a need to evaluate (1.2) for certain ϕ which are not necessarily C^2; for a specific class of random ϕ (which correspond, roughly, to $\phi(t) = \int^t v_s ds$, where $v.$ is a Brownian motion which is independent of w.), it was shown by probabilistic methods in [7] that still $J(\phi) = \lim J(\phi,\varepsilon)$, a.s. P_v.

Our goal in this paper is to evaluate $\lim J(\phi,\varepsilon)$ for ϕ which are not in $C^2[0,T]$. That will allow, in the estimation problem considered in [7], to include feedback in the observation model. The main result is collected in the theorem below.

Theorem 1. For $\phi\in C^{1+\alpha}$, $\alpha>0$ deterministic, $\lim J(\phi,\varepsilon) = J(\phi)$ where $J(\phi)$ is defined by (1.5).

We remark that, in the case of a diffusion evolving on a manifold (or, more specifically, in the case of state-dependent diffusion coefficients), the functional $J(\phi)$ involves an additional term, related to the scalar curvature; however, the result $J(\phi,\varepsilon) \to J(\phi)$ still holds, c.f. the remark in the end of section 3.

We note that Takahashi [13, remark 1, page 379] has claimed a stronger version of theorem 1 and it's converse. However, no proof is given, nor has one been published since. We did not succeed to prove the theorem in the stronger form appearing in [13].

We conclude this introduction by a "cheap" proof of our results for $\phi \in C^{1+\alpha}, \alpha > 1/2$, of a converse result when $\phi \in \Lambda_\alpha^{2,\infty}$ where $\Lambda_\alpha^{2,\infty}$ denotes the fractional Sobolev space (cf. [8]), $\alpha < 1/2$, and by some notation conventions. Section 2 includes a description of the problem in terms of a PDE approximation problem, and section 3 includes the proof of our main theorem.

Let $\phi \in C^{1+\alpha}$, $1>\alpha > 1/2$, and let $\phi^{(\delta)}$ denote the mollification of ϕ by a δ-mollifier. By extending appropriately $\phi(t)$ for $t<0$, let $\phi(0) = \phi^{(\delta)}(0)$. Then (c.f. [8]), $\|\phi-\phi^{(\delta)}\| \leq c\delta^{1+\alpha}$, $\|\phi^{(\delta)}\| < c\delta^{\alpha-1}$, and

$$P(\|x-\phi\| < \varepsilon) \leq P(\|x-\phi^{(\delta)}\|<\varepsilon+c\,\delta^{1+\alpha}) \leq P(\|w\|<\varepsilon+\delta)J(\phi^{(\delta)}\exp(K(\varepsilon)(\|\dot{\phi}^{(\delta)}\|+1)+c\varepsilon\delta^{\alpha-1})$$

(1.6)

but

$$P(\|w\|<\gamma) = K(\gamma,T)\exp -\frac{\lambda_1}{(\gamma)^2}T, \tag{1.7}$$

where λ_1 is the first eigenvalue of the Dirichlet problem in the unit ball (c.f.[4] and also below, and $K(\gamma,T)\overset{\gamma\to 0}{\to}K$. Therefore,

$$\frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} < J(\phi)\exp(K(\varepsilon)(\|\dot{\phi}^{(\delta)}\| + 1))\exp(\lambda_1 T(\frac{1}{\varepsilon^2} - \frac{1}{(\varepsilon+c\delta^{1+\alpha})^2}) + c\varepsilon\delta^{\alpha-1})$$

(1.8)

By choosing $\delta = \varepsilon^\gamma$, one gets that in order to demonstrate theorem 1 we need

$$(\frac{1}{\varepsilon^2} - \frac{1}{(\varepsilon+c\varepsilon^{\gamma(1+\alpha)})^2})\overset{\varepsilon\to 0}{\to} 0 \Rightarrow \gamma > \frac{3}{1+\alpha} \tag{1.9a}$$

$$\varepsilon^{1+\gamma(\alpha-1)} \overset{\varepsilon\to 0}{\to} 0 \Rightarrow \gamma < \frac{1}{1-\alpha} \tag{1.9b}$$

and therefore,

$$\frac{3}{1+\alpha} < \gamma < \frac{1}{1-\alpha},$$

and a solution for γ exists if $\alpha > 1/2$. A similar argument holds also for the lower bound, and the "cheap" proof is completed. Note also that a weak version of a converse to the theorem holds for $\phi \notin \Lambda_\alpha^{2,\infty}$, $\alpha<1/2$ but $\phi \in \Lambda_{\alpha-\alpha'}^{2,\infty}$, all $\alpha' > 0$, where $\Lambda_\alpha^{2,\infty}$ denotes the fractional (p=2) Sobolev space, c.f. [8]: indeed, let $\phi^{(\delta)}$, denote the mollification of ϕ by a δ-mollifier. Again, (c.f. [8]),

$\int_0^T |\dot{\phi}^{(\delta)}|^2 ds \geq c(\delta^{2(\alpha-1)})$, $\|\phi^{(\delta)}\| < \delta^{\alpha-2-\alpha'}$; plugging into (1.7), one has that

$$\frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} < C\exp(-c\delta^{2(\alpha-1)} + \lambda_1 T(\frac{1}{\varepsilon^2}) + \varepsilon\delta^{\alpha-2-\alpha'}) \tag{1.10}$$

To show that the ratio of probabilities in theorem 1 converges to zero as $\varepsilon \to 0$, we need to show that the R.H.S. of (1.10) $\to 0$, for $\delta = \varepsilon^\gamma$. But, similarly as above, one gets the pair of conditions:

$$\frac{1}{1-\alpha} < \gamma < \frac{1}{\alpha}$$

which possess a solution for $\alpha < 1/2$.

Our goal will be therefore to "close the gap" left by the cheap proof; we do that by reducing the problem to the case of $f \equiv 0$ (no drift), following [4], and then transforming the problem to a PDE one. This will allow us to get much tighter bounds on the distance between the "regularized" solution (with $\phi^{(\delta)}$) and the solution to the original problem, and that will yield the sharp estimates announced in the theorem above.

Notations

Throughout, Ω denotes the unit ball in R^n, and $\varepsilon\Omega$ denotes the ball in R^n with radius ε. $\|\ \ \|_k$ denotes the k-th, p=2 Sobolev norm in Ω, i.e.

$$\|\phi\|_k = (\sum_{|\alpha|\leq k} \int (D^\alpha\phi)^2 dx)^{1/2}$$

where the domain of the integration (Ω, $\varepsilon\Omega$) will be clear from the function involved.

v^* denotes the transpose of a vector v.

$u * v$ denotes the composition of u and v, c.f. section 3.

$\|\ \ \|$ denotes the sup-norm, and $|\ \ |$ denotes the Euclidean norm in R^n.

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II. An associated PDE formulation.

In this section, we reformulate (1.2) in terms of an associated PDE. A similar approach can be found also in [5].
We start by noting, following [4], that, for $\phi \in C^1[0,T]$, $x_t - \phi(t)$ satisfies:

$$d(x_t - \phi(t)) = \dot{\phi}(t)dt + f(x_t - \phi(t) + \phi(t))dt + dw_t \tag{2.1}$$

By Girsanov's transformation, one has:

$$\frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} = E(\exp(\int_0^T (f^*(w_t+\phi(t)) - \dot{\phi}^*(t))\,dw_t - \frac{1}{2}\int_0^T |f(\phi(t)+w_t) - \dot{\phi}(t)|^2 dt) \| \|w\|<\varepsilon) \tag{2.2}$$

Note that

$$\int_0^T f^*(w_t+\phi(t))dw_t = \int_0^T f^*((\phi(t))dw_t + \int_0^T w_t^* \nabla f^*(\phi(t))dw_t + \int_0^T 0(w^2)dw_t,$$

where ∇f denotes here the matrix of partial derivatives of f and also

$$\int_0^T f^*(\phi(t))dw_t = w_T f^*(\phi(T)) - \int_0^T \sum_{i,j} w_t^{(i)} \frac{\partial f^{(i)}}{\partial x_j}(\phi(t))\dot{\phi}^j(t)dt \tag{2.3}$$

and, by Ito's lemma

$$\int_0^T w_t^* \nabla f(\phi(t))dw_t = \nabla\bullet f(\phi(t))(\frac{|w_T|^2-T}{2}) - \int_0^T \frac{(|w_T|^2-T)}{2}(\nabla\nabla\bullet f(\phi(t))^*\dot\phi(t)dt$$

$$+\int_0^T \sum_{i\neq j}(\frac{\partial f_i}{\partial x_j})(\phi(t))w_t^i dw_t^j = (\|\dot\phi\| + \|\phi\|)O(\varepsilon^2) + \frac{1}{2}\int_0^T \nabla\bullet f(\phi(t))dt +$$

$$+\int_0^T \sum_{i\neq j}(\frac{\partial f_i}{\partial x_j})(\phi(t))w_t^i dw_t^j \tag{2.4}$$

where $\nabla\bullet f$ denotes here the divergent of f; combining (2.2), (2.3) and (2.4), one has:

$$\frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} = \exp(-\frac{1}{2}\int_0^T |f(\phi(t)) - \dot\phi(t)|^2 dt - \frac{1}{2}\int_0^T \nabla\bullet f(\phi(t))dt) .$$

$$E(\exp(O(\varepsilon^2)(\|\dot\phi\| + \|\phi\|) + \int_0^T \sum_{i\neq j}\frac{\partial f_i}{\partial x_j}(\phi(t))w_t^i dw_t$$

$$+\int_0^T O(|w|^2)dw_t - \int_0^T \dot\phi^*(t)dw_t) \mid \|w\| < \varepsilon) \tag{2.5}$$

By lemmas of [4, pg. 451] (see also [6]), (which are the main part of the proof in [4]),

$$E(\exp c\int_0^T O(|w|^2)dw_t \mid \|w\|<\varepsilon) \overset{\varepsilon\to 0}{\to} 1 \qquad \forall c \tag{2.6a}$$

$$E(\exp c\int_0^T k_y(\phi)w_t^i dw_t^j \mid \|w\|<\varepsilon) \overset{\varepsilon\to 0}{\to} 1 \quad \forall c \tag{2.6b}$$

and therefore, to compute (2.5) we need only compute

$$E(\exp - \int_0^T \dot{\phi}^*(t)dw_t \mid \|w\|<\varepsilon) \; .$$

and show that it converges to 1 as $\varepsilon \to 0$. Let us define

$$A \triangleq E(\exp - \int_0^T \dot{\phi}^*(t)dw_t \mid \|w\|<\varepsilon)P(\|w\|<\varepsilon).$$

Then, by Girsanov's theorem

$$A = \exp\frac{1}{2}(\int_0^T |\dot{\phi}(t)|^2 dt)P(\|w-\phi\| < \varepsilon)$$

Let u(z,t,x,s) be the fundamental solution of

$$u_t = \frac{1}{2}\Delta u + \dot{\phi}(t)\nabla u + \frac{1}{2}|\dot{\phi}(t)|^2 u$$

$$u(z,t,x,s)|_{|z|=\varepsilon} = 0 \tag{2.7}$$

i..e. the solution of (2.7) such that, for each continuous f(x),

$$\lim_{t\to s} \int_{\varepsilon\Omega} u(z,t,x,s)f(x)\, dx = f(z)$$

Such a solution exists and is unique by the maximum principle (cf. [9 , ch. 1-2]). Then

$$A = \int_{\varepsilon\Omega} u(z,T,0,0)dz \tag{2.8}$$

Our goal will therefore be to compute bounds on the fundamental solution of (2.7). It turns out that one can find explicitly the solution to a related equation (eq. 2.3a), and then by perturbation techniques relate the two. Towards this end, let $\phi^{(\delta)}$ be a δ mollification of ϕ (for example, with a Bessel potential, or otherwise, cf. [8]), and let $u^{(\delta)}(z,t,x,s)$ be the fundamental solution of:

$$u_t^{(\delta)} = \frac{1}{2} \Delta u^{(\delta)} + \dot{\phi}(t)\nabla u^{(\delta)} + (\dot{\phi}^{(\delta)}(t) - \dot{\phi}(t))\nabla u^{(\delta)}(t)u^{(\delta)} \tag{2.9a}$$

$$u^{(\delta)}(z,t,x,s)\Big|_{|z|=\varepsilon} = 0 \tag{2.9b}$$

In the sequel, let $j^{(\delta)}(t) = \phi^{(\delta)}(t) - \phi(t)$. We will assume throughout, without mentioning it, that $||j^{(\delta)}(t)|| < 1$.

Our line of attack will be as follow: we first show below that $\int_{\varepsilon\Omega} \frac{dzu^{(\delta)}(z,t,0,s)}{p(||w||<\varepsilon)} \overset{\varepsilon\to 0}{\to} 1$ for $t - s \ge \tau_0 > 0$ uniformly in $\delta>0$, i.e. that if in (2.8) one substitutes $u^{(\delta)}$ instead of u one has the required convergence (lemma 2.2). We then show in section 3 that

$$\frac{\int_{\varepsilon\Omega} dz(u^{(\delta)}(z,t,0,s) - u(z,t,0,s))}{P(||w||<\varepsilon)} \underset{\delta(\varepsilon)\to 0}{\overset{\varepsilon\to 0}{\to}} 0,$$

where $\delta(\varepsilon) \to 0$ in an appropriate way, thus establishing the required convergence. To demonstrate this last convergence, note that the solution to (2.7) can be represented by the classical parametrix method in terms of an infinite series involving the solution of (2.9) (theorem 3.1). Estimates on $u^{\delta}(z,t,x,s)$ which we prepare in the remaider of this section are crucial in obtaining the required convergence.

<u>We use the following classical result</u>:

Lemma 2.1

$u^{(\delta)}(z,t,x,s)$ exists and is unique. Moreover, there exists a c independent of ε, δ, such that

$$|u^{(\delta)}(z,t,x,s)| \leq \frac{c}{(t-s)^{n/2}} \exp -\left(\frac{(z-x)^2}{c(t-s)}\right) \tag{2.10a}$$

$$|\nabla u^{(\delta)}(z,t,x,s)| \leq \frac{c}{(t-s)^{\frac{n+1}{2}}} \exp -\left(\frac{(z-x)^2}{c(t-s)}\right) \tag{2.10b}$$

In particular,

$$|\nabla u^{(\delta)}(z,t,x,s)| \leq \frac{c}{(t-s)^{\mu}|z-x|^{n-2\mu}} \;\forall\; 1/2 < \mu \leq 1 \tag{2.10c}$$

$$|\nabla u^{(\delta)}(z.t.x.s)| \leq \frac{c}{(t-s)^{\mu}|z-x|^{n+1-2\mu}} \tag{2.10d}$$

Proof. The estimates (2.10a) and (2.10b) are the well know Arronson estimates. For an easy derivation of them, we refer to [12] and references there. (Note that in [12], only (2.10a) is proved, however (2.10b) follows easily by differentiating throughout in the proof). (2.10c), (2.10d), which are the only estimates we will need, follow easily from (2.10a), (2.10b). A different, more cumbersome proof of (2.10c), (2.10d) via the parametrix method appears in [9, ch. 1, section 4-5]. Finally, uniqueness follows from the maximum principle.

The usefulness of equation (2.9) lies in the fact that it's solution is easily represented; to do that we need some auxilliary results, which are regrouped in (a)-(c) below; (2.13) is the representation of the solution we will use in the sequel.

Lemma 2.2

Let $(\gamma_m(x), \lambda_m)$ denote the normalized (w.r.t. $L^2(\Omega)$) eigenfunctions and eigenvalues of the Dirichlet problem in the unit ball in R^n, i.e.

$$\Delta\gamma_m(x) = -\lambda_m\gamma_m(x), \ x \in \Omega \tag{2.11a}$$

$$\gamma_m(x)\big|_{|x|=1} = 0 \tag{2.11b}$$

Then:

(a) There exists a unique eigenvector associated with the <u>minimal</u> eigenvalue λ_0, and $\lambda_0 > 0$.

(b) The set $\{\lambda_m\}$ is discrete, and, if $N(\lambda)$ denotes the number of eigenvalues s.t. $\lambda_m < \lambda$ (including multiplicity), then

$$N(\lambda) = K\lambda^{n/2} + o(\lambda^{n/2}), \text{ as } \lambda \to \infty \tag{2.12}$$

(c) $\gamma_m(x) \; \varepsilon \; L_2(\Omega)$; Moreover, $\forall k$, $||\gamma_m(x)||_k < \infty$ and $|\gamma_m(x)| \leq c(\lambda_m)^{n/2}$:

Finally, $\gamma_m(x)$ spans $L_2(\Omega)$.

(d) The following limit exists (pointwise, uniformly in (z,x) for t-s > $\tau_0\varepsilon^2$, for all ε and in $L^2(\varepsilon\Omega x[0,T])$) and is the fundamental solution of (2.9):

$$u^{(\delta)}(z,t,x,s) = \lim_{j\to\infty} u_j^{(\delta)}(z,t,x,s) \triangleq \lim_{j\to\infty} \sum_{m=0}^{j} \frac{1}{\varepsilon^n} \exp -(\frac{\lambda_m(t-s)}{\varepsilon^2})\gamma_m(\frac{z}{\varepsilon})\gamma_m(\frac{x}{\varepsilon})$$

$$\exp(-(\dot{\phi}^{(\delta)}(t)(z) - \dot{\phi}^{(\delta)}(s)x)) \exp\ (\frac{1}{2}\int_s^t |\phi^{(\delta)}(\tau) - \phi(\tau)/^2 d\tau \tag{2.13}$$

<u>Proof</u>: For (a), c.f. [10]. (b) is theorem (14.6) of [11]. That $\gamma_m(x) \in L_2(\Omega)$ follows from theorem (16.5) of [11]. To see that $||\gamma_m(x)|| < \infty$, note that

$$||\Delta\gamma_m(x)||_0 = |\lambda_m|, \ ||\Delta^k\gamma_m(x)||_0 = |\lambda_m|^k \tag{2.14}$$

Therefore, by the Sobolev lemma (c.f., e.g., theorem (3.8) of [11],

$$\|\gamma_m\|_{2k} \leq C_{m,k}(\|\Delta^k\gamma_m\|_0 + \|\gamma_m\|_0) \leq C_{m,k}(|\lambda_m|^k+1) < \infty \tag{2.15}$$

Moreover, since $w_\Omega^{[n/2]+1} \hookrightarrow C(\Omega)$, one has also

$$|\gamma_m(x)| \leq k\lambda_m^{([\frac{n}{4}]+1/2)} \tag{2.16}$$

Finally, we show (d). Note that

$$\sum_{m=1}^{\infty}\left|\frac{1}{\varepsilon^n}\exp\frac{\lambda_m(t-s)}{\varepsilon^2}\gamma_m(\frac{z}{\varepsilon})\gamma_m(\frac{x}{\varepsilon})\exp-(\dot\phi_t^{(\delta)}z-\dot\phi_s^{(\delta)}x)\exp(\frac{1}{2}\int_s^t|\phi^{(\delta)}(\tau)-\phi(\tau)|^2d\tau)\right|$$

$$\leq\sum_{m=1}^{\infty}\frac{k}{\varepsilon^n}\exp\frac{-\lambda_m(t-s)}{\varepsilon^2}\lambda_m^{(\frac{n}{2}+1)}$$

$$\leq\frac{k^1}{\varepsilon^n}\sum_{j=1}^{\infty}j^{n/2}\,j^{(\frac{n}{2}+1)}\exp^{-(\frac{j(t-s)}{\varepsilon^2})}<\infty \tag{2.17}$$

where we have used (2.12); note that the convergence is uniform for $t-s>\varepsilon^2\tau_0$, is independent of ε for $t-s>\varepsilon^2\tau_0$ and also that it holds even after scalling by exp $-(\lambda_0(t-s)/\varepsilon^2)$. The convergence in $L^2(\varepsilon\Omega x[0,T])$ is very similar, and will not be used in the sequel.

It is easy to check , similarly, that the convergence holds also for the derivatives of $u_j^{(\delta)}(z,t,x,s)$ (w.r.t. t (once) and w.r.t. z twice), and that lim $u_j^{(\delta)}(z,t,x,s)$ satisfies (2.9). It remains to check therefore that it is indeed a fundamental solution.

Let f(x) be a C_0^∞ (on $\varepsilon\Omega$) function (and in particular, $f(x) = \sum_{i=1}^{\infty}\gamma_i(\frac{x}{\varepsilon})f_i$ with $\sum_{i=1}^{\infty}f_i^2<\infty$). Let

$$\Theta(t,s,z) \stackrel{\Delta}{=} \int_{\varepsilon\Omega}u^{(\delta)}(z,t,x,s)f(x)dz - f(z) \tag{2.18}$$

we have then

$$\|\Theta(t,s,z)\|^2_{0,\varepsilon\,\Omega} \leq \sum_{i=1}^{\infty} f_i^2[\exp\frac{-\lambda_i(t-s)}{\varepsilon^2} - 1]$$

Let k_o be such that

$$\sum_{i=k_o}^{\infty} f_i^2 < \gamma\,,$$

and τ_0 such that

$$\exp\,(\frac{\lambda_{k_0}\tau_0}{\varepsilon^2}) > 1-\gamma$$

one has then

$$\|\Theta\|^2_{0,\varepsilon,\Omega} < 2\gamma$$

for $t\text{-}s<\tau_0$, and since γ is arbitrary, we have

$$\Theta \underset{t\to s}{\overset{L_2(\varepsilon\Omega)}{\longrightarrow}} 0 \tag{2.19}$$

Similarly,

$$\|\Delta^k\Theta(t,s,z)\|_{0,\varepsilon\Omega} \leq K \sum_{i=1}^{\infty} g_i^2(k)\,[\exp\frac{-\lambda_i(t-s)}{\varepsilon^2} - 1]$$

where

$$\Delta^k g(x) = \sum_{i=1}^{\infty} \gamma_i\,(\frac{z}{\varepsilon})g_i(k) \text{ and } \sum_i g_i^2(k) < \infty.$$

As above, one has then that $||\Delta^k \Theta(t,s,x)||_{0,\varepsilon\Omega} \to 0$, which implies by the Sobolev lemma that $\Theta(t,s,z) \to 0$ pointwise. Therefore, one has that, in the sense of distributions in $D'(\varepsilon\Omega$, lim $u_j^{(\delta)}(z,t,x,s)$ is equal to the (unique, by [9, ch. 2]) fundamental solution of (2.9). Since, as is easily checked, for (t-s) > 0 both this limit and the fundamental solution are continuous in z,x, they are equal everywhere, which concludes the demonstration of the theorem.

We establish below some estimates which will turn out to be useful in the perturbation analysis of section 3;

<u>Lemma 2.3</u>

$$u^{(\delta)}(z,t,x,s) = \frac{1}{\varepsilon^n} \exp \frac{-\lambda_0(t-s)}{\varepsilon^2} [\exp(-(\dot{\phi}^{(\delta)}(t)z-\dot{\phi}^{(\delta)}(s)x))]\gamma_0(\frac{z}{\varepsilon})\, \gamma_0(\frac{x}{\varepsilon}) + A(z,t,x,s)] \tag{2.20a}$$

where

$$|A(z,t,x,s)| \leq \begin{cases} k \exp \dfrac{-\Delta\lambda(t-s)}{\varepsilon^2} & \text{if } (t-s) > \tau_0\varepsilon^2, \quad \Delta\lambda \triangleq \lambda_1 - \lambda_0 \\ \dfrac{k\varepsilon^n}{(t-s)^{\mu}|z-x|^{n-2\mu}} & \text{if } (t-s) \leq \tau_0\varepsilon^2, \quad 1/2 < \mu < 1 \end{cases} \tag{2.20b}$$

and k is independent of ε,δ.

Similarly,

$$\nabla u^{(\delta)}(z,t,x,s) = \frac{1}{\varepsilon^{n+1}} \exp \frac{-\lambda_0(t)}{\varepsilon^2} [\exp -(\dot{\phi}^{(\delta)}(t)z - \dot{\phi}^{(\delta)}(s)x(\nabla\gamma_0(\frac{z}{\varepsilon})\gamma_0(\frac{x}{\varepsilon}) -$$

$$- \dot{\phi}^{(\delta)}(s)\varepsilon\, \gamma_0(\frac{z}{\varepsilon})\gamma_0(\frac{x}{\varepsilon})) + B(z,t,s,x)] \tag{2.21a}$$

where

$$|B(z,t,x,s)| \le \begin{cases} k \exp \dfrac{-\Delta\lambda(t-s)}{\varepsilon^2} & \text{if } (t-s) > \tau_0\varepsilon^2 \\ \dfrac{k\,\varepsilon^{n+1}}{(t-s)^{\mu}|z-x|^{n+1-2\mu}} & \text{if } (t-s) \le \tau_0\varepsilon^2,\ 1/2 < \mu < 1 \end{cases}$$

(2.21b)

Proof. The upper bound in (2.20b) and (2.21b) follows immediately from the representation (2.13) and the method of proof of lemma (2.2). The short time estimates (the lower line of (2.20b) and (2.21b)), follow directly from the derivation of [9, ch. 1, section 3,4], or from [12].

Lemma 2.4

Let $C_i(z,t,x,s)$, $i=1,2$ satisfy

$$|C_i| \le \begin{cases} k_i \exp \dfrac{-\Delta\lambda(t-s)}{\varepsilon^2} & t-s > \varepsilon^2\tau_0 \\ k_i \dfrac{\varepsilon^{n+\beta_i}}{(t-s)^{\mu_i}|z-x|^{n+\gamma_i}} & 0 \le t-s \le \tau_0\varepsilon^2,\ 1 > \mu_i > 0,\ n+1 > n+\gamma_i > 0 \end{cases}$$

(2.22)

Then

$$\left|\int^{t}\!\!\int_{s\varepsilon\Omega} C_i(z,t,x,s)\,dx\,ds\right| \le k\ k_i\varepsilon^{(n+\beta_i-\gamma_i+2(1-\mu_i))\wedge(n+2)} \qquad (2.23)$$

(with similar bound when the integration w.r.t. x,s is replaced by an integration w.r.t. z,t), and

$$|C_i*C_j(z,t,x,s)| \triangleq |\int_\tau \int_{\varepsilon\Omega} C_i(z,t,x',s')C_j(x',s',x,s)dx'ds'| \leq$$

$$\begin{cases} kk_ik_j\,\varepsilon^{(n+(0\wedge\beta_i-\gamma_i+2(1-\mu_i)\wedge(\beta_j-\gamma_j+2(1-\mu_j))}\exp\dfrac{-\Delta\lambda(t-s)}{\varepsilon^2} & t-s>\varepsilon^2\tau_0 \\ \dfrac{kk_ik_j\,\varepsilon^{\beta_i+\beta_j+n-\gamma_i-\gamma_j-2\mu_i-2\mu_j+2}}{\left(\frac{t-s}{\varepsilon^2}\right)^{ov(\mu_i+\mu_j-1)}\left|\frac{z-x}{\varepsilon}\right|^{ov(\gamma_i+\gamma_j+n)}}, & t-s\leq\varepsilon^2\tau_0 \end{cases} \tag{2.24}$$

<u>Proof</u>

First, note that

$$\int_s^t \int_{\varepsilon\Omega} |C_i(z,t,x,s)|dxds \leq k_i \int_s^{t-\tau_0\varepsilon^2}\int_{\varepsilon\Omega} \frac{-\Delta\lambda(t-s)}{\varepsilon^2}dsdx + k_i\int_{t-\tau_0\varepsilon^2}\int_{\varepsilon\Omega}\frac{\varepsilon^{n+\beta_i}}{(t-s)^{\mu_i}|z-x|^{n+\gamma_i}}dsdx$$

$$\leq (kk_i\varepsilon^{n+2} + kk_i\varepsilon^{n+\beta_i+2(1-\mu_i)-\gamma_i})$$

from which (2.23) follows.

Considering (2.24), let first $(t-s) \leq \varepsilon^2\tau_0$; one has then

$$|I_1(t-s)| \triangleq \iint_{s\,\varepsilon\Omega}^{t} |C_i(z,t,x',s')||C_j(x',s',x,s)|dx'ds \leq$$

$$\leq k_ik_j\int_s^t \frac{ds'}{(t-s')^{\mu_i}(s'-s)^{\mu_j}}\int_\Omega \frac{\varepsilon^{\beta_i+\beta_j+n-\gamma_i-\gamma_j}}{\left|\frac{z}{\varepsilon}-x'\right|^{n+\gamma_i}\left|x'-\frac{x}{\varepsilon}\right|^{n+\gamma_j}}dx' \tag{2.25}$$

We recall the following (c.f., e.g., [9, pg. 14]):

$$\int_\Omega \frac{dx'}{|a-x'|^{\alpha}|x'-b|^{\beta}} \le \begin{cases} k\ |a-b|^{n-\alpha-\beta} & \text{if} \quad n < \alpha + \beta \\ k & \text{if} \quad n > \alpha + \beta \end{cases} \tag{2.26}$$

Applying (2.26), one has from (2.25)

$$|I_1(t-s)| \le kk_ik_j \frac{1}{(t-s)^{\mu_i+\mu_j-1}} \frac{\varepsilon^{\beta_i+\beta_j+2n}}{|z-x'|^{n+\gamma_i+\gamma_j}} \qquad (\text{if } \gamma_i+\gamma_j+n>0,\ \mu_i+\mu_j>1,\ t-s\le\varepsilon^2\tau_0) \tag{2.27}$$

with similar bounds for the other cases of $t\text{-}s\le\varepsilon^2\tau_0$. We consider therefore now $t\text{-}s\ge\varepsilon^2\tau_0$. In this case, we get:

$$|I_1(t-s)| \le \int_s^{s+\varepsilon^2\tau_0}\int_{\varepsilon\Omega} k_jk_i \exp^{\frac{-\Delta\lambda(t-s')}{\varepsilon^2}} \frac{\varepsilon^{n+\beta_j}}{(s'-s)^{\mu_j}|x'-x|^{n+\gamma_j}}\, dx'ds'$$

$$+ \int_{s+\varepsilon^2\tau_0}^{t-\varepsilon^2\tau_0}\int_{\varepsilon\Omega} k_ik_j \exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\, dx'ds'$$

$$+ \int_{t-\varepsilon^2\tau_0}^{t}\int_{\varepsilon\Omega} k_ik_j \exp\frac{-\Delta\lambda(s'-s)}{\varepsilon^2} \frac{\varepsilon^{n+\beta_i}}{(t-s')^{\mu_i}|z-x'|^{n+\gamma_i}} \tag{2.28}$$

Using (2.22), one gets

$$|I_1(t-s)| \le kk_ik_j \exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2} (\varepsilon^n+\varepsilon^{(n+2)\wedge(n+\beta_i-\gamma_i+2(1-\mu_i))\wedge(n+\beta_j-\gamma_j+2(1-\mu_j))})$$

$$\le kk_ik_j \exp\frac{-\Delta\lambda(t-s)}{\varepsilon^2} (\varepsilon^{n+(0\wedge(\beta_i-\gamma_i+2(1-\mu_i))\wedge(\beta_j-\gamma_j+2(1-\mu_j))}) \tag{2.29}$$

and the lemma is proved.

III. A Solution to the Original PDE

In this section, we construct, by a perturbation method, a solution to eq. (2.7), based on $u^{(\delta)}(z,t,x,s)$.

Let L denote the operator:

$$L_{t,\delta}v(z,t,x,s) \triangleq -j^{(\delta)}(t)*\nabla v(z,t,x,s) + z\phi^{(\delta)}(t)v(z,t,x,s) \tag{3.1}$$

As before, let * denote the composition of two functions in the following form:

$$f_1(z,t,x,s) * f_2(z,t,x,s) \triangleq \int_s^t \int_{\varepsilon\Omega} f_1(z,t,x',s')f_2(x',s',x,s)dx'ds \tag{3.2}$$

Define

$$L^1_{t,\delta}\, u^{(\delta)}(z,t,x,s) \triangleq L_{t,\delta}u^{(\delta)}(z.t.x.s) \tag{3.3a}$$

and

$$L^k_{t,\delta}u^{(\delta)}(z,t,x,s) \triangleq L_{t,\delta}\, u^{(\delta)} * L^{k-1}_{t,\delta}\, u^{(\delta)} \tag{3.3b}$$

Let

$$u^j(z,t,x,s) \triangleq u^{(\delta)}(z,t,x,s) + \sum_{i=1}^{j} u^{(\delta)}(z,t,x',s)*L^i_{t,\delta}u^{(\delta)}(z,t,x,s) \tag{3.4}$$

Finally, assume that $\varepsilon||\ddot{\phi}^{(\delta)}||<1$ (which is possible if δ is chosen not too small). We will show that:

Theorem 3.1

(a) For any (t-s) > 0, $u^j(z,t,x,s)$ converges (uniformly in $z,x \in \varepsilon\Omega$) to a limit u(z,t,x,s);

(b) u(z,t,x,s) is the fundamental solution of (2.7).

(c) Let $\gamma(\varepsilon) = \varepsilon||\ddot{\phi}^{(\delta)}||^{\varepsilon}\overset{\to 0}{\to} 0$, and let $||j^{(\delta)}(t)|| = 0(\varepsilon\chi)$ for some $\chi>0$.

Then, for any $\tau_0>0$ and $(t-s)\geq\tau_0$,

$$\exp \frac{\lambda_0(t-s)}{\varepsilon^2} \varepsilon^n \,|\, u(z,t,x,s) - u^{(\delta)}(z,t,x,s) | \xrightarrow{\varepsilon \to 0} 0 \tag{3.5}$$

uniformly in z,s, $\in \varepsilon\Omega$, and the rate of convergence is controlled by

$$\varepsilon \|\ddot{\phi}^{(\delta)}\| + \exp\left(\frac{\|\dot{\phi}^{(\delta)} - \dot{\phi}\|^{\frac{1}{1-\mu}}}{\varepsilon^{1/(1-\mu)}} - \frac{\Delta\lambda\tau_0}{\varepsilon^2}\right), \quad 1/2 < \mu < (1+\chi)/2 \tag{3.6}$$

Proof

Part a: Note that, by lemma (2.3),

$$L_{t,\delta} u^{(\delta)}(z,t,x,s) = \exp(-(\dot{\phi}^{(\delta)}(t)\, z - \dot{\phi}^{(\delta)}(s) x)) \frac{\exp \frac{-\lambda_0(t-s)}{\varepsilon^2}}{\varepsilon^n} [\tilde{\alpha}_1(t,z)\gamma_0(\frac{z}{\varepsilon})\gamma_0(\frac{x}{\varepsilon}) +$$

$$+ \beta_1(t)\nabla\gamma_0(\frac{z}{\varepsilon})\gamma_0(\frac{x}{\varepsilon})] + \exp \frac{-\lambda_0(t-s)}{\varepsilon^2} E_1(z,t,x,s) \tag{3.7}$$

where

$$|\tilde{\alpha}_1| \le \varepsilon \|\ddot{\phi}^{(\delta)}\| + \|\dot{\phi}^{(\delta)}\| \, \|j^{(\delta)}\| \triangleq 1^{(\delta)} \xrightarrow{\varepsilon \to 0} 0 \tag{3.8a}$$

$$|\beta_1| = \|\frac{j^{\delta}(t)}{\varepsilon}\| \tag{3.8b}$$

$$|E_1| \leq \begin{cases} \dfrac{Kk^{(\delta)}}{(t-s)^{\mu}|z-x|^{n+1-2\mu}} & t - \tau \leq \tau_0 \varepsilon^2 \\[2ex] \dfrac{K}{\varepsilon^{n+1}} \exp\left(\dfrac{-\Delta\lambda(t-\tau)}{\varepsilon^2}\right) k^{(\delta)} & t-s > \tau_0 \varepsilon^2 \end{cases} \tag{3.8c}$$

where

$$k^{(\delta)} \triangleq \|j^{(\delta)}\| + \varepsilon^2 \|\ddot{\phi}^{(\delta)}\| \overset{\varepsilon \to 0}{\Rightarrow} 0$$

by our assumptions.

Using lemma 2.4, one obtains:

$$L_{t,\delta} u^{(\delta)} * L_{t,\delta} u^{(\delta)} = \frac{\exp^{-\left(\frac{\lambda_0 (t-s)}{\varepsilon^2}\right)}}{\varepsilon^n} \exp -(\dot{\phi}^{(\delta)}(t) z - \dot{\phi}^{(\delta)}(s) x)\left[\tilde{\alpha}_2 \gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right) + \beta_2 \nabla\gamma_0\left(\frac{z}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right) + \tilde{\gamma}_2\right]$$

$$+ \exp \frac{-\lambda_0 (t-s)}{\varepsilon^2} E_2(z,t,x,s) \tag{3.9a}$$

where

$$|\tilde{\alpha}_2| \leq (K'Kl^{(\delta)})^2 \tag{3.9b}$$

$$|\beta_2| \leq (K'K) l^{(\delta)} \left\| \frac{j^{\delta}(t)}{\varepsilon} \right\| \tag{3.9c}$$

$$|\tilde{\gamma}_2| \leq \|j^{\delta}(t)\| K' k^{(\delta)} \tag{3.9d}$$

$$|E_2(z,t,x,s)| \le \begin{cases} \left(\frac{K'Kk^{(\delta)}}{\varepsilon}\right)^2 \dfrac{\exp \frac{-\Delta\lambda(t-s)}{\varepsilon^2}}{\varepsilon^n} & t-s>\varepsilon^2\tau_0 \\ (K'Kk^{(\delta)})^2 \dfrac{1}{(t-s)^{2\mu-1}|z-x|^{n+2(1-2\mu)}} & t-s\le\varepsilon^2\tau_0 \end{cases}$$

(3.9e)

Since $2>2\mu>1$, the singularity in (3.9e) is weaker than that of (3.8c). By the same reasoning, one obtains that there exists a k_0 such that

$$L^{k_0}_{t,\delta}\, u^\delta(z,t,x,s) = \frac{1}{\varepsilon^n} \exp -(\dot\phi^{(\delta)}(t)z - \dot\phi^{(\delta)}(s)x) \exp -\left(\frac{\lambda_0(t-s)}{\varepsilon^2}\right) [\tilde\alpha_{k_0}\, \gamma_0\left(\frac{z}{\varepsilon}\right) +$$

$$+ \beta_{k_0} \nabla\gamma_0\left(\frac{z}{\varepsilon}\right) + \tilde\gamma_{k_0}]$$

$$+ \exp \left(\frac{-\lambda_0(t-s)}{\varepsilon^2}\right) \exp -\left(\frac{\Delta\lambda(t-s)}{\varepsilon^2}\right) E_{k_0}$$

(3.10)

where

$$|\tilde\alpha_{k_0}| \le (K'Kl^{(\delta)})^{k_0} \overset{\varepsilon\to 0}{\to} 0 \qquad (3.11a)$$

$$|\beta_{k_0}| \le m^{(\delta)}(K'K)^{k_0} \left\|\frac{\dot j^{(\delta)}(t)}{\varepsilon}\right\|^{k_0}, \text{ with } \|m^{(\delta)}\| \overset{\varepsilon\to 0}{\to} 0, \text{ at least as fast as } k^{(\delta)}$$

(3.11b)

$$|\tilde\gamma_{k_0}| \le n^{(\delta)}(K'K)^{k_0} \quad \text{with } \|n^{(\delta)}\| \overset{\varepsilon\to 0}{\to} 0 \text{ at least as fast as } k^{(\delta)}$$

(3.11c)

and

$$|E_{k_0}| \leq \frac{1}{\varepsilon^n}\left(\frac{K'Kk^{(\delta)}}{\varepsilon}\right)^{k_0}\frac{1}{k_0!}$$

(3.11d)

Therefore, by the same argument as in [9,ch.1, eq. 4.7], one has that, for $k \geq k_0$

$$L^k_{t,\delta}(u^{(\delta)}(z,t,x,s)) = \frac{1}{\varepsilon^n}\exp -(\dot{\phi}^{(\delta)}(t)z - \dot{\phi}^{(\delta)}(s)x)\exp\frac{-\lambda_0(t-s)}{\varepsilon^2}[\tilde{\alpha}_k\gamma_0(\frac{z}{\varepsilon}) +$$

$$+ \tilde{\beta}_k\nabla\gamma_0(\frac{z}{\varepsilon}) + \tilde{\gamma}_k] + \exp\left(\frac{-\lambda_0(t-\tau)}{\varepsilon^2}\right)\exp\left(\frac{-\Delta\lambda(t-s)}{\varepsilon^2}\right)E_k \tag{3.12}$$

with

$$|\tilde{\alpha}_k| \leq (K'K1^{(\delta)})^k \tag{3.12a}$$

$$|\tilde{\beta}_k| \leq (1^{(\delta)})^k \left\|\frac{j^{(\delta)}(t)}{\varepsilon}\right\|^k (K'K)^k \tag{3.13b}$$

$$|\tilde{\gamma}_k| \leq (\|j^\delta(t)\|K'Kk^{(\delta)})^k \tag{3.13c}$$

and

$$|E_k| \leq \frac{(K'K(t-s)^{1-\mu})^k}{\Gamma((1-\mu)k+1}\left(\frac{k^{(\delta)}}{\varepsilon}\right)^k\frac{1}{\varepsilon^n} \tag{3.13d}$$

By computing $u^{(\delta)}(z,t,x,s) * L^k_{t,\delta}(z,t,x,s)$, the $\tilde{\beta}_k$ term drops out due to the integration

$$\int_{\varepsilon\Omega}\frac{\partial}{\partial x_i}\gamma'_0\left(\frac{x}{\varepsilon}\right)\gamma_0\left(\frac{x}{\varepsilon}\right)dx = 0;$$

one has therefore, for t-s>0:

$$|\sum_{j=k^1}^{\infty} u^{(\delta)} * L^{j}_{t,\delta}\, u^{(\delta)}(z.t.x.s)| \quad \leq \tilde{K} \exp \frac{-\lambda_0(t-s)}{\varepsilon^2} \sum_{j=1^1}^{\infty} (a(\varepsilon)^j + \exp \frac{-\Delta\lambda(t-s)}{\varepsilon^2} \; \frac{b^j}{\Gamma((1-\mu)j+1)})$$

(3.14)

where $|a(\varepsilon)| \leq k^{(\delta)} + 1^{(\delta)} \overset{\varepsilon \to 0}{\rightarrow} 0$, $|b| \leq (\frac{j^{(\delta)}}{\varepsilon})$, and (3.14) is easily seen to converge, uniformly in z,x.

(Remark: a similar proof holds also for the first two z derivatives of u, for details, cf. e.g. [9]).

Part b: The proof is identical to the one given in Lemma (2.2), due to the fact that $u^{(\delta)} * L_{t,\delta}{}^{k} u^{(\delta)}$ has a weaker singularity in the origin than $u^{(\delta)}$; we ommit the details.

Part c: By (3.10) (3.13) and the fact that by comparing with $u^{(\delta)}$, the β_k term drops from (3.10), (3.12), one has that, for t-s $\geq \tau_0$, and ε, δ small enough,

$$|u^{(\delta)}(z,t,x,s) * L^{k}_{t,\delta}(u^{(\delta)}(z,t,x,s))| \leq \frac{1}{\varepsilon^n} \; (\tilde{K}(k^{(\delta)}+1^{(\delta)})^k \; \exp \frac{-\lambda_0(t-s)}{\varepsilon^2}$$

$$+ \exp \frac{-\lambda_0(t-s)}{\varepsilon^2} \; \exp^{-}(\frac{\Delta\lambda(t-s)}{\varepsilon^2}) \, [(\frac{j^{(\delta)}\tilde{K}}{\varepsilon})^{.k} \bullet \frac{1}{\Gamma((1-\mu)k+1)}]$$

Therefore,

$$\varepsilon^n |u(z,t,x,s) - u^{(\delta)}(z,t,x,s)| \exp \frac{\lambda_0(t-s)}{\varepsilon^2} \leq \sum_{k=1}^{\infty} [(\tilde{K}\, (k^{(\delta)} + 1^{(\delta)}))^k +$$

$$\exp^{-}(\frac{\Delta\lambda(t-s)}{\varepsilon^2}) \; (\frac{(\frac{j^{(\delta)}K}{\varepsilon})^{\frac{k}{(1-\mu)}}}{k!}) \; \leq$$

$$\leq (1^{(\delta)} + k^{(\delta)})K + \exp\left(\frac{-\Delta\lambda\tau_0}{\varepsilon^2}\right) \exp\left(\frac{\tilde{K}j^{(\delta)}}{\varepsilon}\right)^{\frac{1}{1-\mu}} \left(\frac{j^{(\delta)}}{\varepsilon}\right)$$

(3.15)

By our assumptions,

$$\frac{\|j^{(\delta)}\|}{\varepsilon} = 0(\varepsilon^{\chi-1}), \ \chi>0;$$

Let $1/2 < \mu < (1+\chi)/2$. In this case, the R.H.S. of (3.15) is bounded by

$$(1^{(\delta)} + k^{(\delta)}) K + \left(\frac{j^{(\delta)}}{\varepsilon}\right) \exp\left(\frac{-\Delta\lambda\tau_0}{\varepsilon}\right) \exp + \left(\frac{\tilde{K}}{\varepsilon^{2-q}}\right), \ q > 0 \qquad (3.26)$$

Therefore, (3.5) holds and moreover the rate of convergence is controlled as in (3.6).

Corollary 3.1. Assume $\phi \in C^{1+\alpha}$, $\alpha>0$; then theorem 1 in the introduction holds.

Proof. Let

$$0 < \gamma < \frac{1.}{1-\alpha}, \text{ and } \delta = \varepsilon^{\gamma}, \text{ then } \|\varepsilon\ddot{\phi}^{(\delta)}\| \leq k\, \varepsilon\, \delta^{\alpha-1} \xrightarrow{\varepsilon\to 0} 0$$

and

$$\|\dot{\phi}^{(\delta)} - \dot{\phi}\| \leq K\varepsilon^{\alpha\gamma},$$

so that the conditions of theorem (3.1) hold. By (2.5) and (2.8), one has

$$\frac{P(\|x-\phi\|<\varepsilon)}{P(\|w\|<\varepsilon)} = \frac{J(\phi)\exp(0(\varepsilon^2)(\|\phi\|+\|\dot{\phi}\|)+K(\varepsilon)) \int_{\varepsilon\Omega} u(z,t,x,s)dz}{P(\|w\|<\varepsilon)}$$

$$= \frac{J(\phi)\exp(0(\varepsilon^2)(\|\phi\|+\|\dot{\phi}\|) + K(\varepsilon)) \int_{\varepsilon\Omega} u^{(\delta)}(z,t,0,0)dz}{P(\|w\|<\varepsilon)}$$

$$+ \frac{J(\phi) \exp (0(\varepsilon^2)(\|\phi\|+\|\dot{\phi}\|)+K(\varepsilon)) \int_{\varepsilon\Omega}(u(z,t,0,0) - u^{(\delta)}(z,t,0,0))dz}{P(\|w\|<\varepsilon)}$$

(3.17)

Note that

$$\frac{\int_{\varepsilon\Omega} u^{(\delta)}(z,t,0,0)dz}{P(\|w\|<\varepsilon)} \overset{\varepsilon \to 0}{\Longrightarrow} \frac{\int_{\Omega} \gamma_0(z)dz\, \gamma_0(0)}{\int_{\Omega} \gamma_0(z)dz\, \gamma_0(0)} = 1$$

combining (3.17) with theorem (3.3) yields the corrollary.

Remark. We remark briefly on the case of diffusions evolving on a manifold (or, more specifically, diffusions with state dependent diffusion coefficients).

In that case, [6] have proved that

$$J(\phi) = \exp -[\frac{1}{2}\int_0^T \frac{|\dot{\phi}(t) - f(\phi(t)|^2 dt}{0} + \frac{1}{2}\int_0^T \text{div } f(\phi(t))dt + \frac{1}{12}\int_0^T R(\phi(t))dt]$$

where R(x) is the scalar curvature at the point x and the divergent is taken on the manifold, we refer to [6] for the definitions involved, and we point out where [6] used the assumption that $\phi(t)$ existed and how that can be avoided.

We recall some notations from [6]: a system of normal coordinates is defined around $\phi(t)$, and, in this system,

$$\sigma^{ij}(t,x) = \delta^{ij} + \frac{1}{3} R_{imlj}\,(t,0)x^m x^l + O(x^3) \qquad (3.18)$$

and we define the y process:

$$dy_t^i = \sum_{k=1}^{n} \sigma^{ik}(t,y_t)dw_t^k + \gamma^i(t,y_t)dt \tag{3.19}$$

where $|\gamma^i(t,y_t)| = O(y_t)$ and the exact form of γ is unimportant to our current needs. We recall again that, by [4, pg. 451], if $|q_s| = O(\varepsilon^2)$ under the conditioning $||w|| < \varepsilon$ is an adapted process, then

$$E(\exp c \int_0^T q_s dw_s | \; |w_s| < \varepsilon) \overset{\varepsilon \to 0}{\to} 1, \quad \forall c . \tag{3.20}$$

Referring now to the proof in [6], we note that the only place one needed the existence and boundedness of $\ddot{\phi}$ was while attempting to use theorem 2.2, pg. 442: using their notations, one has to compute

$$A \triangleq E(\exp c \int F_i(t) \, dw_s^i | \; ||y|| < \varepsilon \tag{3.21}$$

where $F(t) = f(t,0) - \dot{\phi}(t)$, and $f(t,0)$ is in C^1 w.r.t. t. Assuming also that $\dot{\phi}$ is in C^1, [6] used the following estimate:

$$\int_0^T F^*(t)dw_t = \int_0^T F^*(t)\sigma^{-1}(t,y_t)(dy_t - \gamma(t,y_t)dt)$$

$$= \int_0^T F^*(t)\sigma^{-1}(t,y_t)dy_t - \int_0^T F^*(t)\sigma^{-1}(t,y_t)\gamma(t,y_t)dt \tag{3.22}$$

since $|\gamma(t,y_t)| = 0(|y_t|) = 0(\varepsilon)$ under the conditioning, the contribution of the second integral is negligible. Considering the first integral, note that in the normal coordinates,

$$\sigma^{-1}(t,y_t) = I + O(y_t^2) \tag{3.23}$$

By (3.20), the contribution of the second term is again negligible, and therefore we are left with

$$E(\exp(c\int_0^T F^*(t)dy_t) \mid \|y\|<\varepsilon)$$

In the case that $F^*(t)$ is C^1 (which result from the assumption $\phi(t) \in C^1$), integration by parts yields the pathwise convergence (under the conditioning $\|y\|<\varepsilon$). In the general case, however, (3.21) reduces to show that

$$B \triangleq E(\exp(c\int_0^T \dot{\phi}^*(t)d\tilde{y}_t)\Big| \|\tilde{y}\| < \varepsilon) \xrightarrow{\varepsilon \to 0} 1, \ \forall c \tag{3.24}$$

where

$$d\tilde{y}_t = (I + c(\tilde{y}_t))\ dw_t \text{ and } c(\tilde{y}_t) = 0(\varepsilon^2)$$

The procedure which led to our estimates for the case $c \equiv 0$ can be repeated, where now the operator L includes in it's first order term an additional term of the form $k\ \varepsilon^2(\phi^{(\delta)})^2$, which turns out to be negligible. There is an even more direct way to see that, based on lemma (2.1) of [6] or again on a version of 3.20 (c.f. [6], pg. 449]; we ommit the details here.

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