Geometry of Ricci-flat Kähler Manifolds and Some Counterexamples

by

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Abstract

In this work, we study geometry of Ricci-flat Kähler manifolds, and also provide some counterexample constructions. We study asymptotic behavior of complete Ricci-flat metrics at infinity and consider a construction of approximate Ricci-flat metrics on quasiprojective manifolds with a divisor with normal crossings removed, by means of reducing torsion of a non-Kähler metric with the right volume form. Next, we study special Lagrangian fibrations using methods of geometric function theory. In particular, we generalize the method of extremal length and prove a generaliziation of the Teichmüller theorem. We relate extremal problems to the existence of special Lagrangian fibrations in the large complex structure limit of Calabi-Yau manifolds. We proceed to some problems in the theory of minimal surfaces, disproving the Schoen-Yau conjecture and providing a first example of a proper harmonic map from the unit disk to a complex plane. In the end, we prove that the union closed set conjecture is equivalent to a strengthened version, giving a construction which might lead to a counterexample.

Thesis Supervisor: Gang Tian
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Chapter 1

Introduction

The geometry of Ricci-flat Kähler manifolds has been of considerable recent interest. One of the reasons for this is the existence of mirror symmetry between Calabi-Yau manifolds and the related Strominger-Yau-Zaslow conjecture which aims to explain it in a geometric way. But Ricci-flat metrics on Kähler manifolds have also been studied on their own, and the existence of complete metrics in the non-compact case is still not completely understood. This is our main topic, but we also present some counterexample constructions in the last section of this work.

The organization of the text is as follows. In the first chapter we are going to review some background results, trying to give a concise and intuitive exposition and set up a relevant context.

Second chapter deals with Ricci-flat metrics on noncompact Kähler manifolds. We consider a construction of approximate Ricci-flat metrics on quasiprojective manifolds with a divisor with normal crossings removed, in the case when the manifold is a pencil, from a non-Kähler metric with the right volume form. For this purpose, we study torsion reducing flows. Next, we study asymptotic behavior of complete Ricci-flat metrics at infinity.

In the third chapter we study special Lagrangian fibrations on compact Calabi-Yau manifolds, using methods of geometric function theory. We derive generalized Reich Strebel Inequalities and estimate quasiconformality constant of extremal maps. In particular, we prove a generalization of the Teichmüller uniqueness theorem. We argue that in the large complex structure limit extremal maps are related to the existence of special Lagrangian fibrations.

The fourth chapter contains a counterexample to the Schoen-Yau conjecture, and describes a proper harmonic map from the unit disk to a complex plane. The last chapter is concerned with the Frankl conjecture about finite lattices, proving a strengthened version and giving a construction which might lead to a counterexample.

1.1 Kähler Manifolds and Lagrangian Fibrations

Symplectic manifolds - $2n$ dimensional manifolds M with a skew symmetric non-degenerate closed two form ω, which have additional compatible complex structure J such that at every point $\omega(v, Jv) > 0$ for all nonzero vectors v, are called Kähler manifolds. They are Riemannian with the metric $g(u, v)$ - we will also use the notation $< u, v >$ - defined as $g(u, v) = \omega(u, Jv)$. Sometimes, the complex structure is not required to be integrable - then we speak of almost Kähler manifolds. The almost complex structure $J : TM \mapsto TM$ is required to satisfy $J^2 = -I$ at every point. Almost complex structure is integrable - i.e. locally looks like the complex structure of $\mathbb{C}^n$ - if and only if the Nijenhuis tensor $N(J) : TM \times TM \mapsto TM$ is zero, where

$$N(J)(u, v) = [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv]$$

On the other hand, the symplectic structure ω always locally looks like the form $\sum_{i=1}^n dx_i \wedge dy_i$ of $\mathbb{R}^{2n}$ - this follows from the condition $d\omega = 0$ by the Darboux' theorem.

The symplectic manifolds originate from the classical mechanics - the symplectic structure defines the classical Poisson bracket operation, which gives the space of smooth functions on M a Lie algebra structure. To each function H on M, a Hamiltonian vector field - symplectic gradient of H, $X = \tilde{\nabla} H$ - is associated uniquely by the formula $dH = \iota_X(\omega)$. Flow under Hamiltonian fields preserves the symplectic structure as well as the corresponding Hamiltonian function H. The point of having a symplectic structure is that the Lie bracket operation on Hamiltonian fields can be computed using the symplectic form ω:

$$[X, Y] = \tilde{\nabla}\omega(X, Y)$$

whenever X and Y preserve ω under local flow. The Poisson bracket $\{H, F\}$ is just $\omega(\tilde{\nabla} H, \tilde{\nabla} F)$. The symplectic structure determines what fields are locally Hamiltonian - the one that preserve it, and indentifies closed one forms with such fields. It essentially gives a Lie algebra structure to closed 1-forms, inducing it from the Lie algebra of local Hamiltonian vector fields with which it is identified - locally dH corresponds to $\tilde{\nabla} H$, and conversely, X to $\iota_X(\omega)$. The fact that this can be done implies that ω is skew symmetric - because Lie bracket is - and also $d\omega = 0$ - because Lie bracket satisfies the Jacobi identity.

In the almost Kähler case we have metric and the ordinary gradient is related to the symplectic gradient by $\nabla F = J\tilde{\nabla} F$. In this case what matters is a property of gradient fields - the almost Kähler manifolds can be understood as Riemannian manifolds equipped with a compatible almost complex structure, which have the following additional property

$$[J\nabla H, J\nabla F] = J\nabla < \nabla H, J\nabla F >$$

This condition is equivalent to g being J invariant and $d\omega = 0$, where the form ω is defined with $\omega(X, Y) =< X, JY >$. If g is J invariant and J is parallel, i.e.

$\nabla J = 0$, this will hold, but these last two conditions are stronger and they imply that J is integrable - they correspond to the case of a Kähler manifold.

A Lagrangian fibration $f : M \mapsto B$ is a smooth map such that $\omega|_{f^{-1}(x)} = 0$ for any fibre $x \in B$. Smooth functions on B give rise to smooth functions on M, which define corresponding Hamiltonian fields, preserving the fibration. Because ω is zero when restricted to each fibre, all these fields commute, and so all the fibers are stratified with invariant subsets of the form $\mathbf{T}^k \times \mathbb{R}^l$.

Conversely, any set of commuting Hamiltonian fields defines a Lagrangian fibration. The question of integrability of H is the question if H can be consistent with some Lagrangian fibration of maximal dimension n - i.e. if the corresponding Hamiltonian field can preserve an n-dimensional Lagrangian fibration, or equivalently if there are n functionally independent integrals for H, i.e. independent functions commuting with H and with each other under the Poisson bracket operation. These integrals are just functions depending on a fibre - they are induced from functions on the base of Lagrangian fibration.

A typical Hamiltonian field on a compact manifold, which preserves some n dimensional Lagrangian fibration, determines this fibration uniquely, because trajectories are dense on the fibres. However, it is not possible to see locally if two points belong to the same fibre directly from the Hamiltonian field, although of course there are other Hamiltonian fields, commuting with the given one, and which together give the distribution corresponding to the fibration. There is a bit more compact local representation for Lagrangian fibrations, depending on a generic locally Hamiltonian field belonging to the fibration, almost complex structure on the manifold and metric on the base. For a Lagrangian fibration on a symplectic manifold, any closed 1-form on a base gives rise to a locally Hamiltonian field on M, which lies in the fibration. But if we have specified an almost complex structure J on M and a metric on the base, then we can lift the corresponding vector field from the base too. In fact, any gradient field on the base gives rise to two vector fields on M - one, which is locally Hamiltonian and belongs to the fibration, and an other which preserves the fibration, acting on fibres as the original field acts on the base.

1.2 Curvature

The Riemannian curvature tensor gives information about the curvature of two dimensional sections. For any two dimensional tangent plane, curvature of the image of that plane under the exponential map is $\mathbf{R}(u \wedge v, u \wedge v)/\|u \wedge v\|^2$, where u and v are vectors spanning the plane. Specifying all such sectional curvatures is equivalent to specifying the Riemannian curvature tensor.

Computing trace of the full Riemannian curvature tensor gives us the Ricci curvature $\mathbf{Rc}(u, v)$, which can be thought of as a Laplacian of the metric tensor, and scalar curvature s. The Ricci curvature can be interpreted in the following way: $\mathbf{Rc}(u, u)/\|u\|^2$ is sum of all the sectional curvatures of all the 2-dimensional sections which are normal to u, spanned by vectors in some orthogonal frame which contains u. Thus, for dimension 3 Ricci curvature gives the same information as the full curvature tensor, and for dimension 2 scalar curvature, which is trace of Ricci curvature, is enough.

For dimensions greater than 3, the relation between the full Riemannian curvature and Ricci and scalar curvatures is given by the Weyl tensor W, so that

$$\mathbf{R} = \frac{s}{n(n-1)} g \circ g + \frac{2}{n-2}(\mathbf{Rc} - \frac{s}{n} g) \circ g + W$$

Here $\circ$ denotes the Kulkarni-Nomizu product, which can be thought of as giving a scalar product for 2-forms starting from two scalar products g and h in a bilinear way, so that $g \circ h(u \wedge v, u \wedge v) = \frac{1}{2}(K + 1/K)\|u \wedge v\|_g \|u \wedge v\|_h$, where K is the quasiconformal distortion under the change of scalar product from g to h, corresponding to the plane spanned by u and v. The Weyl tensor has to do with conformal structure: if we change g conformally to fg, the Weyl tensor changes to fW.

If W is zero, then g is conformal to the Euclidian metric - in particular, there is only one conformal structure in dimensions 2 and 3. If the second term also vanishes, i.e. if $\mathbf{Rc} = \frac{s}{n} g \circ g$, then the curvature has to be constant. Constant curvature spaces are uniformized - any complete metric of constant curvature has universal cover $\mathbb{R}^n$, hyperbolic space $\mathbb{H}^n$ or sphere $\mathbb{S}^n$. The case when $\mathbf{R} = \lambda g \circ g + W$ is that of Einstein manifolds, when the Ricci tensor is proportional to the metric tensor.

Another aspect of curvature is its relation to the parallel transport. The 2-forms can be understood as generators of the group of linear transformations of the tangent space, and the curvature operator $\mathbf{R}$ can be thought of as specifying an infinitesimal transformation of the tangent space corresponding to the parallel transport around a small loop, spanned by vectors u and v - this generator only depends on $u \wedge v$. This point of view is much more general, as it is not restricted to the tangent space and allows one to define curvature for any vector bundle with a specified connection.

In the Cartan formalism, the Riemannian connection is represented as a skew symmetric matrix γ of 1-forms, which acts on columns $\theta = (\theta_1, \ldots, \theta_n)^t$ of 1-forms specifying an orthonormal coframing - up to a gauge, specifying an orthonormal framing gives the same information as specifying a Riemannian metric. The connection determines the first derivatives of the coframing, $d\theta = -\gamma \wedge \theta$ - this is called the first structural equation. The curvature is represented as a skew symmetric matrix Ω of 2-forms, and we have the second structural equation $\Omega = d\gamma + \gamma \wedge \gamma$. The relation to the curvature operator $\mathbf{R}$ is that (i, j)-th entry of Ω is $\mathbf{R}\theta_i \wedge \theta_j$. The formalism is well suited for computations - one can easily check the first and second Bianchi identities, which read $\gamma \wedge \Omega = 0$ and $d\Omega = \Omega \wedge \gamma - \gamma \wedge \Omega$, by differentiating the structural equations.

Holonomy is the full group of transformations of the tangent space which arise from parallel transports around closed loops. For Riemannian manifolds, it is always a subgroup of the special orthogonal group. If it is a subgroup of the unitary group, then we can get a parallel almost complex structure and the manifold is thus Kähler. If it is a subgroup of the special unitary group, we can get a holomorphic $(n, 0)$-form by parallel transport - this is the case of Ricci-flat Kähler manifolds.

Curvature depends on the second derivatives of the metric. At any point, we can get local coordinates - called normal coordinates - in which the first derivatives of the metric tensor vanish. In the Kähler case we can also get all the higher derivatives with respect to z_i coordinates to be zero in the normal coordinates.

When we have an almost complex structure J, the complexified tangent bundle has a natural decomposition into i and $-i$ eigenspaces of J. The vector fields $\partial/\partial z_i$ and $\partial/\partial \bar{z}_i$ and forms dz_i and $d\bar{z}_i$ are natural in any coordinate system for the complex case. The curvature tensor is also complexified, and there are some special symmetries which hold for Kähler manifolds, reflecting the fact that $\nabla J = 0$. In this case, the parallel transport preserves the complex structure, and so the possible transformations are invariant under the conjugation with J - indeed, the holonomy is a subgroup of the unitary group. On generators, this means that $\mathbf{R}(t \wedge u, v \wedge w) = \mathbf{R}(t \wedge u, Jv \wedge Jw)$. Since on forms $dz_i \wedge dz_j$ and $d\bar{z}_i \wedge d\bar{z}_j$ the sides of this equality are opposite, the only nonzero terms for the curvature tensor are on forms $dz_i \wedge d\bar{z}_j$ in the Kähler case. A consequence is that the Ricci tensor is Hermitian, just like the metric tensor, and we can associate a $(1, 1)$-form ρ with it. From the second Bianchi identity it follows that it is a closed form. The matrix representation Ω for curvature has also entries of type $(1, 1)$.

By the $\partial\bar{\partial}$-Lemma, any closed form of type $(1, 1)$ can locally be represented as $\partial\bar{\partial} f$ for some function f - if the form is cohomologous to zero, there is a global representation. Such a function corresponding to the symplectic form ω is called a Kähler potential. There is an expression for the Ricci form ρ in terms of volume,

$-\partial\bar{\partial}$ log det v, where $dV = v dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$, but this representation depends on the choice of local coordinates. However, since allowed changes of coordinates are holomorphic, all choices give the same Ricci form.

The first Chern class is represented by the Ricci curvature ρ. A canonical representative for the first Chern class as a Ricci curvature would correspond to the Einstein metric, if it exists, and in particular Ricci-flat metrics represent the zero first Chern class.

When the Ricci curvature is zero only the Weyl curvature remains, and in the Kähler case the local holonomy will be contained in $SU(n)$. By transporting parallelly the form $dz_1 \wedge \ldots \wedge dz_n$ we can get a closed holomorphic $(n, 0)$-form. Assuming that the global holonomy is also in $SU(n)$, i.e. that the canonical line bundle is trivial, the Ricci flat Kähler case will be distinguished by the existence of a nonvanishing holomorphic $(n, 0)$-form φ, which can be normalized to the volume, so that $\varphi \wedge \bar{\varphi} = dV$.

The necessary condition for the existence of a Ricci-flat metric on a Kähler manifold is that the first Chern class is zero. When the manifold is compact, only the constant functions are holomorphic, and the volume form is fixed up to a constant factor. By the Yau's celebrated proof of the Calabi conjecture, in every Kähler class then there is a Ricci-flat representative. Any representative of a Kähler class of ω has the form $\omega + \partial\bar{\partial}f$, and the corresponding Monge-Ampere equation $(\omega + \partial\bar{\partial}f)^n = e^{-g}\omega^n$ can be solved for any smooth real valued function g, and since then the Ricci curvature form will change by $\partial\bar{\partial}g$, any representative of the first Chern class can be achieved by this method.

However, in the noncompact case there are some differences. Noncompact manifolds generally allow nonconstant holomorphic functions, and so the form φ has to be given first, i.e. we need to have a manifold with a specified canonical line bundle. If $M = \bar{M} \setminus D$, where $\bar{M}$ is a compact complex manifold, and D is a divisor of $\bar{M}$, we can consider a meromorphic $(n, 0)$-form φ which is singular on D, or more generally a line bundle locally represented with such a form. Then we can ask for a complete Ricci-flat metric corresponding to this form, holomorphic on M. The problem is to find a complete metric with the symplectic form ω, such that $\varphi \wedge \bar{\varphi}$ is well approximated by ω^n, before we can proceed as in the compact case. The case when D has normal crossings is still open in general. We have considered the case when M is a pencil of curves, which is an instance where D has normal crossings.

The Strominger-Yau-Zaslow conjecture requires that Calabi-Yau manifolds have special Lagrangian fibrations. These are Lagrangian fibrations for which $\mathrm{Im}\varphi$ vanishes on fibres. The analogous question can be asked for noncompact Ricci-flat Kähler manifolds too. The known examples where special Lagrangian fibrations exist include $K3$-surfaces and some noncompact manifolds. In the case of $K3$-surfaces, which are hyper-Kähler manifolds, i.e. have a quanternionic structure compatible with the metric, special Lagrangian fibrations are obtained as Lefschetz fibrations with respect to an other complex structure, which has special Lagrangian submanifolds as complex submanifolds.

A special Lagrangian submanifold is necessarily volume minimizing in its homology class. This is a consequence of the fact that $\mathrm{Re}\varphi$ provides a calibration: for any choice of an n-plane V in the tangent space, $\mathrm{Re}\varphi|_V \leq volV$. The equality holds for tangent spaces of special Lagrangian submanifolds, and therefore they are minimal submanifolds. The local theory of deformations of SL submanifolds is well developed. If a deformation field X preserves the special Lagrangian condition, then necessarily JX is dual to a harmonic form in the corresponding submanifold, for instance. This condition is also sufficient, as there are no obstructions to deforming a compact SL submanifold in such a direction by the theorem of McLean.

1.3 Analytic Tools

In this section, we recall some theorems which we are going to use in our work, paying attention to the technical details.

First we quote some standard elliptic regularity results, interior Schauder estimates, taken for instance from [25].

Theorem 1.3.1. *Let be a $C^2(\Omega)$ solution of the equation $Lu = f$ in an open set Ω, where f and the coefficients of the elliptic operator L are in $C^{k,\alpha}(\Omega)$. Then $u \in C^{k+2,\alpha}(\Omega)$. If f and the coefficients of L lie in $C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$*

We quote a more precise version of the interior Schauder estimates.

Theorem 1.3.2. *Suppose B_1 and B_2 are balls of radius 1 and 2 in $\mathbb{R}^n$. Let L be a linear elliptic operator of order 2 on functions on B_2 defined by*

$$Lu(x) = a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + b^i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x)$$

Suppose the coefficients a^{ij}, b^i and c lie in $C^{0,\alpha}(B_2)$ and there are constants $\lambda, \Lambda > 0$ such that $|a^{ij}(x)\xi_i\xi_j| \geq \lambda|\xi|^2$ for all $x \in B_2$ and $\xi \in \mathbb{R}^n$, and $\|a^{ij}\|_{C^{0,\alpha}} \leq \Lambda$, $\|b^i\|_{C^{0,\alpha}} \leq \Lambda$, $\|c\|_{C^{0,\alpha}} \leq \Lambda$ on B_2 for all $i, j = 1 \ldots n$. Then there are constants C and D depending only on n, α, λ and Λ, such that whenever $u \in C^2(B_2)$ and $f \in C^{0,\alpha}(B_2)$ with $Lu = f$, we have $u|_{B_1} \in C^{2,\alpha}(B_1)$ and

$$\|u|_{B_1}\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{0,\alpha}} + \|u\|_{C^0})$$

and whenever $u \in C^2(B_2)$ and f is bounded, then $u|_{B_1} \in C^{1,\alpha}(B_1)$ and

$$\|u|_{B_1}\|_{C^{1,\alpha}} \leq D(\|f\|_{C^0} + \|u\|_{C^0})$$

Moreover, if a^{ij}, b^i and c lie in $C^{l,\alpha}(B_2)$and there are constants $\lambda, \Lambda > 0$ such that $|a^{ij}(x)\xi_i\xi_j| \geq \lambda|\xi|^2$ for all $x \in B_2$ and $\xi \in \mathbb{R}^n$, and $\|a^{ij}\|_{C^{l,\alpha}} \leq \Lambda$, $\|b^i\|_{C^{l,\alpha}} \leq \Lambda$,

$\|c\|_{C^{l,\alpha}} \leq \Lambda$ on B_2 for all $i, j = 1 \ldots n$. Then there is a constant C depending only on n, l, α, λ and Λ such that whenever $u \in C^2(B_2)$ and $f \in C^{l,\alpha}(B_2)$ with $Lu = f$, we have $u|_{B_1} \in C^{l+2,\alpha}(B_1)$ and

$$\|u|_{B_1}\|_{C^{l+2,\alpha}} \leq C(\|f\|_{C^{l,\alpha}} + \|u\|_{C^0})$$

A graded Fréchet space is a vector space together with a choice of grading - a sequence of seminorms $\|\cdot\|_n$ of increasing strength, which define the topology. Then a tame linear map $L : F \mapsto G$ of one graded space to another is a map that satisfies a tame estimate $\|Lf\|_n \leq C\|f\|_{n+r}$ for each $n \geq b$ and some r, with constant C possibly depending on n. Then a space is said to be tame if it is a tame direct summand (i.e. there are tame extension and projection maps) of a space $\Sigma(B)$ of exponentially decreasing sequences in some Banach space B. The Nash-Moser inverse function theorem we quote in the version of R.S. Hamilton, [24]:

Theorem 1.3.3. *Let F and G be tame spaces and $P : U \subseteq F \mapsto G$ a smooth tame map. Suppose that the equation for the derivative $DP(f)h = k$ has a unique solution $h = VP(f)k$ for all f in U and all k, and that the family of inverses $VP : U \times G \mapsto F$ is a smooth tame map. Then P is locally invertible and each local inverse P^{-1} is a smooth tame map.*

Here a tame map means that $\|P(g)\|_n \leq C(1 + \|g\|_{n+r})$ for all $f \in U$ and all $n \geq b$, where C may depend on n.

Next, we quote several results about quasiconformal maps, using [31] as a reference.

A map $f : \Omega \mapsto \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$, is said to be of bounded distortion if there is a constant K such that the differential of f is bounded in terms of the Jacobian, $1/KJ(x, f) \leq |Df(x)\zeta|^n \leq KJ(x, f)$ for any unit vector ζ almost everywhere in Ω. Alternatively, the condition is that eccentricity of the ellipsoid images of small balls is bounded

$$\limsup_{r\to 0} \max_{|\zeta|,|\eta|=r} \frac{|f(x+\zeta)-f(x)|}{|f(x+\eta)-f(x)|} = H(x,f) \leq H \ a.e.\ \Omega$$

This is called a linear distortion. Unlike K, the essential supremum of H is not a lower semicontinuous functional of quasiconformal maps - for dimensions $n \geq 3$ there are maps with linear distorton H converging to a map with a higher linear distortion. A map f of bounded distortion is quasiregular if its Jacobian is locally integrable, does not change sign in Ω and $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. The last condition can be replaced with the absolute continuity on the lines condition for f, as we already know that $Df \in L^n_{loc}$ and this in conjunction with the ACL condition is equivalent to $f \in W^{1,n}_{loc}$. By a theorem of Gehring, bounded distortion implies that in fact $f \in W^{1,n+\epsilon}_{loc}$ with some positive ϵ which depends on K and n. If f is a homeomorphism with bounded linear distortion it is quasiregular and is called a quasiconformal map. But quasiregular maps have nice topological properties too - they are always open and discrete, and can be understood as a branched generalization of quasiconformal maps.

We are mostly interested in the compactness properties of quasiconformal maps - any reasonably normalized family of quasiconformal maps forms a normal family. First, we state a theorem describing limits of quasiconformal maps.

Theorem 1.3.4. *Let $f_j : \Omega \mapsto \mathbb{R}^n$ be a sequence of K-quasiconformal maps converging pointwise to $f : \Omega \mapsto \mathbb{R}^n$. Then one of the following occurs:*

- *f is a K-quasiconformal embedding and the convergence is uniform on compact subsets*
- *$f(\Omega)$ consists of two values one of which is attained only once*
- *f is constant*

Next, we state a result about dominated compactness, with a pointwise bound on the distortion.

Theorem 1.3.5. *Let f_j be a sequence of mappings with finite distortion which is bounded in $W^{1,n}(\Omega, \mathbb{R}^n)$ and which satisfies $K(x, f_j) \leq K(x) < \infty$ almost everywhere on Ω. Then it contains a subsequence converging weakly in $W^{1,n}_{loc}$ and locally uniformly on Ω to a mapping $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ of finite distortion, satisfying $K(x, f) \leq K(x)$ almost everywhere on Ω.*

The Montel's theorem for quasiconformal maps gives a sufficient condition for a family of quasiconformal maps to be normal.

Theorem 1.3.6. *Suppose that Ω is a domain in $\mathbb{R}^n$, and that $K \geq 1$ and $\epsilon > 0$ are some fixed constants. If $\mathcal{F}$ is a family of K-quasiconformal maps $f : \Omega \mapsto \mathbb{R}^n \backslash \{a_f, b_f\}$, such that a_f and b_f have spherical distance greater than ϵ, then $\mathcal{F}$ is a normal family.*

The Beltrami equation in the higher dimensional case can be understood as a system

$$D^t f(x) Df(x) = |Df(x)|^{2/n} G(x)$$

where $G(x)$ is a symmetric matrix of determinant one, specifying the change of conformal structure. This system admits nonconstant local solutions for dimensions 4 and up if and only if the Weyl curvature of G vanishes. Matrix G represents a linear transformation of $\mathbb{R}^n$ and can be lifted to the level of exterior algebra, i.e. G induces a linear map $G_\sharp : \Lambda^l \mapsto \Lambda^l$ represented as a matrix of determinant one having entries the $l \times l$ minors of G. For an even dimension $n = 2m$, there are analogies to the complex case. Introducing $\mu = \frac{G_\sharp - I}{G_\sharp + I} : \Lambda^m(\mathbb{R}^n) \mapsto \Lambda^m(\mathbb{R}^m)$ we can write an equation on the level of m forms, analogous to the one in the two dimensional case. Let $d^+ = \frac{1}{2}(d + (-i)^m * d)$, $d^- = \frac{1}{2}(d - (-i)^m * d)$, $\alpha \in C^\infty(\Omega, \Lambda^{m-1}) \cap \mathrm{Ker}(d^-)$ and ω be a pushforward of α under f, then we can write $d^- \omega = \mu d^+ \omega$. The theory in even dimensions has further analogies with the two dimensional theory, but the main difference is that in higher dimensions the Weyl curvature tensor is an obstruction to the existence of local solutions of the Beltrami equation.

Chapter 2

Ricci-flat Metrics on Noncompact Manifolds

In this section we study noncompact Ricci-flat Kähler manifolds. The question of constructing a complete Ricci-flat metric on quasiprojective complex manifold $M \setminus D$, where D is a divisor of a compact algebraic variety D, has been solved in [1] and [2] for a class of smooth divisors D. The case when D has normal crossings is still open. The method used in [1] and [2] was to first construct a complete Kähler metric which is asymptotically flat at infinity, and then apply a variant of the Yau's method for Monge-Ampere equation on non-compact manifolds. Getting an approximate metric for the more general case is the main issue here. In the case when the manifold is a pencil, our approach is to use cylindrical metric on the projective space and a Ricci-flat metric on the fibres. This metric has the right volume form but is not Kähler. The idea is to deform it using a flow which reduces torsion, and we study such flows. Also, we consider behavior of metrics from [1] at infinity, showing that they are in fact very well aproximated by the initial metric that was used in construction.

2.1 Introduction

To apply Yau's method for solving the Calabi conjecture, [3, 4], to the case of non-compact manifolds, a preliminary step of constructing a Kähler metric which has asymptotically the right volume form is needed. Having such an asymptotic metric, the following result of Tian and Yau from [1] can be applied to finish the construction.

Theorem 2.1.1. *Let (M,g) be a complete Kähler manifold of quasi-finite geometry of order* $2+\frac{1}{2}$ *and with* $(K, 2, \beta)$*-polynomial growth. Let* f *be a smooth function such that* $\int_M (e^f - 1)\omega_g^n = 0$ *and for some constant* C

$$\sup_M\{|\nabla_g f|, |\Delta_g f|\} \leq C, \quad |f(x)| \leq \frac{C}{(1+\rho(x))^N}, \quad x \in M$$

where $N \geq 4 + \beta$ *and* $\rho(x)$ *is a distance from some fixed point* x_0 *with respect to* g.

Then the complex Monge-Ampere equation on M

$$(\omega_g + \frac{i}{2\pi}\partial\bar{\partial}\varphi)^n = e^f\omega_g^n, \quad \omega_g + \frac{i}{2\pi}\partial\bar{\partial}\varphi > 0, \quad \varphi \in C^\infty(M, \mathbb{R})$$

has a bounded, smooth solution φ *such that* $\omega_g + \frac{i}{2\pi}\partial\bar{\partial}\varphi$ *defines a complete Kähler metric equivalent to* g. *The supreme norms of* φ *and its derivatives can be bounded by constants depending only on* f, C, N, K, β *and the order of the derivative.*

Here (K, α, β)-polynomial growth means that sectional curvature is bounded by K, volume of a ball centered at x_0 of radius R by CR^α and volume of a ball of radius one centered at x by $C^{-1}(1+\rho(x))^{-\beta}$. Quasi-finite geometry of order $l+\delta$ means that there are $r > 0$ and $r_1 > r_2 > 0$ such that each geodesic ball of radius r admits a holomorphic chart sandwiched between balls of radii r_1 and r_2 centered at the origin in $\mathbb{C}^n$, with a pullback metric bounded in the Hölder space $C^{l,\delta}$. In particular, the injectivity radius is greater than r.

The asymptotic behavior at infinity makes up for the non-compactness of a manifold. The solution φ in the above theorem which decays to zero at infinity sufficiently rapidly is unique, as can be proved in the same way as in the compact case. To see this, one needs to go over the proof in the compact case and observe that the only place where compactness assumption was used was an application of the Stokes theorem to argue that $\int_M d((\varphi_1-\varphi_2)((\bar{\partial}-\partial)(\varphi_1-\varphi_2))\wedge(\omega_1^{n-1}+\omega_1^{n-2}\wedge\omega_2+\ldots+\omega_2^{n-1}))$ is zero, where φ_1 and φ_2 are two solutions, and ω_1 and ω_2 the corresponding Kähler forms. But the conclusion is still valid in the noncompact case under the assumption that φ_1 and φ_2 decay to zero at infinity faster than the size of boundaries of an exhausting sequence of compacts subsets, since the term with which $(\varphi_1-\varphi_2)$ is multiplied is bounded by assumption that ω_1 and ω_2 give equivalent metric to g and that derivatives of φ are bounded.

We are especially interested in the case of a quasiprojective manifold with a divisor with normal crossings removed, which comes from a pencil. Suppose we have a pencil $\{M_\lambda\}_{\lambda\in\mathbb{P}^1}$ with a base locus B, and let $f:\bar{M}\setminus B\mapsto\mathbb{C}^*$ be the associated holomorphic mapping to the Riemann sphere, with fibres $M_\lambda\setminus B=f^{-1}(\lambda)$. Then we consider the case when a supporting manifold of a divisor D is a union of M_0 and M_∞, which intersect along B. Since we are considering the Ricci-flat case, we suppose that we have a meromorphic $(n,0)$ form φ which is singular at D, and for which we want to find a complete metric ω_g such that $\varphi\wedge\bar{\varphi}$ is proportional to the corresponding volume form.

Proposition 2.1.2. *Suppose that φ has a pole of first order at D. Then the form φ induces meromorphic $(n-1,0)$ forms φ_λ on each fibre M_λ, so that $\varphi=\frac{d\lambda}{\lambda}\wedge\varphi_\lambda$. In the case of a pencil of curves, the flat metric of a line bundle corresponding to φ induces a flat metric on the fibres.*

Proof. This is a straightforward application of the second adjunction formula - we can consider $\frac{\varphi}{\lambda-\lambda_0}$ to get the form corresponding to λ_0. In the case of a pencil of curves, a norm of the line bundle is sufficient to determine a conformal factor on the fibres, and hence a metric. □

This suggests a method for constructing an approximate Kähler metric for the desired volume form - combining a metric which is Ricci-flat on the fibres and cylindrical metric on the projective space, we get a metric which has the right volume form, but is not Kähler, and we try to reduce its torsion. In the case of a pencil of curves the flat, and possibly singular, metric on the fibres can be computed from the form φ directly, while in the higher dimensional case we might use the theorems from [1] and [2].

2.2 Torsion Reducing Flows

Suppose that we have a Hermitian form ω on a complex manifold, with an associated Riemannian metric $< u, v >_\omega = \omega(u, Jv)$, but such that $d\omega$ is not zero. We can try to reduce torsion by means of some flow.

A natural thing to do is to consider a gradient flow with respect to the functional $\int_M < d\omega, d\omega >_\omega$. We are going to assume that derivatives of ω are vanishing at infinity sufficiently fast, so that it is possible to work essentially as if M were compact. We will consider a flow unrestricted to $(1,1)$-forms, with an associated Riemannian metric coming from a $(1,1)$ part of ω, which we assume only to be a real 2-form.

We first compute the first variation and the corresponding gradient.

Proposition 2.2.1. *The Euler-Lagrange equations for the torsion reducing flow at* ω*, gradient with respect to the functional* $\int_M < d\omega, d\omega >_\omega$*, can be written as*

$$\omega_t = -d^* d\omega - (n-1) * (Pd^* d\omega \wedge (P\omega)^{n-2})/n! - (I - P)d^* d\omega$$

where P *is a projector onto* $(1,1)$ *forms.*

Proof. We compute the first variation. All the following equalities are modulo terms of second order in $\delta\omega$. First consider the case when $\delta\omega$ is a (1,1) form.

$$< d(\omega + \delta\omega), d(\omega + \delta\omega) >_{\omega+\delta\omega} - < d\omega, d\omega >_\omega =$$

$$2 < d\omega, d\delta\omega >_\omega + < d\omega, d\omega >_{\omega+\delta\omega} - < d\omega, d\omega >_\omega$$

The integration by parts step gives an equivalent term

$$2 < d^* d\omega, \delta\omega >_\omega + < d^* d\omega, \omega >_{\omega+\delta\omega} - < d^* d\omega, \omega >_\omega =$$

$$< d^* d\omega, \delta\omega >_\omega + < d^* d\omega, \omega + \delta\omega >_{\omega+\delta\omega} - < d^* d\omega, \omega >_\omega =$$

$$< d^* d\omega, \delta\omega >_\omega + d^* d\omega \wedge ((P\omega + \delta\omega)^{n-1} - (P\omega)^{n-1})/n! =$$

$$< d^* d\omega, \delta\omega >_\omega + (n-1) d^* d\omega \wedge (P\omega)^{n-2} \wedge \delta\omega/n! =$$

$$< d^* d\omega + (n-1) * (d^* d\omega \wedge (P\omega)^{n-2})/n!, \delta\omega >_\omega$$

When $\delta\omega$ is in the complement space, the variation of metric is zero, and so we have a total variation term

$$< d(\omega + \delta\omega), d(\omega + \delta\omega) >_\omega - < d\omega, d\omega >_\omega = 2 < d\omega, d\delta\omega >$$

Partial integration step then gives the term $< 2d^* d\omega, \delta\omega >$.

The assertion then follows from the fact that P and $I - P$ project onto orthogonal spaces, since metric is J compatible. □

Next, we show that the stationary points of the flow are ω that are torsion free

Proposition 2.2.2. *If ω satisfies*

$$d^* d\omega + (n-1) * (P d^* d\omega \wedge (P\omega)^{n-2})/n! + (I - P) d^* d\omega = 0$$

then $d\omega = 0$

Proof. Note that

$$< *(P d^* d\omega \wedge (P\omega)^{n-2})/n!, P\omega >_\omega = P d^* d\omega \wedge (P\omega)^{n-1}/n! = < P d^* d\omega, \omega >_\omega$$

Applying this formula we get

$$0 = \int_M < P d^* d\omega + (n-1) * (P d^* d\omega \wedge (P\omega)^{n-2})/n!, P\omega >_\omega = \int_M n < P d^* d\omega, \omega >_\omega$$

But we also know that $(I-P)d^*d\omega = 0$ and hence

$$0 = \int_M < d^*d\omega, \omega >_\omega = \int_M < d\omega, d\omega > = 0$$

proving that $d\omega = 0$ □

Note that our flow can be written in the form $\omega_t = -A(\omega)d^*d\omega$, where $A(\omega)$ is some linear operator. We can consider various flows with different factors $A(\omega)$.

We can write $A(\omega)$ in another form, defined for $(1,1)$ forms as

$$A(\omega)\varphi = \frac{\mathrm{Tr}_\omega\varphi}{n} P\omega + \frac{n-1}{n}\varphi$$

and for $(2,0)$ and (0,2) forms $A(\omega)\varphi = 2\varphi$.

Here Tr_ω is a linear operator mapping forms $\frac{i}{2}dz_k \wedge d\bar{z}_k$ to one and all other forms to zero, where $dz_k = dx_k + idy_k$ comes from a diagonal basis of the associated Riemannian metric, in the cotangent space of the corresponding point. We can check that on Hermitian forms φ factor $A(\omega)$ acts in the same way as $\varphi + (n-1) * (\varphi \wedge (P\omega)^{n-2})/n!$

$$(n-1) * ((\frac{i}{2}dz_k \wedge d\bar{z}_k) \wedge (P\omega)^{n-2})/n! = \frac{1}{n}(P\omega - \frac{i}{2}dz_k \wedge d\bar{z}_k)$$

and also

$$(n-1) * ((\frac{i}{2}dz_k \wedge d\bar{z}_l) \wedge (P\omega)^{n-2})/n! = \frac{i}{2n}d\bar{z}_k \wedge dz_l = -\frac{1}{n}\overline{(\frac{i}{2}dz_k \wedge d\bar{z}_l)}$$

The flow $\omega_t = -A(\omega)d^*d\omega$ is not strictly parabolic, but only weakly so. Thus, to prove local existence even for compact manifolds, we need to use a following theorem of R.S. Hamilton, whose proof relies on the Nash-Moser inverse function theorem, from his paper on Ricci-flow, [23]:

Theorem 2.2.3. *Let $\varphi_t = E(\varphi)$ be an evolution equation with integrability condition $L(\varphi)$, such that*

- *$L(\varphi)E(\varphi)$ has degree at most one*

• *all the eigenvalues of the eigenspaces of $\sigma DE(\varphi)(\xi)$ in Ker $\sigma L(\varphi)(\xi)$ have strictly positive real parts*

Then the initial value problem $\varphi = \varphi_0$ at $t = 0$ has a unique smooth solution for a short time $0 \le t \le \varepsilon$ where ε may depend on φ_0.

Here φ is a smooth section of a vector bundle, integrability condition is a first order operator in φ and ψ, from the same vector bundle and with a value $L(\varphi)\psi$ in an other, possibly different, vector bundle.

For $L(\omega)\psi$ we can take $d^*A(\omega)^{-1}\psi$, and also $E(\omega) = -A(\omega)d^*d\omega$. Then $L(\omega)E(\omega) = -d^*d^*d\omega = 0$, and $L(\omega)$ has degree one.

To write an operator $L(\omega)$ more explicitly, note that on $(1,1)$ forms $A(\omega)^{-1} = \frac{n}{n-1}(I - \frac{P\omega}{2n-1}\mathrm{Tr}_\omega)$. Indeed, applying this to $A(\omega)\varphi = \frac{n-1}{n}\varphi + \frac{P\omega}{n}\mathrm{Tr}_\omega\varphi$ and using $\mathrm{Tr}_\omega(P\omega) = n$ we get

$$\frac{n}{n-1}(I - \frac{P\omega}{2n-1}\mathrm{Tr}_\omega)(\frac{n-1}{n}\varphi + \frac{P\omega}{n}\mathrm{Tr}_\omega\varphi) = \varphi +$$
$$+\frac{P\omega}{n-1}\mathrm{Tr}_\omega\varphi - \frac{P\omega}{2n-1}\mathrm{Tr}_\omega\varphi - \frac{P\omega}{2n-1}\frac{n}{n-1}\mathrm{Tr}_\omega\varphi = \varphi$$

Thus, we can consider

$$L(\omega)\psi = \frac{n}{n-1}d^*(P\psi - \frac{P\omega}{2n-1}\mathrm{Tr}_\omega P\psi) + \frac{1}{2}d^*(I-P)\psi$$

We proceed with the computation of principal symbols of E and L. We will use notation $dt_k = dx_k, dt_{k+n} = dy_k$, and write $\varphi = f_{ij}dt_i \wedge dt_j$ with $1 \le i < j \le n$. Also, we will work in the basis at a point in which a metric tensor has diagonal form. The principal symbol for E we can then compute from the principal symbol of d^*d, as $E(\omega) = -A(\omega)d^*d\omega$. We have

$$d^*d\varphi = \sum_{k \neq i,j}(\frac{\partial^2 f_{ij}}{\partial t_i \partial t_k}dt^k \wedge dt^j + \frac{\partial^2 f_{ij}}{\partial t_j \partial t_k}dt^i \wedge dt^k - \frac{\partial^2 f_{ij}}{\partial t_k^2}dt^i \wedge dt^j) + \ldots$$

Here lower order terms depend on the connection, but we are interested in principal symbol, and this computation suffices.

Thus, the principal symbol of $E(\omega)$ reads in this local basis

$$\sigma DE(\omega)(\xi)\sum_{i,j} f_{ij} dt_i \wedge dt_j = \sum_{i,j} A(\omega)(dt_i \wedge dt_j) \sum_{k\neq i,j} (\xi_k^2 f_{ij} - \xi_i\xi_k f_{kj} - \xi_j\xi_k f_{ik})$$

But we may choose this basis further and rescale ξ for convenience so that $\xi_1 = 1$ and $\xi_k = 0$ for $k > 1$. In this basis, we compute, with indices modulo $2n$

$$\sigma DE(\omega)(\xi) dt_1 \wedge dt_j = 0$$

$$\sigma DE(\omega)(\xi)(dt_i \wedge dt_j + dt_{i+n} \wedge dt_{j+n}) = \frac{n-1}{n}(dt_i \wedge dt_j + dt_{i+n} \wedge dt_{j+n}) \quad i, j \neq 1, n+1, \ |i-j| \neq n$$

$$\sigma DE(\omega)(\xi)(dt_i \wedge dt_j - dt_{i+n} \wedge dt_{j+n}) = 2(dt_i \wedge dt_j - dt_{i+n} \wedge dt_{j+n}) \quad i, j \neq 1, n+1, \ |i-j| \neq n$$

$$\sigma DE(\omega)(\xi)(dt_{n+1} \wedge dt_j - \frac{n+1}{3n-1} dt_1 \wedge dt_{j+n}) = \frac{3n-1}{2n}(dt_{n+1} \wedge dt_j - \frac{n+1}{3n-1} dt_1 \wedge dt_{j+n}) \quad j \neq 1$$

$$\sigma DE(\omega)(\xi) dt_i \wedge dt_{i+n} = dt_i \wedge dt_{i+n} + \frac{1}{n} \sum_{1 \leq j \leq n, j \neq i} dt_j \wedge dt_{j+n} \qquad 1 < i \leq n$$

From this we see that E is weakly parabolic. Zero eigenvalues come from the action of d^*d, whose symbol maps $dt_1 \wedge dt_i$ to zero and acts as identity on all other basis forms, with the conventions as above. We ought to show that the principal symbol of L has kernel which intersects transversally the kernel of the principal symbol of E. To compute the principal symbol of E, we start with the principal symbol of d^*. Note that

$$d^*\varphi = \sum_{i,j} (\frac{\partial f_{ij}}{\partial t_j} dt_i - \frac{\partial f_{ij}}{\partial t_i} dt_j) + \dots$$

Here lower order terms do not involve derivatives of φ. Thus, the principal symbol of d^*, with the conventions as before, has kernel spanned by forms $dt_i \wedge dt_j$ with $i, j \neq 1$. The principal symbol of L is a composition of $A(\omega)^{-1}$ with the principal symbol of

d^*, and hence kernel consists of forms spanned by $A(\omega)dt_i \wedge dt_j$ with $i, j \neq 1$. But this intersects transversally the kernel of $\sigma DE(\omega)$ - if $\sigma DE(\omega)A(\omega)\sum_{i,j\neq 1} f_{ij}dt_i \wedge dt_j = 0$ then we see that $f_{ij} = 0$ for $|i-j| \neq n$, and for the terms $dt_i \wedge dt_{i+n}$ the kernel of $\sigma DE(\omega)$ is spanned by $dt_1 \wedge dt_{n+1}$, but $A(\omega)^{-1}dt_1 \wedge dt_{n+1}$ has a component with $dt_1 \wedge dt_{n+1}$.

This shows that our flow has local existence on compact manifolds, since all eigenvalues of $\sigma DE(\omega)$ are zero or positive. Note that if $A(\omega)$ is a scalar, then the condition for symbols is also satisfied, and such flows also have local existence.

To see what is needed in the case of noncompact manifolds, we need to review the proof of the theorem from [23]. This proof first notes that the flow existence follows from the Nash-Moser inverse function theorem applied to the operator $\mathcal{P}(f) = (\partial f/\partial t - E(f), f|_{t=0})$, defined on smooth sections of the corresponding bundles. When a manifold is compact, this is automatically a tame space, but in the non-compact case even this point needs some care. The proof further uses the tamely equivalent Sobolev norms, slightly modified so that the time derivatives are counting twice as the space derivatives, but in the non-compact case we might as well work directly in this graded space, or a weighted version which does not change anything in the proof, rather than with the smooth sections, because we do not have the Sobolev inequality at our disposal.

The proof proceeds by checking the solvability of the linearized equation. This is shown to be equivalent to the existence and uniqueness of the system

$$\frac{\partial \tilde{f}}{\partial t} - P(f)\tilde{f} + L^*(f)\tilde{g} = \tilde{h}$$

$$\frac{\partial \tilde{g}}{\partial t} - M(f)\tilde{f} = \tilde{k}$$

This is a system in $\tilde{f}$ and $\tilde{g}$, and $P(f)$ is elliptic of order two, while $L^*(f)$ and $M(f)$ are of the first order. To solve this, a time lag δ for the second equation is introduced, and a-priori estimates derived which do not depend on δ, with the solution obtained as a limit when δ tends to zero.

The main difference in the non-compact case is that ellipticity of P does not imply necessarily the strong ellipticity with an uniform bound of eigenvalues from zero. Apart from this, all the estimates from [23] work, because the C^∞ bounds on coefficients are used. Thus, if ω is of quasi-finite geometry of infinite order, the conditions on the coefficients will be satisfied.

However, the operator $P(f)$ also needs to have eigenvalues uniformly bounded away from zero. This operator is equal to $DE(f) + L^*(f)L(f)$ and hence $\sigma P(f)(\xi) = \sigma DE(f)(\xi) + \sigma L^*(f)(\xi)\sigma L(f)(\xi)$. It is thus necessary that $\sigma DE(f)$ has non-zero eigenvalues with real part uniformly bounded away from zero, and in addition to this, that $\sigma L(f)(\xi)\sigma L^*(f)(\xi)$ has nonzero eigenvalues from the image of $\sigma L(f)(\xi)$ uniformly bounded away from zero - any eigenvalue of P that is not in the kernel of L is a positive eigenvalue of $\sigma L(f)(\xi)\sigma L^*(f)(\xi)$, which can be seen by multiplying P with L, but we also want it to be uniformly bounded away from zero. These are the additional conditions which we need if we want to use the Hamilton method in the non-compact case to prove the local existence. To obtain an approximate metric, however, we also need global existence for initial conditions which will give the desired metric in the limit, and thus results from this section serve more as an illustration and a first step in this direction.

2.3 Metrics at Infinity

Vanishing at infinity of the volume correction function f to the approximating volume form, which appears in Theorem 2.1.1, and its derivatives implies vanishing of the corresponding derivatives of the solution φ. This is a consequence of elliptic regularity, as we are going to see. Thus, having a good approximating metric enables us to compute behavior of the solution at infinity. For instance, the asymptotic behavior of curvature tensor for the solution will be the same as for the initial metric when C^2 norm of f vanishes at infinity.

Proposition 2.3.1. *If f has derivatives vanishing at infinity up to an order $k+\alpha$, and metric ω_g is quasi-finite of order $k+\alpha$, then the solution φ from the Theorem 2.1.1 has derivatives vanishing up to an order $k+2+\alpha$ at infinity.*

Proof. To prove this assertion, consider the operator

$$L\psi = \frac{i}{2\pi}\partial\bar{\partial}\psi \wedge (\omega_g^{n-1} + \omega_g^{n-2}\wedge\omega + \ldots + \omega^{n-1})/\omega_g^n$$

where $\omega = \omega_g + \frac{i}{2\pi}\partial\bar{\partial}\varphi$. Note that

$$L\varphi = (\omega - \omega_g)\wedge(\omega_g^{n-1} + \omega_g^{n-2}\wedge\omega + \ldots + \omega^{n-1})/\omega_g^n = (\omega^n - \omega_g^n)/\omega_g^n = e^f - 1$$

Since metric g is quasi-finite, we can map neighborhood of any point to a corresponding ball in $\mathbb{C}^n$ and consider pullback of L restricted to this chart. Then because ω and ω_g are equivalent the symbol will have eigenvalues uniformly bounded away from 0. Because φ is smooth with all derivatives bounded by Theorem 2.1.1, and geometry is quasi-finite of order $k+\alpha$, the coefficients will be uniformly bounded in $C^{k,\alpha}$, and so we can apply Theorem 1.3.2 to estimate $k+2+\alpha$ derivatives of φ. Thus we conclude that in a half ball B_1

$$\|\varphi\|_{C^{k+2,\alpha}(B_1)} < C(\|f\|_{C^{k,\alpha}(B_1)} + \|\varphi\|_{C^0(B_1)})$$

with some uniform C. But φ vanishes at infinity, and also the derivatives up to order $k+\alpha$ of f, and the statement follows. □

To construct an approximate metric in [1], the following approach was used. A form ω which has approximately the right volume form at infinity, and is positive near the divisor, is constructed, and away from divisor it was combined with another form which gives positive metric, to give a positive metric on the whole manifold. The second form, which is positive on the whole manifold, is the curvature of some norm of line bundle defining the divisor. When such a form exists, we need only to worry about constructing approximating form which is positive near the divisor. Since a convex combination of two positive forms is positive, we can add a multiple of the second form to the first one, with a factor which takes over away from divisor and vanishes near it. But the problem is that we need a closed form, which is obtained from some Kähler potential. The fact that the second form is obtained as $i\partial\bar{\partial}s$ where $s = -log\|S\|^2$, can be used to get around this - taking $q(s)$ instead of s, with some appropriately chosen q, we get $q'(s)\omega + q''(s)\partial s \wedge \bar{\partial} s$, and by rescaling and multiplying by a constant, we can get to neglect the second term, while form of $q'(s)$ is at our will. We can in fact choose it to be zero around a divisor, which was not used in [1], but a function s^{-N} instead. The error term for the volume correction function is dominated by this part in [1], improving when one increases N, but there is no need for this as we can simply drop the second part near the divisor, using the same idea of rescaling and normalizing with two free parameters, as in [1], and a function q' that vanishes at infinity. In fact, we will show that all the derivatives of f vanish. Thus, near infinity, we can assume that approximating metric is given as

$$\omega = i\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}F^{(n+1)/n} = (-nF)^{1/n}\omega_{\varphi} + i\frac{\partial F \wedge \bar{\partial} F}{(-nF)^{(n-1)/n}}$$

The form $\omega_{\varphi} = i\partial\bar{\partial}F$ is a form which is flat when restricted to a divisor, and positive near it in the orthogonal direction too. Here $F = -log\|S\|^2_{\varphi} = -log(e^{-\varphi}\|S\|^2)$, and φ is a smooth function on $\bar{M}$, so that the curvature condition for the restriction is satisfied.

To see what f is and how it behaves near the divisor, we choose a local coordinates, which can be always done so that $dz_1 \wedge \ldots \wedge dz_{n-1} \wedge dz_n/z_n$ is a section of the holomorphic $(n, 0)$ form and thus a volume form that we aim at is $dV/|z_n|^2$ where dV is the local Euclidean volume. Suppose that the line bundle norm is represented as

$\|S\|_\varphi^2 = a(z)|z_n|^2$ with smooth positive function a. Then we compute

$$\omega_\varphi = -i\partial\bar{\partial}loga$$

$$\partial F = -\partial loga - \frac{dz_n}{z_n} \qquad \bar{\partial} F = -\bar{\partial} loga - \frac{d\bar{z}_n}{\bar{z}_n}$$

Therefore

$$e^{-f} dV = |z_n|^2 \omega^n = \Phi_0 + z_n \Phi_1 + \bar{z}_n \Phi_{\bar{1}} - nF|z_n|^2(-i\partial\bar{\partial}loga)^n$$

Here Φ_0, Φ_1 and $\Phi_{\bar{1}}$ are forms that can be written as nonsingular real analytic expressions in $z, a, \partial a, \bar{\partial} a, \partial\bar{\partial} a$ and $F^{-1/n}$ in the given local chart. The condition for a on the divisor means that Φ_0 tends to dV when z_n tends to zero, and so f tends to zero as we approach the divisor. The metric corresponding to ω expands transversally with a factor $(-\log|z_n|)^{1/n}$ and with a factor $1/|z_n|$ in the normal direction, and so from this expression it follows that all the derivatives of f tend to zero too. It also follows that the curvature tends to zero because of this expanding property, and hence curvature of the solution also tends to zero at infinity.

We can use the obtained formula to solve for a for which $f = 0$ near the divisor, by means of a power series. The variables are z_n and $\bar{z}_n$, and also $q = (-log|z_n|^2)^{-1/n}$ can be thought of as an independent variable except when computing the derivatives. The solution is of the form $\sum a_{i,\bar{j},k} z_i^i \bar{z}_j^j q^k$ where i and j are positive integers, while k can be negative too, starting with $-n$. Equalizing coefficients will give a sequence of linear partial differential equations of second order on the norm of the corresponding line bundle restricted to the divisor. Solving such an equation in each iteration step leads to a solution in the neighborhood of D.

Chapter 3

Special Lagrangian Fibrations and a Change of Complex Structure

Our approach to special Lagrangian fibrations is global. Lagrangian fibrations can be represented with a locally Hamiltonian field and a fibration preserving resolving vector field, which are paired in a natural way, with an associated singular almost complex structure. We generalize the method of extremal length and drive analogies with the two dimensional Teichmüller theory. In particular, we relate extremal problems to the existence of special Lagrangian fibrations, proposing a method for constructing them in the large complex structure limit of Calabi-Yau manifolds.

3.1 Representing Lagrangian Fibrations

In the presence of an almost complex structure J, Lagrangian fibrations have at each point x two mutually orthogonal Lagrangian tangent subspaces specified, the tangent space V_x of a fibre and its orthogonal complement, equal to JV_x when the fibre is of maximal dimension n. This enables one to pair locally Hamiltonian fields with fibration preserving fields, provided that we have a metric on the base too.

Suppose that we have a fibration $f : M \mapsto B$, and that B is equipped with a metric. Then to any locally gradient field on B two vector fields on M correspond - a locally Hamiltonian vector field X, the symplectic gradient of the lifted local potential from B, and a fibration preserving vector field Y which is lifted from the base field directly. To define the first vector field we need a metric on B, and for the second, a metric on M, which comes from the almost complex structure.

If $dH = \iota_X(\omega)$ locally, then the derivatives $L_Y^m(H)$, for $m = 0 \ldots n-1$, are generically functionally independent, and will determine the distribution corresponding to the fibration. Thus, the pair (X, Y) generically determines the fibration, and describes it in terms of local data.

The choice of a metric on the base is arbitrary, and it determines the pairing. We can suppress this metric, and consider all possible pairs (X, Y) where X is a locally Hamiltonian field belonging to the fibration, and Y preserves the fibration, but is orthogonal to all the fibres.

Definition 3.1.1. Suppose X is a locally Hamiltonian field on an almost Kähler manifold (M, ω, J), then we say that Y is a resolving field for X if for any corresponding local Hamiltonian H satisfying $dH = \iota_X(\omega)$, the derivatives $H^{(m)} = L_Y^m(H)$ commute with respect to the Poisson bracket, are functionally independent for $0 \leq m \leq n-1$ and have symplectic gradients orthogonal to Y. If the induced base field corresponding to Y is the gradient of H with respect to a metric g on the base of a related fibration, we will say that X and Y form a g-pair, and use a notation $(X, Y)_g$.

Thus, only integrable Hamiltonian fields will have a resolving field. As we have seen above, a generic locally Hamiltonian field, belonging to some Lagrangian fibration with a generic fibre of dimension n, has a resolving field. Once a metric on the base is specified, the resolving field can be associated in a canonical way. Conversely, a generic vector field which is orthogonal to all the fibres is a resolving field for some locally Hamiltonian field, belonging to the fibration.

Kähler Ricci-flat manifolds have a specified holomorphic $(n,0)$-form φ, and a Lagrangian fibration described with a pair (X,Y) will be special Lagrangian when $\mathrm{Im}\varphi(\tilde{\nabla}H^{(0)},\ldots,\tilde{\nabla}H^{(n-1)}) = 0$, where $X = \tilde{\nabla}H$ is some local representation of X as a symplectic gradient and $H^{(m)} = L_Y^m(H)$. Pairs (X,Y) satisfying these requirements always exist, and the real issue is to get the dimension of a fibration to be n with some additional topological constraints. But constructing a fibration along these lines has problems with convergence, as fields need to be regular because higher derivatives are involved, and with uniqueness, as there are many possible pairs representing the same fibration. However, the pairing itself is a canonical object which might be used to encode a fibration.

Suppose that we have a metric g_1 specified on the base of our fibration. Then using the associated g_1-pairing we can define an almost complex structure J_1 on all nonsingular points of the fibration, such that $Y = J_1X$. Indeed, both X and Y can be expressed in terms of dH, and depend only on the value of dH at a point. This induces a linear map which is an isomorphism of spaces V_x and JV_x whenever x is a point on a nonsingular fibre, and we can extend this map to get an almost complex structure J_1. When we have a Lagrangian fibration whose all fibres have finite n-volume, we can define a natural metric on the base, obtained by averaging the original dot product over the fibre. In particular, special Lagrangian torus fibrations have a naturally defined metric on the base. We are going to normalize it so that the volume of the fibres remains the same when we change a metric coming from the original complex structure J to the one corresponding to the almost complex structure JJ_1J.

Proposition 3.1.1. *Let J_1 be an almost complex structure corresponding to a special Lagrangian torus fibration, obtained from the metric conformal to the averaged metric on the base. Then the metric $g_1(u,v) = \omega(u, J_1v)$ is J_1 invariant and agrees with the metric on the base. The metric $g_2(u,v) = g_1(Ju, Jv)$ is flat on the fibres and we can normalize metrics and complex structure J_1 so that the fibres have the same volume* *for both g and g_2. The map from the original manifold with a Ricci-flat metric g to the same manifold with the metric g_2 induced from the identity map is harmonic when restricted to nonsingular fibres.*

Proof. If we define metric g_1 as the J_1 invariant metric consistent with the metric on the base, then it suffices to check that $g(X, JY) = g_1(X, J_1Y)$. But from the properties of J and J_1 we see that this is true if only we can check it for Y in the fibration, and X orthogonal to the fibration - since spaces JV_x and J_1V_x are the same by construction, for a tangent space V_x of a fibre at x. But then we may assume that Y is a symplectic gradient of H, and so $g(X, JY) = dH(X)$. On the other hand, by definition of J_1 we have $dH(X) = g_1(X, J_1Y)$. To prove the last assertion, note that there are functions $H_1 \dots H_n$ induced from functions on a base such that $dH_1 \dots dH_n$ is a g_1 orthonormal basis of the space JV_x^*. But then $JdH_1 \dots JdH_n$ form an orthonormal coframe in V_y^* with respect to g_2 for any y in the same fibre as x, and are harmonic forms with respect to g, by the easier direction of the McLean theorem. The normalization changes metric on a base conformally and complex sturctures J_1 and $J_2 = JJ_1J$ in a consistent way. □

Maps corresponding to a change of metric from g to g_1 and g_2 do not have bounded quasiconformality constants in general, and it can tend to infinity as we approach singular fibres. The Jacobian determinants of these two maps are the same. However, of special interest is the Jacobian determinant of a harmonic map on the fibres, corresponding to a change of metric from g to g_2. It defines a nonnegative function ρ_0 on the manifold, which is an extremal metric with respect to the fibration, as we are going to see. This is the reason we normalized metrics g_1 and g_2, with the correspoding almost complex structures J_1 and J_2, to the g_2-volume of the fibres.

3.2 Quadratic Differentials and Higher Dimensional Singular Flat Examples

When $n = 1$, Ricci-flat condition coincides with the flat condition, and we have a flat metric on a Riemann surface. Such a metric is equal to hdz^2, where h is a harmonic function in any isothermal coordinates. These metrics are generally not uniformized, because the completion can have singularities, corresponding to the zeroes of h. In the natural parameter the metric has the form dw^2, and $h = |\Psi|^2$ where $\Psi = \partial w/\partial z$. This leads to study of holomorphic quadratic differentials, and special Lagrangian fibrations correspond to trajectories of such quadratic differentials. The theory of trajectories of quadratic diferentials is well developed, for instance we refer to the book of Strebel, [7]

The Teichmüller theorem for surfaces of finite analytic type links quadratic differentials to extremal quasiconformal maps. Any quasiconformal map of minimal dilatation in its class has a Beltrami differential of a form $k\bar{\varphi}/|\varphi|$, where $k < 1$ is a constant and φ is an integrable holomorphic quadratic differential, and is uniquely extremal.

The situation for surfaces of infinite analytic type is strikingly different. Extremal maps do not necessarily have Beltrami differentials of constant absolute value, for instance the so called Strebel chimney is an example. The similar question for uniquely extremal maps was long open, and it was thought that they have to be of the form $k\bar{\varphi}/|\varphi|$ with analytic, but not necessarily integrable quadratic differential φ - this was conjectured by Teichmüller. This conjecture was disproved in my joint work with N. Lakic, V. Marković and M. Mateljević, [9], where uniquely extremal condition was characterized. Examples of uniquely extremal maps which do not have constant modulus and which disprove the Teichmuller conjecture, a part of this joint research with my particular contribution, were first announced in [10], and [11] provides account of the first counterexample. The counterexample theorem from [9] provides uniquely extremal maps with essentially arbitrary choice of Beltrami differential outside some set of small measure on any Riemann surface from which a set with a cluster point is removed. Thus, while extremality sometimes grants nice regularity properties, this is not always the case.

The higher dimensional singular flat examples are important to understand, as they are conjectured to correspond to the large complex structure limits of Calabi-Yau metrics. One simple class of such examples can be obtained by the analogy with the two dimensional billiards. Such a billiard is a polygonal domain in $\mathbb{C}$, with trajectories extending by reflection. The angles at vertices are rational, so that reflections induce a compact Riemann surface, with a corresponding branched covering and quadratic differential. We can do a similar construction in $\mathbb{C}^n$, using hyper-planes of reflection instead. To any polygonal domain in $\mathbb{C}^n$ with rational hyperplane angles we can associate an orbifold, obtained as a formal union of multiply reflected domains along the polygonal hyperplanes. Such an orbifold might have a resolution of singularities which gives a Calabi-Yau manifold, and we can hope to represent its large complex structure limit with the singular metric induced from the flat orbifold that we started with.

Alternatively, we may combine singular two-dimensional flat examples in a Cartesian product. The special Lagrangian fibres in the product will be products of trajectories of quadratic differentials. Note that in higher dimensions the parameter θ corresponding to the flat special Lagrangian fibration is not enough to determine the direction - there are many special Lagrangian tangent n-planes with the same θ, which is essential problem in constructing an SL fibration. The angle θ even together with the homology condition does not generally determine the direction uniquely, for instance in the case of $\mathbb{C}^n$. A flat complex torus might not have a special Lagrangian fibration at all. We believe that these monodromy problems are an issue with the existence of special Lagrangian fibrations, rather than the convergence problems.

3.3 Extremal Methods

Here we are going to generalize the method of extremal length and adopt it to the problem of special Lagrangian fibrations. This method in Teichmüller theory is also known as the Grötzsch's argument.

Definition 3.3.1. Let Γ be a family of rectifiable n-submanifolds of a Riemannian $2n$ dimensional manifold M. The generalized extremal length of this family is

$$\Lambda(\Gamma) = \sup_{\rho \geq 0} \frac{\lambda(\Gamma, \rho)^2}{\|\rho\|^2}$$

where $\rho \in L^2(M)$ and $\lambda(\Gamma, \rho) = \inf_{S \in \Gamma} \int_S \rho d\sigma$.

The ρ for which the supremum is attained is called extremal metric. Modulus of Γ is defined as a reciprocal, $\mathcal{M}(\Gamma) = 1/\Lambda(\Gamma)$. Like the ordinary modulus of curves, it is conformal invariant and is quasi-preserved under quasiconformal maps - the quasiconformality constant gives a bound on the linear distortion, and hence a bound on the distortion of n-plane volumes too.

The comparison and composition properties of generalized extremal length are the same as in the classical case, [6]:

• If every $S \in \Gamma$ contains a $S' \in \Gamma'$ then $\Lambda(\Gamma) \geq \Lambda(\Gamma')$

• If Ω_1 and Ω_2 are disjoint open sets, all $S_1 \in \Gamma_1$ belong to Ω_1 and all $S_2 \in \Gamma_2$ to Ω_2 and every $S \in \Gamma$ contains a $S_1 \in \Gamma_1$ and a $S_2 \in \Gamma_2$ then

$$\Lambda(\Gamma) \geq \Lambda(\Gamma_1) + \Lambda(\Gamma_2)$$

$$1/\mathcal{M}(\Gamma) \geq 1/\mathcal{M}(\Gamma_1) + 1/\mathcal{M}(\Gamma_2)$$

• If Ω_1 and Ω_2 are disjoint open sets, all $S_1 \in \Gamma_1$ belong to Ω_1 and all $S_2 \in \Gamma_2$ to Ω_2 and every $S_1 \in \Gamma$ and every $S_2 \in \Gamma_2$ contains a $S \in \Gamma$ then

$$1/\Lambda(\Gamma) \geq 1/\Lambda(\Gamma_1) + 1/\Lambda(\Gamma_2)$$

$$\mathcal{M}(\Gamma) \geq \mathcal{M}(\Gamma_1) + \mathcal{M}(\Gamma_2)$$

The link between extremal length and volume minimizing fibrations is given by the following Beurling criterion:

Proposition 3.3.1. *A nonnegative $\rho_0 \in L^2(M)$ is extremal for Γ if Γ contains a subfamily Γ_0 with the following properties:*

- $\int_S \rho_0 d\sigma = \lambda(\Gamma, \rho_0)$ *for all* $S \in \Gamma_0$
- *If a real valued $h \in L^2(M)$ satisfies $\int_S h d\sigma \geq 0$ for all $S \in \Gamma_0$ then*

$$\int_M h\rho_0 dV \geq 0$$

Proof. Take any other nonnegative $\rho \in L^2(M)$ and normalize it so that $\lambda(\Gamma, \rho) = \lambda(\Gamma, \rho_0)$. Then $\int_S \rho d\sigma \geq \int_S \rho_0 d\sigma$ for any $S \in \Gamma_0$, and so by our assumption

$$\int_M (\rho - \rho_0)\rho_0 dV \geq 0$$

But this means that $\int_M \rho\rho_0 dV \geq \int_M \rho_0^2 dV$, and thus by the Cauchy-Schwarz inequality $\|\rho\| \geq \|\rho_0\|$. It follows that

$$\frac{\lambda(\Gamma, \rho_0)^2}{\|\rho_0\|^2} \geq \frac{\lambda(\Gamma, \rho)^2}{\|\rho\|^2}$$

□

The application of the Cauchy-Schwarz inequality here is essentially the original Grötzsch's argument.

Proposition 3.3.2. *Let Γ_0 be a family of non-degenerate Lagrangian torus fibres corresponding to some SL torus fibration. Then the Jacobian determinant ρ_0 of the map on fibres induced by the identity map under the change of metric from the original Ricci-flat metric g to the fibrewise flat g_2 is an extremal metric for Γ_0.*

Proof. We employ the Beurling criterion. By the definition of a metric g_1 follows that, with respect to the original metric g, $\int_S \rho_0 d\sigma$ is constant for all $S \in \Gamma_0$, as it equals $\int_S d\sigma_2$ with respect to g_2. Now suppose that we have h such that $\int_S h d\sigma \geq 0$ for all $S \in \Gamma_0$. We can use the compatibility of a metric g_1 with the metric on the base to get the second condition:

$$\int_M h\rho_0 dV = \int_M h\rho_0 d\sigma \wedge Jd\sigma = \int_M hd\sigma \wedge Jd\sigma_2 = \int_B d\tau_1 \int_{S(\tau_1)} hd\sigma \geq 0$$

Here τ_1 denotes base coordinate and $d\tau_1$ base volume form with respect to the metric g_1. □

Suppose $\psi = \text{Re } \varphi$ is a calibration form for our SL fibration. Then in general $d\rho_0\psi = d\rho_0 \wedge \psi$ is not zero, and therefore we cannot argue that ρ_0 is an extremal metric for a more general family of homologous n-manifolds Γ. However, if it is extremal, then we can use modulus to test weather a certain set of n-manifolds can belong to a fibration: If $M \setminus \Omega_1 = \Omega_2$ where Ω_1 is an open set obtained as a variation of such a set of n-manifolds, which we want to test, then necessarily $\mathcal{M}(M) = \mathcal{M}(\Omega_1) + \mathcal{M}(\Omega_2)$.

To get around the fact that $d\rho_0\psi$ is not always zero, and still be able to get some estimates about the extremal maps, we are going to use ψ to compare lengths, and thus a variation of ρ_0, the number $\sup_M \rho_0 / \inf_M \rho_0$, will measure how far we are from the sharp estimates.

We first derive a generalization of the Reich-Strebel Inequality, which estimates quasiconformality constant of a self-map homotopic to identity for Riemann surfaces.

Theorem 3.3.3. *Let Γ be a special Lagrangian fibration with the corresponding extremal metric ρ having mean value $\bar{\rho}$ with respect to the Calabi-Yau volume form on M. Then for any quasiconformal self-map f of M homotopic to identity the following inequality holds:*

$$\int_M dV \leq \int_M K_{\Gamma,f}(x)^2 (\frac{\rho}{\bar{\rho}})^2 dV$$

Here $K_{\Gamma,f}(x)$ denotes quasiconformal distortion of the n-plane which is tangent to the fibration Γ at x, i.e. the Jacobian of the map f restricted to this n-plane is $K_{\Gamma,f}(x)|Df(x)|^{1/2}$.

Proof. Let $\rho_1(x) = K_{\Gamma,f}(x)|Df(x)|^{1/2}$. Then we have

$$\int_M \rho_1 \rho dV = \int_B d\tau_1 \int_{S(\tau_1)} \rho_1 d\sigma = \int_B d\tau_1 \int_{f(S(\tau_1))} d\sigma \geq \int_B d\tau_1 \int_{S(\tau_1)} d\sigma = \int_M \rho dV$$

where we used minimality property of the special Lagrangian fibres in their homology class, and the fact that f is homotopic to the identity.

On the other hand,

$$\int_M \rho_1 \rho dV = \int_M K_{\Gamma,f}(x)|Df(x)|^{1/2} \rho dV \leq (\int_M K_{\Gamma,f}(x)^2 \rho^2 dV)^{1/2} (\int_M |Df(x)| dV)^{1/2}$$

by the Cauchy-Schwarz inequality. But $\int_M |Df(x)| dV$ is the volume of the manifold, by the change of variables formula and the fact that f is a homeomorphism. Squaring and combining the two inequalities then proves the assertion. ☐

Here the only substantial difference from the classical Reich-Strebel Inequality is that we used a non-extremal metric, but which has a closed calibration form, to compare volume of the fibres. So $\bar{\rho}$ is average, instead of quadratic mean as in the original Reich-Strebel Inequality, and the inequality is not sharp if there is a variation of ρ. We can write a simpler inequality which emphasizes this difference, which will not be sharp unless the variation of extremal metric is one.

Corollary 3.3.4. *Suppose that $\Gamma, \rho, \bar{\rho}$ and f are as before, and let $C = \sup_M \rho/\bar{\rho}$. Then*

$$\int_M dV \leq C^2 \int_M K_{\Gamma,f}(x)^2 dV$$

Next, we proceed with the Main Inequality, which deals with maps $f : M \mapsto M_1$ and $f_1 : M_1 \mapsto M$, such that $f_1 \circ f$ is homotopic to the identity

Theorem 3.3.5. *Let $\Gamma, \rho, \bar{\rho}$ be as before, and let $f : M \mapsto M_1$ and $f_1 : M_1 \mapsto M$ be quasiconformal maps such that $f_1 \circ f$ is homotopic to the identity. Then the following inequality holds:*

$$\int_M dV \leq \int_M K_{\Gamma,f}(x)^2 K_{\Gamma',f_1}(y)^2 (\frac{\rho}{\bar{\rho}})^2 dV$$

where $\Gamma' = f(\Gamma)$ and $y = f(x)$

Proof. We apply the generalized Reich-Strebel Inequality, using the fact that

$$K_{\Gamma,f_1 \circ f}(x) = K_{\Gamma,f}(x) K_{\Gamma',f_1}(y)$$

□

The importance of the Main Inequality is that it enables one to estimate the quasiconformality constant of an extremal map. The quasiconformality constants in higher dimensions are many and are all intertwined, and we have in mind the n-plane quasiconformality constant. The following corollary corresponds to the First Fundamental Inequality in Teichmüller theory.

Corollary 3.3.6. *Let K_0 be the minimal n-plane quasiconformality constant for all maps $f_1 : M \mapsto M_1$ such that $f_1^{-1} \circ f$ is homotopic to the identity. Then*

$$(1/K_0)^2 \int_M dV \leq \int_M K_{\Gamma,f}(x)^2 (\frac{\rho}{\bar{\rho}})^2 dV$$

This illustrates why our inequality isn't sharp. Having a constant dilatation on the right hand side would imply that the map is extremal, if $\bar{\rho}$ was quadratic mean, like it is in the original Reich-Strebel Inequality. We can get the same type of inequality when the form $\rho\psi$ is closed, and in fact a variation from such a form is what matters.

3.4 Generalized Teichmüller Theorem and Extremal Fibrations

We can prove the following result about a uniquely extremal map, corresponding to the Teichmüller uniqueness theorem, using the inequalities like those that we have obtained

Theorem 3.4.1. *Suppose that* $d\rho \wedge \psi = 0$*, where* ρ *is an extremal metric of a special Lagrangian fibration* Γ *with a calibrating form* $\psi = Re\, \varphi$*. Then the map* $f : (M, g) \mapsto (M, g_k)$ *induced by the identity, and corresponding to the shrinking of the Calabi-Yau metric* g *on fibres by a homothety with a constant factor* k*, is uniquely extremal in the class of all qc-maps* $f_1 : (M, g) \mapsto (M, g_k)$ *such that* $f_1^{-1} \circ f$ *is homotopic to the identity. The extremality is understood as minimizing first the n-plane quasiconformality constant, and then the one dimensional qc-constant, and uniqueness is understood up to conformal self-maps of the target.*

Proof. Note that the extremality here means double minimization - although qc-constants are intertwined, and families compact with respect to any one of them, the particular minimizers might differ for various qc-classes. The second minimizing will ensure that our map is in fact uniquely extremal, as the n-plane constant has to be $k^{n/2}$ and therefore pointwise we can get uniqueness, up to conformal self-maps of the target. The extremality of the map would follow from the equality analogous to the First Fundamental Inequality, only with $\bar{\rho}$ being replaced with a quadratic mean. We will prove this as our next proposition. □

Proposition 3.4.2. *If* $d\rho \wedge \psi = 0$ *then we can replace* $\bar{\rho}$ *with a quadratic mean in all the theorems above.*

Proof. In this case, the proof of the generalized Reich-Strebel inequality goes along essentially the same lines as in the classical case. We repeat the two steps with the necessary changes. Let $\rho_1(x) = \rho(f(x)) K_{\Gamma,f}(x) |Df(x)|^{1/2}$. Then

$$\int_M \rho_1 \rho dV = \int_B d\tau_1 \int_{S(\tau_1)} \rho_1 d\sigma = \int_B d\tau_1 \int_{f(S(\tau_1))} \rho d\sigma \geq \int_B d\tau_1 \int_{S(\tau_1)} \rho d\sigma = \int_M \rho^2 dV$$

and

$$\int_M \rho_1 \rho dV = \int_M K_{\Gamma,f}(x) |Df(x)|^{1/2} \rho(f(x)) \rho dV \leq (\int_M K_{\Gamma,f}^2 \rho^2 dV)^{1/2} (\int_M \rho^2 dV)^{1/2}$$

by the Cauchy-Schwarz inequality and using $\int_M |Df(x)|\rho(f(x))^2 dV = \int_M \rho^2 dV$. Squaring and combining the two inequalities then shows that we can replace the average $\bar{\rho}$ with the quadratic mean. □

Note that that what matters to us is a fibration which is given as a family of n-submanifolds Γ, such that an extremal metric for this family ρ is also extremal for a larger family of homologous submanifolds. Such an object is what seems to correspond to integrable holomorphic quadratic differentials in the higher dimensional case. In fact, we can define extremal fibrations of arbitrary dimension m, which correspond to quadratic differentials in the higher dimensional case in this sense.

Definition 3.4.1. Suppose (M, g) is a compact Riemannian manifold, with an m-dimensional fibration given as a projection $\pi : M \mapsto B$ with singular locus Δ and $\Gamma_0 = \{\pi^{-1}(x) : x \in B \setminus \Delta\}$ consisting of compact fibres. Let Γ be a family of m-submanifolds homotopic to fibres from Γ_0, and let $\rho : M \mapsto \mathbb{R}^+$ be a conformal factor, such that all fibres from Γ_0 have the same m-volume, and such that there is a volume form on a base that can be pullbacked to M in a way consistent with ρ on orthogonal complements of fibre tangent spaces. Then ρ is extremal metric for Γ_0 by the analog of the Beurling criterion, and if it is extremal metric for Γ, i.e. if m-volume of all elements of Γ is greater or equal to the volume of fibres from Γ_0, then we say that the fibration (π, Γ_0, ρ) is extremal.

Note that in this definition Γ_0 depends only on the conformal structure specified on M. We use a conformal factor ρ, and in the case that the dimension of a manifold $n = 2m$ that we have considered so far, ρ^m was used instead. Correspondingly, now $\rho \in L^n$ instead of L^2.

The analog of the Beurling criterion we can state as two corresponding conditions

- $\int_S \rho_0^m d\sigma = \lambda(\Gamma, \rho_0)$ for all $S \in \Gamma_0$
- If a real valued $h \in L^{n/m}(M)$ satisfies $\int_S h d\sigma \geq 0$ for all $S \in \Gamma_0$ then

$$\int_M h\rho_0^{n-m} dV \geq 0$$

As before, these conditions imply the extremality of ρ_0. Indeed, for any other nonnegative $\rho \in L^n(M)$, normalized so that $\lambda(\Gamma,\rho) = \lambda(\Gamma,\rho_0)$, we have

$$\int_S \rho^m d\sigma \geq \int_S \rho_0^m d\sigma, \quad S \in \Gamma_0$$

So by our assumption

$$\int_M (\rho^m - \rho_0^m)\rho_0^{n-m} dV \geq 0$$

But this means that $\int_M \rho^m \rho_0^{n-m} dV \geq \int_M \rho_0^n dV$. By the Hölder inequality with $p = n/m$ and $q = n/(n-m)$

$$\int_M \rho^m \rho_0^{n-m} dV \leq \|\rho\|_{L_n}^m \|\rho_0\|_{L_n}^{n-m}$$

Thus $\|\rho\|_{L_n} \geq \|\rho_0\|_{L_n}$, and it follows that

$$\frac{\lambda(\Gamma,\rho_0)^{1/m}}{\|\rho_0\|_{L_n}} \geq \frac{\lambda(\Gamma,\rho)^{1/m}}{\|\rho\|_{L_n}}$$

The last expression is of course the corresponding defining term for extremal metric in our generalization.

Next, we prove the generalized Reich-Strebel inequality for an extremal fibration of dimension m.

Theorem 3.4.3. *Let (π, Γ, ρ) be an extremal fibration of dimension m on a n dimensional manifold M. Then for any quasiconformal self-map f of M homotopic to the identity the following inequality holds:*

$$\|\rho\|_{L_n}^n \leq \int_M K_{\Gamma,f}(x)^{n/(n-m)} \rho^n dV$$

Here $K_{\Gamma,f}(x)$ denotes quasiconformal distortion of the m-plane which is tangent to the fibration Γ at x, i.e. the Jacobian of the map f restricted to this n-plane is $K_{\Gamma,f}(x)|Df(x)|^{m/n}$.

Proof. Let $\rho_1(x) = \rho(f(x))K_{\Gamma,f}(x)^{1/m}|Df(x)|^{1/n}$. Then

$$\int_M \rho_1^m \rho^{n-m} dV = \int_B d\tau_1 \int_{S(\tau_1)} \rho_1^m d\sigma = \int_B d\tau_1 \int_{f(S(\tau_1))} \rho^m d\sigma \geq \int_B d\tau_1 \int_{S(\tau_1)} \rho^m d\sigma = \int_M \rho^n dV$$

and

$$\int_M \rho_1^m \rho^{n-m} dV = \int_M K_{\Gamma,f}(x)|Df(x)|^{m/n}\rho(f(x))^m \rho^{n-m} dV \le (\int_M K_{\Gamma,f}^p \rho^n dV)^{\frac{1}{p}} (\int_M \rho^n dV)^{\frac{1}{q}}$$

by the Hölder inequality with $p = n/(n-m)$ and $q = n/m$ and using $\int_M |Df(x)|\rho(f(x))^n dV = \int_M \rho^n dV$. Combining the two inequalities then finishes the proof, as before. □

The Teichmüller uniqueness theorem follows.

Theorem 3.4.4. *To any extremal fibration of dimension m, the corresponding map $f : (M, g) \mapsto (M, g_k)$, induced by the identity, and corresponding to the shrinking of the metric g on fibres by a homothety with a constant factor k, is uniquely extremal in the class of all qc-maps $f_1 : (M, g) \mapsto (M, g_k)$ such that $f_1^{-1} \circ f$ is homotopic to the identity. The extremality is understood as minimizing first the m-plane quasiconformality constant, and then the one dimensional qc-constant, and uniqueness is understood up to conformal self-maps of the target.*

Proof. The proof is exactly the same as before, starting from the generalized Reich-Strebel inequality. □

In the classical Teichmüller theory, infinitesimally trivial Beltrami differentials μ are characterized as those that anihilate quadratic differentials, i.e. such that $\int \mu\varphi = 0$ for all integrable holomorphich quadratic differentials φ. Here is the corresponding condition for extremal fibrations.

Theorem 3.4.5. *Let (ρ, Γ) be an m-dimensional extremal fibration of an n-dimensional Riemannian manifold M, and let δK_Γ be some infinitensimal variation of an m-dimensional quasiconformality constant with respect to Γ, corresponding to a variation of the identity map, homotopic to identity. Then*

$$\int_M \delta K_\Gamma \rho^n dV = 0$$

Proof. This follows from the fact that the first variation of the integral $\int_M K_{\Gamma,f}^{n/(n-m)}\rho^n dV$ is zero when f is an identity, since identity is an extremal map. We can show this more directly, using the fact that the first variation of an m-area of any $S \in \Gamma$, measured with respect to conformal factor ρ, is zero by the assumption that (ρ, Γ) is an extremal fibration, but this proof goes essentially along the lines of the proof of the generalized Reich-Strebel inequality as well. If δJ is the corresponding variation of the Jacobian determinant and $\delta\rho$ of ρ, then for any $S \in \Gamma$

$$\int_S (\delta K_\Gamma \rho + \frac{m}{n}\delta J \rho + m\delta\rho)\rho^{m-1} d\sigma = 0$$

Now, we can use this and the fact that $\int_M (\delta J \rho^n + n\rho^{n-1}\delta\rho)dV = 0$, since variation comes from a homeomorphism, to prove the assertion:

$$\int_M \delta K_\Gamma \rho^n dV = \int_B d\tau \int_{S(\tau)} \delta K_\Gamma \rho^m d\sigma =$$

$$= -\frac{m}{n}\int_B d\tau \int_{S(\tau)} (\delta J\rho + n\delta\rho)\rho^{m-1} d\sigma = -\frac{m}{n}\int_M (\delta J \rho + n\delta\rho)\rho^{n-1} dV = 0$$

□

Special Lagrangian fibrations are in some cases also extremal fibrations. One such case comes from the examples constructed by E. Goldstein in [26], where SL fibrations come from structure preserving torus action on a noncompact Ricci-flat Kähler manifold.

Proposition 3.4.6. *Suppose SL fibration is obtained from a structure preserving torus action with the corresponding vector fields $X_1, \ldots, X_{n-1}$ as in [26], such that (n-1)-volume spanned by $X_1, \ldots X_{n-1}$ is constant along the fibres. Then the fibration is extremal.*

Proof. The fibration is obtained by contracting the form φ with vector fields $X_1 \ldots X_{n-1}$ to get a holomorphic 1-form $d\eta + id\xi$, and then the fibres correspond to the condition $\xi = c$ in addition to the fixed moments corresponding to the torus action. Then

$\nabla\xi = J\nabla\eta$, and so $d\xi = i_{\nabla\eta}\omega$ and both $\nabla\xi$ and $\nabla\eta$ preserve ω. Then if $d\mu_k = i_{X_k}\omega$, we have the form $d\tau = d\mu_1 \wedge \ldots \wedge d\mu_{n-1} \wedge d\xi$, which is a pullback of the corresponding form on the base, and we ought to show that $d\tau \wedge Jd\tau$ is some function on the base times the volume form. But

$$d\tau \wedge Jd\tau = i_{X_1}\omega \wedge \ldots i_{X_{n-1}}\omega \wedge i_{\nabla\eta}\omega \wedge i_{JX_1}\omega \wedge \ldots i_{JX_{n-1}}\omega \wedge i_{J\nabla\eta}\omega$$

This is proportional to the volume form for points on nonsingular fibres, and the coefficient of proportionality is constant times volume spanned by vectors $X_1, \ldots, X_{n-1}, \nabla\eta$ $JX_1, \ldots, JX_{n-1}$ and $J\nabla\eta$. But the volume spanned by these vectors is $|\nabla\eta|^4$, since Re φ is the volume form on a fibre by the calibration property, and

$$\text{Re } \varphi(X_1, \ldots X_{n-1}, \nabla\eta) = d\eta(\nabla\eta) = |\nabla\eta|^2$$

But since $\nabla\eta$ is orthogonal to the orbits of the torus action, the volume spanned by $X_1, \ldots, X_{n-1}$ is just $|\nabla\eta|$, and so it follows that $|\nabla\eta|^4$ is constant along the fibres too, and the fibration is extremal. □

The additional condition is satisfied for instance in the case of an $n-1$ torus action on $\mathbb{C}^n$ given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}}) \cdot (z_1, \ldots, z_n) = (e^{i\theta_1}z_1, \ldots, e^{i\theta_{n-1}}z_{n-1}, e^{-i(\theta_1+\ldots+\theta_{n-1})}z_n)$$

The fibration is given by

$$|z_i|^2 - |z_n|^2 = c_i, \ \ \text{Im}(i^{n-1}z_1 \cdots z_n) = c_n$$

This is a classical example of Harvey and Lawson, [12].

It is also true that SL fibrations are extremal when the condition $d\rho \wedge \psi = 0$ is satisfied, and in particular when ρ is a constant. When a variation $\sup_M \rho/\bar{\rho}$ of extremal metric becomes close to 1 we can expect that the metric g_k will be close to extremal. The metric g_k can be obtained by changing a complex structure J to some almost complex structure. It will not be integrable, but we can expect it to be close to the integrable almost complex structure in the large complex structure limit of Calabi-Yau manifolds. In this limit, we expect that variation of extremal metric ρ will also tend to one, and so extremal maps corresponding to the change of complex structure should be related to the existence of special Lagrangian fibrations in this limit. The Strominger-Yau-Zaslow conjecture can be understood as relating the geometry of mirror manifolds only in such a limit. It is possible that one can relate the extremal qc-maps obtained by varying the complex structure and the symplectic structure of the mirror in the large complex and large symplectic structure limits more directly. In any case, the study of extremal problems in this context seems to make sense.

The question which naturally arises is how are we to understand the large complex structure limit. We again recall the two-dimensional case where there are various possible compactifications of the Teichmüller space. The Teichmüller compactification is obtained by adding points at infinity to the rays corresponding to Teichmüller type Beltrami differentials, $k\bar{\varphi}/|\varphi|$. But this compactification depends on the base point. A more natural Thurston compactification is obtained by gluing projective measured foliations onto Teichmüller space. By a theorem of M. Wolf, this compactification can be related to quadratic differentials which are obtained as Hopf differentials of harmonic maps, in an alternative representation for Teichmüller space, [5]. The Hopf differential gives a corresponding fibration which has the property that the harmonic map shrinks maximally along the fibres, although the quasiconformality constant varies from point to point for this harmonic representative. However, in the large complex structure limit this map varies little from the affine stretch in the natural parameter, as shown in the proof of the Wolf's theorem. This seems similar to the asymptotic behavior which might exist in the large complex structure limit of Calabi-Yau manifolds. It is possible that energy minimizing representatives in the higher dimensional case can be used to represent the corresponding compactifications in some way, and for constructing SL fibrations in the large complex structure limit. Some conjectures about how the right compactification should look like have been proposed by Kontsevich and Soibelman in [16].

As opposed to the two dimensional case, in higher dimensions the Weyl curvature is an obstruction to representing conformal structures as local deformations. Thus, it seems natural to study only such changes of conformal structure which satisfy a certain curvature condition, corresponding to the vanishing of the Weyl curvature in the flat case. It would be interesting to characterize infinitesimally trivial Beltrami differentials in the higher dimensional case in some way analogous to the two dimensional Teichmüller theory, with extremal fibrations playing the role of quadratic differentials. We have seen that extremal fibrations have many of the corresponding properties of the quadratic differentials, but there might be some differences. By a theorem of Hubbard and Masur every measured foliation on a Riemann surface is measure equivalent to exactly one measure foliation corresponding to a quadratic differential. Weather such strong existence results hold for higher dimensional extremal fibrations, in some cases at least, remains to be seen.

Chapter 4

The Schoen-Yau Conjecture

In this section we give a counterexample to a conjecture posed by Richard Schoen and S.T. Yau in [5], which says that there is no proper harmonic map from the unit disk to the complex plane with the flat metric. This conjecture is connected with some problems in the theory of minimal surfaces.

4.1 Connection with Minimal Surfaces

The Schoen-Yau conjecture is related to the question weather there is a hyperbolic minimal surface in $\mathbb{R}^3$ which properly projects onto $\mathbb{R}^2$. There are no known examples of such a surface - all known minimal surfaces that are graphs over $\mathbb{R}^2$ project properly.

The link is given by the following proposition.

Proposition 4.1.1. *A hyperbolic minimal surface in $\mathbb{R}^3$ which projects properly onto xy-plane exists if and only if there is a proper harmonic map from the unit disk onto the complex plane with a flat metric, given as $Re\ f + iIm\ g$ with analytic functions of the unit disk f and g satisfying $f' = m^2 + n^2$ and $g' = m^2 - n^2$ for some analytic m and n*

Proof. If $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is the first fundamental form of our minimal surface $(x(u,v), y(u,v), z(u,v))$, then $E = x_u^2 + y_u^2 + z_u^2$, $F = x_u x_v + y_u y_v + z_u z_v$ and $G = x_v^2 + y_v^2 + z_v^2$. For conformal coordinates u, v we have that $F = 0$ and $E = G$. Then the Euler-Lagrange equations for the minimal surface boil down to the additional condition that the coordinates x, y and z are harmonic, given as real parts of some analytic functions f_1, f_2 and f_3 of $u + iv$. If the surface is hyperbolic, there is a conformal change of coordinates such that domain of $w = u + iv$ is the unit disk. Then the conditions $E = G$ and $F = 0$ are equivalent to $f_1'^2 + f_2'^2 + f_3'^2 = 0$, and so the functions f_1', if_2' and if_3' form a Pythagorean triangle. In other words, we can restate this condition in terms of functions m and n so that $f_1' = m^2 + n^2$, $if_2' = m^2 - n^2$ and $if_3' = 2mn$, proving the assertion. □

Our counterexample does not satisfy this restrictive condition, but it shows that there could be a related construction in minimal surfaces, or that there is no such minimal surface for some much more subtle reason.

4.2 The Counterexample

By a classical result of E. Heinz, [17, 18, 19], there is no harmonic diffeomorphism from the unit disk $\Delta = \{z : |z| < 1\}$ to $\mathbb{C}$ with flat metric. However, there is a proper harmonic map, contrary to a conjecture from [5].

The counterexample is given by the following function $u : \Delta \mapsto \mathbb{C}$

$$u(z) = \sum_{k=1}^{\infty} \text{Re } (\frac{z}{r_k})^{m_k} + i \sum_{k=1}^{\infty} \text{Im } (\frac{z}{\rho_k})^{n_k}$$

$$r_k = e^{-\frac{1}{k}}, \rho_k = e^{-\frac{1}{k+\frac{1}{2}}}$$

$$m_1 = n_1 = 1, m_{k+1} = (4n_k + 1)m_k, n_{k+1} = (4m_{k+1} + 1)n_k$$

Proposition 4.2.1. *Map $u : \Delta \mapsto \mathbb{C}$ is proper harmonic.*

Proof. Let us consider a map $v : \mathbb{H} \mapsto \mathbb{C}$, where $\mathbb{H} = \{z : \text{Im } z > 0\}$, given by $v(z) = u(e^{iz})$, i.e. $(z = x + iy)$

$$v(z) = \sum_{k=1}^{\infty} e^{m_k(\frac{1}{k}-y)} \cos\,(m_k x) + i \sum_{k=1}^{\infty} e^{n_k(\frac{1}{k+\frac{1}{2}}-y)} \sin\,(n_k x)$$

We are to show that $v(z) \mapsto \infty$ when $y \mapsto 0$.

Notice that $m_k, n_k \geq \frac{1}{4} 2^{2^k}$, $m_{k+1} > n_k$, $n_k \geq m_k$

Consider the case $\frac{1}{k+\frac{1}{2}} \leq y \leq \frac{1}{k}$ $(k > 2)$

$$\text{Re }(v) = e^{m_{k-1}(\frac{1}{k-1}-y)} \cos\,(m_{k-1}x) + e^{m_k(\frac{1}{k}-y)} \cos\,(m_k x) + \Delta_1(z)$$

$$\text{Im }(v) = e^{n_{k-1}(\frac{1}{k-\frac{1}{2}}-y)} \sin\,(n_{k-1}x) + \Delta_2(z)$$

We have

$$|\Delta_1(z)| < \sum_{l=1}^{k-2} e^{\frac{m_l}{l}} + \sum_{l=k+1}^{\infty} e^{m_l(\frac{1}{l}-\frac{1}{k+\frac{1}{2}})} < \sum_{l=1}^{k-2} e^{\frac{m_l}{l}} + \sum_{l=k+1}^{\infty} e^{-m_l \frac{1}{(2k+1)(k+1)}}$$

Since $m_{l+1} > 2^{2^l} m_l$, we have $2e^{\frac{m_l}{l}} < e^{\frac{m_{l+1}}{l+1}}$ and $e^{-m_l} > 2e^{-m_{l+1}}$, so $|\Delta_1(z)| < 2e^{\frac{m_{k-2}}{k-2}} + 2e^{-\frac{m_{k+1}}{(2k+1)(k+1)}}$, and when $k \mapsto \infty$ we have

$$|\Delta_1(z)| = e^{m_{k-1}(\frac{1}{k-1}-y)} o(\frac{1}{m_{k-1}^2})$$

Similarly

$$|\Delta_2(z)| < \sum_{l=1}^{k-2} e^{\frac{n_l}{l+\frac{1}{2}}} + \sum_{l=k}^{\infty} e^{n_l(\frac{1}{l+\frac{1}{2}}-\frac{1}{k+\frac{1}{2}})} < \sum_{l=1}^{k-2} e^{\frac{n_l}{l+\frac{1}{2}}} + 1 + \sum_{l=k+1}^{\infty} e^{-\frac{n_l}{(k+\frac{1}{2})(k+\frac{3}{2})}} < 2e^{\frac{n_{k-2}}{k-\frac{3}{2}}} + 2$$

so when $k \mapsto \infty$

$$|\Delta_2(z)| = e^{n_{k-1}(\frac{1}{k-\frac{1}{2}}-y)} o(\frac{1}{n_{k-1}^2})$$

Since sin $(n_{k-1}x_1) = 0 \Leftrightarrow x_1 = \frac{p\pi}{n_{k-1}}$, $p \in \mathbb{Z}$, cos $(m_{k-1}x_2) = 0 \Leftrightarrow x_2 = \frac{(2q+1)\pi}{2m_{k-1}}$, $q \in \mathbb{Z}$, and both n_{k-1} and m_{k-1} are odd, we have $x_1 \neq x_2$ and $|x_1 - x_2| > \frac{\pi}{2n_{k-1}m_{k-1}}$. Similarly if cos $(m_k x_3) = 0$ then $|x_1 - x_3| > \frac{\pi}{2n_{k-1}m_k}$.

Also notice that cos $(m_k x_1)$ = cos $(m_{k-1}x_1)$ since $m_k = (4n_{k-1}+1)m_{k-1}$, so $m_k x_1 = m_k \frac{p\pi}{n_{k-1}} = (4n_{k-1}+1)m_{k-1}(\frac{p\pi}{n_{k-1}}) \equiv \frac{p\pi}{n_{k-1}}m_{k-1}(\text{mod } 2\pi)$. Similarly sin $(n_{k-1}x_3)$ = sin $(n_k x_3)$ - here we need 4 since $\frac{\pi}{2}$ appears.

For given x let x_1 be the closest zero of sin $n_{k-1}x$. When $|x - x_1| \leq \frac{\pi}{20n_{k-1}^3}$ we have that cos $(m_k x)$cos $(m_{k-1}x) > 0$ because $\frac{\pi}{20n_{k-1}^3} < \frac{1}{2}\frac{\pi}{2n_{k-1}m_{k-1}}, \frac{1}{2}\frac{\pi}{2n_{k-1}m_k}$, for $m_k \leq (4n_{k-1}+1)n_{k-1}$. Also $|\cos m_{k-1}x| > \frac{1}{20}\frac{1}{m_{k-1}^2}$, since the closest zero is at least $\frac{\pi}{4n_{k-1}m_{k-1}}$ away, and $n_{k-1} \leq (4m_{k-1}+1)m_{k-1}$.

Thus, we have in this case

$$|v| > |\text{Re } v| = e^{m_{k-1}(\frac{1}{k-1}-y)}(\frac{\theta}{20m_{k-1}^2} + a + o(\frac{1}{m_{k-1}^2}))$$

with $a > 0$ and $\theta > 1$, and clearly tends to ∞ as $k \mapsto \infty$.

When $|x - x_1| \geq \frac{\pi}{20n_{k-1}^3}$, then $|\sin n_{k-1}x| > \frac{1}{20n_{k-1}^2}$, so

$$|v| > |\text{Im } v| = e^{n_{k-1}(\frac{1}{k-\frac{1}{2}}-y)}(\frac{\theta}{20n_{k-1}^2} + o(\frac{1}{n_{k-1}^2}))$$

with $\theta > 1$, which also tends to ∞ as $k \mapsto \infty$.

The case $\frac{1}{k+1} \leq y \leq \frac{1}{k+\frac{1}{2}}$ is completely analogous to the discussed one, with the roles of Im v and Re v interchanged. In particular, in this case we have

$$\text{Im } (v) = e^{n_{k-1}(\frac{1}{k-\frac{1}{2}}-y)}\sin\,(n_{k-1}x) + e^{n_k(\frac{1}{k+\frac{1}{2}}-y)}\sin\,(n_k x) + \Delta_1(z)$$

$$\text{Re } (v) = e^{m_k(\frac{1}{k}-y)}\cos(m_k x) + \Delta_2(z)$$

Where, by the similar computation as above, when $k \mapsto \infty$

$$|\Delta_1(z)| = e^{n_{k-1}(\frac{1}{k-\frac{1}{2}}-y)}o(\frac{1}{n_{k-1}^2})$$

$$|\Delta_2(z)| = e^{m_k(\frac{1}{k}-y)} o(\frac{1}{m_k^2})$$

For given x let x_1 be the closest zero of cos $m_k x$. As above, when $|x - x_1| \leq \frac{\pi}{20m_k^3}$ we have that sin $(n_k x)$sin $(n_{k-1}x) > 0$. Also $|\sin n_{k-1}x| > \frac{1}{20}\frac{1}{n_{k-1}^2}$, since the closest zero is at least $\frac{\pi}{4n_{k-1}m_k}$ away, and $m_k \leq (4n_{k-1}+1)n_{k-1}$.

Thus, we have in this case

$$|v| > |\text{Im } v| = e^{n_{k-1}(\frac{1}{k-\frac{1}{2}}-y)}(\frac{\theta}{20n_{k-1}^2} + a + o(\frac{1}{n_{k-1}^2}))$$

with $a > 0$ and $\theta > 1$, and clearly tends to ∞ as $k \mapsto \infty$.

When $|x - x_1| \geq \frac{\pi}{20m_k^3}$, then $|\cos m_k x| > \frac{1}{20m_k^2}$, so

$$|v| > |\text{Re } v| = e^{n_{k-1}(\frac{1}{k-1}-y)}(\frac{\theta}{20m_k^2} + o(\frac{1}{m_k^2}))$$

with $\theta > 1$, which also tends to ∞ as $k \mapsto \infty$.

Thus, we can conclude that $|v| \mapsto \infty$ when $y \mapsto 0$, so our map is proper.

□

Chapter 5

The Frankl Conjecture

In this chapter, we describe an approach for disproving the Frankl conjecture, [20, 21, 22], and study the structure of finite lattices. We give a stronger, equivalent statement to the Frankl conjecture, which can be used to construct a counterexample or, if the conjecture is indeed true, might be easier to prove by induction.

5.1 Introduction

A widely believed hypothesis in the general theory of finite lattices, Frankl conjecture states that every finite lattice contains a join irreducible element which is below no more than half of the elements in the lattice. The hypothesis can be restated in terms of finite families of sets closed under intersection (or union) - every such family is supposed to have an element occurring in at most (or at least in the dual formulation with unions) one half of the sets. It was first stated by Frankl in 1979, [22] and appears in [20] and [21]. There have been many partial results and attempts to prove it, and the conjecture holds in many special cases.

We are going to consider the version with intersections, and consider finite families of sets $\mathcal{A} \subseteq \mathbb{P}(T)$ with some finite total set T.

To study such families, we will introduce three operations corresponding to an element $a \in T$: projecting $\mathcal{P}_a$, enhancing $\mathcal{E}_a$ and diluting $\mathcal{D}_a$ of a family. The projecting operation $\mathcal{P}_a$, mapping one set family closed under intersection to another, is the simplest of the three, and we define it as removing of an element a from all the sets in the family. Projections are connected with the surjective meet product homomorphisms of underlying lattices, and obviously all commute one with another. We can so define a projection operation $\mathcal{P}_A$ corresponding to a set of elements A as a sequence of corresponding projections $\mathcal{P}_a$ for $a \in A$. This should be distinguished from the projection onto set A, which we will denote by $\Pi_A = \mathcal{P}_{T \setminus A}$. Similar thing will not possible to define for operations $\mathcal{E}_a$ and $\mathcal{D}_a$, as the order of enhancing and diluting will matter. The unity of the underlying lattice plays somehow artificial role in this setting, and we will assume it is removed.

We shall define operations $\mathcal{E}_a$ and $\mathcal{D}_a$ so that they map one set family closed under intersection to another, keeping the cardinality of the set families unchanged and adding or removing an element a whenever possible. To understand these operations, we need to introduce a notion of a blinking subfamily, corresponding to a.

Suppose that $\mathcal{A} \subseteq \mathbb{P}(T)$. The "blinking" subfamily $\mathcal{B}_a \subseteq \mathcal{A}$ is a family of sets $X \subseteq T$ with the property that both $X \cup \{a\}$ and $X \setminus \{a\}$ are in $\mathcal{A}$. This family is also closed under intersections. All other sets of $\mathcal{A}$ can be distinguished from each other by the other elements alone. In particular, if $\mathcal{B}_a$ is empty, then we might add a to all the sets of $\mathcal{A}$, or remove it, without changing the cardinality of $\mathcal{A}$. However, if $\mathcal{B}_a$ is not empty, then there is the minimal blinking element $E_a = \bigcap \mathcal{B}_a$. As easily seen, $E_a \cup \{a\}$ is the minimal element of $\mathcal{A}$ containing a. Then we enhance all the sets that contain E_a, and are not in $\mathcal{B}_a$, by adding a to them whenever that does not contradict closeness of a family under intersections. In other words, whenever we have a set $X \supseteq E_a$, for which $X \cap Y \in \mathcal{B}_a$ for all $Y \in \mathcal{B}_a$, we will add a to X. In this way we will obtain a new set family $\mathcal{E}_a\mathcal{A}$, which is closed under intersection, has the same cardinality as $\mathcal{A}$, and has frequency of occurrence of all elements at least as great as in $\mathcal{A}$. Similarly, if we remove a from all the elements not in $\mathcal{B}_a$, we will get a diluted family $\mathcal{D}_a\mathcal{A}$

We say that a family $\mathcal{A}$ is saturated for a if $\mathcal{A} = \mathcal{E}_a\mathcal{A}$. By repeating the procedure of enhancing, we can get a maximally enhanced family of sets which will be saturated for all elements of T. It will have the same cardinality as $\mathcal{A}$ and it will be closed under intersections. The enhancing process is not canonical, and such a lattice in general depends on the sequence of enhancing steps we perform, as well as the underlying enhanced lattice. But any maximally enhanced family will have frequencies of all the elements at least as great as in $\mathcal{A}$, and so we may restrict our attention to the families of sets which are saturated for all the elements of T.

The diluting operation $\mathcal{D}_a$ is a bit simpler than $\mathcal{E}_a$. The maximally diluted families are just ideals of the Boolean algebra, i.e. families of sets closed for subsets. Such families are easily described by specifying the set of maximal elements of the lattice with the unity removed. The diluting process also depends on the sequence of steps performed and there is no canonical maximally diluted family of $\mathcal{A}$, and we first need to specify ordering in which dilution is to be done. The diluting operation can only possibly decrease cardinality of projections, as the following proposition shows.

Proposition 5.1.1. *Let $\mathcal{A} \subseteq \mathbb{P}(T)$ be a family of sets closed under intersections. Then for any $a, b \in T$ we have $\mathcal{P}_a\mathcal{D}_b\mathcal{A} \subseteq \mathcal{D}_b\mathcal{P}_a\mathcal{A}$.*

Proof. We have to deal only with the case when a is different from b. We first note that the projection of E_b is E_b for the projected family, because $\mathcal{P}_aE_b$ and $\mathcal{P}_aE_b \cup \{b\}$ are both present, and in every case when b is present in some projection, it has to be present in the original, which hence lies above E_b and so the projection is above the projection for E_b. Also, the blinking sublattice of $\mathcal{P}_a\mathcal{D}_b\mathcal{A}$ is contained in the one for $\mathcal{D}_b\mathcal{P}_a\mathcal{A}$, which is possibly larger. Indeed, $\mathcal{D}_b\mathcal{P}_a\mathcal{A}$ has the same blinking sublattice as $\mathcal{P}_a\mathcal{A}$ by the definition of the diluting operation, and if both X and $X \cup \{b\}$ are present in $\mathcal{P}_a\mathcal{D}_b\mathcal{A}$ then they must be present in $\mathcal{P}_a\mathcal{A}$ too, because we can pass to the originals and use the fact that dilution $\mathcal{D}_b$ will keep b only if a set with b removed is already present. The rest of the elements of $\mathcal{P}_a\mathcal{D}_b\mathcal{A}$, which are not in blinking lattices, are present in $\mathcal{D}_b\mathcal{P}_a\mathcal{A}$ because they come from some originals from which a and b are removed. □

To assess the Frankl conjecture, we may restrict our attention to the maximally enhanced lattices, saturated for all elements of T in some representation as a set family. Maximally enhanced families of sets are determined by the blinking sublattices $\mathcal{B}_a$. Lets assume that the blinking sublattices are ideals of the interval $[E_a, 1]$, and that they only depend on some other underlying lattice L'.

Such families can be described in the following way. The set E_a will be an element α of L'. Now $\mathcal{B}_a$ will correspond to an ideal of the interval $[\alpha, 1_{L'}]$, which can be described by an antichain of minimal β_i, which are not in the ideal. There might be more than one a with the same ideal, and hence we add a multiplicity index l to the description.

Definition 5.1.1. Let $T \subseteq \{a_{\alpha,\beta_1\ldots\beta_k,l} : \alpha, \beta_1 \ldots \beta_k \in L'\}$. For $S \in L'$ let $F_S = \{a_{\alpha,\beta_1\ldots\beta_k,l} \in T : \alpha \leq S\}$, $B_S = \{a_{\alpha,\beta_1\ldots\beta_k,l} \in T : \alpha \leq S, \beta_1 \ldots \beta_k \not\leq S\}$, and $E_S = F_S \setminus B_S$. Next, let $\mathcal{B}_S = [E_S, F_S] = \{X : E_S \subseteq X \subseteq F_S\}$, which is a Boolean algebra isomorphic to $\mathbb{P}(B_S)$. Then we define lattice $\mathcal{A}_{T,L'} = \{(S, X) : S \in L', X \in \mathcal{B}_S\}$ with the meet product operation $(S_1, X_1) \wedge (S_2, X_2) = (S_1 \wedge S_2, X_1 \cap X_2)$.

This construction is not quite the most general that can be considered, as one generally needs several extension steps and also a possibility that $\mathcal{B}_a$ is not an ideal, but only a sublattice of $[E_a, 1]$, to construct an arbitrary saturated family.

Let us consider families $\mathcal{A}_{m,n,M,N} = \mathcal{A}_{T(m,n,M,N),\mathbb{P}(\{1\ldots N\})}$, depending on integer parameters m, n - index subset sizes, index space size N, and multiplicity M, in more detail.

The set $T(m,n,M,N)$ is a set of elements $a_{\alpha,\beta,l}$ where $\alpha,\beta \subset [1\dots N]$, such that $|\alpha| = m$, $|\beta| = n$ and $l \in [1\dots M]$.

For every $i \in [1\dots N]$ we have $F_i = \{a_{\alpha,\beta,l} : i \notin \alpha\}$, and $E_i = \{a_{\alpha,\beta,l} : i \notin \beta\}$. Next, $\mathcal{B}_i = \{F_i \cap X : E_i \subseteq X\}$. Note that this is a Boolean algebra under intersection, with $E_i \cap F_i$ playing the role of an empty set, and F_i role of a full set.

Since all $\mathcal{B}_i$ are closed under intersection, each nonempty member of a family $\mathcal{A}_{m,n,M,N}$ is going to be intersection of sets from $\mathcal{B}_i$'s with no repetitions. Let $S \subseteq [1\dots N]$, then we define

$$\mathcal{B}_S = \{\bigcap_{i \notin S} X_i : X_i \in \mathcal{B}_i\}$$

Note that $\mathcal{B}_S$ is a Boolean algebra with $F_S = \bigcap_{i \notin S} F_i = \{a_{\alpha,\beta,l} : \alpha \subseteq S\}$ playing the role of a full set and $E_S = F_S \cap \bigcap_{i \notin S} E_i = \{a_{\alpha,\beta,l} : \alpha, \beta \subseteq S\}$ playing the role of an empty set. It is isomorphic to $\mathbb{P}(B_S)$ where $B_S = \{a_{\alpha,\beta,l} : \alpha \subseteq S, \beta \nsubseteq S\}$ is the set of "blinking" elements of $\mathcal{B}_S$. We will call the size of B_S the "volume" $V(S)$ of S.

Note that volume is proportional to multiplicity, $V(S) = MV'(S)$. The idea is that when M is very large, the family $\mathcal{A}_{m,n,M,N}$ is going to be dominated by families $\mathcal{B}_S$ maximizing the volume $V(S)$. We can roughly think of our family of sets as an enhanced Boolean algebra over $[1\dots N]$ with each $S \subseteq [1\dots N]$ given multiplicity $V(S)$. It then clear that, by making M large, it is sufficient to restrict attention only to those S which maximize the volume.

The volume $V(S)$ of a set of indices $S \subseteq [1\dots N]$ depends on the size of S only. Define $p = |S|/N$. It is useful to normalize the volume by considering $V(S)/|T|$ - the fraction of all elements which are in B_S, i.e. the probability that an element will be "blinking" in $\mathcal{B}_S$. This does not depend on the multiplicity M, and we can consider the limit as $N \to \infty$, which will depend on p, n and m alone. So, we define the limit probability of blinking

$$v(p) = \lim_{N\to\infty} V(S)/|T|, \text{ where } |S| = pN$$

Proposition 5.1.2. *The limit probability of blinking $v(p) = p^m(1-p^n)$*

Proof. The set of "blinking" elements is $F_S \setminus E_S = F_S \setminus \bigcap_{i \notin S} E_i$. This means that an element $a_{\alpha,\beta,l}$ is blinking precisely when $\alpha \subseteq S$ and $\beta \not\subseteq S$. Obviously, when $N \to \infty$, the limit probability of the first condition is p^m, and of the second is $1-p^n$, and since α and β run independently, the statement follows. □

Now it is easy to find p, $0 \leq p \leq 1$, which maximizes this probability. We have $m - (m+n)p^n = 0$, i.e. $p = \sqrt[n]{\frac{m}{m+n}}$. For large enough N, and M large enough (depending on N), only the sets which belong to $\mathcal{B}_S$ with the size of S closest to $(1 - \sqrt[n]{\frac{m}{m+n}})N$ will matter.

An element $a_{\alpha,\beta,l}$ occurs in a set from $\mathcal{B}_S$ with probability 1 if it belongs to E_S, with probability $\frac{1}{2}$ if it belongs to B_S and with probability 0 otherwise. Note that all families $\mathcal{B}_S$ with S of the same size contain the same number of sets. It is also true that these families are disjoint, provided the size of S is not less than n nor m (if it is, the families will contain an empty set).

Proposition 5.1.3. *Let $S, S' \subset [1 \ldots N]$ be different sets of indices of size not less than n, m. Then the families $\mathcal{B}_S$ and $\mathcal{B}_{S'}$ are disjoint.*

Proof. For any $X \in \mathcal{B}_S$, we can reconstruct S as the set $\{i : X \not\subseteq F_i\}$. Indeed, if $i \notin S$ then $X \subseteq F_S = \bigcap_{i \notin S} F_i \subseteq F_i$. But if $i \in S$, then because of our assumption about the size of S, we can find α, β such that $i \in \alpha$, $\alpha \subseteq S$, $\beta \subseteq S$, and we have that $a_{\alpha,\beta,l} \notin F_i$ and $a_{\alpha,\beta,l} \in E_S \subseteq X$. This proves the assertion. □

In particular, for n,m fixed, sets of maximal volume will always have disjoint families $\mathcal{B}_S$, provided that N is large enough. Thus, the frequency of an element in all the sets of maximal volume is going to be equal to the probability that it belongs to E_S, plus one half of the probability that it belongs to B_S, where probability is taken over all the sets S of maximal volume.

In the limit $N \to \infty$ the first probability is going to tend to p^{m+n} and it is the same for all elements $a_{\alpha,\beta,l}$. However, the second probability depends on the size of the set $\alpha \cap \beta$ - it is largest when these sets are disjoint. Indeed, the probability is of the event $\alpha \subseteq S$ and $\beta \not\subseteq S$, and it will be $p^m(1-p^{n-d})$ in the limit of large N, where $d = |\alpha \cap \beta|$. The fraction of elements with $d > 0$ is tends to zero as $N \to \infty$, and the total probability equals the expected size of the set in $\mathcal{B}_S$, asymptotically as $N \to \infty$. The frequency of all elements thus tends to $p^{m+n} + \frac{1}{2}p^m(1-p^n)$.

Setting $t = m/n$, and recalling that for the maximal volume $p^n = \frac{m}{m+n}$, we can rewrite this limit as

$$(\frac{t}{t+1})^{t+1}(1+\frac{1}{2t})$$

Proposition 5.1.4. *For any t, $0 < t < 1$, there is a sequence of families of sets $\mathcal{A}_{m,n,M,N}$, such that the minimal frequency of occurrence of an element tends to*

$$(\frac{t}{t+1})^{t+1}(1+\frac{1}{2t})$$

Proof. For $t = \frac{m}{n}$ we let $N \to \infty$. For each such N, letting $M \to \infty$ we get that the minimal frequency of an element is going to tend to the minimal frequency of an element over the union of families corresponding to S with the maximal volume, because non-maximal volume differs at least by M from the maximal, and so the size of $\mathcal{B}_{S'}$ is at least 2^M times smaller than $\mathcal{B}_S$ when $V(S') < V(S)$. Passing to a subsequence and using the asymptotic estimate derived above, we get the desired sequence. □

The function $(\frac{t}{t+1})^{t+1}(1+\frac{1}{2t})$ has a limit $\frac{1}{2}$ when t goes to zero, and e^{-1} when t goes to infinity. The situation does not much improve if we consider multiple β and more general T - we can get frequencies of occurrence of some elements to be above one half for the maximal volume, but there is always one element which has to pay for the adjustment of p. In fact, it can be shown that for elements of maximal volume in any $\mathcal{A}_{T,L'}$ the Frankl conjecture is true.

5.2 The Equivalence Theorem

From the considered families $\mathcal{A}_{T,L}$ it is evident that there is a link between the frequencies of occurrence of an element $a \in T$ and volume contributions. This can be generalized to any lattice, and we can show that a generalization of the weighted sum $\sum_{a\in T} f_a v_a$, where v_a is the volume contribution of an element a, is less than equal to $1/2$ if the Frankl conjecture holds. This is a strengthened version of the conjecture.

Theorem 5.2.1. *The Frankl conjecture holds if and only if for any finite family of sets $\mathcal{A} \subseteq \mathbb{P}(T)$ closed for intersections, any probability measure $p : \mathbb{P}(T) \mapsto [0,1]$ satisfying $p(a \in X) \leq f_a$ for all $a \in T$, where f_a is the frequency of occurrence of a in $\mathcal{A}$, has an expected value of mass $Em(X) \leq 1/2$. Here the random variable $m(X)$, a mass of a set, is defined as $m(X) = 1 - log|\mathcal{A}_X|/log|\mathcal{A}|$, where $\mathcal{A}_X$ is a family of sets obtained by removing all the elements of X from all the sets of $\mathcal{A}$.*

Proof. One direction is straightforward - if $\mathcal{A}$ has all frequencies of occurrence greater than $q > 1/2$, then define p to be T with probability q and zero set with probability $1-q$ - it satisfies the required condition and the expected value of mass is equal to q.

To prove the converse, assume that there is a family $\mathcal{A}$ and a probability measure such that $Em(X) > 1/2$, $p(a \in X) \leq f_a$. By taking complements and boosting a bit the probability of a full set on expense of all other, we get a probability measure such that $p(a \in X) > 1 - f_a$ and $Em(T \setminus X) > 1/2$. But the mass of $T \setminus X$ measures the size of a projection to X, i.e. we have that $E(log|\Pi_X \mathcal{A}|/log|\mathcal{A}|) < 1/2$.

Let $\mathcal{A}_1 = \mathcal{A}^{K_1}$ be a cartesian product of K_1 copies of $\mathcal{A}$. Define the corresponding probability measure p_1 to be nonzero only on multiple sets X^{K_1} and define it to be equal to $p(X)$. Then the desired properties will hold for the new family $\mathcal{A}_1$ as well. In this way, taking K_1 to be large, we can get $\mathcal{A}_1$ with arbitrarily large cardinality, but with the same expected value of mass, which we denote by m, and the frequencies the same as in $\mathcal{A}$, only each one repeated K_1 times. We can also assume that $\mathcal{A}_1$ contains an empty set.

Then again we take a cartesian product $\mathcal{A}_2 = \mathcal{A}_1^{2K_2-1}$. We will take constants $K_2 \gg \frac{1}{m-1/2}$ and K_1 so that $|\mathcal{A}_1| \gg 2^{K_2}$. Let us define the family $\mathcal{A}_3$ in the following way. Except for the elements of a total set of $\mathcal{A}_2$, sets from $\mathcal{A}_3$ will also have elements c_i with $i \in [1, 2K_2 - 1]$, and $\mathcal{A}_3$ will be generated by the sets from $\mathcal{A}_2$ having j-th component an empty set, to which a set $\{c_i : j \leq i \leq j + K_2 - 1 \vee i \leq j - K_2\}$ is added, for $j \in [1, 2K_2 - 1]$. The role of the added elements c_i is to cut the volume contribution of the family $\mathcal{B}$, which we will soon describe, in almost a half. To this end, the frequency of occurrence of c_i, which tends to $K_2/(2K_2 - 1)$ as K_1 goes to infinity, is much closer to one half than the expected value of mass m, by our assumption about K_2. By increasing K_1 and K_2 we can get frequencies of all the other elements to be close to the initial f_a frequencies in $\mathcal{A}_3$ as well, since only the described generating sets of $\mathcal{A}_3$ will matter for K_1 large enough.

We are going to construct a family of sets $\mathcal{B}$ which will have frequencies of occurrence of corresponding elements from $\mathcal{A}_3$ greater than $1 - f_a$, be invariant under the intersection with $\mathcal{A}_3$, and have frequencies of elements c_i slightly under $1/2$. We will then use a convex combination of the two families $\mathcal{A}_3$ and $\mathcal{B}$, with the first having just above $1/2$ and second just under $1/2$ contribution, so that all frequencies are above $1/2$. We will do that by adding a multiplicity to the second family using a linearly ordered set $[1 \dots MAX]$, where MAX will be present only with $\mathcal{A}_3$ and all other weighting elements from $[1 \dots MAX]$ present as maximums with $\mathcal{B}$.

To get the family $\mathcal{B}$ we first approximate p_1 with a very fine rational measure, taking an integer K_3 large enough so that all events are with probability $1/K_3$, but with repetition. Let M be a multiplicity parameter, and each c_i we replace with a block $c_{i,j,k}$ where $j \in [1 \dots K_3]$ and $k \in [1 \dots M]$. Then for each event ω_l with $l \in [1 \dots K_3]$ we define a family of sets $\mathcal{C}_l = \{\{c_{i,j,k} : j \neq l, k \in B\} : B \subseteq [1 \dots M]\}$. Corresponding to every event there is a set X of $\mathcal{A}_1$ and $A_l = X^{2K_2-1}$ of $\mathcal{A}_2$. Define $\mathcal{B}_0$ to be a family of sets $A_l \cup C_l$ with $C_l \in \mathcal{C}_l$ and $l \in [1 \dots K_3]$. The idea is that when M is large enough, closure of $\mathcal{B}_0$ under intersections with itself and under intersections with $\mathcal{A}_3$, in which every element c_i is interpreted as a whole block of $c_{i,j,k}$, is going to be dominated by $\mathcal{B}_0$ in size. But we need to adjust M and K_1 in such a way that $log|\mathcal{B}_0|/log|\mathcal{A}_3|$ is less than, but very close to one. So we first choose K_3 and K_2 much larger than $\frac{1}{m-1/2}$ and so large as not to affect the condition on the frequencies f_a and probability p_1. Next, we let M and K_1 go to infinity simultaneously, but so that $log|\mathcal{B}_0|/log|\mathcal{A}_3|$ tends to $1 - \varepsilon$ with $0 < \varepsilon \ll m - 1/2$. This is easily achieved since the size of $\mathcal{B}_0$ does not depend on K_1 and the size of $\mathcal{A}_3$ on M.

In this limit, where K_2 and K_3 are fixed with $K_3 \gg K_2$, the number of intersections of $\mathcal{B}_0$ with $\mathcal{A}_3$ is going to be less than the size of $\mathcal{B}_0$ by a factor which tends to infinity, because the contribution of elements from the total set of $\mathcal{A}_2$ is going to be less than $1/2$ and of those of $\mathcal{C}_l$ almost $1/2$ of the total logarithm of $\mathcal{B}_0$ for every one of the K_3 events ω_l. The same holds for intersection closure of $\mathcal{B}_0$, and so in this limit we are going to have a family of sets $\mathcal{B}$, closed for intersections and for intersections with $\mathcal{A}_3$, having size less than $\mathcal{A}_3$, but frequencies of occurrence over $1-f_a$ for the corresponding elements, according to the property of our probability measure. For elements $c_{i,j,k}$ the frequency will tend to $1/2(1 - 1/K_3)$ in $\mathcal{B}$. Now because we took $K_3 \gg K_2$ - a choice which matters only here - we will have that frequency of $c_{i,j,k}$, averaged for $\mathcal{A}_3$ and $\mathcal{B}$, will be over $1/2$. Then taking a parameter MAX corresponding to a convex linear combination with slightly greater contribution of $\mathcal{A}_3$ than of $\mathcal{B}$ - we need this slight difference in order to get weighting element MAX to have a frequency over one half too - we get that all the frequencies of elements in the obtained combined family are greater than $1/2$. This contradicts the Frankl conjecture, and proves the converse assertion. $\square$

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