Guaranteed Properties for Nonlinear Gain Scheduled Control Systems[†]

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Abstract

Gain scheduling has proven to be a successful design methodology in many engineering applications. However in the absence of a sound analysis, these designs come with no guarantees on the robustness, performance, or even nominal stability of the overall gain scheduled design. This paper presents such an analysis for two types of nonlinear gain scheduled control systems : (1) scheduling on a reference trajectory and (2) scheduling on the plant output. Conditions are given which guarantee stability, robustness, and performance properties of the global gain scheduled designs. These conditions confirm and formalize popular notions regarding gain scheduled designs, such as the scheduling variable should "vary slowly" and "capture the plant's nonlinearities." These results extend previous work by the authors which addressed the case of linear plants whose dynamics depend on exogenous parameters.

Section 1. Introduction

Gain scheduling is a popular engineering method used to design controllers for systems with widely varying nonlinear and/or parameter dependent dynamics. The idea is to select several operating points which cover the range of the plants dynamics. Then, at each of these points, the designer makes a linear time-invariant approximation to the plant and designs a linear compensator for each linearized plant. In between operating points, the parameters (gains) of the compensators are then interpolated, or scheduled, thus resulting in a global compensator.

Despite the lack of a sound theoretical analysis, gain scheduling [11] is a design methodology which is known to work in a myriad of operating control systems (e.g. jet engines , submarines , and aircraft). However in the absence of such an analysis, these designs come with no guarantees. More precisely, even though the local operating point designs may have excellent feedback properties, the global gain scheduled design need not have any of these properties (even nominal stability). In other words, one typically cannot assess *a priori* the guaranteed stability, robustness, and performance properties of gain scheduled designs. Rather, any such properties are inferred from extensive computer simulations.

In the place of a sound theoretical analysis, a collection of intuitive ideas have developed into heuristic guidelines for gain scheduled designs. For example, two common rules of thumb are "the scheduling variable should vary slowly" and "the scheduling variable should capture the plant's nonlinearities." Thus, a sound analysis would prove very useful in better understanding these designs.

In an earlier paper [10], such an analysis was performed for a special class of gain scheduled control systems, namely linear plants whose dynamics depend on exogenous parameters. In this paper, the results in [10] are extended to analyze two nonlinear gain scheduling situations : (1) a nonlinear plant scheduling on a reference trajectory and (2) a nonlinear plant scheduling on the plant output. In both cases, the analysis confirms and formalizes the popular notions regarding the design of gain scheduled control systems and enables one to give guarantees on the stability, robustness, and performance of gain scheduled designs. In this sense, the analysis can be used towards the ultimate goal to develop a complete and systematic gain scheduling design framework.

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The remainder of the paper is organized as follows. In Section 2, the notation and some mathematical preliminaries are given. Furthermore, a brief review of the results found in [10] regarding stability of linear time-varying Volterra integrodifferential equations is presented. Sections 3 and 4 discuss the two gain scheduling situations to be addressed. They are, respectively, (1) a nonlinear plant scheduling on a reference trajectory and (2) a nonlinear plant scheduling on the plant output. Conditions are given which guarantee stability and robustness, performance properties of the global gain scheduled designs. Finally, concluding remarks are given in Section 5.

For the sake of brevity, only sketches of proofs are given throughout.

Section 2. Background Material

A. *Notation and Mathematical Preliminaries*

$\mathcal{R}^+$ denotes the set $\{ t \in \mathcal{R} \mid t \geq 0\}$. $|\cdot|$ denotes both the vector norm on $\mathcal{R}^n$ and its induced matrix norm.

$\mathcal{D}f$ denotes the derivative of $f : \mathcal{R}^n \rightarrow \mathcal{R}^m$. $\mathcal{D}_i f$ denotes the derivative with respect to the i^{th} variable of

$$f : \mathcal{R}^{n1} \times \mathcal{R}^{n2} \times ... \times \mathcal{R}^{ni} \times ... \times \mathcal{R}^{nk} \rightarrow \mathcal{R}^m \tag{2-1}$$

Let $f : \mathcal{R} \rightarrow \mathcal{R}$. $\mathcal{D}^+ f$ denotes the Dini derivative of f defined by

$$\mathcal{D}^+ f(x_o) \equiv \limsup_{h \rightarrow 0^+} \frac{f(x_o + h) - f(x_o)}{h} \tag{2-2}$$

Let $f : \mathcal{R}^+ \rightarrow \mathcal{R}^n$. $\hat{f}$ denotes the Laplace transform of f. $\mathcal{P}_T$ denotes the standard truncation operator on f. $\mathcal{W}_{T,\sigma}$ denotes the truncation and exponential weighting operator on f defined by

$$(\mathcal{W}_{T,\sigma} f)(t) = \begin{cases} e^{-\sigma(T-t)} f(t), & t \leq T \\ 0, & t > T \end{cases} \tag{2-3}$$

$\mathcal{L}_p$ and $\mathcal{L}_{pe}$, $p \in [1, \infty]$, denote the standard Lebesgue and extended Lebesgue function spaces. Similarly, ℓ_p, $p \in [1, \infty]$, denote the appropriately summable sequence spaces. $\mathcal{B}$ denotes the set of functions such that

$$\| f \|_{\mathcal{B}} \equiv \sup_{t \in \mathcal{R}^+} | f(t) | < \infty \tag{2-4}$$

$\mathcal{B}_e$ denotes the set of functions such that $\mathcal{P}_T f \in \mathcal{B}$, $\forall T \in \mathcal{R}^+$.

$\mathcal{A}(\sigma)$ denotes the set whose elements are of the form

$$f(t) = \begin{cases} f_a(t) + \sum_{i=0}^{\infty} f_i\, \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2-5}$$

where $f_a : \mathcal{R}^+ \to \mathcal{R}$, $t_i \geq 0, f_i \in \mathcal{R}$, and

$$\|f\|_{\mathcal{A}(\sigma)} \equiv \int_0^\infty |f_a(t)\, e^{-\sigma t}|\, dt + \sum_{i=0}^{\infty} |f_i\, e^{-\sigma t_i}| < \infty \qquad (2\text{-}6)$$

For any two elements of $\mathcal{A}(\sigma)$, $f * g$ denotes the convolution of f and g. $\mathcal{A}^{nxm}(\sigma)$ denotes the set of n by m matrices whose elements are in $\mathcal{A}(\sigma)$. Let $\Delta \in \mathcal{A}^{nxm}(\sigma)$ and let $\Delta' \in \mathcal{R}^{nxm}$ as $\Delta'_{ij} = \|\Delta_{ij}\|_{\mathcal{A}(\sigma)}$.

Then define $\|\Delta\|_{\mathcal{A}(\sigma)} \equiv |\Delta'|$. Finally, $\hat{\mathcal{A}}(\sigma)$ and $\hat{\mathcal{A}}^{nxm}(\sigma)$ are defined as the set of Laplace transforms of elements of $\mathcal{A}(\sigma)$ and $\mathcal{A}^{nxm}(\sigma)$, respectively. For further details on $\mathcal{A}(\sigma)$ and $\hat{\mathcal{A}}(\sigma)$, see [2, 6].

B. Linear Volterra Integrodifferential Equations

This section presents background material for equations of the form

$$\dot{x}(t) = A(t)\, x(t) + \int_0^t B(t)\, \Delta(t-\tau)\, C(\tau)\, x(\tau)\, d\tau, \quad t \geq t_o \qquad (2\text{-}7)$$

with initial condition

$$\begin{cases} x(t) = \phi(t), & 0 \leq t \leq t_o,\ \phi \in \mathcal{B} \\ x(t_o^+) = \phi(t_o) \end{cases} \qquad (2\text{-}8)$$

where it is assumed that for some $\sigma \geq 0$, $\Delta \in \mathcal{A}(\sigma)$.

These equations are known as linear time-varying Volterra integrodifferential equations (VIDE's). VIDE's and their stability have been studied in [3, 4, 8, 10] and references contained in [5]. In this section, the assumptions on (2-7) are given, a definition of exponential stability is presented, and sufficient conditions for exponential stability are given in both cases of time-invariant or time-varying A, B, and C matrices. Further details may be found in [10].

The following assumption is made on (2-7) :

Assumption 2.1 A, B, and C are bounded and globally Lipschitz continuous. Thus, there exist constants $k_{A,B,C}$ and $L_{A,B,C} \geq 0$ such that $\forall\, t, \tau \in \mathcal{R}^+$

$$|A(t)| \leq k_A, \quad |A(t) - A(\tau)| \leq L_A |t - \tau| \qquad (2\text{-}9)$$

$$|B(t)| \leq k_B, \quad |B(t) - B(\tau)| \leq L_B |t - \tau| \qquad (2\text{-}10)$$

$$|C(t)| \leq k_C, \quad |C(t) - C(\tau)| \leq L_C |t - \tau| \qquad (2\text{-}11)$$

A definition of exponential stability is now presented.

Definition 2.1 The VIDE (2-7) with initial condition (2-8) is said to be *exponentially stable* if there exist constants m, λ, and $\beta > 0$ where $\beta \geq \lambda$ such that for $t \geq t_o$

$$|x(t)| \leq m e^{-\lambda(t - t_o)} \|\mathcal{W}_{t_o,\beta}\, \phi\|_{\mathcal{B}} \qquad (2\text{-}12)$$

It is stressed that the constants m, λ, and β are *independent* of the initial condition (ϕ, t_o). The convention $\beta \geq \lambda$ follows from the reasoning that solutions to (2-7) cannot decay faster than they are forgotten.

In the case of *time-invariant* A, B, and C matrices, one has the following condition for exponential stability.

Theorem 2.1[10] Consider the time-invariant VIDE

$$\dot{x}(t) = A\, x(t) + \int_0^t B\, \Delta(t-\tau)\, C\, x(\tau)\, d\tau, \quad t \geq t_o \qquad (2\text{-}13)$$

with initial condition

$$\begin{cases} x(t) = \phi(t), & 0 \leq t \leq t_o,\ \phi \in \mathcal{B} \\ x(t_o^+) = \phi(t_o) \end{cases} \qquad (2\text{-}14)$$

A sufficient condition for exponential stability is that there exist a constant $\beta > 0$ such that

$$s \mapsto (sI - A - B\, \hat{\Delta}(s)\, C)^{-1} \in \hat{\mathcal{A}}^{nxn}(-2\beta) \qquad (2\text{-}15)$$

$$\hat{\Delta} \in \hat{\mathcal{A}}(-2\beta) \qquad (2\text{-}16)$$

where the rate of decay is $\beta/2$.

Finally, the following theorem gives a sufficient condition for exponential stability of (2-7) in the case where (2-7) is exponentially stable for all *frozen-values* of time. This generalizes a standard result for ordinary differential equations (e.g. [6]).

Theorem 2.2[10] Consider the VIDE (2-7) with initial condition (2-8) under Assumption 2.1. Now define the following measure of the time-variations of (2-7) :

$$K \equiv L_A + L_B \|\Delta\|_{\mathcal{A}(-\beta)} k_C + k_B \|\Delta\|_{\mathcal{A}(-\beta)} L_C \qquad (2\text{-}17)$$

Finally, assume that there exists a constant $\beta > 0$ such that

$$s \mapsto (sI - A(\tau) - B(\tau)\, \hat{\Delta}(s)\, C(\tau))^{-1} \in \hat{\mathcal{A}}^{nxn}(-2\beta),\ \forall\, \tau \in \mathcal{R}^+ \qquad (2\text{-}18)$$

$$\hat{\Delta} \in \hat{\mathcal{A}}(-2\beta) \qquad (2\text{-}19)$$

Under these conditions, given any $\eta \in (0, \beta)$, (2-7) is exponentially stable with a rate of decay $\eta/2$ for sufficiently small K, or equivalently for sufficiently slow time-variations.

Note that (2-18)-(2-19) imply exponential stability for all *frozen-values* of time using Theorem 2.1.

Section 3. Scheduling on a Reference Trajectory

A. Problem Statement

Consider the block diagram of Fig. 3.1. This figure shows a standard unity feedback configuration in which the command trajectory, **r***, is generated by passing a reference control signal, **u***, through a model of the plant, $\mathbf{P}_m$. This may be the outcome of a nonlinear optimal control problem, or some other off-line design process. The control input, u, to the actual plant, P, then consists of the reference control, **u***, and a small perturbational control, δu. In the ideal situation of no modeling errors, disturbances, or other uncertainties, the perturbational control δu = 0, and perfect command tracking is achieved, i.e. y = r*.

Such perfect knowledge is rare, hence the need for feedback and compensator design. Now consider the block diagram of Fig. 3.2. This diagram represents the feedback system of Fig. 3.1 in the presence of three modeling errors : (1) Δ_y, unmodeled sensor dynamics, (2) Δ_u, unmodeled actuator dynamics, and (3) Δ_p, an artificial uncertainty which corresponds to a performance specification (see [7] for a detailed discussion on how various performance specifications can be put into the form of artificial uncertainties).

A gain scheduled approach to control design for Fig. 3.2 would be as follows. Let the plant model, $\mathbf{P}_m$, be given by

$$\dot{x}(t) = f(x(t)) + B\, u(t), \quad x(0) = x_o \in \mathcal{R}^n \qquad (3\text{-}1)$$

$$y(t) = C\, x(t) \qquad (3\text{-}2)$$

Equations (3-1)-(3-2) are quite general since many systems may be put into

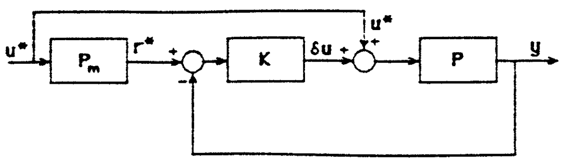

Figure 3.1 Scheduling on a Reference Trajectory

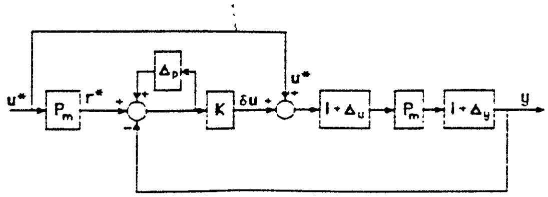

Figure 3.2 Scheduling in Presence of Robustness / Performance Uncertainties

the above form by selecting state variables as outputs and augmenting dynamics at the plant input. Applying the reference command input, u^*,

$$\dot{x}^*(t) = f(x^*(t)) + B\,u^*(t), \qquad x^*(0) = x_o^* \in \mathcal{R}^n \tag{3-3}$$

$$r^*(t) = y^*(t) = C\,x^*(t) \tag{3-4}$$

Now, define

$$\delta x(t) = x(t) - x^*(t) \tag{3-5}$$

$$\delta y(t) = y(t) - y^*(t) \tag{3-6}$$

$$\delta u(t) = u(t) - u^*(t) \tag{3-7}$$

Then, subtracting (3-3) from (3-1) and linearizing about $x^*(t)$,

$$\delta\dot{x}(t) = Df(x^*(t))\,\delta x(t) + B\,\delta u(t) + \delta f(t, \delta x(t)), \tag{3-8a}$$

$$\delta x(0) = x_o - x_o^* \in \mathcal{R}^n \tag{3-8b}$$

$$\delta y(t) = C\,\delta x(t) \tag{3-9}$$

where

$$\delta f(t, \delta x(t)) = f(x(t)) - \{\, f(x^*(t)) + Df(x^*(t))\,\delta x(t) \,\} \tag{3-10}$$

These equations may be decomposed into (1) a linear time-varying plant and (2) a nonlinear residual from the linearization. Let δP denote the *nonlinear time-varying* perturbational plant (3-8)-(3-10). Furthermore, let δP_τ denote the *linear frozen-time* plant

$$\delta\dot{x}(t) = Df(x^*(\tau))\,\delta x(t) + B\,\delta u(t), \quad \delta x(0) = x_o - x_o^* \in \mathcal{R}^n \tag{3-11}$$

$$\delta y(t) = C\,\delta x(t) \tag{3-12}$$

Then a gain scheduled approach would be to design a compensator for (3-11)-(3-12) so that for all *frozen-values* of time, the feedback system of Fig. 3.3 achieves robust stability and robust performance.

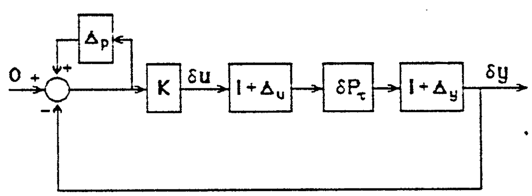

Figure 3.3 Diagram for Frozen-Time Compensator Design

Since the original plant model, δP, is nonlinear and time-varying, none of the desired feedback properties - including nominal stability - of the frozen-time designs may be present in the overall gain-scheduled system. In Section 3-B, conditions are given which *guarantee* the robust stability and robust performance of the global gain-scheduled design.

B. Stability, Robustness, and Performance Analysis

Suppose that one has carried out the gain scheduled design procedure outlined in Section 3-A. Then at each instant of time, one has designed a finite-dimensional compensator which stabilizes the feedback configuration of Fig. 3.3. Let the resulting *time-varying* compensator have the following state-space realization

$$\dot{x}_k(t) = A_k(t)\,x_k(t) + B_k(t)\,e(t) \tag{3-13}$$

$$\delta u(t) = C_k(t)\,x_k(t) \tag{3-14}$$

Using (3-8)-(3-10) along with (3-13)-(3-14), the feedback equations of Fig. 3.2 are given by

$$\begin{bmatrix} \delta\dot{x}(t) \\ \dot{x}_k(t) \end{bmatrix} = \begin{bmatrix} Df(x^*(t)) & B\,C_k(t) \\ -B_k(t)\,C & A_k(t) \end{bmatrix} \begin{bmatrix} \delta x(t) \\ x_k(t) \end{bmatrix} + \begin{bmatrix} \delta f(t, \delta x(t)) \\ 0 \end{bmatrix} + \begin{bmatrix} B\,(\Delta_u u^*)(t) \\ -B_k(t)\,(\Delta_{\tilde{py}} r^* + \Delta_y r^*)(t) \end{bmatrix} +$$

$$\int_0^t \begin{bmatrix} B & 0 \\ 0 & -B_k(t) \end{bmatrix} \begin{bmatrix} \Delta_u(t-\tau) & 0 \\ 0 & \Delta_{\tilde{py}}(t-\tau) + \Delta_{\tilde{p}}(t-\tau) + \Delta_y(t-\tau) \end{bmatrix} \begin{bmatrix} 0 & C_k(\tau) \\ C & 0 \end{bmatrix} \begin{bmatrix} \delta x(\tau) \\ x_k(\tau) \end{bmatrix} d\tau \tag{3-15}$$

where

$$\Delta_{\tilde{p}} = (I - \Delta_p)^{-1} * \Delta_p \tag{3-16}$$

$$\Delta_{\tilde{py}} = \Delta_{\tilde{p}} * \Delta_y \tag{3-17}$$

Rewriting (3-15),

$$\dot{z}(t) = A(t)\,z(t) + \int_0^t B(t)\,\Delta(t-\tau)\,C(\tau)\,z(\tau)\,d\tau + \delta F(t, z(t)) + d(t) \tag{3-18}$$

where A, B, C, δF, Δ, and d are defined in the obvious manner. Note that the feedback equations may be decomposed into (1) a linear time-varying VIDE, (2) a nonlinear residual of the linearization, and (3) an exogenous disturbance.

The stability of (3-18) will be shown as follows. Recall that the compensator (3-13)-(3-14) was designed so that the VIDE

$$\dot{z}(t) = A(t)\,z(t) + \int_0^t B(t)\,\Delta(t-\tau)\,C(\tau)\,z(\tau)\,d\tau \tag{3-19}$$

is stable for all *frozen* A, B, and C. Using results from Section 2-B, it is shown that (3-19) is exponentially stable for sufficiently slow time-variations. Given this time-varying exponential stability, a *Lyapunov functional* for (3-19) is constructed. This generalizes the concept of 'Converse Theorems of Lyapunov' for ordinary differential equations (e.g. [1, 9]). This Lyapunov functional is then used to give guaranteed stability margins for (3-18).

Step 1 Slowly Time-Varying Stability of (3-19)

Since (3-19) is precisely the class of equations addressed in Section 2-B, one can use Theorem 2.2 to guarantee stability for sufficiently slow time-variations as follows :

Assumption 3.1 The matrices A, B, and C satisfy the boundedness and Lipschitz continuity conditions of Assumption 2.1.

Assumption 3.2 There exists a constant $\beta > 0$ such that $\forall\, \tau \in \mathcal{R}^+$

$$s \mapsto (sI - A(\tau) - B(\tau)\,\hat{\Delta}(s)\,C(\tau))^{-1} \in \hat{\mathcal{A}}^{n \times n}(-2\beta) \tag{3-20}$$

$$\hat{\Delta} \in \hat{\mathcal{A}}(-2\beta) \tag{3-21}$$

The following theorem is a direct consequence of Theorem 2.2.

Theorem 3.1 Consider the linear time-varying VIDE (3-19) under Assumptions 3.1-3.2. Under these conditions, (3-19) is exponentially stable for sufficiently slow time-variations in A, B, and C.

In terms of the reference state-trajectory, x^*, this slowness condition on the dynamics of (3-19) states that x^* itself should vary slowly.

Step 2 Construction of a Lyapunov Functional

Assume now that one has satisfied Theorem 3.1 to guarantee the time-varying stability of (3-19). Let

$$\mathbf{s}(t\,;\,\phi,\,t_o) \qquad (3\text{-}22)$$

denote the solution to (3-19) with initial conditions (ϕ, t_o). From the definition of exponential stability, there exist constants m, λ, and β where $\beta \geq \lambda$ such that for any initial condition (ϕ, t_o)

$$|\,\mathbf{s}(t\,;\,\phi,\,t_o)\,| \leq m e^{-\lambda(t-t_o)} \|\, \mathcal{W}_{t_o,\beta}\, \phi \,\|_{\mathbf{B}} \qquad (3\text{-}23)$$

Theorem 3.2 Consider the linear time-varying VIDE (3-19). Suppose that (3-19) is exponentially stable and satisfies (3-23). Under these conditions, there exists a function $V : \mathbf{B}_e \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ which satisfies

$$\|\, \mathcal{W}_{t,\beta}\, \mathbf{x} \,\|_{\mathbf{B}} \leq V(\mathbf{x}, t) \leq m \,\|\, \mathcal{W}_{t,\beta}\, \mathbf{x} \,\|_{\mathbf{B}} \qquad (3\text{-}24)$$

$$|\, V(\mathbf{x}, t) - V(\mathbf{x}', t) \,| \leq m \,\|\, \mathcal{W}_{t,\beta}\, (\mathbf{x} - \mathbf{x}') \,\|_{\mathbf{B}} \qquad (3\text{-}25)$$

Furthermore, let $\tilde{V}_{(3\text{-}19)}$ denote V evaluated along trajectories of (3-19), i.e.

$$\tilde{V}_{(3\text{-}19)}(t) \equiv V(\mathbf{s}(t\,;\,\phi,\,t_o),\,t), \qquad t \geq t_o \qquad (3\text{-}26)$$

Then for some $\gamma \in (0, 1)$, V satisfies

$$D^+ \tilde{V}_{(3\text{-}19)}(t) \leq -\gamma\lambda \,\|\, \mathcal{W}_{t,\beta}\, \mathbf{s}(\cdot\,;\,\phi,\,t_o) \,\|_{\mathbf{B}}, \quad t \geq t_o \qquad (3\text{-}27)$$

Proof Let $\gamma \in (0, 1)$. Then define

$$V(\mathbf{x}, t) \equiv \sup_{\tau \geq t} \left\{ e^{\gamma\lambda(\tau - t)} \,\|\, \mathcal{W}_{\tau,\beta}\, \mathbf{s}(\cdot\,;\,\mathcal{P}_t\, \mathbf{x},\, t) \,\|_{\mathbf{B}} \right\} \qquad (3\text{-}28)$$

Then Theorem 3.2 follows using standard Lyapunov techniques (e.g. [9, 13]) generalized to linear VIDE's.

As mentioned earlier, Theorem 3.2 represents a type of 'converse theorem of Lyapunov' [1, 9]. It is noted that the existence of a function which satisfies (3-24)-(3-27) can be used to prove exponential stability of (3-19). Thus, Theorem 3.2 is also a statement of the equivalence of exponential stability and existence of Lyapunov functions. Finally, it is noted that Theorem 3.2 *does not* require that the exponential stability of (3-19) is due to slow time-variations.

Step 3 Stability of the Overall Gain Scheduled System

Recall that the feedback configuration of Fig. 3.2 leads to dynamics of the form

$$\dot{\mathbf{z}}(t) = A(t)\, \mathbf{z}(t) + \int_0^t B(t)\, \Delta(t - \tau)\, C(\tau)\, \mathbf{z}(\tau)\, d\tau + \delta\mathbf{F}(t, \mathbf{z}(t)) + \mathbf{d}(t) \quad (3\text{-}29)$$

In light of Steps 1 and 2, these equations may be viewed as perturbations ($\delta\mathbf{F}$ and $\mathbf{d}$) on an exponentially stable time-varying VIDE. Using the Lyapunov functional of Theorem 3.2, conditions will be placed on $\delta\mathbf{F}$ and $\mathbf{d}$ to guarantee the boundedness of solutions to (3-29).

First, the following assumption is made on $\delta\mathbf{F}$.

Assumption 3.3 There exists a constant $k_{\delta F} \geq 0$ such that

$$|\,\delta\mathbf{F}(t, \mathbf{z})\,| \leq k_{\delta F}\, |\,\mathbf{z}\,|^2, \qquad \forall\, t \in \mathcal{R}^+,\ \forall\, \mathbf{z} \in \mathcal{R}^n \qquad (3\text{-}30)$$

This quadratic bound reflects that $\delta\mathbf{F}$ is a residual from a linearization.

The stability of (3-29) is now addressed. Let $\mathbf{s}'(t\,;\,\phi,\,t_o)$ denote the solution to (3-29) with initial condition (ϕ, t_o).

Theorem 3.3 Consider the nonlinear VIDE (3-29). Let the linear time-varying VIDE (3-19) be exponentially stable. Let V be defined as in Theorem 3.2. Then given any $\gamma' \in (0, 1)$,

$$\|\, \mathcal{W}_{t_o,\beta}\, \phi \,\|_{\mathbf{B}} \leq \frac{\gamma\lambda}{m^2 k_{\delta F}}\, \gamma' \qquad (3\text{-}31)$$

$$\|\, \mathbf{d} \,\|_{L_\infty} \leq \frac{(\gamma\lambda)^2}{m^2 k_{\delta F}}\, (1 - \gamma')\gamma' \qquad (3\text{-}32)$$

together imply

$$|\,\mathbf{s}'(t\,;\,\phi,\,t_o)\,| \leq \frac{\gamma\lambda}{m^2 k_{\delta F}}\, \gamma', \qquad t \geq t_o \qquad (3\text{-}33)$$

Proof Let $\tilde{V}_{(3\text{-}29)}$ denote V evaluated along trajectories of (3-29):

$$\tilde{V}_{(3\text{-}29)}(t) \equiv V(\mathbf{s}'(t\,;\,\phi,\,t_o),\,t), \qquad t \geq t_o \qquad (3\text{-}34)$$

Then one can show that for $t \geq t_o$,

$$D^+ \tilde{V}_{(3\text{-}29)}(t) \leq -\gamma\lambda \,\|\, \mathcal{W}_{t,\beta}\, \mathbf{s}'(\cdot\,;\,\phi,\,t_o) \,\|_{\mathbf{B}} + m k_{\delta F}\, |\,\mathbf{s}'(\tau\,;\,\phi,\,t_o)\,|^2 + m \,\|\, \mathbf{d} \,\|_{L_\infty} \qquad (3\text{-}35)$$

The desired result then follows from appropriate manipulation of (3-35).

Theorem 3.3 can be interpreted as a type of small signal finite gain stability result [1, 12]. It states that provided the disturbance, $\mathbf{d}$, is sufficiently small, then the mapping $\mathbf{d} \mapsto \mathbf{s}'(\cdot\,;\,\phi,\,t_o)$ is finite-gain stable. However, recall the definition of $\mathbf{d}$

$$\mathbf{d}(t) \equiv \begin{bmatrix} B\,(\Delta_u \mathbf{u}^*)(t) \\ -\,\mathbf{B}_k(t)\,(\Delta_{\tilde{py}}\, \mathbf{r}^* + \Delta_y \mathbf{r}^*)(t) \end{bmatrix} \qquad (3\text{-}36)$$

Thus, the condition "$\mathbf{d}$ sufficiently small" essentially states the intuitive condition that the reference trajectories $\mathbf{u}^*$ and $\mathbf{r}^*$ should not excite the unmodeled actuator or sensor dynamics.

To summarize, it has been shown that a gain scheduled approach applied to the feedback system of Fig. 3.2 has guaranteed robustness and performance properties under the following conditions. First of all, it is required that the reference trajectory $\mathbf{x}^*$ is sufficiently slow. this comes as no suprise since the gain scheduled designs are based on *linear time-invariant* approximations to the plant. The restriction of slow variations simply states that such a frozen-time approximation should be accurate. Since the system is actually nonlinear, the internal stability is only local. As the nonlinearities approach zero (i.e. $k_{\delta F} \rightarrow 0$), one has that the internal stability approaches global internal stability. Again, the restricition that nonlinearities impose are reminders that the design plants are linear time-invariant. The nonlinearities place another restriction on feedback system, this time on the reference trajectories $\mathbf{u}^*$ and $\mathbf{r}^*$. Namely, from (3-36) it is required that these reference trajectories do not excite the unmodeled dynamics. For example, if the reference control trajectory, $\mathbf{u}^*$, has significant frequency content in the region of unmodeled actuator dynamics, then one cannot make demands on the resulting stability and performance of the closed loop gain-scheduled system. In fact, since the reference control trajectory is fedforward to the plant, it is unlikely that any control strategy can remedy this situation.

Section 4. Scheduling on the Plant Output

A. Problem Statement

Consider the plant model given by

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = f(y, z) + B\,u\ ,\quad y(t) \in \mathcal{R}^{m},\ z(t) \in \mathcal{R}^{n-m},\ u(t) \in \mathcal{R}^{m} \tag{4-1}$$

where the plant output, y, is explicitly a state variable. The following assumption is made on (4-1).

Assumption 4.1 $f : \mathcal{R}^m \times \mathcal{R}^{n-m} \to \mathcal{R}^m$ satisfies

$$f(0, 0) = 0 \tag{4-2}$$

Assumption 4.2 There exist unique continuously differentiable functions

$$u_{eq} : \mathcal{R}^m \to \mathcal{R}^m \tag{4-3}$$

$$z_{eq} : \mathcal{R}^m \to \mathcal{R}^{n-m} \tag{4-4}$$

which satisfy

$$0 = f(y, z_{eq}(y)) + B\,u_{eq}(y) \tag{4-5}$$

Assumption 4.2 essentially states that one has a family of equilibrium conditions parameterized by the output, y. In terms of gain scheduling, each of these equilibrium conditions is a possible "operating condition."

A gain scheduled approach to controlling (4-1) would be as follows. The plant linearized about a possible operating point, y_o, is given by

$$\frac{d}{dt}\begin{bmatrix} y - y_o \\ z - z_{eq}(y_o) \end{bmatrix} = Df(y_o, z_{eq}(y_o))\begin{bmatrix} y - y_o \\ z - z_{eq}(y_o) \end{bmatrix} + B\,(u - u_{eq}(y_o)) \tag{4-6}$$

Thus at each operating point, one would design a compensator based on a local linear time-invariant approximation (4-6). This procedure would result in a *family* of linear time-invariant compensators $\{A_k(y_o), B_k(y_o), C_k(y_o)\}$ parameterized by the operating condition y_o. These compensators are then used as in Fig. 4.1.

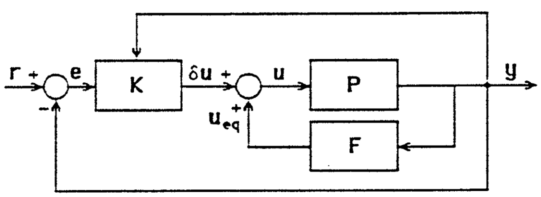

Figure 4.1 Scheduling on the Plant Output

In this figure, the current operating condition is *instantaneously* updated as the current plant output. Thus, the compensator dynamics would evolve as

$$\dot{x}_k(t) = A_k(y(t))\,x_k(t) + B_k(y(t))\,e(t) \tag{4-7}$$

$$\delta u(t) = C_k(y(t))\,x_k(t) \tag{4-8}$$

This procedure leads to feedback equations of the form

$$\frac{d}{dt}\begin{bmatrix} y \\ z - z_{eq}(y) \\ x_k \end{bmatrix} = \begin{bmatrix} 0 & D_2 f_y(y, z_{eq}(y)) & B_y C_k(y) \\ 0 & D_2 f_z(y, z_{eq}(y)) & B_z C_k(y) \\ -B_k(y) & 0 & A_k(y) \end{bmatrix}\begin{bmatrix} y \\ z - z_{eq}(y) \\ x_k \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ B_k(y) \end{bmatrix} r + \begin{bmatrix} \delta f_y(y, z) \\ \delta f_z(y, z) \\ 0 \end{bmatrix} - \frac{d}{dt}\begin{bmatrix} 0 \\ z_{eq}(y) \\ 0 \end{bmatrix} \tag{4-9}$$

where

$$\delta f(y, z) = f(y, z) - \Big(f(y, z_{eq}(y)) + D_2 f(y, z_{eq}(y))\,(z - z_{eq}(y)) \Big) \tag{4-10}$$

and the subscripts y and z denote decomposition of the matrix functions into their y and z components, respectively. Explicitely evaluating the time derivative of $z_{eq}(y)$, (4-9) takes the form

$$\frac{d}{dt}\begin{bmatrix} y \\ z - z_{eq}(y) \\ x_k \end{bmatrix} = \begin{bmatrix} \delta f_y(y, z) \\ \delta f_z(y, z) - Dz_{eq}(y)\,\delta f_y(y, z) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_k(y) \end{bmatrix} r(t) + \tag{4-11}$$

$$\begin{bmatrix} 0 & Df_y(y, z_{eq}(y)) & B_y C_k(y) \\ 0 & Df_z(y, z_{eq}(y)) - Dz_{eq}(y)\,Df_y(y, z_{eq}(y)) & B_z C_k(y) - Dz_{eq}(y)\,B_y C_k(y) \\ -B_k(y) & 0 & A_k(y) \end{bmatrix}\begin{bmatrix} y \\ z - z_{eq}(y) \\ x_k \end{bmatrix}$$

which represents the nominal feedback equations for Fig. 4.1. It is noted that the 'dynamics matrix' A(y) of (4-11) *differs* from the closed loop dynamics matrix which would occur from applying the compensator dynamics $\{A_k(y_o), B_k(y_o), C_k(y_o)\}$ to linearized plant dynamics (4-6).

In the following sections, conditions are given for the nominal stability and robust stability of Fig. 4.1.

B. Nominal Stability

The nominal gain scheduled feedback equations (4-11) of Fig. 4.1 may be put in the form

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v}(t) \end{bmatrix} = A(y(t))\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + \delta F(y, v) + B(y(t))\,r(t) \tag{4-12}$$

where A, B, δF, and v are defined in the obvious manner. Furthermore, define

$$x = \begin{bmatrix} y \\ v \end{bmatrix} \tag{4-13}$$

The following assumptions are made on (4-12)

Assumption 4.3 The matrix A is bounded with constant k_A and Lipschitz continuous with constant L_A.

Assumption 4.4 The constant eigenvalues of A(y) are uniformly bounded away from the closed complex RHP for all constant y.

Assumption 4.5 The linearization residual satisfies

$$|\,\delta F(y, v)\,| \le k_{\delta F}\,|\,v\,|^2 \tag{4-14}$$

It is important to note that in (4-10), the linearization is proportional only to $z - z_{eq}(y)$. This implies that if the dynamics of (4-1) are *linear* in z, then $k_{\delta F} \equiv 0$. In this sense, the notion of "the scheduling variable capturing the plant's nonlinearities" is precisely quantified.

Before examining the stability of (4-12), consider the unforced equations

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = A(y)\begin{bmatrix} y \\ v \end{bmatrix} \tag{4-15}$$

From Assumption 4.4, it follows that solutions of (4-15) will decay provided that the time-variations in (4-15) are sufficiently slow. This is quantified in the following theorem.

Theorem 4.1[10] Consider (4-15) under Assumptions 4.3-4.4. Under these conditions, there exist constants m, λ, and $\varepsilon > 0$ such that

$$|\,\dot{y}(t)\,| \le \varepsilon,\ \forall\, t \in [0, T] \Rightarrow |\,x(t)\,| \le m e^{-\lambda t}\,|\,x_o\,|,\ \forall\, t \in [0, T] \tag{4-16}$$

This is used in showing exponential stability of (4-15) as follows

Theorem 4.2 Consider (4-15) under Assumptions 4.3-4.4. Let L_y be such that

$$|\dot{y}| \le L_y |x| \tag{4-17}$$

Under these conditions,

$$|x_o| \le \rho \equiv \frac{\varepsilon}{m^{k_\lambda/\lambda} L_y} \tag{4-18}$$

implies

$$|x(t)| \le m e^{-\lambda t} |x_o|, \quad \forall t \in \mathcal{R}^+ \tag{4-19}$$

Proof Condition (4-18) implies that $|\dot{y}| \le \varepsilon$ for some interval T. During this interval, Theorem 4.1 assures that the state decays exponentially. If the interval T is sufficiently long, the state at time T is small enough so that $|\dot{y}| \le \varepsilon$ for all time, which in turn implies (4-19).

In Theorem 4.2, the notion of a slow scheduling variable plays an important role. Namely, the slower the time-variations of the output y (i.e. as $L_y \to 0$), then from (4-18) the larger the neighborhood for local exponential stability. Furthermore, the size of this neighborhood increases as the size of allowable time-variations increases (i.e. as $\varepsilon \to \infty$). This dependence is important since the parameter ε is a function of the frozen operating point closed loop designs.

The stability of the nominal dynamics (4-12) is now addressed. First note that (4-12) may be decomposed into (1) locally exponentially stable dynamics A(y), (2) a nonlinear residual δF, and (3) an exogenous input r. This is precisely the same sort of decomposition which existed of the Volterra equation (3-18) in Section 3-B. Thus, stability of (4-12) may be shown in the same manner; namely use the local exponential stability of (4-15) to construct a Lyapunov function and use this Lyapunov function to prove local exponential stability in the presence of δF and small-signal finite-gain stability in the presence of r. Unlike Section 3-B, the equations of interest are ordinary differential equations.

Step 1 Construction of Lyapunov Function

Let $s(t\,;x_o)$ denote the solution to (4-15) under initial conditions $x(0) = x_o$.

Assumption 4.6 There exists a constant L_{Ax} such that $\forall |x|, |x'| \le m\rho$

$$|A(y)x - A(y')x'| \le L_{Ax} |x - x'| \tag{4-20}$$

Theorem 4.3 Consider the locally exponentially stable system (4-15) under Assumptions 4.3-4.6. Under these conditions given any $\gamma \in (0, 1)$, there exists a Lyapunov function $V: \mathcal{R}^n \to \mathcal{R}^+$ such that

$$|x| \le V(x) \le m|x|, \quad \forall |x| \le \rho \tag{4-21}$$

$$|V(x) - V(x')| \le L_V |x - x'|, \quad \forall |x|, |x'| \le \rho \tag{4-22}$$

where

$$L_V \equiv e^{(\gamma\lambda + L_{Ax})T} \tag{4-23}$$

$$T = \frac{\ln m}{(1-\gamma)\lambda} \tag{4-24}$$

Furthermore, let $\tilde{V}_{(4\text{-}15)}$ denote V evaluated along $s(t\,;x_o)$. Then

$$D^+\tilde{V}_{(4\text{-}15)}(t) \le -\gamma\lambda \tilde{V}_{(4\text{-}15)}(t), \quad \forall |x_o| \le \rho \tag{4-25}$$

Proof Define V as

$$V(x) \equiv \sup_{t \ge 0} e^{\gamma\lambda t} |s(t\,;x)| \tag{4-26}$$

The theorem then follows from arguments found in [1, 9].

Step 2 Nominal Stability

Let $s'(t\,;x_o)$ denote the solution to (4-12) under initial conditions $x(0) = x_o$.

Theorem 4.4 Consider the nominal gain scheduled feedback equations (4-12) under Assumptions 4.3-4.6. Let V be defined as in Theorem 4.3. Let γ' be such that

$$\frac{\gamma\lambda}{L_V k_{\delta F}} \gamma' < \rho \tag{4-27}$$

for any $\gamma' \in (0, 1)$. Under these conditions,

$$|x_o| \le \frac{\gamma\lambda}{m L_V k_{\delta F}} \gamma' \tag{4-28}$$

$$\| B(y) r \|_{L_\infty} \le \frac{(\gamma\lambda)^2}{L_V^2 k_{\delta F}} (1-\gamma')\gamma' \tag{4-29}$$

together imply

$$|s'(t\,;x_o)| \le \frac{\gamma\lambda}{m L_V k_{\delta F}} \gamma', \quad t \ge 0 \tag{4-30}$$

Proof Let $\tilde{V}_{(4\text{-}12)}$ denote V evaluated along $s'(t\,;x_o)$. Then

$$D^+\tilde{V}_{(4\text{-}12)}(t) \le -\gamma\lambda\tilde{V}_{(4\text{-}12)}(t) + L_V k_{\delta F} \tilde{V}^2_{(4\text{-}12)}(t) + L_V \| B(y) r \|_{L_\infty}, \quad \text{for } \tilde{V}_{(4\text{-}12)}(t) < \rho \tag{4-31}$$

Theorem 4.4 then follows from arguments found in [1, 9].

C. Robustness to Plant Input Unmodeled Dynamics

Consider the block diagram of Fig. 4.2. This represents the nonlinear gain scheduled system of Fig. 4.1 in the presence of input unmodeled dynamics. In this gain scheduling framework, such unmodeled dynamics not only limit the bandwidth but also destroy the linearization through u_{eq}. In this section, it is outlined how one can guarantee that the robust stability of the frozen operating condition designs carries over to the full gain scheduled system.

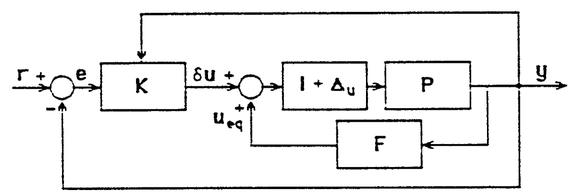

Figure 4.2 Unmodeled Dynamics at the Plant Input

The feedback equations of Fig. 4.2 are given by

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = A(y) \begin{bmatrix} y \\ v \end{bmatrix} + \delta F(y, v) + B(y) r + \int_0^t \begin{bmatrix} B_y \\ B_z - Dz_{eq}(y(t))B_y \\ 0 \end{bmatrix} \Delta_u(t-\tau)\Big(C_k(y(\tau))\, x_k(\tau) + u_{eq}(y(\tau)) \Big) d\tau \tag{4-32}$$

or equivalently

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = A(y) \begin{bmatrix} y \\ v \end{bmatrix} + \delta F(y, v) + B(y)\, r + (gy)(t) + \tag{4-33}$$

$$\int_0^t \begin{bmatrix} B_y \\ \begin{bmatrix} B_z - Dz_{eq}(y(t))B_y \\ 0 \end{bmatrix} \end{bmatrix} \Delta_u(t-\tau) \begin{bmatrix} 0 & \begin{bmatrix} 0 & C_k(y(\tau)) \end{bmatrix} \end{bmatrix} \begin{bmatrix} y(\tau) \\ v(\tau) \end{bmatrix} d\tau$$

where g is defined as

$$(gy)(t) \equiv \int_0^t \begin{bmatrix} B_y \\ \begin{bmatrix} B_z - Dz_{eq}(y(t))B_y \\ 0 \end{bmatrix} \end{bmatrix} \Delta_u(t-\tau)\, u_{eq}(y(\tau))\, d\tau \tag{4-34}$$

Thus, it is seen that (4-33) may be decomposed into (1) an output-varying VIDE, (2) an exogenous input r, (3) a linearization residual δF, and (4) a nonlinear perturbation g due to the feedback of the equilibrium control, u_{eq}.

An *outline* to proving stability of (4-33) now follows. The procedure essentially parallels that of Section 4-B with the exception that Section 4-B deals with ordinary differential equations.

First, since the output-varying VIDE, is a product of a gain-scheduled design

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = A(y) \begin{bmatrix} y \\ v \end{bmatrix} + \int_0^t \begin{bmatrix} B_y \\ \begin{bmatrix} B_z \\ 0 \end{bmatrix} \end{bmatrix} \Delta_u(t-\tau) \begin{bmatrix} 0 & \begin{bmatrix} 0 & C_k(y(\tau)) \end{bmatrix} \end{bmatrix} \begin{bmatrix} y(\tau) \\ v(\tau) \end{bmatrix} d\tau \tag{4-35}$$

is guaranteed to be a stable VIDE for all *frozen* values of the output. Using Theorem 2.2, it follows that if $|\dot{y}|$ is sufficiently small over some interval T then the state decays exponentially during this interval (*cf.* Theorem 4.1). Thus, one can place restrictions on the initial function of (4-35) to guarantee local exponential stability (*cf.* Theorem 4.2). Given the local exponential stability of (4-35), one can construct a Lyapunov functional using solutions of (4-35), (*cf.* Theorem 3.2 and Theorem 4.3). One can then use this Lyapunov functional prove the local exponential stability and small-signal finite-gain stability of (4-33) (*cf.* Theorem 3.3 and Theorem 4.4). It is important to note that as in the case of nominal stability, the ideas of "a slow scheduling variable" and "capturing the nonlinearities" explicitely affect the resulting stability conditions.

Section 5. Concluding Remarks

This paper has presented a formal analysis of two types of nonlinear gain-scheduled control systems : (1) scheduling on a reference trajectory and (2) scheduling on the plant output. In both cases, conditions were given which guarantee that certain stability, robustness, and performance properties of the frozen operating condition designs carry over to the overall gain scheduled design.

The main results may be summarized as follows. In the case of scheduling on a reference trajectory, given that the feedback system (3-15) is stable for all *frozen* values of time, then robust stability and robust performance is maintained provided that (1) the reference trajectory varies slowly and (2) the reference trajectory does not excite unmodelled dynamics. In the case of scheduling on the plant output, conditions were given which essentially verify and formalize the two standard gain scheduling guidelines of "scheduling on a slow variable" and "capturing the plant's nonlinearities." That is, in the limiting cases where the rate of the output time-variations approaches zero and the non-output nonlinearities approach zero, then the feedback properties of the gain scheduled designs approach those of the frozen-time designs.

Finally, the methods used here generalize the concepts of Lyapunov stability / exponential stability equivalence [1] and small-signal finite-gain stabilty [1, 12] to Volterra integrodifferential equations.

The main limitations of these results are

(1) In the theorems dealing with infinite-dimensional unmodeled dynamics, verification of the sufficient conditions requires hard-to-obtain information on the uncertainties, such as the exponentially weighted input / output norm $\|\Delta\|_{\mathcal{A}(-\beta)}$ in (2-17).

(2) Even if the sufficient conditions are verified, they are apt to be conservative, which is typical of Lyapunov analyses of nonlinear systems. However, the conservatism of the stability conditions is simply a reminder that the original gain scheduled designs were based on *linear time-invariant approximations* to the nonlinear plant. If these approximations are inaccurate, then one should not demand guarantees on the overall gain scheduled system.

In spite of these limitations, the theorems are useful in that they help to identify various parameters which in turn improve the feedback properties of the gain scheduled design. That is although the sufficient conditions may not be explicitely verified, one can still use them to gain new insights for design purposes.

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