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On the Bit Error Rate of Lightwave Systems with Optical Amplifiers

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Abstract

We revisit the problem of evaluating the performances of communication systems with optical amplifiers and a wideband optical filter. We compute the exact probability of error and the optimal threshold and compare them with those predicted by Gaussian approximations for ASK, FSK or DPSK modulations, both for ideal photodetectors and for the case where shot noise is significant.

1 Introduction

We consider the problem of receiving modulated signals that have passed through optical amplifiers. An optical amplifier with power gain G can be modeled as a linear optical field amplifier together with a source of white Gaussian noise over the bandwidth of interest. The noise has a two sided spectral density $N_0/2$, with $N_0 = N_{sp}h\nu(G-1)$ [21]. There h is Planck's constant, ν is the frequency of interest, and N_{sp} is a factor that takes various imperfections into account, its value is ideally 1.

In the case of amplitude shift keying (ASK) the signal is either 0 or an optical pulse of duration T. A possible receiver structure consists of a polarizer passing only the signal component, an optical bandpass filter of bandwidth B, a photodetector and an integrator over time T.

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In the case of frequency shift keying (FSK) there are two orthogonal signals, two filters whose frequency response (each of bandwidth B) do not overlap, two photodetectors and a circuit integrating the difference of the photodetector currents over time T. Very similar structures can be used for other binary orthogonal signals, such as in polarization modulation and in binary pulse position modulation.

In the case of differential phase shift keying (DPSK) the signal during a bit time is a replica of the signal sent during the previous bit, with phase possibly shifted by π. The receiver consists of an optical filter of bandwidth B, a Mach-Zehnder filter with differential delay T (adjusted so that νT is an integer), two photodetectors (one for each output branch of the Mach-Zehnder filter) and the same photodetectors and integrator as for FSK. Half the sum of the inputs at times t and $t - T$ appears at time t at one output of the Mach-Zehnder filter, while half the difference appears at the other output. Thus depending on whether the signal phase was shifted, the information signal will appear only at one of the outputs.

Due to the importance of optical amplifiers there have been many recent of papers analyzing such systems, often using approximations, e.g. [12], [20], [10] [17], [8], [4], [18], [6], as well as many other references cited therein.

If the amplifier spontaneous noise dominates the receiver shot and thermal noises, the performance of such an optical system will be identical to that of a radio system with square law detection. This is a well known problem in communication theory [14] and the goal of this paper is to give a self-contained account of the key exact results in a format appropriate for optical communication. We also give some exact results that can be obtained when the shot noise is considered, using the theory described in [2].

2 Analysis

The dimensionality of the space of finite energy signals with a bandwidth B and a time spread T is about $2BT + 1$ [5], [19]. For mathematical convenience we assume that $2BT + 1$ is an even integer, $2M$. For large M one can accurately write the filtered noise process at the photodetector (in each polarization) over an interval of time T as $\sum_{i=1}^{2M} n_i\phi_i(t)$. Here the ϕ_i's are orthonormal functions over the pulse interval. The n_i's are zero mean independent Gaussian random variables with variance $N_0/2$. If an information signal is present at the photodetector in the filter band we can expand it in the same basis, and write it as $\sum_{i=1}^{2M} s_i\phi_i(t)$. The energy E of the signal satisfies $E = \sum_{i=1}^{2M} s_i^2$, assuming that the signal passes undistorted through the filter. This will be the case if M is larger than 1, as already assumed. The signal to noise ratio E/N_0 can be regarded as the number of signal photons at the input of the ideal high gain optical amplifier that produces the noise.

For convenience we use units such that the photodetector has unit gain. The integral of the output of the photodetector is then given by

$$x = \int_0^T (\sum_{i=1}^{2M}(s_i + n_i)\phi_i(t))^2 dt = \sum_{i=1}^{2M}(n_i + s_i)^2 \tag{1}$$

By completing the square in the integral, it is easy to see that the Laplace Transform of the probability density x is given by

$$F_E(s) = E[\exp(-sx)] = \frac{1}{(1+sN_0)^M}\exp(-\frac{sE}{1+sN_0}),\ \Re(s) > -\frac{1}{N_0} \tag{2}$$

The subscript of the function denotes the signal energy. $F_E(\cdot)$ is the transform of a non-central Chi Square distribution with $2M$ degrees of freedom [1], [14]. Note that it does not depend on the individual s_i's, but only on the sum of their squares, which is E. This is due to the spherical symmetry of the noise.

Inverting $F_E(s)$ yields the probability density

$$f_E(x) = \frac{1}{N_0}\left(\frac{x}{E}\right)^{\frac{M-1}{2}} \exp(-\frac{x+E}{N_0}) I_{M-1}(2\frac{\sqrt{xE}}{N_0}),\ x > 0 \tag{3}$$

Here I_n denotes the nth modified Bessel function of the first kind . By differentiating $F_E(s)$ one finds that the mean of the distribution is $MN_0 + E$ and that its variance is $MN_0^2 + 2EN_0$. The two components of the variance are often denoted by "noise/noise beat" and "signal/noise beat" respectively.

If no signal is present then $E = 0$ and

$$F_0(s) = \frac{1}{(1+sN_0)^M}.$$

This is the transform of the familiar central Chi Square density with $2M$ degrees of freedom (also called Gamma or Erlang) [1], [19], [14], with probability density

$$f_0(x) = \frac{1}{N_0}\frac{(x/N_0)^{M-1}\exp(-x/N_0)}{(M-1)!},\ x > 0$$

These statistics will be used in obtaining the performances of different modulation formats.

When $M = 1$ the previous result are outside the range of validity of the model. However they then give the statistics of ideal direct detection systems with an optical matched filter, and of coherent reception systems with an IF matched filter and envelope detection.

3 ASK modulation

In the case of ASK with equally likely signals, the ON signal will have energy $2\mathcal{E}$, where $\mathcal{E}$ denotes the average signal energy. The receiver compares the output from the integrator with a threshold γ, deciding ON if the output is greater than γ. The probability that an ON is decided when an OFF is sent is given by

$$P_0 = \int_\gamma^\infty f_0(x)dx = \exp(-\frac{\gamma}{N_0})\sum_{i=0}^{M-1}\frac{1}{i!}\left(\frac{\gamma}{N_0}\right)^i \tag{4}$$

Similarly the probability that an OFF is decided when an ON is sent can be expressed in terms of the generalized Marcum Q function of order M,

$$P_1 = \int_0^\gamma f_{2\mathcal{E}}(x)dx = 1 - Q_M(\sqrt{4\mathcal{E}/N_0}, \sqrt{2\gamma/N_0})$$

[14]. $Q_M(a,b)$ is defined as $\int_b^\infty x^M/a^{M-1}\exp(-(x^2+a^2)/2)I_{M-1}(ax)\,dx$. Numerical methods to compute it as well as many references to papers about it appear in [16].

The probability of error is given by $P_e = (P_0 + P_1)/2$. The optimal threshold γ satisfies the equation $f_0(\gamma) = f_{2\mathcal{E}}(\gamma)$ or $(2\mathcal{E}\gamma)^{(M-1)/2} = N_0^{M-1}(M-1)!\exp(-2\mathcal{E}/N_0)I_{M-1}(2\sqrt{2\mathcal{E}\gamma}/N_0)$.

This can be solved iteratively, but it is as easy to directly search for the value of γ that minimizes P_e. Using the asymptotic expansion of I_M reveals that the optimal normalized threshold $(\gamma - MN_0)/2\mathcal{E}$ approaches 1/4 for large $\mathcal{E}/(N_0M^2)$.

3.1 Gaussian Approximation

Because x is a sum of $2M$ independent random variables one is tempted to assume that its probability distribution is Gaussian. If no signal is present the mean of x is MN_0 and the variance MN_0^2. If a signal is present the mean is $MN_0+2\mathcal{E}$ while the variance is $MN_0^2+4\mathcal{E}N_0$. P_0 and P_1 can be written as $Q((\gamma - MN_0)/\sqrt{MN_0^2})$ and $Q((MN_0+2\mathcal{E}-\gamma)/\sqrt{MN_0^2+4\mathcal{E}N_0})$ respectively. Here Q denotes the complementary distribution function of a zero mean, unit variance Gaussian random variable. The optimal threshold can be found numerically. Customarily [13] the threshold is set so that P_0 and P_1 are equal, or $\gamma = MN_0 + 2\mathcal{E}\sqrt{MN_0^2}/(\sqrt{MN_0^2}+\sqrt{MN_0^2+4\mathcal{E}N_0})$. We then have

$$P_e \approx Q(\frac{2\mathcal{E}/N_0}{\sqrt{M}+\sqrt{M+4\mathcal{E}/N_0}}) \tag{5}$$

3.2 Results

In figures 1 and 2 we show the exact and approximate values of P_0, P_1 and P_e as functions of the normalized threshold. We see that they do not match at all, and that the optimal thresholds are very different. Using the threshold predicted by the Gaussian approximation would result in dismal actual performances. However the exact probability of error at the optimal threshold is unexpectedly close to the approximate probability of error at its best threshold.

The figures also show that the minimum of the approximate P_e occurs close to the point where $P_0 = P_1$, validating the customary choice of threshold referred to above.

The exact P_0 and P_1 are also very close to each other when the threshold minimizes P_e. This is known for $M = 1$ [15], but many authors use a Gaussian approximation for $\sqrt{x}$ when a signal is present and incorrectly claim that P_0 dominates P_1.

In figure 3 we plot the exact and approximate values of the probability of error as functions of $\mathcal{E}/N_0$, with M as a parameter. The Gaussian approximation and the exact method agree within a factor of 5 when $P_e \simeq 10^{-9}$, confirming the coincidence just noted. Again the normalized thresholds differ widely as seen in figure 4. Some recently published approximations [8] underestimate the exact error probability, making the Gaussian approximation appear worse than it really is. They also fail to predict the optimal threshold by a wide margin.

By inverting the customary Gaussian approximation (5), the signal to noise ratio necessary to give a probability of error P_e is found to be

$$\frac{\mathcal{E}}{N_0} \approx q^2 + q\sqrt{M} \tag{6}$$

where q is such that $P_e = Q(q)$ [4]. This curve (for $P_e = 10^{-9}$, i.e. $q = 6$) is displayed in figure 5, together with the exact values. One sees that $\mathcal{E}/N_0$ values produced by the Gaussian approximation are too large by about 3 or 4 "photons", as already noticed in [17] and [18].

The curve given in [3] follows our exact answer for small M, but it is too low by about 5 "photons" for $M = 40$. Curves such as those in figure 5 but expressed in dB with respect to the value for $M = 1$ are known as "noncoherent combining loss" in the communication theory literature.

Using the value of M given by the previous formula reveals that the noise/noise beat MN_0^2 and the signal/noise beat term $4\mathcal{E}N_0$ become equal when $M = (2q(1+\sqrt{2}))^2$, or $M = 839$ for $q = 6$, corresponding to $P_e = 10^{-9}$.

As noted by [8], if the polarizer is omitted then twice as many noise modes are admitted on the photodetector. Thus the probability of error and the optimal threshold of such a system using a filter with $2BT + 1 = 2M'$ can be read on figures 3, 4 and 5 with $M = 2M'$ as the parameter value. This also holds for the other modulation formats considered below.

4 FSK and orthogonal modulations

In the case of orthogonal modulation where each signal has energy $\mathcal{E}$ one filter output will contain only noise, while the other has both signal and noise. If the integral of square of the latter output is u, an error occurs if the integral of the square of the output of the first filter is greater than u. As in (4) this event has probability $P(u) = \exp(-u/N_0)\sum_{i=0}^{M-1}(u/N_0)^i/i!$. The probability of error P_e is obtained by taking the average of $P(u)$, recalling that u has density $f_{\mathcal{E}}(\cdot)$ given in (3). One sees that the results can be expressed in terms of the first M derivatives of $F_{\mathcal{E}}(s) = E[\exp(-su)]$ given in (2), yielding

$$P_e = \sum_{i=0}^{M-1} \frac{(-1)^i}{i!} \left[\frac{d^i}{ds^i} F_{\mathcal{E}}(s/N_0)\right]_{s=1} \tag{7}$$

One checks by induction on i that

$$\frac{d^i F_{\mathcal{E}}(s)}{ds^i} = \frac{i!(-N_0)^i}{(1+sN_0)^{M+i}} \exp(\frac{-\mathcal{E}}{1+sN_0}) L_i^{M-1}(\frac{-\mathcal{E}}{N_0(1+sN_0)}) \tag{8}$$

where $L_i^D(x)$ is the generalized Laguerre polynomial given by

$$L_i^D(x) = \sum_{k=0}^{i} \binom{D+i}{D+k} \frac{(-x)^k}{k!}$$

After replacing in (7) and interchanging the order of summation

$$P_e = \frac{1}{2^M} \exp(-\frac{\mathcal{E}}{2N_0}) \sum_{k=0}^{M-1} c_k \left(\frac{\mathcal{E}}{2N_0}\right)^k \tag{9}$$

where $c_k = 1/k! \sum_{i=k}^{M-1} 2^{-i} \binom{M+i-1}{i-k}$ [7], [14].

4.1 Gaussian Approximation and results

The variance of the difference of the integrals is simply the sum of the individual variances (as the noises are statistically independent), or $2MN_0^2 + 2\mathcal{E}N_0$. The Gaussian approximation thus yields $P_e \approx Q(\mathcal{E}/\sqrt{2MN_0^2 + 2\mathcal{E}N_0})$. Values of P_e given by the exact and approximate formulas are plotted in figure 6. One sees that the Gaussian approximation is inaccurate for low M, e.g. by 2.6 dB for $M = 1$ and $P_e = 10^{-9}$.

Using the approximation, the signal to noise ratio $\mathcal{E}/N_0$ necessary to give a probability of error $P_e = Q(q)$ is

$$\mathcal{E}/N_0 \approx q^2 + q\sqrt{q^2 + 2M} \tag{10}$$

This expression as well as the exact values for $P_e = 10^{-9}$ are displayed in figure 5. One sees that FSK is only slightly worse than ASK when $M = 1$ (for $P_e = 10^{-9}$ $\mathcal{E}/N_0$ is 40 vs. 38.3). Comparing (10) with (6) reveals that ASK has a 1.5 dB average energy advantage over FSK when M is very large. The value of M where the noise/noise beat term $2MN_0^2$ is equal to the signal/noise beat $2\mathcal{E}N_0$ is given by $M = 4q^2$. When $q = 6$ this occurs when M is as small as 144.

5 DPSK modulation

In the case of DPSK it is necessary to expand the output of the broadband filter over time intervals $(-T, 0)$ and $(0, T)$ as both will influence the output of the Mach-Zehnder over time $(0, T)$. We use the same basis functions as before and denote by n'_i and s'_i the coefficients of the expansion on $(-T, 0)$ and by n''_i and s''_i those on $(0, T)$.

Taking into account the field scaling in the Mach-Zehnder filter, the integrals of the power of the two inputs will be respectively $\sum_{i=1}^{2M}((n'_i + n''_i)/2 + (s'_i + s''_i)/2)^2$ and $\sum_{i=1}^{2M}((n'_i - n''_i)/2 + (s'_i - s''_i)/2)^2$.

The correlation of n'_i and n''_j is weak as the correlation of the noise process extends only over time T/M. Assuming it is 0 [1] all the noise components $(n'_i \pm n''_i)/2$ are mutually independent and have variance $N_0/4$, half of what they are in the case of FSK. As in FSK one of the signal energies is 0 and the other is $\mathcal{E}$, depending on the data bit. We can thus conclude that DPSK has a 3 dB advantage over FSK for all values of M, ignoring error propagations. This applies to Gaussian approximations as well.

The signal to noise ratio needed by DPSK to achieve $P_e = 10^{-9}$ is displayed in figure 5 as a function of M. It is not always recognized [11] that the penalty incurred by DPSK reception with a broadband filter and an integrate and dump postdetection filter is related to the "noncoherent combining loss".

Our previous results show that for small M DPSK has about a 3 dB advantage over ASK

[1] It is 0 if the n'_i and n''_j are obtained by sampling at the Nyquist rate

(with respect to average energy), but this advantage is reduced for large M. Other comparisons with ASK and FSK appear in figure 9 and are discussed in the next section which considers the effect of the shot noise.

6 Shot Noise

In this section we consider the fact that the photodiode is not a perfect squarer. We use the semi-classical model where, conditioned on the envelope of the input field, the photodiode generates electrons following a nonhomogeneous Poisson process with rate equal to the square of the field envelope. We choose energy units such that it takes an average of one unit of energy to generate a photoelectron.

The total number y of photoelectrons generated over a bit time by the photodiode, conditioned on the x defined in (1) has a Poisson distribution with mean x. Thus the conditional Z-transform of the distribution of y given x is

$$E[z^y|x] = \exp((z-1)x),\ |z| \leq 1$$

The expectation over x of that last expression is just the Laplace transform of the Chi-Square density of x given in (2), evaluated at $s = 1 - z$:

$$G_E(z) = E[z^y] = F_E(1-z) = (\frac{1}{1+N_0(1-z)})^M \exp(\frac{E(z-1)}{1+N_0(1-z)}),\ |z| < \frac{1+N_0}{N_0}$$

Inverting this transform (by using (8)) yields a Laguerre distribution [2]

$$g_E(k) = \frac{1}{k!}\frac{d^k G_E(z)}{dz^k}\bigg|_{z=0} = (\frac{1}{1+N_0})^M (\frac{N_0}{1+N_0})^k \exp(-\frac{E}{1+N_0}) L_k^{M-1}(-\frac{E}{N_0(1+N_0)}),\ k = 0, 1, \ldots.$$

The mean of y is $MN_0 + E$, i.e. the mean of x, but the variance $MN_0(1+N_0) + E(1+2N_0)$ has increased by a quantity equal to the mean. This increase corresponds to the shot noise; it is relatively large when N_0 is small. If $E = 0$ the value of the Laguerre polynomial in $g_0(\cdot)$ is $\binom{k+M-1}{k}$ and the distribution is negative binomial.

Again evaluating the performances of various modulation formats is straightforward. For ASK the receiver should compare y with a threshold γ to make its decision. The optimal value of the threshold is the largest integer γ that satisfies the inequality $g_0(\gamma) > g_{2\mathcal{E}}(\gamma)$.

The conditional error probabilities are

$$P_1 = \sum_{i=0}^{\gamma} g_{2\mathcal{E}}(i)$$

and

$$\begin{aligned} P_0 &= \sum_{k=\gamma+1}^{\infty} (\frac{1}{1+N_0})^M (\frac{N_0}{1+N_0})^k \binom{k+M-1}{k} \\ &= (\frac{N_0}{1+N_0})^{\gamma+1} \sum_{j=0}^{M-1} \binom{\gamma+j}{\gamma} \left(\frac{1}{1+N_0}\right)^j \end{aligned} \quad (11)$$

The previous equality has a simple justification using Bernoulli trials with success probability $1/1+N_0$. The first expression is the probability that the number of failures until M successes exceeds γ. The second expression is the probability that less than M successes occur before the $\gamma+1$'st failure. The exact error probability is $P_e = (P_0 + P_1)/2$. The Gaussian approximation yields in this case

$$P_e \approx Q\left(\frac{2\mathcal{E}/N_0}{\sqrt{M(1+1/N_0)} + \sqrt{M(1+1/N_0) + 4\mathcal{E}/N_0(1+1/2N_0)}}\right)$$

The approximate $\mathcal{E}/N_0$ necessary to achieve $P_e = Q(q)$ is

$$\mathcal{E}/N_0 \approx q^2(1+\frac{1}{2N_0}) + q\sqrt{M(1+\frac{1}{N_0})}$$

The same problem has recently been considered in [6] [2], leading to conclusions similar to ours regarding the inaccuracy of the Gaussian approximation for $M = 1$.

In figure 7 we show the necessary $\mathcal{E}/N_0$ to achieve $P_e = 10^{-9}$, for various values of N_0. To interpret these curves recall that an ideal amplifier of gain G followed by an ideal photodiode would result in $N_0 = G - 1$ in our system of units. N_0 is reduced by losses between the amplifier and the photodiode (e.g. in the filters or in the photodiode). To avoid clutter, we have not

[2] That work assumes $L_k^{M-1}(0) = 1$, even though the equality only holds for $M = 1$ or $k = 0$.

displayed the values predicted by the Gaussian approximation. Their match is slightly worse than corresponding values in figure 5.

In the case of FSK an error certainly occurs if the photon count from the "noise only" filter exceeds that from the "signal plus noise" filter; a tie occurs when those two counts are equal. Let u express the count from the filter with the signal. Using (11) we express the conditional probability of error given u as

$$\left(\frac{N_0}{1+N_0}\right)^{u+1} \sum_{j=0}^{M-1} \frac{(u+1)(u+2)..(u+j)}{j!} \left(\frac{1}{1+N_0}\right)^j \left[1+\frac{1}{2N_0}\delta_{M-1-j}\right].$$

δ denotes the delta function, its presence is to take the ties into account. The average over u of the previous expression can be written in terms of the derivatives of the generating function of u, $G_{\mathcal{E}}(z)$ to yield

$$P_e = \sum_{j=0}^{M-1} \frac{1}{j!}(\frac{1}{1+N_0})^{j+1} \left[N_0 + \frac{1}{2}\delta_{M-1-j}\right] \frac{d^j}{dz^j}\left[z^j G_{\mathcal{E}}(z)\right]_{z=N_0/1+N_0}.$$

Of course

$$\frac{d^j h(x)k(x)}{dx^j} = \sum_{i=0}^{j} \binom{j}{i} \frac{d^{j-i}h(x)}{dx^{j-i}} \frac{d^i k(x)}{dx^j}$$

and as in (8)

$$\frac{d^i G_{\mathcal{E}}(z)}{dz^i} = \frac{i! N_0^i}{(1+N_0(1-z))^{M+i}} \exp(\frac{\mathcal{E}(z-1)}{1+N_0(1-z)}) L_i^{M-1}(\frac{-\mathcal{E}}{N_0(1+N_0(1-z))})$$

After replacement and interchange of summations one obtains

$$P_e = (\frac{1+N_0}{1+2N_0})^M \exp(-\frac{\mathcal{E}}{1+2N_0}) \sum_{k=0}^{M-1} c_k \left(\frac{\mathcal{E}(1+N_0)}{N_0(1+2N_0)}\right)^k$$

where

$$c_k = \frac{1}{k!}\sum_{i=k}^{M-1} \left(\frac{N_0^2}{(1+N_0)(1+2N_0)}\right)^i \binom{M+i-1}{i-k} \sum_{j=0}^{M-i-1} \left(\frac{1}{1+N_0}\right)^{j+1} \binom{i+j}{i} (N_0 + \frac{1}{2}\delta_{M-i-1-j})$$

For $M = 1$ this gives the known [2] result $P_e = .5\exp(-\mathcal{E}/(1+2N_0))$. The previous formulas reduce to (9) when N_0 gets large.

The Gaussian approximation yields

$$P_e \approx Q\left(\frac{\mathcal{E}/N_0}{\sqrt{2M(1+1/N_0) + 2\mathcal{E}/N_0(1+1/2N_0)}}\right)$$

The approximate $\mathcal{E}/N_0$ necessary to achieve $P_e = Q(q)$ is

$$\mathcal{E}/N_0 \approx q^2(1+\frac{1}{2N_0}) + q\sqrt{q^2(1+\frac{1}{2N_0})^2 + 2M(1+\frac{1}{N_0})}$$

In figure 8 we show the necessary $\mathcal{E}/N_0$ to achieve $P_e = 10^{-9}$ as a function of M, for various values of N_0. Again we have not displayed the values predicted by the Gaussian approximation, their match is slightly worse than corresponding values in figure 5.

The probability of error for DPSK is the same as for FSK, except that N_0 is reduced by a factor of 2. In figure 9 we display the ratio of the energy $\mathcal{E}$ required by DPSK to that required by ASK and also to that required by FSK, for $P_e = 10^{-9}$. One sees that DPSK rapidly loses its advantage in presence of shot noise. This is not surprising: in absence of background noise (in the "quantum limited" regime), FSK and DPSK have identical performances, but ASK is twice as efficient in terms of average energy.

7 Conclusion

We have given exact probability of error expressions for optical signals in presence of amplifier spontaneous noise and photodetector shot noise, and we have compared these results with Gaussian approximations. The quality of the approximation is questionable when it is used to evaluate the optimal threshold for ASK, or the probability of error for small values of M. Post detection receiver thermal noise is often also significant. Exact closed form results are not available for that case, although receiver statistics can be given in the transform domain. The probability of error can be found by numerical inversion, approximated by the steepest descent method, as was done recently in [9], or bounded by Chernoff's technique, as Personick [12] did two decades ago. One expects Gaussian approximations to become accurate as the thermal noise becomes more dominant. Thermal noise will also reduce the advantage of DPSK over ASK and FSK.

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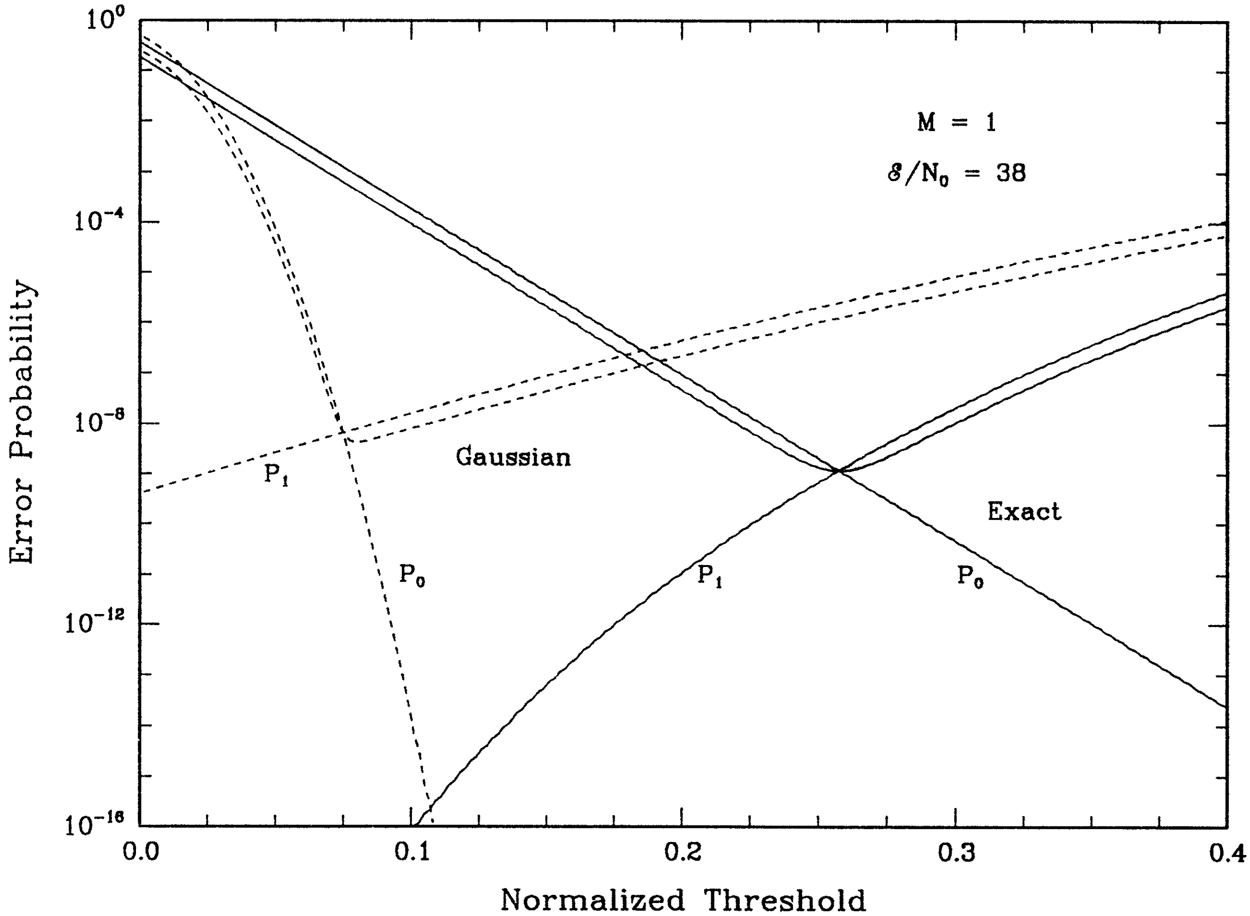

Figure 1: The exact and approximate probabilities of error P_0, P_1 and P_e as a function of the normalized threshold for $M = 1$ and $\mathcal{E}/N_0 = 38$.

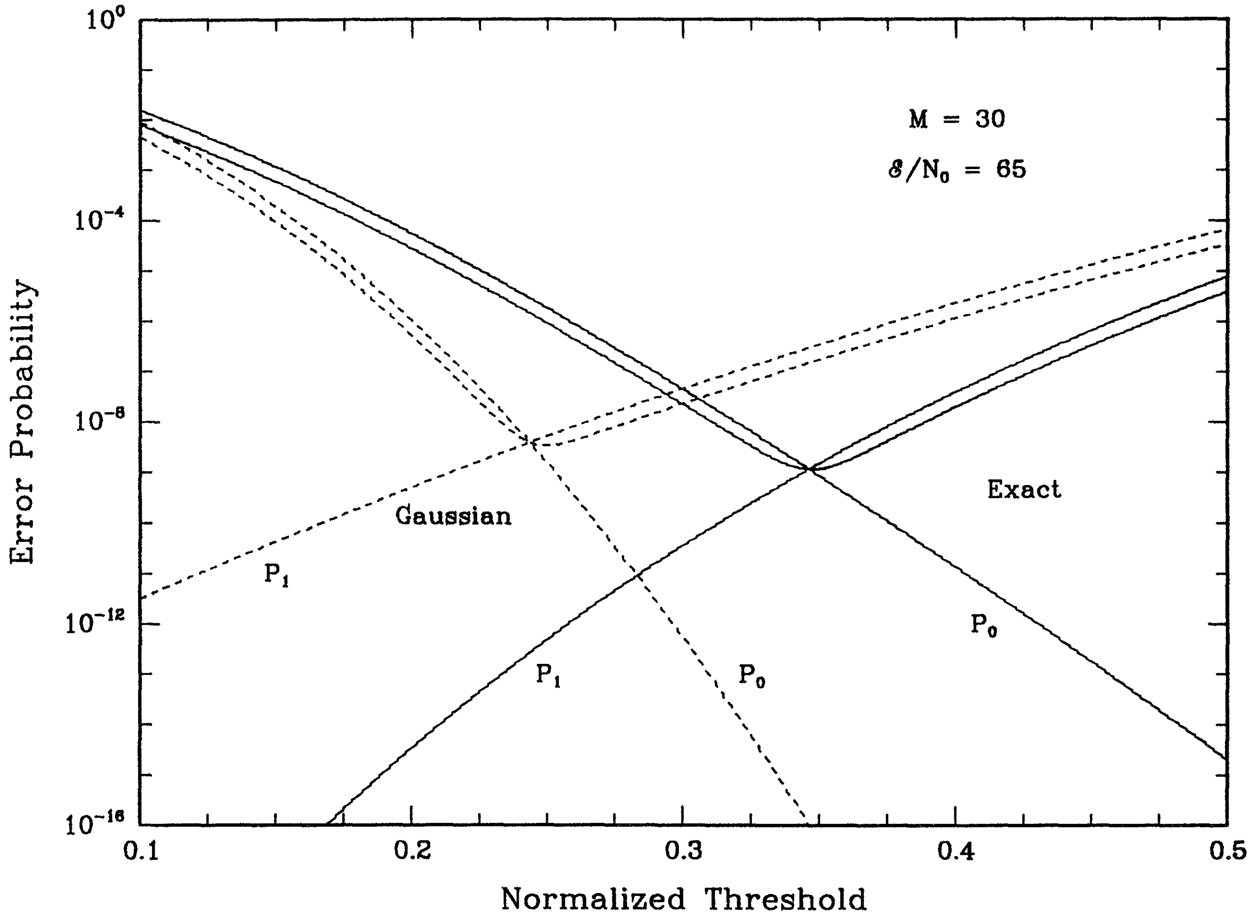

Figure 2: The exact and approximate probabilities of error P_0, P_1 and P_e as a function of the normalized threshold for $M = 30$ and $\mathcal{E}/N_0 = 65$.

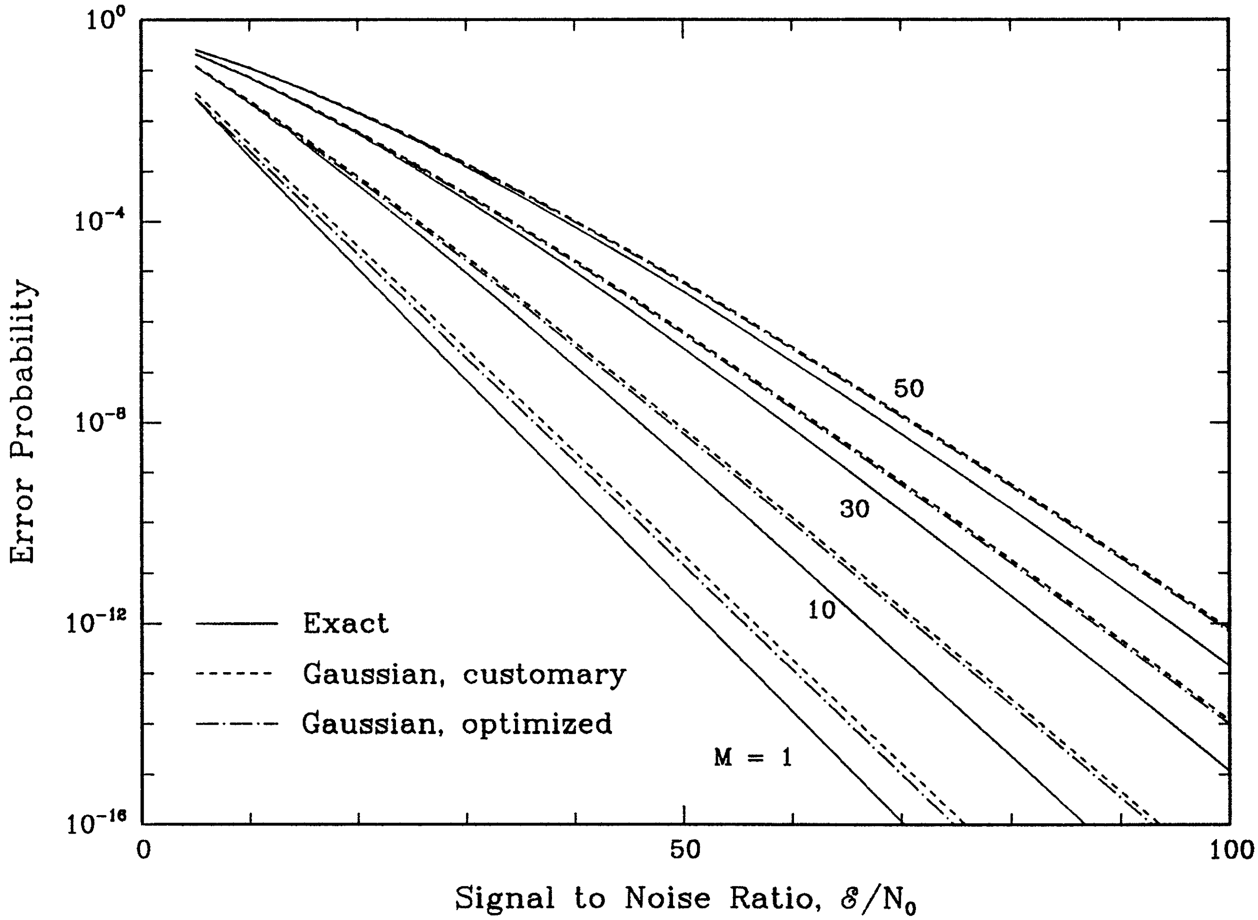

Figure 3: The probability of error for ASK as a function of signal to noise ratio $\mathcal{E}/N_0$ for various M values. For each M, the exact result as well as the Gaussian approximation with both customary and optimal threshold setting are shown.

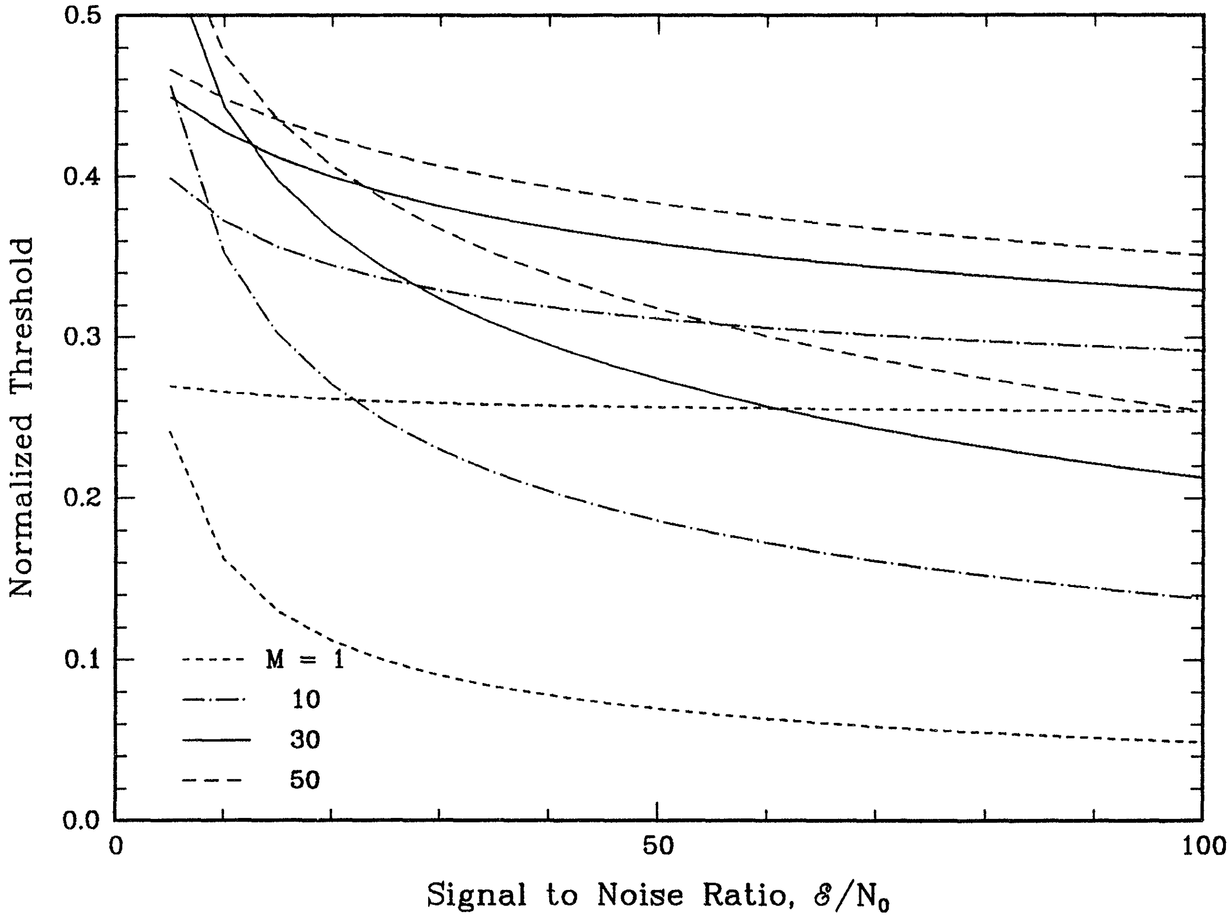

Figure 4: The optimal normalized threshold as a function of signal to noise ratio $\mathcal{E}/N_0$ for various M values. For each M, the exact result as well as the Gaussian approximation are shown, with the approximation being generally too low.

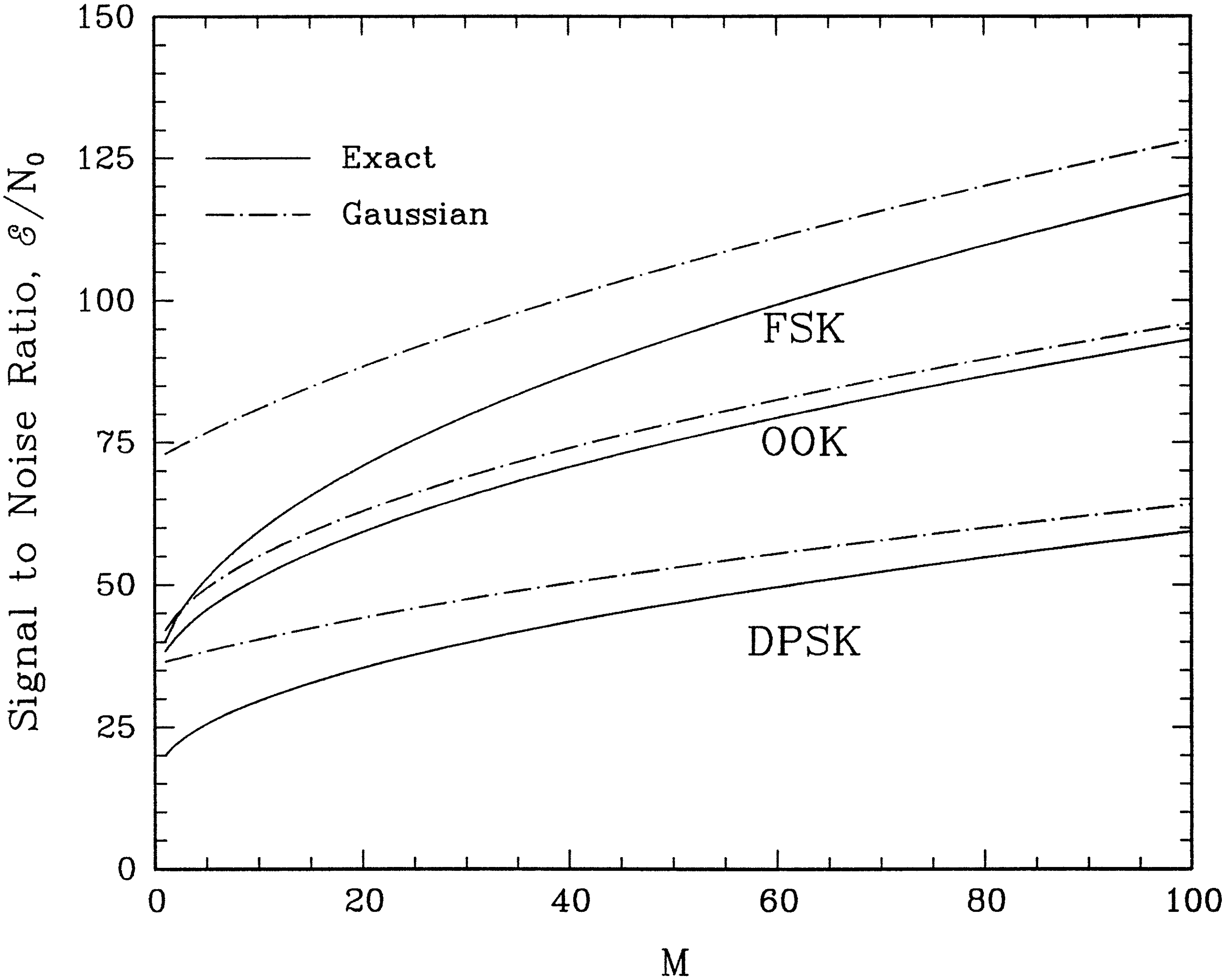

Figure 5: Required signal to noise ratio to insure a probability of error of 10^{-9}.

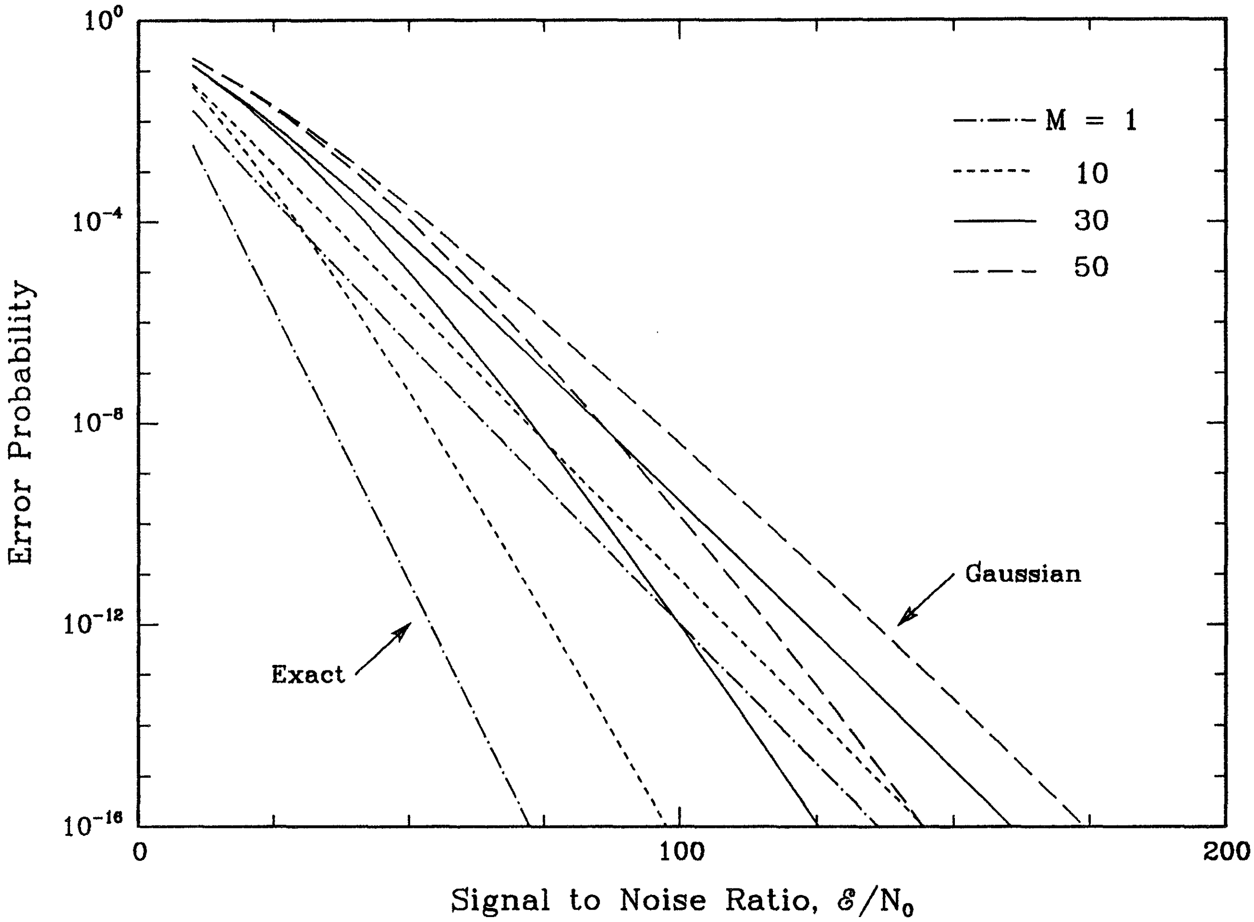

Figure 6: The probability of error for FSK as a function of signal to noise ratio $\mathcal{E}/N_0$ for various values of M. For each M, the exact result as well as the Gaussian approximation are shown.

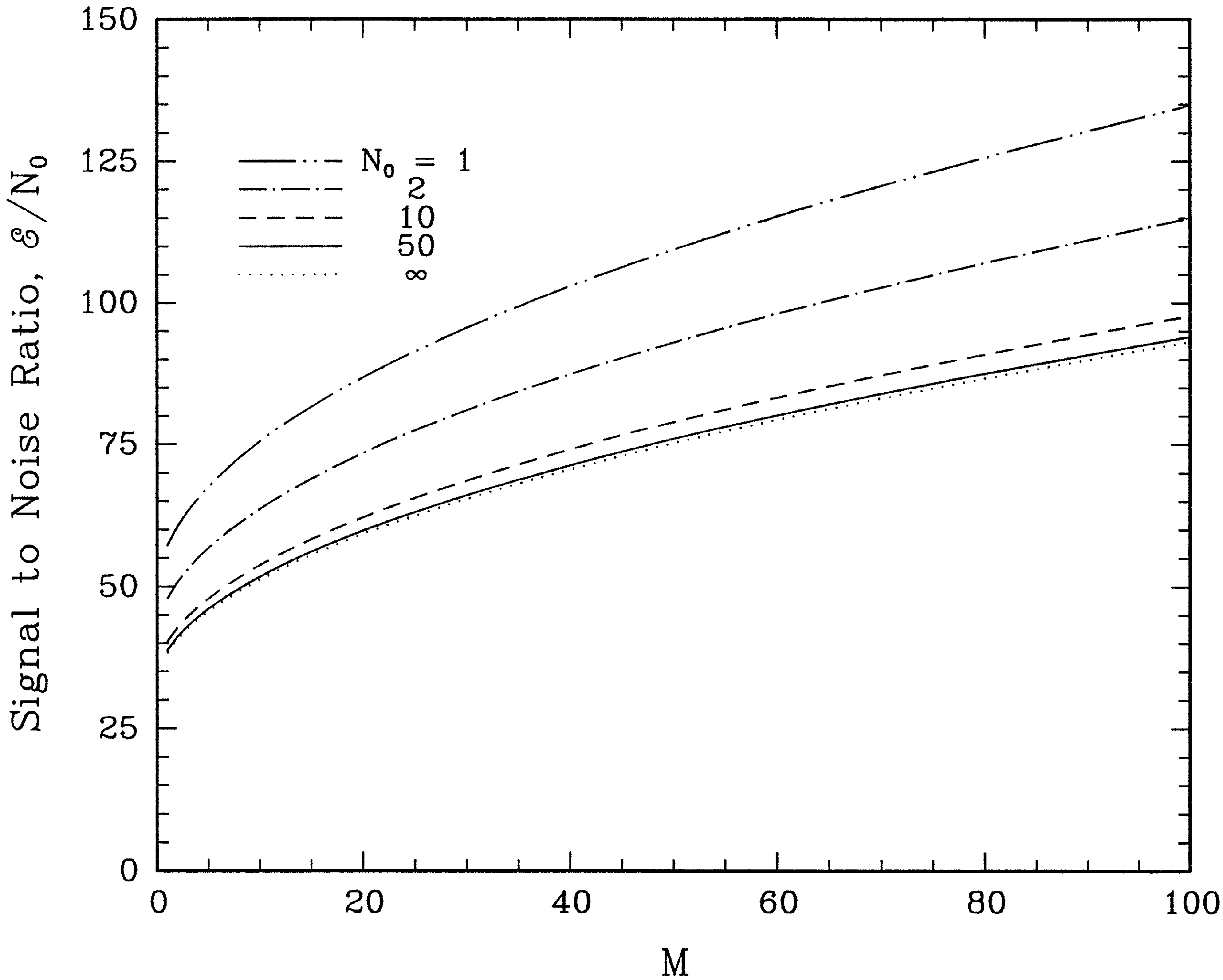

Figure 7: Required $\mathcal{E}/N_0$ for ASK to insure $P_e = 10^{-9}$ for various values of N_0.

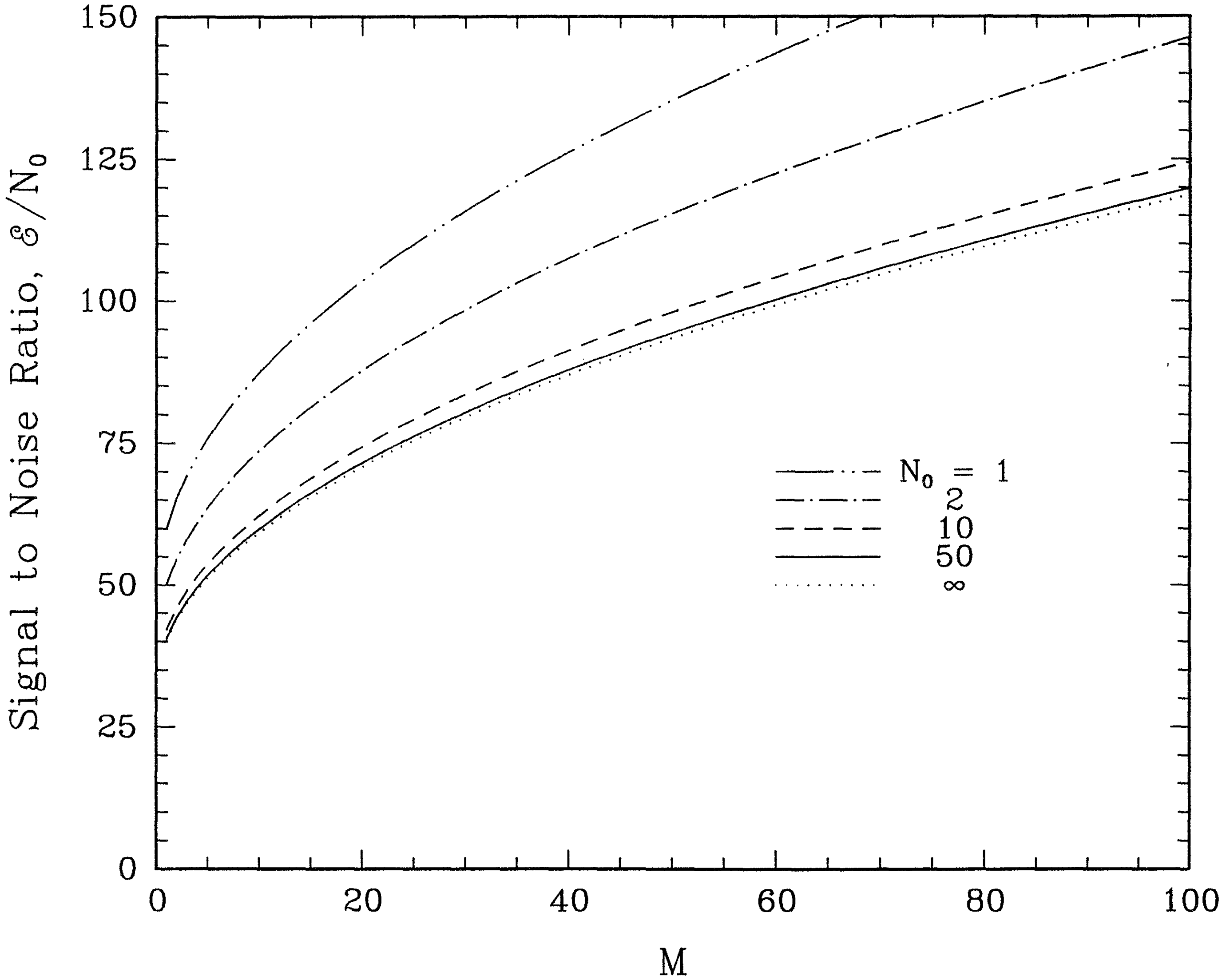

Figure 8: Required $\mathcal{E}/N_0$ for FSK to insure $P_e = 10^{-9}$ for various values of N_0.

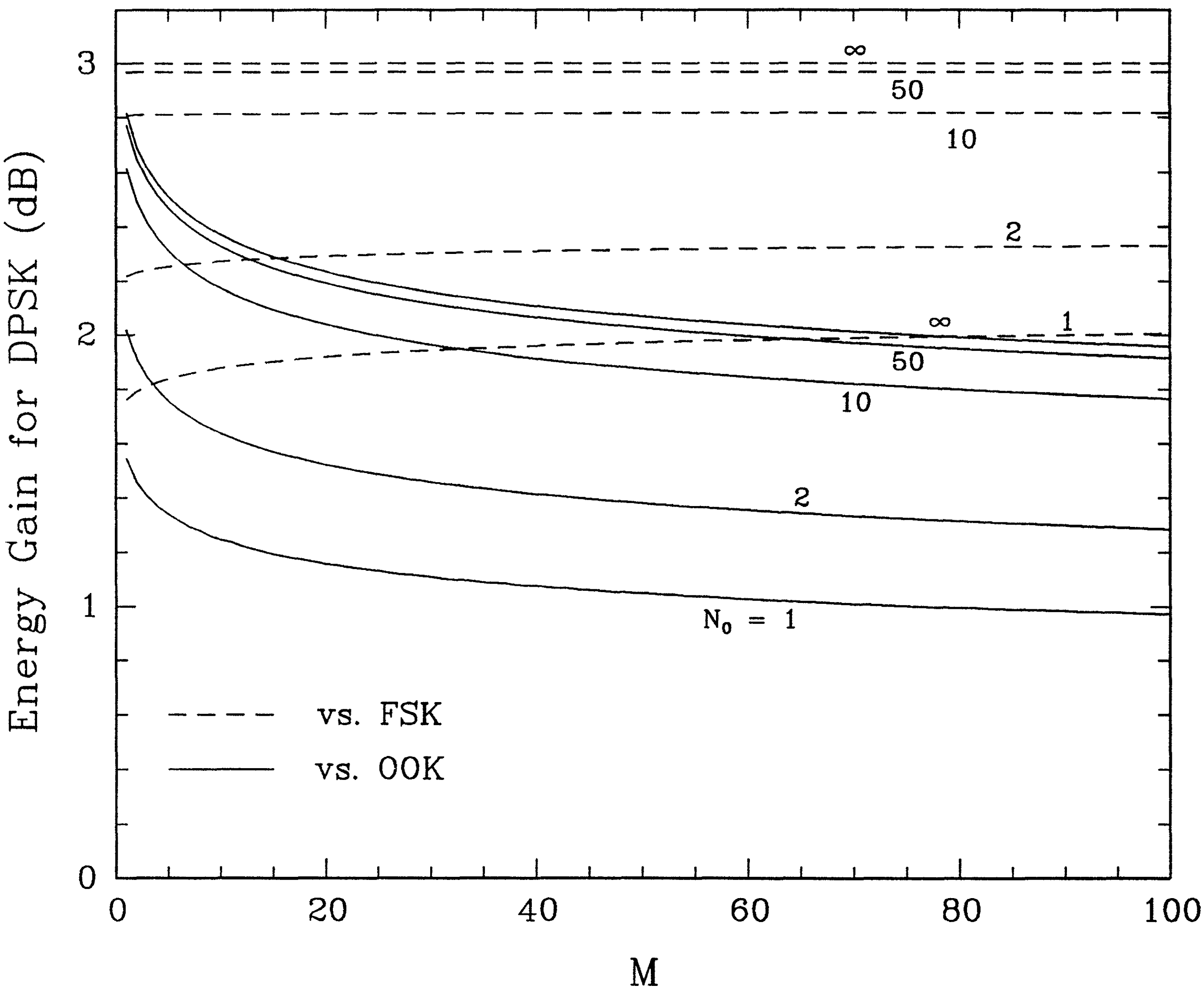

Figure 9: Energy gain of DPSK vs. ASK (solid curves) and of DPSK vs. FSK (dashed curves) to insure $P_e = 10^{-9}$ for various values of N_0, as functions of M.