Computer Science and Artificial Intelligence Laboratory

Technical Report

MIT-CSAIL-TR-2005-007
AIM-2005-003

February 2, 2005

Functional Differential Geometry

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Abstract

Differential geometry is deceptively simple. It is surprisingly easy to get the right answer with unclear and informal symbol manipulation. To address this problem we use computer programs to communicate a precise understanding of the computations in differential geometry. Expressing the methods of differential geometry in a computer language forces them to be unambiguous and computationally effective. The task of formulating a method as a computer-executable program and debugging that program is a powerful exercise in the learning process. Also, once formalized procedurally, a mathematical idea becomes a tool that can be used directly to compute results.

A *manifold* is a generalization of our idea of a smooth surface embedded in Euclidean space. The critical feature of an n-dimensional manifold is that locally (near any point) it looks like n-dimensional Euclidean space: around every point there is a simply-connected open set, the *coordinate patch*, and a bijective continuous function, the *coordinate function* or *chart* mapping every point in that open set to a tuple of n real numbers, the *coordinates*. In general, several coordinate systems are needed to label all points on a manifold. It is required that if a region is in more than one coordinate patch then the coordinates are consistent in that the function mapping one set of coordinates to another is continuous (and perhaps differentiable to some degree). A consistent system of coordinate patches and coordinate functions that covers the entire manifold is called an *atlas*.[1]

An example of a two-dimensional manifold is the surface of a sphere or of a coffee cup. The space of all configurations of a planar double pendulum is a more abstract example of a two-dimensional manifold. A manifold that looks locally Euclidean may not look like Euclidean space globally: it may not be simply connected. The surface of the coffee cup is not simply connected.

An example of a coordinate function is the function that maps points in a simply-connected open neighborhood of a sphere to the tuple of latitude and longitude.[2] If we want to talk about motion on the Earth, we can identify the space of configurations to a 2-sphere. The map from the 2-sphere to the three-dimensional coordinates of a point on the surface of the Earth captures the shape of the Earth.

Two angles specify the configuration of the planar double pendulum. The manifold of configurations is a torus, and each angle requires two coordinate patches to cover it (because angles wrap around). Each point on the torus corresponds to a configuration of the double pendulum, specifying the position of each constituent in space. The constraints, such as the lengths of the pendulum

[1] We may impose a non-Euclidean metric on the coordinates. For example, the coordinates used in relativity have a metric structure that does not obey the requirement that distinct points have non-zero distance between them.

[2] The open set for a latitude-longitude coordinate system may not include either pole (because longitude is not defined at the poles) or on the 180° meridian (where the longitude is discontinuous). Other coordinate systems are needed for these places.

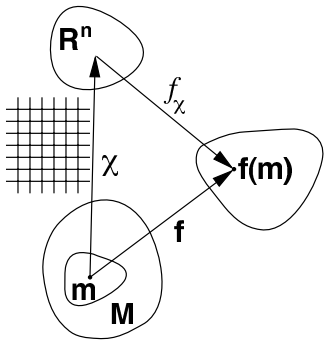

Figure 0.1 The coordinate function maps points on the manifold in the coordinate patch to a tuple of coordinates. A function f on the manifold M can be represented in coordinates by $f_\chi = \mathsf{f} \circ \chi^{-1}$.

rods, are built into the map between the torus and the arrangements of masses in space.

Coordinate Functions

A coordinate function χ maps points in a coordinate patch of a manifold to a coordinate tuple:[3]

$$x = \chi(\mathsf{m}), \tag{0.1}$$

where x may have a convenient tuple structure. The coordinates are arranged as an "up structure"; the coordinates are selected with superscripts:

$$x^i = \chi^i(\mathsf{m}). \tag{0.2}$$

The number of independent components of x is the dimension of the manifold.

Assume we have two coordinate functions χ and χ'. The coordinate transformation from χ' coordinates to χ coordinates is just the composition $\chi \circ \chi'^{-1}$, where χ'^{-1} is the functional inverse of

[3] In the text that follows we will use sans-serif names, such as f, v, m, to refer to objects defined on the manifold. Objects that are defined on coordinates (tuples of real numbers) will be named with symbols like f, v, x.

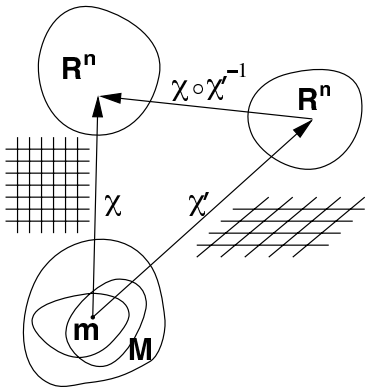

Figure 0.2 Here there are two overlapping coordinate patches that are the domains of the two coordinate functions χ and χ'. It is possible to represent manifold points in the overlap using either coordinate system. The coordinate transformation from χ' coordinates to χ coordinates is just the composition $\chi \circ \chi'^{-1}$.

χ' (see figure 0.2). We assume that the coordinate transformation is continuous and differentiable to any degree we require.

For example, we can have both rectangular and polar coordinates on a plane, and

```
(define R2 (rectangular 2))
(define P2 (polar/cylindrical 2))
```

For each of these coordinate systems we obtain the coordinate functions and their inverses.

```
(define R2-chi (R2 '->coords))
(define R2-chi-inverse (R2 '->point))
(define P2-chi (P2 '->coords))
(define P2-chi-inverse (P2 '->point))
```

The coordinate transformations are then just compositions:

```
(print-expression
  ((compose P2-chi R2-chi-inverse)
   (up 'x0 'y0)))
(up (sqrt (+ (expt x0 2) (expt y0 2))) (atan y0 x0))

(print-expression
  ((compose R2-chi P2-chi-inverse)
   (up 'r0 'theta0)))
(up (* r0 (cos theta0)) (* r0 (sin theta0)))
```

And we can obtain the Jacobian of the transformation by taking its derivative[4]

```
(print-expression
  ((D (compose R2-chi P2-chi-inverse))
   (up 'r0 'theta0)))
(down (up (cos theta0) (sin theta0))
      (up (* -1 r0 (sin theta0)) (* r0 (cos theta0))))
```

Manifold Functions:
Let f be a function on a manifold M. This function has a coordinate representation f_χ with respect to the coordinate function χ:

$$f_\chi = \mathsf{f} \circ \chi^{-1}. \tag{0.3}$$

The value of $f_\chi(x)$ is independent of coordinates:

$$f_\chi(x) = (\mathsf{f} \circ \chi^{-1})(\chi(\mathsf{m})) = \mathsf{f}(\mathsf{m}). \tag{0.4}$$

The subscript χ may be dropped when it is unambiguous.

The manifold point m is represented in rectangular coordinates by a pair of real numbers,

$$(x, y) = \chi(\mathsf{m}), \tag{0.5}$$

and the manifold function f is represented in rectangular coordinates by f that takes a pair of real numbers and produces a real number

$$f : \mathsf{R}^2 \to \mathsf{R}$$

[4]See *SICM*, Appendix 8, for an introduction to tuple arithmetic and a discussion of derivatives of functions with structured input or output.

$$f : (x, y) \mapsto f(x, y). \tag{0.6}$$

We define our manifold function

$$\mathsf{f} : \mathsf{M} \to \mathsf{R}$$
$$\mathsf{f} : \mathsf{m} \mapsto (f \circ \chi)(\mathsf{m}). \tag{0.7}$$

We can illustrate the coordinate independence with a program. We will show that an arbitrary manifold function `f`, when defined by its coordinate representation in rectangular coordinates, has the same behavior when applied to a manifold point independent of whether the point is specified in rectangular or polar coordinates.

First define a signature for functions that map an up structure of two reals to a real, then define a manifold function by specifying its behavior in rectangular coordinates:[5]

```
(define R2->R (-> (UP Real Real) Real))

(define f
  (compose (literal-function 'f-rect R2->R)
           R2-chi))
```

A typical point can be specified using each coordinate system:

```
(define R2-point (R2-chi-inverse (up 'x0 'y0)))
(define P2-point (P2-chi-inverse (up 'r0 'theta0)))
```

The definition of the function `f` is independent of the coordinate system used to specify the point of evaluation:

```
(print-expression (f R2-point))
(f-rect (up x0 y0))

(print-expression (f P2-point))
(f-rect (up (* r0 (cos theta0)) (* r0 (sin theta0))))
```

[5] Alternatively, we can define the same function in a shorthand

```
(define f (literal-manifold-function 'f-rect R2))
```

And, similarly, a function defined in terms of its behavior in polar coordinates makes sense when applied to a point specified in rectangular coordinates.

```
(define g (literal-manifold-function 'g-polar P2))

(print-expression (g P2-point))
(g-polar (up r0 theta0))

(print-expression (g R2-point))
(g-polar
  (up (sqrt (+ (expt x0 2) (expt y0 2)))
      (atan y0 x0)))
```

To make things a bit easier, we can give names to the individual coordinate functions associated with a coordinate system. Here we name the coordinate functions for the `R2` coordinate system `x` and `y` and for the `P2` coordinate system `r` and `theta`.

```
(instantiate-coordinates R2 '(x y))
(instantiate-coordinates P2 '(r theta))
```

This allows us to extract the coordinates from a point, independent of the coordinate system used to specify the point.

```
(print-expression (x R2-point))
x0

(print-expression (x P2-point))
(* r0 (cos theta0))

(print-expression (r P2-point))
r0

(print-expression (r R2-point))
(sqrt (+ (expt x0 2) (expt y0 2)))

(print-expression (theta R2-point))
(atan y0 x0)
```

We can work with the coordinate functions in a natural manner, defining new manifold functions in terms of them:[6]

[6]This is actually a nasty, but traditional, abuse of notation. The problem is that $\cos r$ really means $\cos \circ r$, or in Scheme notation `(square r)` really means `(compose square r)`. Both have been made to work in our system. This

```
(define h (+ (* x (square r)) (cube y)))

(print-expression (h P2-point))
(+ (* (expt r0 3) (expt (sin theta0) 3))
   (* (expt r0 3) (cos theta0)))

(print-expression (h R2-point))
(+ (expt x0 3)
   (* x0 (expt y0 2))
   (expt y0 3))
```

We can't take the derivative of a manifold function; we can only take derivatives of functions defined with real number inputs and outputs. But we can take a real valued function on a manifold and compose it with the inverse of a coordinate function to obtain a function we can differentiate:

```
(print-expression
 ((D (compose h R2-chi-inverse)) R2-point))
(down (+ (* 3 (expt x0 2)) (expt y0 2))
      (+ (* 2 x0 y0) (* 3 (expt y0 2))))
```

Vector Fields

A *vector field* on a manifold is an assignment of a vector to each point on the manifold. Probably you were taught that a vector is an arrow with a length and a direction. But we will think about vectors in a different way.

A function on a manifold takes on various values for different points on the manifold. A vector field takes directional derivatives of manifold functions: it measures how each function varies as the argument of the function is moved in the direction specified by the vector field. Thus a vector field is an operator on functions.

Let m be a point on a manifold, v be a vector field on the manifold, and f be a real-valued function on the manifold. Then v(f)(m) is the directional derivative of the function f at the point m

---

usually does not cause trouble, but it complicates the understanding of the derivatives, because there is a functional dependency that is implicit rather than explicit.

in the direction specified by v.[7] Note that there is no mention here of any coordinate system. The vector field specifies a direction and length at each manifold point that is independent of how it is described using any coordinate system.

The direction can be specified with respect to a coordinate system, with coordinate function χ, by giving a coefficient function $b_{\chi,\mathsf{v}}$ that maps the coordinates of a point to a tuple of coefficients corresponding to the direction in which the derivative is to be taken. The vector field is then

$$\begin{aligned}\mathsf{v}(\mathsf{f})(\mathsf{m}) &= ((D(\mathsf{f}\circ\chi^{-1})\ b_{\chi,\mathsf{v}})\circ\chi)(\mathsf{m}) && (0.8)\\ &= D(\mathsf{f}\circ\chi^{-1})(\chi(\mathsf{m}))\ b_{\chi,\mathsf{v}}(\chi(\mathsf{m}))\\ &= \sum_i \partial_i(\mathsf{f}\circ\chi^{-1})(\chi(\mathsf{m}))\ b^i_{\chi,\mathsf{v}}(\chi(\mathsf{m})). && (0.9)\end{aligned}$$

We implement the definition of a vector field (0.8) as the following Scheme procedure

```
(define (vector-field-procedure components coordinate-system)
  (define (the-procedure f)
    (compose (* (D (compose f (coordinate-system '->point)))
                components)
             (coordinate-system '->coords)))
  the-procedure)
```

The vector field is an operator, like derivative, made as follows:[8]

```
(define (components->vector-field components coordinate-system)
  (procedure->vector-field
   (vector-field-procedure components coordinate-system)))
```

Given a vector field v and a chart χ we can find the coefficient function $b_{\chi,\mathsf{v}}$ by applying the vector field to the chart:

$$\begin{aligned}\mathsf{v}(\chi)(\chi^{-1}(x)) &= D(\chi\circ\chi^{-1})(x)\ b_{\chi,\mathsf{v}}(x)\\ &= D(I)(x)\ b_{\chi,\mathsf{v}}(x)\end{aligned}$$

[7] The vector field is a function that takes a real-valued manifold function and a manifold point and produces a number. The order of arguments is chosen to make $\mathsf{v}(\mathsf{f})$ be a new manifold function that can be manipulated further.

[8] An operator is just like a procedure except that multiplication is interpreted as composition. For example, the derivative procedure is made into an operator `D` so that we can say `(expt D 2)` and expect it to compute the second derivative.

$$= b_{\chi,\mathsf{v}}(x) \tag{0.10}$$

So we can use a vector field v to measure the rate of change of the coordinates χ in the direction specified by v at each point m on the manifold. The result is a function $b_{\chi,\mathsf{v}}$ of the coordinates of the point in the manifold. Each component of $b_{\chi,\mathsf{v}}$ specifies the rate of change of a coordinate in the direction and rate specified by v.

Because the coordinates are an up structure, the derivative makes a down structure, and the coefficient tuple $b_{\chi,\mathsf{v}}(x)$ is an up structure compatible for addition to the coordinates. Note that for any vector field v the coefficients $b_{\chi,\mathsf{v}}(x)$ are different for different coordinate functions χ. In the text that follows we will usually drop the subscripts on b, understanding that it is dependent on the coordinate system and the vector field.

Given a coordinate system and coefficient functions that map coordinates to real values, we can make a vector field. For example, a general vector field can be defined by giving components relative to the coordinate system R2 by[9]

```
(define v
  (components->vector-field
   (up (literal-function 'vx (-> (UP Real Real) Real))
       (literal-function 'vy (-> (UP Real Real) Real)))
   R2))
```

When this vector field is applied to an arbitrary manifold function we see that it gives the directional derivative of that manifold function in the direction specified by the components vx and vy.

```
(print-expression
  ((v (literal-manifold-function 'f R2)) R2-point))
(+ (* (((partial 0) f) (up x0 y0)) (vx (up x0 y0)))
   (* (((partial 1) f) (up x0 y0)) (vy (up x0 y0))))
```

[9]To make it convenient to define literal vector fields we provide a shorthand:

```
(define w (literal-vector-field 'v R2))
```

This makes a vector field with component functions named v^0 and v^1 and names the result w.

The vector field v has a coordinate representation v:

$$\begin{aligned}\mathsf{v}(\mathsf{f})(\mathsf{m}) &= D(\mathsf{f}\circ\chi^{-1})(\chi(\mathsf{m}))\ b(\chi(\mathsf{m})) \\ &= Df(x)\ b(x) \\ &= v(f)(x),\end{aligned} \tag{0.11}$$

with the definitions $f = \mathsf{f}\circ\chi^{-1}$ and $x = \chi(\mathsf{m})$. The function b is the coefficient function for the vector field v. It provides a scale factor for the component in each coordinate direction. However, v is the coordinate representation of the vector field f in that it takes directional derivatives of coordinate representations of manifold functions.

Given a vector field v and a coordinate system `coordsys` we can construct the coordinate representation of the vector field.[10]

```
(define (coordinatize vector-field coordsys)
  (define ((v f) x)
    (let ((b (compose (vector-field (coordsys '->coords))
                      (coordsys '->point))))
      (* ((D f) x) (b x)))))
  (make-operator v))
```

We can apply a coordinatized vector field to a function of coordinates to get the same answer as before.

```
(print-expression
 (((coordinatize v R2)
   (literal-function 'f (-> (UP Real Real) Real)))
  (up 'x0 'y0)))
(+ (* (((partial 0) f) (up x0 y0)) (vx (up x0 y0)))
   (* (((partial 1) f) (up x0 y0)) (vy (up x0 y0))))
```

For the manifold $\mathbf{R}^n$ we can let $\chi = I$. Then $x = \mathsf{m}$, $f = \mathsf{f}$, and

$$v(f)(x) = Df(x)\ b(x), \tag{0.12}$$

which is the usual directional derivative.

Vector Field Properties:
The vector fields on a manifold form a vector space over the reals, with the sum and scalar multiplication defined as follows. Let u

[10]The `make-operator` procedure takes a procedure and returns an operator.

and v be vector fields and let α be a real number. Then

$$\begin{aligned}(\mathsf{u}+\mathsf{v})(\mathsf{f}) &= \mathsf{u}(\mathsf{f})+\mathsf{v}(\mathsf{f})\\ (\alpha\mathsf{u})(\mathsf{f}) &= \alpha(\mathsf{u}(\mathsf{f})).\end{aligned} \tag{0.13}$$

Vector fields have the following properties. Assume f and g are functions on the manifold, a and b are real constants, and F is a function on the range of f.

$$\mathsf{v}(a\mathsf{f}+b\mathsf{g})(\mathsf{m}) = a\mathsf{v}(\mathsf{f})(\mathsf{m})+b\mathsf{v}(\mathsf{g})(\mathsf{m})$$

$$\mathsf{v}(a\mathsf{f})(\mathsf{m}) = a\mathsf{v}(\mathsf{f})(\mathsf{m})$$

$$\mathsf{v}(\mathsf{f}\mathsf{g})(\mathsf{m}) = \mathsf{v}(\mathsf{f})(\mathsf{m})\,\mathsf{g}(\mathsf{m})+\mathsf{f}(\mathsf{m})\,\mathsf{v}(\mathsf{g})(\mathsf{m})$$

$$\mathsf{v}(F\circ\mathsf{f})(\mathsf{m}) = DF(\mathsf{f}(\mathsf{m}))\,\mathsf{v}(\mathsf{f})(\mathsf{m})$$

$$(\mathsf{f}\mathsf{v})(\mathsf{g})(\mathsf{m}) = \mathsf{f}(\mathsf{m})\,\mathsf{v}(\mathsf{g})(\mathsf{m})$$

Coordinate Basis Vector Fields

For an n-dimensional manifold any set of n linearly independent vector fields form a *basis* in that any vector field can be expressed as a linear combination of the basis fields. Given a coordinate system we can construct a basis as follows: we choose the component tuple $b_i(x)$ to be the ith unit tuple $u_i(x)$—an up tuple with one in the ith position and zeroes in all other positions—selecting the partial derivative in that direction. We then define the basis vector field X_i by

$$\begin{aligned}\mathsf{X}_i(\mathsf{f})(\mathsf{m}) &= D(\mathsf{f}\circ\chi^{-1})(\chi(\mathsf{m}))\ u_i(\chi(m))\\ &= \partial_i(\mathsf{f}\circ\chi^{-1})(\chi(\mathsf{m})).\end{aligned} \tag{0.14}$$

In terms of X_i the vector field of equation (0.9) is

$$\mathsf{v}(\mathsf{f})(\mathsf{m}) = \sum_i \mathsf{X}_i(\mathsf{f})(\mathsf{m})\ b^i(\chi(\mathsf{m})). \tag{0.15}$$

We can also write

$$\mathsf{v}(\mathsf{f})(\mathsf{m}) = \mathsf{X}(\mathsf{f})(\mathsf{m})\ b(\chi(\mathsf{m})), \tag{0.16}$$

letting the tuple algebra do its job.

The basis vector field is often written

$$\frac{\partial}{\partial x^i} = \mathsf{X}_i, \tag{0.17}$$

to call to mind that it is an operator that computes the directional derivative in the ith coordinate direction.

In addition to making the coordinate functions the procedure `instantiate-coordinates` also makes the traditional named basis vectors:

```
(instantiate-coordinates R2 '(x y))
(instantiate-coordinates P2 '(r theta))
```

Using these we can examine the application of a rectangular basis vector to a polar coordinate function:

```
(print-expression ((d/dx (square r)) R2-point))
(* 2 x0)
```

More general functions and vectors can be made as combinations of these simple pieces:

```
(print-expression
  (((+ d/dx (* 2 d/dy)) (+ (square r) (* 3 x))) R2-point))
(+ 3 (* 2 x0) (* 4 y0))
```

Coordinate transformations:
Consider a coordinate change $\chi = \phi \circ \chi'$. So if $x = \chi(\mathsf{m})$, $x' = \chi'(\mathsf{m})$, then $x = \phi(x')$, and

$$\begin{aligned}
\mathsf{X}(\mathsf{f})(\mathsf{m}) &= D(\mathsf{f} \circ \chi^{-1})(\chi(\mathsf{m})) \\
&= D(\mathsf{f} \circ (\chi')^{-1} \circ \phi^{-1})((\phi \circ \chi')(\mathsf{m})) \\
&= D(\mathsf{f} \circ (\chi')^{-1})((\phi^{-1} \circ \phi \circ \chi')(\mathsf{m}))\ D(\phi^{-1})((\phi \circ \chi')(\mathsf{m})) \\
&= D(\mathsf{f} \circ (\chi')^{-1})(\chi'(\mathsf{m}))\ D(\phi^{-1})(\chi(\mathsf{m})) \\
&= \mathsf{X}'(\mathsf{f})(\mathsf{m})\ D(\phi^{-1})(\chi(\mathsf{m})) \\
&= \mathsf{X}'(\mathsf{f})(\mathsf{m})\ D(\chi' \circ \chi^{-1})(\chi(\mathsf{m})).
\end{aligned} \tag{0.18}$$

This is the rule for the transformation of basis vector fields. The second factor can be recognized as "$\partial x'/\partial x$".[11]

The vector field does not depend on coordinates. So, from equation (0.16), we have

$$\begin{aligned}\mathsf{v}(\mathsf{f})(\mathsf{m}) &= \mathsf{X}(\mathsf{f})(\mathsf{m})\ b(\chi(\mathsf{m})) \\ &= \mathsf{X}'(\mathsf{f})(\mathsf{m})\ b'(\chi'(\mathsf{m}))\end{aligned} \tag{0.19}$$

Using equation (0.18), we deduce

$$D(\phi^{-1})(x)\ b(x) = b'(x'). \tag{0.20}$$

Because $\phi^{-1} \circ \phi = I$, we have

$$D(\phi^{-1})(x) = (D\phi(x'))^{-1}, \tag{0.21}$$

and so

$$b(x) = D\phi(x')\ b'(x'), \tag{0.22}$$

as expected.[12]

Integral Curves

A vector field gives a direction and rate for every point on a manifold. We can start at any point and go in the direction specified by the vector field, tracing out a parametric curve on the manifold. This curve is an integral curve of the vector field.

More formally, let v be a vector field on the manifold M. An integral curve $\gamma^{\mathsf{v}}_{\mathsf{m}} : \mathsf{R} \to \mathsf{M}$ of v is a parametric path on M satisfying

$$D(\mathsf{f} \circ \gamma^{\mathsf{v}}_{\mathsf{m}})(t) = \mathsf{v}(\mathsf{f})(\gamma^{\mathsf{v}}_{\mathsf{m}}(t)) = (\mathsf{v}(\mathsf{f}) \circ \gamma^{\mathsf{v}}_{\mathsf{m}})(t) \tag{0.23}$$

$$\gamma^{\mathsf{v}}_{\mathsf{m}}(0) = \mathsf{m}, \tag{0.24}$$

---

[11] This notation helps one remember the transformation rule:

$$\frac{\partial f}{\partial x^i} = \sum_j \frac{\partial f}{\partial x'^j}\frac{\partial x'^j}{\partial x^i},$$

which is the relation in the usual Leibnitz notation. Notice that f means something different on each side of the equation.

[12] For coordinate paths q and q' related by $q(t) = \phi(q'(t))$ the velocities are related by $Dq(t) = D\phi(q'(t))Dq'(t)$. Abstracting off paths, we get $v = D\phi(x')v'$.

for arbitrary functions f on the manifold, with real or structured real values. The rate of change of a function along an integral curve is the vector field applied to the function evaluated at the appropriate place along the curve. Often we will simply write γ, rather than $\gamma^{\mathsf{v}}_{\mathsf{m}}$.

We can recover the differential equations satisfied by a coordinate representation of the integral curve by letting $\mathsf{f} = \chi$, the coordinate function, and letting $\sigma = \chi \circ \gamma$ be the coordinate path corresponding to the curve γ. Then the derivative of the coordinate path σ is

$$\begin{aligned} D\sigma(t) &= D(\chi \circ \gamma)(t) \\ &= (\mathsf{v}(\chi) \circ \gamma)(t) \\ &= (\mathsf{v}(\chi) \circ \chi^{-1} \circ \chi \circ \gamma)(t) \\ &= (b \circ \sigma)(t), \end{aligned} \tag{0.25}$$

where $b = \mathsf{v}(\chi) \circ \chi^{-1}$ is the coefficient function for the vector field v for coordinates χ. So the coordinate path σ satisfies the differential equations

$$D\sigma = b \circ \sigma. \tag{0.26}$$

Differential equations for the integral curve can only be expressed in a coordinate representation, because we cannot go from one point on the manifold to another by addition of an increment. We can only do this by adding the coordinates to an increment of coordinates and then finding the corresponding point on the manifold.

Iterating the process described by equation (0.23) we can compute higher-order derivatives of functions along the integral curve:

$$\begin{aligned} D(\mathsf{f} \circ \gamma) &= \mathsf{v}(\mathsf{f}) \circ \gamma \\ D^2(\mathsf{f} \circ \gamma) &= D(\mathsf{v}(\mathsf{f}) \circ \gamma) = \mathsf{v}(\mathsf{v}(\mathsf{f})) \circ \gamma \\ &\cdots \\ D^n(\mathsf{f} \circ \gamma) &= \mathsf{v}^n(\mathsf{f}) \circ \gamma \end{aligned} \tag{0.27}$$

Thus, the evolution of $\mathsf{f} \circ \gamma$ can be written formally as a Taylor series in the parameter

$$(\mathsf{f} \circ \gamma)(t)$$

$$
\begin{aligned}
&= (\mathsf{f} \circ \gamma)(0) + t\,D(\mathsf{f} \circ \gamma)(0) + \frac{1}{2}t^2\,D^2(\mathsf{f} \circ \gamma)(0) + \cdots \\
&= (e^{tD}(\mathsf{f} \circ \gamma))(0) \\
&= (e^{t\mathsf{v}}\mathsf{f})(\gamma(0)).
\end{aligned} \tag{0.28}
$$

In particular, let $\mathsf{f} = \chi$, then

$$\sigma(t) = (\chi \circ \gamma)(t) = (e^{tD}(\chi \circ \gamma))(0) = (e^{t\mathsf{v}}\chi)(\gamma(0)), \tag{0.29}$$

a Taylor series representation of the solution to the differential equation (0.26).

For example, a vector field `J` that generates a rotation about the origin is:

```
(define J (- (* x d/dy) (* y d/dx)))
```

We can exponentiate the `J` vector field, to generate an evolution in a circle around the origin starting at `(1, 0)`:

```
(series:for-each print-expression
                 (((exp (* 'a J)) R2-chi)
                  ((R2 '->point) (up 1 0)))
                 6)
(up 1 0)
(up 0 a)
(up (* -1/2 (expt a 2)) 0)
(up 0 (* -1/6 (expt a 3)))
(up (* 1/24 (expt a 4)) 0)
(up 0 (* 1/120 (expt a 5)))
```

These are the first six terms of the series expansion of the coordinates of the position for parameter `a`.

This calculation can also be done, slightly differently, in coordinates. We recover the differential equation for the coordinate path σ, from the coordinate representation of v by letting $f = I$,

$$
\begin{aligned}
D\sigma(t) &= v(I)(\sigma(t)) \\
&= \mathsf{v}(\chi)(\chi^{-1}(\sigma(t))) \\
&= (b \circ \sigma)(t).
\end{aligned} \tag{0.30}
$$

Consider the evolution of $f \circ \sigma$.

$$
\begin{aligned}
D(f \circ \sigma) &= (Df \circ \sigma)\ D\sigma \\
&= (Df \circ \sigma)\ (b \circ \sigma)
\end{aligned}
$$

$$= v(f) \circ \sigma. \tag{0.31}$$

This restates equation (0.23), the constraint that γ is an integral curve, in coordinates.

Taking another derivative

$$\begin{aligned} D^2(f \circ \sigma) &= D(v(f) \circ \sigma) \\ &= (D(v(f)) \circ \sigma)\ D\sigma \\ &= (D(v(f)) \circ \sigma)\ (b \circ \sigma) \\ &= v(v(f)) \circ \sigma. \end{aligned} \tag{0.32}$$

So successive derivatives of $f \circ \sigma$ correspond to successive applications of v to f composed with σ

$$D^n(f \circ \sigma) = v^n(f) \circ \sigma. \tag{0.33}$$

And thus we can exponentiate the coordinate representation of the vector field.

We can define an evolution operator $\mathsf{E}_{t,\mathsf{v}}$

$$(\mathsf{E}_{t,\mathsf{v}}\mathsf{f})(\gamma(0)) = (\mathsf{f} \circ \gamma)(t) = e^{t\mathsf{v}}\mathsf{f}(\gamma(0)). \tag{0.34}$$

The evolution operator also has a coordinate representation:

$$\mathsf{E}_{t,\mathsf{v}}\mathsf{f}(\gamma(0)) = E_{t,v}f(\sigma(0)) = e^{tv}f(\sigma(0)). \tag{0.35}$$

But evolution is well defined whether or not coordinates are used, because it only depends on there being an integral curve γ. Also, evolution may be well defined for t beyond the point at which the Taylor series converges, even when using coordinates.

We can approximate the evolution operator by summing the series up to a given order:

```
(define ((((evolution order)
           delta-t vector-field)
          manifold-function)
         manifold-point)
  (series:sum
    (((exp (* delta-t vector-field))
      manifold-function)
     manifold-point)
    order))
```

We can evolve J from the initial point up to the parameter a, and accumulate the first six terms as follows:

```
(print-expression
 ((((evolution 6) 'a J) R2-chi)
  ((R2 '->point) (up 1 0))))
(up (+ (* -1/720 (expt a 6))
       (* 1/24 (expt a 4))
       (* -1/2 (expt a 2))
       1)
    (+ (* 1/120 (expt a 5))
       (* -1/6 (expt a 3))
       a))
```

One-form fields

One-form fields are linear functions of vector fields that produce real-valued functions on the manifold. A one-form field is linear in vector fields: if ω is a one-form field, v and w are vector fields, and c is a manifold function, then

$$\omega(\mathsf{v} + \mathsf{w}) = \omega(\mathsf{v}) + \omega(\mathsf{w})$$

and

$$\omega(\mathsf{cv}) = \mathsf{c}\omega(\mathsf{v}).$$

We can verify these identities in three dimensions:

```
(define omega (literal-1form-field 'omega R3))
(define v (literal-vector-field 'v R3))
(define w (literal-vector-field 'w R3))
(define c (literal-manifold-function 'c R3))

(print-expression
  ((- (omega (+ v w)) (+ (omega v)  (omega w)))
   R3-point))
0

(print-expression
  ((- (omega (* c v)) (* c (omega v))) R3-point))
0
```

The space of one-form fields on a manifold is a vector space over the reals. Sums and scalar products have the expected properties,

with the following definitions: if ω and θ are one-form fields, and if α is real number, then:

$$\begin{aligned}(\omega+\theta)(\mathsf{v}) &= \omega(\mathsf{v})+\theta(\mathsf{v})\\ (\alpha\omega)(\mathsf{v}) &= \alpha(\omega(\mathsf{v}))\end{aligned} \tag{0.36}$$

Given a coordinate function χ, we may define the basis one-form fields $\widetilde{\mathsf{X}}^i$ by

$$\widetilde{\mathsf{X}}^i(\mathsf{v})(\mathsf{m}) = \mathsf{v}(\chi^i)(\mathsf{m}) \tag{0.37}$$

or collectively

$$\widetilde{\mathsf{X}}(\mathsf{v})(\mathsf{m}) = \mathsf{v}(\chi)(\mathsf{m}). \tag{0.38}$$

The tuple of basis one-form fields $\widetilde{\mathsf{X}}(\mathsf{v})(\mathsf{m})$ is an up structure like that of χ.

The basis one-form fields are dual to the basis vector fields in the following sense:[13]

$$\widetilde{\mathsf{X}}^i(\mathsf{X}_j)(\mathsf{m}) = \mathsf{X}_j(\chi^i)(\mathsf{m}) = \partial_j(\chi^i\circ\chi^{-1})(\chi(\mathsf{m})) = \delta^i_j. \tag{0.39}$$

The general one-form field ω is a linear combination of basis one-form fields:

$$\omega(\mathsf{v})(\mathsf{m}) = a(\chi(\mathsf{m}))\ \widetilde{\mathsf{X}}(\mathsf{v})(\mathsf{m}) = \sum_i a_i(\chi(\mathsf{m}))\widetilde{\mathsf{X}}^i(\mathsf{v})(\mathsf{m}), \tag{0.40}$$

with coefficient tuple $a(x)$, for $x = \chi(\mathsf{m})$. We can write this more simply

$$\omega(\mathsf{v}) = (a\circ\chi)\ \widetilde{\mathsf{X}}(\mathsf{v}), \tag{0.41}$$

because everything is evaluated at m.

The coefficient tuple can be recovered from the one-form:

$$a(x) = \omega(\mathsf{X})(\chi^{-1}(x)). \tag{0.42}$$

This follows from the dual relationship (0.39). We can see this as a program:[14]

---

[13]The Kronecker delta δ^i_j is one if $i=j$ and zero otherwise.

[14]Again, we provide a shortcut for this construction:

```
(define omega
  (components->1form-field
   (down (literal-manifold-function 'a_0 R3)
         (literal-manifold-function 'a_1 R3)
         (literal-manifold-function 'a_2 R3))
   R3))

(pe ((omega (down d/dx d/dy d/dz)) R3-point))
(down (a_0 (up x0 y0 z0))
      (a_1 (up x0 y0 z0))
      (a_2 (up x0 y0 z0)))
```

The coordinate basis one-form fields can be used to find the coefficients of vector fields on the dual coordinate vector-field basis:

$$\widetilde{\mathsf{X}}^i(\mathsf{v}) = \mathsf{v}(\chi^i) = b^i \circ \chi \tag{0.43}$$

or collectively,

$$\widetilde{\mathsf{X}}(\mathsf{v}) = \mathsf{v}(\chi) = b \circ \chi \tag{0.44}$$

A coordinate basis one-form field is often written

$$\mathsf{dx}^i = \widetilde{\mathsf{X}}^i \tag{0.45}$$

The `instantiate-coordinates` procedure also makes the basis one-form fields with these traditional names inherited from the coordinates.

The duality of the coordinate basis vector fields and the coordinate basis one-form fields is illustrated:

```
(define R3 (rectangular 3))
(instantiate-coordinates R3 '(x y z))
(define R3-point
  ((R3 '->point) (up 'x0 'y0 'z0)))

(print-expression ((dx d/dy) R3-point))
0

(print-expression ((dx d/dx) R3-point))
1
```

```
(define omega (literal-1form-field 'a R3))
```

We can use the coordinate basis one-form fields to extract the coefficients of the J vector field on the rectangular vector basis:

```
(print-expression ((dx J) R2-point))
(* -1 y0)

(print-expression ((dy J) R2-point))
x0
```

But we can also find the coefficients on the polar vector basis:

```
(print-expression ((dr J) R2-point))
0

(print-expression ((dtheta J) R2-point))
1
```

So J is the same as d/dtheta, as we can see by applying them both to the general function f:

```
(print-expression (((- J d/dtheta) f) R2-point))
0
```

Differential:
An example of a one-form field is the differential df of a manifold function f, defined as follows. If df is applied to a vector field v we obtain

$$\mathsf{df}(\mathsf{v}) = \mathsf{v}(\mathsf{f}), \tag{0.46}$$

which is a function of a manifold point. The differential of a function is linear in the vector fields. The differential is also a linear operator on functions: if f_1 and f_2 are manifold functions, and if c is a real constant, then

$$\mathsf{d}(\mathsf{f}_1 + \mathsf{f}_2) = \mathsf{df}_1 + \mathsf{df}_2$$

and

$$\mathsf{d}(c\mathsf{f}) = c\mathsf{df}.$$

The traditional notation for the coordinate basis one-form fields is justified by the relation:

$$\mathsf{dx}^i = \widetilde{\mathsf{X}}^i = \mathsf{d}(\chi^i) \tag{0.47}$$

All one-form fields can be constructed as linear combinations of basis one-form fields, but not all one-form fields are differentials of functions. This is why we started with the basis one-form fields and built the general one-form fields in terms of them.

Coordinate transformations:
Under a coordinate change $\chi = \phi \circ \chi'$

$$\begin{aligned}\widetilde{\mathsf{X}}(\mathsf{v}) &= \mathsf{v}(\phi \circ \chi') \\ &= (D\phi \circ \chi')\ \mathsf{v}(\chi') \\ &= (D\phi \circ \chi')\ \widetilde{\mathsf{X}}'(\mathsf{v}),\end{aligned} \tag{0.48}$$

where the second line follows from the chain rule for vector fields.

One-form fields are independent of coordinates. So,

$$\begin{aligned}\omega(\mathsf{v}) &= (a \circ \chi)\ \widetilde{\mathsf{X}}(\mathsf{v}) \\ &= (a' \circ \chi')\ \widetilde{\mathsf{X}}'(\mathsf{v}).\end{aligned} \tag{0.49}$$

With equation (0.48) this is

$$a(\chi(\mathsf{m}))\ D\phi(\chi'(\mathsf{m})) = a'(\chi'(\mathsf{m})), \tag{0.50}$$

or

$$a(\chi(\mathsf{m})) = a'(\chi'(\mathsf{m}))\ (D\phi(\chi'(\mathsf{m})))^{-1}. \tag{0.51}$$

The coefficient tuple $a(x)$ is a down structure compatible for contraction with $b(x)$. Let v be the vector with coefficient tuple $b(x)$, and ω be the one-form with coefficient tuple $a(x)$. Then, by equation (0.41),

$$\omega(\mathsf{v}) = (a \circ \chi)\ (b \circ \chi). \tag{0.52}$$

As a program:

```
(print-expression
  ((omega (literal-vector-field 'v R3)) R3-point))
(+ (* (a_0 (up x0 y0 z0)) (v^0 (up x0 y0 z0)))
   (* (a_1 (up x0 y0 z0)) (v^1 (up x0 y0 z0)))
   (* (a_2 (up x0 y0 z0)) (v^2 (up x0 y0 z0))))
```

Comparing equation (0.51) with equation (0.22) we see that one-form components and vector components transform oppositely, so that

$$a(x)\ b(x) = a'(x')\ b'(x'). \tag{0.53}$$

This shows that $\omega(\mathsf{v})(\mathsf{m})$ is independent of coordinates.

Basis Fields

A vector field may be written as a linear combination of basis vector fields. If n is the dimension, then any set of n linearly independent vector fields may be used as a basis. The coordinate basis X is an example of a basis.[15]

Let e be a tuple of basis vector fields, such as the coordinate basis X. The general vector field v applied to an arbitrary manifold function f can be expressed as a linear combination

$$\mathsf{v}(\mathsf{f})(\mathsf{m}) = \mathsf{e}(\mathsf{f})(\mathsf{m})\ \mathsf{b}(\mathsf{m}) = \sum_i \mathsf{e}_i(\mathsf{f})(\mathsf{m})\, \mathsf{b}^i(\mathsf{m}), \tag{0.54}$$

where b is a tuple-valued coefficient function on the manifold.[16] When expressed in a coordinate basis, the coefficients that specify the direction of the vector are naturally expressed as functions b^i of the coordinates of the manifold point. Here, the coefficient function b is more naturally expressed as a tuple-valued function on the manifold.

The coordinate basis forms have a simple definition in terms of the coordinate basis vectors and the coordinates (equation 0.38). With this choice, the dual property, equation (0.39), holds without further fuss. More generally, we can define a basis of one-forms $\tilde{\mathsf{e}}$ that is dual to e in that they satisfy the property

$$\tilde{\mathsf{e}}^i(\mathsf{e}_j)(\mathsf{m}) = \delta^i_j, \tag{0.55}$$

analogous to property (0.39).

---

[15] We cannot say if the basis vectors are orthogonal or normalized until we introduce a metric.

[16] If b is the coefficient function expressed as a function of coordinates, then $\mathsf{b} = b \circ \chi$ is the coefficient function as a function on the manifold.

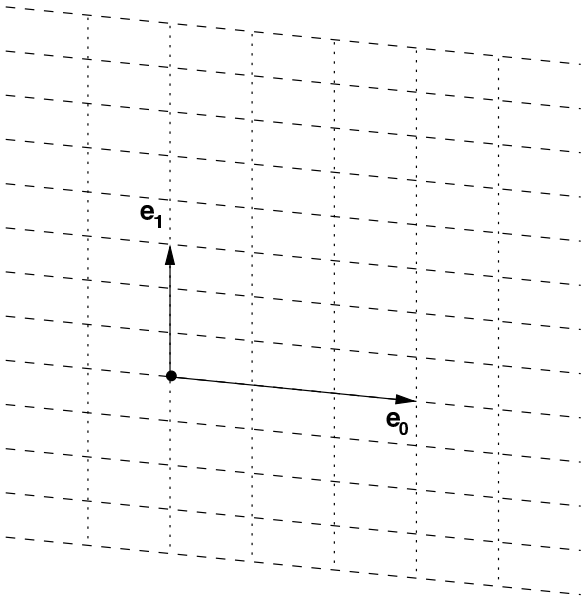

Figure 0.3 Let arrows e_0 and e_1 depict the vectors of a basis vector field at a particular point. Then the foliations shown by the parallel lines depict the dual basis one-form fields at that point. The dotted lines represent the field $\tilde{\mathsf{e}}^0$ and the dashed lines represent the field $\tilde{\mathsf{e}}^1$. The spacings of the lines are 1/3 unit. That the vectors pierce three of the lines representing their duals and do not pierce any of the lines representing the other basis elements is one way to see the relationship $\tilde{\mathsf{e}}^i(\mathsf{e}_j)(\mathsf{m}) = \delta^i_j$.

To solve for the dual basis $\tilde{\mathsf{e}}$ given the basis e express the basis vectors e in terms of a coordinate basis

$$\mathsf{e}_j(\mathsf{f}) = \sum_k \mathsf{X}_k(\mathsf{f})\, \mathsf{c}^k_j \tag{0.56}$$

and the dual one-forms $\tilde{\mathsf{e}}$ in terms of the dual coordinate one-forms

$$\tilde{\mathsf{e}}^i(\mathsf{v}) = \sum_l \mathsf{d}^i_l\, \widetilde{\mathsf{X}}^l(\mathsf{v}), \tag{0.57}$$

then

$$\tilde{\mathsf{e}}^i(\mathsf{e}_j) = \sum_l \mathsf{d}^i_l \widetilde{\mathsf{X}}^l(\mathsf{e}_j)$$

$$
\begin{aligned}
&= \sum_l \mathsf{d}_l^i \mathsf{e}_j(\chi^l) \\
&= \sum_l \mathsf{d}_l^i \sum_k \mathsf{X}_k(\chi^l)\mathsf{c}_j^k \\
&= \sum_{kl} \mathsf{d}_l^i \delta_k^l \mathsf{c}_j^k \\
&= \sum_k \mathsf{d}_k^i \mathsf{c}_j^k. \qquad (0.58)
\end{aligned}
$$

Applying this at m we get

$$\tilde{\mathsf{e}}^i(\mathsf{e}_j)(\mathsf{m}) = \delta_j^i = \sum_k \mathsf{d}_k^i(\mathsf{m})\mathsf{c}_j^k(\mathsf{m}). \qquad (0.59)$$

So the d coefficients can be determined from the c coefficents (essentially by matrix inversion).

The dual form fields can be used to determine the coefficients b of a vector field v relative to a basis e, by applying the dual basis form fields $\tilde{\mathsf{e}}$ to the vector field. Let

$$\mathsf{v}(\mathsf{f}) = \sum_i \mathsf{e}_i(\mathsf{f})\,\mathsf{b}^i. \qquad (0.60)$$

Then

$$\tilde{\mathsf{e}}^j(\mathsf{v}) = \mathsf{b}^j. \qquad (0.61)$$

Define two general vector fields:

```
(define e0
  (+ (* (literal-manifold-function 'e0x R2) d/dx)
     (* (literal-manifold-function 'e0y R2) d/dy)))

(define e1
  (+ (* (literal-manifold-function 'e1x R2) d/dx)
     (* (literal-manifold-function 'e1y R2) d/dy)))
```

We use these as a vector basis and compute the dual:

```
(define e-vector-basis (down e0 e1))
(define e-dual-basis
  (vector-basis->dual e-vector-basis P2))
```

Then we can verify that they satisfy the dual relationship (equation 0.39) by applying the dual basis to the vector basis:

```
(print-expression ((e-dual-basis e-vector-basis) m))
(up (down 1 0) (down 0 1))
```

Note that the dual basis was computed relative to the polar coordinate system, showing how the resulting objects are independent of the coordinates in which they were expressed!

Or we can make a general vector field with this basis and then pick out the coefficients by applying the dual basis:

```
(define v
  (* e-vector-basis
     (up (literal-manifold-function 'bx R2)
         (literal-manifold-function 'by R2))))

(print-expression ((e-dual-basis v) R2-point))
(up (bx (up x0 y0)) (by (up x0 y0)))
```

Commutators

The commutator of two vector fields is defined as

$$[\mathsf{v},\mathsf{w}](\mathsf{f}) = \mathsf{v}(\mathsf{w}(\mathsf{f})) - \mathsf{w}(\mathsf{v}(\mathsf{f})) \tag{0.62}$$

The commutator of two coordinate basis fields is zero:

$$\begin{aligned}[\mathsf{X}_i,\mathsf{X}_j](\mathsf{f}) &= \mathsf{X}_i(\mathsf{X}_j(\mathsf{f})) - \mathsf{X}_j(\mathsf{X}_i(\mathsf{f})) \\ &= \partial_i\partial_j(\mathsf{f}\circ\chi^{-1})\circ\chi - \partial_j\partial_i(\mathsf{f}\circ\chi^{-1})\circ\chi \\ &= 0,\end{aligned} \tag{0.63}$$

because the individual partial derivatives commute.

The commutator of two vector fields is a vector field. Let v be a vector field with coefficient function $\mathsf{c} = c\circ\chi$, and u be a vector field with coefficient function $\mathsf{b} = b\circ\chi$, both with respect to the coordinate basis X. Then

$$\begin{aligned}[\mathsf{u},\mathsf{v}](\mathsf{f}) &= \mathsf{u}(\mathsf{v}(\mathsf{f})) - \mathsf{v}(\mathsf{u}(\mathsf{f})) \\ &= \mathsf{u}(\sum_i \mathsf{X}_i(\mathsf{f})\mathsf{c}^i) - \mathsf{v}(\sum_i \mathsf{X}_i(\mathsf{f})\mathsf{b}^i)\end{aligned}$$

$$
\begin{aligned}
&= \sum_j \mathsf{X}_j(\sum_i \mathsf{X}_i(\mathsf{f})\mathsf{c}^i)\mathsf{b}^j - \sum_j \mathsf{X}_j(\sum_i \mathsf{X}_i(\mathsf{f})\mathsf{b}^i)\mathsf{c}^j \\
&= \sum_{ij} [\mathsf{X}_i, \mathsf{X}_j](\mathsf{f})\mathsf{c}^i\mathsf{b}^j \\
&\quad + \sum_i \mathsf{X}_i(\mathsf{f}) \sum_j (\mathsf{X}_j(\mathsf{c}^i)\mathsf{b}^j - \mathsf{X}_j(\mathsf{b}^i)\mathsf{c}^j) \\
&= \sum_i \mathsf{X}_i(\mathsf{f})\mathsf{a}^i, \qquad (0.64)
\end{aligned}
$$

where the coefficient function a of the commutator vector field is

$$
\begin{aligned}
\mathsf{a}^i &= \sum_j \left(\mathsf{X}_j(\mathsf{c}^i)\mathsf{b}^j - \mathsf{X}_j(\mathsf{b}^i)\mathsf{c}^j\right) \\
&= \mathsf{u}(\mathsf{c}^i) - \mathsf{v}(\mathsf{b}^i). \qquad (0.65)
\end{aligned}
$$

We used the fact, shown above, that the commutator of two coordinate basis fields is zero.

We can check this formula for the commutator for the general vector fields `e0` and `e1` in polar coordinates:

```
(let ((polar-vector-basis (basis->vector-basis polar-basis))
      (polar-dual-basis (basis->1form-basis polar-basis)))
  (print-expression
    ((- ((commutator e0 e1) f)
        (* (- (e0 (polar-dual-basis e1))
              (e1 (polar-dual-basis e0)))
           (polar-vector-basis f)))
     R2-point)))
0
```

Let e be a tuple of basis vector fields. The commutator of two basis fields can be expressed in terms of the basis vector fields:

$$
[\mathsf{e}_i, \mathsf{e}_j](\mathsf{f}) = \sum_k \mathsf{d}_{ij}^k \mathsf{e}_k(\mathsf{f}), \qquad (0.66)
$$

where d_{ij}^k are functions of m, called the *structure constants* for the basis vector fields. The coefficients are

$$
\mathsf{d}_{ij}^k = \tilde{\mathsf{e}}^k([\mathsf{e}_i, \mathsf{e}_j]). \qquad (0.67)
$$

Define the vector fields Jx, Jy, and Jz that generate rotations about the three rectangular axes in three dimensions:[17]

```
(define Jz (- (* x d/dy) (* y d/dx)))
(define Jx (- (* y d/dz) (* z d/dy)))
(define Jy (- (* z d/dx) (* x d/dz)))

(print-expression (((+ (commutator Jx Jy) Jz) g) R3-point))
0
(print-expression (((+ (commutator Jy Jz) Jx) g) R3-point))
0
(print-expression (((+ (commutator Jz Jx) Jy) g) R3-point))
0
```

We see that

$$[J_x, J_y] = -J_z$$
$$[J_y, J_z] = -J_x$$
$$[J_z, J_x] = -J_y. \tag{0.68}$$

You can tell if a set of basis vector fields is a coordinate basis by calculating the commutators. If they are non-zero, then the basis is not a coordinate basis. If they are zero then the basis vector fields can be integrated to give the coordinate system.

Take a point 0 in M as the origin. Then, presuming $[\mathsf{e}_i, \mathsf{e}_j] = 0$, the coordinates x of the point m in the coordinate system corresponding to the e basis satisfy[18]

$$\mathsf{m} = E_{x,\mathsf{e}}(0) = \chi^{-1}(x), \tag{0.69}$$

where χ is the coordinate function being defined. Because the elements of e commute, the terms in the exponential can be simply factored into separate exponentials if needed.

---

[17] Using

```
(define R3 (rectangular 3))
(instantiate-coordinates R3 '(x y z))
(define R3->R (-> (UP Real Real Real) Real))
(define g
  (compose (literal-function 'g R3->R)
           (R3 '->coords)))
(define R3-point ((R3 '->point) (up 'x 'y 'z)))
```

[18] Here x is an up-tuple structure of components, and e is down-tuple structure of basis vectors. The product of the two contracts to make a scaled vector.

Higher-Rank Forms

A one-form field applied to a vector field is a function on the manifold. We can extend this idea to k-form fields. A k-form field is an antisymmetric multilinear function that takes k vector fields and produces a function on the manifold.

Given two one-form fields ω and τ we can form a two-form field $\omega \wedge \tau$ as follows:

$$(\omega \wedge \tau)(\mathsf{v}, \mathsf{w}) = \omega(\mathsf{v})\tau(\mathsf{w}) - \omega(\mathsf{w})\tau(\mathsf{v}) \tag{0.70}$$

In the special case where the one forms are the coordinate basis forms in rectangular coordinates the wedge of them measures the oriented area enclosed by a parallogram whose sides are described by the vectors given as inputs:

```
(define R2 (rectangular 2))
(instantiate-coordinates R2 '(x y))
(define R2-point ((R2 '->point) (up 'x0 'y0)))

(define v (+ (* 'a d/dx) (* 'b d/dy)))

(define w (+ (* 'c d/dx) (* 'd d/dy)))

(print-expression (((wedge dx dy) v w) R2-point))
(+ (* a d) (* -1 b c))
```

More generally we can form the wedge of higher-rank forms. Let ω be a k-form field and τ be an l-form field. We can form a $k+l$-form field $\omega \wedge \tau$ as follows:

$$\omega \wedge \tau = \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(\omega \otimes \tau) \tag{0.71}$$

where, if η is a m-form then

$$\begin{aligned}&\mathrm{Alt}(\eta)(\mathsf{v}_0, \ldots, \mathsf{v}_{m-1}) \\ &\quad = \frac{1}{m!}\sum_{\sigma \in \mathrm{Perm}(m)} \mathrm{Parity}(\sigma)\eta(\mathsf{v}_{\sigma(0)}, \ldots, \mathsf{v}_{\sigma(m-1)})\end{aligned} \tag{0.72}$$

and where

$$\begin{aligned}&\omega \otimes \tau(\mathsf{v}_0, \ldots, \mathsf{v}_{k-1}, \mathsf{v}_k, \ldots, \mathsf{v}_{k+l-1}) \\ &\quad = \omega(\mathsf{v}_0, \ldots, \mathsf{v}_{k-1})\tau(\mathsf{v}_k, \ldots, \mathsf{v}_{k+l-1}).\end{aligned} \tag{0.73}$$

The wedge product is associative, and thus we need not specify the order of a multiple application.

The 3-form that is the wedge product of the rectangular coordinate-basis forms measures volumes:

```
(define R3 (rectangular 3))
(instantiate-coordinates R3 '(x y z))
(define R3-point ((R3 '->point) (up 'x0 'y0 'z0)))

(define u (+ (* 'a d/dx) (* 'b d/dy) (* 'c d/dz)))
(define v (+ (* 'd d/dx) (* 'e d/dy) (* 'f d/dz)))
(define w (+ (* 'g d/dx) (* 'h d/dy) (* 'i d/dz)))

(print-expression
  (((wedge dx dy dz) u v w) R3-point))
(+ (* a e i)
   (* -1 a f h)
   (* -1 b d i)
   (* b f g)
   (* c d h)
   (* -1 c e g))
```

This last expression is, if you don't recognize it, the determinant of a 3×3 matrix:

```
(print-expression
  (- (((wedge dx dy dz) u v w) R3-point)
     (determinant
      (matrix-by-rows (list 'a 'b 'c)
                      (list 'd 'e 'f)
                      (list 'g 'h 'i)))))
0
```

In general, if the rank of a form is greater than the dimension of the manifold then the form is identically zero.

Exterior Derivative

The intention of introducing the exterior derivative is to unify all of the classical theorems of "vector analysis" into one unified Stokes's Theorem, which asserts that the integral of a form on the

boundary of a manifold is the integral of the exterior derivative of the form on the interior of the manifold:

$$\int_{\partial M} \omega = \int_M \mathsf{d}\omega \tag{0.74}$$

In order for this to work in a way independent of the coordinate system, the exterior derivative has to incorporate a Jacobian determinant to scale the line elements, area elements, or volume elements.

As we have seen in equation (0.46), the differential of a function on a manifold is a one-form field. If a function on a manifold is considered to be a form field of rank zero[19] then the differential operator increases the rank of the form by one. We can generalize this to k-form fields with the exterior derivative operation.

Consider a one-form ω. We define

$$\mathsf{d}\omega(\mathsf{v}_1, \mathsf{v}_2) = \mathsf{v}_1(\omega(\mathsf{v}_2)) - \mathsf{v}_2(\omega(\mathsf{v}_1)) - \omega([\mathsf{v}_1, \mathsf{v}_2]). \tag{0.75}$$

The exterior derivative of a k-form field is a $k+1$-form field, given by:[20]

$$\begin{aligned}
&\mathsf{d}\omega(\mathsf{v}_0, \ldots, \mathsf{v}_k) \\
&\quad = \sum_{i=0}^{k} ((-1)^i \mathsf{v}_i(\omega(\mathsf{v}_0, \ldots, \mathsf{v}_{i-1}, \mathsf{v}_{i+1}, \ldots, \mathsf{v}_k)) + \\
&\qquad \sum_{j=i+1}^{k} (-1)^{i+j} \omega([\mathsf{v}_i, \mathsf{v}_j], \mathsf{v}_0, \ldots, \mathsf{v}_{i-1}, \mathsf{v}_{i+1}, \ldots, \mathsf{v}_{j-1}, \mathsf{v}_{j+1}, \ldots, \mathsf{v}_k))
\end{aligned} \tag{0.76}$$

This formula is coordinate-system independent. This is the way we compute the exterior derivative in our software.

---

[19] A manifold function f induces a form field $\hat{\mathsf{f}}$ of rank 0 as follows:

$\hat{\mathsf{f}}()(\mathsf{m}) = \mathsf{f}(\mathsf{m})$.

[20] See Spivak, *Differential Geometry*, Volume 1, p.289.

If the form field ω is represented in a coordinate basis

$$\omega = \sum_{i_0=0,\ldots,i_{k-1}=0}^{n-1} \omega_{i_0,\ldots,i_{k-1}} \mathsf{d}x^{i_0} \wedge \cdots \wedge \mathsf{d}x^{i_{k-1}} \tag{0.77}$$

then the exterior derivative can be expressed as

$$\mathsf{d}\omega = \sum_{i_0=0,\ldots,i_{k-1}=0}^{n-1} \mathsf{d}\omega_{i_0,\ldots,i_{k-1}} \wedge \mathsf{d}x^{i_0} \wedge \cdots \wedge \mathsf{d}x^{i_{k-1}}. \tag{0.78}$$

Though this formula is expressed in terms of a coordinate basis, the result is independent of the choice of coordinate system.

We can test that the computation indicated by equation (0.76) is equivalent to the explicit computation indicated by equation (0.78) in three dimensions

```
(define R3 (rectangular 3))
(instantiate-coordinates R3 '(x y z))
(define R3-point ((R3 '->point) (up 'x0 'y0 'z0)))
```

with a general one-form field:

```
(define a (literal-manifold-function 'alpha R3))
(define b (literal-manifold-function 'beta R3))
(define c (literal-manifold-function 'gamma R3))

(define theta (+ (* a dx) (* b dy) (* c dz)))
```

The test will require two arbitrary vector fields

```
(define X (literal-vector-field 'X R3))
(define Y (literal-vector-field 'Y R3))

(print-expression
 (((- (d theta)
      (+ (wedge (d a) dx)
         (wedge (d b) dy)
         (wedge (d c) dz)))
   X Y)
  R3-point))
0
```

We can also try a general 2-form field in three-dimensional space: Let

$$\omega = a\mathsf{dy}\wedge\mathsf{dz} + b\mathsf{dz}\wedge\mathsf{dx} + c\mathsf{dx}\wedge\mathsf{dy}, \tag{0.79}$$

where $a = \alpha\circ\chi$, $b = \beta\circ\chi$, $c = \gamma\circ\chi$, and α, β, and γ are real-valued functions of three real arguments. As a program,

```
(define omega
  (+ (* a (wedge dy dz))
     (* b (wedge dz dx))
     (* c (wedge dx dy))))
```

Here we need another vector field because our result will be a 3-form field.

```
(define Z (literal-vector-field 'Z R3))

(print-expression
 (((- (d omega)
      (+ (wedge (d a) dy dz)
         (wedge (d b) dz dx)
         (wedge (d c) dx dy)))
   X Y Z)
  R3-point))
0
```

A form field ω that is the exterior derivative of another form field $\omega = \mathsf{d}\theta$ is called *exact*. A form field whose exterior derivative is zero is called *closed*.

Every exact form field is a closed form field, i.e. applying the exterior derivative operator twice always yields zero:

$$\mathsf{d}^2\omega = 0. \tag{0.80}$$

This is equivalent to the statement that partial derivatives with respect to different variables commute.[21]

Consider the general one-form field θ defined on 3-dimensional rectangular space. Two exterior derivatives of θ yields a 3-form field, which must be proportional to the volume element.[22] It turns out to be zero:

---

[21] See Spivak, *Calculus on Manifolds*, p.92

[22] The dimension of the space of k-forms in an n-dimensional space is $n-k+1$.

```
(print-expression
 (((d (d theta)) X Y Z) R3-point))
0
```

Relationship to Vector Calculus:
In three-dimensional Euclidean space the traditional vector derivative operations are gradient, curl, and divergence. If $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ are the usual orthonormal vector basis, f a function on the space, and $\vec{\mathrm{v}}$ is a vector field on the space, then

$$\mathrm{grad}(\mathsf{f}) = \frac{\partial \mathsf{f}}{\partial x}\hat{\mathrm{x}} + \frac{\partial \mathsf{f}}{\partial y}\hat{\mathrm{y}} + \frac{\partial \mathsf{f}}{\partial z}\hat{\mathrm{z}}$$

$$\mathrm{curl}(\vec{\mathrm{v}}) = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathrm{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathrm{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathrm{x}}$$

$$\mathrm{div}(\vec{\mathrm{v}}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

These vector calculus operations are subsumed by exterior derivatives of form fields, as follows. Let θ be a one-form field and let ω be a two-form field:

$$\theta = \theta_x \mathsf{dx} + \theta_y \mathsf{dy} + \theta_z \mathsf{dz}$$

$$\omega = \omega_x \mathsf{dy} \wedge \mathsf{dz} + \omega_y \mathsf{dz} \wedge \mathsf{dx} + \omega_z \mathsf{dx} \wedge \mathsf{dy}$$

The exterior-derivative expressions corresponding to the vector-calculus expressions are:

$$\mathsf{df} = \frac{\partial \mathsf{f}}{\partial x}\mathsf{dx} + \frac{\partial \mathsf{f}}{\partial y}\mathsf{dy} + \frac{\partial \mathsf{f}}{\partial z}\mathsf{dz}$$

$$\begin{aligned}\mathsf{d}\theta = &\left(\frac{\partial \theta_z}{\partial y} - \frac{\partial \theta_y}{\partial z}\right)\mathsf{dy} \wedge \mathsf{dz} + \left(\frac{\partial \theta_x}{\partial z} - \frac{\partial \theta_z}{\partial x}\right)\mathsf{dz} \wedge \mathsf{dx} \\ &+ \left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right)\mathsf{dx} \wedge \mathsf{dy}\end{aligned}$$

$$\mathsf{d}\omega = \left(\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z}\right)\mathsf{dx} \wedge \mathsf{dy} \wedge \mathsf{dz}.$$

Vector Integral Theorems:
Green's Theorem states that for an arbitrary compact set $M \subset \mathrm{R}^2$

$$\int_{\partial M} (\alpha \circ \chi)\,\mathsf{dx} + (\beta \circ \chi)\,\mathsf{dy} = \int_M ((\partial_0 \beta - \partial_1 \alpha) \circ \chi)\,\mathsf{dx} \wedge \mathsf{dy} \quad (0.81)$$

We can test this. By Stokes's theorem, the integrands are related by an exterior derivative. First we need a plane.

```
(define R2 (rectangular 2))
(instantiate-coordinates R2 '(x y))
(define R2-chi (R2 '->coords))
(define R2->R (-> (UP Real Real) Real))
```

We also need some vectors to test our forms:

```
(define v (literal-vector-field 'v R2))
(define w (literal-vector-field 'w R2))
```

We can now test our integrands:

```
(define alpha (literal-function 'alpha R2->R))
(define beta (literal-function 'beta R2->R))

(print-expression
  (((- (d (+ (* (compose alpha R2-chi) dx)
             (* (compose beta R2-chi) dy)))
       (* (compose (- ((partial 0) beta)
                      ((partial 1) alpha))
                   R2-chi)
          (wedge dx dy)))
    v w)
   R2-point))
0
```

We can also compute the integrands for the Divergence Theorem: For an arbitrary compact set $M \subset \mathrm{R}^3$ and a vector field w

$$\int_M \mathrm{div}(\mathsf{w}) dV = \int_{\partial M} \mathsf{w} \cdot \mathsf{n}\, dA \tag{0.82}$$

where n is the outward pointing normal to the surface ∂M. Again, the integrands should be related by an exterior derivative, if this is an instance of Stokes's Theorem.

Let

$$\mathsf{w} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \tag{0.83}$$

We interpret $\mathsf{w} \cdot \mathsf{n} dA$ as the two-form

$$\omega = a\mathsf{dy} \wedge \mathsf{dz} + b\mathsf{dz} \wedge \mathsf{dx} + c\mathsf{dx} \wedge \mathsf{dy}, \tag{0.84}$$

and $\mathrm{div}(\mathsf{w})dV$ as the three-form

$$\mathsf{d}\omega = \left(\frac{\partial}{\partial x}a + \frac{\partial}{\partial y}b + \frac{\partial}{\partial z}c\right)\mathsf{dx}\wedge\mathsf{dy}\wedge\mathsf{dz}. \tag{0.85}$$

Let's compute this to make sure it is true:

```
(define domega
  (* (+ (d/dx a) (d/dy b) (d/dz c))
     (wedge dx dy dz)))

(print-expression
 (((- (d omega) domega) X Y Z) R3-point))
0
```

Over a Map

To deal with motion on manifolds we need to think about paths on manifolds and vectors along these paths. Vectors along paths are not vector fields on the manifold because they are only defined on the path. And the path may even cross itself, which would give more than one vector at a point. Here we introduce the concept of *over a map*, which solves this problem.[23]

Let μ map points n in the manifold N to points m in the manifold M. A vector over the map μ takes directional derivatives of functions on M at points $\mathsf{m} = \mu(\mathsf{n})$. The vector over the map applied to the function on M is a function on N.

Let v be a vector field on M, and f a function on M. Then

$$\mathsf{v}_\mu(\mathsf{f}) = \mathsf{v}(\mathsf{f})\circ\mu, \tag{0.86}$$

is a vector over the map μ. Note that $\mathsf{v}_\mu(\mathsf{f})$ is a function on N, not M:

$$\mathsf{v}_\mu(\mathsf{f})(\mathsf{n}) = \mathsf{v}(\mathsf{f})(\mu(\mathsf{n})). \tag{0.87}$$

We can implement this definition as:

[23] see Bishop and Goldberg, *Tensor Analysis on Manifolds*.

```
(define ((vector-field-over-map mu:N->M) v-on-M)
  (procedure->vector-field
   (lambda (f-on-M)
     (compose (v-on-M f-on-M) mu:N->M))))
```

Given a one-form ω, the one-form over the map μ is constructed as follows:

$$\omega^{\mu}(\mathsf{v}_{\mu})(\mathsf{n}) = \omega(\mathsf{u})(\mu(\mathsf{n})), \text{ where } \mathsf{u}(\mathsf{f})(\mathsf{m}) = \mathsf{v}_{\mu}(\mathsf{f})(\mathsf{n}). \tag{0.88}$$

The object u is not really a vector field on M even though we have given it that shape so that the dual vector can apply to it; $\mathsf{u}(\mathsf{f})$ is only evaluated at images $\mathsf{m} = \mu(\mathsf{n})$ of points n in N. If we were defining u as a vector field we would need the inverse of μ to find the point $\mathsf{n} = \mu^{-1}(\mathsf{m})$, but this is not required to define the object u in a context where there is already an m associated with the n of interest. To extend this idea to k-forms, we carry each vector argument over the map.

The procedure that implements u, that is, that makes a vector field over the map $\mu : \mathsf{N} \to \mathsf{M}$ appear as a vector field on M is

```
(define (vector-field-over-map->vector-field V-over-mu n)
  (procedure->vector-field
   (lambda (f)
     (lambda (m) ((V-over-mu f) n)))))
```

Using this, the procedure that constructs a k-form over the map from a k-form is:

```
(define ((form-field-over-map mu:N->M) w-on-M)
  (let ((k (get-rank w-on-M)))
    (procedure->nform-field
     (lambda vectors-over-map
       (lambda (n)
         ((apply w-on-M
                 (map (lambda (V-over-mu)
                        (vector-field-over-map->vector-field
                          V-over-mu n))
                      vectors-over-map))
          (mu:N->M n))))
     k)))
```

Let e be a tuple of basis vector fields, and $\tilde{\mathsf{e}}$ be the tuple of basis one-forms that is dual to e:

$$\tilde{\mathsf{e}}^i(\mathsf{e}_j)(\mathsf{m}) = \delta^i_j. \tag{0.89}$$

The *basis vectors over the map*, e^μ, are particular cases of vectors over a map:

$$\mathsf{e}^\mu(\mathsf{f}) = \mathsf{e}(\mathsf{f}) \circ \mu. \tag{0.90}$$

And the elements of the *dual basis over the map*, $\tilde{\mathsf{e}}_\mu$, are particular cases of one-forms over the map. The basis and dual basis over the map satisfy

$$\tilde{\mathsf{e}}^i_\mu(\mathsf{e}^\mu_j)(\mathsf{n}) = \delta^i_j. \tag{0.91}$$

For example, let μ map the time line to the unit sphere. We use colatitude θ and longitude ϕ as coordinates on the sphere

```
(define sphere (S2 1))  ; sphere with R=1
(instantiate-coordinates sphere '(theta phi))
(define sphere-basis (coordinate-system->basis sphere))
```

and let t be the coordinate of the real line.

```
(instantiate-coordinates the-real-line 't)
```

A general path on the sphere is:

```
(define mu
  (compose (sphere '->point)
           (up (literal-function 'alpha)
               (literal-function 'beta))
           (the-real-line '->coords)))
```

The basis over the map is constructed from the basis on the sphere:

```
(define sphere-basis-over-mu
  (basis->basis-over-map mu sphere-basis))

(define h
  (compose (literal-function 'h R2->R)
           (sphere '->coords)))
```

```
(print-expression
  (((basis->vector-basis sphere-basis-over-mu) h)
   ((the-real-line '->point) 't0)))
(down
 (((partial 0) h) (up (alpha t0) (beta t0)))
 (((partial 1) h) (up (alpha t0) (beta t0))))
```

The basis vectors over the map compute derivatives of the function h evaluated on the path at the given time.

We can check that the dual basis over the map does the correct thing

```
(print-expression
  (((basis->1form-basis sphere-basis-over-mu)
    (basis->vector-basis sphere-basis-over-mu))
   ((the-real-line '->point) 't0)))
(up (down 1 0) (down 0 1))
```

Pullback of a Function:
The *pullback* of a function f on M over the map μ is defined as

$$\mu^*\mathsf{f} = \mathsf{f} \circ \mu. \tag{0.92}$$

This allows us to take a function defined on M and use it to define a new function on N.

For example, the integral curve of v evolved for time t as a function of the initial manifold point m generates a map ϕ_t^{v} of the manifold onto itself. This is a simple Currying of the integral curve of v from m as a function of time: $\phi_t^{\mathsf{v}}(\mathsf{m}) = \gamma_{\mathsf{m}}^{\mathsf{v}}(t)$. The evolution of the function f along an integral curve, equation (0.34), can be written in terms of the pullback over ϕ_t^{v}:

$$(\mathsf{E}_{t,\mathsf{v}}\mathsf{f})(\mathsf{m}) = \mathsf{f}(\phi_t^{\mathsf{v}}(\mathsf{m})) = ((\phi_t^{\mathsf{v}})^*\mathsf{f})(\mathsf{m}). \tag{0.93}$$

Pushforward of a Vector Field:
We can also *pushforward* a vector field over the map μ. The pushforward takes a vector field v defined on N. The result takes directional derivatives of functions on M at a place determined by a point in M:

$$\mu_*\mathsf{v}(\mathsf{f})(\mathsf{m}) = \mathsf{v}(\mu^*\mathsf{f})(\mu^{-1}(\mathsf{m})) = \mathsf{v}(\mathsf{f} \circ \mu)(\mu^{-1}(\mathsf{m})). \tag{0.94}$$

We can pushforward a vector field over the map generated by an integral curve, because the inverse is always available.[24]

$$((\phi_t^{\mathsf{v}})_* \mathsf{v})(\mathsf{f})(\mathsf{m}) = \mathsf{v}((\phi_t^{\mathsf{v}})^* \mathsf{f})(\phi_{-t}^{\mathsf{v}}(\mathsf{m})) = \mathsf{v}(\mathsf{f} \circ \phi_t^{\mathsf{v}})(\phi_{-t}^{\mathsf{v}}(\mathsf{m})). \tag{0.95}$$

Differential of a Map:
For more general maps the pushforward of a vector field is not very useful because it requires inverting the map μ. A more useful construct in this context is the *differential* of the map

$$d\mu(\mathsf{v})(\mathsf{f})(\mathsf{n}) = \mathsf{v}(\mu^* \mathsf{f})(\mathsf{n}) = \mathsf{v}(\mathsf{f} \circ \mu)(\mathsf{n}), \tag{0.96}$$

which takes its argument in the source manifold N. The differential of a map applied to a vector yields a vector over the map. A procedure to compute the differential is:

```
(define (((differential mu) v) f)
  (v (compose f mu)))
```

The nomenclature of this subject is confused. The "differential of a map" between manifolds $d\mu$ takes one more argument than the "differential of a real-valued function on a manifold" df, but the two are related:

$$d\mu(\mathsf{v})(I)(\mathsf{n}) = \mathsf{d}\mu(\mathsf{v})(\mathsf{n}), \tag{0.97}$$

where the target manifold of μ is the reals and I is the identity function on the reals.

Let μ be the map from the time line to the manifold M, and $\partial/\partial t$ be a basis vector on the time line. Then $d\mu(\partial/\partial t)$ is the vector over the map μ that computes the rate of change of functions on M along the path μ. This is the velocity vector.

Let χ be a tuple of coordinates on M, with associated basis vectors X_i, and dual basis elements dx^i. The vector basis and dual basis over the map μ are X_i^μ and dx_μ^i. The components of the velocity (rates of change of coordinates along the path μ) are obtained by applying the dual basis over the map to the velocity

$$v^i(t) = \mathsf{dx}_\mu^i(d\mu(\partial/\partial t))(\mathsf{t}), \tag{0.98}$$

[24]The map ϕ_t^{v} is always invertible: $(\phi_t^{\mathsf{v}})^{-1} = \phi_{-t}^{\mathsf{v}}$ because of the uniqueness of the solutions of the initial value problem for ordinary differential equations.

where t is the coordinate for the point t.

For example, the velocities on a sphere are

```
(print-expression
  (((basis->1form-basis sphere-basis-over-mu)
    ((differential mu) d/dt))
   ((the-real-line '->point) 't0)))
(up ((D alpha) t0) ((D beta) t0)))
```

as expected.

Pullback of a Vector Field:
Given a vector field v on manifold M we can pullback the vector field through the map $\mu : \mathsf{N} \to \mathsf{M}$ as follows:

$$\mu^*\mathsf{v} = (\mu^{-1})_*\mathsf{v}, \tag{0.99}$$

or

$$\mu^*\mathsf{v}(\mathsf{f}) = (\mathsf{v}(\mathsf{f} \circ \mu^{-1})) \circ \mu \tag{0.100}$$

This may be useful when the map is invertible, as in the flow generated by a vector field.

Pullback of a Form Field:
We can also pullback a one-form field ω defined on M, but an honest definition is rarely written. The pullback of a one-form field applied to a vector field is intended to be the same as the one-form field applied to the pushforward of the vector field.

The pullback of a one-form field is often described by the relation

$$\mu^*(\omega)(\mathsf{v}) = \omega(\mu_*(\mathsf{v})), \tag{0.101}$$

but this is wrong, because the two sides are not functions of points in the same manifold. The one-form field ω must apply to a vector field on the manifold M, which takes a directional derivative of a function g defined on M and is evaluated at a place m on M, but the left-hand side is evaluated at a place in the manifold N.

A more precise description would be

$$\mu^*(\omega)(\mathsf{v})(\mathsf{n}) = \omega(\mu_*(\mathsf{v}))(\mu(\mathsf{n})). \tag{0.102}$$

Although this is accurate, it may not be effective, because computing the pushforward requires the inverse of the map μ. Again, this is available when the map is the flow generated by a vector field.

In fact it is possible to compute the pullback of a one-form field without having the inverse of the map. Using the Church $\boldsymbol{\lambda}$-calculus we can define a function that looks like a vector field on M, but which evaluates the vector field over the map at a point in N:

$$\mathcal{C}(\mathsf{v}_\mu, \mathsf{n}) = \boldsymbol{\lambda}(\mathsf{g})(\boldsymbol{\lambda}(\mathsf{m})\mathsf{v}_\mu(\mathsf{g})(\mathsf{n})). \tag{0.103}$$

Note that the bound variable m is ignored and there are no inverses. This is cumbersome in traditional mathematical notation, because of the unusual routing of arguments. But, unwinding the $\boldsymbol{\lambda}$-expressions:

$$\mathcal{C}(\mathsf{v}_\mu, \mathsf{n})(\mathsf{g})(\mathsf{m}) = \mathsf{v}_\mu(\mathsf{g})(\mathsf{n}). \tag{0.104}$$

Using this idea we can write the pullback of a one-form field as follows:

$$\mu^*(\omega)(\mathsf{v})(\mathsf{n}) = \omega(\mathcal{C}(d\mu(\mathsf{v}), \mathsf{n}))(\mu(\mathsf{n})) \tag{0.105}$$

We can show that the definition (0.105) implements the requirement (0.102). This only requires showing

$$\mu_*(\mathsf{v}) = \mathcal{C}(d\mu(\mathsf{v}), \mathsf{n}), \tag{0.106}$$

at $\mathsf{m} = \mu(\mathsf{n})$. The latter condition is always satisfied in the context (0.105).

The pullback of a k-form generalizes equation 0.102:

$$\mu^*(\omega)(\mathsf{u}, \mathsf{v}, \ldots)(\mathsf{n}) = \omega(\mu_*(\mathsf{u}), \mu_*(\mathsf{v}), \ldots)(\mu(\mathsf{n})). \tag{0.107}$$

Properties of Pullback:
The pullback through a map has many nice properties: it distributes through addition and through wedge product:

$$\mu^*(\theta + \phi) = \mu^*\theta + \mu^*\phi \tag{0.108}$$
$$\mu^*(\theta \wedge \phi) = \mu^*\theta \wedge \mu^*\phi \tag{0.109}$$

The pullback also commutes with the exterior derivative:

$$\mathsf{d}(\mu^*(\theta)) = \mu^*(\mathsf{d}(\theta)), \tag{0.110}$$

for θ a function or k-form field.

We can verify this by computing an example. Let μ map the rectangular plane to rectangular 3-space:[25]

```
(define mu
  (compose R3-chi-inverse
           (up (literal-function 'mux R2->R)
               (literal-function 'muy R2->R)
               (literal-function 'muz R2->R))
           R2-chi))
```

First, let's compare the pullback of the exterior derivative of a function with the exterior derivative of the pullback of the function.

```
(define f (compose (literal-function 'f R3->R) R3-chi))

(print-expression
 (((- ((pullback mu) (d f)) (d ((pullback mu) f)))
   X2)
  R2-point))
0
```

More generally, we can consider what happens to a form field. For a one-form field the result is as expected.

---

[25] We use the following definitions:

```
(define R3 (rectangular 3))
(instantiate-coordinates R3 '(x y z))
(define R3-chi (R3 '->coords))
(define R3-chi-inverse (R3 '->point))
(define R3->R (-> (UP Real Real Real) Real))
(define R3-point (R3-chi-inverse (up 'x0 'y0 'z0)))

(define R2 (rectangular 2))
(instantiate-coordinates R2 '(u v))
(define R2-chi (R2 '->coords))
(define R2-chi-inverse (R2 '->point))
(define R2->R (-> (UP Real Real) Real))
(define R2-point (R2-chi-inverse (up 'u0 'v0)))
(define X2 (literal-vector-field 'X R2))
(define Y2 (literal-vector-field 'Y R2))
```

```
(define theta (literal-1form-field 'theta R3))

(print-expression
 (((- ((pullback mu) (d theta)) (d ((pullback mu) theta)))
   X2 Y2)
  R2-point))
0
```

Lie Derivative

The Lie derivative is a measure of how things change along the integral curves of a vector field. The Lie derivative of any object is another object of the same kind. The form of the Lie derivative depends on the kind of object that is being differentiated. However, we will see that there are uniform ways of specifying the Lie derivative.[26]

Functions:
The simplest kind of object is a function on the manifold. Let f be a function on the manifold, and $\phi_t^{\mathsf{v}}(\mathsf{m})$ be the point along the integral curve of the vector field v beginning at m, advanced for interval t. Then the Lie derivative of f with respect to v is a function defined by:

$$\mathcal{L}_{\mathsf{v}}(\mathsf{f})(\mathsf{m}) = \lim_{t\to 0}\left(\frac{((\phi_t^{\mathsf{v}})^*\mathsf{f})(\mathsf{m}) - \mathsf{f}(\mathsf{m})}{t}\right) \tag{0.111}$$

$$= \lim_{t\to 0}\left(\frac{\mathsf{f}(\phi_t^{\mathsf{v}}(\mathsf{m})) - \mathsf{f}(\mathsf{m})}{t}\right)$$

$$= \mathsf{v}(\mathsf{f})(\mathsf{m}) \tag{0.112}$$

So the Lie derivative of a function is just the derivative of the function along the integral curve of v.

Vector Fields:
The Lie derivative of a vector field y with respect to a vector field v is a vector field that is defined by its behavior when applied to an arbitrary manifold function f. The Lie derivative compares

[26]Thanks to Will Farr, who showed us a unified framework for understanding Lie derivatives.

the pullback of the vector field y along the integral curves of the vector field v with the original vector field y:

$$\mathcal{L}_{\mathsf{v}}\mathsf{y}(\mathsf{f})(\mathsf{m}) = \lim_{t\to 0}\left(\frac{((\phi_t^{\mathsf{v}})^*\mathsf{y})(\mathsf{f})(\mathsf{m}) - \mathsf{y}(\mathsf{f})(\mathsf{m})}{t}\right) \tag{0.113}$$

$$= \lim_{t\to 0}\left(\frac{((\phi_{-t}^{\mathsf{v}})_*\mathsf{y})(\mathsf{f})(\mathsf{m}) - \mathsf{y}(\mathsf{f})(\mathsf{m})}{t}\right) \tag{0.114}$$

$$= \lim_{t\to 0}\left(\frac{(\phi_t^{\mathsf{v}})^*(\mathsf{y}((\phi_{-t}^{\mathsf{v}})^*\mathsf{f}))(\mathsf{m}) - \mathsf{y}(\mathsf{f})(\mathsf{m})}{t}\right) \tag{0.115}$$

$$= \lim_{t\to 0}\left(\frac{\mathsf{y}((\phi_{-t}^{\mathsf{v}})^*\mathsf{f})(\mathsf{m}) + t\mathsf{v}(\mathsf{y}((\phi_{-t}^{\mathsf{v}})^*\mathsf{f}))(\mathsf{m}) - \mathsf{y}(\mathsf{f})(\mathsf{m})}{t}\right)$$

$$= \lim_{t\to 0}\left(\mathsf{v}(\mathsf{y}((\phi_{-t}^{\mathsf{v}})^*\mathsf{f}))(\mathsf{m})\right) - \mathsf{y}(\mathsf{v}(\mathsf{f}))(\mathsf{m})$$

$$= \mathsf{v}(\mathsf{y}(\mathsf{f}))(\mathsf{m}) - \mathsf{y}(\mathsf{v}(\mathsf{f}))(\mathsf{m}) \tag{0.116}$$

$$= [\mathsf{v},\mathsf{y}](\mathsf{f})(\mathsf{m}) \tag{0.117}$$

So the Lie derivative of one vector field with respect to another is just their commutator. Note that the tricky step (0.115) in the derivation above depends on Hadamard's Lemma,[27] which allows us to deduce that: $(\phi_t^{\mathsf{v}})^*\mathsf{g} = \mathsf{g}\circ\phi_t^{\mathsf{v}} = \mathsf{g} + t\mathsf{v}(\mathsf{g}) + O(t^2)$. We apply this in equation (0.115), defining $\mathsf{g} = \mathsf{y}((\phi_{-t}^{\mathsf{v}})^*\mathsf{f})$.

The Lie derivative of a vector field can also be seen as a comparison of the advance of the application of the vector field to the function with the application of the vector field to the advance of the function:

$$\mathcal{L}_{\mathsf{v}}\mathsf{y}(\mathsf{f})(\mathsf{m}) = \lim_{t\to 0}\left(\frac{(\phi_t^{\mathsf{v}})^*(\mathsf{y}(\mathsf{f}))(\mathsf{m}) - \mathsf{y}((\phi_t^{\mathsf{v}})^*\mathsf{f})(\mathsf{m})}{t}\right)$$

$$= \lim_{t\to 0}\left(\frac{\mathsf{y}(\mathsf{f})(\phi_t^{\mathsf{v}}(\mathsf{m})) - \mathsf{y}(\mathsf{f}\circ\phi_t^{\mathsf{v}})(\mathsf{m})}{t}\right) \tag{0.118}$$

We can write this as a derivative[28]

$$\mathcal{L}_{\mathsf{v}}\mathsf{y}(\mathsf{f})(\mathsf{m}) = Dg(0), \tag{0.119}$$

---

[27] See Theodore Frankel, *The Geometry of Physics*, Cambridge University Press 1997, pp. 126–127.

[28] Using L'Hospital's rule.

with

$$g(t) = \mathsf{y}(\mathsf{f})(\phi_t^{\mathsf{v}}(\mathsf{m})) - \mathsf{y}(\mathsf{f} \circ \phi_t^{\mathsf{v}})(\mathsf{m}). \tag{0.120}$$

The derivative of the first term of g is of the form of the derivative of a function $\mathsf{h} = \mathsf{y}(\mathsf{f})$ along the integral curve of v starting at m, so this term is just $\mathsf{v}(\mathsf{y}(\mathsf{f}))(\mathsf{m})$.

Let $h(t) = \mathsf{y}(\mathsf{f} \circ \phi_t^{\mathsf{v}})(\mathsf{m})$, the second term of $g(t)$. We need to calculate $Dh(0)$. Now,[29]

$$(\mathsf{f} \circ \phi_t^{\mathsf{v}}) = \mathsf{f} + t\,\mathsf{v}(\mathsf{f}) + \cdots, \tag{0.121}$$

and since y is linear

$$\mathsf{y}(\mathsf{f} \circ \phi_t^{\mathsf{v}}) = \mathsf{y}(\mathsf{f}) + t\,\mathsf{y}(\mathsf{v}(\mathsf{f})) + \cdots. \tag{0.122}$$

So $Dh(0) = \mathsf{y}(\mathsf{v}(\mathsf{f}))(\mathsf{m})$. Combining these two terms we get the commutator: so

$$\mathcal{L}_{\mathsf{v}}\mathsf{y}(\mathsf{f}) = [\mathsf{v}, \mathsf{y}](\mathsf{f}), \tag{0.123}$$

as before.

Exponentiating Lie Derivatives:
We can exponentiate the Lie derivative and apply it to a vector field:

$$\begin{aligned} e^{\mathcal{L}_{\mathsf{v}}}\mathsf{y} &= \mathsf{y} + \mathcal{L}_{\mathsf{v}}\mathsf{y} + \frac{1}{2}\mathcal{L}_{\mathsf{v}}{}^2\mathsf{y} + \cdots \\ &= \mathsf{y} + [\mathsf{v}, \mathsf{y}] + \frac{1}{2}[\mathsf{v}, [\mathsf{v}, \mathsf{y}]] + \cdots \end{aligned} \tag{0.124}$$

Consider a simple case. We advance the coordinate basis vector $\partial/\partial y$ by an angle a around the circle. Let $\mathsf{J} = x\,\partial/\partial y - y\,\partial/\partial x$, the circular vector field. We recall

```
(define J (- (* x d/dy) (* y d/dx)))
```

We can apply the exponential of Lie derivative with respect to J to $\partial/\partial y$. We examine how the result affects a general function on the manifold:

[29] Again, by Hadamard's Lemma.

```
(series:for-each print-expression
  ((((exp (* 'a (Lie-derivative J))) d/dy)
    (literal-manifold-function 'f R2))
   ((R2 '->point) (up 1 0)))
  5)
(((partial 1) f) (up 1 0))
(* a (((partial 0) f) (up 1 0)))
(* -1/2 (expt a 2) (((partial 1) f) (up 1 0)))
(* -1/6 (expt a 3) (((partial 0) f) (up 1 0)))
(* 1/24 (expt a 4) (((partial 1) f) (up 1 0)))
;Value: ...
```

Apparently the result is

$$\exp\left(\mathcal{L}_{a\,(\mathsf{x}\,\partial/\partial y - \mathsf{y}\,\partial/\partial x)}\right)\frac{\partial}{\partial y} = \sin(a)\frac{\partial}{\partial x} + \cos(a)\frac{\partial}{\partial y}. \tag{0.125}$$

Form fields:
We can also define the Lie derivative of a form field. Consider a one-form field ω. Its Lie derivative with respect to a vector field v is a one-form field defined by its action on an arbitrary vector field y:

$$\begin{aligned}
\mathcal{L}_{\mathsf{v}}\omega(\mathsf{y})(\mathsf{m}) &= \lim_{t\to 0}\left(\frac{(\phi_t^{\mathsf{v}})^*\omega(\mathsf{y})(\mathsf{m}) - \omega(\mathsf{y})(\mathsf{m})}{t}\right) &(0.126)\\
&= \lim_{t\to 0}\left(\frac{\omega((\phi_t^{\mathsf{v}})_*\mathsf{y})(\phi_t^{\mathsf{v}}(\mathsf{m})) - \omega(\mathsf{y})(\mathsf{m})}{t}\right)\\
&= \lim_{t\to 0}\left(\mathsf{v}(\omega(\phi_t^{\mathsf{v}})_*\mathsf{y})(\mathsf{m}) + \frac{\omega((\phi_t^{\mathsf{v}})_*\mathsf{y})(\mathsf{m}) - \omega(\mathsf{y})(\mathsf{m})}{t}\right)\\
&= \mathsf{v}(\omega(\mathsf{y}))(\mathsf{m}) - \omega(\mathcal{L}_{\mathsf{v}}\mathsf{y})(\mathsf{m}) &(0.127)
\end{aligned}$$

The Lie derivative of a k-form field ω with respect to a vector field v is a k-form field that is defined by its behavior when applied to k arbitrary vector fields $\mathsf{w}_0, \ldots, \mathsf{w}_{k-1}$. Using the fact that the pullback of a k-form is the form applied to the pushforward of all its arguments evaluated at the right place, we can generalize the derivation for one-form fields to k-form fields, by the following formula:

$$\begin{aligned}
&\mathcal{L}_{\mathsf{v}}\omega(\mathsf{w}_0, \ldots, \mathsf{w}_{k-1})\\
&\quad = \mathsf{v}(\omega(\mathsf{w}_0, \ldots, \mathsf{w}_{k-1})) - \sum_{i=0}^{k-1}\omega(\mathsf{w}_0, \ldots, \mathcal{L}_{\mathsf{v}}\mathsf{w}_i, \ldots, \mathsf{w}_{k-1}) &(0.128)
\end{aligned}$$

Uniform Interpretation
Notice that in each case, the Lie derivative can be written in a form reminiscent of an elementary derivative. Consider equations (0.111), (0.113), and (0.126). Each is a limit of a quotient of a difference of a pullback of a thing and the thing. If we understand the pullback as the advance along the integral curve then this is the ordinary derivative.

Another uniform way of writing the Lie derivative is the vector field applied to the composite function minus the function applied to the Lie derivative of its arguments. For a function, the Lie derivative is just

$$\mathcal{L}_{\mathsf{v}}(\mathsf{f}) = \mathsf{v}(\mathsf{f}). \tag{0.129}$$

For a vector field (see equation 0.116)

$$\mathcal{L}_{\mathsf{v}}\mathsf{y}(\mathsf{f}) = \mathsf{v}(\mathsf{y}(\mathsf{f})) - \mathsf{y}(\mathcal{L}_{\mathsf{v}}(\mathsf{f})). \tag{0.130}$$

For a one-form field (see equation 0.127)

$$\mathcal{L}_{\mathsf{v}}\omega(\mathsf{y}) = \mathsf{v}(\omega(\mathsf{y})) - \omega(\mathcal{L}_{\mathsf{v}}\mathsf{y}). \tag{0.131}$$

The generalization for k-form fields is just equation (0.128).

Properties of the Lie Derivative:
For any k-form field ω and any vector field v the exterior derivative commutes with the Lie derivative with respect to the vector field:

$$\mathcal{L}_{\mathsf{v}}(\mathsf{d}\omega) = \mathsf{d}(\mathcal{L}_{\mathsf{v}}\omega) \tag{0.132}$$

We can verify this in three-dimensional rectangular space for our general one form:

```
(define V (literal-vector-field 'V R3))

(print-expression
  (((- ((Lie-derivative V) (d theta))
       (d ((Lie-derivative V) theta)))
    X Y)
   R3-point))
0
```

and for the general 2-form:

```
(print-expression
  (((- ((Lie-derivative V) (d omega))
       (d ((Lie-derivative V) omega)))
    X Y Z)
   R3-point))
0
```

The Lie derivative satisfies a another nice elementary relationship. If v and w are two vector fields then

$$[\mathcal{L}_{\mathsf{v}}, \mathcal{L}_{\mathsf{w}}] = \mathcal{L}_{[\mathsf{v},\mathsf{w}]}. \tag{0.133}$$

Again, for our general one-form θ:

```
(print-expression
 ((((- (commutator (Lie-derivative X) (Lie-derivative Y))
       (Lie-derivative (commutator X Y)))
    theta)
   Z)
  R3-point))
0
```

and for the two-form ω:

```
(print-expression
 ((((- (commutator (Lie-derivative X) (Lie-derivative Y))
       (Lie-derivative (commutator X Y)))
    omega)
   Z V)
  R3-point))
0
```

Interior Product

There is a simple, but useful operation available between vector fields and form fields called *interior product*. This is the substitution of a vector field v into the first argument of an p-form field ω to produce an $p-1$-form field:

$$(i_{\mathsf{v}}\omega)(\mathsf{v}_1, \ldots \mathsf{v}_{\mathsf{p}-1}) = \omega(\mathsf{v}, \mathsf{v}_1, \ldots \mathsf{v}_{\mathsf{p}-1}) \tag{0.134}$$

There is a rather nice identity for the Lie derivative in terms of the interior product called *Cartan's formula*:

$$\mathcal{L}_{\mathsf{v}}\omega = i_{\mathsf{v}}(\mathsf{d}\omega) + \mathsf{d}(i_{\mathsf{v}}\omega) \tag{0.135}$$

We can verify Cartan's formula in a simple case with a program:

```
(define R3 (rectangular 3))
(instantiate-coordinates R3 '(x y z))

(define X (literal-vector-field 'X R3))
(define Y (literal-vector-field 'Y R3))
(define Z (literal-vector-field 'Z R3))

(define alpha (literal-manifold-function 'alpha R3))
(define beta (literal-manifold-function 'beta R3))
(define gamma (literal-manifold-function 'gamma R3))

(define omega
  (+ (* alpha (wedge dx dy))
     (* beta (wedge dy dz))
     (* gamma (wedge dz dx))))

(define ((L1 X) omega)
  (+ ((interior-product X) (d omega))
     (d ((interior-product X) omega))))

(print-expression
  ((- (((Lie-derivative X) omega) Y Z)
      (((L1 X) omega) Y Z))
   ((R3 '->point) (up 'x0 'y0 'z0))))
0
```

Note that $i_{\mathsf{v}} \circ i_{\mathsf{u}} + i_{\mathsf{u}} \circ i_{\mathsf{v}} = 0$. One consequence of this is that $i_{\mathsf{v}} \circ i_{\mathsf{v}} = 0$.

Covariant Derivative of Vector Fields

Vectors at the same manifold point can be directly compared, but vectors at different points cannot. Parallel transport allows vectors at different points to be compared by transporting them to the same place. To define parallel transport, and the associated idea of covariant derivative, we need to introduce a *connection*. The connection cannot be derived, and there is freedom in the

choice of a connection. If there is a metric, then there is a special connection determined by it.

The covariant derivative with respect to the vector field u of the vector field v is

$$\begin{aligned}(\nabla_{\mathsf{u}}\mathsf{v})(\mathsf{f}) &= \sum_i \mathsf{e}_i(\mathsf{f})\left(\mathsf{u}(\tilde{\mathsf{e}}^i(\mathsf{v})) + \sum_j \omega^i_j(\mathsf{u})\tilde{\mathsf{e}}^j(\mathsf{v})\right)\\ &= \sum_i \mathsf{e}_i(\mathsf{f})\left(\mathsf{u}(v^i) + \sum_j \omega^i_j(\mathsf{u})v^j\right)\\ &= \mathsf{e}(\mathsf{f})(\mathsf{u}(v) + \omega(\mathsf{u})v),\end{aligned} \tag{0.136}$$

where ω is a structure-valued one-form and ω^i_j is a one-form for each selector i and j. These one-forms are called the Cartan one-forms, or the connection one-forms. The Cartan forms are defined with respect to the basis e and its dual $\tilde{\mathsf{e}}$. In these expressions v is the structure of coefficient manifold functions for the vector v with respect to the basis e, and v^i is a component of v. Though the covariant derivative is expressed with respect to a basis, the result is independent of the basis.

As a program, the covariant derivative is:[30]

[30]This program is slightly incomplete, for two reasons. It must construct a vector field, and it must make a differential operator. The full program is a bit longer:

```
(define (covariant-derivative Cartan)
  (let ((basis (Cartan->basis Cartan))
        (Cartan-forms (Cartan->forms Cartan)))
    (let ((vector-basis (basis->vector-basis basis))
          (1form-basis (basis->1form-basis basis)))
      (define (nabla X)
        (define (nabla_X V)
          (let ((v-components (1form-basis V)))
            (let ((deriv-components
                   (+ (X v-components)
                      (* (Cartan-forms X) v-components))))
              (define (the-derivative f)
                (* (vector-basis f) deriv-components))
              (procedure->vector-field the-derivative
                `((nabla ,(diffop-name X)) ,(diffop-name V))))))
        (procedure->vector-field nabla_X
                                 `(nabla ,(diffop-name X))))
      nabla)))
```

```
(define ((((covariant-derivative Cartan) X) V) f)
  (let ((basis (Cartan->basis Cartan))
        (Cartan-forms (Cartan->forms Cartan)))
    (let ((vector-basis (basis->vector-basis basis))
          (1form-basis (basis->1form-basis basis)))
      (let ((v-components (1form-basis V)))
        (* (vector-basis f)
           (+ (X v-components)
              (* (Cartan-forms X) v-components)))))))
```

The Cartan forms can be constructed from the dual basis one-forms:

$$\omega^i_j(\mathsf{v})(\mathsf{m}) = \sum_k \Gamma^i_{jk}(\mathsf{m})\,\tilde{\mathsf{e}}^k(\mathsf{v})(\mathsf{m}). \tag{0.137}$$

The connection coefficient functions Γ^i_{jk} are sometimes called the *Christoffel symbols*.[31] Making use of the structures, the Cartan forms are

$$\omega(\mathsf{v}) = \Gamma\,\tilde{\mathsf{e}}(\mathsf{v}). \tag{0.138}$$

Conversely, the Christoffel coefficients may be obtained from the Cartan forms

$$\Gamma^i_{jk} = \omega^i_j(\mathsf{e}_k). \tag{0.139}$$

The covariant derivative is independent of the coordinate system. We can show this in a simple case, using rectangular and

[31] This terminology may be restricted to the case in which the basis is a coordinate basis.

polar coordinates in the plane.[32] The Christoffel coefficients for rectangular coordinates in a (flat) plane are all zero:[33]

```
(define rectangular-Christoffel
  (make-Christoffel
   (let ((zero (lambda (m) 0)))
     (down (down (up zero zero)
                 (up zero zero))
           (down (up zero zero)
                 (up zero zero))))
   rectangular-basis))
```

The Christoffel coefficients for polar coordinates on the plane are:[34]

```
(define polar-Christoffel
  (make-Christoffel
   (let ((zero (lambda (m) 0)))
     (down (down (up zero zero)
                 (up zero (/ 1 r)))
           (down (up zero (/ 1 r))
                 (up (* -1 r) zero))))
   polar-basis))
```

From these we can make the Cartan forms in both coordinate systems:[35]

---

[32] We will need a few definitions:

```
(define R2 (rectangular 2))
(instantiate-coordinates R2 '(x y))
(define rectangular-basis (coordinate-system->basis R2))
(define R2-chi (R2 '->coords))
(define R2-chi-inverse (R2 '->point))

(define P2 (polar/cylindrical 2))
(instantiate-coordinates P2 '(r theta))
(define polar-basis (coordinate-system->basis P2))
(define P2-chi (P2 '->coords))
(define P2-chi-inverse (P2 '->point))
```

[33] Since the Christoffel coefficients are basis dependent they are packaged with the basis.

[34] We will derive these later.

[35] The code for making the Cartan forms is as follows:

```
(define rectangular-Cartan
  (Christoffel->Cartan rectangular-Christoffel))

(define polar-Cartan
  (Christoffel->Cartan polar-Christoffel))
```

The vector field J_z generates a rotation in the plane (the same as J). The covariant derivative with respect to $\partial/\partial x$ of J_z applied to an arbitrary manifold function is:

```
(define f
  (compose (literal-function 'f-rect R2->R) R2-chi))

(print-expression
 (((((covariant-derivative rectangular-Cartan) d/dx)
    J)
   f)
  R2-point))
(((partial 1) f-rect) (up x0 y0))
```

Note that this is the same thing as $\partial/\partial y$ applied to the function:

```
(print-expression ((d/dy f) R2-point))
(((partial 1) f-rect) (up x0 y0))
```

In rectangular coordinates, where the Christoffel coefficients are zero, the covariant derivative $\nabla_{\mathsf{u}}\mathsf{v}$ is the vector whose coefficents are obtained by applying u to the coefficients of v. Here, only one coefficient of J_z depends on x, the coefficient of $\partial/\partial y$, and it depends linearly on x. So $\nabla_{\partial/\partial x}\mathsf{J}_z = \partial/\partial y$.

Note that we get the same answer if we use polar coordinates to compute the covariant derivative:

```
(print-expression
 (((((covariant-derivative polar-Cartan) d/dx)
    J)
   f)
  R2-point))
(((partial 1) f-rect) (up x0 y0))
```

```
(define (Christoffel->Cartan Christoffel)
  (let ((basis (Christoffel->basis Christoffel))
        (Christoffel-symbols (Christoffel->symbols Christoffel)))
    (make-Cartan
     (* Christoffel-symbols (basis->1form-basis basis))
     basis)))
```

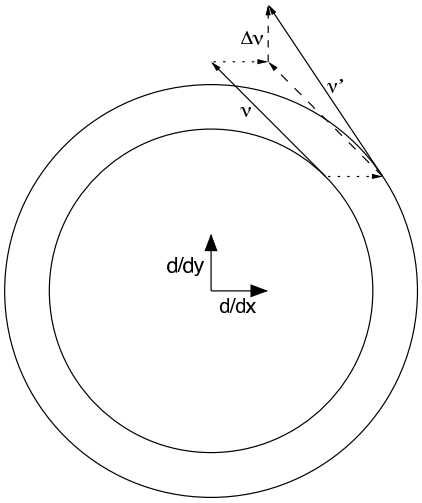

Figure 0.4 If v and v' are "arrow" representations of vectors in the circular field and we parallel transport v to v' in the $\partial/\partial x$ direction, then the difference is in the $\partial/\partial y$ direction.

In rectangular coordinates the Christoffel coefficients are all zero; in polar coordinates there are non-zero coefficients, but the value of the covariant derivative is the same. In polar coordinates the basis elements vary with position, and the Christoffel coefficients compensate for this.[36]

Of course, this is a pretty special situation. Let's try something more general.

```
(define v (literal-vector-field 'v R2))
(define w (literal-vector-field 'w R2))

(print-expression
 (((((- (covariant-derivative rectangular-Cartan)
        (covariant-derivative polar-Cartan))
     v)
    w)
   f)
  R2-point))
0
```

[36]Non-zero Christoffel coefficients also appear if the manifold has curvature, as we will see.

Parallel Transport

Given a connection we can define the operation of parallel transport of a vector along a path. Let γ be a path, a map from the real line to the manifold M. We would like to parallel transport w, a vector over the map γ, along the path γ.

We can specify w in terms of a basis over the map e^γ as follows:

$$\mathsf{w}(\mathsf{f})(\mathsf{t}) = \sum_i \mathsf{e}_i^\gamma(\mathsf{f})(\mathsf{t})\mathsf{c}^i(\mathsf{t}) = (\mathsf{e}^\gamma(\mathsf{f})\,\mathsf{c})(\mathsf{t}). \tag{0.140}$$

The coefficients c are functions on the time line and have structured values that are compatible for contraction with a basis on M.

The covariant derivative is expressed with respect to a basis and a set of Cartan forms. If we replace these with a basis over a map and the Cartan forms over the map then we get a covariant derivative over the map. The resulting covariant derivative is independent of the particular basis chosen, but now depends on the map, notated ∇^γ. The construction of the basis over the map was shown previously. The Cartan forms over the map are the pullback over the map of the Cartan forms: $\omega^\gamma = \gamma^*\omega$.

The equation governing the parallel transport of w is

$$\nabla^\gamma_{\partial/\partial t}\mathsf{w} = 0. \tag{0.141}$$

Let's figure out what this equation is for transport of an arbitrary vector w along an arbitrary path γ on a sphere. We start by constructing the necessary manifolds.

```
(instantiate-coordinates the-real-line 't)

(define M (rectangular 2))
(instantiate-coordinates M '(theta phi))
(define M-basis (coordinate-system->basis M))
```

We specify a connection by giving the Christoffel symbols.[37]

[37] We will show how to get these Christoffel symbols from a metric.

```
(define G-S2-1
  (make-Christoffel
   (let ((zero  (lambda (point) 0)))
     (down (down (up zero zero)
                 (up zero (/ 1 (tan theta))))
           (down (up zero (/ 1 (tan theta)))
                 (up (- (* (sin theta) (cos theta))) zero))))
   M-basis))
```

Next, we need the path γ, which we represent as a map from the real line to M, and w, the parallel-transported vector over the map:

```
(define gamma:N->M
  (compose (M '->point)
           (up (literal-function 'alpha)
               (literal-function 'beta))
           (the-real-line '->coords)))
```

where `alpha` is the colatitude and `beta` is the longitude.

```
(define basis-over-gamma
  (basis->basis-over-map gamma:N->M M-basis))

(define w
  (basis-components->vector-field
   (up (compose (literal-function 'w0)
                (the-real-line '->coords))
       (compose (literal-function 'w1)
                (the-real-line '->coords)))
   (basis->vector-basis basis-over-gamma)))

(define sphere-Cartan-over-gamma
  (Christoffel->Cartan-over-map G-S2-1 gamma:N->M))
```

Finally, we compute the residual of the equation (0.140) that governs parallel transport for this situation:

```
(print-expression
 (s:map/r
  (lambda (omega)
    ((omega
      (((covariant-derivative sphere-Cartan-over-gamma)
        d/dt)
       w))
     ((the-real-line '->point) 'tau)))
  (basis->1form-basis basis-over-gamma)))
(up (+ (* -1
           (sin (alpha tau))
           (cos (alpha tau))
           ((D beta) tau)
           (w1 tau))
        ((D w0) tau))
    (/ (+ (* (w0 tau) (cos (alpha tau)) ((D beta) tau))
           (* ((D alpha) tau) (cos (alpha tau)) (w1 tau))
           (* ((D w1) tau) (sin (alpha tau))))
        (sin (alpha tau)))))
```

Thus the equations governing the evolution of the components of the transported vector are:

$$Dw^0(\tau) = \sin(\alpha(\tau))\cos(\alpha(\tau))D\beta(\tau)w^1(\tau)$$
$$Dw^1(\tau) = -\frac{\cos(\alpha(\tau))}{\sin(\alpha(\tau))}\left(D\beta(\tau)w^0(\tau) + D\alpha(\tau)w^1(\tau)\right) \tag{0.142}$$

These equations describe the transport on a sphere, but more generally they look like

$$Dw(\tau) = f(\sigma(\tau), D\sigma(\tau))\, w(\tau), \tag{0.143}$$

where σ is the tuple of the coordinates of the path on the manifold and w is the tuple of the components of the vector. The equation is linear in w and driven by the path σ, as in a variational equation.

Suppose that the path is not known explicitly, but rather that it is the integral curve of a vector field v on the manifold. If b is the tuple of components of v (recall that $b = (\mathsf{v}(\chi)) \circ \chi^{-1}$, see equation 0.26) then the evolution of σ and w is governed by

$$D\sigma(\tau) = b(\sigma(\tau)) \tag{0.144}$$
$$Dw(\tau) = f(\sigma(\tau), D\sigma(\tau))\, w(\tau), \tag{0.145}$$

which we can collectively write as

$$Ds(\tau) = g(s(\tau)), \tag{0.146}$$

by defining the state $s = (\sigma, w)$ and substituting the right-hand side of equation (0.144) into equation (0.145). We can evolve this state by exponentiating the vector field defined by g on the Euclidean space of twice the dimension of our manifold.

Geodesic Motion

In geodesic motion the velocity vector is parallel transported by itself.[38] Recall (equation 0.98) that the velocity is the differential of the vector $\partial/\partial t$ over the map γ. More precisely, the equation of geodesic motion is

$$\nabla^{\gamma}_{\partial/\partial t} d\gamma(\partial/\partial t) = 0. \tag{0.147}$$

In coordinates, this is

$$D^2\sigma^i(t) - \sum_{jk} \Gamma^i_{jk}(\gamma(t)) D\sigma^j(t) D\sigma^k(t) = 0, \tag{0.148}$$

where $\sigma(t)$ is the coordinate path corresponding to the manifold path γ.

For example, let's consider geodesic motion on the surface of a unit sphere. We let

```
(define gamma:N->M
  (compose (M '->point)
           (up (literal-function 'alpha)
               (literal-function 'beta))
           (the-real-line '->coords)))
```

[38] The geodesic equation is often written

$$\nabla_{\mathsf{v}}\mathsf{v} = 0,$$

but this is nonsense. The geodesic equation is a constraint on the path, but the path does not appear in this equation. Further, the velocity along a path is not a vector field, so it cannot appear in either argument to the covariant derivative.

be a map from the real line to the sphere, with colatitude `alpha` and longitude `beta`. The geodesic equation is:

```
(print-expression
 (((((covariant-derivative sphere-Cartan-over-gamma) d/dt)
    ((differential gamma:N->M) d/dt))
   (M '->coords))
  ((the-real-line '->point) 't)))
```

$$\begin{pmatrix} -\cos(\alpha(t))\sin(\alpha(t))\,(D\beta(t))^2 + D^2\alpha(t) \\ \dfrac{2D\beta(t)\cos(\alpha(t))\,D\alpha(t)}{\sin(\alpha(t))} + D^2\beta(t) \end{pmatrix}$$

Notice that the program for ∇^γ uses the program for ∇ but uses a basis over the map and Cartan forms over the map for the basis and Cartan forms.

The geodesic equation is the same as the Lagrange equation for free motion constrained to the surface of the unit sphere. The Lagrangian for motion on the sphere is the composition of the free particle Lagrangian and the state transformation induced by the coordinate constraint:[39]

```
(define (Lfree s)
  (* 1/2 (square (velocity s))))

(define (sphere->R3 s)
  (let ((q (coordinate s)))
    (let ((theta (ref q 0)) (phi (ref q 1)))
      (up (* (sin theta) (cos phi))
          (* (sin theta) (sin phi))
          (cos theta)))))

(define Lsphere
  (compose Lfree (F->C sphere->R3)))
```

Then the Lagrange equations are:

[39]The method of formulating a system with constraints by composing a free system with the state-space coordinate transformation that represents the constraints can be found in *SICM*, Section 1.6.3.

```
(print-expression
  (((Lagrange-equations Lsphere)
    (up (literal-function 'alpha)
        (literal-function 'beta)))
   't))
```

$$\begin{bmatrix} -(D\beta(t))^2 \sin(\alpha(t)) \cos(\alpha(t)) + D^2\alpha(t) \\ 2D\alpha(t)\, D\beta(t) \sin(\alpha(t)) \cos(\alpha(t)) + D^2\beta(t) (\sin(\alpha(t)))^2 \end{bmatrix}$$

The Lagrange equations are true of the same paths as the geodesic equations. The second Lagrange equation is the second geodesic equation multiplied by $(\sin(\theta_0(t)))^2$, and the Lagrange equations are arranged in a down tuple, whereas the geodesic equations are arranged in an up tuple. The two systems are equivalent unless $\theta_0(t) = 0$, where the coordinate system is singular.[40]

Curvature

If the intrinsic curvature is not zero a vector parallel transported around a small loop will end up different from the vector that started. The Riemann tensor encapsulates this idea.

The Riemann curvature operator is

$$\mathcal{R}(\mathsf{u},\mathsf{v}) = [\nabla_{\mathsf{u}}, \nabla_{\mathsf{v}}] - \nabla_{[\mathsf{u},\mathsf{v}]} \tag{0.149}$$

or

```
(define ((Riemann-curvature Cartan) u v)
  (let ((nabla (covariant-derivative Cartan)))
    (- (commutator (nabla u) (nabla v))
       (nabla (commutator u v)))))
```

This allows us to compute the *intrinsic curvature* of a manifold at a point. The Riemann curvature takes a vector field as an argument and returns a vector field as a result.

The traditional Riemann tensor is

$$\mathsf{R}(\omega,\mathsf{x},\mathsf{u},\mathsf{v}) = \omega((\mathcal{R}(\mathsf{u},\mathsf{v}))(\mathsf{x})), \tag{0.150}$$

[40]The geodesic equations and the Lagrange equations are related by a contraction with the metric.

where ω is a one form field that measures the negative of the incremental change in the vector field x caused by parallel transporting it around the loop defined by the vector fields u and v.

So, for example,

```
(define sphere-Cartan (Christoffel->Cartan G-S2-1))

(print-expression
 (((Riemann sphere-Cartan) dphi d/dtheta d/dphi d/dtheta)
  ((M '->point) (up 'theta0 'phi0))))
+1
```

Here we have computed the ϕ component of the result of carrying a $\partial/\partial\theta$ basis vector around the parallelogram defined by $\partial/\partial\phi$ and $\partial/\partial\theta$. The result shows a net rotation in the ϕ direction.

Most of the coefficients of the Riemann tensor for the sphere are zero. The following are the nonzero coefficients:

$$\begin{aligned}
\mathsf{R}\left(\mathsf{d}\theta, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right)(\chi^{-1}(q^\theta, q^\phi)) &= \left(\sin(q^\theta)\right)^2 \\
\mathsf{R}\left(\mathsf{d}\theta, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right)(\chi^{-1}(q^\theta, q^\phi)) &= -\left(\sin(q^\theta)\right)^2 \\
\mathsf{R}\left(\mathsf{d}\phi, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right)(\chi^{-1}(q^\theta, q^\phi)) &= -1 \\
\mathsf{R}\left(\mathsf{d}\phi, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right)(\chi^{-1}(q^\theta, q^\phi)) &= 1
\end{aligned} \tag{0.151}$$

Explicit Transport

The result of the Riemann calculation is what we expect by explicitly doing the transport.

Equation (0.145) governs parallel transport along an integral curve σ of the vector field v with coefficient function b. The collective equation (0.146) is

$$Ds(\tau) = g(s(\tau)), \tag{0.152}$$

where $s = (\sigma, w)$. The differential equations describing the evolution of a function h of state s along the state path are

$$D(h \circ s) = (Dh \circ s)(g \circ s) = L_g h \circ s. \tag{0.153}$$

Exponentiation gives a finite evolution:

$$h(s(\tau+\epsilon)) = (e^{\epsilon L_g} h)(s(\tau)). \tag{0.154}$$

We want to parallel transport a vector w around a parallelogram defined by two vectors u and v. The vector w is really a vector over a map, where the map is the parametric curve describing our parallelogram. This map is implicitly defined in terms of the vector fields u and v. Let g_u and g_v be the right-hand sides of the differential equations for parallel transport along u and v respectively. Then evolution along u for interval ϵ, then along v for interval ϵ, then reversing u, and reversing v, brings σ back to where it started to second order in ϵ.

The state $s = (\sigma, w)$ after transporting s_0 around the loop is

$$\begin{aligned}
&(e^{-\epsilon L_{g_v}} I) \circ (e^{-\epsilon L_{g_u}} I) \circ (e^{\epsilon L_{g_v}} I) \circ (e^{\epsilon L_{g_u}} I)(s_0) \\
&\quad = (e^{\epsilon L_{g_u}} e^{\epsilon L_{g_v}} e^{-\epsilon L_{g_u}} e^{-\epsilon L_{g_v}} I)(s_0) \\
&\quad = (e^{\epsilon^2 [L_{g_u}, L_{g_v}] + \cdots} I)(s_0).
\end{aligned} \tag{0.155}$$

So the lowest order change in the transported vector is

$$\epsilon^2 W(([L_{g_u}, L_{g_v}] I)(s_0)), \tag{0.156}$$

where $W(\sigma, w) = w$.

Recall the differential equations (0.142) for parallel transport by the vector v of the components w of the vector w on the sphere are

$$\begin{aligned}
Dw^0(\tau) &= \sin(\alpha(\tau)) \cos(\alpha(\tau)) D\beta(\tau) w^1(\tau) \\
Dw^1(\tau) &= -\frac{\cos(\alpha(\tau))}{\sin(\alpha(\tau))} \left(D\beta(\tau) w^0(\tau) + D\alpha(\tau) w^1(\tau) \right),
\end{aligned}$$

where α and β are functions giving the coordinates of the path for each value of the parameter τ.

To integrate these equations of the coordinates of the vector being transported along a path $\sigma(\tau) = (\alpha(\tau), \beta(\tau))$, defined by differential equations we need to make a state space that represents both the path and the coordinates of the vector being transported. To represent these states we make a new manifold with four coordinates. The first two coordinates are the coordinates of the path. The second two coordinates are the components of the vector to

be transported, relative to the coordinate directions in the original manifold M.[41] The states are $s = (\sigma, w) = ((\theta, \phi), (w^0, w^1))$

```
(define M (rectangular 2))
(instantiate-coordinates M '(theta phi))

(define states (rectangular 4))
(instantiate-coordinates states '(Theta Phi w0 w1))
```

The differential equations (0.144) for the path are $D\sigma(\tau) = b(\sigma(\tau))$. The differential equations for the coordinates of the vector are driven by the path σ. The right-hand side of this composite differential equation is a vector field on this manifold. The vector field for a path through this bigger state space, given a constant vector field `v` for the path segment is constructed by:

```
(define (G v)
  (let ((alphadot (dTheta v)) (betadot (dPhi v)))
    (+ v
       (* (compose (* sin cos) Theta) betadot w1 d/dw0)
       (* -1
          (compose (/ cos sin) Theta)
          (+ (* w0 betadot) (* w1 alphadot))
          d/dw1))))
```

We can make vector fields which drag a vector in the θ and the ϕ directions by specifying the components of the vector field.

```
(define Gu (G d/dTheta))
(define Gv (G d/dPhi))
```

We also need to make initial states for the integration. These are the initial coordinates, and the initial vector to be transported.

```
(define (initial-state initial-coords w)
  (let ((Theta0 (ref initial-coords 0))
        (Phi0 (ref initial-coords 1)))
    (let ((m ((M '->point) (up Theta0 Phi0))))
      ((states '->point)
       (up Theta0 Phi0 ((dtheta w) m) ((dphi w) m))))))
```

[41] In the programs below there are both capitalized and lower-case `theta` and `phi`. This is to avoid a symbol conflict among the coordinate functions defined for the two manifolds. We need to distinguish them.

If we project the commutator of these vector fields in the θ and ϕ directions, applied to initial vectors that are in the ϕ and θ directions we get the same discrepancy predicted by the Riemann tensor!

```
(print-expression
 ((dw0 (commutator Gu Gv))
  (initial-state (up 'Theta0 'Phi0) d/dphi)))
(* -1 (expt (sin Theta0) 2))

(print-expression
 ((dw1 (commutator Gu Gv))
  (initial-state (up 'Theta0 'Phi0) d/dtheta)))
1
```

Torsion

There are many connections that describe the local properties of any particular manifold. A connection has a property called *torsion*, which is computed as follows:

$$\mathcal{T}(\mathsf{u},\mathsf{v}) = \nabla_{\mathsf{u}}\mathsf{v} - \nabla_{\mathsf{v}}\mathsf{u} - [\mathsf{u},\mathsf{v}] \tag{0.157}$$

The torsion takes two vector fields and produces a vector field. As a program this shows the dependence on the Cartan forms, which specify the connection:

```
(define ((torsion Cartan) u v)
  (let ((nabla (covariant-derivative Cartan)))
    (+ ((nabla u) v)
       (* -1 ((nabla v) u))
       (* -1 (commutator u v)))))
```

The torsion for the connection for the 2-sphere specified by the Christoffel symbols `G-S2-1` above is zero. We demonstrate this by applying the torsion to the basis vector fields:

```
(for-each
 (lambda (x)
   (for-each
    (lambda (y)
      (print-expression
       ((((torsion sphere-Cartan) x y) a-function)
        ((M '->point) (up 'theta0 'phi0)))))
    (list  d/dtheta d/dphi)))
 (list  d/dtheta d/dphi))
0
0
0
0
```

Metrics

We often want to impose further structure on a manifold to allow us to define lengths and angles. This is done by generalizing the idea of the Euclidean dot product, which allows us to compute lengths of vectors and angles between vectors in traditional vector algebra.

For vectors $\vec{u} = u^x\hat{x} + u^y\hat{y} + u^z\hat{z}$ and $\vec{v} = v^x\hat{x} + v^y\hat{y} + v^z\hat{z}$ the dot product is $\vec{u} \cdot \vec{v} = u^x v^x + u^y v^y + u^z v^z$. The generalization is to provide coefficients for these terms and to include cross terms, consistent with the requirement that the function of two vectors is symmetric. This symmetric, bilinear, real-valued function of two vector fields is called a metric field. Given a metric field g and a vector field v there is a scalar field $\mathsf{g}(\mathsf{v},\mathsf{v})$ that is the squared length of the vector at each point of the manifold.

For example, the natural metric on a sphere of radius R is

$$\mathsf{g}(\mathsf{u},\mathsf{v}) = R^2(\mathsf{d}\theta(\mathsf{u})\mathsf{d}\theta(\mathsf{v}) + (\sin\circ\theta)^2\mathsf{d}\phi(\mathsf{u})\mathsf{d}\phi(\mathsf{v})), \tag{0.158}$$

and the Lorentz metric on the four-dimensional space of special relativity is

$$\mathsf{g}(\mathsf{u},\mathsf{v}) = \mathsf{d}x(\mathsf{u})\mathsf{d}x(\mathsf{v}) + \mathsf{d}y(\mathsf{u})\mathsf{d}y(\mathsf{v}) + \mathsf{d}z(\mathsf{u})\mathsf{d}z(\mathsf{v}) - c^2\mathsf{d}t(\mathsf{u})\mathsf{d}t(\mathsf{v}). \tag{0.159}$$

Although these examples are expressed in terms of a coordinate basis, the metric is invariant under coordinate transformations.

The metric can be used to construct a one-form field ω_{u} from a vector field u, such that for any vector field v we have

$$\omega_{\mathsf{u}}(\mathsf{v}) = \mathsf{g}(\mathsf{v}, \mathsf{u}). \tag{0.160}$$

The operation of constructing a one-form field from a vector field using a metric is called "lowering" the vector field.

There is also an inverse metric that takes two one-form fields. It is defined by the relation

$$\delta^i_k = \sum_j \mathsf{g}^{-1}(\tilde{\mathsf{e}}^i, \tilde{\mathsf{e}}^j)\mathsf{g}(\mathsf{e}_j, \mathsf{e}_k), \tag{0.161}$$

where e and $\tilde{\mathsf{e}}$ are any basis and its dual basis.

The inverse metric can be used to construct a vector field v_ω from a one-form field ω, such that for any one-form field τ we have

$$\tau(\mathsf{v}_\omega) = \mathsf{g}^{-1}(\omega, \tau). \tag{0.162}$$

The operation of constructing a vector field from a one-form field using a metric is called "raising" the one-form field ω.

This vector field can be explicitly computed from the one-form field, with respect to a basis as follows:

$$\mathsf{v}_\omega = \sum_i \mathsf{g}^{-1}(\omega, \tilde{\mathsf{e}}^i)\mathsf{e}_i \tag{0.163}$$

The raising and lowering operations allow one to interchange the vector fields and the one-form fields. However they should not be confused with the dual operation that allows one to construct a dual one-form basis from a vector basis or construct a vector basis from a one-form basis. The dual operation that interchanges bases is defined without assigning a metric structure on the space.

There is a unique torsion-free connection induced by a metric. The Christoffel coefficients of the first kind are computed from the metric by the following

$$\bar{\Gamma}_{ijk} = \frac{1}{2}(\mathsf{e}_k(\mathsf{g}(\mathsf{e}_i, \mathsf{e}_j)) + \mathsf{e}_j(\mathsf{g}(\mathsf{e}_i, \mathsf{e}_k)) - \mathsf{e}_i(\mathsf{g}(\mathsf{e}_j, \mathsf{e}_k))), \tag{0.164}$$

for the coordinate basis e. We can then construct the Christoffel coefficients of the second kind (the ones used previously to define

a connection) by "raising the first index." To do this we define a function of three vectors, with a wierd currying:

$$\tilde{\Gamma}(\mathsf{v},\mathsf{w})(\mathsf{u}) = \sum_{ijk} \bar{\Gamma}_{ijk}\tilde{\mathsf{e}}^i(\mathsf{u})\tilde{\mathsf{e}}^j(\mathsf{v})\tilde{\mathsf{e}}^k(\mathsf{w}) \tag{0.165}$$

This function takes two vector fields and produces a one-form field. We can use it with equation (0.163) to construct a new function that takes two vector fields and produces a vector field:

$$\hat{\Gamma}(\mathsf{v},\mathsf{w}) = \sum_i \mathsf{g}^{-1}(\tilde{\Gamma}(\mathsf{v},\mathsf{w}),\tilde{\mathsf{e}}^i)\mathsf{e}_i \tag{0.166}$$

We can now construct the Christoffel coefficients of the second kind:

$$\Gamma^i_{jk} = \tilde{\mathsf{e}}^i(\hat{\Gamma}(\mathsf{e}_j,\mathsf{e}_k)) = \sum_m \bar{\Gamma}_{mjk}\mathsf{g}^{-1}(\tilde{\mathsf{e}}^m,\tilde{\mathsf{e}}^i) \tag{0.167}$$

The Cartan forms are then just

$$\omega^i_j = \sum_k \Gamma^i_{jk}\tilde{\mathsf{e}}^k = \sum_k \tilde{\mathsf{e}}^i(\hat{\Gamma}(\mathsf{e}_j,\mathsf{e}_k))\tilde{\mathsf{e}}^k \tag{0.168}$$

So, for example, we can compute the Christoffel coefficients for the sphere from the metric for the sphere. First, we need the metric

```
(define 2-sphere (rectangular 2))
(instantiate-coordinates 2-sphere '(theta phi))
(define 2-sphere-basis (coordinate-system->basis 2-sphere))

(define ((g-sphere R) u v)
  (* (square R)
     (+ (* (dtheta u) (dtheta v))
        (* (compose (square sin) theta)
           (dphi u)
           (dphi v)))))
```

The Christoffel coefficients of the first kind are a complex structure with all three indices down

```
(print-expression
 ((Christoffel->symbols
   (metric->Christoffel-1 (g-sphere 'R) 2-sphere-basis))
  ((2-sphere '->point) (up 'theta0 'phi0))))
(down
 (down (down 0 0)
       (down 0 (* (* (cos theta0) (sin theta0)) (expt R 2))))
 (down (down 0 (* (* (cos theta0) (sin theta0)) (expt R 2)))
       (down (* (* -1 (cos theta0) (sin theta0)) (expt R 2))
             0)))
```

And the Christoffel coefficients of the second kind have the innermost index up

```
(print-expression
 ((Christoffel->symbols
   (metric->Christoffel-2 (g-sphere 'R)
                          2-sphere-basis)))
 ((2-sphere '->point) (up 'theta0 'phi0))))
(down (down (up 0 0)
            (up 0 (/ (cos theta0) (sin theta0))))
      (down (up 0 (/ (cos theta0) (sin theta0)))
            (up (* -1 (cos theta0) (sin theta0)) 0)))
```

Hodge Star

The vector space of p-form fields on an n-dimensional manifold has dimension $n!/((n-p)!p!)$. This is the same dimension as the space of n–p-form fields. So these vector spaces are isomorphic. If we have a metric there is a natural isomorphism; for each p-form field ω on an n-dimensional manifold with a metric there is a n–p-form field $*\omega$, called its *Hodge dual*. The Hodge dual should not be confused with the duality of vector bases and one-form bases, which is defined without reference to a metric. The Hodge dual is useful for the elegant formalization of electrodynamics.

In Euclidean 3-space, if we think of a one form as a foliation of the space, then the dual is a two form of two-dimensional tubes, whose axes are perpendicular to the leaves of the foliation. The original one-form divides these tubes up into volume elements. For example, the dual of the basis one form $\mathsf{d}x$ is the two form $*\mathsf{d}x = \mathsf{d}y \wedge \mathsf{d}z$. We may think of $\mathsf{d}x$ as a set of planes perpendicular to the x axis. Then $*\mathsf{d}x$ is a set of tubes parallel to the x axis. In higher dimensional spaces the visualization is more complicated,

but the basic idea is the same. The Hodge dual of a two-form in two dimensions is a two-form that is perpendicular to the given two form. However, if the metric is indefinite (e.g. the Lorentz metric) there is an added complication with the signs.

The Hodge dual is a linear operator, so it can be defined by its action on the basis elements. Let $\{\frac{\partial}{\partial x^0}, \ldots, \frac{\partial}{\partial x^{n-1}}\}$ be an orthonormal basis of vector fields[42] and let $\{\mathsf{d}x^0, \ldots, \mathsf{d}x^{n-1}\}$ be the ordinary dual basis for the one-forms, then the n–p-form $*\omega$ that is the Hodge dual of the p-form ω can be defined by its coefficients with respect to the basis, using indices, as

$$(*\omega)_{j_p \ldots j_{n-1}} = \sum_{i_0 \ldots i_{p-1} j_0 \ldots j_{p-1}} \frac{1}{p!} \omega_{i_0 \ldots i_{p-1}} g^{i_0 j_0} \ldots g^{i_{p-1} j_{p-1}} \epsilon_{j_0 \ldots j_{n-1}}, \qquad (0.169)$$

where g^{ij} are the coefficients of the inverse metric and $\epsilon_{j_0 \ldots j_{n-1}}$ is either -1 or $+1$ if the permutation $\{0 \ldots n-1\} \to \{j_0 \ldots j_{n-1}\}$ is odd or even, respectively.

Electrodynamics

Using Hodge duals we can represent electrodynamics in an elegant way. Maxwell's electrodynamics is invariant under Lorentz transformations. We use four-dimensional rectangular coordinates for the flat spacetime of special relativity.

```
(define SR (rectangular 4))
(instantiate-coordinates SR '(t x y z))
(define an-event ((SR '->point) (up 't0 'x0 'y0 'z0)))
(define c 'c)          ; We like units.
```

The Lorentz metric is

$$\mathsf{g}(\mathsf{u}, \mathsf{v}) = \mathsf{dx}(u)\,\mathsf{dx}(v) + \mathsf{dy}(u)\,\mathsf{dy}(v) + \mathsf{dz}(u)\,\mathsf{dz}(v) - c^2\,\mathsf{dt}(u)\,\mathsf{dt}(v) \qquad (0.170)$$

As a program

[42] We have a metric, so we can define "orthonormal" and use it to construct an orthonormal basis given any basis. The Gram-Schmidt procedure does the job.

```
(define (g-Lorentz u v)
  (+ (* (dx u) (dx v))
     (* (dy u) (dy v))
     (* (dz u) (dz v))
     (* -1 (square c) (dt u) (dt v))))
```

The invariant length of a vector is described in terms of the metric.

$$\|\mathsf{v}\| = \mathsf{g}(\mathsf{v}, \mathsf{v}) \tag{0.171}$$

If the length is positive the vector is *spacelike*. If the length is negative the vector is *timelike*. If the length is zero the vector is *lightlike* or *null*.

The vector-field basis that is convenient for describing electrodynamics is orthonormal with respect to the metric.

```
(define SR-vector-basis
  (down (* (/ 1 c) d/dt) d/dx d/dy d/dz))

(define SR-1form-basis
  (up (* c dt) dx dy dz))

(define SR-basis
  (make-basis SR-vector-basis
              SR-1form-basis))
```

Indeed, the vector-field basis is orthonormal:

```
(print-expression
 ((SR-1form-basis SR-vector-basis)
  an-event))
(up (down 1 0 0 0)
    (down 0 1 0 0)
    (down 0 0 1 0)
    (down 0 0 0 1))
```

In this formulation of electrodynamics the electric and magnetic fields are represented together as a two-form field, the *Faraday tensor*. We will see that under Lorentz transformation the individual components are mixed. The Faraday tensor is:[43]

[43]This representation is from Misner, Thorne, and Wheeler, *Gravitation*, p.108

```
(define (Faraday Ex Ey Ez Bx By Bz)
  (+ (* Ex c (wedge dx dt))
     (* Ey c (wedge dy dt))
     (* Ez c (wedge dz dt))
     (* Bx (wedge dy dz))
     (* By (wedge dz dx))
     (* Bz (wedge dx dy))))
```

The Hodge dual of the Faraday tensor exchanges the electric and magnetic fields, negating the components that will involve time. The result is called the *Maxwell tensor*:

```
(define (Maxwell Ex Ey Ez Bx By Bz)
  (+ (* -1 Bx c (wedge dx dt))
     (* -1 By c (wedge dy dt))
     (* -1 Bz c (wedge dz dt))
     (* Ex (wedge dy dz))
     (* Ey (wedge dz dx))
     (* Ez (wedge dx dy))))
```

We make a Hodge dual operator for this situation

```
(define SR-star
  (Hodge-star g-Lorentz SR-basis))
```

and indeed, it transforms the Faraday tensor into the Maxwell tensor:

```
(print-expression
 (((- (SR-star (Faraday 'Ex 'Ey 'Ez 'Bx 'By 'Bz))
      (Maxwell 'Ex 'Ey 'Ez 'Bx 'By 'Bz))
   (literal-vector-field 'u SR)
   (literal-vector-field 'v SR))
  an-event))
0
```

One way to get electric fields is to have charges; magnetic fields can arise from motion of charges. In this formulation we combine the charge density and the current to make a 4-vector field:

```
(define (J charge-density Ix Iy Iz)
  (- (* (/ 1 c) (+ (* Ix dx) (* Iy dy) (* Iz dz)))
     (* charge-density c dt)))
```

The placement of `c` makes the components of the 4-vector uniform with respect to units.

To develop Maxwell's equations we need a general Faraday field and a general current-density field:

```
(define F
  (Faraday (literal-manifold-function 'Ex SR)
           (literal-manifold-function 'Ey SR)
           (literal-manifold-function 'Ez SR)
           (literal-manifold-function 'Bx SR)
           (literal-manifold-function 'By SR)
           (literal-manifold-function 'Bz SR)))

(define 4-current
  (J (literal-manifold-function 'rho SR)
     (literal-manifold-function 'Ix SR)
     (literal-manifold-function 'Iy SR)
     (literal-manifold-function 'Iz SR)))
```

Maxwell's equations in this language are

$$\mathsf{dF} = 0 \tag{0.172}$$

$$\mathsf{d}(*\mathsf{F}) = 4\pi\ *\mathsf{J}. \tag{0.173}$$

The first equation gives us what would be written in vector notation as

$$\mathrm{div}\vec{B} = 0, \tag{0.174}$$

and

$$\mathrm{curl}\vec{E} = -\frac{1}{c}\frac{d\vec{B}}{dt}. \tag{0.175}$$

The second equation gives us what would be written in vector notation as

$$\mathrm{div}\vec{E} = 4\pi\rho, \tag{0.176}$$

and

$$\mathrm{curl}\vec{B} = \frac{1}{c}\frac{d\vec{E}}{dt} + \frac{4\pi}{c}\vec{I}. \tag{0.177}$$

To see how these work out, we must evaluate each component. Since Faraday is a two-form field its exterior derivative is a three-form field, so we have to provide three basis vectors; so there are four possibilities for each equation; each possibility will yield

one of Maxwell's equations, written in coordinates, without vector notation.

```
(print-expression
 (((d F) d/dx d/dy d/dz) an-event))
(+ (((partial 1) Bx) (up t0 x0 y0 z0))
   (((partial 2) By) (up t0 x0 y0 z0))
   (((partial 3) Bz) (up t0 x0 y0 z0)))
```

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{0.178}$$

```
(print-expression
 (((d F) (* (/ 1 c) d/dt) d/dy d/dz) an-event))
(+ (((partial 2) Ez) (up t0 x0 y0 z0))
   (* -1 (((partial 3) Ey) (up t0 x0 y0 z0)))
   (/ (((partial 0) Bx) (up t0 x0 y0 z0)) c))
```

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = \frac{1}{c}\frac{\partial B_x}{\partial t} \tag{0.179}$$

```
(print-expression
 (((d F) (* (/ 1 c) d/dt) d/dz d/dx) an-event))
(+ (((partial 3) Ex) (up t0 x0 y0 z0))
   (* -1 (((partial 1) Ez) (up t0 x0 y0 z0)))
   (/ (((partial 0) By) (up t0 x0 y0 z0)) c))
```

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = \frac{1}{c}\frac{\partial B_y}{\partial t} \tag{0.180}$$

```
(print-expression
 (((d F) (* (/ 1 c) d/dt) d/dx d/dy) an-event))
(+ (((partial 1) Ey) (up t0 x0 y0 z0))
   (* -1 (((partial 2) Ex) (up t0 x0 y0 z0)))
   (/ (((partial 0) Bz) (up t0 x0 y0 z0)) c))
```

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{1}{c}\frac{\partial B_z}{\partial t} \tag{0.181}$$

```
(print-expression
 (((- (d (SR-star F)) (* '4pi (SR-star 4-current)))
   d/dx d/dy d/dz)
  an-event))
(+ (* -1 4pi (rho (up t0 x0 y0 z0)))
   (((partial 1) Ex) (up t0 x0 y0 z0))
   (((partial 2) Ey) (up t0 x0 y0 z0))
   (((partial 3) Ez) (up t0 x0 y0 z0)))
```

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \tag{0.182}$$

```
(print-expression
 (((- (d (SR-star F)) (* '4pi (SR-star 4-current)))
   (* (/ 1 c) d/dt) d/dy d/dz)
  an-event))
(+ (/ (* 4pi (Ix (up t0 x0 y0 z0))) c)
   (* -1 (((partial 2) Bz) (up t0 x0 y0 z0)))
   (((partial 3) By) (up t0 x0 y0 z0))
   (/ (((partial 0) Ex) (up t0 x0 y0 z0)) c))
```

$$\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} = -\frac{1}{c}\frac{\partial E_x}{\partial t} - \frac{4\pi}{c} I_x \tag{0.183}$$

```
(print-expression
 (((- (d (SR-star F)) (* '4pi (SR-star 4-current)))
   (* (/ 1 c) d/dt) d/dz d/dx)
  an-event))
(+ (/ (* 4pi (Iy (up t0 x0 y0 z0))) c)
   (* -1 (((partial 3) Bx) (up t0 x0 y0 z0)))
   (((partial 1) Bz) (up t0 x0 y0 z0))
   (/ (((partial 0) Ey) (up t0 x0 y0 z0)) c))
```

$$\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = -\frac{1}{c}\frac{\partial E_y}{\partial t} - \frac{4\pi}{c} I_y \tag{0.184}$$

```
(print-expression
 (((- (d (SR-star F)) (* '4pi (SR-star 4-current)))
   (* (/ 1 c) d/dt) d/dx d/dy)
  an-event))
(+ (/ (* 4pi (Iz (up t0 x0 y0 z0))) c)
   (* -1 (((partial 1) By) (up t0 x0 y0 z0)))
   (((partial 2) Bx) (up t0 x0 y0 z0))
   (/ (((partial 0) Ez) (up t0 x0 y0 z0)) c))
```

$$\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = -\frac{1}{c}\frac{\partial E_z}{\partial t} - \frac{4\pi}{c} I_z \tag{0.185}$$

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