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ON BOUNDARY IMPLICATIONS OF STABILITY AND POSITIVITY PROPERTIES OF MULTIDIMENSIONAL SYSTEMS

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Abstract

The paper considers multidimensional generalizations of various 1-D results on the robustness of Hurwitz, Schur and positivity properties on polynomials and rational functions. More specifically, the convexity property of the stable region in the coefficient space of multivariable polynomials is studied. Multidimensional generalizations of Kharitonov-type results are reviewed and further extensions, including that of the 1-D Edge Theorem, are discussed. Interval positivity property of multivariable rational functions are also characterized in terms of ratios of a finite number of Kharitonov-type polynomials constructed from the extreme values of the intervals of perturbation.

I. Introduction

IA. Overview:

Recent interest in the study of robustness of stability property of polynomials was spurred by the remarkable results [1],[10] of Kharitonov which showed that only a finite number (4 for real and 8 for complex) of polynomials are needed to characterize the interval stability property of an entire family of polynomials. In view of potential application of these results in robust system design the last few years have witnessed an explosion of activity in this field of research. Various alternate ways of viewing Kharitonov's results have been suggested and further generalizations in different directions have been carried out. A summary of these developments has most recently been documented in [21]. While the characterization of interval Hurwitz property has potential applications in studying robust stability of systems, the problem of characterization of interval positive (real) property of rational functions potentially arises in studying the robustness of convergence properties of recursive parameter identification or adaptive filtering schemes. Motivated by these latter class of problems a characterization of interval (strictly) positive real property of rational functions in one variable in terms of a set of sixteen Kharitonov polynomials has also been given in [12],[13].

On the other hand, problems associated with the zero-distribution of polynomials in more than one variable have been studied over sometime primarily due to their relevance in stable multidimensional (k-D) filter design [16],[45]. The question of characterizing robustness of stability of multidimensional polynomials in terms of stability of a finite number of polynomials as in Kharitonov's result thus arises naturally. Work in this field was initiated by Bose [6] by demonstrating that Kharitonov's theorem can be extended to bivariate (k=2) real polynomials, while the validity of such a result in the case of multidimensions has also been subsequently verified [38]. A key to this development is that a large number of results on robust stability, including Kharitonov's original theorem, can be viewed quite transparently in terms of results from passive network theory. In particular, the one-to-one correspondence between reactance functions and Hurwitz polynomials along with the fact that a reactance function can be characterized by an alternating pole zero pattern in 1-D has been crucially exploited in [2],[3] in providing elementary proofs of Kharitonov's theorem. The relationship between multidimensional reactance functions and multidimensional stable polynomials (specifically, scattering Hurwitz polynomials) was first established in [4], and the theory was subsequently further elaborated both in continuous and discrete domain in [5], [9]. While the first results on the characterization of robust stability of multidimensional polynomials in [6],[38] rests on these facts, and the 1-D theory [21] continues to mature, it is becoming increasingly more obvious that a larger variety of 1-D results can be extended to the multidimensional context. The present paper attempts to outline such a program of research. It is thus concievable that many of the results to be discussed are only preliminary and can be developed much further in various different directions.

IB. Notation and terminology:

The k-tuple of variables $(p_1,p_2,\ldots p_k)$ (or $(z_1,z_2,\ldots z_k)$) will be denoted by $\underline{p}$ (or $\underline{z}$). The symbol $\underline{\omega}$ is always assumed to be real valued. The partial <u>degree</u> n_i of a polynomial $g=g(\underline{p})$ in p_i will be denoted by $\deg_i g$. We also say that $\underline{n}=(n_1,n_2,\ldots n_k)$ is the (partial) degree of g. The notation $\underline{p}^{\underline{i}}$, with $\underline{i}=(i_1,i_2,\ldots i_k)$ will indicate the monomial $\underline{p}^{\underline{i}}=p_1^{i_1}p_2^{i_2}\ldots p_k^{i_k}$. Thus, $|\underline{i}|=i_1+i_2+\ldots+i_k$ is the total dgeree of $\underline{p}^{\underline{i}}$. The <u>paraconjugate</u> g_* of a polynomial g is defined as: $g_*=g^*(-\underline{p}^*)$ (or $g^*(\underline{z}^{*-1})x\underline{z}^{\underline{n}}$, where $\underline{n}$ is the (partial) degree of g) with superscript * denoting complex conjugation. If $g_*=\gamma g$, where $\gamma=+1$ or -1 then g is said to be <u>paraeven</u> or <u>paraodd</u> respectively. The polynomials denoted by the symbols 'e' or 'o' are always taken to be paraeven or paraodd respectvely. Any polynomial g can be decomposed into paraeven and paraodd parts as in (1.1) through (1.3).

$$g = e + o \tag{1.1}$$

$$e = (g+g_*)/2 \tag{1.2}$$

$$o = (g-g_*)/2 \tag{1.3}$$

The symbol ω is always assumed to have real value. Accordingly, we write $\underline{p}=j\underline{\omega}$

with $\underline{\omega}=(\omega_1,\omega_2,\ldots\omega_k)$ to indicate $p_i=j\omega_i$ for all i=1 to k. A region in the k-dimensional real space of $\underline{\omega}$, where each ω_i is sign definite will be called an orthant. Clearly, there are $N=2^k$ orthants.

The notation $\text{Re}\underline{p}>0$ (or $|\underline{z}|<1$) will denote the cartesian product of half-planes (or discs $|z_i|<1$) for all i=1 to k. Obvious variations of this notation with $>$ (or $<$) replaced by $<$ (or $>$), $\leq$, $\geq$, $=$, etc. will also be used.

A polynomial is equivalently represented by its set of coefficients. The real space spanned by the real and imaginary parts coefficients of polynomials with (partial) degree $\underline{n}$ is to be denoted by S. With abuse of notation we will often write $g \in S$ and treat g as a point in S having coordinates specified by coefficients of g.

A rational function $F=b/a$ will be called irreducible if its numertor b and denominator a are relatively prime polynomials.

In order to precisely pin down the classes of stable multivariable polynomials of interest to us the following terminology from [4],[5],[9] will be recalled. A polynomial is scattering Hurwitz (Schur) if it has no zeros in $\text{Re}\underline{p}>\underline{0}$ (or $|\underline{z}|<1$) and is relatively prime with its paraconjugate (see [4],[5],[9] for other definitions). It is this class of polynomials that is tied to the multidimensional reactances (see Fact 2.1 to follow). Strictest sense Hurwitz (Schur) polynomials are those devoid of zeros in $\text{Re}\underline{p}\geq\underline{0}$ (or $|\underline{z}|\leq 1$). In the continuous case, points at (multiple) infinity are also points of forbidden zero locations. In fact, the strictest sense Hurwitz and Schur poilynomials are in one-to-one correspondence via the Cayley (bilinear) transform and its inverse.

In Section II we study multidimensional extensions of some 1-D results on convexity property of the stable region in the space S of polynomial coefficients. Section III deals first with various forms of generalizations of Kharitonov's theorem and then considers the problem of characterizing robustness of Schur property of multidimensional polynomials. All discussions upto this point may be essentially traced back to connections with reactance functions. Multidimensional counterpart of an alternate method of study

[28],[29] is examined and further generalizations are obtained in Section IV. In Section V we return to network theory based methods and study the characterization of interval positivity property of multivariable rational functions. Conclusions are drawn in Section VI.

II. Convex Combinations of Stable Multidimensional Polynomials:

Since one of the major issues of concern in the present paper is to identify regions in coefficient space of polynomials having the property of stability, a natural question is whether or not these properties hold for a convex combination of two or more polynomials if the property holds individually for them. Availability of such results are indeed useful in this context because the said convexity property allows us to infer stability of domains of regular shapes from an examination of their boundary (or even vertex) points only.

Unfortunately, it is known that a convex combination of two arbitrary 1-D Hurwitz polynomials is not necessarily Hurwitz [23],[26]. A necessary and sufficient condition for this to be true is available in 1-D [27], but not in k-D ($k>1$). However, the following results can be conveniently obtained by appealing to the close relationship between k-D scattering Hurwitz (Schur) polynomials and k-D (discrete) reactance functions as is mentioned next.

Fact 2.1 [4],[9]: A polynomial $g=e+o$ in k variables, where e is the paraeven part and o is the paraodd part of g, is scattering Hurwitz (Schur) if and only if e/o, or equivalently, o/e is an irreducible (discrete) reactance function.

We then have the following sequence of results essentially based on the above Fact.

Theorem 2.1: Let $g_i=e_i+o_i$; $i=1,2,...m$ be a set of scattering Hurwitz (Schur) polynomials in k variables with corresponding paraeven and paraodd parts being denoted by e_i and o_i.

(i) If g_i's have common paraeven part (or paraodd part) i.e., if $e_i=e$ (or $o_i=o$) for all i, then a convex combination of g_i's i.e.,

$$\sum_{i=1}^{m} \lambda_i g_i \text{ with } \sum_{i=1}^{m} \lambda_i = 1 \qquad (2.1a,b)$$

is also scattering Hurwitz (Schur).

(ii) In fact, the above result holds true even without the restriction (2.1b).

Remark: Obviously in the statement of the above theorem the terms paraeven and paraodd have to be interpreted properly in the context of Hurwitz or Schur polynomials as mentioned in the introduction.

Remark: Part (i) of the above theorem for the case when $k=2$ and $m=2$ was stated first in [6]. A 1-D version of part (ii) appears in [7].

Furthermore, it is possible to prove the following less obvious result. We will need the Schur version of this result in further discussions to follow. Once again, 1-D Hurwitz and Schur versions of these appeared in [7] and [18] respectively.

Theorem 2.2:

(a) The set of polynomials $g_{ij}=e_i+o_j$; $i,j=1,2$ are each scattering Hurwitz (Schur) if and only if all polynomials in the following four families (i.e., convex combinations of g_{ij} and g_{kl} except when $(i,j)=(k,l)$) are scattering Hurwitz (Schur):

$$[\lambda e_1 + (1-\lambda)e_2] + o_i;\ i=1,2 \text{ with } 0\leq\lambda\leq 1 \tag{2.2a,b}$$

$$e_i + [\lambda o_1 + (1-\lambda)o_2];\ i=1,2 \text{ with } 0\leq\lambda\leq 1 \tag{2.3a,b}$$

(b) With the hypothesis of part (a) all polynomials of the following family are scattering Hurwitz (Schur).

$$[\lambda e_1 + (1-\lambda)e_2] + [\nu o_1 + (1-\nu)o_2] \tag{2.4a,b}$$

where $0 \leq \lambda \leq 1$ and $0 \leq \nu \leq 1$.

In particular, if $\lambda=\nu$ then we have that any convex combination of g_{11} and g_{22} is scattering Hurwitz (Schur).

As an obvious extension of the above theorem we have the following corollary:

Corollary 2.1: Consider the set of m^2 polynomials $g_{ij}=e_i+o_j$ for all $i,j=1,2,\ldots m$. If each of these m^2 polynomials is scattering Hurwitz (Schur) then each member of the family g parametrized by λ and ν is scatteirng Hurwitz (Schur):

$$g=e+o;\ e=\sum_{i=1}^{m} \lambda_i e_i;\ o=\sum_{i=1}^{m} \nu_i o_i \qquad (2.5a,b,c)$$

$$\text{where } \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} \nu_i = 1$$

Proof of Theorem 2.1:

(i) Let $o_i=o$. Due to Fact 2.1, e_i/o, and thus $(\lambda_1e_1+\lambda_2e_2+\ldots+\lambda_ke_k)/o$ are reactance functions. The proof would be complete if it is shown that the last mentioned reactance function is in irreducible form. This latter point is somewhat nonobvious and has not appeared anywhere including [6]. Since it requires considerations rather technical, we will dwell on it separately in Appendix A.

(ii) As in 1-D [7], but by following the strategy outlined in (i) above.

Proof of Theorem 2.2:

(a) Only necessity is nontrivial. Due to Fact 2.1, scattering Hurwitz (Schur) property of (e_j+o_i) imply that e_j/o_i for j=1,2 and thus $[\lambda e_1 + (1-\lambda)e_2]/o_i$ is a reactance function. Scattering Hurwitz (Schur) property of (2.2) then follows by appealing to arguments of the type in Appendix A and the 1-D version of the present theorem [7]. Similarly, for (2.3).

(b) Clearly, $o_i/[\nu e_1 + (1-\nu)e_2]$; i=1,2 are reactances as above. Thus, $[\lambda e_1+(1-\lambda)e_2]/[\nu e_1+(1-\nu)e_2]$, being a convex combination two reactance functions, is itself a reactance function. The result then follows from its irreducibility, which can be ascertained via arguments as in Appendix A and validity of the present result in 1-D (see [18] for the Schur case, for example).

We next return to the issue of convexity of the stable region in coefficient space of k-D polynomials raised at the beginning of this section. Although such a region is non-convex its intersection with certain affine spaces are indeed so, as is elaborated in the following Theorem. The discussions to follow are multidimensional generalizations of those appearing in [23] and [32]. We first need some terminolgy.

In the space **S** of coefficients of complex polynomials of (partial) degrees $\underline{n}=(n_1,n_2,\ldots n_k)$ the subspaces spanned by coefficients of paraeven (or paraodd) polynomials will be called even (odd) subspaces and will be denoted by $\mathbf{S}_e$ (or $\mathbf{S}_o$). Also, in what follows we denote the scattering Hurwitz (or Schur) region of **S** by **H**. We then have:

<u>Theorem</u> <u>2.3</u>: Given any $g \varepsilon \mathbf{S}$ the sets $\mathbf{H}\cap\{g+\mathbf{S}_o\}$ and $\mathbf{H}\cap\{g+\mathbf{S}_e\}$ are both convex.

<u>Proof</u>: Consider two arbitrary members $g_1=g+e_1$, $g_2=g+e_2$ of $\mathbf{H}\cap\{g+\mathbf{S}_e\}$. Clearly, a convex combination of g_1 and g_2 is in $\{g+\mathbf{S}_e\}$. Also, since g_1 and g_2 have identical paraodd parts and are scattering Hurwitz, by Theorem 2.1 a convex combination of them is in **H**. Similarly for $\mathbf{H}\cap\{g+\mathbf{S}_o\}$.

Given $g_1,g_2 \varepsilon \mathbf{S}$ where $g_1=e_1+o_1$ and $g_2=e_2+o_2$ with obvious decomposition in paraeven and paraodd parts, we consider the even or odd ordering between them as defined in the following.

$$g_1 \underset{e}{<} g_2 \text{ if } e_1=e_1(j\underline{\omega}) \leq e_2(j\underline{\omega})=e_2 \tag{2.6}$$

$$g_1 \underset{o}{<} g_2 \text{ if } o_1=o_1(j\underline{\omega}) \leq o_2(j\underline{\omega})=o_2 \tag{2.7}$$

in the n-th orthant of the $\underline{\omega}$-hyperspace. Note that g_1 and g_2 may be differently ordered in different orthants.

With the above terminology one can state the following result, the 1-D version [32] of which hinges on the Hermite-Bieler theorem [42]. Since a multidimensional Hermite-Bieler theorem is not known it can be considered to be a nontrivial generalization.

Theorem 2.4: Let $g \varepsilon S$ as well as g_n^+, $g_n^- \varepsilon$ $H \cap \{g+S_e\}$ for $n=1,2,\ldots N$ (N=total numer of orthants = 2^k) be given such that for all $n=1$ to N we have (2.8) in the n-th orthant of the $\underline{\omega}$-hyperspace.

$$g_n^- \underset{e}{<} g_m^-, g_m^+ \underset{e}{<} g_n^+;\ 0 \leq m \leq N \tag{2.8}$$

Then the set:

$$\{x \ \varepsilon\ \{g+S_e\};\ g_n^- \underset{e}{<} x \underset{e}{<} g_n^+ \text{ in the n-th orthant, } n=1 \text{ to } N\}$$

is a subset of $H \cap \{g+S_e\}$. The obvious odd counterpart also holds.

Proof: Same as in 1-D [32], but by invoking Theorem 3.1 of following section instead of Hermite-Bieler theorem.

This motivates the following definition of certain special kinds of vertices of a convex polytope in S and a consequent theorem which characterizes scattering Hurwitz (Schur) property of these polytopes in terms of Hurwitz (Schur) properties of their vertices.

Given $g \varepsilon S$, consider a polytope $\Lambda_e \subset \{g+S_e\}$ with vertex set $\{u_1, u_2, \ldots u_\nu\}$. If for each $n=1$ to N (2.9) is satisfied in the n-th orthant

$$u_n^- \underset{e}{<} u_m^-, u_m^+, u \underset{e}{<} u_n^+ \quad;\ 0 \leq m \leq N \tag{2.9}$$

where u, u_m^+, u_m^-, u_n^+, $u_n^- \varepsilon \{u_1, u_2, \ldots u_\nu\}$ then the vertex u is noncritical. All other vertices are to be called critical. Analogous definition holds for $\Lambda_o \subset \{g+S_o\}$.

We then naturally arrive at :

Theorem 2.5: Given $g \varepsilon S$ and a convex polytope $\Lambda_e \subset \{g+S_e\}$, we have $\Lambda_e \subset H$ if and only if the critical vertices of Λ_e belong to H. The obvious odd counterpart also holds true.

Proof: Only sufficiency is nontrivial. It follows from the preceeding definition and Theorem 2.4 that the noncritical vertices also belong to H if the critical vertices do. i.e., the entire set of vertices belong to $H \cap \{g+S_e\}$. Since due to Theorem 2.3 $H \cap \{g+S_e\}$ is a convex set, the convex hull of these vertices, namely Λ_e, also is in $H \cap \{g+S_o\}$. Thus, in particular, $\Lambda_e \subset H$.

The methodology outlined above can be carried much further. It may be noted that in 1-D such investigations have resulted in new generalizations of Kharitonov's celebrated theorem [32]. It is, therefore, concievable that such an approach is expected to yield yet unknown Kharitonov-type results in the k-D context. However, such a discussion will not be pursued here.

III. Multidimensional Extension of Kharitonov Type Results:

IIIA. Continuous domain results:

By exploiting stability results on the scattering description of passive multidimensional systems a characterization for robustness of scattering Hurwitz property of a given multidimensional (complex) polynomial in terms of the scattering Hurwitz property of a finite number of multidimensional (complex) polynomials has been recently [38] established. The result thus completely verifies a recent conjecture [6] extending Kharitonov's celebrated theorem on the characterization of interval Hurwitz property of real as well as complex polynomials to multidimensions. The multidimensional versions of both weak (16 point box [1]) and strong (8 point box [10]) forms of Kharitonov's 1-D results are thus presented in this subsection. Since details of proofs appear in [38] we will be content with statements of the main results only.

The first important result in this category, which in many respects (e.g., in the proof of Theorem 2.4 and the generalized Kharitonov theorem to follow) play the role of Hermite-Bieler theorem [42] of 1-D theory, is as follows. Note that results of this type when k=1 or 2 has been discussed in [3],[11] or [6] respectively.

<u>Theorem</u> <u>3.1</u>: Let $g_n^+=g_n^+(\underline{p})$ and $g_n^-=g_n^-(\underline{p})$ for n = 1,2,...N as in (3.1) be a set of 2N scattering Hurwitz polynomials such that for each n:

$$g_n^+(\underline{p}) = e_n^+(\underline{p}) + o(\underline{p});\; g_n^-(\underline{p}) = e_n^-(\underline{p}) + o(\underline{p}) \qquad (3.1a,b)$$

Also, assume that for all $\underline{\omega}$ in the n-th orthant the following ordering in (3.2) holds in which m can assume any value between 1 to N.

$$g_n^- \underset{e}{<} g_m^-,\; g_m^+ \underset{e}{<} g_n^+ \qquad (3.2)$$

Then the polynomial $g(\underline{p})=e(\underline{p})+o(\underline{p})$ is a scattering Hurwitz polynomial if for each n=1 to N (3.3) holds in the n-th orthant.

$$g_n^- \underset{e}{<} g \underset{e}{<} g_n^+ \qquad (3.3)$$

A dual version of the above theorem in which the roles of paraeven and paraodd polynomials are interchanged is also true.

For statements of generalized k-D versions of Kharitonov's theorem it will be convenient to introduce the following notation. Let $\underline{A}_{\underline{i}} \leq A_{\underline{i}} \leq \overline{A}_{\underline{i}}$ and $\underline{B}_{\underline{i}} \leq B_{\underline{i}} \leq \overline{B}_{\underline{i}}$ be two sets of closed intervals respectively around $A_{\underline{i}}$ and $B_{\underline{i}}$, where $C_{\underline{i}} = A_{\underline{i}} + jB_{\underline{i}}$ is the coefficient of $\underline{p}^{\underline{i}}$ in $g(\underline{p})$. Note that the coefficient of $\underline{p}^{\underline{i}}$ in $e(\underline{p})$ is $A_{\underline{i}}$ if $|\underline{i}|$ = even and is $jB_{\underline{i}}$ if $|\underline{i}|$ = odd.

Consider next a set of 2N 'extreme' paraeven polynomials $e_n^+(\underline{p})$, $e_n^-(\underline{p})$ for n = 1 to N formed from the polynomial $e(\underline{p})$ by observing the following rule.

<u>Rule</u> <u>3.1</u>: In forming the coefficient of $\underline{p}^{\underline{i}}$ in $e_n^+(\underline{p})$ if $|\underline{i}|$ = even (odd) replace the coefficient $A_{\underline{i}}$ (corresp. $jB_{\underline{i}}$) of $\underline{p}^{\underline{i}}$ in $e(\underline{p})$ by either $\underline{A}_{\underline{i}}$ or $\overline{A}_{\underline{i}}$ (corresp. $j\underline{B}_{\underline{i}}$ or $j\overline{B}_{\underline{i}}$), whichever maximizes the value of $A_{\underline{i}}(j\underline{\omega})^{\underline{i}}$ (corresp. $jB_{\underline{i}}(j\underline{\omega})^{\underline{i}}$) for any given real value of the k-tuple $\underline{\omega} = (\omega_1, \omega_2, \ldots \omega_k)$ in the n-th orthant of the $\underline{\omega}$-hyperspace. The coefficients of $e_n^-(\underline{p})$ are obtained by observing the same rule with the term 'maximizes' changed to 'minimizes'.

<u>Example</u>: Consider a complex polynomial $g(\underline{p})$ of total degree 2 in two variables (i.e., k=2) p_1, p_2. We then have from (1.2) and (1.3):

$$e(\underline{p}) = A_{00} + j(B_{10}p_1 + B_{01}p_2) + (A_{20}p_1^2 + A_{11}p_1p_2 + A_{02}p_2^2)$$

$$o(\underline{p}) = jB_{00} + (A_{10}p_1 + A_{01}p_2) + j(B_{20}p_1^2 + B_{11}p_1p_2 + B_{02}p_2^2)$$

Thus, if n = 1,2,3,4 are taken to correspond to the respective quadrants in the $(\omega_1 - \omega_2)$ plane then we have:

$$e_1^+(\underline{p}) = \overline{A}_{00} + j(\underline{B}_{10}p_1 + \underline{B}_{01}p_2) + (\underline{A}_{20}p_1^2 + \underline{A}_{11}p_1p_2 + \underline{A}_{02}p_2^2)$$

$$e_2^+(\underline{p}) = \overline{A}_{00} + j(\overline{B}_{10}p_1 + \underline{B}_{01}p_2) + (\underline{A}_{20}p_1^2 + \overline{A}_{11}p_1p_2 + \underline{A}_{02}p_2^2)$$

$$e_3^+(\underline{p}) = \overline{A}_{00} + j(\overline{B}_{10}p_1 + \overline{B}_{01}p_2) + (\underline{A}_{20}p_1^2 + \underline{A}_{11}p_1p_2 + \underline{A}_{02}p_2^2)$$

$$e_4^+(\underline{p}) = \bar{A}_{00} + j(\underline{B}_{10}p_1 + \bar{B}_{01}p_2)+(\underline{A}_{20}p_1^2 + \underline{A}_{11}p_1p_2 + \underline{A}_{02}p_2^2)$$

and

$$e_1^-(\underline{p}) = \bar{A}_{00} + j(\bar{B}_{10}p_1 + \bar{B}_{01}p_2)+(\bar{A}_{20}p_1^2 + \bar{A}_{11}p_1p_2 + \bar{A}_{02}p_2^2)$$

$$e_2^-(\underline{p}) = \bar{A}_{00} + j(\underline{B}_{10}p_1 + \bar{B}_{01}p_2)+(\bar{A}_{20}p_1^2 + \underline{A}_{11}p_1p_2 + \bar{A}_{02}p_2^2)$$

$$e_3^-(\underline{p}) = \bar{A}_{00} + j(\underline{B}_{10}p_1 + \underline{B}_{01}p_2)+(\bar{A}_{20}p_1^2 + \bar{A}_{11}p_1p_2 + \bar{A}_{02}p_2^2)$$

$$e_4^+(\underline{p}) = \bar{A}_{00} + j(\bar{B}_{10}p_1 + \underline{B}_{01}p_2)+(\bar{A}_{20}p_1^2 + \underline{A}_{11}p_1p_2 + \bar{A}_{02}p_2^2)$$

Next, note that the coefficient of $\underline{p}^{\underline{i}}$ in $o(\underline{p})$ as in (1.3) is $jB_{\underline{i}}$ if $|\underline{i}|$ = even and is $A_{\underline{i}}$ if $|\underline{i}|$ = odd, and consider another of 2N 'extreme' paraodd polynomials $o_n^+(\underline{p})$, $o_n^-(\underline{p})$ for = 1 to N formed from the polynomial $o(\underline{p})$ by observing the following Rule.

<u>Rule 3.2</u>: In forming the coefficient of $\underline{p}^{\underline{i}}$ in $o_n^+(\underline{p})$ if $|\underline{i}|$ = even (odd) replace the coefficient $jB_{\underline{i}}$ (corresp. $A_{\underline{i}}$) or $\underline{p}^{\underline{i}}$ in $o(\underline{p})$ by either $j\underline{B}_{\underline{i}}$ or $j\bar{B}_{\underline{i}}$ (corresp. $\underline{A}_{\underline{i}}$ or $\bar{A}_{\underline{i}}$), whichever maximizes the value of $B_{\underline{i}}(j\underline{\omega})^{\underline{i}}$ (corresp. $A_{\underline{i}}(j\underline{\omega})^{\underline{i}}/j$) for any given real value of the k-tuple $\underline{\omega} = (\omega_1,\omega_2 \ldots \omega_k)$ in the n-th orthant of the $\underline{\omega}$-hyperspace. The coefficients of $o_n^-(\underline{p})$ are obtained by observing the same rule with the term 'maximizes' changed to 'minimizes'.

<u>Example</u>: Consider the polynomial $g(\underline{p})$ and the same assignment of integers n for the quadrants in $(\omega_1-\omega_2)$ plane as in the last example. Then we have:

$$o_1^+(\underline{p}) = j\bar{B}_{00} + (\bar{A}_{10}p_1 + \bar{A}_{01}p_2) + j(\underline{B}_{20}p_1^2 + \underline{B}_{11}p_1p_2 + \underline{B}_{02}p_2^2)$$

$$o_2^+(\underline{p}) = j\bar{B}_{00} + (\underline{A}_{10}p_1 + \bar{A}_{01}p_2) + j(\underline{B}_{20}p_1^2 + \bar{B}_{11}p_1p_2 + \underline{B}_{02}p_2^2)$$

$$o_3^+(\underline{p}) = j\bar{B}_{00} + (\underline{A}_{10}p_1 + \underline{A}_{01}p_2) + j(\underline{B}_{20}p_1^2 + \underline{B}_{11}p_1p_2 + \underline{B}_{02}p_2^2)$$

$$o_4^+(\underline{p}) = j\bar{B}_{00} + (\bar{A}_{10}p_1 + \underline{A}_{01}p_2) + j(\underline{B}_{20}p_1^2 + \bar{B}_{1}p_1p_2 + \underline{B}_{02}p_2^2)$$

and

$$o_1^-(\underline{p}) = j\underline{B}_{00} + (\underline{A}_{10}p_1 + \underline{A}_{01}p_2) + j(\bar{B}_{20}p_1^2 + \bar{B}_{11}p_1p_2 + \bar{B}_{02}p_2^2)$$

$$o_2^-(\underline{p}) = j\underline{B}_{00} + (\bar{A}_{10}p_1 + \underline{A}_{01}p_2) + j(\bar{B}_{20}p_1^2 + \underline{B}_{11}p_1p_2 + \bar{B}_{02}p_2^2)$$

$$o_3^-(\underline{p}) = j\underline{B}_{00} + (\bar{A}_{10}p_1 + \bar{A}_{01}p_2) + j(\bar{B}_{20}p_1^2 + \bar{B}_{11}p_1p_2 + \bar{B}_{02}p_2^2)$$

$$o_4^-(\underline{p}) = j\underline{B}_{00} + (\underline{A}_{10}p_1 + \bar{A}_{01}p_2) + j(\bar{B}_{20}p_1^2 + \underline{B}_{11}p_1p_2 + \bar{B}_{02}p_2^2)$$

The above construction immediately suggests the following fact, which is crucial in this development.

<u>Fact 3.1</u>: The extreme paraeven and paraodd polynomials constructed by using Rules 3.1 and 3.2 satisfy (3.4a,b) for all $\underline{\omega}$ in the n-th orthant.

$$e_n^-(j\underline{\omega}) \le e(j\underline{\omega}) \le e_n^+(j\underline{\omega}); \quad o_n^-(j\underline{\omega}) \le o(j\underline{\omega}) \le o_n^+(j\underline{\omega}) \tag{3.4a,b}$$

A very important consequence of the above fact and Theorem 3.1 is the following step (see [3],[11] for 1-D and [6] for 2-D versions) towards establishing generalized Kharitonov's theorem.

<u>Lemma 3.1</u>: If the set of 2N polynomials $(e_n^+(\underline{p})+o(\underline{p}))$, $(e_n^-(\underline{p})+o(\underline{p}))$, n=1 to N, where $o_n^+(\underline{p})$ and $o_n^-(\underline{p})$ are as defined via Rule 3.1, are each scattering Hurwitz then the polynomial $g(\underline{p})= e(\underline{p})+o(\underline{p})$ given in (1.1) is also scattering Hurwitz. The obvious dual with the roles of paraeven and paraodd polynomials interchanged also hold.

The above lemma and its dual version combined together then yield the generalized (weak) form of Kharitonov's theorem [38].

<u>Theorem 3.2</u>: Let the set of $4N^2$ polynomials in k-variales $\underline{p} = (p_1,p_2,\ldots,p_k)$: $(e_r^+(\underline{p})+o_s^+(\underline{p}))$, $(e_r^+(\underline{p})+o_s^-(\underline{p}))$, $(e_r^-(\underline{p})+o_s^+(\underline{p}))$ and $(e_r^-(\underline{p})+o_s^-(p))$ for all possible combinations of r,s = 1,2,...,N be scattering Hurwitz, where the $e_n^{\pm}(p)$'s and $o_n^{\pm}(p)$'s have been constructed by observing Rules 3.1 and 3.2. Then the polynomial $g(\underline{p}) = e(\underline{p}) + o(\underline{p})$ in (1.1) is scattering Hurwitz.

Remark: Note that the above theorem requires a total of $4N^2 = 2^{2(k+1)}$ extreme polynomials for characterizing the interval scattering Hurwitz property of k-variable polynomials. In the 1-D case i.e., when $k = 1$, we have $4N^2 = 16$, and thus, coincides with weak version [1] of Kharitonov's theorem for complex polynomials. The k-D generalization of strong form of Kharitonov's theorem next follows.

Theorem 3.3: Let the set of 4N polynomials in k-variables $\underline{p} = (p_1, p_2, \ldots p_k)$: $(e_n^+(\underline{p})+o_n^+(\underline{p}))$, $(e_n^+(\underline{p})+o_n^-(\underline{p}))$, $(e_n^-(\underline{p})+o_n^+(\underline{p}))$ and $(e_n^-(\underline{p})+o_n^-(\underline{p}))$ for $n = 1,2,\ldots N$ be scattering Hurwitz where $e_n^{\pm}(p)$'s and $o_n^{\pm}(p)$'s have been constructed using Rules 3.1 and 3.2. Then the polynomial $g(\underline{p})=e(\underline{p})+o(\underline{p})$ in (1.1) is scattering Hurwitz.

Example: We consider the robustness of scattering Hurwitz property of the following 3-variable ($k=3$) polynomial $g=g(p_1,p_2,p_3)$ as a function of the real coefficients α, β, γ and δ. It is easy to show that g is scattering Hurwitz when $\alpha=\beta=\gamma=\delta=1$.

$$g(p_1,p_2,p_3) = \alpha p_1p_2p_3 + \beta p_1p_2 + \gamma p_2p_3 + \delta p_3p_1$$

Thus,

$$e(p_1,p_2,p_3)= \beta p_1p_2 + \gamma p_2p_3 + \delta p_3p_1; \; o(p_1,p_2,p_3) = \alpha p_1p_2p_3$$

In order to construct the generalized Kharitonov polynomials of Theorem 3.3 it is first necessary to construct the polynomials e_n^+, e_n^-, o_n^+ and o_n^- for each of the $2^k=2^3=8$ orthants from Rules 3.1 and 3.2. Let us denote the orthants corresponding to the following signs of $(\omega_1,\omega_2,\omega_3)$ as (+,+,+), (+,+,-), (+,-,+), (+,-,-), (-,+,+), (-,+,-), (-,-,+), (-,-,-,) by the respective natural numbers from 1 to 8. We then have the tabular assignment of coefficients for the polynomials e_n^+, e_n^-, o_n^+, o_n^-, as shown in Table 3.1, where α, β, γ, δ are restricted to lie in real intervals $[\underline{\alpha},\bar{\alpha}]$,$[\underline{\beta},\bar{\beta}]$,$[\underline{\gamma},\bar{\gamma}]$ and $[\underline{\delta},\bar{\delta}]$ respectively.

At this point, the $4N=4(2)^k=32$ generalized Kharitonov polynomials, in general, would be constructed as the polynomials: $(e_n^++o_n^+)$, $(e_n^++o_n^-)$, $(e_n^-+o_n^+)$ and $(e_n^-+o_n^-)$. However, in the present case, it is easily seen that at least one half of this set of 32 polynomials coincide with the other half. Thus, one needs to consider only a set of 16 polynomials at most. This, in fact, is a consequence of the realness of the coefficients of g. This observation can, in fact, be justified in general as in the following remark.

Remark: Note that for real polynomials $B_{\underline{i}}=\underline{B}_{\underline{i}}=\overline{B}_{\underline{i}}=0$. Thus, from Rules 3.1 and 3.2 if follows that $e_n^{\pm}(p)$ (or $o_n^{\pm}(p)$) must have only monomials of even (corresp. odd) total degree $|\underline{i}|$. Now, consider two orthants n_1 and n_2 such that the sign of each component of $\underline{\omega}$ in n_1-th orthant is exactly opposite of the sign of corresponding component of $\underline{\omega}$ in n_2-th orthant. Then a close examination of Rule 3.1 and Rule 3.2 show that the generalized Kharitonov polynomials corresponding to the n_1-th orthant are replicated by the polynomials corresponding to the n_2-th orthant in Theorems 3.2 and 3.3. Consequently, in the case of real polynomials the total number of polynomials required for characterization of interval scattering Hurwitz property is reduced by a factor of two.

Remark: In 1-D Theorem 3.3 requires eight extreme (complex) polynomials, which coincides with strongest form of Kharitonov's theorem [10]. If the polynomials concerned are all real then the number of extreme polynomials are reduced by a factor of two to $2N = 2^{k+1}$. If $k = 2$ then this result coincides with that in [6], whereas if $k = 1$ then $2^{k+1} = 4$, which is Kharitonov's 1-D result for the real case.

IIIB. Discrete domain results:

Given Kharitonov's theorem a natural question is if the corresponding obvious discrete counterpart of it is true for Schur polynomials. Unfortunately, it is known [20],[24],[25] that the four Kharitonov polynomials are not enough to guarantee the stability of the corresponding discrete domain polynomial family even in 1-D. The most straightforward approach to studying discrete domain stability, namely, via the Cayley transform on the continuous domain results is not effective due to the distortions of the faces of the rectangular parallelopiped introduced by the transform (although some progress can be made as in [24]). Nevertheless, 1-D results which are aesthetically somewhat less pleasing and perhaps computationally more burdensome have been recently obtained [18],[20] primarily by using convexity arguments and the geometry of the coefficient space (cf. discussions in Section II). In the present subsection it will only be indicated that the major techniques exploited in these studies do carry over to multivariable polynomials with complex coefficients. The details of further developments, as in [18], can then be derived in an exactly analogous manner.

As before, we will denote the space spanned by the paraeven (paraodd) polynomials of degree $\underline{n}$ by S_e (S_o). Consider then a family of polynomials g such that their paraeven (paraodd) coefficients are contained in the convex hull $\Lambda_e \subset S_e$ (or $\Lambda_o \subset S_o$) of fixed paraeven or paraodd polynomials e_i; i=1 to m (or o_i; i=1 to m).

As a consequence of Corollary 2.1 the following result is then almost immediate:

Theorem 3.4: Consider the polyhedron $\Lambda \subset S$ consisting of coefficients of polynomials g=e+o, where the coefficients of e belong to Λ_e and those of o belong to Λ_o as described above. Then any $g \in \Lambda$ is scattering Schur (Hurwitz) if and only if the set of m^2 polynomials (e_i+o_j); i,j=1,2,...m are each scattering Schur (Hurwitz).

Proof: Necessity is trivial. To show sufficiency it is only needed to realize that for any $g=e+o \in \Lambda$ we have $e \in \Lambda_e$ and $o \in \Lambda_o$, and thus e (also o) is a convex combination of m vertices of Λ_e (correspondingly Λ_o). Thus, due to Corollary 2.1 and the scattering Schur (Hurwitz) property of (e_i+o_j), g is scattering Schur (Hurwitz).

The statement of a natural result, which in 1-D has been claimed to be the closest known discrete analog of Kharitonov's theorem, thus follows.

Theorem 3.5: Consider the family of polynomials in k variables $\underline{z}=(z_1,z_2,\ldots z_k)$:

$$g = \sum_{\underline{i}} (A_{\underline{i}} + B_{\underline{i}})\underline{z}^{\underline{i}} \qquad (3.5)$$

of (partial) degree $\underline{n}=(n_1,n_2,\ldots n_k)$ and with complex coefficients, where we have used the multi-index notation introduced earlier, and where for each $\underline{i}$ the coefficients $A_{\underline{i}}$, $A_{\underline{n}-\underline{i}}$ and $B_{\underline{i}}$, $B_{\underline{n}-\underline{i}}$ vary inside boxes shown in Figure 3.1(a),(b) (if $A_{\underline{i}}=A_{\underline{n}-\underline{i}}$ for some $\underline{i}$ then $A_{\underline{i}}$ varies over then real interval $[-A_{\underline{i}}, +A_{\underline{i}}]$;

similarly for $B_{\underline{i}}$'s). Then the polynomial family in (3.5) is scattering Schur if and only if each member of a finite number of g's defined by all possible combination of corner points from the sets $R_{\underline{i}j}$, $I_{\underline{i}j}$; j=1 to 4, $\underline{0} \le \underline{i} \le \underline{n}$ (or corresponding real intervals when $A_{\underline{i}}=A_{\underline{n}-\underline{i}}$ or $B_{\underline{i}}=B_{\underline{n}-\underline{i}}$) is scattering Schur.

Proof: Note first that the coefficient of $\underline{z}^{\underline{i}}$ in the paraeven and paraodd parts of g are respectively given by:

$$(A_{\underline{i}} + A_{\underline{n}-\underline{i}}) + j(B_{\underline{i}} - B_{\underline{n}-\underline{i}}) \tag{3.6}$$

$$(A_{\underline{i}} - A_{\underline{n}-\underline{i}}) + j(B_{\underline{i}} + B_{\underline{n}-\underline{i}}) \tag{3.7}$$

and as $A_{\underline{i}}$, $A_{\underline{n}-\underline{i}}$ and $B_{\underline{i}}$, $B_{\underline{n}-\underline{i}}$ vary inside the boxes in Figure 3.1(a),(b) we actually have that:

$$\underline{a}_{\underline{i}} \le A_{\underline{i}} + A_{\underline{n}-\underline{i}} \le \bar{a}_{\underline{i}};\ \underline{\beta}_{\underline{i}} \le B_{\underline{i}} - B_{\underline{n}-\underline{i}} \le \bar{\beta}_{\underline{i}} \tag{3.8}$$

$$\underline{\alpha}_{\underline{i}} \le A_{\underline{i}} - A_{\underline{n}-\underline{i}} \le \bar{\alpha}_{\underline{i}};\ \underline{b}_{\underline{i}} \le B_{\underline{i}} + B_{\underline{n}-\underline{i}} \le \bar{b}_{\underline{i}} \tag{3.9}$$

Thus, the coefficients of the paraeven and paraodd parts of g vary within the the rectangular parallelopiped in the coefficient spaces, which we may identify with Λ_e and Λ_o of Theorem 3.4. The proof is then completed by invoking Theorem 3.4 after recognizing that the family of g's in (3.5) is actually the polyhedron Λ in Theorem 3.4.

Remark: Although the above theorem bears resemblance with the generalized k-D Kharitonov's theorem in that the coefficients are allowed to vary inside boxes (albeit with sides skewed with respect to the axis) it is far less powerful because the number of extreme polynomials i.e., the number of vertices of the polytope Λ in the above theorem grows with the (partial) degrees of the polynomial family under consideration. In particular, we need as many as 4^M, where

$$M = \prod_{i=1}^{k} (n_i + 1) \tag{3.10}$$

extreme polynomials for a polynomial family of partial degree $\underline{n}=(n_1,n_2,\ldots n_k)$.

For real polynomials we need only 2^M polynomials, the 1-D version of which coincide with the result in [18].

Remark: Note further that the 1-D version of this result has been used [18] to derive various sets of necessary conditions as well as sufficient conditions (but not both simultaneously) for the robust stability of the polynomial family g in (3.5). However, since Theorem 3.5 can form the basis for such a study in multidimensions the rest of the development can be considered to be routinely carried out in a manner exactly similar to the one reported in [18] for 1-D.

It has thus been shown that most of the 1-D results on discrete domain robustness of stability, which rely on convexity arguments can be naturally extended to multidimensions essentially by virtue of Theorems 2.1 and 2.2. This statement, in fact, holds even for results which we have not explicitly dealt with here, and includes the ones in [20], for example.

IV: Alternate Methods of Analysis and Further Generalizations of Kharitonov-type Results in Multidimensions:

In this section we report recent investigations [40] by the present author into multidimensional counterpart of a vast range of 1-D generalizations of Kharitonov's theorem as has been worked out by Anagnost, Desoer and Minnichelli in [28], [29]. Essentially by using some analytical results along with convexity arguments discussed earlier, we show that Kharitonov-type results can be obtained in a very broad setting even for multidimensional polynomials. The method to be presented and the resulting consequences are so broad that both discrete and continuous domain results including k-D generalization of Kharitonov's theorem (i.e., Theorem 3.3) as well as the k-D generalization of the Edge Theorem of Bartlett, Hollot and Lin [22], among other results, fall out as corollaries of the present discussion. As indicated in [29] in the 1-D context these results can also be used for graphical implementation of tests for k-D robust stability. However, as is always the case in higher dimensions [16], the implementation of such tests are much more complex than in 1-D, and it is concievable that these preliminary results can be improved much further. The primary objective of this section is, therefore, to indicate how the methodology of investigation advanced in [28],[29] can be adapted to incorporate into the k-D context, albeit in a somewhat nontrivial manner.

Let $U = U_1 \times U_2 \times \ldots \times U_k$, where each U_i, i=1 to k is an open subset of the complex plane. Roughly speaking U is going to be the domain of undesirable or forbidden zero locations of multivariable polynomials $g=g(\underline{z})$ in $\underline{z}=(z_1,z_2,\ldots z_k)$. U_i's may be half-planes or unit-discs, or even sets of mixed types (as arising in certain formulations of the lumped distributed network problems e.g., [39]) in special cases of interest. Let ∂U_i denote the boundary of U_i for each i=1 to k and $\partial U = \partial U_1 \times \partial U_2 \times \ldots \times \partial U_k$ be the distinguished boundary of U.

A polynomial $g=g(\underline{z})$ in k variables may be called widest sense U-Hurwitz if $g \neq 0$ for $\underline{z} \in U$. Similarly, other variants of multivariable U-Hurwitz property can also be naturally considered. For example, if g is widest sense U-Hurwitz and the set of zeros of g in ∂U form a sequentially almost complete set [5] of k-tuples of order at most (k-1) then g is <u>scattering</u> <u>U-Hurwitz</u>; g is <u>strictest</u> <u>sense</u> <u>U-Hurwitz</u> if $g \neq 0$ for all $\underline{z}$ in $\bar{U} = \bar{U}_1 \times \bar{U}_2 \times \ldots \times \bar{U}_k$, where each U_i is a compact domain (a corresponding modification, to include noncompact U_i's by taking into account the points at infinity is also possible as in the case of conventional strictest sense multidimensional Hurwitz and Schur polynomials [17],[35],[5]).

All results in this section and their proofs will be given only for the restricted class of strictest sense U-Hurwitz polynomials. It must be mentioned that analogous results can also be proved for the wider class of scattering U-Hurwitz polynomials by exploiting the methods to be presented in the following. However, exposition in such a general context involves rather technical considerations and will be reported separately elsewhere. Apart from considerations of simplicity of exposition, this will also establish in the sequel the result that a strictest sense Hurwitz counterpart of the generalized Kharitonov's theorem (Thoerem 3.3 in the present paper) hold true.

One of the main tools of the analysis to follow is a multidimensional extension of the notion of <u>evaluation</u> <u>map</u> E(.) associated with the polynomial g, as a map from the space **S** of polynomial coefficients to the complex numbers **C** defined as $E(g) = g(\underline{z})$, where $\underline{z}$ is a k-tuple of complex numbers. Clearly, for a fixed $\underline{z}$, E(.) so defined is a <u>linear</u> map from **S** to **C**.

Evaluation map E(.) was apparently first introduced by Dasgupta [30] in the present area of study and has been later exploited extensively in [28],[29].

Consider next a set of m polynomials $g_i=g_i(\underline{z})$; i=1 to m and the family Λ of polynomials obtained by convex combinations of g_i, i=1 to m. Thus, in the space S of polynomial coefficients Λ is a <u>convex</u> <u>polyhedron</u> (i.e., convex hull of vertices, which correspond to coefficients of g_i's). Let $Ed(\Lambda)$ denote the set of <u>exposed</u> <u>edges</u> (i.e., the one dimensional edges along with the vertices) of Λ.

Also, for a fixed $\underline{z}$ let H_z denote the set of complex numbers $H_z = \{g(\underline{z});\ g \in \Lambda\}$. Since the evaluation map is linear, H_z can be seen to be the convex hull of the set of points $\{E(g_i);\ i=1 \text{ to } m\}$, and is thus a convex polygon.

We then have our first result which can be stated as follows.

<u>Theorem 4.1</u>: The family Λ of polynomials is strictest sense U-Hurwitz if and only if the following two conditions simultaneously hold true:

(a) g is strictest sense U-Hurwitz for some g in Λ

(b) $0 \not\in H_{\underline{z}}$ for all $\underline{z}$ in the distinguished boundary ∂U of U

<u>Sketch of proof</u>: For simplicity we restrict ourselves to the case k=2.
Necessity: If the family Λ is strictest sense U-Hurwitz then clearly (a) follows. To show (b) assume for contradiction that there is a $\underline{z}_0 \in \partial U$ such that $0 \in H_{\underline{z}_0}$. Then there would exist a $g \in \Lambda$ such that $g(\underline{z}_0)=0$. Since $\underline{z}_0 \in \partial U \subset \bar{U}$, this contradicts with strictest sense U-Hurwitz property of g.

Sufficiency: Given that $g=\hat{g} \in \Lambda$ is strictest sense U-Hurwitz and (b) holds assume for contradiction that there is a $\bar{g} \in \Lambda$ which is not strictest sense U-Hurwitz i.e., has at least one zero in $\bar{U}$, say, $\underline{z}_0=(z_{10}, z_{20})$. Since Λ is pathwise connected, as we continuously deform g along a path in Λ emanating from $\bar{g}$ and ending in $\hat{g}$, the zero (z_{10},z_{20}) of $\bar{g}$, due to the continuity property of zeros of polynomials as a function of coefficients, must float over the boundaries ∂U_1 and ∂U_2 into domains outside of U_1 and U_2 respectively. For the purpose of the present proof, however, this will be allowed to happen in a controlled manner as described in the following (see Figure 4.1).

First we shall maintain z_1 fixed at $z_1=z_{10}$ and deform g continuously from $\bar{g}$ towards $\hat{g}$ as explained above. Clearly, then there must be a $g' \in \Lambda$ on the continuous path connecting $\bar{g}$ to $\hat{g}$ such that $g'=0$ for $z=z_{10}$ and some $z_2=z'_{20} \in \partial U_2$. We then maintain z_2 fixed at $z_2=z'_{20} \in \partial U_2$ and continue on with the deformation process from g' further towards $\hat{g}$. Invoking the same argument as above (but with the roles of z_1 and z_2 reversed) it then follows that there must exist a $g'' \in \Lambda$ on the continuous path connecting g' to $\hat{g}$ such that $g''=0$ for $z_1=z'_{10} \in \partial U_1$. However, this last situation is impossible in view of the fact that $(z'_{10},z'_{20}) \in \partial U$ and (b) holds true. Thus, sufficiency of (a) and (b) has been demonstrated.

<u>Remark</u>: It must be mentioned that apart from the possibility of incorporating the wider class of scattering U-Hurwitz polynomials, Theorem 4.1 can be further

broadened to include the case when Λ is not just a polytope but is an arbitrary connected subset in the space of polynomial coefficients. Such a generalization in 1-D has been worked out in [29].

Remark: Other than considering the special case of $k=2$ the proof of Theorem 4.1 can be considered incomplete due to the valid criticism that the continuity property of zeros of a polynomial in k variables apparently ceases to hold when the polynomial becomes identically zero for a fixed value of one of the k variables. However, such degeneracies can be adequately handled by means of detailed but elementary arguments as shown in a different context in [15].

Our next result is a multidimensional analog of the 1-D 'edge theorem' of [22], which has been proven via alternate techniques in [29]. It will be shown that via argument of the type exploited in the last theorem, we can conveniently obtain a k-D generalization, which roughly states that in order to test for the strictest sense U-Hurwitz property of a polyhedral family of polynomials, it suffices to test the exposed edges.

Theorem 4.2: The family Λ of polynomials is strictest sense U-Hurwitz if and only if the following two conditions simultaneously hold true:

(a) The set of polynomials belonging to the exposed edges $Ed(\Lambda)$ of Λ are all strictest sense U-Hurwitz.

(b) If ∂U_i^j, $j=1,2,\ldots m_i$ are connected components of ∂U_i then each of components of the distinguished boundary ∂U, namely

$$\partial U_1^{j_1} \times \partial U_2^{j_2} \times \ldots \times \partial U_k^{j_k} \tag{4.1}$$

for all possible choices of j_i from $\{1,2,\ldots m_i\}$, contain a $\underline{z}$ such that $0 \notin H_{\underline{z}}$.

For a proof of the above theorem we need the following lemma.

Lemma 4.1: $\partial H_{\underline{z}} \subset E(Ed(\Lambda))$ i.e., the boundary of $H_{\underline{z}}$ is a subset of the image of exposed edges of Λ under the evaluation map.

A 1-D version of this lemma is stated and proved in [29]. Since the proof in [29] essentially exploits the linearity property of the evaluation map - a fact valid irrespective of the number of dimensions - a separate proof in the present context would be repetition of exactly same arguments.

Sketch of proof of Theorem 4.2 for k=2:

Necessity: If Λ is strictest sense U-Hurwitz then (a) trivially holds, whereas (b) follows from Theorem 4.1.

Sufficiency: Clearly, condition (a) of the present Theorem implies condition (a) of Theorem 4.1. The proof would be completed by showing that (b) of Theorem 4.1 is also implied by conditions (a) and (b) of the present theorem. For this, suppose for contradiction there exits a $\underline{z}_0$ in ∂U such that $0 \in H_{\underline{z}_0}$ and that $\underline{z}_0$ belongs to

$$(\partial U_1^{j_1} \times \partial U_2^{j_2}) \tag{4.2}$$

for some fixed j_1 and j_2. Then due to condition (b) of the present theorem there exists a $\hat{\underline{z}}$ belonging the boundary component (4.2) such that $0 \notin H_{\hat{\underline{z}}}$. Since $\underline{z}_0$ and $\hat{\underline{z}}$ belong to the same boundary component it is possible to continuously move from $\underline{z}_0$ to $\hat{\underline{z}}$, as a result of which $H_{\underline{z}}$ must move continously. Thus, there must exist a $\underline{z}'$ in the boundary component (4.2) above such that $0 \in \partial H_{\underline{z}'} \subset E(Ed(\Lambda))$ (the last inclusion follows from Lemma 4.1) contradicting strictest sense U-Hurwitz property of $Ed(\Lambda)$. Consequently, a $\underline{z}_0 \in \partial U$ with $0 \in H_{\underline{z}_0}$ cannot exist.

As an application of the above analysis, we now show that Theorem 4.1 can, in fact, be used to prove a strictest sense U-Hurwitz version of generalized Kharitonov theorem (our Theorem 3.3).

Corollary 4.1: If the set of 4N polynomials mentioned in Theorem 3.3 are all strictest sense Hurwitz then g in Theorem 3.3 is also so.

We first need to recall an elementary fact in the follwing Lemma for a proof of the above Corollary. For the purpose of the proof of Corollary 4.1 we will replace the notation $\underline{z}$ of the present section with the notation $\underline{p}$ as is conventionally used in continuous domain.

Lemma 4.2: If $g=g(\underline{p})$ is a strictest sense Hurwitz polynomial in k-variables $\underline{p}=(p_1,p_2,\ldots p_k)$ then $\arg(g(j\underline{\omega}))$ is a strictly increasing continuous function of each ω_i, where $\underline{\omega}=(\omega_1,\omega_2,\ldots\omega_k)$. It is assumed that g involves the variable p_i.

The lemma follows from the fact that the 1-D polynomial $g_1(p_i)$ obtained by freezing $p_\nu=i\omega_\nu$, $\nu=1$ to k except i in g is Hurwitz, and the well known result that the phase function of an 1-D Hurwitz polynomial is a strictly increasing continuous function of the frequency variable [29].

Proof of Corollary 4.1: It follows from the construction of generalized Kharitonov polynomials (cf. Fact 3.1) that for each n=1,2...k we have

$$e_n^-(j\underline{\omega}) \le e(j\underline{\omega}) \le e_n^+(j\underline{\omega}) \qquad (4.3)$$

$$o_n^-(j\underline{\omega})/j \le o(j\underline{\omega})/j \le o_n^+(j\underline{\omega})/j \qquad (4.4)$$

which can be interpreted to mean that for all $\underline{p}=j\underline{\omega}$ with $\underline{\omega}$ in the n-th orthant the polygon $H_{\underline{p}}$ is a rectangle with horizontal and vertical sides in the complex plane C with its four corner points as given in Figure 4.2.

We will show that conditions (a) and (b) of Theorem 4.1 are satisfied. Clearly, (a) is satisfied because any one of the 4N generalized Kharitonov polynomials is in Λ and is strictest sense Hurwitz.

For (b), we first claim that there exists at least one value of $\underline{p}=j\underline{\omega}$ such that $0 \notin H_{\underline{p}}$. For this, set $p_i=j\omega$ for each i=1 to k in the 4N generalized Kharitonov polynomials each of which then produces a 1-D Hurwitz plynomial with degree equal to their common total degree, say, t. Then, if $\omega \to \infty$ then the rectangle $H_{\underline{p}}$ travels to infinity uniformly [28] at an asymptotic angle of $\frac{1}{2}t\pi$ (mod2π). Ôherefore, $0 \notin H_{\underline{p}}$ for $p_i=j\omega$ and $\omega \to \infty$.

Now suppose $0 \in H_{\underline{p}}$ for some $\underline{p}=j\hat{\underline{\omega}}$. Then since $H_{\underline{p}}$ is a continuous function of $\underline{p}$ there must exist a $\underline{\omega}'$ such that $0 \in \partial H_{\underline{p}}$ for $\underline{p}=j\underline{\omega}'$. We assume without loss of generality that the bottom edge of $H_{\underline{p}}$ with $\underline{p}=j\underline{\omega}$ contains 0 (note that a corner point may not be the origin, because they are the images of generalized

Kharitonov polynomials, which are assumed strictest sense Hurwitz) i.e., we have the situation shown in Figure 4.3, where n denotes the orthant to which $\underline{\omega}'$ belongs. Now if we increase any one of the component variables in $\underline{\omega}'=(\omega_1',\omega_2',\ldots\omega_k')$, say, ω_1' by an infinitesimal amount by keeping ω_i', i=2 to k fixed, due to Lemma 4.2, the images of the generalized Kharitonov polynomials $[e_n^+(j\underline{\omega})+o_n^-(j\underline{\omega})]$ and $[e_n^-(j\underline{\omega})+o_n^-(j\underline{\omega})]$ will move respectively into the first and the third quadrant. However, this is impossible because they must have the same imaginary parts by construction. The proof of Corollary 4.1 is thus complete.

Note further that in analogy with the 1-D case [29] if the domains U_i are each connected, unbounded then condition (b) of Theorem 4.2 can, in fact, be dropped. This fact deserves a separate statement as in the following.

<u>Corollary 4.2</u>: Assuming U_i's to be each connected unbounded domain, the polyhedral family Λ is strictest sense U-Hurwitz if and only if $Ed(\Lambda)$ i.e., the set of exposed edges of Λ are so.

<u>Proof</u>: Only sufficiency needs to be demonstrated. First note that by using techniques used in the proof of Corollary 4.1 it is easily shown that if $|z_i| \to \infty$ for each i then $0 \notin H_{\underline{z}}$ for some $\underline{z}$. The existence of such a $\underline{\hat{z}}=(\hat{z}_1,\hat{z}_2,\ldots\hat{z}_k)$ in $U=U_1\times U_2\times\ldots\times U_k$ is guaranteed due to the unboundedness of U_i's. Next, for contradiction suppose there is a $\underline{z}'$ in U with $g(\underline{z}')=0$ for some g in Λ. Consider then domains $U_i^* \subset U_i$ such that $z_i' \in U_i^*$ and $\hat{z}_i \in \partial U_i^*$. This can be done due to the connectedness of U_i's. Thus, conditions (a),(b) of Theorem 4.2 are both satisfied by $U^*=U_1^*\times U_2^*\times\ldots\times U_k^*$. Consequently, Λ is strictest sense U-Hurwitz, which in turn shows (since $g \in \Lambda$ and $\underline{z}' \in U^*$) that $g(\underline{z}') \neq 0$ - a contradiction.

From computational point of view Theorems 4.1 and 4.2 provide two different kinds of tests for strictest sense U-Hurwitz property of Λ. The edge theorem can be implemented by considering convex combinations of two vertex polynomials parametrized by a single parameter $0\leq\alpha\leq 1$. The extra variable α, although adds to computational burden, nevertheless poses a problem solvable in finite number of steps (at least in principle), because they fall into the category of problems of decision algebra [16]. In particular, if k=2 it is known that tests for discrete domain stability of polynomials can be implemented via Schur-Cohn method [36],[41], the essential computational core of which is to check the sign pattern of a set of real 1-D polynomials in x in the interval [-1,+1] (see also [19] for similar calculations). In the present context the extra parameter α which enters into calculations in a rational manner makes it necessary to determine the sign pattern of a set of polynomials in x and α for $x \in [-1,+1]$ and $0 \leq \alpha \leq 1$. Computational algorithms for solving this latter problem exist, for example, in [16].

On the other hand, multidimensional versions of graphical test procedure suggested in [29] can be given in the same spirit as that of [34], for example. To demonstrate this we confine attention to the discrete domain problem i.e., where U_i's are unit discs, their boundaried being parametrized as $z_i = \exp(j\theta_i)$ for i=1 to k. Again for the sake of simplicity we treat the case k=2 only. Consider the nearest point function [29]:

$$Nr(\theta_1,\theta_2) = \arg\min_{s \in H_{\underline{z}}} \{||s||\}; \quad \underline{z}=(z_1,z_2), \quad z_i=\exp(j\theta_i) \tag{4.5}$$

which, in fact can be shown to be a continuous function of both θ_1 and θ_2. To test for condition (b) of Theorem 4.1, therefore, one needs to determine whether or not $Nr(\theta_1,\theta_2)$ becomes zero for any θ_1, θ_2 in $[0,2\pi]$. One way of implementing this test would be to hold $\hat{\theta}_2$ fixed and make a Nyquist-like plot of the closed contour traced out by $Nr_{\theta_2}(\theta_1) = Nr(\theta_1,\hat{\theta}_2)$ as θ_1 changes from 0 to 2π (see Figure 4.4). Repeat next the plot for values of θ_2 with small increments which will correspond approximately to continuous deformation of the contour just mentioned. We then have the following:

<u>Fact</u> <u>4.1</u>: $Nr(\theta_1,\theta_2) \neq 0$ for all θ_1, θ_2 if and only if the number of encirclement of the origin by the plot of $Nr_{\theta_2}(\theta_1)$ for $0 \leq \theta_1 \leq 2\pi$ is constant as a function of the parameter θ_2.

Of course, the accuracy of such a test concievably depends on the smallness of the increments in θ_2. Nevertheless, since it has been noted [29] that when Λ is a convex polyhedron, thus $H_{\underline{z}}$ is a convex polygon, computation of $Nr(\theta_1,\theta_2)$ is not a laborious task, the method suggests the principles underlying a computationally feasible (though nonfinite) graphical test for determining whether or not the polyhedral region Λ is strictest sense U-Hurwitz.

V. Interval Positive Property of Multivariable Rational functions:

In the present section it will be indicated that multidimensional extensions of recent 1-D [12],[13] results on robustness of positivity property of rational functions with real as well as complex coefficients are also feasible. In particular, the class of strict(est) sense positive k-D rational functions is introduced, and the interval strict(est) sense positive property as well as the interval positive property is characterized in terms of the corresponding property of a set of $2^{2(k+2)}$ rational functions, which in fact, are ratios of Kharitonov-like k-D polynomials formed from the extreme values of the intervals of coefficient perturbations. For this, we will first briefly recall and reformulate, whenever necessary, some basic notions of multivariate positive functions from our recent discussions on the topic [5].

A rational function F of k-variables $\underline{p} = (p_1, p_2, \ldots, p_k)$ is said to be <u>positive</u> if $\mathrm{Re}F \geq 0$ for $\mathrm{Re}\underline{p} > \underline{0}$. A characterization of F=n/d, where n and d are coprime polynomials, to be a positive function is provided by the following result proved in [5].

<u>Fact 5.1</u>: The irreducible rational function F=b/a is a positive function if and only if:

(i) a+b is a scattering Hurwitz polynomial

(ii) $\mathrm{Re}F \geq 0$ for $\underline{p} = j\underline{\omega}$, whenever F is well defined.

Let us recall that the numerators as well as the denominators of irreducible rational positive functions are necessarily products of scattering Hurwitz and reactance Hurwitz [5] polynomials. Furthermore, irreducible factors of the latter type may occur only with multiplicity equal to one. We will also need the notion of strict positivity of multivariable rational functions, which is introduced in the following:

A rational function F of k-variables is said to be <u>strictly positive</u> if (i) F is positive and (ii) neither its denominator nor its numerator in irreducible rational form contains a reactance Hurwitz factor (i.e., it is a ratio of scattering Hurwitz polynomials).

The following comments regarding the motivational contents of the above definition are in order. Clearly, in 1-D our definition coincides with the standard definition of strict positivity [12],[13]. It is classically known in network synthesis theory that if a 1-D positive function contains a reactance Hurwitz factor in its numerator or denominator (i.e., if it is not strictly positive) then a pure reactance can be extracted from it in such a way that the residual function is a strictly positive function. In multidimensions, however, if a rational positive function has reactance Hurwitz factors in its numerator or denominator, it is not known [5],[37] whether or not a reactance containing these reactance Hurwitz factors can always be extracted in such a way that the residual is strictly positive. Nevertheless, there exist k-D positive functions with reactance Hurwitz factors in its numerator or denominator from which a reactance extraction of the above type can be carried out. Consequently, in keeping with the notion of 1-D strict positivity, k-D reactance Hurwitz factors is not allowed in the numerators and denominators of multivariable strict positive functions either.

Example: The function F_1 in the following is a positive function with scattering Hurwitz denominator and numerator. Thus, F_1 is strictly positive.

$$F_1 = (p_1+p_2+p_1p_2)/(p_1+1)$$

Next, note that if a (strictly) positive rational function F has a nonzero difference in partial degrees of its numerator and denominator (in fact, this difference may not be larger than 1) in one of the variables, say p_i, then (see Theorem 36b in [5]) a reactance can be extracted from F; thus producing a strictly positive function with equal p_i-degrees in its numerator and denominator. This situation is no different from the one discussed in the last paragraph. In both cases the rational function F has a zero or a pole for a fixed p_i (in this case $p_i=\infty$) and arbitrary values of the rest of the variables, thus allowing for a reactance extraction to be possible. We will, therefore, also be concerned with multivariable positive functions devoid of singularities of this latter kind . Consequently, we have the following definition:

A rational function F in k-variables $\underline{p} = (p_1,p_2,\ldots,p_k)$ is said to be strictest sense positive if it is strictly positive and the partial degrees of its numerator and denominator polynomials in each p_i; $i=1$ to k are the same (i.e., relative partial degree of F in each variable p_i is zero).

Various other ways of characterizing this latter notion of positivity (including those analogous to Fact 5.1), which are all mathematically equivalent, exist, but we will not undertake this discussion here.

Example: Note that F_1 is strictly positive but not strictest sense positive, because a reactance p_2 can be trivially extracted from F_1 as mentioned in the preceeding discussion. Interestingly, it is possible for a strictest sense positive function to have zeros (or singularities of the 2nd kind) on the distinguished boundary $\mathrm{Re}\,\underline{p}=\underline{0}$ as is demonstrated by $F_2=(1+F_1^{-1})^{-1}$ in the following:

$$F_2 = (p_1+p_2+p_1p_2)/(1+p_1+2p_2+p_1p_2)$$

in which the positive function F_2 has a zero at $p_1=p_2=0$.

For the purpose of the present section we will adopt a slightly different notation than in Section 3. The extreme paraeven (paraodd) parts of g associated with the n-th orthant will be denoted by g_e^- and g_e^+ (g_o^- and g_o^+) respectively (obviously, these depend on n, but this dependence is not reflected in our choice of notation for the sake of clarity). Analogous notations for polynomials other than g will also be used.

Given $\underline{n}=(n_1,n_2,\ldots,n_k)$, we consider the real interval sets as in (5.1) for each $\underline{i}=(i_1,i_2,\ldots,i_k)$ such that $0\le i_j\le n_j$; $j=1$ to k.

$$I:\ \underline{\alpha}_{\underline{i}} \le \alpha_{\underline{i}} \le \overline{\alpha}_{\underline{i}} \ ;\ \underline{\beta}_{\underline{i}} \le \beta_{\underline{i}} \le \overline{\beta}_{\underline{i}} \qquad (5.1)$$

The $\alpha_{\underline{i}}$'s and $\beta_{\underline{i}}$'s will, along with their upper and lower bounds in (5.1), denote the real parts and imaginary parts of coefficients of polynomials in our ensuing discussions. We then consider for each $n=1,2,\ldots,N$ a set of four polynomials:

$$K(I,n) = \{(g_e^+ + g_o^+),(g_e^+ + g_o^-),(g_e^- + g_o^+),(g_e^- + g_o^-)\} \qquad (5.2)$$

where g_e^-, g_e^+, g_o^- and g_o^+ are the extreme polynomials for the n-th orthant associated with the interval set **I** in (5.1), and are constructed via Rules 3.1 and 3.2. The set of 4N generalized Kharitonov polynomials associated with the interval set **I** is, thus, $K(\mathbf{I}) = \{g;\ g \in K(\mathbf{I},n),\ n=1,2,\ldots,N\}$.

We next state a multidimensional version of a weak form of Dasgupta's 1-D result [12]. For a specified degree $\underline{n}=(n_1,n_2,\ldots,n_k)$ consider for each $\underline{i}=(i_1,i_2,\ldots,i_k)$ with $0 \le i_j \le n_j$ for for all j=1 to k, two sets of real intervals designated by **N** and **D**, (such as (5.1)) in the following:

$$\mathbf{N}:\ \underline{\alpha}_{\underline{i}} \le \alpha_{\underline{i}} \le \overline{\alpha}_{\underline{i}}\quad ;\quad \underline{\beta}_{\underline{i}} \le \beta_{\underline{i}} \le \overline{\beta}_{\underline{i}} \qquad (5.4)$$

$$\mathbf{D}:\ \underline{\gamma}_{\underline{i}} \le \gamma_{\underline{i}} < \overline{\gamma}_{\underline{i}}\quad ;\quad \underline{\delta}_{\underline{i}} \le \delta_{\underline{i}} \le \overline{\delta}_{\underline{i}} \qquad (5.5)$$

Theorem 5.1: Let F=b/a be a rational function in k-variables $\underline{p}=(p_1,p_2,\ldots,p_k)$, where a and b are each polynomial of (partial) degree $\underline{n}=(n_1,n_2,\ldots,n_k)$ as in (5.6), in which the coefficients satisfy (5.4) and (5.5).

$$b = \Sigma_{\underline{i}}\ (\alpha_{\underline{i}}+j\beta_{\underline{i}})\underline{p}^{\underline{i}}\ ,\ a = \Sigma_{\underline{i}}\ (\gamma_{\underline{i}}+j\delta_{\underline{i}})\underline{p}^{\underline{i}} \qquad (5.6)$$

Then F is a positive function in the strictest sense if each member of the set $P(\mathbf{N},\mathbf{D})$ of $16N^2$ rational functions in (5.7)

$$P(\mathbf{N},\mathbf{D}) = \{h/g\ ;\ h \in K(\mathbf{N})\ ,\ g \in K(\mathbf{D})\} \qquad (5.7)$$

is strictest sense positive and is in irreducible rational form.

The converse statement i.e., if all F as in (5.6) with their coefficients satisfying (5.4), (5.5) are (irreducible) strictest sense positive functions, then each member of $P(\mathbf{N},\mathbf{D})$ is also so, is trivial.

For proofs of Theorems 5.1 and 5.2 to follow we will need the following elementary fact [13].

Lemma 5.1:

Let $a=a_e+a_o$ and $b=b_e+b_o$ be two complex numbers decomposed into their real and imaginary parts respectively, which satisfy the following inequalities:

$$\underline{g}_e \le a_e \le \bar{g}_e \text{ , } \underline{g}_o/j \le a_o/j \le \bar{g}_o/j \tag{5.8}$$

$$\underline{h}_e \le b_e \le \bar{h}_e, \; \underline{h}_o/j \le b_o/j \le \bar{h}_o/j \tag{5.9}$$

where $\underline{g}_e$, $\bar{g}_e$, $\underline{h}_e$ and $\bar{h}_e$ are each real and $\underline{g}_o$, $\bar{g}_o$, $\underline{h}_o$ and $\bar{h}_o$ are each imaginary number satisfying $(AB-CD)\geq 0$ with each choice of A, B, C and D as follows:

$$A = \underline{h}_e \text{ or } \bar{h}_e; \; B = \underline{g}_e \text{ or } \bar{g}_e \tag{5.10a,b}$$

$$C = \underline{h}_o \text{ or } \bar{h}_o; \; D = \underline{g}_o \text{ or } \bar{g}_o \tag{5.11a,b}$$

Then we must have

$$a_e b_e - a_o b_o \ge 0 \tag{5.12}$$

Proof of Theorem 5.1:

Since the numerators and denominators of strictest sense positive functions in irreducible rational form are necessarily scattering Hurwitz, we have that each $h \in K(\mathbf{N})$ and $g \in K(\mathbf{D})$ is scattering Hurwitz. Thus, from Theorem 3.3 it follows that both b and a, as in (5.6), are scattering Hurwitz polynomials if their coefficients are restricted by (5.4) and (5.5) respectively.

Next, note that since a is scattering Hurwitz $a(j\underline{\omega})\neq 0$ for almost all $\underline{\omega}$ (specifically, if $\underline{\omega} \in \Omega$, where Ω is a sequentially almost complete set [5] of k tuples of order k-1). Thus, F is well defined for any $\underline{\omega} \in \Omega$. In what follows we fix $\underline{\omega} \in \Omega$ in the μ-th orthant, where μ is arbitrary.

Let $h=h_e+ h_o$, $g=g_e+ g_o$ be decompositions of h and g in (5.7) into paraeven and paraodd parts. Then since $h \in K(\mathbf{N})$, $g \in K(\mathbf{D})$ implies that h/g is a positive function, it follows that for $\underline{p}=j\underline{\omega}$ we have

$$(h_e g_e - h_o g_o) \ge 0 \tag{5.13}$$

In particular, (5.13) holds for all $h \varepsilon K(N,\mu)$ and for all $g \varepsilon K(D,\mu)$. Thus, we have $(AB-CD)\geq 0$ for each choice of A, B, C and D as follows:

$$A = h_e^- \text{ or } h_e^+ \ , \ B = g_e^- \text{ or } g_e^+ \tag{5.14a,b}$$

$$C = h_o^- \text{ or } h_o^+ \ , \ D = g_o^- \text{ or } g_o^+ \tag{5.15a,b}$$

where g_e^-, g_e^+, h_e^- and h_e^+ are the extreme paraeven polynomials for g and h in the μ-th orthant as defined before in (5.2). Similarly, g_o^-, g_o^+, h_o^- and h_o^+ are the corresponding set of paraodd polynomials.

Furthermore, since $h \varepsilon K(N,\mu)$ and $g \varepsilon K(D,\mu)$, it follows from Fact 3.1 that for the chosen $\underline{p}=j\underline{\omega}$, which belongs to the μ-th orthant, we have (5.16) and (5.17).

$$g_e^- \leq a_e \leq g_e^+ \ ; \ g_o^-/j \leq a_o/j \leq g_o^+/j \tag{5.16a,b}$$

$$h_e^- \leq b_e \leq h_e^+ \ ; \ h_o^-/j \leq b_o/j \leq h_o^+/j \tag{5.17a,b}$$

Thus, invoking Lemma 5.1 it follows that we have $(a_e b_e - a_o b_o)\geq 0$ i.e., $\mathrm{ReF}(j\underline{\omega})\geq 0$ for arbitrary $\underline{\omega} \varepsilon \Omega$. Next, since the denominator a of F is scattering Hurwitz and $\deg_i a = \deg_i b = n_i$ for each i=1 to k (i.e., F is regular at infinity) it can be shown (by invoking Theorem 35 in [5]) that F=b/a is a positive function. Since a and b are both scattering Hurwitz with equal partial degrees in each variable p_i, F is also strictest sense positive.

Remark: Note that if $\deg_i b > \deg_i a$ for some i=1 to k in Theorem 5.1 then Theorem 35 from [5] becomes inapplicable to F. Thus, the proof of Theorem 5.1 does not extend to the broader class of (strictly) positive functions which allows for nonzero difference in degree between the numerator and denominator. However, for the subclass of (strictly) positive functions which satisfies $\deg_i a \geq \deg_i b$ for all i=1 to k the proof of Theorem 5.1 can be seen to hold true. Thus, the statement of the theorem can be slightly broadened to include this latter class without substantially changing the proof.

Remark: Various special cases e.g., the ones for real polynomials as well for 1-D polynomials can be inferred from Theorem 5.1. As discussed in Section 3, in the case of real rational functions i.e., if polynomials h and g have real coefficients then K(I) consists of 2N distinct members. Thus, the cardinality of the set P(N,D) in this case is $|P(N,D)|=4N^2$. If k=1 then $N=2^k=2$, thus we have $|P(N,D)|=16$, which coincides with Dasgupta's result in [12].

We next address the problem of characterization of interval positive property of rational functions. In fact, a result exactly similar to Theorem 5.1 holds, but the proof requires a different strategy and makes use of Fact 5.1.

Given two multi-indices $\underline{m}=(m_1,m_2,\ldots,m_k)$ and $\underline{n}=(n_1,n_2,\ldots,n_k)$ such that the corresponding members of $\underline{m}$ and $\underline{n}$ do not differ by more than one, we now consider the inequalities (5.4) for all $\underline{i}=(i_1,i_2,\ldots,i_k)$ where $0\le i_j\le n_j$; j=1 to k. Similarly, we also consider (5.5) for all $\underline{i}=(i_1,i_2,\ldots,i_k)$ where $0\le i_j\le m_j$, j=1 to k. We then have the following result.

<u>Theorem 5.2</u>: Let F=b/a be any rational function in k-variables $\underline{p}=(p_1,p_2,\ldots,p_k)$, where a and b are polynomials of respective (partial) degrees $\underline{m}=(m_1,m_2,\ldots,m_k)$ and $\underline{n}=(n_1,n_2,\ldots,n_k)$ and are otherwise given as in (5.6), in which the real and imaginary coefficients satisfy (5.4) and (5.5). Then F is a positive function if each member of the set P(N,D) of $16N^2$ rational functions as in (5.7) is a positive function in irreducible rational form.

The converse statement that if every F prescribed via (5.6) with its coefficients satisfying (5.4) and (5.5) are (irreducible) positive functions then each member of P(N,D) is also so, is trivial.

<u>Proof</u>: Consider the set of intervals as in (5.18) designated by **S+N**.

$$\mathbf{S+N}:\ \underline{\eta}_{\underline{i}} = \underline{\alpha}_{\underline{i}} + \underline{\gamma}_{\underline{i}} \le \eta_{\underline{i}} \le \overline{\alpha}_{\underline{i}} + \overline{\gamma}_{\underline{i}} = \overline{\eta}_{\underline{i}} \qquad (5.18a)$$

$$\underline{\xi}_{\underline{i}} = \underline{\beta}_{\underline{i}} + \underline{\delta}_{\underline{i}} \le \xi_{\underline{i}} \le \overline{\beta}_{\underline{i}} + \overline{\delta}_{\underline{i}} = \overline{\xi}_{\underline{i}} \qquad (5.18b)$$

In (5.18) if any of the coefficients α, β, γ or δ is not defined for a certain subscript $\underline{i}$ then we assume that coefficient to be zero. We next observe that the polynomial (a+b), due to (5.6), can be written as in (5.19) below.

$$a+b = \Sigma_{\underline{i}} \; (\eta_{\underline{i}} + j\xi_{\underline{i}})p^{\underline{i}} \tag{5.19a}$$

where

$$\eta_{\underline{i}} = \alpha_{\underline{i}} + \gamma_{\underline{i}} \; ; \; \xi_{\underline{i}} = \beta_{\underline{i}} + \delta_{\underline{i}} \tag{5.19b,c}$$

and the summation in (5.19a) ranges over all possible $\underline{i}$.

Since each member of P(**N**,**D**) is positive, we assert, due to Fact 5.1, that each member of the set of $16N^2$ polynomials $\{(h+g);\ h \in K(\mathbf{N}),\ g \in K(\mathbf{D})\}$ is scattering Hurwitz. A closer examination of (5.4), (5.5), (5.18) and the coefficients of members of K(**N**) and K(**D**) reveals that K(**S+N**) (i.e., the generalized Kharitonov polynomials associated with the interval set **S+N**) is, in fact, a subset of $\{(h+g);\ h \in K(\mathbf{N}),\ g \in K(\mathbf{D})\}$. Thus, in particular, each member of K(**S+N**) is scattering Hurwitz. Consequently, in view of (5.18) and (5.19), it follows from (generalized Kharitonov) Theorem 3.3 that (a+b) as in (5.19) is scattering Hurwitz if coefficients of a and b individually satisfy (5.4) and (5.5), thus if $\eta_{\underline{i}}$, $\delta_{\underline{i}}$ satisfy (5.18).

Next, since P(**N**,**D**) is a positive set, if $h/g \in$ P(**N**,**D**) then $\mathrm{Re}(h/g) \geq 0$ for all $\underline{p}=j\underline{\omega}$, whenever h/g is well defined. Via a use of Lemma 5.1 it can then be shown in an exactly same manner as that in the proof of Theorem 5.1 that $\mathrm{Re}(b/a) \geq 0$ for an $\underline{p}=j\underline{\omega}$, whenever b/a is well defined. If d=gcd(b,a), b=b′d, a=a′d then F=b/a=b′/a′ and since (b+a)=d(b′+a′) is scattering Hurwitz (due to Theorem 2 in [14]) the factor (b′+a′) is also so. Furthermore, $\mathrm{Re}F=\mathrm{Re}(b'/a') \geq 0$ for $\underline{p}=j\underline{\omega}$, wherever F is well defined. Thus, invoking Fact 5.1 it can be concluded that F is a positive function.

<u>Remark</u>: Theorem 5.1 can also be proved via the technique used in proving Theorem 5.2 if slightly modified forms of Lemma 5.1 and Fact 5.1 are exploited. This is sketched in the following. If P(**N**,**D**) is a strictest sense positive set then we can conclude that for $\underline{p}=j\underline{\omega}$ except possibly for $\underline{\omega}$ from a sequentially almost complete set of order less than (k-1) [5], (AB-CD)>0 (where A, B, C and D are defined in (5.14), (5.15)), which from a modified form of Lemma 5.1 yields that $a_e b_e - a_o b_o > 0$ i.e., ReF>0. This latter conclusion along with the scattering Hurwitz property of (a+b) and a modified form of Fact 5.1 can be used to show that F is strictest sense positive.

Remark: Exactly similar remarks as those made after Theorem 5.1 on the number of polynomials needed to characterize interval positive property of real as well as 1-D rational functions can also be made here in the context of Theorem 5.2.

The results of the present section are fairly recent even in the 1-D case. It has been shown that much like in the study of stability property of polynomials, the positivity property of multivariable rational functions belonging to a certain region of the coefficient space can be guaranteed by requiring that a finite set of 'extreme rational functions' have the same property. The cardinality of this set is independent of the degree of the rational functions being considered and depends only on the number of dimensions (i.e., k) involved. Analogous results for discrete transfer functions are not known. An approach mimicking that described in the present section is bound to run into the same difficulties which plague the corresponding problem for studying stability of polynomials. The question of efficiently identifying regions (polytopic, polyhedral) of discrete domain positivity in the space of coefficients of rational functions thus remains open.

VI. Conclusions

Multidimensional extensions of robustness of stability property of polynomials as well as the positivity property of rational functions have been discussed. The motivation for dealing with such problems essentially derives from increasing need for studying digital filters [16] as well as feedback control systems [8] described by more than one transform variable. A very broad set of techniques, which largely rely on passive network theory, have been shown to generalize from 1-D to higher dimensions. It is thus fair to speculate that our discussions here should open up ways for many other results in the area. We have mostly been concerned with only one specific class of stable multidimensional polynomials, namely the scattering Hurwitz (Schur) class. There exist [5],[9],[38] other classes of stable multidimensional polynomials such as the reactance Hurwitz (which in some sense is diametrically opposite to scattering Hurwitz), immittance Hurwitz (i.e., pruduct of scattering and reactance Hurwitz) polynomials etc. , each having meaningful interpretations in terms of passive system theory. The potential applicability of the techniques presented here in studying robustness of these properties remains to be seen (recent claim of a Kharitonov like result for 1-D modified Hurwitz polynomials [13] can be taken to be a positive step in this direction). There are, of course, other issues of concern even for scattering Hurwitz polynomials which we have not addressed in the current paper. For example, characterization of Kharitonov poly-domains (roughly speaking, Kharitonov domains are domains in the complex plane for which Kharitonov-like results hold) generalizing the results of [33], study of stability domains for multivariable polynomial families described by linear inequalities as in [31] (network ideas once again surface here) and the study of robust stability of polynomial matrices involving more than one variable remain completely open.

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Appendix A:

Completion of the proof of Theorem 2.1:

Our exposition is for Hurwitz polynomials. Exactly similar arguments apply for Schur polynomials by using facts from [9].

Let d be the gcd between $(\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_m e_m)$ and o, assumed nonconstant, thus involving, say, the variable p_k. Then $(\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_m e_m)+o$ would be the product of a scattering Hurwitz polynomial and d. However, since o is reactance Hurwitz [5], so is its factor d, which thus would have a sequentially almost complete set Ω of zeros of k-tuples of order (k-1) on the distinguished boundary $\underline{p}=j\underline{\omega}$ [5]. Let Ω_i be a sequentially almost complete set of (k-1) tuples $(\omega_1,\omega_2,\ldots\omega_k)$ of order k such that the 1-D polynomials obtained by freezing in g_i the variables $\underline{\omega}'=(\omega'_1,\omega'_2,\ldots\omega'_{k-1})$ from Ω is scattering Hurwitz (i.e., simply Hurwitz). Choose then any $\underline{\omega}'_0$ from the nonempty intersection of Ω_i's and Ω such that $d_1=d(j\underline{\omega}'_0,p_k)$ still contains the variable p_k (the feasibility of such a choice can be confirmed in view of discussions in [5]). Substituting $\underline{\omega}'=\underline{\omega}'_0$ in $(\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_m e_m)$ we then have a convex combination of 1-D Hurwitz polynomials with a common paraodd part (thus itself a Hurwitz polynomial, due to the validity of present theorem in 1-D) containing the nonconstant polynomial d_1. But this contradicts with the fact that d and thus d_1 is reactance Hurwitz (i.e., d_1 contains zeros on $j\omega$ axis).

Appendix B.

Proof for strictest sense Hurwitz (Schur) polynomials:

Many of the statements of results in Sections II and III, in fact, remain valid if the term scattering Hurwitz (Schur) is replaced by strictest sense Hurwitz (Schur). We consider the proof for Theorem 2.1(i) as a typical instance. Other results follow from analogous arguments.

If the polynomials g_i are strictest sense Hurwitz then they are also scattering Hurwitz [5]; thus the conclusion of Theorem 2.1 still apply. However, we need to show that $(\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_m e_m)$ is strictest sense Hurwitz in addition to being scattering Hurwitz. For this, as in Appendix A substitute $\underline{p}=j\underline{\omega}_0'$ in each g_i, where $\underline{\omega}_0'$ is now arbitrary. Then the resulting 1-D polynomials are Hurwitz; consequently, due to the validity of Theorem 2.1 in 1-D, the 1-D polynomial obtained from $(\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_m e_m)$ with $\underline{p}=j\underline{\omega}_0'$ must also be so. However, this is impossible if $(\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_m e_m)$ is only scattering Hurwitz but not strictest sense Hurwitz, because such a polynomial must have zeros on the boundary (possibly including points at multiple infinity [4]) of $\mathrm{Re}\,\underline{p}\geq\underline{0}$ [4].

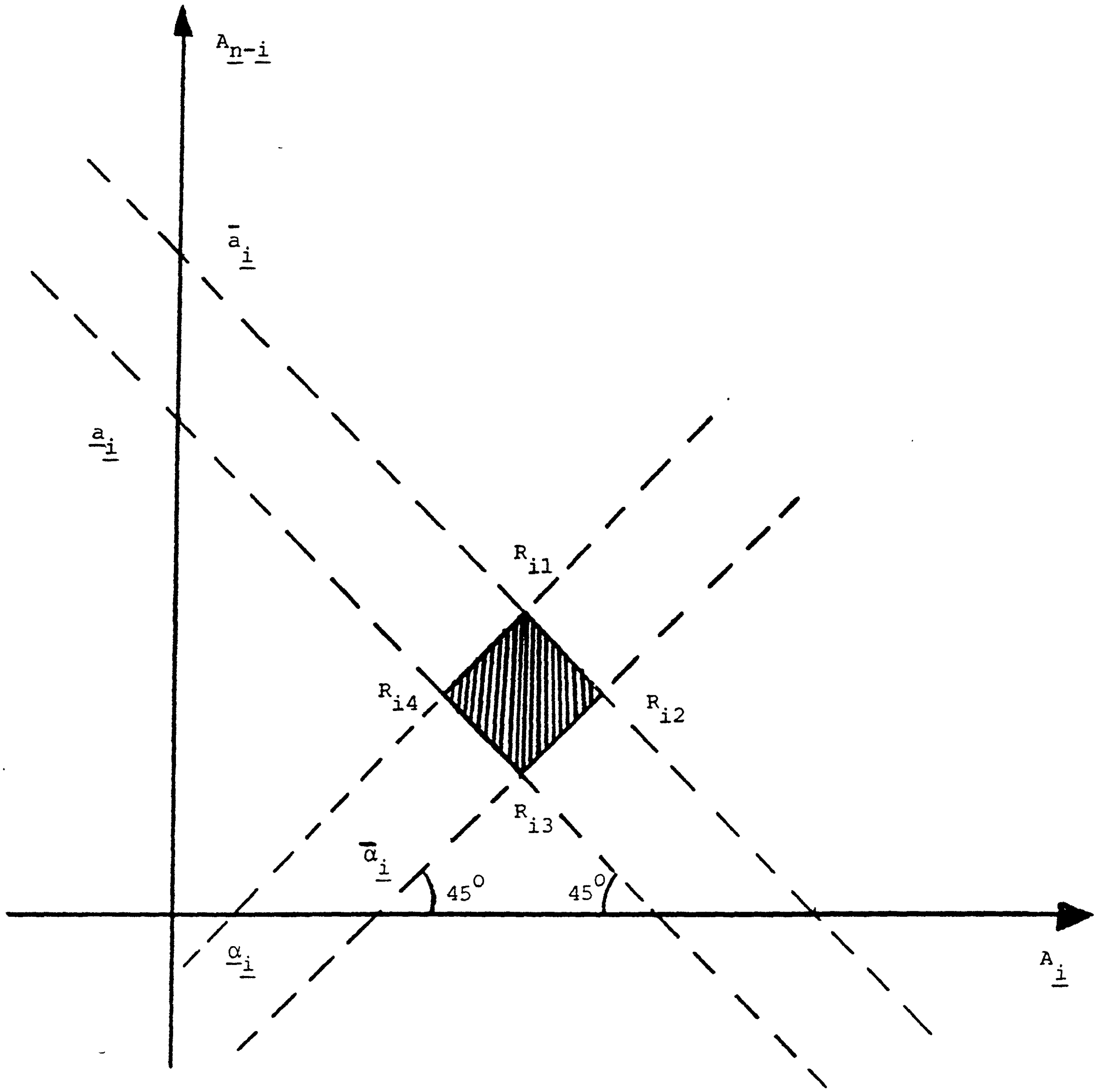

Figure 3.1(a): Proof of Theorem 3.5

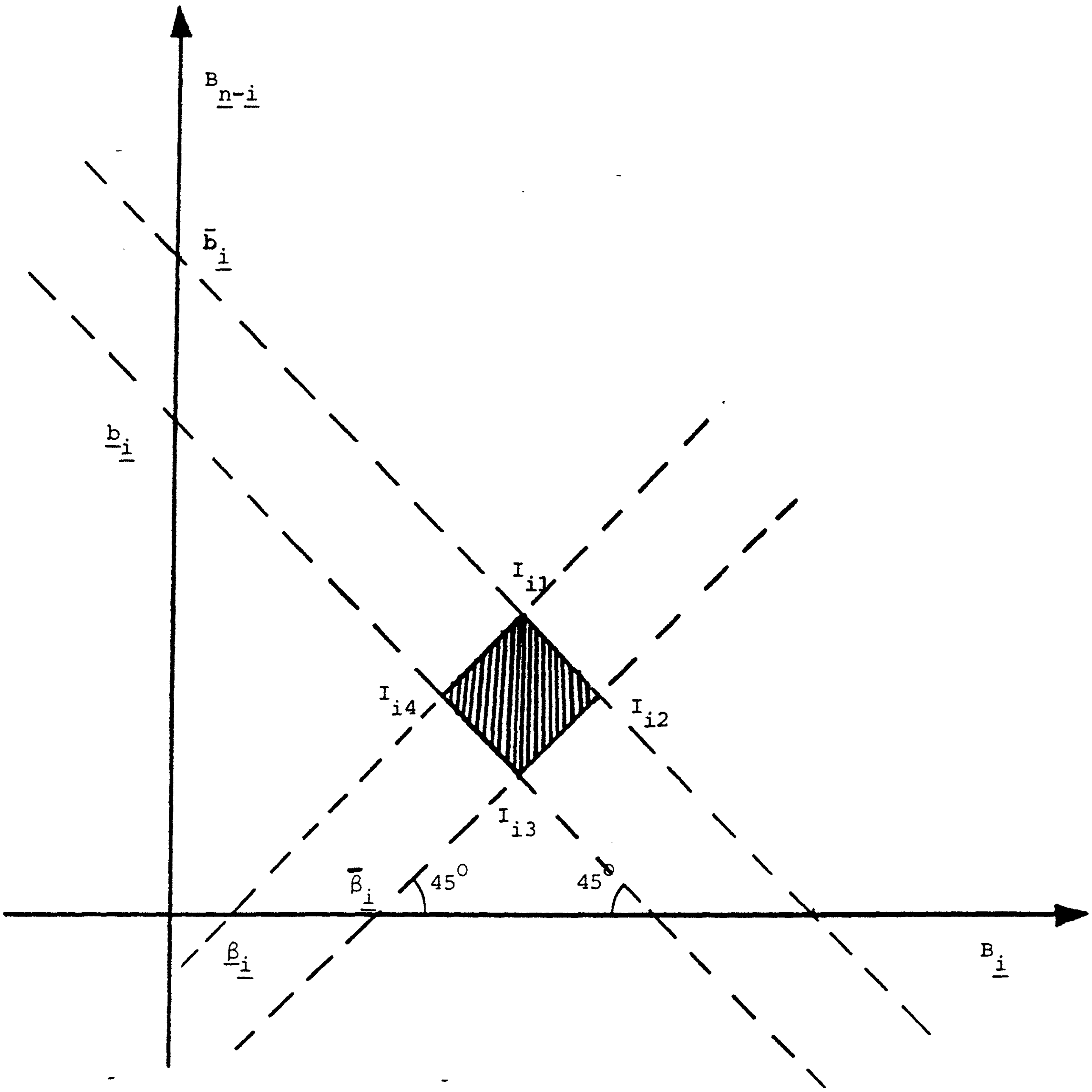

Figure 3.1(b): Proof of Theorem 3.5

Table 3.1

	e_1^+	e_2^+	e_3^+	e_4^+	e_5^+	e_6^+	e_7^+	e_8^+	e_1^-	e_2^-	e_3^-	e_4^-	e_5^-	e_6^-	e_7^-	e_8^-
p_1p_2	$\underline{\beta}$	$\underline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	$\underline{\beta}$	$\underline{\beta}$	$\overline{\beta}$	$\overline{\beta}$	$\underline{\beta}$	$\underline{\beta}$	$\underline{\beta}$	$\underline{\beta}$	$\overline{\beta}$	$\overline{\beta}$
p_2p_3	$\underline{\gamma}$	$\overline{\gamma}$	$\overline{\gamma}$	$\underline{\gamma}$	$\underline{\gamma}$	$\overline{\gamma}$	$\overline{\gamma}$	$\underline{\gamma}$	$\overline{\gamma}$	$\underline{\gamma}$	$\underline{\gamma}$	$\overline{\gamma}$	$\overline{\gamma}$	$\underline{\gamma}$	$\underline{\gamma}$	$\overline{\gamma}$
p_3p_1	$\underline{\delta}$	$\overline{\delta}$	$\underline{\delta}$	$\overline{\delta}$	$\overline{\delta}$	$\underline{\delta}$	$\overline{\delta}$	$\underline{\delta}$	$\overline{\delta}$	$\underline{\delta}$	$\overline{\delta}$	$\underline{\delta}$	$\underline{\delta}$	$\overline{\delta}$	$\underline{\delta}$	$\overline{\delta}$
	o_1^+	o_2^+	o_3^+	o_4^+	o_5^+	o_6^+	o_7^+	o_8^+	o_1^-	o_2^-	o_3^-	o_4^-	o_5^-	o_6^-	o_7^-	o_8^-
$p_1p_2p_3$	$\underline{\alpha}$	$\overline{\alpha}$	$\overline{\alpha}$	$\underline{\alpha}$	$\overline{\alpha}$	$\underline{\alpha}$	$\underline{\alpha}$	$\overline{\alpha}$	$\overline{\alpha}$	$\underline{\alpha}$	$\underline{\alpha}$	$\overline{\alpha}$	$\underline{\alpha}$	$\overline{\alpha}$	$\overline{\alpha}$	$\underline{\alpha}$

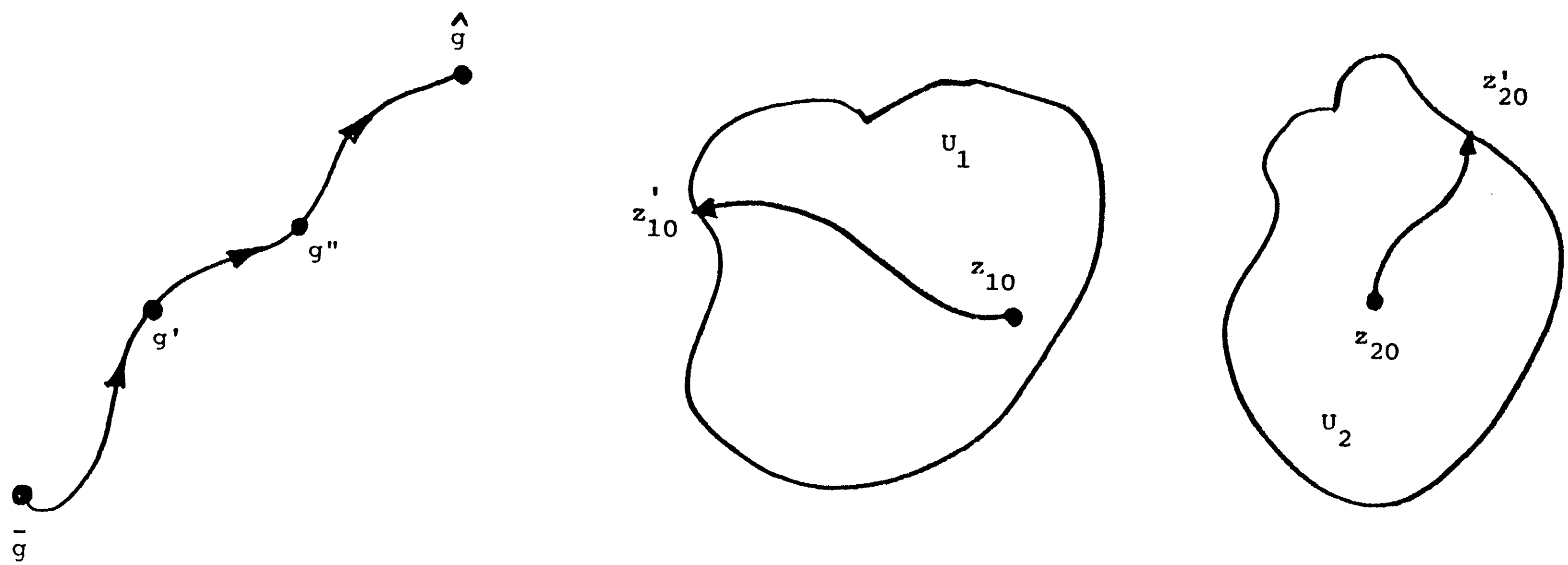

Figure 4.1: Proof of Theorem 4.1.
$\hat{g}(z_1,z_2) \neq 0$ in $\bar{U}_1 \times \bar{U}_2$, but $\bar{g}(z_{10},z_{20}) = 0$
$g'(z_{10},z'_{20}) = 0$ and $g''(z'_{10},z'_{20}) = 0$.

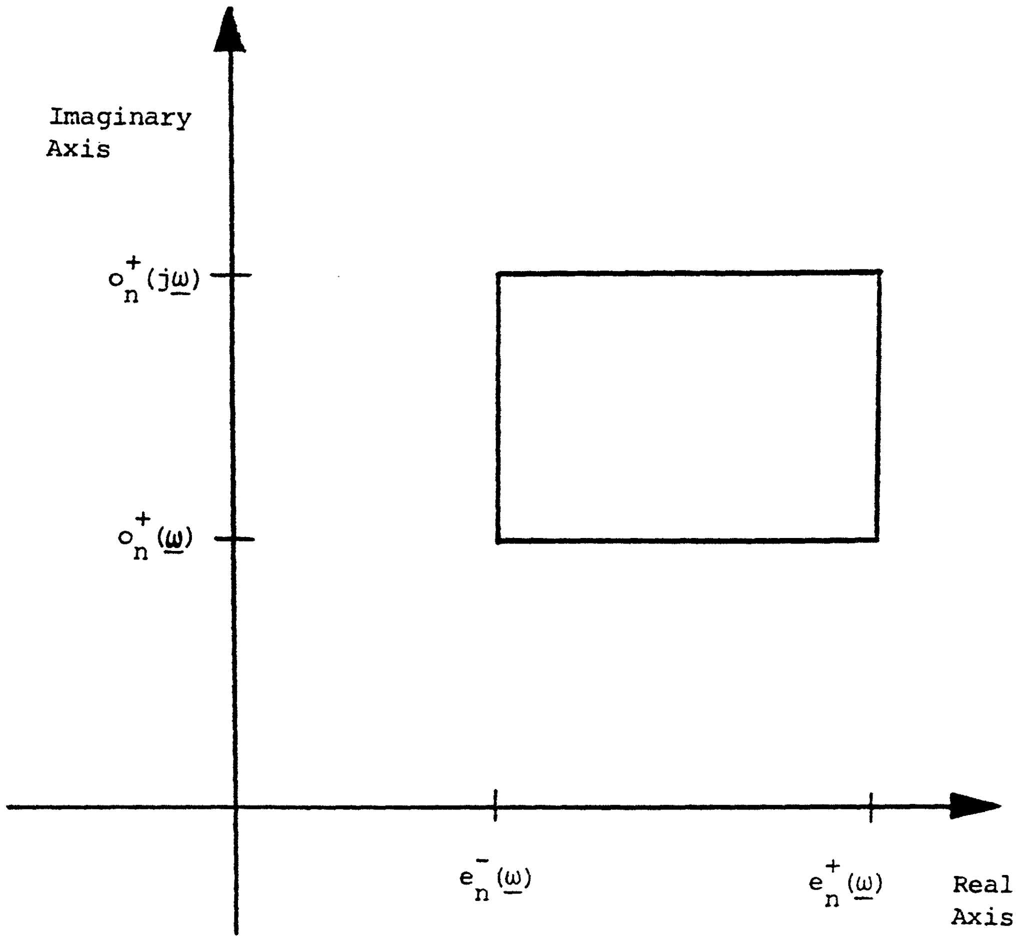

Figure 4.2: The rectangle $H_{\underline{p}}$ under the evaluatiom map

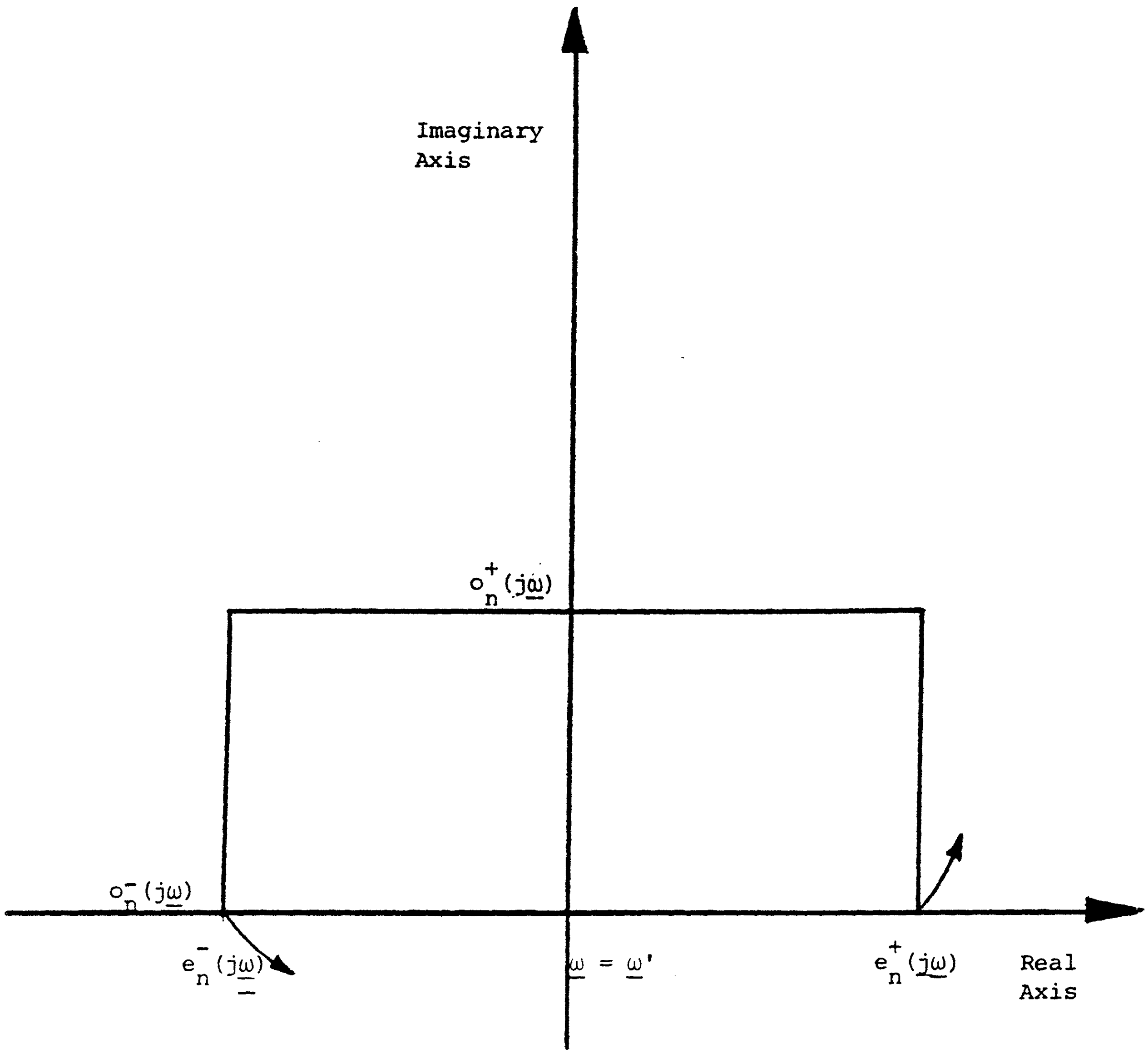

Figure 4.3: Proof of generalized Kharitonov's theorem

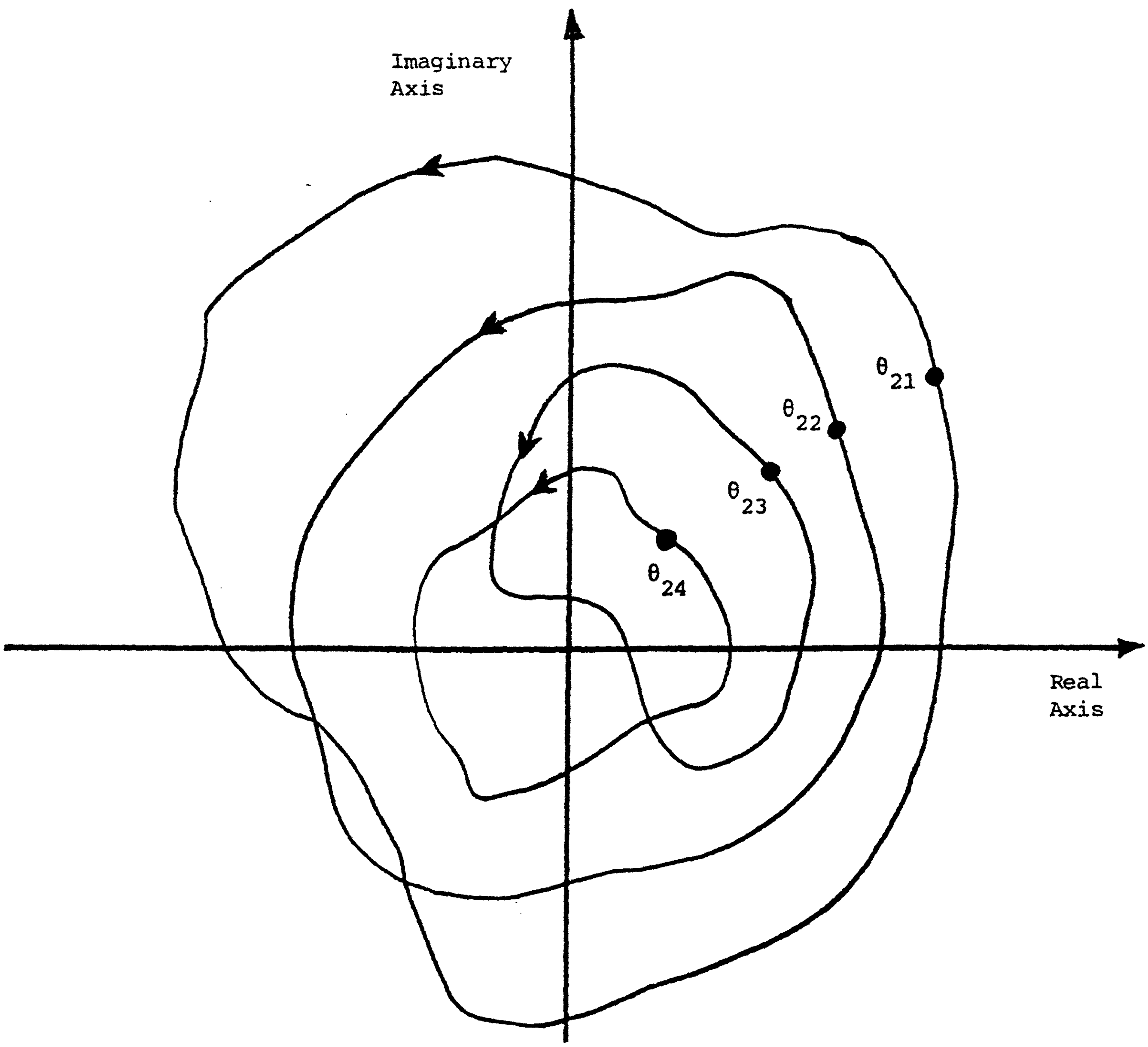

Figure 4.4: Plots of $Nr(\theta_1, \theta_2)$ for $0 \leq \theta_1 \leq 2\pi$ and for different (fixed) $\theta_2 = \theta_{21}, \theta_{22}, \ldots$etc. Dots correspond to points with $\theta_1 = 0$ or 2π