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LATTICE APPROXIMATION IN THE STOCHASTIC QUANTIZATION
OF $(\phi^4)_2$ FIELDS

by

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LATTICE APPROXIMATION IN THE STOCHASTIC QUANTIZATION OF \( \phi^4 \) FIELDS

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I. INTRODUCTION

The Parisi-Wu program of stochastic quantization [8] involves construction of a stochastic process which has a prescribed Euclidean quantum field measure as its invariant measure. This program was rigorously carried out for a finite volume \( \phi^4 \) measure by G. Jona-Lasinio and P. K. Mitter in [6]. These results were extended in [2], which also proves a finite to infinite volume limit theorem. The aim of this note is to prove a related limit theorem, viz., that of the finite dimensional processes obtained by stochastic quantization of the lattice \( \phi^4 \) fields to their continuum limit, i.e., the \( \phi^4 \) process of [2], [6]. The proof imitates that of the limit theorem of [2] in broad terms, though the technical details differ. Note that this limit theorem can also be construed as an alternative construction of the \( \phi^4 \) process in finite volume.

The next section recalls the finite volume \( \phi^4 \) process. Section III summarizes the relevant facts about the lattice approximation to the \( \phi^4 \) field from Sections 9.5 and 9.6 of [4]. Section IV proves the limit theorem.

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II. THE ($\phi_2^\gamma$) PROCESS

Let $ICR^2$ be a finite rectangle which, for simplicity, we take to be the unit cube $x = (x_1, x_2) = (0, 1) \times (0, 1)$. Let $A$ denote the Dirichlet Laplace operator on $A$. It is diagonalized by the basis $e_k(x) = \sin(kx_1) \sin(kx_2)$, $x = (x_1, x_2)$, $k \in \mathbb{Z}^2$, $k^2 = k_1^2 + k_2^2$. In fact, $-A e_k = k^2 e_k$ where $k^2 = k_1^2 + k_2^2$. For $\gamma \in \mathbb{R}$, let $H^\gamma$ denote the Hilbert space obtained by completing $H^0$ with respect to the inner product $<f, g> = \sum_{k \in \mathbb{Z}^2} (k^2)^\gamma <f, e_k> <g, e_k>$ where $<\cdot, \cdot>$ is the $L^2$ scalar product. Topologize $Q = \cup H^\gamma$ by the countable family of seminorms $||\cdot||_n = <\cdot, \cdot>_n^\gamma$ and $Q = \cup H^\gamma$ via duality.

Let $C = (-\Delta + 1)^{-1}$, $C(\cdot, \cdot)$ its integral kernel, $C^\gamma$ its $\gamma$th operator power, and $\mu_C$ the centered Gaussian measure on $H^1$ with covariance $C$ [2], [6]. Let $\cdot : \cdot$ denote the Wick ordering with respect to $C$ (see [4], Ch. 3; for a definition). The ($\phi_2^\gamma$) measure on $H^1$ is defined by

$$\frac{d\mu}{d\mu_C} = \exp \left(-\frac{1}{4} \int :\phi^2 : \mu_C \right) \frac{d\mu_C}{Z} \quad [2.1]$$

where $Z = \int \exp \left(-\frac{1}{4} \int :\phi^2 : \mu_C \right) d\mu_C <\infty$.

See [4], Section 8.6, for details.

Let $0 < \epsilon < 1$ and $\mathcal{B}_k(\cdot)$, $k \in \mathbb{Z}^2$, a collection of independent standard Brownian motions. Define

$$W(t) = \sum_{k \in \mathbb{Z}^2} (k^2)^{-\gamma} \mathcal{B}_k(t) e_k(\cdot), \ t \geq 0.$$ 

This defines an $H^1$-valued Wiener process with covariance $C^{1-\gamma} [2], [6]$. The equation

$$d\phi(t) = -\frac{1}{2} (C^{-\gamma} \phi(t) + C^{1-\gamma} \phi^3(t, \cdot)) dt + dW(t). \quad [2.2]$$

with initial law $\mu$ can be shown to have a unique stationary weak solution as an $H^1$-valued process, defining an ergodic process called the ($\phi_2^\gamma$) process. See [2], [6] for details.
III. LATTICE APPROXIMATION

Let \( \Lambda = \{2^{-n}, \ n \in \mathbb{Z}\} \) and pick \( \delta \Lambda \). The finite lattice \( \Lambda_{\delta} \) with spacing \( \delta \) is defined as follows: Let \( \delta \mathbb{Z} = \{\delta n | n \in \mathbb{Z}\} \) and \( \int \Lambda = \int \Lambda \cap \delta \mathbb{Z} \).

\[ \Lambda_{\delta} = \Lambda \cap \delta \mathbb{Z}, \quad \Lambda_{\delta} = \operatorname{int} \Lambda_{\delta} \cup \delta \Lambda_{\delta} = \Lambda \cap \delta \mathbb{Z}, \quad \ell_2(\int \Lambda_{\delta}) \] is the Hilbert space with inner product

\[ \langle f, f \rangle_{\int \Lambda_{\delta}} = \sum_{x \in \int \Lambda_{\delta}} \delta^2 |f(x)|^2, \]

viewed as a subspace of \( L_2(\Lambda_{\delta}) \). On \( L_2(\delta \mathbb{Z}) \), define the forward gradient \( \delta_{\alpha} \) in direction \( \alpha \) by \( \delta_{\alpha} f(x) = \delta^{-1} [f(x + \delta v_\alpha) - f(x)] \) where \( v_\alpha \) is the unit vector in the \( \alpha \)-th direction for \( \alpha = 1, 2 \). The backward gradient \( \delta^*_{\alpha} \) is its adjoint with respect to the \( L_2(\delta \mathbb{Z}) \) inner product.

Let \( -\Delta_{\delta} = \delta_{\alpha, 1}^* \delta_{\alpha, 1} + \delta_{\alpha, 2}^* \delta_{\alpha, 2} \). Then \( (-\Delta_{\delta} f)(x) = \delta^{-2} (-4f(x) + \sum f(y)) \)

where the summation is over the nearest neighbours of \( x \). Let \( \Pi \) be the projection \( L_2(\delta \mathbb{Z}) \rightarrow L_2(\int \Lambda_{\delta}) \). The Dirichlet difference Laplacian \( \Delta_{\delta} \) is defined as \( \Pi \Delta_{\delta} \Pi \) and agrees with \( \Delta_{\delta} \) on \( \int \Lambda_{\delta} \).

Choose as a basis on \( L_2(\int \Lambda_{\delta}) \) the \((\delta^{-1} - 1)^2\) functions

\[ \{e_k(x) = e_k(x) | x \in \int \Lambda_{\delta} \}, \quad k = \pi, 2\pi, \ldots, (\delta^{-1} - 1)\pi; \quad \alpha = 1, 2 \} \]

Lemma 3.1 ([4], p. 221) \( \{e_k^\alpha \} \) diagonalize \( -\Delta_{\delta} \) with

\[ -\Delta_{\delta} e_k^\alpha = \lambda_{\delta}^\alpha e_k^\alpha, \quad \lambda_{\delta}^\alpha = 4\delta^{-2} \sum_{i=1}^{\delta^{-1} - 1} \sin^2 \left( \frac{\pi i}{\delta} \right) \]

Also, \( \langle e_k^\alpha, e_l^\beta \rangle_{\int \Lambda_{\delta}} = 1 \) if \( k = l \), \( 0 \) otherwise

Lemma 3.2 ([4], p. 222) The map \( i : e_k^\alpha \rightarrow e_k^\alpha \) defines an isometric imbedding of \( L_2(\int \Lambda_{\delta}) \rightarrow L_2(\Lambda) \).

Let \( \Pi_{\delta} \) be the projection operator on \( L_2(\Lambda) \) which truncates the Fourier series at \( k/\pi = \delta^{-1} \), so that

\[ \Pi_{\delta} \sum_{k} a_k e_k = \sum_{k} a_k e_k \quad \text{where} \quad \sum_{k} \text{denotes the summation over} \]

\[ B_{\delta} = \{k = (k_1, k_2) | 1 \leq k_1, k_2 \leq \delta^{-1} - 1, i = 1, 2 \}. \quad \text{Then} \quad i_{\delta}^\alpha = \Pi_{\delta} f |_{\Lambda_{\delta}} \]

We can consider \( C_{\delta}(-\Delta_{\delta} + 1)^{-1} : L_2(\int \Lambda_{\delta}) \rightarrow L_2(\int \Lambda_{\delta}) \) as an operator on \( L_2(\Lambda) \), via the above isometry, i.e., let \( C_{\delta} = \Pi_{\delta} C_{\delta}^\alpha \Pi_{\delta}^\alpha \) where the \( C_{\delta} \) on the right (resp. left) acts on \( L_2(\int \Lambda_{\delta})(\text{resp.} L_2(\Lambda)) \). As an operator on \( L_2(\Lambda) \), its kernel \( C_{\delta}(x, y) = \sum_{(k_1, k_2) \in B_{\delta}} (\delta^{-1} - 1)^2 e_k^\alpha(x) e_k^\beta(y) \), when restricted to the lattice points in \( \int \Lambda_{\delta} \), coincides with the matrix entries of \( C_{\delta} \) as an operator on \( L_2(\int \Lambda_{\delta}) \).

Lemma 3.3 ([4], pp. 222-224) \( \| C_{\delta} - C \| \leq (0 \delta^2) \) as operators on \( L_2(\Lambda) \),

Moreover, \( \sup_{x, y \in \Lambda} \| C_{\delta}(x, \cdot) \|_{L^p(\Lambda)} \leq (0 \delta^2) \) for \( p < (2\delta^{-1}, 1) \).
If \( \phi \) is a Gaussian field with covariance \( \mathbf{C} \)
\( \phi_{\delta}(x) = (i_{\delta}^{*} \phi)(x) \) for \( x \in \text{int} \Lambda_{\delta} \) defines a Gaussian lattice field with covariance \( \mathbf{C}_{\delta} = i_{\delta}^{*} \mathbf{C}_{\delta} i_{\delta} \).

The field \( \phi_{\delta} \) can be realized by a Gaussian measure on \( L_{2}(\text{int} \Lambda_{\delta}) \).

Explicitly, letting \( x \in \text{int} \Lambda_{\delta} \), \( d\phi_{\delta}(x) \) denote the Lebesgue measure on \( \mathbb{R} \cap \text{int} \Lambda_{\delta} \), the above measure is given by

\[
d\mu_{\delta} = (\det \mathbf{C}_{\delta})^{-\frac{1}{2}} \pi^{-\frac{1}{2}} \int_{\text{int} \Lambda_{\delta}} \exp \left( -\frac{1}{2} \sum_{x,y \in \text{int} \Lambda_{\delta}} \phi_{\delta}(x)^{\dagger} \mathbf{C}_{\delta}^{-1}(x,y) \phi_{\delta}(y) \right) \prod_{x} d\phi_{\delta}(x).
\]

This is the lattice analog of \( \mu_{\mathbf{C}} \). The lattice analog of \( \mu \) can now be defined as follows: Define for \( f \in L_{2}(\text{int} \Lambda_{\delta}) \),

\[
\phi_{\delta}^{n}(f) = \delta^{2} \sum_{x \in \text{int} \Lambda_{\delta}} \phi_{\delta}(x) \cdot \mathbf{F}_{\delta} f(x).
\]

The lattice analog \( \nu_{\delta} \) is given by

\[
d\nu_{\delta} = \exp \left( -\frac{1}{4} \phi_{\delta}^{n}(1) \right) d\mu_{\delta} \left\{ \left( \int \exp \left( -\frac{1}{4} \phi_{\delta}^{n}(1) \right) d\mu_{\delta} \right) \right\}^{1/2} [3.1]
\]

For \( k \in B_{\delta} \), let \( \{ \mathbf{e}_{k}(\cdot) \} \) be a collection of independent standard Brownian motions. For \( 0 < \varepsilon < 1 \), define

\[
\mathbf{B}_{\varepsilon}(t) = \delta^{2} \sum_{x \in \text{int} \Lambda_{\delta}} (\lambda_{k}^{\delta} + 1)^{-\varepsilon} \mathbf{B}_{k}(t) \mathbf{e}_{k}(\cdot), \quad t > 0.
\]

This defines an \( L_{2}(\Lambda) \)-valued Wiener process with covariance \( \mathbf{C}_{\delta}^{1-\varepsilon} \). The analog of [2.2] in the lattice case is

\[
d\phi_{\delta}(t) = \frac{1}{2} \left( \mathbf{C}_{\delta}^{-\varepsilon} \phi_{\delta}(t) + \mathbf{C}_{\delta}^{1-\varepsilon} : \phi_{\delta}^{n}(t) : \right) dt + dB_{\varepsilon}(t) \quad [3.2]
\]

where the operators act on \( L_{2}(\Lambda) \). \( \phi_{\delta}(\cdot) \) is viewed here as an \( L_{2}(\Lambda) \)-valued process. However, letting \( \phi_{\delta}(t) = \sum_{k} \phi_{\delta k}(t) \mathbf{e}_{k} \), [3.2] translates into an equivalent stochastic differential equation for finitely many scalar processes \( \phi_{\delta k}(\cdot) \) with locally Lipschitz (in fact, polynomial) coefficients. This ensures the existence of an \( a.s. \) unique strong solution to [3.2] up to an explosion time. That it does not explode \( a.s. \) is proved by a standard application of Khasminskii's test for non-explosion exactly as in [G], Section 3.

By identifying the vector \( \{ \phi_{\delta}(x), x \in \text{int} \Lambda_{\delta} \} \) with \( \phi_{\delta}(\cdot) \in L_{2}(\text{int} \Lambda_{\delta}) \), \( \nu_{\delta} \) can be considered as a probability measure on \( L_{2}(\text{int} \Lambda_{\delta}) \) and via the isometry \( i_{\delta}^{*} \), as a probability measure on \( L_{2}(\Lambda) \). We retain the notation \( \nu_{\delta} \) for the latter interpretation, as only this interpretation will be used henceforth. A computation similar to that of [2], Section 3, shows that the generator of the Markov process described by [3.2] is self-adjoint on \( L_{2}(\nu_{\delta}) \). By Theorem 2.3 of [3], the same holds for the associated transition semigroup of \( \{ \mathbf{T}_{t}, t \geq 0 \} \) of operators on \( L_{2}(\nu_{\delta}) \).

Thus for \( f, g \in L_{2}(\mu_{\delta}) \),

\[
\int f \mathbf{T}_{t} g \ d\mu_{\delta} = \int (T_{t} f) g \ d\mu_{\delta}.
\]

Letting \( f(\cdot) \equiv 1 \),

\[
\int T_{t} g \ d\mu_{\delta} = \int g \ d\mu_{\delta}, \quad \text{implying that } \mu_{\delta} \text{ is an invariant probability measure.}
\]
for $\phi_\delta(\cdot)$. In fact, the resulting process will be ergodic. We won't need this fact here, so we omit the details. From now on, [3.2] will always be considered with initial law $\nu_0$.

IV. THE CONTINUUM LIMIT

This section establishes the main result of this paper, viz., the convergence of $\phi_\delta(\cdot)$ to the $(\phi^t)$ process as $\delta \to 0$ in $A$, in the sense of weak convergence of $Q'$-valued processes. Thus we consider $\phi_\delta(\cdot)$ as a $Q'$-valued process and $\nu_\delta$ as a measure on $Q'$ via the injection of $L^2(A)$ into $Q'$. From theorem 9.6.4, p. 228, [4], it follows that the finite dimensional marginals of the collection $\{\phi_\delta(e_k), k \in B\}$ under $\nu_\delta$ converge weakly to the corresponding ones under $\nu$ as $\delta \to 0$ in $A$. Since $\nu_\delta, \nu$ are supported on $H^1$, it follows that $\nu_\delta + \nu$ weakly as probability measures on $Q'$. (A proof of the former assertion would go as follows: Since $H^1$ is Polish, it is homeomorphic to a $\mathcal{G}_0$ subset of $[0,1]^{\infty}$ whose closure $\overline{H^1}$ can be considered a compactification of $H^1$. As a measure on $\overline{H^1}$, $\{\nu_\delta\}$ are tight and for any weak limit point $\nu$ thereof, its restriction $\nu'$ to $H^1$ must yield the same finite dimensional marginals for $\{\phi(e_k), k \in B\}$ as $\nu$. Thus $\nu = \nu' = \nu$.)

As a first step towards proving the continuum limit, we prove some tightness results.

Let

\[ \phi_1(t) = \phi_\delta(t) \]
\[ \phi_\delta(t) = \frac{1}{2} \int t \to (s) \, ds \]
\[ \phi_\delta(t) = \frac{1}{2} \int t \to (s) \, ds \]
\[ \phi_\delta(t) = B_\delta(t) \]

for $t \leq 0$. Pick $t \leq t_0$ in $[0,T]$, $\omega > T > 0$. In what follows, $K$ denotes a positive constant (not always the same) that may depend on $T$, but not on $\delta$. Let $f \in Q$.

Lemma 4.1

\[ E\left[ \left( \int_{t_0}^{t_2} C_\delta^{t_2} \phi_\delta(t)(f) \, dt \right)^2 \right] \leq K \left| t_2 - t_1 \right|^2 \]  

[4.1]

Proof: Using Jensen's inequality and stationarity of $\phi_\delta(\cdot)$, one obtains

\[ E\left[ \left( \int_{t_0}^{t_2} C_\delta^{t_2} \phi_\delta(t)(f) \, dt \right)^2 \right] \leq K \left| t_2 - t_1 \right|^2 E\left[ C_\delta^{t_2} \phi_\delta(0)(f)^2 \right]. \]

Letting $M_\delta = \mu_\delta / \mu_\delta^{C_\delta}$, the expectation on the right is bounded by

\[ \left[ \int C_\delta^{t_2} \phi(f)^2 \mu_\delta^{C_\delta}(\phi) \right]^{1/2} \left[ \int L_\delta^2 \mu_\delta^{C_\delta} \right]^{1/2}. \]

By Lemma 9.6.2, p. 227, [4], the second term above is bounded uniformly in $\delta$. Using Feynman graph calculations, as in Theorem 9.5.3, p. 191, [4], one has

\[ \text{Text should end on this page.} \]
\[ \int |C_0^\phi (f)|^2 \, d\mu_\phi (\phi) \leq K \| C_0^\phi f \|_2^2. \]

Now
\[ \| C_0^\phi f - C_0^\phi f \|_2^2 = 0 \sum_{e_k \in E} \langle f, e_k \rangle^2 \left( \lambda_k^2 + 1 \right)^2 I\{ k \in \mathbb{B}_0 \} - (\lambda_k + 1)^2. \]

The summand on the right can be dominated in absolute value by \( K_1 \langle f, e_k \rangle^2 \lambda_k^2 \) which is summable for \( f \in \mathcal{F}_0 \). By the dominated convergence theorem,
\[ \lim_{\delta \to 0} \| C_0^\phi f - C_0^\phi f \|_2^2 = 0, \]
implying \( \sup_\delta \| C_0^\phi f \|_2 < \infty \). [4.1] follows. QED

**Lemma 4.2**
\[ E \left[ \int_{t_0}^{t_1} [\phi (f)]^2 \, dt \right] \leq K \| t_2 - t_1 \|^2. \quad [4.2] \]
This follows along similar lines.

**Lemma 4.3**
\[ E \left[ \left( B_0 (t_2) (f) - B_0 (t_1) (f) \right)^4 \right] \leq K \| t_2 - t_1 \|^2. \quad [4.3] \]

**Proof** The lefthand side equals
\[ 3 \| C_0^\phi (f) \|_2^2 \| t_2 - t_1 \|^2 \leq 3 \sup_\delta \| C_0^{(1-\delta)/2} f \|_2^2 \| t_2, t_1 \|^2. \]
As in the proof of Lemma 4.1, one can prove
\[ \lim_{\delta \to 0} \| C_0^{(1-\delta)/2} f - C_0 f \|_2^2 = 0. \]
Thus \( \sup_\delta \| C_0 f \|_2 < \infty \) and the claim follows. QED

**Corollary 4.1**
\[ E \left[ \left| \phi (t_2) (f) - \phi (t_1) (f) \right|^4 \right] \leq K \| t_2 - t_1 \|^2. \quad [4.4] \]

**Proof** Follows from [3.2] and [4.1] - [4.3]. QED

**Lemma 4.4**
The laws of the processes \( \phi_1 (\cdot), \phi_2 (\cdot), \phi_3 (\cdot), \phi_4 (\cdot) \)
viewed as \( (C([0, \omega]; Q)) )^4 \)-valued random variables remain tight as \( \delta \)
varies over \( A \).

**Proof** By Theorem 3.1 of [7], it suffices to establish the tightness
of \( \phi_1 (\cdot) (f), \phi_2 (\cdot) (f), \phi_3 (\cdot) (f), \phi_4 (\cdot) (f) \) on \([0, T] \) as \( (C([0, T]; R)) )^4 \)-valued random variables for arbitrary \( T > 0 \) and \( f \in \mathcal{F}_0 \).
This, however, is immediate from the tightness of \( \{ \nu_0 \} \) (since \( \mu_0 \to \mu \)
weakly as a measure on \( H^3 \)), the estimates [4.1] - [4.4] and the
criterion of [1], p. 95. QED

Recall that a family of probability measures on a product of
Polish spaces is tight if and only if its images under projection onto
each factor space are. Letting \( \{ e_i \} \) denote an enumeration of \( \{ e_k \} \).
This implies, in view of the foregoing, that \( \phi_1 (\cdot) (e_i), \ldots, \phi_4 (\cdot) (e_i), \ldots, \phi_1 (\cdot) (e_2), \ldots, \phi_2 (\cdot) (e_2), \ldots, \phi_3 (\cdot) (e_2), \ldots, \phi_4 (\cdot) (e_2), \ldots \)
are tight as \( (C([0, \omega]; R)) )^4 \)-valued random variables. By dropping to a subsequence
of \( A \), denoted by \( A \) again, we may assume that they converge in law as \( \delta \to 0 \) along \( A \). Then for any finite subset \( \{ t_1, \ldots, t_k \} \) of \([0, \omega] \) and a
collection \( \{ g_1, \ldots, g_k \} \) of finite linear combinations of \( \{ e_i \} \), the
Consider a collection \( f_1, \ldots, f_k \) in \( Q \). Using the kind of estimates used in the proofs of Lemmas 4.1-4.3, we have

\[
E[|\hat{c}_{ij}(t_j) - g_j|^2] \leq M \left| f_j - g_j \right|^2_{L^2},
\]

for a suitable constant \( M \) depending on \( \max(t_1, \ldots, t_k) \). As \( \delta \to 0 \) in \( A \), the righthand sides of [4.6] - [4.8] converge to the corresponding quantities with \( C \) replacing \( C_0 \). Since \( g_j \) can be obtained by suitably truncating the Fourier series of \( f_j \) in \( \{e_i \} \), each of these limiting expressions and the righthand side of [4.5] can be made smaller than any prescribed \( \eta > 0 \) uniformly in \( 1 \leq j \leq k \) by a suitable choice of \( \{g_j\} \). It follows that the righthand sides of [4.5] - [4.8] can be made smaller than any prescribed \( \eta > 0 \) uniformly in \( \delta \in A \) and \( 1 \leq j \leq k \) by a suitable choice of \( \{g_j\} \).

Let \( \{h_k\} \) be an enumeration of finite linear combinations of \( \{\hat{e}_i\} \) with rational coefficients. By a well-known theorem of Skorohod ([5], p. 9), we can construct on some probability space random variables \( X_{ijl}^\delta, Y_{ijl}^\delta \) \( \delta \in A, 1 \leq i \leq 4, 1 \leq j \leq k, l > 1, \) such that \( \{X_{ijl}\} \) agrees in law with \( \{\hat{c}_{ij}(t_j)(h_k)\} \) for each fixed \( \delta \) and \( X_{ijl} \to Y_{ijl} \) a.s. as \( \delta \to 0 \) in \( A \). By augmenting this probability space, if necessary, we may construct on it random variables \( Z_{ijl}^\delta, (\delta, i, j) \) as above, such that the joint law of \( \{\hat{c}_{ij}(t_j)(f_j), \hat{c}_{ij}(t_j)(h), \hat{c}_{ij}(t_j)(h_k), \ldots\} \) agrees with that of \( \{Z_{ijl}^\delta, X_{ijl}^\delta, X_{ijl}^\delta, \ldots\} \) for each \( \delta, i, j \). Since \( X_{ijl} \to Y_{ijl} \) a.s. and \( E[|X_{ijl}^\delta|^2] = E[|\hat{c}_{ij}(t_j)(h_k)|^2] \) can be bounded uniformly in \( \delta \) as \( \delta \to 0 \) in \( A \) for each \( i, j, l \). On the other hand, given \( \eta > 0 \), we can pick \( \delta, \delta' > 0 \) such that setting \( g_j = h_k(\delta) \) in [4.5] - [4.8] makes all the quantities on the righthand side there less than \( \eta \). Thus

\[
\lim_{\delta, \delta' \to 0} E[|Z_{ijl}^\delta - Z_{ijl}^\delta'|^2] \leq 2n + \lim_{\delta, \delta' \to 0} E[|X_{ijl}^\delta(i) - X_{ijl}^\delta'(i)|^2] = 2n.
\]

Thus \( Z_{ijl}^\delta \) converge in mean square for each \( i, j \) as \( \delta \to 0 \) in \( A \). It follows that the joint laws of \( \{\hat{c}_{ij}(t_j)(f_j), 1 \leq i \leq 4, 1 \leq j \leq k\} \) converge. Theorem 5.3, [7], now implies that \( \{\hat{c}_{ij}^\delta(\cdot), \ldots, \hat{c}_{ij}^\delta(\cdot)\} \) converge as \( C([0,\infty); \mathbb{R}^n) \)-valued random variables. Let \( \{\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot), \phi_4(\cdot)\} \) denote its limit in law (abbreviated as "l.i.l." henceforth). By taking the l.i.l. in [3.2] along an appropriate subsequence,
Theorem 4.1 $\phi_1(\cdot)$ is the $(\phi^*)_2$ process.

Proof We prove the theorem by identifying each term of [4.9]. Let $f \in \mathcal{Q}$.

By Jensen's inequality and stationarity, $E[\int \phi_\delta(s)(C^\delta f)ds]$

$$\leq t E[|\phi_\delta(0)(C^\delta f-C^\delta f)|^2] < t E[|C^\delta f-C^\delta f|^2].$$

The right-hand side tends to zero as $\delta \to 0$ by arguments similar to those employed in the proof of Lemma 4.1. Thus

$$\lim_{\delta \to 0} \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^\delta f)ds\right) = \lim_{\delta \to 0} \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^\delta f)ds\right)$$

$$= (\phi_1(\cdot), \int_0^t \phi_1(s)(C^\delta f)ds).$$

It follows that

$$\phi_2(t)(\cdot) = \frac{1}{2} \int_0^t \phi_1(s)(C^\delta f)ds a.s.$$

Similarly

$$\lim_{\delta \to 0} \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^\delta f)ds\right) = \lim_{\delta \to 0} \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^\delta f)ds\right)$$

$$= (\phi_1(\cdot), \int_0^t \phi_1(s)(C^\delta f)ds).$$

Let $\alpha > \delta$ in $A$. Then

$$\lim_{\delta \to 0} \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^\delta f)ds\right) = \lim_{\delta \to 0} \left(\phi_\delta(\cdot), \int_0^t \phi_\delta(s)(C^\delta f)ds\right)$$

$$= (\phi_1(\cdot), \int_0^t \phi_1(s)(C^\delta f)ds).$$

where $\phi_\alpha(\cdot)$ is defined by

$$\phi_\alpha(t)(h) = \sum_{k} \phi_1(t)(e_k) < e_k, h>, h \in \mathcal{Q}.$$
The above limit equals
\[ (\phi_1(\cdot), \int_0^t \phi_1^3(s) (C^{-1} f) \, ds), \]
Thus
\[ \phi_3(t)(f) = -\frac{1}{2} \int_0^t \phi_1^3(s) (C^{-1} f) \, ds a.s. \]

Finally, it is easy to check that \( \phi_\delta(\cdot) \) will be a Wiener process with covariance \( C^{1-\delta} \). Thus \( \phi_1(\cdot) \) satisfies [3.2] with initial law \( \mu \). By the uniqueness in law of this equation (proved in [2], Section IV), we conclude that \( \phi_1(\cdot) \) is the \( (\phi_\delta)^2 \) process.

QED

Corollary 4.2 \( \phi_\delta(\cdot) \) converge in law to \( \phi(\cdot) \) as \( \delta \to 0 \) in A, as defined originally.

Proof A careful look at the foregoing shows that any subsequence of \( A \) will have a further subsequence along which the above convergence holds.

QED

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