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Optimal Rejection of Bounded Persistent Disturbances in Periodic Systems*

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Abstract

In this paper the problem of optimal rejection of bounded persistent disturbances is solved in the case of linear discrete-time periodic systems. The solution consists of solving an equivalent time invariant standard ℓ^1 optimization problem subject to an additional constraint. This constraint assures the causality of the resulting periodic controller. By the duality theory, the problem is shown to be equivalent to a linear programming problem, which is no harder than the standard ℓ^1 problem. The solution to this problem has an immediate application to multi-rate sampled systems.

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1 Introduction

The study of periodically time varying systems is a topic of growing research. In [8] an equivalence between m-input, p-output, linear, N-periodic, discrete systems and a class of discrete linear time invariant systems was established. Namely, this class consists of mN-input, pN-output, linear time invariant (LTI) systems with z transforms $\hat{C}(z)$ such that $\hat{C}(0)$ is a lower triangular matrix. This equivalence is strong in the sense that it preserves the algebraic structure (isomorphism) and the norm (isometry). Hence, we can effectively use the theory of LTI systems to study periodic ones. In fact, the authors in [8] use this equivalence to prove that although the performance is not improved , periodic compensators for LTI plants offer significant advantages in terms of robustness. Moreover, they argue that the optimal (in ℓ^2 to ℓ^2 sense) compensator for a N-periodic system is N-periodic. Indeed, as it is proved in [1] the above argument is true also in the worst case ℓ^∞ to ℓ^∞ sense. Hence, it can be easily inferred that the optimal controller for the N-periodic system can be obtained by solving the equivalent LTI problem. This problem however, includes a constraint on the optimal LTI compensator $\hat{C}(z)$, namely $\hat{C}(0)$ should be lower triangular so that C corresponds to a causal N-periodic controller. It is exactly this problem we solve in this paper in an optimal ℓ^∞ to ℓ^∞ sense. The unconstrained problem is solved in [2]. In [2] the problem is transformed to a tractable linear programming problem, via duality theory. In this paper, we show that the same approach can be extended. In section 2 we present some mathematical preliminaries together with some background on periodic systems. In section 3 the problem is defined and in section 4 we present the solution following two approaches. Moreover, in section 4 we comment on the relation between this problem and the problem of optimal disturbance rejection for multi-rate sampled systems. Finally, in section 5 we summarize and draw conclusions.

2 Notation and Preliminaries

In this paper the following notation is used:

$\ell^1_{m\times n}$: The normed linear space of all $m \times n$ matrices H each of whose entries is a right sided, absolutely summable real sequence $H_{ij} = (H_{ij}(k))_{k=0}^{\infty}$. The norm is defined as:

$$\|H\|_{\ell^1_{m\times n}} = \max_i \sum_{j=1}^{n} \sum_{k=0}^{\infty} |H_{ij}(k)|$$

$\ell^{\infty}_{m\times n}$: The normed linear space of all $m \times n$ matrices H each of whose entries is a right sided, magnitude bounded real sequence $H_{ij} = (H_{ij}(k))_{k=0}^{\infty}$. The norm is defined as:

$$\|H\|_{\ell^{\infty}_{m\times n}} = \sum_{i=1}^{m} \max_{j}(\sup_{k} |H_{ij}|)$$

$c^{0}_{m\times n}$: The subspace of $\ell^{\infty}_{m\times n}$ consisting of all elements each of whose converges to zero.
$\hat{H}(z)$: The z transform of a right sided $m \times n$ real sequence $H = (H(k))_{k=0}^{\infty}$ defined as:

$$\hat{H}(z) = \sum_{k=0}^{\infty} H(k) z^{k}$$

$A_{m\times n}$: The real normed linear space of all $m \times n$ matrices $\hat{H}(z)$ such that $\hat{H}(z)$ is the z transform of an ℓ^{1} sequence H.
$\mathcal{L}_{TV}^{m\times n}$: The space of all linear bounded and causal maps from $\ell^{\infty}_{n\times 1}$ to $\ell^{\infty}_{m\times 1}$. We refer to these operators as stable.
$\mathcal{L}_{TI}^{m\times n}$: The subspace of $\mathcal{L}_{TV}^{m\times n}$ consisting of the maps that commute with the shift operator (i.e. the time ivariant maps). This space is isometrically isomorphic to $A_{m\times n}$

We will often drop the m and n in the above notation when the dimension is not important or clear from the context. Also, subscripts on the norms are dropped when there is no ambiguity.
X^{*} : The dual space of the normed linear space X.
BX : The closed unit ball of X.
${}^{\perp}S$: The left annihilator of $S \subset X$.
$\langle x, x^{*}\rangle$: The value of the bounded linear functional x^{*} at point $x \in X$.
Fact 0[2]
Every linear functional on $\ell^{1}_{m\times n}$ is representable uniquely in the form

$$f(H) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{\infty} Y_{ij}(k) H_{ij}(k)$$

where $Y = (Y_{ij}) \in \ell^{\infty}_{m\times n}$ and $H = (H_{ij}) \in \ell^{1}_{m\times n}$. Hence, $(\ell^{1}_{m\times n})^{*} = \ell^{\infty}_{m\times n}$. It can be also shown that $(c^{0}_{m\times n})^{*} = \ell^{1}_{m\times n}$ where the linear functionals are defined as above.

We now present some background on periodic systems following [8].
Let $\ell^{\infty,e}_{m}$ denote the space of real $m \times 1$ vector valued sequences. Let Λ denote the right shift

operator on $\ell_m^{\infty,e}$ i.e.

$$\Lambda(a) = \{0, a(0), a(1), \ldots\}$$

where $a = \{a(0), a(1), \ldots\} \in \ell_m^{\infty,e}$. Finally, let f represent the input-output map from $\ell_m^{\infty,e}$ to $\ell_p^{\infty,e}$of a linear causal time varying system.

Definition : The map f is N-periodic if and only if it commutes with the Nth power of the right shift i.e.

$$f\Lambda^N = \Lambda^N f$$

Let W represent the isomorphism

$$W : \ell_m^{\infty,e} \to \ell_{mN}^{\infty,e} : a = \{a(0), a(1), \ldots\} \to \begin{pmatrix} a(0) & a(N) & a(2N) & \ldots \\ a(1) & a(N+1) & a(2N+1) & \ldots \\ . & . & . & . \\ a(N-1) & a(2N-1) & a(3N-1) & \ldots \end{pmatrix}$$

Define the map L as

$$L(f) = WfW^{-1}$$

then $L(f)$ represents a system with inputs in $\ell_{mN}^{\infty,e}$ and outputs in $\ell_{pN}^{\infty,e}$. Moreover, as shown in [8] $L(f)$ is LTI and the following hold:

Fact 1

Given a m-input p-output linear causal N-periodic system f, one can associate via the map L a unique causal $(pN \times mN)$ LTI system $L(f)$ with a transfer matrix $\hat{F}(z)$. Conversely, any $(pN \times mN)$ proper transfer matrix $\hat{F}(z)$ with $\hat{F}(0)$ lower triangular can be associated by L^{-1} to a unique m-input, p-output, linear, causal, N-periodic system.

Fact 2

L preserves the algebraic properties and the norm. In particular,

$$\sup_{u \in B\ell_m^\infty} \|fu\|_{\ell_p^\infty} = \sup_{w \in B\ell_{mN}^\infty} \|Fw\|_{\ell_{pN}^\infty}$$

Hence, f is input-output stable if and only if $F = L(f)$ is stable.

Suppose in addition, that f is finite dimensional. In [1] it is shown that we can obtain a doubly coprime factorization (dcf) of f by obtaining a dcf of the lifted system F. The key observation in [1] is that the factors of F obtained using the standard formulas in [6] possess the property of being lower triangular at $z = 0$. Hence, since L is an isomorphism, we can obtain a dcf of f, the factors being the images of the inverse map of the lifted LTI factors of F ; therefore N periodic. In summary we have:

Fact 3

Let $F = N_l D_l^{-1} = D_r^{-1} N_r$ and

$$\begin{pmatrix} X_r & -Y_r \\ -N_r & D_r \end{pmatrix} \begin{pmatrix} D_l & Y_l \\ N_l & X_l \end{pmatrix} = I$$

represent a dcf of F where the factors are given as in [6]. Then, the following represent a dcf of f:

$$f = n_l d_l^{-1} = d_r^{-1} n_r$$

$$\begin{pmatrix} x_r & -y_r \\ -n_r & d_r \end{pmatrix} \begin{pmatrix} d_l & y_l \\ n_l & x_l \end{pmatrix} = I$$

where $n_l = L^{-1}(N_l)$, $d_l = L^{-1}(D_l)$, $x_l = L^{-1}(X_l)$, $y_l = L^{-1}(Y_l)$, $n_r = L^{-1}(N_r)$, $d_r = L^{-1}(D_r)$, $x_r = L^{-1}(X_r)$, $y_r = L^{-1}(Y_r)$ are in $\mathcal{L}_{TV}$ and N-periodic.

Also, along the lines of [12], it is shown in [1] that the optimal performance in periodic plants is achieved with periodic controllers, namely:

Fact 4

Let H_p, U_p, V_p be N -periodic and stable causal linear operators. Then

$$\inf_{Q \in \mathcal{L}_{TV}} \|H_p - U_p Q V_p\| = \inf_{Q_p} \|H_p - U_p Q_p V_p\|$$

where Q_p is N-periodic and in $\mathcal{L}_{TV}$.

3 Problem Definition

The standard block diagram for the disturbance rejection problem is depicted in Figure 1. In this figure , P_p denotes some fixed linear causal N-periodic plant, C_p denotes a time varying compensator (not necessarily periodic), and the signals w, v, y, and u are defined as follows: w, exogenous disturbance; v, signals to be regulated; y, measured plant output; and u, control inputs to the plant. Let T_{vw} represent the resulting map from w to v for a given compensator C_p. Our objective can be now stated as:

Find C_p such that the resulting closed loop system is stable and also the induced norm $\|T_{vw}\|$ over ℓ^∞ is minimized.(OBJ)

In the sequel we show that this problem can be turned to an equivalent LTI problem . For this purpose , consider the same problem as defined above, for the lifted plant $P = L(P_p)$ and let $\tilde{T}_{vw}$ denote the map from $\tilde{w} = Ww$ to $\tilde{v} = Wv$. Then it is well known [14, 5, 13, 4] that all the feasible maps are given as $\tilde{T}_{vw} = H - UQV$ where $H, U, V \in \mathcal{L}_{TI}$ and $Q \in \mathcal{L}_{TV}$.

Moreover, H, U, V are determined by P. Now, the following lemma shows the aforementioned equivalence.

Lemma

The (OBJ) is equivalent to the problem

$$\inf_{Q \in \mathcal{L}_{TI}} \|H - UQV\| \tag{OPT}$$

subject to $Q(0)$ is lower triangular

Proof

From facts 3 and 4 we obtain

$$\inf_{C_p} \|T_{vw}\| = \inf_{Q_p} \|H_p - U_p Q_p V_p\|$$

where Q_p is N-periodic, stable, and $H_p = L^{-1}(H), U_p = L^{-1}(U), V_p = L^{-1}(V)$. Now, by facts 1 and 2 it follows that

$$\inf_{Q_p} \|H_p - U_p Q_p V_p\| = \inf_{Q = L(Q_p)} \|H - UQV\|$$

QED

4 Problem Solution

Clearly, if in (OPT) we remove the constraint on $Q(0)$ then the problem becomes the standard ℓ^1 optimization problem [2]. In [2] the authors solve the problem by solving the dual problem with linear programming methods. We can solve (OPT) by simply extending the method in [2] to account for the constraint on $Q(0)$. To view this let $\{P_n\}_{n=1}^{N_s}$ be as in [2] the basis for the functionals in $\ell^\infty_{m\times n}$ that annihilate the space

$$S_s = \{UQV : Q \in \ell^1_{m\times n}\}$$

These functionals are attributed to the unstable zeros of U and V. Moreover, assume $\{P_n\}_{n=1}^{N_s} \in c^0_{m\times n}$, i.e. U, V have no zeros on the unit circle. Suppose now, that we are able to find functionals $\{X_j\}_{j=1}^{r}$ in $c^0_{m\times n}$ such that

$$\langle UQV, X_j\rangle = 0 \quad \forall j = 1, 2, \ldots, r \quad \text{iff} \quad Q(0) \quad \text{lower triangular} \tag{1}$$

then the annihilator subspace $^{\perp}S$ of

$$S = \{UQV : Q \in \ell^1_{m\times n}, Q(0) \quad \text{lower triangular}\}$$

can be characterized as

$$^{\perp}S = span(\{P_n\}_{n=1}^{N_s} \cup \{X_j\}_{j=1}^{r})$$

Hence, we have a complete characterization of $S^{\perp}$ and we can proceed exactly as in [2] to solve (OPT). So the problem can be stated as follows: Find $\{X_j\}_{j=1}^{r}$ such that (1) holds.

In the sequel we show how to obtain these $\{X_j\}_{j=1}^{r}$. First, define the functionals R_j, $j = 1, 2, \ldots, r$ as follows: Let r be the number of elements of Q with indices (k, l) that do not belong in the lower triangular portion of Q , i.e. $k < l$. For each $j = 1, 2, \ldots, r$ define

$$(R_j(0))_{i_1 j_1} = \begin{cases} 1 & i_1 = k \text{ and } j_1 = l; \\ 0 & \text{elsewhere.} \end{cases}$$

$$R_j(q) = 0 \text{ for } q = 1, 2, \ldots$$

then $R_j \in c_{m \times n}^{0}$ and also

$$\langle Q, R_j \rangle = 0 \quad \forall j = 1, 2, \ldots, r \quad \text{iff} \quad Q(0) \quad \text{lower triangular}$$

Therefore, the problem can be reformulated as : Find $\{X_j\}_{j=1}^{r}$ such that

$$\langle UQV, X_j \rangle = 0 \quad \text{iff} \quad \langle Q, R_j \rangle = 0 \quad \forall j = 1, 2, \ldots, r \tag{2}$$

We approach this problem in two ways:

Approach 1

By performing an inner outer factorization [6] for U, V we obtain

$$U = U_i U_o, \quad V = V_o V_i$$

where the subscript i stands for "inner" and o for "outer". Let U_{or}, V_{ol} denote the right and left inverses of U_o, V_o respectively. Also, let $Z = U_o Q V_o$ then

$$\inf_{Q \in \mathcal{L}_{TI}} \|H - UQV\| = \inf_{Z \in \mathcal{L}_{TI}} \|H - U_i Z V_i\|$$

Hence, if $R \in \{R_j\}_{j=1}^{r}$, the condition $\langle Q, R \rangle = 0$ becomes

$$\langle U_{or} Z V_{ol}, R \rangle = 0$$

with $Q = U_{or} Z V_{ol}$

Let $T_{U_{or}}, T_{V_{ol}}$ be the bounded operators on ℓ^1 defined as

$$(T_{U_{or}} X)(t) = \sum_{\tau=0}^{t} U_{or}(\tau) X(t - \tau)$$

$$(T_{V_{ol}}X)(t) = \sum_{\tau=0}^{t} X(\tau)V_{ol}(t-\tau)$$

where $X \in \ell^1$ Then, as it can be easily checked, the weak* adjoint (bounded) operators $T^*_{U_{or}}, T^*_{V_{ol}}$ on c^0 are given as

$$(T^*_{U_{or}}Y)(t) = \sum_{\tau=0}^{\infty} U^T_{or}(\tau+t)Y(\tau)$$

$$(T^*_{V_{ol}}Y)(t) = \sum_{\tau=0}^{\infty} Y(\tau)V^T_{ol}(\tau+t)$$

where $Y \in c^0$

Now, the interpretation

$$U_{or}ZV_{ol} = T_{U_{or}}(T_{V_{ol}}(Z))$$

allows us to verify that

$$\langle U_{or}ZV_{ol}, R\rangle = \langle Z, T^*_{U_{or}}(T^*_{V_{ol}}(R))\rangle$$

Define as $R_z = T^*_{U_{or}}(T^*_{V_{ol}}(R))$ then

$$\langle Q, R\rangle = 0 \text{ iff } \langle Z, R_z\rangle = 0$$

Moreover, since the only nonzero element of R is $R(0)$ we have that

$$R_z = \{U^T_{or}(0)R(0)V^T_{ol}(0), 0, 0, \ldots\} \in c^0$$

In a similar fashion as before define the bounded operators T_{U_i}, T_{V_i} on ℓ^1 as

$$(T_{U_i}X)(t) = \sum_{\tau=0}^{t} U_i(\tau)X(t-\tau)$$

$$(T_{V_i}X)(t) = \sum_{\tau=0}^{t} X(\tau)V_i(t-\tau)$$

where $X \in \ell^1$ and their weak* adjoints $T^*_{U_i}, T^*_{V_i}$ bounded on c^0 as

$$(T^*_{U_i}Y)(t) = \sum_{\tau=0}^{\infty} U^T_i(\tau)Y(\tau+t)$$

$$(T^*_{V_i}Y)(t) = \sum_{\tau=0}^{\infty} Y(\tau)V^T_i(\tau+t)$$

where $Y \in c^0$

Notice, that since U_i, V_i inner then $\hat{U}_i^T(z^{-1})\hat{U}_i(z) = I$ and $\hat{V}_i(z^{-1})\hat{V}_i^T(z) = I$. Note also that $T_{U_i}^*, T_{U_i}$ represent multiplication from the right whereas $T_{V_i}^*, T_{V_i}$ represent multiplication from the left. Hence, it follows that $T_{U_i}^* T_{U_i} = I$ and $T_{V_i}^* T_{V_i} = I$

Interpreting

$$U_i Z V_i = T_{U_i}(T_{V_i}(Z))$$

and

$$U_i R_z V_i = T_{U_i}(T_{V_i}(R_z))$$

we can verify that

$$\langle U_i Z V_i, U_i R_z V_i \rangle = \langle Z, R_z \rangle$$

Hence, if $X = U_i R_z V_i$ then

$$\langle UQV, X \rangle = 0 \text{ iff } \langle Q, R \rangle = 0, \quad j = 1, 2, \ldots, r$$

It then follows that the functionals we are looking for are

$$X_j = U_i R_{z_j} V_i$$

with

$$R_{z_j} = \{U_{or}^T(0) R_j(0) V_{ol}^T(0), 0, 0, \ldots\} \in c^0$$

Approach 2

For simplicity, we indicate the method in the case where $V = I$. The extention in the case where $V \neq I$ should be immediate to the reader.

Let U be written in the Smith form [7] as:

$$U = S_1 \Sigma S_2$$

with S_1, S_2 square, stable transfer matrices with stable inverses and Σ can be partitioned as

$$\Sigma = \begin{pmatrix} \Sigma_U & 0 \end{pmatrix}$$

with

$$\Sigma_U = diag(G_0, z g_1, \ldots, z^k g_k)$$

where G_0 is a diagonal stable transfer matrix with no zeros at $z = 0$, and $g_1, \ldots, g_k$ are scalar stable transfer functions also with no zeros at $z = 0$.

Since, S_2 has a stable inverse then, if we partition S_2 as $S_2 = \begin{pmatrix} S_{21} \\ S_{22} \end{pmatrix}$ we guarantee that S_{21} has a stable right inverse S_r. Letting, $\overline{Q} = S_{21}Q$ then

$$\inf_{Q \in \ell^1} \|H - UQ\| = \inf_{\overline{Q} \in \ell^1} \left\|H - S_1 \Sigma_U \overline{Q}\right\|$$

with $Q = S_r\overline{Q}$.

Hence, if $R \in \{R_j\}_{j=1}^r$ then

$$\langle Q, R\rangle = 0 \text{ iff } \langle S_r\overline{Q}, R\rangle = 0$$

Using the same reasoning as in the previous approach, it is easy to check that

$$\langle Q, R\rangle = 0 \text{ iff } \langle \overline{Q}, F\rangle = 0$$

where

$$F = \{S_r^T(0)R(0), 0, 0, \ldots\}$$

Now, define $D \in c^0$ as

$$D = diag(G_0^{-T}(0), zg_1^{-1}(0), \ldots, z^k g_k^{-1}(0))$$

then we can verify

$$\langle \Sigma_U \overline{Q}, D\rangle = \langle \overline{Q}, F\rangle$$

As in approach 1, let $T_{S_1^{-1}}$ be the operator on ℓ^1 associated with the stable transfer function S_1^{-1} and let $T^*_{S_1^{-1}}$ be its adjoint on c^0. Then by defining

$$X = T^*_{S_1^{-1}}(D)$$

we can show that

$$\langle \Sigma_U \overline{Q}, D\rangle = \langle S_1 \Sigma_U \overline{Q}, X\rangle$$

Hence

$$\langle S_1 \Sigma_U \overline{Q}, X\rangle = 0 \text{ iff } \langle \overline{Q}, F\rangle = 0$$

It then follows that the functionals we are seeking for, are given as

$$X_j = T^*_{S_1^{-1}}(diag(G_0^{-T}(0), zg_1^{-1}(0), \ldots, z^k g_k^{-1}(0))F_j)$$

where

$$F_j = \{S_r^T(0)R_j(0), 0, 0, \ldots\}$$

Remarks

Sofar in this section, we implicitly assumed that we were dealing with the "good" rank case [3, 9]. Namely we assumed that $\hat{U}(z), \hat{V}(z)$ have full row and collumn rank respectively. However, there is no loss of generality since in the "bad" rank case [3, 9] it is shown that in order to solve the unconstrained problem it is necessary to solve a square (unconstrained) subproblem. In particular, we can partition U, V as

$$U = \begin{pmatrix} \bar{U} \\ U_2 \end{pmatrix}, V = (\bar{V} \quad V_2)$$

where $\bar{U}, \bar{V}$ are square and invertible. Let $K = UQV$ then

$$K = \begin{pmatrix} \bar{K} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

A necessary condition for the existence of solution is that $\bar{K}$ interpolates $\bar{U}, \bar{V}$ which is the aforementioned subproblem.

The functionals $X_j, j = 1 \ldots, r$ obtained by Approach 2 will contain finitely many nonzero components. Namely, $X_j(m) = 0 \; \forall \; m > k$. In Approach 1 however, the obtained functionals will not, in general, have this property. They will decay though, (since they lie in c^0) and the rate of decay will be dictated by the poles of U_i and V_i. Hence, it seems that the linear programming problem in the dual space [2] will be simpler when Approach 2 is used, provided that a simple way to obtain a Smith decomposition of a transfer matrix exists.

The situation becomes very simple when $U(0)$ and $V(0)$ are invertible matrices. To realize that, let $K = UQV$ then by Fact 3 both $U(0), V(0)$ are lower triangular. Hence, since $K(0) = U(0)Q(0)V(0)$, we have that $Q(0)$ is lower triangular if and only if $K(0)$ lower triangular. Therefore, the functionals we are looking for are simply the R_j's as defined in the begining of the section.

Example

Consider an optimization problem in a 2-periodic single-input single-output system with the equivalent LTI problem being as follows:

$$\inf_{Q\in\ell^1} \|H - UQ\|$$

such that $Q(0)$ is lower triangular, where $H = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, U = \begin{pmatrix} z & -z \\ 0 & 1 \end{pmatrix}$.

We first solve the unconstrained problem following [2]:

The basis for the functionals that annihilate $S_s = \{UQ : Q \in \ell^1\}$ consists of the following:

$$F_1 = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0, \ldots\}$$

$$F_2 = \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0, \ldots\}$$

The resulting optimal solution $\Phi_u = H - UQ_u$ is:

$$\Phi_u = \begin{pmatrix} 1 & 0 \\ \phi_{12} & \phi_{22} \end{pmatrix}$$

where ϕ_{12}, ϕ_{22} arbitrary in ℓ^1 such that

$$\|\phi_{12}\|_{\ell^1} + \|\phi_{22}\|_{\ell^1} \leq 1$$

and

$$Q_u = \begin{pmatrix} -\phi_{21} & 2 - \phi_{22} \\ -\phi_{21} & 2 - \phi_{22} \end{pmatrix}$$

Also,

$$\|\Phi_u\|_{\ell^1} = 1$$

.

Now, in the constrained case, we obtain by using Approach 2 the following extra functional:

$$F_3 = \{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0, \ldots\}$$

The resulting optimal solution is:

$$\Phi = \begin{pmatrix} 1 & .5z \\ 0 & 1.5 \end{pmatrix}$$

with

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & .5 \end{pmatrix}$$

Also,

$$\|\Phi\|_{\ell^1} = 1.5$$

First, note that the optimal Q obtained for the constrained case does have the lower triangular structure at $z = 0$. Also, notice that the optimal performance is (as expected) worse

in the constrained case than in the unconstrained one. Moreover, we demonstrate that if we just "project" Q_u so that it corresponds to a causal periodic controller we do not necessarily obtain optimal performance:
Let Q_{up} denote the causal projection of Q_u i.e.

$$Q_{up} = \begin{pmatrix} -\phi_{21} & -\phi_{22} + \phi_{22}(0) \\ -\phi_{21} & 2 - \phi_{22} \end{pmatrix}$$

and let $\Phi_{up} = H - UQ_{up}$ then

$$\Phi_{up} = \begin{pmatrix} 1 & z(2 - \phi_{22}(0)) \\ -\phi_{21} & -\phi_{22} \end{pmatrix}$$

Hence, since $\|\phi_{21}\|_{\ell^1} + \|\phi_{22}\|_{\ell^1} \leq 1$ we have that

$$\|\Phi_{up}\|_{\ell^1} \geq 2$$

thus $\|\Phi_{up}\|_{\ell^1} > \|\Phi\|_{\ell^1}$.

The connection to MRSD systems

An important feature of the solution presented in this paper is its immediate applicability in the case of multi-rate sampled data (MRSD) systems. As it was shown in [10] MRSD systems belong in a more general class of periodic systems the so-called (P_i, M_i) shift varying systems. Furthermore, these systems are equivalent in the strong sense (just as in the case of N-periodic) with a class of LTI systems. The z transform of this class of LTI systems should satisfy an analogous to the N-periodic case constraint. Namely, let $\hat{G}(z)$ represent the equivalent LTI system of an MRSD one. Let $D = \hat{G}(0)$ and let as in [11] $\{P_q\}_{q=1}^{Q}, \{M_r\}_{r=1}^{R}$ be the sets of integers, relative prime, which are associated with this MRSD system. Then the structure of the matrix D is determined by these sets as follows:

$$D = \begin{pmatrix} D_{11} & D_{12} & \dots & D_{1R} \\ . & . & . & . \\ D_{Q1} & D_{Q2} & \dots & D_{QR} \end{pmatrix}$$

where each D_{qr} is a $P_q \times M_r$ matrix with

$$(D_{qr})_{\alpha\beta} = 0 \;\text{ when }\; (\alpha - 1)\frac{N}{P_q} - (\beta - 1)\frac{N}{M_r} < 0;\; 1 \leq \alpha \leq P_q, 1 \leq \beta \leq M_r$$

where N is the least common multiple of $(P_1, P_2, \dots, P_Q, M_1, \dots, M_R)$. Hence, we can proceed exactly as in the N-periodic case by simply modifying the functionals R_j defined in the begining of this section as follows: Let J be the number of elements in D that are (by the above property) necessarilly zero. To each $j \in \{1, 2, \ldots, J\}$ we associate the indices of the element of D that is necessarily zero. Also, for each $j \in \{1, 2, \ldots, J\}$ consider a matrix L_j with the same dimension as D that has all its entries but one equal to zero. The nonzero entry is taken to be one and its indices are taken to be the ones that correspond to j in the D matrix. Define the functionals $\{R_j\}_{j=1}^{J}$ as

$$R_j(0) = L_j; \ R_j(k) = 0, \quad k = 1, 2, \ldots$$

It is now clear that the solution of the N-periodic case applies immediately. Thus, if we constrain ourselves to (P_i, M_i) compensators to solve the optimal disturbance rejection problem, then exactly the same methods of this section apply. Moreover, the results in [1] can be extended to show that optimal performance in a (P_i, M_i) system can be achieved by a (P_i, M_i) compensator. Therefore, we can obtain a complete solution to the optimal disturbance rejection problem for MRSD systems.

5 Conclusions

In this paper we presented the solution to the problem of optimal ℓ^∞ to ℓ^∞ disturbance rejection in the case of periodic systems. We showed how we can simply extend the method in [2] to obtain the solution. The key observation was that we can obtain a simple set of functionals to account for the constraint on $Q(0)$ when considering the equivalent LTI problem. The case is even easier when there are no zeros at $z = 0$. We have shown that the complete solution involves a tractable linear programming problem. Finally, we indicated how we can obtain optimal compensators for MRSD systems using an identical method of solution to the one presented .

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