Time-Varying vs. Time-Invariant Compensation for Rejection of Persistent Bounded Disturbances and Robust Stabilization

Jeff S. Shamma   Munther A. Dahleh

Massachusetts Institute of Technology
Laboratory for Information and Decision Systems
Cambridge, MA 02139 MA
Room 35~~409~~ 402
(617) 253-5992
LIDS-P-1858

Abstract

This paper considers time-varying compensation for linear time-invariant discrete-time plants subject to persistent bounded disturbances. In the context of certain feedback objectives, it is shown that time-varying compensation offers no advantage over time-invariant compensation. These results complement similar existing results for feedback systems subject to finite-energy disturbances.

First, it is shown that time-varying compensation does not improve the optimal rejection of persistent bounded disturbances. This result is obtained by exploiting a key observation that any time-varying compensator which yields a given degree of disturbance rejection must do so uniformly over time, thereby removing any advantage of time-variation. This key observation is further exploited to show that time-varying compensation does not improve the optimal rejection of disturbances regardless of the norm used to measure the disturbances. Thus, absolutely summable, finite-energy, or persistent bounded disturbances may be treated in the same manner.

It is then shown that time-varying compensation does not help in the bounded-input bounded-output robust stabilization of time-invariant plants with unstructured uncertainty. In doing so, it is also shown that the small-gain theorem is both necessary and sufficient for the bounded-input bounded-output stability of certain linear time-varying plants subject to unstructured linear time-varying perturbations.

1. Introduction

This paper addresses the possible advantage of time-varying compensation for time invariant plants in order to achieve certain feedback objectives, namely, optimal disturbance rejection and robust stabilization with unstructured uncertainty. These objectives are informally summarized as follows.

The problem of optimal disturbance rejection is to find some compensator which stabilizes a given linear time-invariant feedback control system and also minimizes the maximum response of certain "error signals" to possible exogenous disturbances. In the case where the disturbances are assumed to have finite energy, and the quantity to be made small is the energy of the resulting error signals, the optimal disturbance rejection problem is also known as $\mathcal{H}^\infty$-optimal control (cf. [7]). In the case where the disturbances are persistent and bounded, and the quantity to be made small is the maximum value of the resulting error signals, the optimal disturbance rejection problem is also known as ℓ^1-optimal control (cf. [1, 15]). Further background and motivation to optimal disturbance rejection problems may be found in [1, 7, 15] and references contained therein.

In [6, 10] it was shown that in the context of optimal rejection of finite-energy disturbances, time-varying compensation offers no advantages over time-invariant compensation. That is, time-varying compensators cannot do better than time-invariant compensators in uniformly reducing the energy of the resulting error responses to exogenous finite-energy disturbances. In [9], this result was strengthened to encompass nonlinear time-varying compensators.

The question of time-varying compensation for minimizing the maximum response to persistent bounded disturbances was addressed in [12], where it was shown that under certain very restrictive assumptions, time-varying compensation offers no advantages over time-invariant compensation.

In this paper, the general problem of time-varying vs. time-invariant compensation

for minimizing the maximum response to persistent bounded disturbances is addressed. As in the case of finite-energy disturbances, it is shown that time-varying compensators cannot do better than time-invariant compensators in uniformly reducing the maximum error responses.

This result is obtained by exploiting a key observation that any time-varying compensator which yields a given degree of disturbance rejection must do so uniformly over time, thereby removing any advantage of time-variation.

This key observation is then further exploited to show that time-varying compensation does not improve the rejection of disturbances *regardless* of the norm used to measure the disturbances. Thus, finite-energy, persistent bounded, and even absolutely summable disturbances may be treated in the same manner. Given this independence of norms, it is only the time-varying vs. time-invariant aspect of the problem which is isolated to lead to the desired results.

The second objective addressed in this paper is the bounded-input/bounded-output robust stabilization of a time-invariant plant with unstructured uncertainty.

One example of unstructured uncertainty is that of "additive plant uncertainty." More precisely, consider the *family* of plants $\mathcal{P}_{\text{add}} = \{P_o + W\Delta\}$ where P_o is a known linear time-invariant plant; Δ, the unstructured uncertainty, is an arbitrary nonlinear time-varying system which is known only to be stable and to satisfy a given norm bound; and W is a known linear time-invariant system which "shapes and normalizes" the effect of Δ (e.g., [4, 5]). Another example of unstructured uncertainty is "multiplicative plant uncertainty," where the family of plants takes the form $\mathcal{P}_{\text{mul}} = \{P_o(I + W\Delta)\}$.

The problem of robust stabilization is then to find a single compensator which not only stabilizes the nominal plant, P_o, but also stabilizes the *entire family* of plants, $\mathcal{P}_{\text{add}}$ or $\mathcal{P}_{\text{mul}}$. In this case, the compensator is said to *robustly stabilize* the family $\mathcal{P}_{\text{add}}$ or $\mathcal{P}_{\text{mul}}$, respectively.

Now depending of the nature of the exogenous disturbances to the perturbed feedback

system, the notion of "stabilization" may take on different interpretations (e.g., [3]). For example, stabilization may mean that finite-energy disturbances lead to finite-energy signals in the feedback loop. Alternatively, one may wish that exogenous disturbances which are bounded in magnitude lead to signals in the feedback loop which are also bounded in magnitude.

In [8, 13], it was shown that in the context of robust stabilization of time-invariant plants with unstructured uncertainty and finite-energy disturbances, nonlinear time-varying compensation offers no advantage over time-invariant compensation. That is, given a nonlinear time-varying compensator which robustly stabilizes a plant with a given unstructured uncertainty (such as either family $\mathcal{P}_{\text{add}}$ or $\mathcal{P}_{\text{mul}}$), then there exists a linear time-invariant compensator which robustly stabilizes the same family of plants.

In this paper, the issue of time-varying compensation for *bounded-input/bounded-output* robust stabilization of time-invariant plants with unstructured uncertainty is addressed. More precisely, it is shown that given a linear time-varying compensator which robustly stabilizes a plant with a given unstructured uncertainty, then there exists a linear time-invariant compensator which robustly stabilizes the same family of plants. However, the notion of stability used here is bounded-input/bounded-output stability rather than finite-energy input/output stability. Thus, time-varying compensation again offers no advantage over time-invariant compensation in achieving this objective of robust stabilization.

This result is obtained by first showing that the small-gain theorem is both necessary and sufficient for the bounded-input/bounded-output stability of certain linear time-varying plants subject to unstructured linear time-varying perturbations. One then exploits the results regarding time-varying compensation for disturbance rejection to lead to the desired conclusion.

The remainder of this paper is organized as follows. Section 2 establishes the notation and definitions used throughout the paper and presents some preliminary facts regard-

4

ing time-varying operators. Sections 3 and 4 contain the precise problem statements and present the main results. Section 3.1 addresses time-varying compensation for optimal rejection of persistent bounded disturbances, while Section 3.2 extends these results to arbitrary disturbances. Section 4 addresses time-varying compensation for robust stabilization with unstructured uncertainty. Finally, concluding remarks are given in Section 5.

2. Mathematical Preliminaries

First, some notation regarding standard concepts for input/ouptput feedback systems (e.g., [3, 16]) is established.

R denotes the field of real numbers, R^n the set of $n \times 1$ vectors with elements in R, and $R^{n \times m}$ the set of all $n \times m$ matrices with elements in R.

Let $x \in R^n$ and $A \in R^{n \times m}$. Then $x(i)$ denotes the ith element of x, $A(i,j)$ denotes the ij^{th} element of A, and $A(\bullet, j)$ denotes the jth column of A. The following norms are defined:

$$|x|_\infty \overset{\text{def}}{=} \max_i |x(i)|$$

$$|A|_\infty \overset{\text{def}}{=} \max_{\substack{x \in R^m \\ x \neq 0}} \frac{|Ax|_\infty}{|x|_\infty} = \max_i \sum_j |A(i,j)|$$

$$|A|_{max} \overset{\text{def}}{=} \sum_j |A(\bullet, j)|_\infty .$$

For $A \in R^{n \times n}$, $tr(A)$ denotes $\sum_{i=1}^n A(i,i)$.

$\ell_{n,e}^\infty$ denotes the extended space of sequences in R^n, $f = \{f_0, f_1, f_2, \ldots\}$. ℓ_n^∞ denotes the set of all $f \in \ell_{n,e}^\infty$ such that

$$\|f\|_{\ell_n^\infty} \overset{\text{def}}{=} \sup_i |f_i|_\infty < \infty.$$

$\ell_{n,e}^\infty \backslash \ell_n^\infty$ denotes the set $\{f : f \in \ell_{n,e}^\infty \text{ and } f \notin \ell_n^\infty\}$. $\ell^p, p \in [1, \infty)$, denotes the set of all

sequences, $f = \{f_0, f_1, f_2, \ldots\}$ in R such that

$$\|f\|_{\ell^p} \stackrel{\text{def}}{=} \left(\sum_i |f_i|^p \right)^{1/p} < \infty.$$

Given $f = \{f_0, f_1, f_2, \ldots\} \in \ell^\infty_{n,e}$, the support of f, denoted $supp(f)$, is defined as

$$supp(f) \stackrel{\text{def}}{=} \{n : f_n \neq 0\}$$

S_k denotes the kth-shift operator on $\ell^\infty_{n,e}$:

$$S_k : \{f_0, f_1, f_2, \ldots\} \longrightarrow \begin{cases} \{\underbrace{0, \ldots, 0}_{k \text{ times}}, f_0, f_1, f_2, \ldots\}, & \text{if } k \geq 0; \\ \{f_{-k}, f_{-k+1}, f_{-k+2}, \ldots\}, & \text{if } k < 0. \end{cases}$$

In the special case where $k = 1$, S_1 is simply denoted as S and is called the shift operator.

P_k denotes the kth-truncation operator on $\ell^\infty_{n,e}$:

$$P_k : \{f_0, f_1, f_2, \ldots\} \longrightarrow \{f_0, \ldots, f_k, 0, \ldots\}$$

Let $H : \ell^\infty_{m,e} \longrightarrow \ell^\infty_{n,e}$ be a nonlinear operator. H is called *causal* if

$$P_k H f = P_k H P_k f, \quad \forall k = 0, 1, 2, \ldots,$$

H is called *strictly causal* if

$$P_k H f = P_k H P_{k-1} f, \quad \forall k = 0, 1, 2, \ldots$$

H is called *time-invariant* if it commutes with the shift operator:

$$HS = SH.$$

Finally, H is called *stable* if

$$\|H\| \stackrel{\text{def}}{=} \sup_k \sup_{\substack{f \in \ell^\infty_{m,e} \\ P_k f \neq 0}} \frac{\|P_k H f\|_{\ell^\infty_n}}{\|P_k f\|_{\ell^\infty_m}} < \infty.$$

The quantity $\|H\|$ is called the *induced operator norm over ℓ^∞*.

$\mathcal{L}_{TV}^{n \times m}$ denotes the set of all *linear* causal stable operators, $T: \ell_{m,e}^\infty \longrightarrow \ell_{n,e}^\infty$. $\mathcal{L}_{TI}^{n \times m}$ denotes the set of all $T \in \mathcal{L}_{TV}^{n \times m}$ which are *time-invariant*.

The remainder of this section is devoted to showing that $\mathcal{L}_{TV}^{n \times m}$ may be viewed as the dual space of a certain normed space, $\mathcal{L}_0^{m \times n}$, to be defined.

First, given any $T \in \mathcal{L}_{TV}^{n \times m}$, it is straightforward to show that its action on any $f \in \ell_m^\infty$ may be given the kernel representation

$$(Tf)_n = \sum_{j=0}^{n} T_{nj} f_j, \quad n = 0, 1, 2, \ldots$$

where the T_{nj} are a collection of matrices in $R^{n \times m}$ uniquely defined by T. It then follows that

$$\|T\| = \sup_i \|[\, T_{i0} \quad \ldots \quad T_{ii} \,]\|_\infty$$

Using this kernel representation, $\mathcal{L}_{TV}^{n \times m}$ may be identified as the set of all infinite lower-triangular block matrices,

$$T = \begin{pmatrix} T_{00} & 0 & 0 & \ldots \\ T_{10} & T_{11} & 0 & \ldots \\ T_{20} & T_{21} & T_{22} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with elements in $R^{n \times m}$ whose rows have uniformly bounded $|\bullet|_\infty$ norms, i.e.

$$\sup_i \|[\, T_{i0} \quad \ldots \quad T_{ii} \,]\|_\infty < \infty.$$

The normed space $\mathcal{L}_0^{m \times n}$ is now defined as the set of all infinite upper-triangular block matrices,

$$G = \begin{pmatrix} G_{00} & G_{01} & G_{02} & \ldots \\ 0 & G_{11} & G_{12} & \ldots \\ 0 & 0 & G_{22} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with elements in $R^{m \times n}$ whose columns have $|\bullet|_{max}$ norms which are summable, i.e.

$$\|G\|_{\mathcal{L}_0} \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \left| \begin{bmatrix} G_{0i} \\ \ldots \\ G_{ii} \end{bmatrix} \right|_{max} < \infty.$$

7

Let $\left(\mathcal{L}_0^{m \times n}\right)^*$ denote the dual space of $\mathcal{L}_0^{m \times n}$. It is now shown that $\left(\mathcal{L}_0^{m \times n}\right)^* = \mathcal{L}_{TV}^{n \times m}$.

Proposition 2.1 *Let $T \in \mathcal{L}_{TV}^{n \times m}$. Then T defines a linear functional on $\mathcal{L}_0^{m \times n}$ whose value at G, denoted $\langle T, G \rangle$, is defined as*

$$\langle T, G \rangle \stackrel{\text{def}}{=} tr \left(\begin{pmatrix} T_{00} & 0 & 0 & \cdots \\ T_{10} & T_{11} & 0 & \cdots \\ T_{20} & T_{21} & T_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} G_{00} & G_{01} & G_{02} & \cdots \\ 0 & G_{11} & G_{12} & \cdots \\ 0 & 0 & G_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right).$$

Conversely, any element of $\left(\mathcal{L}_0^{m \times n}\right)^*$ takes the form of $\langle T, G \rangle$ with $T \in \mathcal{L}_{TV}^{n \times m}$. Furthermore, one has that

$$\sup_{\|G\|_{\mathcal{L}_0} \leq 1} \langle T, G \rangle = \|T\|.$$

Proof The proof of Proposition 2.1 involves straightforward arguments, hence the details are ommitted here. First note that by the summability of the columns of G, the "infinite trace" present in the definition of $\langle T, G \rangle$ is well defined. It is then easy to see that any $T \in \mathcal{L}_{TV}^{n \times m}$ defines an element of $\left(\mathcal{L}_0^{m \times n}\right)^*$. To see that $\left(\mathcal{L}_0^{m \times n}\right)^*$ is precisely $\mathcal{L}_{TV}^{n \times m}$, one simply exploits that the summability of the columns of any element of $\mathcal{L}_0^{m \times n}$ implies that $\mathcal{L}_0^{m \times n}$ has a Schauder basis (e.g., [11]). Thus, any element of $\left(\mathcal{L}_0^{m \times n}\right)^*$ is uniquely defined by evaluating the functional on the basis elements. This evaluation process in turn uniquely defines an element of $\mathcal{L}_{TV}^{n \times m}$. ∎

Finally, the following proposition regarding a composition of operators as a linear functional is presented.

Proposition 2.2 *Let $T_1 \in \mathcal{L}_{TV}^{n \times m}$, $T_2 \in \mathcal{L}_{TV}^{m \times p}$, $T_3 \in \mathcal{L}_{TV}^{p \times q}$, and $G \in \mathcal{L}_0^{q \times n}$. Let $\tilde{G}$ be the upper block triangular portion of $(T_3 G T_1)$ viewed as a product of infinite-matrices. Then*

(1) $\tilde{G} \in \mathcal{L}_0^{p \times m}$,

(2) $\langle T_1 T_2 T_3, G \rangle = \langle T_2, \tilde{G} \rangle.$

Proof As in Proposition 2.1, the proof of Proposition 2.2 involves straightforward arguments, hence the details are ommitted here. First, the summability of the columns of G guarantees that $\tilde{G}$ is well-defined and belongs to $\mathcal{L}_0^{p \times m}$. To see (2), note that any $G \in \mathcal{L}_0^{q \times n}$ can be approximated arbitrarily closely by a $G' \in \mathcal{L}_0^{q \times n}$ which has a finite number of non-zero elements. Thus, replacing G by G' above makes all products of infinite-matrices a finite-matrix product. Statement (2) then follows since $tr(AB) = tr(BA)$ for finite matrices A, B. ∎

Notational Convention In order to avoid a proliferation of notation, the following convention is adopted. In Section 3.1, all operators are assumed to be multi-input/multi-output without explicit reference to the dimension of the inputs and outputs, and in Sections 3.2 and 4, all operators are assumed to be single-input/single-output. Furthermore, all subscripts on norms will be dropped throughout. This informality results in no loss of clarity.

3. Optimal Disturbance Rejection

The standard block diagram for the optimal disturbance rejection problem (e.g., [7]) is shown in Fig. 3.1. In this figure, P denotes some fixed time-invariant discrete-time plant, K denotes a possibly *time-varying* compensator, and the signals w, z, y, and u are defined as follows: w, exogenous disturbances; z, signals to be regulated; y, measured plant outputs; and u, control inputs to the plant. For technical simplicity, it is assumed that the transfer function from u to y is strictly causal.
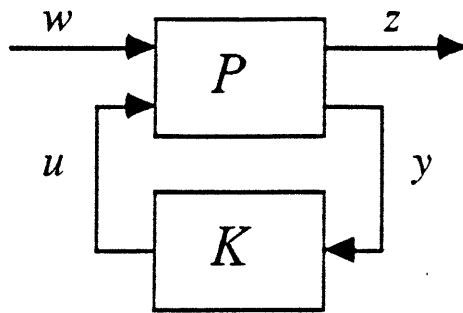
9

Fig. 3.1 Block Diagram for Disturbance Rejection

Let $T_{zw}(K)$ denote the resulting closed-loop dynamics from w to z for a given compensator K. The objective of optimal disturbance rejection can then be stated as minimizing over all admissible compensators the resulting input/output norm of $T_{zw}(K)$.

3.1 ℓ^∞ Disturbance Rejection

In the case where the disturbances w are persistent and bounded, the pertinent input/output norm of $T_{zw}(K)$ is its induced operator norm over ℓ^∞.

The cost resulting from such a minimization can be stated more precisely as follows:

$$\mu_{TV} \stackrel{\text{def}}{=} \inf \left\{ \|T_{zw}(K)\| : K \text{ is any stabilizing linear } \textit{time-varying} \text{ controller} \right\}.$$

It is stressed here that the phrase "time-varying" should be interpreted as meaning "not necessarily time-invariant." If one indeed wishes to restrict the compensation to be time-invariant, then the class of admissible compensators is reduced, and the following cost is defined:

$$\mu_{TI} \stackrel{\text{def}}{=} \inf \left\{ \|T_{zw}(K)\| : K \text{ is any stabilizing linear } \textit{time-invariant} \text{ controller} \right\}.$$

The main result is now stated:

Theorem 3.1 $\mu_{TV} = \mu_{TI}$.

10

Clearly, one has that $\mu_{TV} \leq \mu_{TI}$. The remainder of this section is devoted to proving the reverse inequality, $\mu_{TI} \leq \mu_{TV}$.

The first step is to employ a parameterization of all stabilizing *and possibly time-varying* compensators [13, 14, 17]. Then it can be shown (e.g., [7]) that

$$\mu_{TV} = \inf_{Q \in \mathcal{L}_{TV}} \|T_1 - T_2 Q T_3\|$$

and

$$\mu_{TI} = \inf_{Q \in \mathcal{L}_{TI}} \|T_1 - T_2 Q T_3\|,$$

where T_1, T_2, and T_3 are stable linear time-invariant operators which belong to $\mathcal{L}_{TI}$ and depend only on the plant P, and $T_1 - T_2 Q T_3$ is the resulting closed-loop dynamics from w to z. Thus in proving Theorem 3.1, it suffices to show that

$$\inf_{Q \in \mathcal{L}_{TI}} \|T_1 - T_2 Q T_3\| \leq \inf_{Q \in \mathcal{L}_{TV}} \|T_1 - T_2 Q T_3\|.$$

First, some preliminary lemmas are presented.

Lemma 3.1 *Let T_1, T_2, and $T_3 \in \mathcal{L}_{TI}$ and $Q \in \mathcal{L}_{TV}$ satisfy*

$$\|T_1 - T_2 Q T_3\| = \mu.$$

Then

$$\|T_1 - T_2(S_{-n} Q S_n) T_3\| \leq \mu, \quad \forall n = 0, 1, 2, \ldots$$

Proof By definition of the induced operator norm,

$$\|T_1 - T_2 Q T_3\| \overset{\text{def}}{=} \sup_{\substack{f \in \ell^\infty \\ f \neq 0}} \frac{\|(T_1 - T_2 Q T_3) f\|}{\|f\|}$$

$$\geq \sup_{\substack{f \in \ell^\infty \\ f \neq 0}} \frac{\|(T_1 - T_2 Q T_3) S_n f\|}{\|f\|}$$

$$= \|(T_1 - T_2 Q T_3) S_n\|$$

$$= \|S_{-n}(T_1 - T_2 Q T_3) S_n\|.$$

Using the time-invariance of T_1, T_2, and T_3,

$$\|S_{-n}(T_1 - T_2 Q T_3) S_n\| = \|T_1 - T_2 (S_{-n} Q S_n) T_3\|,$$

which yields the desired result. ∎

Lemma 3.1 essentially states that given any $Q \in \mathcal{L}_{TV}$ which yields some closed-loop induced norm, one can find a *family* of operators in $\mathcal{L}_{TV}$, namely $\{S_{-n} Q S_n\}$, which yield at most the same closed-loop induced norm. However, a closer inspection shows that this family is simply "delayed versions" of the original Q. This fact becomes clear using the matrix identification of Q. More precisely, if

$$Q = \begin{pmatrix} Q_{00} & 0 & 0 & \cdots \\ Q_{10} & Q_{11} & 0 & \cdots \\ Q_{20} & Q_{21} & Q_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then

$$S_{-n} Q S_n = \begin{pmatrix} Q_{nn} & 0 & 0 & \cdots \\ Q_{(n+1)n} & Q_{(n+1)(n+1)} & 0 & \cdots \\ Q_{(n+2)n} & Q_{(n+2)(n+1)} & Q_{(n+2)(n+2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is this uniformity in time of the closed-loop norm which will ultimately remove the advantage of time-variation in Q.

Lemma 3.2 *Let T_1, T_2, T_3, and Q be as in Lemma 3.1, and define the following sequence of averages*

$$\overline{Q}_n \stackrel{\text{def}}{=} \frac{1}{n+1} \sum_{k=0}^{n} S_{-k} Q S_k, \quad n = 0, 1, 2, \ldots$$

Then

$$\|T_1 - T_2 \overline{Q}_n T_3\| \le \mu, \quad \forall n = 0, 1, 2, \ldots$$

Furthermore, one has that

$$\|\overline{Q}_n S - S \overline{Q}_n\| \le \frac{2}{n+1} \|Q\|, \quad \forall n = 0, 1, 2, \ldots$$

12

Proof The first inequality follows from Lemma 3.1 and the convexity of $\|T_1 - T_2 Q T_3\|$ in Q. The second inequality follows from the easily obtained identity

$$S\overline{Q}_n - \overline{Q}_n S = \frac{1}{n+1} S \left(Q - S_{-(n+1)} Q S_{n+1} \right).$$

∎

As in Lemma 3.1, Lemma 3.2 states that given any $Q \in \mathcal{L}_{TV}$, one can find a *family* of operators in $\mathcal{L}_{TV}$, namely $\{\overline{Q}_n\}$, which yield at most the same closed-loop induced norm as Q. A major difference between these two lemmas is that the sequence of operators $\{\overline{Q}_n\}$ asymptotically approaches commuting with the shift operator S. Thus, the operators $\{\overline{Q}_n\}$ become asymptotically close to being time-invariant, in some sense.

Now suppose that there exists a $\tilde{Q} \in \mathcal{L}_{TV}$ such that the $\{\overline{Q}_n\}$ converge to $\tilde{Q}$ in norm, i.e., in the uniform operator topology. Then

$$\left\|\overline{Q}_n - \tilde{Q}\right\| \longrightarrow 0.$$

From Lemma 3.2, it follows that $\tilde{Q}$ is time-invariant, i.e., $\tilde{Q} \in \mathcal{L}_{TI}$, and achieves the same closed-loop induced norm as Q. Given this time-invariant $\tilde{Q}$, the proof of Theorem 3.1 is then complete.

Now in the special case where Q is periodically time-varying (i.e., $S_k Q = Q S_k$ for some $k \geq 0$), it is easy to show that the sequence of averages $\{\overline{Q}_n\}$ does converge. Unfortunately, however, this sequence $\{\overline{Q}_n\}$ need not converge for a general $Q \in \mathcal{L}_{TV}$.

It turns out that this condition of $\{\overline{Q}_n\}$ converging in norm is unnecessarily demanding. Using the identification of $\mathcal{L}_{TV}$ as the dual space of $\mathcal{L}_0$, the weaker condition of the $\{\overline{Q}_n\}$ converging in the weak* topology (e.g., [11]) on $\mathcal{L}_{TV}$ is sufficient. This is captured in the following lemma.

Lemma 3.3 *Let T_1, T_2, T_3, and Q be as in Lemma 3.1, and let $\{\overline{Q}_n\}$ form a sequence of operators as defined in Lemma 3.2. Since the sequence $\{\overline{Q}_n\}$ is bounded, it has a weak**

13

convergent subsequence, say $\{\overline{Q}_{n_k}\}$. Let $\tilde{Q}$ be the weak* limit of $\{\overline{Q}_{n_k}\}$, i.e. $\{\overline{Q}_{n_k}\} \xrightarrow{\text{wk}*} \tilde{Q}$. Then

(1) $\tilde{Q} \in \mathcal{L}_{TI}$,

(2) $(T_1 - T_2 \overline{Q}_{n_k} T_3) \xrightarrow{\text{wk}*} \left(T_1 - T_2 \tilde{Q} T_3\right)$,

(3) $\left\| T_1 - T_2 \tilde{Q} T_3 \right\| \le \mu$.

Proof (1) Let $E_{n_k} = \tilde{Q} - \overline{Q}_{n_k}$. Then

$$S\tilde{Q} = S\overline{Q}_{n_k} + SE_{n_k}$$

and

$$\tilde{Q}S = \overline{Q}_{n_k}S + E_{n_k}S.$$

Thus for any $G \in \mathcal{L}_0$,

$$\langle S\tilde{Q} - \tilde{Q}S, G \rangle = \langle S\overline{Q}_{n_k} - \overline{Q}_{n_k}S, G \rangle + \langle SE_{n_k}, G \rangle - \langle E_{n_k}S, G \rangle.$$

Since n_k is arbitrary, it follows from Lemma 3.2, Proposition 2.2, and $E_{n_k} \xrightarrow{\text{wk}*} 0$ that

$$\langle S\tilde{Q} - \tilde{Q}S, G \rangle = 0, \quad \forall G \in \mathcal{L}_0$$

Thus $S\tilde{Q} = \tilde{Q}S$ which proves (1).

(2) Since $\tilde{Q} - \overline{Q}_{n_k} \xrightarrow{\text{wk}*} 0$, it follows from Proposition 2.2 that $T_2(\tilde{Q} - \overline{Q}_{n_k})T_3 \xrightarrow{\text{wk}*} 0$ which proves (2).

(3) Using $\left\| T_1 - T_2 \overline{Q}_{n_k} T_3 \right\| \le \mu$ and $(T_1 - T_2 \overline{Q}_{n_k} T_3) \xrightarrow{\text{wk}*} \left(T_1 - T_2 \tilde{Q} T_3\right)$, statement (3) then follows from a standard result on weak* convergent sequences. See [11, Sec. 4.9, problem 9]. ∎

With Lemma 3.3 in hand, the proof of Theorem 3.1 is now complete. In words, Lemma 3.3 states that given any $Q \in \mathcal{L}_{TV}$ which yields some closed-loop induced norm,

14

one can find a *time-invariant* operator, namely $\tilde{Q} \in \mathcal{L}_{TI}$, which yields the same closed-loop induced norm, which is the desired result.

3.2 ℓ^p Disturbance Rejection

In this section, it is shown how to exploit Lemma 3.1 and Lemma 3.2 to show that time-varying compensation does not improve the optimal rejection of general ℓ^p-disturbances, $p \in [1, \infty]$, with the operator norms induced over ℓ^p. Thus, both finite-energy and persistent bounded disturbances may be treated in the same manner. Since the multi-input/multi-output case is rather cumbersome, only the single-input/single-output case is discussed.

First, some specialized notation for this purpose is established. X^* denotes any *one* of the spaces $\ell^p, p \in [1, \infty]$, and X denotes the space such that X^* is its dual (e.g., [11]). $e_j, j = 0, 1, 2, \ldots$, denotes the jth standard coordinate vector of X^*, i.e.

$$e_j = \{\overbrace{0, \ldots, 0}^{j\ 0's}, 1, 0, \ldots\}.$$

$\mathcal{L}_{TV}(X^*)$ denotes the set of all linear causal operators $T : X^* \longrightarrow X^*$ such that

$$\|T\| \stackrel{\text{def}}{=} \sup_k \sup_{\substack{f \in X^* \\ P_k f \neq 0}} \frac{\|P_k T f\|}{\|P_k f\|} < \infty.$$

$\mathcal{L}_{TI}(X^*)$ denotes the set of all $T \in \mathcal{L}_{TV}(X^*)$ which are time-invariant. Given any $T \in \mathcal{L}_{TV}(X^*)$, T_* denotes the bounded linear operator $T_* : X \longrightarrow X$ such that T is its adjoint, i.e., $(T_*)^* = T$. (It is easy to see that T_* is well-defined using a matrix representation of T.)

The main result is now stated as follows:

Theorem 3.2 *Let $T_1, T_2, T_3 \in \mathcal{L}_{TI}(X^*)$ and $Q \in \mathcal{L}_{TV}(X^*)$ be such that*

$$\|T_1 - T_2 Q T_3\| = \mu.$$

15

Then, there exists a $\tilde{Q} \in \mathcal{L}_{TI}(X^*)$ such that

$$\left\| T_1 - T_2 \tilde{Q} T_3 \right\| \leq \mu.$$

Proof Following Lemma 3.1 and Lemma 3.2, define

$$\overline{Q}_n \overset{\text{def}}{=} \frac{1}{n+1} \sum_{k=0}^{n} S_{-k} Q S_k, \quad n = 0, 1, 2, \ldots$$

Then using the same arguments as in Lemma 3.1 and Lemma 3.2, one has that

$$\left\| T_1 - T_2 \overline{Q}_n T_3 \right\| \leq \mu, \quad \forall n = 0, 1, 2, \ldots$$

and

$$\left\| \overline{Q}_n S - S \overline{Q}_n \right\| \leq \frac{2}{n+1} \left\| Q \right\|, \quad \forall n = 0, 1, 2, \ldots$$

Since it is unclear whether $\mathcal{L}_{TV}(X^*)$ is the dual of some vector space (as is the case for $\mathcal{L}_{TV}$), one cannot follow the same route as Lemma 3.3.

Given this predicament, consider the sequence in X^* given by $\{\overline{Q}_n e_0\}$. Since it is a bounded sequence, it has a weak* convergent subsequence. Thus, let

$$\overline{Q}_{n_k} e_0 \overset{\text{wk*}}{\longrightarrow} v_0.$$

Then

$$\overline{Q}_{n_k} e_1 = \overline{Q}_{n_k} S e_0 = S \overline{Q}_{n_k} e_0 + \left(\overline{Q}_{n_k} S - S \overline{Q}_{n_k} \right) e_0 \overset{\text{wk*}}{\longrightarrow} S v_0.$$

Similarly, for any finite integer N,

$$\overline{Q}_{n_k} \left(\sum_{j=0}^{N} \alpha_j e_j \right) \overset{\text{wk*}}{\longrightarrow} \sum_{j=0}^{N} \alpha_j S_j v_0.$$

This motivates the definition of $\tilde{Q}$ as

$$P_N \tilde{Q} f \overset{\text{def}}{=} \text{weak}^* \lim \overline{Q}_{n_k} P_N f, \quad f \in X^*, N = 0, 1, 2, \ldots$$

16

The above expression clearly defines a unique causal time-invariant linear operator on X^*.
Using a standard result on weak* convergent sequences [11], one has that

$$\left\| P_N \tilde{Q} f \right\| \le \liminf \left\| \overline{Q}_{n_k} P_N f \right\|,$$

which then implies $\tilde{Q}$ is also bounded, hence $\tilde{Q} \in \mathcal{L}_{TI}(X^*)$.

Thus, it remains to be shown that $\left\| T_1 - T_2 \tilde{Q} T_3 \right\| \le \mu$. First, let $\langle f, x \rangle$ denote the value of $f \in X^*$ acting on $x \in X$. Then for any integer $N < \infty$, $f \in X^*$, and $x \in X$,

$$\langle P_N T_2 \overline{Q}_{n_k} T_3 f, x \rangle = \langle \overline{Q}_{n_k} P_N T_3 f, (T_2)_* (P_N)_* x \rangle$$

$$\longrightarrow \langle \tilde{Q} P_N T_3 f, (T_2)_* (P_N)_* x \rangle$$

$$= \langle P_N T_2 \tilde{Q} P_N T_3 f, x \rangle$$

$$= \langle P_N T_2 \tilde{Q} T_3 f, x \rangle.$$

Thus, for any integer $N < \infty$ and $f \in X^*$,

$$P_N (T_1 - T_2 \overline{Q}_{n_k} T_3) f \xrightarrow{\text{wk}*} P_N (T_1 - T_2 \tilde{Q} T_3) f$$

which implies

$$\left\| P_N (T_1 - T_2 \tilde{Q} T_3) f \right\| \le \left\| P_N (T_1 - T_2 \overline{Q}_{n_k} T_3) f \right\|$$

which completes the proof. ∎

4. Robust Stabilization

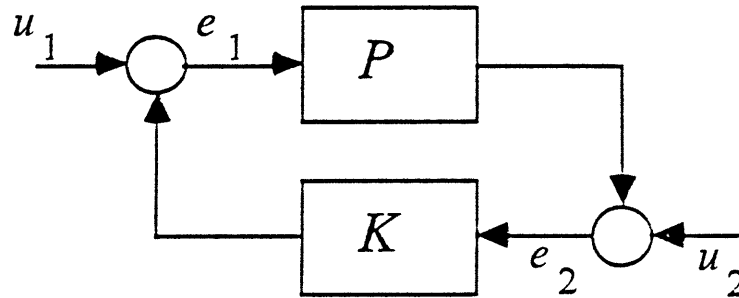To set up the problem of robust stabilization, consider the block diagram of Fig. 4.1.

Fig. 4.1 Block Diagram for Robust Stabilization

In this figure, the plant, P, and compensator, K, are viewed as single-input/single-output causal operators on ℓ_e^∞. This feedback system is said to be *well-posed* (e.g., [16]) if given any $(u_1, u_2) \in \ell_e^\infty \times \ell_e^\infty$, there exist unique $(e_1, e_2) \in \ell_e^\infty \times \ell_e^\infty$ which satisfy

$$e_1 = u_1 + K e_2$$

$$e_2 = u_2 + P e_1.$$

such that the mapping $(u_1, u_2) \mapsto (e_1, e_2)$ is causal. Assuming well-posedness, the compensator, K, is then said to *stabilize* the plant, P, if the mapping $(u_1, u_2) \mapsto (e_1, e_2)$ is stable.

Now, define the following *families* of plants:

$$\mathcal{P}_{\mathrm{add}} \overset{\mathrm{def}}{=} \{P : P = P_o + W\Delta\}$$

where

(1) $P_o : \ell_e^\infty \longrightarrow \ell_e^\infty$ is linear and strictly causal,

(2) $\Delta : \ell_e^\infty \longrightarrow \ell_e^\infty$ is strictly causal and stable with $\|\Delta\| < 1$,

(3) $W : \ell_e^\infty \longrightarrow \ell_e^\infty$ is linear, causal, and stable,

and

$$\mathcal{P}_{\mathrm{mul}} \overset{\mathrm{def}}{=} \{P : P = P_o(I + W\Delta)\}$$

where

(1) $P_o : \ell_e^\infty \longrightarrow \ell_e^\infty$ is linear and strictly causal,

(2) $\Delta : \ell_e^\infty \longrightarrow \ell_e^\infty$ is causal and stable with $\|\Delta\| < 1$,

(3) $W : \ell_e^\infty \longrightarrow \ell_e^\infty$ is linear, causal, and stable.

The assumptions of strict-causality simply assure that any causal compensator results in a well-posed feedback system for every $P \in \mathcal{P}_{\mathrm{add}}$ or $\mathcal{P}_{\mathrm{mul}}$ [16].

The problems of robust stabilization addressed in this paper can now be stated as follows:

(1) Find a *single* compensator, K, which stabilizes every $P \in \mathcal{P}_{\mathrm{add}}$,

or

(2) Find a *single* compensator, K, which stabilizes every $P \in \mathcal{P}_{\mathrm{mul}}$.

In either case, the compensator, K, is said to robustly stabilize the family $\mathcal{P}_{\mathrm{add}}$ or $\mathcal{P}_{\mathrm{mul}}$, respectively.

In this section, it is shown that there exists a linear time-varying robustly stabilizing compensator if and only if there exists a linear time-invariant robustly stabilizing compensator. Thus, time-varying compensation offers no advantage over time-invariant compensation for these particular objectives of robust stabilization.

First, some preliminary lemmas which generalize the results of [2] are presented.

Lemma 4.1 *Let* $H \in \mathcal{L}_{TV}$ *satisfy*

$$\inf_k \|S_{-k} H S_k\| = \delta > 1.$$

Then there exist an $n^* \geq 0$, $m > 1$, *and* $f \in \ell_\epsilon^\infty \backslash \ell^\infty$ *such that*

$$\frac{\|P_{n-1} H f\|}{\|P_n f\|} \geq m, \quad \forall n > n^*.$$

Proof Choose constants m, M, and γ such that

$$1 < m < M < \delta,$$

$$1 < \gamma < M/m,$$

and set

$$\varepsilon = (M - \gamma m)\gamma,$$

$$M' = \max(M, \delta - \varepsilon/2).$$

Now using arguments as in Lemma 3.1, one can show that for some $N \geq 0$,

$$\|S_{-N} H S_N\| \leq \delta + \varepsilon/2.$$

19

Setting $\tilde{H} = S_{-N} H S_N$, it follows that

$$1 < \delta \leq \left\| S_{-k} \tilde{H} S_k \right\| \leq \delta + \varepsilon/2, \quad \forall k = 0, 1, 2, \ldots$$

Given this inequality, there exists an $e^0 \in \ell^\infty$ and $n_0 \geq 0$ such that

(1) $supp(e^0) \subset [0, n_0]$

(2) $\left\| e^0 \right\| = 1$

(3) $\left| (\tilde{H} e^0)_{n_0} \right| \geq M'$

Similarly, there exists an $e^1 \in \ell^\infty$ and $n_1 \geq n_0 + 1$ such that

(1) $supp(e^1) \subset [n_0 + 1, n_1]$

(2) $\left\| e^1 \right\| = M'/m$

(3) $\left| (\tilde{H} e^1)_{n_1} \right| \geq (M')^2/m$

Now since $M' \geq \delta - \varepsilon/2$ and $\left\| S_{-k} \tilde{H} S_k \right\| \leq \delta + \varepsilon/2$ for all k, it follows that

$$\left| (\tilde{H} e^0)_{n_1} \right| \leq \varepsilon \left\| e^0 \right\|.$$

Thus,

$$\left| \left(\tilde{H}(e^0 + e^1) \right)_{n_1} \right| \geq \frac{(M')^2}{m} - \varepsilon.$$

In general, the above construction of e^1 based on e^0 may be given the following recursive form. Let $\alpha_1(k)$ denote the lower bound

$$\left| \left(\tilde{H} \sum_{j=0}^{k} e^j \right)_{n_k} \right| \geq \alpha_1(k),$$

and let

$$\alpha_2(k) = \left\| e^k \right\|.$$

Then given signals $e^0, \ldots, e^{k-1}$ and constants $n_0, \ldots, n_{k-1}$, there exists an $e^k \in \ell^\infty$ and $n_k \geq n_{k-1} + 1$ such that

(1) $supp(e^k) \subset [n_{k-1} + 1, n_k]$

(2) $\left\| e^k \right\| = \alpha_2(k) = \alpha_1(k-1)/m$

20

(3) $\left| (\tilde{H} e^k)_{n_k} \right| \geq M' \alpha_2(k)$.

Again, since $M' \geq \delta - \varepsilon/2$, it follows that

$$\left| \left(\tilde{H} \sum_{j=0}^{k-1} e^j \right)_{n_k} \right| \leq \varepsilon \alpha_2(k-1)$$

provided that the $\alpha_2(k)$ are non-decreasing. If so, then

$$\left| \left(\tilde{H} \sum_{j=0}^{k} e^j \right)_{n_k} \right| \geq M' \alpha_2(k) - \varepsilon \alpha_2(k-1) = \alpha_1(k)$$

Thus, it is seen that the variables α_1 and α_2 satisfy the recursion equation

$$\begin{bmatrix} \alpha_1(k+1) \\ \alpha_2(k+1) \end{bmatrix} = \begin{bmatrix} M'/m & -\varepsilon \\ 1/m & 0 \end{bmatrix} \begin{bmatrix} \alpha_1(k) \\ \alpha_2(k) \end{bmatrix}, \quad \begin{bmatrix} \alpha_1(0) \\ \alpha_2(0) \end{bmatrix} = \begin{bmatrix} M' \\ 1 \end{bmatrix}$$

Furthermore, using the selection of M' and ε, it is straightforward to show that if for some k,

$$\alpha_1(k) \geq \gamma m \alpha_2(k)$$

then

$$\alpha_1(k+1) \geq \gamma m \alpha_2(k+1),$$

$$\alpha_1(k+1) \geq \gamma \alpha_1(k),$$

$$\alpha_2(k+1) \geq \gamma \alpha_2(k).$$

Since $\alpha_1(0) \geq \gamma m \alpha_2(0)$, it follows by induction that the sequences $\alpha_1(k)$ and $\alpha_2(k)$ are exponentially increasing.

Now, let $g = \sum_j e^j$. Since the sequence $\alpha_2(k)$ is exponentially increasing, $g \in \ell_e^\infty \backslash \ell^\infty$. Furthermore, for any $n \in [n_k + 1, n_{k+1}]$,

$$\left\| P_{n-1} \tilde{H} g \right\| \geq \left| \left(\tilde{H} g \right)_{n_k} \right| \geq \alpha_1(k) = m \alpha_2(k+1) = m \left\| P_{n_{k+1}} g \right\| \geq m \left\| P_n g \right\|$$

Thus,

$$\frac{\left\| P_{n-1} \tilde{H} g \right\|}{\left\| P_n g \right\|} \geq m, \quad \forall n > n_0.$$

21

Similarly,

$$\frac{\|P_{N+n-1}HS_N g\|}{\|P_{N+n}S_N g\|} \geq m, \quad \forall n > n_0 + N$$

which completes the proof with $f = S_N g$. ∎

It is noted that [2] proved a less general version of Lemma 4.1 in which the operator H is restricted to be time-invariant. Since [2] used the time-invariance of H extensively, the methods cannot be directly applied to Lemma 4.1.

Lemma 4.2 *Let $H \in \mathcal{L}_{TV}$ be as in Lemma 4.1. Then there exists a strictly causal $\Delta \in \mathcal{L}_{TV}$ such that $\|\Delta\| < 1$ and the operator $(I + \Delta H)^{-1}$ is not stable.*

Proof The proof essentially follows the example in [2]. First, choose f and n^* as in Lemma 4.1 and define the integer function $\phi(n)$ for $n > n^*$ as follows: $\phi(n) \stackrel{\text{def}}{=}$ an integer less than n such that $\left|(Hf)_{\phi(n)}\right| = \|P_{n-1}Hf\|$. Now define the strictly causal operator $\Delta \in \mathcal{L}_{TV}$ by

$$(\Delta e)_n = \begin{cases} 0 & 0 \leq n \leq n^*; \\ \dfrac{-f_n}{(Hf)_{\phi(n)}} e_{\phi(n)}, & n > n^*. \end{cases}$$

By the construction of f, it is clear that $\|\Delta\| < 1$.

To see that $(I + \Delta H)^{-1}$ is not stable, let

$$v = (I + \Delta H)f = \{f_0, \ldots, f_{n^*}, 0, \ldots\}.$$

Now the strict causality of Δ guarantees the invertibility of $(I + \Delta H)$ [16]. Thus, one has that $v \in \ell^\infty$ while $f = (I + \Delta H)^{-1}v \in \ell_e^\infty \backslash \ell^\infty$, which proves the lack of stability. ∎

In words, Lemma 4.2 may be given the interpretation that the small-gain theorem (e.g., [3]) is actually necessary for the linear time-varying operators considered in Lemma 4.1.

It is noted that [2, 13] show how to construct a nonlinear *time-invariant* Δ which is destabilizing.

The next theorem give a necessary and sufficient conditions for the existence of a linear compensator to robustly stabilize either family $\mathcal{P}_{\text{add}}$ or $\mathcal{P}_{\text{mul}}$.

Theorem 4.1 *Let $S(P_o)$ denote the set of all linear, possibly time-varying, compensators which stabilize the plant P_o. Then there exists a $K \in S(P_o)$ which robustly stabilizes $\mathcal{P}_{\text{add}}$ if and only if*

$$\inf_{K \in S(P_o)} \left\| K(I - P_o K)^{-1} W \right\| \leq 1.$$

Similarly, there exists a $K \in S(P_o)$ which robustly stabilizes $\mathcal{P}_{\text{mul}}$ if and only if

$$\inf_{K \in S(P_o)} \left\| K P_o (I - K P_o)^{-1} W \right\| \leq 1.$$

Proof   First consider the family $\mathcal{P}_{\text{add}}$. To prove necessity, let $H(K) = K(I - P_o K)^{-1} W$, and suppose

$$\inf_{K \in S(P_o)} \| H(K) \| \geq \delta > 1.$$

Then for any $K \in S(P_o)$,

$$\left\| S_{-k} H(K) S_k \right\| = \left\| S_{-k} K S_k (I - P_o S_{-k} K S_k)^{-1} W \right\| \geq \delta > 1,$$

where it is used that $K \in S(P_o)$ implies $S_{-k} K S_k \in S(P_o)$. (This fact is easily shown using arguments similar to those found in Lemma 3.1.)

Thus, for any $K \in S(P_o)$, $H(K)$ satisfies the hypothesis of Lemma 4.2. However, writing the feedback equations for Fig. 4.1 with $u_2 = 0$, one has that

$$e_1 = (I - H(K)\Delta)^{-1}(I - K P_o)^{-1} u_1.$$

Since $H(K)$ satisfies the hypothesis of Lemma 4.2, it follows that one can construct an admissible Δ which makes $(I - \Delta H(K))^{-1}$ unstable, hence $(I - H(K)\Delta)^{-1}$ is unstable [18, Proposition 2.1].

The proof of necessity for $\mathcal{P}_{\text{mul}}$ essentially follows the same line of reasoning. Namely, redefine $H(K) = KP_o(I - KP_o)^{-1}W$. Again, for any $K \in S(P_o)$, $H(K)$ satisfies the hypothesis of Lemma 4.2. Then with $u_2 = 0$, one has that

$$e_1 = (I - H(K)\Delta)^{-1}(I - KP_o)^{-1}u_1,$$

which, via Lemma 4.2, leads to the desired result.

The proofs of sufficiency are straightforward, hence omitted. Briefly, they simply involve choosing a K such that either $\|K(I - P_oK)^{-1}W\| \leq 1$ or $\|KP_o(I - KP_o)^{-1}W\| \leq 1$ and performing standard manipulations of the feedback equations of Fig. 4.1 along with an application of the small-gain theorem. ∎

The main results regarding robust stabilization are now presented.

Theorem 4.2 *There exists a linear time-varying compensator which robustly stabilizes the family $\mathcal{P}_{\text{add}}$ (resp., $\mathcal{P}_{\text{mul}}$) if and only if there exists a linear time-invariant robustly stabilizing compensator for $\mathcal{P}_{\text{add}}$ (resp., $\mathcal{P}_{\text{mul}}$).*

Proof With Theorem 4.1 in hand, the proof of Theorem 4.2 is essentially complete. More precisely, it is easy to show that either optimization

$$\inf_{K \in S(P_o)} \|K(I - P_oK)^{-1}W\|$$

or

$$\inf_{K \in S(P_o)} \|KP_o(I - KP_o)^{-1}W\|$$

is equivalent to an optimal disturbance rejection problem.

Thus from Theorem 3.1, there exists a stabilizing time-varying compensator satisfying either $\|K(I - P_oK)^{-1}W\| \leq 1$ or $\|KP_o(I - KP_o)^{-1}W\| \leq 1$ if and only if a stabilizing time-invariant compensator satisfies the same bound. ∎

24

It is noted that the methods in this section do not seem to be restricted to the classes $\mathcal{P}_{add}$ and $\mathcal{P}_{mul}$. Rather, they should apply to any class of unstructured uncertainty for which a necessary and sufficient condition for the existence of a robustly stabilizing compensator takes a form equivalent to some optimal disturbance rejection problem (cf. Theorem 4.1).

5. Concluding Remarks

This paper has considered linear time-varying compensation for linear time-invariant discrete time plants subject to persistent bounded disturbances. For both objectives of optimal disturbance rejection and robust stabilization, it was shown that time-varying compensation offers no advantage over time-invariant compensation.

In the analysis of optimal disturbance rejection, the key observations are really those of Lemma 3.1 and Lemma 3.2. It is these lemmas which exploit the original time-invariance of the plant to intuitively show why time-varying compensation does not improve optimal disturbance rejection. Furthermore, as used in Section 3.2, their proofs are really independent of the norms used to measure the signals and operators. Given this independence of norms, it is only the time-varying vs. time-invariant aspect of the problem which is isolated to lead to the desired results.

In the discussion of robust stabilization, the key observation was Lemma 4.2 which essentially stated that the small-gain theorem is also necessary for the stability of certain classes of time-varying plants. However, it is still unknown whether time-varying compensation improves multi-objective robust stabilization problems (e.g., robust performance).

As mentioned earlier, these results complement existing results regarding time-varying compensation for time-invariant plants subject to *finite-energy* disturbances. Since induced operator norms over ℓ^∞ disturbances are more closely related to time-domain feedback specifications (e.g., overshoot), it is interesting that these results remain true.

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