Feedback Control of Separation in Unsteady Flows

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Submitted to the Department of Mechanical Engineering
in partial fulfillment of the requirements for the degree of

Master of Science in Mechanical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2005

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Abstract

Prandtl (1904) showed that in a steady flow past a bluff body, streamlines separate from the boundary where the skin friction (or wall shear) vanishes and admits a negative gradient. Despite initial suggestions, however, it was recognized that Prandtl’s zero-skin-friction criterion for separation is invalid for unsteady flows.

Employing a Lagrangian approach, Haller (2004) derived an exact kinematic theory for unsteady separation in two-dimensional flows. This theory predicts separation at points where a weighted average of the skin-friction vanishes. The weight function in this criterion depends on quantities measured along the wall, and hence can be used in an active feedback control of separation. Feedback control has been shown to lead to performance improvement in a range of aerodynamic applications, but no rigorous feedback law has been constructed for lack of a detailed understanding of separation.

In this work, we use a wall-reduced form of the vorticity-transport equation to design a feedback controller that enforces Haller’s criteria—and hence induces separation—at prescribed boundary points. We also present a stability analysis of the controller, and explore alternative control strategies for separation. We use FLUENT to validate our controller numerically on a range of flows, including steady and unsteady channel flows and backward-facing step flows.

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Acknowledgments

I would like to express my sincere gratitude to professor George Haller for being an outstanding advisor, excellent professor and a wonderful friend. His constant encouragement, support, and invaluable suggestions made this work possible. I am also grateful to my friends Amit Surana and Mustafa Sabri Kilic for valuable discussions and memorable moments we spent together. I would like to thank Professor Ahmed Ghoniem, Professor Thomas Peacock, Professor Jan Hesthaven, Dr. Weijiu Liu, Dr. Francois Lekien, Dr. Guus Jacobs, Olivier Grunberg, Yildiray Yildiz and Raul Coral for their help and valuable advice.

I dedicate this work to my younger and only brother, Mahdi, for all contributions he has made to my life. Mahdi! You have to be jealous of me; I have a very nice and devoted brother like you, while you don’t...
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Introduction

Flow separation is the detachment of fluid particles from a solid surface caused by skin friction and adverse pressure gradient presented in the free stream (Gad-el-Hak and Bushnell 1991 [7]). Although separation is an unwelcome phenomenon in many applications (say, it is blamed for Karman vortex shedding behind a circular cylinder, most of the pressure drag on airplanes, stall on turbine blades and flow-induced noise), it is found to be useful for some other applications such as improving mixing efficiency. Indeed a separated turbulent flow can greatly enhance mixing. In both cases, control is necessary to prevent adverse separation effects and optimize the use of its positive influences.

Many types of active and passive control approaches have been defined and utilized to overcome the strange behavior of separation. Among passive control devices, backward facing steps and diffusers are widely used to generate flow separation. In the design of a stainless cylindrical micro-combustor, a critical component for micro-power systems using hydrogen and hydrocarbon fuels as an energy source, Yang et al. (2002)[23] used the backward facing step to provide a simple yet effective solution to enhance the mixing of fuel mixture, prolong the residence time, control of the flame and widen the operational range of the flow rate and $H_2$/air ratio. For active separation control, blowing and suction are the most common ways of actuation. Although suction can be used to control separation as a more energy efficient device, it usually needs larger interior ducting (Gad-el-Hak, 2000[6]). Using wall jets and, in general, blowing to control separation is an interesting idea particularly for aircraft because high pressure fluid is readily provided by the engine compressors (Chang, 1976 [4]).

On the analytical side of flow control, a group of researches have focused on optimal control theory (Hinze and Kunisch 2001 [12] for example). In another series of work (Christofides
2001 [5] for example), Proper Orthogonal Decomposition is used to find an estimation of equations and to derive a control law for lower order equation. Aamo et al. [1] use a state space representation of discretized linearized channel flow to design different type of controllers for stabilization of the channel flow by feedback.

Feedback control of the point and/or strength of separation has been considered only by few researchers. Wang et al. (2003) [22] considered a shear flow model around a bluff body and used potential flow formulation and a point vortex model to derive a control law. They then showed using this model that some mixing measures improve. For viscous flows, Haller et al. (2002) [11] used Blasius boundary layer solution as a model for the background flow and Glauert wall jet solution as a model for two counteracting wall jets. They superimposed three solutions and derived a feedback control law for the combined flow to create an unsteady separation with a specified strength. They found their controller unstable for Navier-Stokes flow, and attributed this instability to a delay in transferring information from the wall jet exit point to the point of separation. This delay arises from the inertial terms in the Navier-Stokes equation. Along the same line of thought, Insperger et al. (2002) [13] assumed a known delay between jet exit area and the intended point of separation. They introduced a gain in the controller and showed that for a specific range of this gain, the controller becomes stable. This controller, incidentally, is confined to a narrow range of Reynolds numbers and is based on a known delay in the flow, which is itself a function of the wall jet velocity and varies with time.

In another approach to resolve this problem, we directly consider the incompressible Navier-Stokes (NS) equation. The two-dimensional vorticity transport equation, when reduced to the wall, is a partial differential equation that depends on three variables: \( x, y \) and \( t \). Terms including derivatives with respect to \( x \) and \( t \) can easily be measured while higher derivatives with respect to \( y \) are very difficult to sense. To overcome this issue and derive the control law based on the solution of this partial differential equation, we have used an observer that determines the term containing higher \( y \) derivatives based on the measurement at other points. Given this quantity, we discretize the solution of the partial differential equation and derive a control law. This control law based on the Navier-Stokes equation, is Reynolds-number independent and can be applied, in principle, to both laminar and turbulent flow.

In a different approach to the same control problem, we also explore model-independent con-
control strategies. Using classical (PI) controllers and nonlinear multi-input multi-output (MIMO) Fuzzy control, we derive a model-free control system for imposing the separation point. Fuzzy controller can be designed much more stably than other mathematical controllers and is independent of compressibility and Newtonian behavior of the fluid. Different examples and comparisons show the performance of each controller.

**Thesis outline**

The outline of this thesis is as follows:

In chapter one, the objective is to control the separation in a two dimensional incompressible Navier-Stokes flow. We first briefly review steady and unsteady criteria for separation. Then the problem and configuration is stated and control laws for steady and unsteady cases are derived. Numerical simulations are presented in the next section and the last section gives a derivation of our main formulae.

In chapter two, after a brief introduction highlighting the importance of model-free approaches, classical (PI) control laws are presented. We then introduce time delay and quantify this concept in our problem. The last section is an introduction to Fuzzy Logic followed by the design of fuzzy separation control.
Chapter 1

Control of Separation Point in Unsteady Incompressible Navier-Stokes Flows

1.1 Steady and Unsteady Separation Point

Steady two dimensional separation first explained by Prandtl (1904) in his famous boundary layer theory. He showed that the streamlines in a steady two dimensional flow separate from the boundary where the wall shear stress (or skin friction) vanishes and admits an adverse pressure gradient. Mathematically, if \( y = 0 \) denoted the flat boundary of a steady incompressible two dimensional velocity field \((u(x, y), v(x, y))\), then the steady separation would take place at a point \((\gamma, 0)\) if

\[
\begin{align*}
\tau_w(\gamma) &= \nu \rho u_y(\gamma, 0) = 0, \\
\tau'_w(\gamma) &= \nu \rho u_{xy}(\gamma, 0) < 0
\end{align*}
\]

where \( \tau_w \) is the wall shear along the wall, \( \nu \) is the kinematic viscosity and \( \rho \) is the density of the fluid. At the point of separation a distinguished streamline attaches to the wall. In Lagrangian point of view, this streamline acts as an unstable manifold for the flow, i.e., collects and ejects material particles from the vicinity of the separation point away from the wall. In
this sense, separation may be defined as the detachment of fluid from a solid surface (Gad and Bushnell 1991 [7] and Greenblatt 2000 [8]). Later, it was shown (see Lachmann 1961 [17]) that the separation angle is given by

$$\tan(\beta) = -3 \frac{\tau_x}{p_x} = -3 \frac{u_{xy}}{u_{yy}}. \quad (1.3)$$

In unsteady flows, instantaneous streamlines and wall shear stress don’t indicate any meaningful property of the path of the fluid particles, but in Lagrangian frame, unstable manifolds for distinguished points on the wall continue to exist. These distinguished points are unsteady separation points and the time-dependent unstable manifold associated with each of these points acts as a separation profile; it attracts and transports fluid particles from the vicinity of the wall. Kinematic theory of unsteady separation, recently presented by Haller (Haller 2004 [10]), states that the unsteady separation points defined in the Lagrangian sense are located at boundary points \((x, y) = (\gamma, 0)\), where

$$\lim_{t \to -\infty} \sup \left| \int_{t_0}^t \frac{u_y(\gamma, 0, \tau)}{\rho(\gamma, 0, \tau)} \right| < \infty \quad (1.4)$$

and

$$\lim_{t \to -\infty} \int_{t_0}^t \left[ \frac{u_{xy}(\gamma, 0, \tau) - v_{yy}(\gamma, 0, \tau)}{\rho(\gamma, 0, \tau)} - 2v_{xy}(\gamma, 0, \tau) \int_{t_0}^t \frac{u_y(\gamma, 0, s)}{\rho(\gamma, 0, s)} ds \right] d\tau = \infty \quad (1.5)$$

where \(\rho(x, y, t)\) denotes density. By dropping the time dependency, these two conditions are equivalent to Prandtl’s two criteria for steady flow separation. For \(T\)-periodic incompressible flows, the unsteady separation criteria simplify dramatically to

$$\int_0^T u_y(\gamma, 0, \tau) = 0, \quad (1.6)$$

$$\int_0^T u_{xy}(\gamma, 0, \tau) > 0. \quad (1.7)$$

It is to be noted that the second condition is as important as the first condition. For example, a flow at rest satisfies the first condition all along the boundaries without having any separation point.
It also turns out to be possible to derive equations for the derivatives of the separation profile (the unstable manifold attached to \((\gamma, 0)\)). As was shown by Haller (Haller 2004 [10]), near the wall the unsteady separation profile can be written as a Taylor expansion in the form
\[
x = y[f_0(t) + yf_1(t) + \frac{1}{2}y^2f_2(t) + ...].
\]

Introducing the quantities
\[
A(x, y, t) = \int_0^1 u_y(x, sy, t) \, ds, \quad C(x, y, t) = \int_0^1 \int_0^1 v_{yy}(x, spy, t) \, dp \, ds,
\]
and
\[
a(t) = A(\gamma, 0, t), \quad a_y(t) = A_y(\gamma, 0, t), \quad a_{yy}(t) = A_{yy}(\gamma, 0, t),
\]
\[
a_{yy}(t) = A_{yy}(\gamma, 0, t), \quad c(t) = C(\gamma, 0, t), \quad c_x(t) = C_x(\gamma, 0, t),
\]
\[
c_{xx}(t) = C_{xx}(\gamma, 0, t), \quad c_{xy}(t) = C_{xy}(\gamma, 0, t), \quad c_{yy}(t) = C_{yy}(\gamma, 0, t),
\]
the first three coefficients in the above Taylor expansion can be written, for any \(t = t_0\)
\[
f_0(t_0) = \lim_{t \to -\infty} \frac{\int_{t_0}^t [a_y(\tau) - 3c(\tau) \int_{t_0}^\tau a(s) \, ds] \, d\tau}{3 \int_{t_0}^t c(\tau) \, d\tau},
\]
\[
f_1(t_0) = \lim_{t \to -\infty} \frac{\int_{t_0}^t \left\{ a_{yy}(\tau) - 8c_y(\tau)f_0(\tau) - 4c_x(\tau)f_0^2(\tau) - 8c(\tau) \int_{t_0}^\tau [a_y(s) - 3c(s)f_0(s)] \, ds \right\} \, d\tau}{8 \int_{t_0}^t c(\tau) \, d\tau},
\]
\[
f_2(t_0) = \lim_{t \to -\infty} \left[ \frac{\int_{t_0}^t \left( \frac{1}{3}a_{yy} - \frac{1}{3}c_{xx}f_0^3 - c_{xy}f_0^2 - c_{yy}f_0 - 2c_yf_1 - 2c_xf_0f_1 \right) \, d\tau}{\int_{t_0}^t c(\tau) \, d\tau} + \frac{\int_{t_0}^t c \left( \int_{t_0}^\tau \left( 4c_xf_0^3 + 8c_yf_0 + 8c_f_1 - a_{yy} \right) \, ds \right) \, d\tau}{\int_{t_0}^t c(\tau) \, d\tau} \right].
\]

For time periodic flow the limit of external integrals changes from 0 to the period \(T\) and the limit of internal integrals changes from 0 to the time \(t_0\).
1.2 Setup

Consider a 2D unsteady incompressible flow \((u, v)\) over a concave rectangle, which is governed by the 2D Navier-Stokes equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1.14}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{1.15}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.16}
\]

In the above equation, \(u\) and \(v\) denote the velocity components of flow in the \(x\) and \(y\) directions respectively, \(p\) denotes the pressure, \(\rho\) the density and \(\nu\) the kinematic viscosity.

In our configuration, we use two counteracting wall jets located at the two ends of the an interval along which separation is desired (see Figure 3-1). There is no restriction on the velocity profile or the temporal behavior of the background flow. The \(y\) direction is measured upward from the wall and \(x\) coordinate is along the wall. The wall jet flow is provided either by ducting flow from other parts of the system (that can be an airplane body or combustion chamber) or by active flow generators.

No slip boundary condition at the wall implies

\[
J_1(y, t) = y j_1(t) + y^2 F_1(y, t) + O(y^3) \quad 0 \leq y \leq \delta \tag{1.17}
\]

\[
J_2(y, t) = y j_2(t) + y^2 F_2(y, t) + O(y^3) \quad 0 \leq y \leq \delta \tag{1.18}
\]

where \(J_1\) and \(J_2\) are velocity profile at \(x = 0\) and \(x = 1\) respectively, \(j_1\) and \(j_2\) are \(y\)-derivative of velocity at the wall, which are corresponded to the wall shear stress for Newtonian fluids. We call \(j_1\) and \(j_2\) the strength of the wall jets. \(\delta\) is a small positive constant determined by the jet parameters. The strengths \((j_1\) and \(j_2\)) depend on the flow velocity profile, its pressure, velocity derivatives and the designated separation location \(x = \gamma\). The problem is to design a
feedback control law that in general is in the form

\[
\begin{align*}
    j_1 &= j_1 \left( t; \gamma, u, v, p, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial p}{\partial x}, \ldots \right) \\
    j_2 &= j_2 \left( t; \gamma, u, v, p, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial p}{\partial x}, \ldots \right)
\end{align*}
\]  

(1.19) \quad (1.20)

such that the separation occurs at the designated location. With wall jets controller, from the partial differential equation point of view, the NS-equation is complemented by the following boundary conditions

\[
\begin{align*}
    u(0, y, t) &= y_j \left( t; \gamma, u, v, p, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial p}{\partial x}, \ldots \right) + y^2 F_1(y, t) \quad 0 \leq y \leq \delta, \\
    u(0, y, t) &= U_{in}(y, t) \quad \delta < y \leq R, \\
    u(1, y, t) &= y_j \left( t; \gamma, u, v, p, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial p}{\partial x}, \ldots \right) + y^2 F_2(y, t) \quad 0 \leq y \leq \delta, \\
    u(x, 0, t) &= v(x, 0, t) = 0, \quad 0 \leq x \leq 1, \\
    u(x, y, t) &= U_{\infty}(x) \quad y \to \infty, \\
    u(x, y, 0) &= u_0(x, y).
\end{align*}
\]  

(1.21) \quad (1.22) \quad (1.23) \quad (1.24) \quad (1.25) \quad (1.26)

In the above equations, \( u_0(x, y) \) is the initial velocity profile and \( U_{\infty}(y) \) and \( U_{\infty} \) are inflow (or background flow) and outer flow respectively.

### 1.3 Feedback Control Laws

#### 1.3.1 Introduction

Using the skin friction equation (see proof of Theorem 1)

\[
\begin{align*}
    \frac{\partial u_y}{\partial t} &= 2\nu \frac{\partial^2 u_y}{\partial x^2} + \nu \frac{\partial^2 u_y}{\partial y^2}, \\
    u_y(0, t) &= j_1(t), u_y(1, t) = j_2(t), \\
    u_y(x, 0) &= u_y(x).
\end{align*}
\]  

(1.27) \quad (1.28) \quad (1.29)

we derive a feedback control law to impose a separation point at a specific location on a no slip boundary. The first theorem presented in this section (Theorem 1) is a feedback control law for
imposing first condition of Prandtl's steady separation criteria. It leaves one degree of freedom, (i.e., one arbitrary constant) which can be used to optimize any secondary objective function for the separation problem. Theorem 4, is a special case of first controller that satisfies second condition of Prandtl's criteria as well. Third Controller (Theorem 5) is to impose the strong necessary condition of unsteady separation and the fourth controller (Theorem 6) is a special case of Theorem 5 that satisfies both necessary and sufficient conditions of unsteady separation.

1.3.2 Observer

The basic idea behind the model-based controller design is the use of skin friction equation and consider the third term as a given function. Replacing the third term in the partial differential equation by a known function of time and location, the equation reduces to a one dimensional heat equation with external forcing function. The solution of this equation can be represented by different methods including use of Green's function or by expansion in Fourier's series. In our case, the problem of control is converted to design of suitable boundary conditions for the heat equation to achieve a desired temperature at a specified point. The main question is how to estimate or measure the third term. Is it negligible or not?

If we use the traditional idea of similarity solution to compare the order of magnitude of terms in the skin friction equation, we easily drop the first term in the right hand side of the equation compared with the second term, because within the boundary layer, the third derivative of the x-component of the velocity with respect to $y$ is of the order $\delta^{-3}$ where $\delta$ is the thickness of boundary layer which is much smaller than the other variables. At the same time, the first term on the right-hand side and the time derivative on the left hand side is of order $\delta^{-1}$. Therefore, if we assume the Reynolds number, that we are interested in, is in the limit for which boundary layer assumption is valid, the third derivative of $u$ with respect to $y$ is not only important but also dominant. If we insert Blasius boundary layer solution to the skin friction equation, the third derivative of velocity with respect to $y$ equals to zero. We expect this result, because Blasius solution is derived for steady flow and it has already neglected the derivative with respect to $x$ compared with the derivatives with respect to $y$. The left hand side of skin friction equation for steady flow is zero and so the $u_{yyy}$ must be zero. But does Blasius equation satisfy the steady skin friction equation? The answer is NO!
The Blasius solution inserted into skin friction equation gives \( u_{xxy} = 0 \), or in terms of Blasius non-dimensional function \( f''(\eta = 0) = 0 \) where \( \eta = y \sqrt{\frac{u}{v}} \) is the independent dimensionless variable. The only solution to this equation is \( f \equiv 0 \) near the wall which is a trivial solution and is not the case even in steady flow field. To summarize, the Blasius similarity solution is not a good approximation for higher derivatives of Navier-Stokes equation. Another problem with similarity solution is its restriction for a uniform flow, and as we know near separation point the uniformity doesn’t exist.

In a feedback control system measuring some quantities is inescapable. One may think of directly measuring the term \( u_{yyy} \) on the wall and interpolating the data to get it as a function of \( x \) and \( t \) along the wall and then insert it into the solution of the heat equation. Although, mathematically speaking, this method works, it is not a practical scheme. Measuring first derivative of \( u \) with respect to \( y \) on the wall is doable. In Newtonian fluids \( u_y \) is proportional to the wall shear stress and there are variety of methods to directly or indirectly measure the skin friction. Shear stress sensors are now fabricated in sizes of \( 200 \times 200 \mu m^2 \) and can be integrated in flexible skins and glued to curved surfaces (Aamo et al [1]). But measurement of higher vertical derivatives is inaccurate. To measure higher vertical derivatives very small sensors must be installed perpendicular to the wall and out of its plane (to measure \( u \) at different levels). They themselves disturb the flow field and, furthermore, there are always limitations in ‘how close to the wall we can install our sensors’ that affects accuracy.

We finally thought of indirect measurement of \( u_{yyy} \). We know that if the flow obeys the full Navier-Stokes equation, it should obey its derivatives and all independent equations extracted from it. The skin friction equation is extracted from three independent equations: Navier-Stokes equation in \( x \) direction, Navier-Stokes equation in \( y \) direction, and continuity equation. We can easily measure wall shear stress along the wall, so we have it in hand as a function of \( x \), also we can measure it in time, so we can compute all \( x \) and \( t \) derivatives of skin friction (or \( u_y \)). Now we use the skin friction equation to get the third \( y \) derivative of \( u \) by one measurement and two estimations. Looking again at the equation (1.27), we can estimate the left-hand side just by measuring \( u_y \) in time, and the first term in right-hand side having the same measurement along the wall. Therefore, the basic idea for our model-based feedback control law is to consider the skin friction equation as a heat equation with an external forcing function provided by
measurements. We discretize the solution of the heat equation and imply our condition to get the control law.

1.3.3 Steady Separation Control

In this section, we present our main results on steady separation control. By steady separation control, we mean the objective function is to satisfy Prandtl’s criteria for steady separation, though the period for reaching to this state is an unsteady period.

Theorem 1. Necessary Condition for Steady Separation

Assume the 2D unsteady flow field \((u, v) = (u(x, y, t), v(x, y, t))\) satisfies NS-equation and boundary conditions given by

\[
\begin{align*}
    u(0, y, t) &= y j_1(t) + O_1(y^2) \\
    u(1, y, t) &= -y j_2(t) + O_2(y^2)
\end{align*}
\]

Then the feedback control law

\[
\begin{align*}
    j_1(t_{m+1}) &= \frac{A(t_m) + B(t_m) + C(t_m)}{D(t_m)} \\
    j_2(t_{m+1}) &= \alpha j_1(t_{m+1})
\end{align*}
\]

where

\[
\begin{align*}
    A(t_m) &= \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m) e^{-\xi \delta t} \sin n\pi y \\
    B(t_m) &= \sqrt{2} \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{\xi} f_n(t_m) \sin n\pi y \\
    C(t_m) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\xi \delta t} (J_1(t_m) + \cos(n\pi) J_2(t_m)) \sin n\pi y \\
    D(t_m) &= 4\nu \pi \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{\xi} (1 + \alpha \cos(n\pi)) \sin n\pi y
\end{align*}
\]
in which

\begin{align*}
  a_n(t_m) &= \sqrt{2} \int_{0}^{1} \psi_0(x, t_m) \sin(n \pi x) dx \\
  \psi_0(x, t_m) &= u_y(x, t_m) - (1 - x) j_1(t_m) + x j_2(t_m) \\
  f_n(t_m) &= \sqrt{2} \int_{0}^{1} \nu \frac{\partial^2 u_y}{\partial y^2} \sin(n \pi x) dx \\
  \xi &= 2 \nu n^2 \pi^2 
\end{align*}

vanishes \( u_y \) at \( x = \gamma \).

Behavior of the controller in the transient and steady state periods highly depends on the external forcing term of the equation. As was discussed earlier, the third derivative of velocity with respect to normal direction can be estimated from measurements. This is the external term of the equation and controller responds based on the evolution of this term in time and space. The behavior of this term, itself, depends on the variation of the background flow field and two wall jets. We know that for each type of the time dependency, the order of a typical controller must be modified to adjust to the rate of compensation. For example, for the type one systems (i.e., the open loop transfer function of the system has one pole at the origin) the closed loop response to the step input has no steady state error while for a ramp input there remains a constant steady state error and for accelerating input the steady state error grows as time grows (see Ogata [19]). In our case, we should have an estimation of the behavior of the external term in hand. Steady state error has not been well formulated in linear PDE control systems. But one - intuitively - can expect the same behavior by adding integrators and differentiators to the control loop. For example for the heat equation with a constant external forcing, at least one integrator is needed to suppress the steady state error, while if the input is a ramp, we need at least two integrators to suppress the steady state error. The final form of the controller with an integrator is

\begin{align*}
  j_1(t_{m+1}) &= \frac{IEr(t_m) - A(t_m) - B(t_m) - C(t_m)}{D(t_m)} \\
  j_2(t_{m+1}) &= \alpha j_1(t_{m+1}),
\end{align*}

(1.34) (1.35)
where \( IEr(t_m) \) is the total error of the integrator(s).

The time step have to be chosen carefully. Although theoretically for a small enough time step the system is under control, practically this is not the case. If the time step is chosen to be very small, the controller tries to suppress the error in the next time step (which is a very short interval) and as a result it requires a very high effort to be applied to the system in a short period. The high effort, although pushes the system toward the set point, does not guarantee convergence of the error. As a result an overshoot may remain at the end of the time step which may easily grow during time. Big time step, on the other hand, reduces the settling time of the control system. Also because there is an external term in the equation, the controller can not work effective with inadequate data sampling rate and it may go easily unstable.

In the above controller \( \alpha \) is an arbitrarily constant. Physically speaking by \( \alpha = C \) we mean that we want to impose the separation at a specific point \( x = \gamma \) while the ratio of the strength of two controllers is a constant number \( C \). It is to be noted that this controller only imposes the necessary condition for steady separation. For example consider zero background flow. One possible output of the controller is \( j_1 = 0 \) and \( j_2 = \alpha j_1 = 0 \) and there would be no separation.

Since our controller is a model-based controller and there is no limitation on the direction and sign of the actuation, the sign of jet outlet velocity may change. For example when you set \( \alpha \) is such a way that the strength of stream-wise jet is many times higher than the strength of counteracting jet, and the desired separation point is somewhere near the high strength jet, the controller set the outputs to two negative numbers in order to impose the zero skin friction at the intended point. As a result, both wall jets have to start sucking the fluid; a zero skin friction point will be made at the desired point, but instead of separation it is a reattachment point!

Parameter \( \alpha \) is left for applications in which one additional parameter must be optimized or specified. This additional constraint can be the settling time, percentage overshoot, any other transient specifications or an optimal objective function of the flow behavior.

Although this controller has been designed to induce separation at a desired point, there is no guarantee that this separation point is unique. Controller may create many separation and reattachment points along the wall, amongst them one is at \( x = \gamma \). Although existence of further separation points can be determined by analyzing skin friction which has already
measured, to avoid this phenomena more modification is needed.

In application, the strength of jets may become negative - as we discussed earlier - and very big number which may not be desirable. In these cases putting bounds on the controller is unavoidable, and as a result a bang-bang analysis is necessary to prove the stability and convergence of the controller (Kirk 1970 [15]).

To show the different possible responses of the controller we have simulated a couple of cases with a bench mark equation: heat equation with external forcing function. As we stated before, to derive the controller we have considered the skin friction equation as a heat equation with external forcing term. In general this external forcing function can be any arbitrary function of time and space. In this simulation we choose \( F(x, t) = t + u_{xy} \), which physically means a time growing background flow and a complicated \( x \)-dependent term that comes from the behavior of the main part of the equation. Figure (3-2) shows the steady state distribution of the wall shear stress for the desired separation point \( \gamma = 0.3 \). In Fig.3-2(a) the coefficient \( \alpha = 1 \) means that two jets act in different directions: if one blows, the other blows as well (which means the output velocities of two jets are in opposite directions). In this case the wall shear before the separation point is negative and after that is positive, so it is a sharp 'reattachment' point at \( \gamma = 0.3 \). In (b) output of both jets are in the same directions, one blows and the other sucks, the result is one separation point in the right and one reattachment point in the left. Figures (c) and (d) are cases with \( \alpha \) two times and half of the two previous cases.

To force the system to behave smoother, we introduce a damping coefficient in front of our control law:

\[
\begin{align*}
    j_1(t_{m+1}) &= \zeta \frac{E_r(t_m) - A(t_m) - B(t_m) - C(t_m)}{D(t_m)} \\
    j_2(t_{m+1}) &= \alpha j_1(t_{m+1})
\end{align*}
\]

greater \( \zeta \) means faster response, sharp acceleration and instantaneous high effort. For big damping ratio (specially greater than one), there is the danger of instability. On the other hand, lower damping coefficient means slower moving toward set point, continuous behavior and more stability.

In Theorem (1) we derived the strengths of wall jets based on the discretization assumption.
which implies that the measurement can only be done in discrete time intervals. We assumed that the external forcing function does not vary much during one time step, and set the jet strength to a constant value during the time step. But in reality external forcing function keeps varying and causes an error which is not zero at the end of a time step. We also discussed that a damping coefficient usually is placed in front of the controller to push the system toward set point smoothly. This damping coefficient itself prevent system to reach to a zero error at one time step. These two effects leave an error at the end of a typical time step. Theorem (2) considers the behavior of error in time and gives the condition for external forcing function under which the controller stabilizes the flow.

**Theorem 2. Stability**

Having $k$ integrator with integration gain $I$ in the feedback rout, necessary condition for stability of the controller (Theorem 1) by a damping ratio $\lambda$ requires that

$$P = \frac{\sqrt{2}(z-1)^{k+1}}{(z-1)^k(z+1) + IT} \left[ \int_0^T e^{-\xi(T-u)} f_n(u + mT)du - \lambda B f_n(mT) \right]$$

(1.37)

$$B = \left( \frac{1 - e^{-\xi T}}{\xi} \right) = \text{Const.} , \quad A = (1 + \lambda)e^{-\xi T}$$

to have no pole outside the unit circle and

$$\lim_{m \to \infty} \epsilon_{m,n} = \lim_{z \to 1} P < \infty.$$  

(1.38)

**Theorem 3. Maximum time step**

The range of time step for the control system (Theorem 1) under which controller momentarily decreases the error is

$$0 < \delta t < \frac{-2f'(t^m)}{f''(t^m)}$$

(1.39)

where $t^m$ is the current time.

We can derive a weaker yet more sensible stability criteria for this controller. We consider the controller stable if it indeed generates separation at the point $\gamma$, i.e., the time-integral of $\phi(\gamma, t)$ remains uniformly bounded. In other words we want the integral of the error of the controller to remains uniformly bounded. Regardless of the particular choice of $j_1(t^m)$ and
\[ j_2(t_m), \text{ the controller with damping ratio } \lambda \text{ gives the actual skin friction value} \]

\[ \phi(\gamma, t_m + \delta t) = (1 - \lambda) \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m)e^{-\xi n \delta t} \sin(n\pi \gamma) \]

\[ + (1 - \lambda) 2 \int_{t_m}^{t_m+\delta t} \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi n \delta t} [J_1(t_m) + \cos(n\pi)J_2(t_m)] \sin(n\pi \gamma) ds \]

\[ + \sqrt{2} \int_{t_m}^{t_m+\delta t} \sum_{n=1}^{\infty} e^{-\xi (t_m+\delta t-s)} f_n(s) \sin(n\pi \gamma) ds \]

\[ - \lambda \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) f_n(t_m) \sin(n\pi \gamma) = \varepsilon(t_{m+1}) = \varepsilon_{m+1} \]

at the intended point of separation at time \( t_{m+1} \).

Suppose that \( \lambda = 1 \), i.e., we select \( j_1(t) \) and \( j_2(t) \) to make \( \phi^*(\gamma, t_m + \delta t) \) exactly zero where \( \phi^* \) is the estimation of \( \phi \) at the end of the time step. Now we have

\[ \phi(\gamma, t_m) = \sum_{n=1}^{\infty} \left\{ \int_{t_m}^{t_m+1} e^{-\xi n (t_{m+1} - s)} \left[ f_n(s) - f_n^*(s) \right] ds \right\} \sin n\pi \gamma. \]

To obtain a separation point at \( x = \gamma \), we must have

\[ \lim \sup_{m \to \infty} \left| \int_{t_0}^{t_{m+1}} \phi(\gamma, \tau) d\tau \right| < \infty. \]

Note that

\[ \lim \sup_{m \to \infty} \left| \sum_{n=1}^{\infty} \int_{t_m}^{t_{m+1}} e^{-\xi n (t_{m+1} - s)} \left[ f_n(s) - f_n^*(s) \right] ds \right| \sin n\pi \gamma \]

\[ \leq \lim \sup_{m \to \infty} \sum_{n=1}^{\infty} \int_{t_m}^{t_{m+1}} e^{-\xi n (t_{m+1} - s)} \left[ f_n(s) - f_n^*(s) \right] ds \]

\[ \leq \lim \sup_{m \to \infty} \sum_{n=1}^{\infty} \left\{ \max_{t \in [t_m, t_{m+1}]} \left| \dot{f}_n(t) \right| \frac{\Delta (1 - e^{-\xi n \Delta})}{\xi_n} \right\} \]

\[ = \sum_{n=1}^{\infty} \lim \sup_{m \to \infty} \left| \dot{f}_n(t) \right| \frac{\Delta (1 - e^{-\xi n \Delta})}{\xi_n} \]

\[ \leq \sum_{n=1}^{\infty} \lim \sup_{m \to \infty} \left| \dot{f}_n(t) \right| \frac{\Delta L^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{\lim \sup_{m \to \infty} \left| \dot{f}_n(t) \right|}{n^2}. \]

If \( F_t(x,t) \) is of class \( C^0 \) in \( x \), then for any fixed \( t \), its Fourier coefficients are of the order
\[ \left| f_n(t) \right| = O(|n|^{-r}), \text{ i.e.,} \]
\[ \left| \dot{f}_n(t) \right| \leq C(t)n^{-r}, \]

where \( C(t) \) is a positive function of \( t \). In that case, we obtain

\[
\limsup_{m \to \infty} \left| \sum_{n=1}^{\infty} \left\{ \int_{t_m}^{t_{m+1}} e^{-\xi_n(t_{m+1}-s)} \left[ f_n(s) - f_n^*(s) \right] \sin n\pi \gamma \right\} ds \right| \leq \frac{C(t)\Delta L^2}{2\nu\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^r \gamma^2} < \infty,
\]

if \( r \geq 0 \) and \( C(t) \) is uniformly bounded in \( t \). Note that for any fixed \( n \), \( \left| f_n(t) \right| \) remains uniformly bounded in \( t \) if \( F_t(x, t) \) remains uniformly bounded in \( t \). We conclude:

The controller (1) is stable (i.e., creates skin friction with bounded asymptotic average at \( x = \gamma \)) if \( \partial_x u_{yyt}(x, 0, t) \) is continuous in \( x \) and uniformly bounded in \( t \) over the \([0, L]\) section of the boundary.

Note, however, that we do not know the above properties of \( u_{yyt}(x, 0, t) \) a priori for the closed-loop system. We may impose bounds on the actuators to prevent any large-scale instability, but doing so we alter the controller that we have studied.

As a special case of the first controller, we use the sufficient condition for steady separation as a restriction equation. In this case \( j_1 \) and \( j_2 \) are determined precisely.

**Theorem 4. Sufficient Condition for Steady Separation**

assume the 2D unsteady flow field \( (u, v) = (u(x, y, t), v(x, y, t)) \) satisfies NS-equation and boundary conditions given by

\[
\begin{align*}
  u(0, y, t) &= yj_1(t) + O_1(y^2) \quad (1.41) \\
  u(1, y, t) &= -yj_2(t) + O_2(y^2) \quad (1.42)
\end{align*}
\]

then the feedback control law

\[
\begin{align*}
  j_1(t_{m+1}) &= \frac{A_1^*C_1^*(t_m) - A_2^*C_1^*(t_m)}{A_1^*B_2^* - A_2^*B_1^*} \\
  j_2(t_{m+1}) &= \frac{C_1^*(t_m)B_2^* - C_2^*(t_m)B_1^*}{A_1^*B_2^* - A_2^*B_1^*}
\end{align*}
\]
where

\[ A_1^*(t_m) = \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{\xi} n^2 \cos{n\pi \gamma} \]

\[ B_1^*(t_m) = \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{\xi} n^2 \cos{(n\pi) \cos{n\pi \gamma}} \]

\[ A_2^*(t_m) = \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{\xi} n \sin{n\pi \gamma} \]

\[ B_2^*(t_m) = \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{\xi} n \cos{(n\pi) \sin{n\pi \gamma}} \]

\[ C_1^*(t_m) = \frac{\beta - A'(t_m) - B'(t_m) - C'(t_m)}{4\nu \pi^2} \]

\[ C_2^*(t_m) = -\frac{A(t_m) + B(t_m) + C(t_m)}{4\nu \pi} \]

\[ A'(t_m) = \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n(t_m) e^{-\xi \delta t} \cos{n\pi \gamma} \]

\[ B'(t_m) = \sqrt{2} \sum_{n=1}^{\infty} n\pi \frac{1 - e^{-\xi \delta t}}{\xi} f_n(t_m) \cos{n\pi \gamma} \]

\[ C'(t_m) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\xi \delta t} \left( j_1(t_m) + \cos(n\pi)j_2(t_m) \right) \cos{n\pi \gamma} \]

where \( A(t_m), B(t_m), C(t_m) \) and \( D(t_m) \) are given by Theorem (1), imposes \( u_y(\gamma, 0, t) = 0 \) and \( u_{xy}(\gamma, 0, t) = \beta \).

Theoretically \( \beta \) can be any positive or negative number, but practically, the range of attainable \( \beta \) is a complicated function of background flow. Assume a hypothetical situation in which there is no background flow and \( \gamma \) is chosen to be at the middle of two wall jets. We expect two wall jets to blow with the same strength to create a separation point at the mid point of the wall, and we know by any other blowing ratio the separation point can not be at that point (in the steady state case). As a result the separation angle is a right angle and \( u_{xy} = \infty \). It is the only amount of \( u_{xy} \) that can be created in this configuration. For the points off the midpoint the situation is better; after reaching a separation point by minimum required strengths, the angle can be increased by increasing the strength of blowers in an appropriate manner. If there
exists a background flow, the possibility of getting wider range of attainable angle of separation increases. In practical use if creating a distinguished separation point is necessary, we found that the best way is to set $\beta$ to be a big number and then set the damping coefficient of its relevant equation with a small number. Physically it means that we set the objective function of the controller to a specific and big strength, but in a long time, so controller first create a zero skin friction at the point $x = \gamma$ and then tries to reach to the required strength.

**1.3.4 Unsteady Separation Control**

In this section we apply unsteady separation criteria to our equations to derive the control law. The unsteady controller is milder than the steady one.

**Theorem 5. Necessary Condition for Unsteady Separation**

Assume the 2D unsteady flow field $(u, v) = (u(x, y, t), v(x, y, t))$ satisfies NS-equation and boundary conditions given by

\begin{align*}
    u(0, y, t) &= yj_1(t) + O_1(y^2) \quad (1.43) \\
    u(1, y, t) &= -yj_2(t) + O_2(y^2) \quad (1.44)
\end{align*}

then the feedback control law

\begin{align*}
    J_1(t_{m+1}) &= \frac{Err(t_m) - A(t_m) - B(t_m) - C(t_m)}{D(t_m)} \\
    J_2(t_{m+1}) &= \alpha J_1(t_{m+1})
\end{align*}

where

\begin{align*}
    A(t_m) &= \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m) \frac{1 - e^{-\xi \delta t}}{\xi} \sin n\pi \gamma \\
    B(t_m) &= \sqrt{2} \sum_{n=1}^{\infty} \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} f_n(t_m) \sin n\pi \gamma \\
    C(t_m) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{n \xi} (j_1(t_m) + \cos(n\pi) j_2(t_m)) \sin n\pi \gamma \\
    D(t_m) &= 4\nu \pi \sum_{n=1}^{\infty} \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} (1 + \alpha \cos(n\pi)) \sin n\pi \gamma
\end{align*}
and Err\( (t_m) = \int_0^{t_m} u_y \, dy \) is the error, satisfies the necessary condition for unsteady separation
\[ \int_0^{t_m} u_y \, dt = 0 \]

For periodic flow, after the transient period, the unsteady separation criteria is simplified to
\[ \int_0^T u_y(\gamma, 0, \tau) \, d\tau = 0 \] (1.45)
in which \( T \) is the period of the motion. To satisfy the simplified criterion, we can just change the limit of the integral in \( Err(t_m) \) to \( Err(t_m) = \int_{t_m-T}^{t_m} u_y \, dy \)

Following the same procedure that we used to study the stability of the first controller, we derive stability condition for this controller. For \( \lambda = 1 \) we have

\[
\limsup_{m \to \infty} \left| \int_{t_0}^{t_{m+1}} \phi(\gamma, t) \, dt \right| = \limsup_{m \to \infty} \sum_{n=1}^{\infty} \left( \int_{t_m}^{t_{m+1}} \frac{1 - e^{-\xi_n(t_{m+1} - s)}}{\xi_n} \left[ f_n(s) - f_n^*(s) \right] \, ds \right) \sin n\pi\gamma
\]

\[
\leq \limsup_{m \to \infty} \sum_{n=1}^{\infty} \int_{t_m}^{t_{m+1}} \frac{1 - e^{-\xi_n(t_m - s)}}{\xi_n} \left| f_n(s) - f_n^*(s) \right| \, ds
\]

\[
\leq \limsup_{m \to \infty} \sum_{n=1}^{\infty} \max_{t \in [t_m, t_{m+1}]} |f_n(t)| \left( \frac{\Delta}{\xi_n} - \frac{1 - e^{-\xi_n \Delta}}{\xi_n^2} \right)
\]

\[
\leq \Delta \sum_{n=1}^{\infty} \limsup_{t \to \infty} |f_n(t)| \left( \frac{\Delta}{\xi_n} - \frac{1 - e^{-\xi_n \Delta}}{\xi_n^2} \right)
\]

If \( F_1(x, t) \) is \( C^r \) in \( x \), then

\[
\Delta \sum_{n=1}^{\infty} \limsup_{t \to \infty} |f_n(t)| \left( \frac{\Delta}{\xi_n} - \frac{1 - e^{-\xi_n \Delta}}{\xi_n^2} \right) \leq \Delta \sum_{n=1}^{\infty} \frac{C(t) C_0 \Delta}{n^r \xi_n} = \frac{C_0 \Delta^2 L^2 C(t)}{2\nu \pi^2} \sum_{n=1}^{\infty} \frac{1}{n^{r+2}} < \infty,
\]

provided that \( C(t) \) remains uniformly bounded for all \( t \).

Thus, we obtain a conclusion identical to that about the stability of the proportional controller. Note, however, that the Fourier coefficients of the error in the integral controller decay faster with \( n \), and hence the response will be smoother.

**Theorem 6. Sufficient Condition for Unsteady Separation**
assume the 2D unsteady flow field \((u, v) = (u(x, y, t), v(x, y, t))\) satisfies NS-equation and boundary conditions given by

\[
\begin{align*}
    u(0, y, t) &= yj_1(t) + O_1(y^2) \quad (1.46) \\
    u(1, y, t) &= -yj_2(t) + O_2(y^2) \quad (1.47)
\end{align*}
\]

then the feedback control law

\[
\begin{align*}
    j_1(t_{m+1}) &= \frac{A_1^* C_2^*(tm) - A_2^* C_1^*(tm)}{A_1^* B_2^* - A_2^* B_1^*} \\
    j_2(t_{m+1}) &= \frac{C_1^*(tm) B_2^* - C_2^*(tm) B_1^*}{A_1^* B_2^* - A_2^* B_1^*}
\end{align*}
\]

where

\[
\begin{align*}
    A_1^*(tm) &= \sum_{n=1}^{\infty} \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} n^2 \cos n\pi \gamma \\
    B_1^*(tm) &= \sum_{n=1}^{\infty} \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} n^2 \cos(n\pi) \cos n\pi \gamma \\
    A_2^*(tm) &= \sum_{n=1}^{\infty} \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} n \sin n\pi \gamma \\
    B_2^*(tm) &= \sum_{n=1}^{\infty} \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} n \cos(n\pi) \sin n\pi \gamma \\
    C_1^*(tm) &= \frac{\text{Err}(tm) - A'(tm) - B'(tm) - C'(tm)}{4\nu\pi^2} \\
    C_2^*(tm) &= -\frac{A(t_m) + B(t_m) + C(t_m)}{4\nu\pi} \\
    A'(tm) &= \sqrt{2} \sum_{n=1}^{\infty} n\pi \alpha_n(t_m) \frac{1 - e^{-\xi \delta t}}{\xi} \cos n\pi \gamma \\
    B'(tm) &= \sqrt{2} \sum_{n=1}^{\infty} n\pi \delta t \xi + e^{-\xi \delta t} - 1 \frac{f_n(t_m)}{\xi^2} \cos n\pi \gamma \\
    C'(tm) &= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - e^{-\xi \delta t}}{n^2 \xi^2} (j_1(t_m) + \cos(n\pi) j_2(t_m)) \cos n\pi \gamma \\
    \text{Err}(tm) &= -(t_{m+1} - t_0)\beta - \int_{t_0}^{t_m} u_{xy} dt
\end{align*}
\]
where \( A(t_m), B(t_m), C(t_m) \) and \( D(t_m) \) are given by Theorem (5) satisfies both necessary and sufficient condition for sharp separation that is:

\[
\int_0^t u_y(\gamma, 0, t) = 0 \quad (1.48)
\]
\[
\int_{t_0}^t u_{xy} = -(t - t_0)\beta \quad (1.49)
\]

The question remains: is the second separation condition indeed satisfied asymptotically with the above controller? The actual value of the skin-friction-gradient integral between \( t_m \) and \( t_{m+1} \) is

\[
\int_{t_0}^{t_{m+1}} \phi_x(\gamma, t) \, dt = \int_{t_0}^{t_m} \phi_x(\gamma, t) \, dt + \int_{t_m}^{t_{m+1}} \phi_x(\gamma, t) \, dt
\]
\[
= -\beta (t_{m+1} - t_0) + \int_{t_m}^{t_{m+1}} [\phi_x(\gamma, t) - \phi_x(\gamma, t)] \, dt
\]
\[
= -\beta (m + 1) \Delta
\]
\[
+ \int_{t_m}^{t_{m+1}} \left( \sum_{n=1}^{\infty} \pi n \left\{ \int_{t_m}^t e^{-\xi_n(t-s)} [f_n(s) - f_n^*(s)] \, ds \right\} \cos n\pi \gamma \right) \, dt. \quad (1.51)
\]

Assuming again that \( F_{yt}(x, t) \) is \( C^r \) in \( x \), we find that

\[
\limsup_{m \to \infty} \left| \int_{t_m}^{t_{m+1}} \left( \sum_{n=1}^{\infty} \pi n \left\{ \int_{t_m}^t e^{-\xi_n(t-s)} [f_n(s) - f_n^*(s)] \, ds \right\} \cos n\pi \gamma \right) \, dt \right|
\]
\[
\leq \limsup_{m \to \infty} \sum_{n=1}^{\infty} \pi n \left| \int_{t_m}^{t_{m+1}} \frac{1 - e^{-\xi_n(t_{m+1}-s)}}{\xi_n} [f_n(s) - f_n^*(s)] \, ds \right|
\]
\[
\leq \limsup_{n \to \infty} \sum_{n=1}^{\infty} \max_{t \in [t_m, t_{m+1}]} \left| f_n(t) \right| \pi n \left( \frac{1 - e^{-\xi_n \Delta}}{\xi_n} \right)
\]
\[
= \Delta \sum_{n=1}^{\infty} \limsup_{t \to \infty} \left| f_n(t) \right| \pi n \left( \frac{1 - e^{-\xi_n \Delta}}{\xi_n} \right)
\]
\[
\leq \Delta \sum_{n=1}^{\infty} \frac{C(t) C_0 \Delta}{n^r \xi_n} = \frac{C_0 \Delta^2 LC(t)}{2\nu \pi} \sum_{n=1}^{\infty} \frac{1}{n^{r+1}} < \infty,
\]

provided that \( r > 1 \) and \( C(t) \) remains uniformly bounded for all \( t \). Under these conditions,
(1.51) gives
\[
\lim_{m \to \infty} \int_{t_0}^{t_{m+1}} \phi_x(\gamma, t) \, dt = \lim_{m \to \infty} [-\beta (m + 1) \Delta] = -\infty,
\]
thus the second condition for separation holds. We conclude:

The controller (6) is stable (i.e., creates skin friction with bounded asymptotic average at \( x = \gamma \), and skin-friction-gradient with asymptotic average tending to \(-\infty\)) if

\[
\partial_x u_{yy}(x, 0, t) \text{ is continuously differentiable in } x \text{ with a derivative that is uniformly bounded in } t \text{ over the } [0, L] \text{ section of the boundary.}
\]

Now we will demonstrate our control law on a model system which is a non-homogeneous heat equation with time and space dependent external forcing function. We perform the simulation on the transformed equation with homogeneous boundary condition (look at proof of the theorems). The skin friction equation can be transformed to the form

\[
\frac{\partial \psi}{\partial t} = 2\nu \frac{\partial^2 \psi}{\partial x^2} + F(x, t) - \frac{\partial G}{\partial t},
\]

\[
\psi(0, t) = 0, \psi(1, t) = 0,
\]

\[
\psi(x, 0) = \psi_0(x)
\]

where \( F = F(x, t) = \nu \frac{\partial^2 u}{\partial y^2}(x, 0, t), G(x, t) = (1-x) j_1(t) - x j_2(t) \) and \( \psi = u_y - G \). To simulate this equation numerically and see the effect of the controller on the solution of this equation we have used MATLAB PDE solver. We generated triangular dense grids in a rectangular area. To avoid the effect of two dimensionality we took rectangle length 10 times its width. The boundary condition in \( x \) direction is Dirichlet and in \( y \) direction is Neuman type boundary condition, as a result the solution does not vary much along \( y \) axis. The time step is \( \delta t = 0.2 \) and we divided this interval into twenty subintervals during which the external forcing function varies but the controller output does not vary. The controller sends signal based on the measured \( u_y \) at the discrete time steps; we assume there is no measurement error except computer roundoff error. To have a more realistic model, we add the effect of background flow to the actuators. We assume that two moving vortex close to the wall pass over the wall and affect the output of the controller. It means, when the controller specifies a specific amount for \( u_y \), actuators blow based on the the blower characteristic curves to create that value at the exit, but the actual
value of $u_y$ is a function of action of both background flow and jet flow. If the jet output is much higher than the background flow this effect can be neglected (which is not usually the case), so in highly unsteady flow it should be considered carefully. To model the effect of this disturbance we have introduced two bump function (Figure 3-3.e) that pass over the controller; this means at time $t = 5$ a disturbance hits the left jet output, passes it and hits the right jet after a time which is proportional to the speed of the modeled vortex. After a while another vortex does the same. To add this effect to simulation we measure the vortex disturbance close to the jet output and adjust the output of the jet in such a way that the total effect be equal to the desired value of the skin friction at two ends of the wall. The effect of vortex continues changing during simulation while modification can be done only at the time steps. The damping ratio is set to one and simulation has been done for 22 seconds. For a more realistic model we put a saturation limit on the controller. The external forcing function is

$$F(x, t) = -\nu u_{yy} + u_y + 1 + \sin(t)$$

The first term push the skin friction equation to the less stability, the second term is chosen because it is anticipated that the external forcing term is dependent on the measured quantities and the last term is to insert a time dependency to the system. We set the strength of separation $\beta = 5$.

Simulation graphs are shown in Figure 3-3. In Figure 3-3.a,b and f you can compare the response of the equation and external forcing function with and without controller (the dash-lines are without). On the uncontrolled flow the effect of two disturbance is appeared and c,d show the effect of the bump on the controller outputs. The controller successfully controls the point and strength of the separation at the specified point $\gamma = 0.3$.

### 1.4 Application to Navier-Stokes Flows

In this section we consider the implementation of the controller on a fluid flow simulator. We first show the behavior of the controller in the steady background flow and then discuss the unsteady case. The fluid in our simulation is air ($\rho = 1.225$ and $\mu = 1.775e-5$) and we assume that the Mach number in small enough for incompressibility condition to be valid.
1.4.1 Steady Channel Flow

A rectangular channel with inflow boundary condition at the left and outflow at the right is considered (see Figure 3-1). Height of the channel is 0.5m and total length of it is 5m where 1m is enclosed by two wall jets. Background flow velocity is set to $v_{bg} = 0.03m/s = constant$, the desired separation point is $\gamma = 0.8$. We use the steady separation controller with strength restriction (Theorem 4). As we discussed in previous section, we set the strength of the separation a big number (12000).

Figure 3-4 shows the system response to this controller. Figure 3-5 shows streamlines colored by velocity magnitude in the channel after the control system has been implemented.

1.4.2 Unsteady Channel Flow

To show the capability of the controller in unsteady flows, we demonstrate an example in which the channel inlet flow varies with time. We set the channel flow velocity $U_{bg} = 0.03 + 0.015 \sin(ft) (m/s)$ where $f = 0.05Hz$ is the frequency of the variation. The desired separation point is $\gamma = 0.7$ and sampling time is $\delta t = 1sec$. We set the damping ratio $\lambda = 0.0001$ and the required strength $\beta = 10000$.

Figure 3-6.a shows variation of $u_y$ with time. In an ideal case, the integral should vanishes if $u_y$ vanishes at $t = \infty$. But in actual control systems because the background flow is an 'unknown' function of time there always exists an error between desired and attained values. If the variation of background flow is known to be a finite terms polynomial of time, by adding enough integrators, one can overcome the steady state error, but in the case of sinusoidal variation, oscillation of controller and error is unavoidable. In this figure also you can compare between uncontrolled values and controlled time history and of $u_y$ and its integral at the separation point (Figures 3-6.a,b). Figures 3-6.c and 3-6.d show the forcing amplitude of our actuators. Figure 3-7 shows behavior of streaklines in the unsteady flow. Although instantaneous streamlines are moving and points of zero skin friction (indicated by black triangles) at each moment change their positions, there is a unique and distinguished point of separation. Figure 3-8 shows a close-up picture of what happens close to the point of separation. Since the Reynolds number is low, the region close to the point of separation is dead-fluid region and numerical errors are considerable. As a result, streaklines in the close-up picture are not very smooth.
To generate required skin friction at both ends of the wall, we blow the actuators and measure the resultant skin friction at the outlet of actuators. Numerical simulations show that usually the most dominant regime on the wall and close to outlet of the actuators is determined by the actuator flow. In our simulations we use a curve fitted to a bunch of stored-data from numerical simulations to determine the blower strength in order to generate the desired skin friction on the wall near the outlet. For the case of suction generating a specific skin friction is much more difficult. For more safety we add a feedback P controller to the skin friction and blower system. This secondary controller acts when the skin friction measured at the outlet of actuators is different from the one that has been set by the controller. Coefficient of this secondary controller must be selected carefully to avoid possible instability in the actuator outlet.

1.4.3 Control of Reattachment in Backward Facing Step Flow

Backward facing step flows have been considered extensively by experimentalists, theoreticians and numericists. This problem is important because it has both separation and reattachment, and the flow behavior is similar to the flow in diffusers, combustors, turbomachinery and flow around airfoils (Lai et al. 2002 [16]). From the control point of view, control of the size of the separation bubble (Bao 2004, [2]) is of great interest. In this section we use our Navier-Stokes based controller to control the position of the reattachment point in a backward facing step flow.

We have used a 0.2m height cubic channel with a 0.1m height step on its left-hand side. The flow comes into the channel from the left side, passes over the step and exits from the right side (Figure 3-9). Flow rate is 0.02 $m^2/s$ for a unite depth of the channel, means the inlet velocity is around 0.2$m/s$. For high enough Reynolds number there is a distinct separation point at the corner of the step. For low Reynolds number and steady flow, there is a distinguished reattachment point on the lower wall as well, while for higher Reynolds number flow becomes unsteady and vortex shedding prevents seeing a non-moving reattachment point. For simulation we have used air ($\rho = 1.225 Kg/m^3, \mu = 1.775e - 5 Kg/(ms)$) and there are two tangentially blowing channels (actuators) which are located 1m far from each other and can blow and/or suck at the angle of $\theta = tan^{-1}0.1$. The intended separation point is $x = 0.2m$. Reynolds
number is of order $10^3$. Like previous simulation we have set the time step of simulation and controller update to 1 sec, it means that the controller is updated every 1 sec while there might be as many as needed (up to 200, but it usually converges in less than 10 iterations) iterations in each time step. Theoretically $\beta$ can be any positive number because at the reattachment point only the sign of $\beta$ must be positive, but as it was mentioned earlier, we can not easily force the controller to get the exact value for $\beta$ in simulation, therefore we set it to a big number and set its coefficient to a small to be sure that there would be a separation point at the intended point. we set $\beta = 100$ and the damping ratio $\lambda = 0.005$. It is to be noted the suction mechanism is different from blowing: in blowing the outlet of actuators usually are the only dominant flow near the wall, while in suction there is no such a dominant flow. Physically speaking, the probability of happening of local vortices and disturbances is very much higher when the fluid is sucked by two actuators compared with when the flow is blown by them. Appearance of local vortices and disturbances makes many separation and reattachment point along the wall and prevents the control procedure to make one strong reattachment point. These issues make the control of reattachment point much more difficult compared to the separation control.

For simulation we have used FLUENT two dimensional incompressible solver for laminar flow. Height of the step is $0.1m$ which we use it as a reference length; so dimensionless height of the step is one. Hereinafter we write all lengths in dimensionless form. The channel width is 2 and the channel has a length of 5 before step (inlet flow) and a length a 40 after step. Long channel lessens the effect of the out flow boundary condition on the flow over the step. The inlet condition is ‘velocity inlet’ and a uniform velocity profile has been selected. The outlet condition is set to ‘outflow’. Other boundaries are ‘wall’. The size of the grid on the face of the wall is 0.1 and on the wall on which controller works is 0.2. For the upper wall and continuation of the lower wall, the grid size grows slowly as it goes toward the outlet. We have used triangular elements for meshing the face of the channel.

Figure 3-10 shows the time history of variables during control. Figure 3-10.a shows $u_y$ as a function of time. In the uncontrolled case (dashed line) since the flow is not much unsteady, $u_y$ remains almost constant at the intended reattachment point which is inside the separation bubble. By applying our controller, $u_y$ changes and after some overshoots reaches to a steady value that must be zero in order to keep the $\int u_y dt$ bounded. Time history of $\int u_y dt$ is shown
in Figure 3-10.b. As it is expected when controller is off (dashed line) \( \int u_y \) grows (or decays) toward infinity, while in the controlled case it remains bounded. Both \( J_1 \) and \( J_2 \) suck the flow and reach to steady state values (no final oscillations). Figure 3-11 shows the evolution of \( u_{yx} \) and \( \int u_{yx} \) in time. The objective function of the controller is to keep the value of \( u_{yx} \) a positive number (which we set it to be 100 \((ms)^{-1}\)). Since we do not force it by adding an integrator, a permanent error is expected. (as it can be seen from the figure, the steady state value of \( u_{yx} \) is 58\((ms)^{-1}\).

Evolution of \( u_y \) on the wall during time is shown in figure 3-12. Although previous graphs show that both necessary and sufficient condition for separation is satisfied, figure 3-12 shows that this point is indeed unique. Initial distribution of \( u_y \) shows a possible separation/reattachment point at the point \( x = 0.65m \). We will see that this is indeed a reattachment point. During control process, the distribution moves toward the configuration that imposes a reattachment point at the intended point of reattachment. It undergoes a short overshoot and then settles into that configuration.

Figures 3-13 and 3-14 compare velocity distribution and pathlines in uncontrolled and controlled cases respectively. In the controlled case, size of the separation bubble is reduces around 3 times. Pressure recovery close to the exit is obvious from the pathlines colored by pressure.

1.5 Conclusion

A model-based nonlocal controller based on the Navier-Stokes equation is proposed. For necessary-sufficient and steady-unsteady flow, four controllers have been derived. Stability of the controller relative to time dependence of the background flow has also been studied. Numerical simulation for a model equation (heat equation with time- and space-dependent external forcing function) and for the Navier-Stokes equation is shown to validate the controller.
1.6 Derivation of formulae

1.6.1 Proof of theorems 1, 4, 5 and 6

Consider vorticity transport equation for two-dimensional unsteady flow

\[
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \quad (1.53)
\]

where \( \omega = v_x - u_y \), setting \( y = 0 \) and using incompressibility relation \( u_x + v_y = 0 \) we get

\[
u u_t(x, 0, t) = \nu(2 u_{yxx}(x, 0, t) + u_{yy}(x, 0, t)) \quad (1.54)
\]

we denote the shear stress along the wall \( y = 0 \) by

\[
\phi(x, t) = \frac{\partial u}{\partial y}(x, 0, t). \quad (1.55)
\]

substituting in the equation 1.54 we get

\[
\frac{\partial \phi}{\partial t} = 2 \nu \frac{\partial^2 \phi}{\partial x^2} + F(x, t) \quad (1.56)
\]

\[
\phi(0, t) = j_1(t), \phi(L, t) = -j_2(t) \quad (1.57)
\]

\[
\phi(x, 0) = \phi_0(x) \quad (1.58)
\]

where

\[
F(x, t) = \nu \frac{\partial^3 u}{\partial y^3}(x, 0, t) \quad \text{and} \quad \phi_0(x) = \frac{\partial u_0}{\partial y}(x, 0, 0) \quad (1.59)
\]

To transfer this non-homogeneous boundary condition problem to a homogeneous one we set

\[
G(x, t) = (1 - x)j_1(t) - xj_2(t) \quad \text{and} \quad \psi = \phi - G \quad (1.60)
\]
it then follows that
\[ \frac{\partial \psi}{\partial t} = 2\nu \frac{\partial^2 \psi}{\partial x^2} + F(x,t) - \frac{\partial G}{\partial t}, \quad (1.61) \]
\[ \psi(0,t) = 0, \, \psi(1,t) = 0, \quad (1.62) \]
\[ \psi(x,0) = \psi_0(x) \quad (1.63) \]

where
\[ \psi_0(x) = \phi_0(x) - G(x,0). \quad (1.64) \]

By using constant variation formula this equation can be solved. For this we denote
\[
\begin{align*}
    a_n &= \sqrt{2} \int_0^1 \psi_0(x) \sin(n\pi x) dx \\
    f_n(s) &= \sqrt{2} \int_0^1 \nu F(x,s) \sin(n\pi x) dx \\
    g_n(s) &= \sqrt{2} \int_0^1 \frac{\partial G}{\partial s}(x,s) \sin(n\pi x) dx \\
        &= \sqrt{2} \int_0^1 \left( (1-x) \frac{dj_1}{ds} + x \frac{dj_2}{ds} \right) \sin(n\pi x) dx \\
        &= \frac{\sqrt{2}}{n\pi} \left( \frac{dj_1}{ds} - \cos(n\pi) \frac{dj_2}{ds} \right) \\
\end{align*}
\]

\[
\begin{align*}
    \psi &= \sqrt{2} \sum_{n=1}^{\infty} a_n e^{-\xi t} \sin(n\pi x) + \sqrt{2} \int_0^t \sum_{n=1}^{\infty} e^{-\xi(t-s)} (f_n(s) - g_n(s)) \sin(n\pi x) ds \\
    &= \sqrt{2} \sum_{n=1}^{\infty} a_n e^{-\xi t} \sin(n\pi x) + \sqrt{2} \int_0^t \sum_{n=1}^{\infty} e^{-\xi(t-s)} f_n(s) \sin(n\pi x) ds \\
    &\quad - 2 \int_0^t \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi(t-s)} \left( \frac{dj_1(s)}{ds} - \cos(n\pi) \frac{dj_2(s)}{ds} \right) \sin(n\pi x) ds
\end{align*}
\]
Integrating by parts we get

$$
\psi = \sqrt{2} \sum_{n=1}^{\infty} a_n e^{-\xi t} \sin(n \pi x) + \sqrt{2} \int_0^t \sum_{n=1}^{\infty} e^{-\xi(t-s)} f_n(s) \sin(n \pi x) ds \\
-2 \int_0^t \sum_{n=1}^{\infty} \frac{1}{n \pi} \left( j_1(t) + \cos(n \pi j_2(t)) \right) \sin(n \pi x) \\
+2 \sum_{n=1}^{\infty} \frac{1}{n \pi} e^{-\xi t} (j_1(0) + \cos(n \pi j_2(0))) \sin(n \pi x) \\
+4 \int_0^t \sum_{n=1}^{\infty} \nu n \pi e^{-\xi(t-s)} j_1(s) + \cos(n \pi j_2(s)) \sin(n \pi x) ds
$$

Now setting $G(x, t) = 2 \sum_{n=1}^{\infty} \frac{1}{n \pi} (j_1(t) + j_2(t)) \sin(n \pi x)$ the solution of the equation is given by

$$
\phi = \sqrt{2} \sum_{n=1}^{\infty} a_n e^{-\xi t} \sin(n \pi x) + \sqrt{2} \int_0^t \sum_{n=1}^{\infty} e^{-\xi(t-s)} f_n(s) \sin(n \pi x) ds \\
+2 \sum_{n=1}^{\infty} \frac{1}{n \pi} e^{-\xi t} (j_1(0) + \cos(n \pi j_2(0))) \sin(n \pi x) \\
+4 \int_0^t \sum_{n=1}^{\infty} \nu n \pi e^{-\xi(t-s)} j_1(s) + \cos(n \pi j_2(s)) \sin(n \pi x) ds \tag{1.65}
$$

the external forcing function term $f_n(s)$ is provided by measurement, so we have no a priori knowledge of its behavior. We measure it at the end of the time step $t_m$ and assume that it is constant during the next time step $t_{m+1}$. Based on this assumption we design the controller for the next time step $t_{m+1}$. The approximate value of the $\phi(\gamma)$ at the end of the time step $t_{m+1}$ would be

$$
\phi^*(\gamma, t_{m+1}) = \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m)e^{-\xi t_{m+1}} \sin(n \pi \gamma) + \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi t_m}}{\xi} \right) f_n(t_m) \sin(n \pi \gamma) \\
+2 \sum_{n=1}^{\infty} \frac{1}{n \pi} e^{-\xi t_m} (j_1(t_m) + \cos(n \pi j_2(t_m))) \sin(n \pi \gamma) \\
+4 \nu \pi \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi t_m}}{\xi} \right) n (j_1(t_{m+1}) + \cos(n \pi j_2(t_{m+1}))) \sin(n \pi \gamma) \tag{1.66}
$$

where by $t_{m+1}$ we mean at the end of the time step $t_{m+1}$ and $\phi^*$ stands for approximation of $\phi$. Now to derive our feedback control law, we set the error at the end of the next time step to
zero, means we set $\phi^*(\gamma, t_{m+1}^b) = 0$ and derive $j_1$ and $j_2$ for time interval $[t_{m+1}^b, t_{m+1}^e]$. The first term is given by initial conditions and the second term is a function of external forcing function. For each time step the third term is initial wall jet strengths and the last term includes the effect of the newly set wall jet strengths. By setting $\phi^*(\gamma, t_{m+1}^e) = 0$, we have:

$$4\nu \pi \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) n[j_1(t_{m+1}) + \cos(n\pi)j_2(t_{m+1})] \sin(n\pi \gamma)$$

$$= -\sqrt{2} \sum_{n=1}^{\infty} a_n(t_m) e^{-\xi \delta t} \sin(n\pi \gamma) - \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) f_n(t_m) \sin(n\pi \gamma)$$

$$-\sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi \delta t}[j_1(t_m) + \cos(n\pi)j_2(t_m)] \sin(n\pi \gamma)$$

(1.67)

We assume $j_2(t_{m+1}) = \alpha j_1(t_{m+1})$ where $\alpha$ is an arbitrary constant. Solving the above equation for $j_1$ (or $j_2$) and keeping in mind that there are some errors in measurements, the most accurate solution would be

$$j_1(t_{m+1}) = \left\{ -\sqrt{2} \sum_{n=1}^{\infty} a_n(t_m) e^{-\xi \delta t} \sin(n\pi \gamma) - \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) f_n(t_m) \sin(n\pi \gamma)$$

$$-\sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi \delta t}[j_1(t_m) + \cos(n\pi)j_2(t_m)] \sin(n\pi \gamma) \right\} /$$

$$\left\{ 4\nu \pi \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) n[1 + \alpha \cos(n\pi \gamma) \sin(n\pi \gamma) \right\}$$

$$j_2(t_{m+1}) = \alpha j_1(t_{m+1}).$$

Proof of Theorem 4

Consider the solution of skin friction equation (equation 1.65). Taking derivative with respect to $x$ we get the $u_{yx}$

$$\phi_x = u_{yx}(x, 0, t) = \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n e^{-\xi \xi} \cos(n\pi \gamma) + \sqrt{2} \int_0^t \sum_{n=1}^{\infty} n\pi e^{-\xi(t-s)} f_n(s) \cos(n\pi \gamma) ds$$

$$+ 2 \sum_{n=1}^{\infty} e^{-\xi \xi}(j_1(0) + \cos(n\pi)j_2(0)) \cos(n\pi \gamma)$$

$$+ 4 \int_0^t \sum_{n=1}^{\infty} n\pi^2 \pi^2 \cos(n\pi \gamma) \sin(n\pi \gamma) ds.$$
Similar to the procedure that we used for proof of the theorem 4, we set $\phi^*(\gamma, t_{m+1}) = \beta$, it gives

$$u_{y2}(\gamma, 0, t_{m+1}) = \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n(t_m)e^{-\xi t} \cos(n\pi \gamma) + \sqrt{2} \sum_{n=1}^{\infty} n\pi \left( \frac{1 - e^{-\xi t}}{\xi} \right) f_n(t_m) \cos(n\pi \gamma)$$

$$+ 2 \sum_{n=1}^{\infty} e^{-\xi t} [j_1(t_m) + \cos(n\pi) j_2(t_m)] \cos(n\pi \gamma)$$

$$+ 4\nu \pi^2 \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi t}}{\xi} \right) n^2 [j_1(t_{m+1}) + \cos(n\pi) j_2(t_{m+1})] \cos(n\pi \gamma) = \beta$$

this equation now can be solved together with equation 1.67 to gives the values of two wall jets strength. Note that both equations after simplification are in the form of $\sum j_1 \sum_j \sum \cdots = \cdots$. Therefore two equations give a system of two equation and two unknown which has a unique solution, as a result there remains no arbitrary constant in the solution.

To impose the unsteady separation condition, we integrate equation 1.65 in time from zero time (origin of the time axis) to the current time $t$ keeping the assumption that external forcing function and wall jets are constant during integration

$$\int_0^t \phi(\gamma, t) dt = \sqrt{2} \sum_{n=1}^{\infty} a_n(0) \left( \frac{1 - e^{-\xi t}}{\xi} \right) \sin(n\pi \gamma)$$

$$+ \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{t\xi + e^{-\xi t} - 1}{\xi^2} \right) f_n(0) \sin(n\pi \gamma)$$

$$+ 2 \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi t}}{\xi} \right) [j_1(0) + \cos(n\pi) j_2(0)] \sin(n\pi \gamma)$$

$$+ 4\nu \pi \sum_{n=1}^{\infty} \left( \frac{t\xi + e^{-\xi t} - 1}{\xi^2} \right) n[j_1(t^*)] + \cos(n\pi) j_2(t^*)] \sin(n\pi \gamma).$$

The assumption is only valid during one time step, we transfer the origin of the time to the beginning of a specific time step $t_{m+1}$ and replace all initial condition by the value of the variables at the end of the previous time step that is $t_{m}$. We know that the integral of $u_y$ is non zero at $t_{m}$ and is equal to $Err(t_{m})$ and it is desired to set it equal to zero at the end of the next time step, so we equalize the integral over the time interval $t_{m+1}$ to the minus of this
error.

\[
\int_{t_m}^{t_{m+1}} \phi(\gamma, t) dt = \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m) \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) \sin(n\pi \gamma) \\
+ \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} \right) f_n(t_m) \sin(n\pi \gamma) \\
+ 2 \sum_{n=1}^{\infty} \frac{1}{n\pi} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) \left[ j_1(t_m) + \cos(n\pi) j_2(t_m) \right] \sin(n\pi \gamma) \\
+ 4\nu \pi \sum_{n=1}^{\infty} \left( \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} \right) n \left[ j_1(t_{m+1}) + \cos(n\pi) j_2(t_{m+1}) \right] \sin(n\pi \gamma)
\] (1.70)

\[
Err(t_m) = \int_0^{t_m} u_y dy
\] (1.71)

Setting \( j_2(t_{m+1}) = \alpha j_1(t_{m+1}) \) and solving the above equation for \( j_1 \) (or \( j_2 \)) gives the Theorem 5. In ideal case, in which external forcing function does not vary over one time interval, the total integral of \( u_y \) from zero till \( t_{m+1} \) will be zero.

Proof of the last theorem is a combination of Proofs of Theorems 4 and 5. Taking derivative with respect to \( x \) from equation 1.69 and equalize \( \int u_{yx} \) to desired strength \( \beta \) we get

\[
\int_{t_m}^{t_{m+1}} \phi_x(\gamma, t) dt = \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n(t_m) \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) \cos(n\pi \gamma) \\
+ \sqrt{2} \sum_{n=1}^{\infty} n\pi \left( \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} \right) f_n(t_m) \cos(n\pi \gamma) \\
+ 2 \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) \left[ j_1(t_m) + \cos(n\pi) j_2(t_m) \right] \cos(n\pi \gamma) \\
+ 4\nu \pi \sum_{n=1}^{\infty} \left( \frac{\delta t \xi + e^{-\xi \delta t} - 1}{\xi^2} \right) n^2 \left[ j_1(t_{m+1}) + \cos(n\pi) j_2(t_{m+1}) \right] \cos(n\pi \gamma)
\] = \( Err(t_m) \)

\[
Err(t_m) = \beta - \int_0^{t_m} u_{xy} dt
\]

which can be solved together with equation 1.70 to give the \( j_1(t_{m+1}) \) and \( j_2(t_{m+1}) \) with which the both separation point and the separation strength is specified.
1.6.2 Proof of theorem 2

We consider one time step. Assume all needed initial conditions are given at the beginning of the time step. We use Theorem 1 to compute the strength required for wall jets to suppress error during one time interval. Plugging these computed strengths into the full solution of the skin friction equation gives the error at the end of the time step. So we take initial condition at time $t^c_m$ and compute the actual value of $\phi(\gamma, t^c_{m+1})$. The error in this case is the deviation of $\phi(-y, t_Mn+1)$ from zero. To do so, consider the actual $\phi(\gamma, t_m + \delta t)$ which is

$$
\phi(\gamma, t_m + \delta t) = \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m)e^{-\xi \delta t} \sin(n\pi \gamma) + \sqrt{2} \int_{t_m}^{t_m+\delta t} \sum_{n=1}^{\infty} e^{-\xi (t_m+\delta t-s)} f_n(s) \sin(n\pi \gamma) ds \\
+ 2 \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi \delta t} (j_1(t_m) + \cos(n\pi) j_2(t_m)) \sin(n\pi \gamma) \\
+ 4 \int_{t_m}^{t_m+\delta t} \sum_{n=1}^{\infty} \nu n\pi e^{-\xi (t_m+\delta t-s)} j_1(t_{m+1}) + \cos(n\pi) j_2(t_{m+1}) ) \sin(n\pi \gamma) ds
$$

(1.72)

Note that this equation is different from equation 1.66. We can take the integral of the last term because $j_1, j_2$ are constant during one time step (note that this is not an assumption, they are really constant because we can change the strength of our actuators only at the time steps). We have:

$$
\phi(\gamma, t_m + \delta t) = \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m)e^{-\xi \delta t} \sin(n\pi \gamma) + \sqrt{2} \int_{t_m}^{t_m+\delta t} \sum_{n=1}^{\infty} e^{-\xi (t_m+\delta t-s)} f_n(s) \sin(n\pi \gamma) ds \\
+ 2 \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi \delta t} (j_1(t_m) + \cos(n\pi) j_2(t_m)) \sin(n\pi \gamma) \\
+ 4 \sum_{n=1}^{\infty} \nu n\pi \frac{1-e^{-\xi \delta t}}{\xi} (j_1(t_{m+1}) + \cos(n\pi) j_2(t_{m+1}) ) \sin(n\pi \gamma)
$$

(1.73)
inserting a $\lambda$ fraction of the $j_1, j_2$ into the exact $\phi$ solution we get

$$\phi(\gamma, t_m + \delta t) = (1 - \lambda)\sqrt{2} \sum_{n=1}^{\infty} a_n(t_m)e^{-\xi \delta t} \sin(n \pi \gamma)$$

$$+ (1 - \lambda)2 \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\xi \delta t} [J_1(t_m) + \cos(n \pi)J_2(t_m)] \sin(n \pi \gamma)$$

$$+ \sqrt{2} \int_{t_m}^{t_m + \delta t} \sum_{n=1}^{\infty} e^{-\xi(t_m + \delta t - s)} f_n(s) \sin(n \pi \gamma) ds \quad (1.74)$$

$$- \lambda \sqrt{2} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) f_n(t_m) \sin(n \pi \gamma) = \epsilon(t_{m+1}) = \epsilon_{m+1}$$

or

$$\sum_{n=1}^{\infty} \left\{ (1 - \lambda)\sqrt{2}a_n(t_m)e^{-\xi \delta t} + (1 - \lambda)2 \frac{1}{n\pi} e^{-\xi \delta t} [J_1(t_m) + \cos(n \pi)J_2(t_m)] \right. $$

$$+ \sqrt{2} \int_{t_m}^{t_m + \delta t} e^{-\xi(t_m + \delta t - s)} f_n(s) ds - \lambda \sqrt{2} \left( \frac{1 - e^{-\xi \delta t}}{\xi} \right) f_n(t_m) \left\} \sin(n \pi \gamma) = \sum_{n=1}^{\infty} \epsilon_{m+1,n} \sin(n \pi \gamma)$$

but we know that

$$\phi(\gamma, t_m) = \sqrt{2} \sum_{n=1}^{\infty} a_n(t_m) \sin(n \pi \gamma)$$

$$+ 2 \sum_{n=1}^{\infty} \frac{1}{n\pi} (J_1(t_m) + \cos(n \pi)J_2(t_m)) \sin(n \pi \gamma)$$

$$= \epsilon(t_{m+1}) = \epsilon_m = \sum_{n=1}^{\infty} \epsilon_{m,n} \sin(n \pi \gamma)$$

Changing the variable $u = s - t_m$ in the first integral and consider the time step of the system $T = \delta t$ and assume the elapsed time $t_m = m\delta t = mT$ we get

$$\Rightarrow (1 - \lambda)e^{-\xi T} \epsilon_{m,n} + \sqrt{2} \int_{0}^{T} e^{-\xi(T-u)} f_n(u + mT) du = \sqrt{2} \lambda \left( \frac{1 - e^{-\xi T}}{\xi} \right) f_n(mT) = \epsilon_{m+1} \quad (1.75)$$

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Now we have a difference equation, taking Z-transform from both sides we get (see Jury 1964 [14] pp. 264-266)

\[(1 - \lambda)e^{-\xi T}e(z) + \sqrt{2}2^\prime \left[\int_0^T e^{-\xi(T-u)}f_n(u + mT)du - \lambda Bf_n(mT)\right] = z(\epsilon(z) - \epsilon(0)) \quad (1.76)\]

\[B = \left(\frac{1 - e^{-\xi T}}{\xi}\right) = \text{Const.} \quad (1.77)\]

Now we have the Z-transform of the error of the system in hand. We define the system stable if the error does not grow up infinitely. We can find the steady state error of the system using final value theorem in z space. It gives

\[
\lim_{m \to \infty} \epsilon_{m,n} = \lim_{z \to 1} \left[\frac{z - 1}{z - (1 - \lambda)e^{-\xi T}}2^\prime \left[\int_0^T e^{-\xi(T-u)}f_n(u + mT)du - \lambda Bf_n(mT)\right]\right] \quad (1.78)
\]

provided that there is no pole outside the unit circle (Vich 1987 [21]). We fast conclude that for stability we should have \(|(1 - \lambda)e^{-\xi T}| < 1\) which makes a bound on the damping ratio \(\lambda\) although by damping ratio we mean \(\lambda < 1\) and by this assumption this inequality is always valid.

**Example-1**

for the external forcing function \(f(t) = t\) and by choosing \(\lambda = 1\) we have:

\[
\lim_{m \to \infty} \epsilon_{m+1,n} = \lim_{z \to 1} \left[\frac{z - 1}{z}\left[\int_0^T e^{-\xi(T-u)}(u + mT)du - B(mT)\right]\right]
\]

\[
\lim_{m \to \infty} \epsilon_{m+1,n} = \lim_{z \to 1} \left[\frac{z - 1}{z}\left[\frac{\xi T - 1 + e^{-\xi T}}{\xi^2}\right]\right]
\]

\[2^\prime 1^n = \frac{z}{(z - 1)}\]

\[
\lim_{m \to \infty} \epsilon_{m+1,n} = \frac{\xi T - 1 + e^{-\xi T}}{\xi^2}
\]

and so there would be a steady state error in the system.

**Example-2** for the external forcing function \(f(t) = \sin(t)\) and \(\lambda = 1\) we have:

\[
\lim_{m \to \infty} \epsilon_{m+1,n} = \lim_{z \to 1} \left[\frac{z - 1}{z}\left[\int_0^T e^{-\xi(T-u)}\sin(u + mT)du - B\sin(mT)\right]\right] \quad (1.79)
\]

taking the integral of the first term and using the relation for Z-transformation of \(\sin(\alpha n)\) and
\[ \cos(\alpha n) \text{ we get} \]

\[ 2_1 = \frac{-z \left( -e^{-\xi^T} z + e^{-\xi^T} \cos(T) + e^{-\xi^T} \xi \sin(T) + z \cos(T) - 1 - \xi z \sin(T) \right)}{z^2 + z^2 - 2 z \cos(T) + \xi^2 + 1} \quad (1.80) \]

\[ 2_2 = \frac{- \left( 1 + e^{-\xi^T} \right) z \sin(T)}{\xi \left( z^2 - 2 z \cos(T) + 1 \right)} \quad (1.81) \]

where \( 2_1, 2_2 \) are the Z-transforms of the first and second terms. Plugging these two transforms into the stability criteria we get the zero limit. Although it does not mean that the error is zero (because poles of the function under transform are on the unit circle), we can conclude that the error would not grow.

effect of adding an integrator

Adding an integrator to the error we get the following error equation

\[ \sum_{n=1}^{\infty} \left( 1 + \lambda \right) e^{-\xi^T} e_{m,n} \sin(n\pi\gamma) + \sqrt{2} \int_{t_m}^{t_m+\delta t} \sum_{n=1}^{\infty} e^{-\xi(t_m+\delta t-s)} f_n(s) ds \sin(n\pi\gamma) \]

\[ - \lambda \sqrt{2} \sum_{n=1}^{\infty} \left( 1 - e^{-\xi t} \right) f_n(t_m) \sin(n\pi\gamma) = e_{m+1} + I \int_0^{t_m} \epsilon(t) dt \quad (1.82) \]

doing the same procedure and looking discrete to the problem we have

\[ (1 + \lambda) e^{-\xi T} e_{m,n} \sin(n\pi\gamma) + \sqrt{2} \int_0^{T} e^{-\xi(T-u)} f_n(u + mT) du \]

\[ - \lambda \sqrt{2} \left( 1 - e^{-\xi t} \right) f_n(mT) = e_{m+1,n} + IT \sum_{i=0}^{m-1} \epsilon_{i,n} \quad (1.83) \]

but we know

\[ \mathcal{Z} \left\{ \sum_{k=0}^{n-1} f_k \right\} = \frac{1}{z - 1} F(z) \quad (1.84) \]

taking Z-transform of the equation and substituting the equivalent integral transform we get

\[ \lim_{m \to \infty} e_{m,n} = \lim_{z \to 1} \frac{\sqrt{2}(z - 1)^2}{(z - 1)(z - A) + IT} \mathcal{Z} \left[ \int_0^T e^{-\xi(T-u)} f_n(u + mT) du - \lambda B f_n(mT) \right] \]

\[ B = \left( 1 - e^{-\xi T} \right) = \text{Const.}, A = (1 + \lambda) e^{-\xi T} \quad (1.85) \]
it is easy to show that by adding $k$ integrator the equation would be

$$
\lim_{m \to \infty} \epsilon_{m,n} = \lim_{z \to 1} \frac{\sqrt{z} \left(z - 1\right)^{k+1}}{\left(z - 1\right)^k (z + A + IT)} \left[ \int_0^T e^{-\xi(T-u)} f_n(u + mT) du - \lambda B f_n(mT) \right]
$$

(1.86)

It can be seen that the integrator vanishes the steady state error in the controller Ex. 1. Based on the estimation of variation of background flow with time, enough integrator(s) must be added to stabilize the controller.

The proof for the second controller is quite similar except for expansion of $\hat{c}$ that must be $\cos(n\pi\gamma)$ expansion, and also there is a constant term which results in a different steady state error (but not total stability of the system). In this case we have equation 1.74 together with a similar equation which is derived for $u_{yx}$.

$$
\phi_x(\gamma, t_{m+\delta t}) = u_{yx}(\gamma, 0, t_{m+1})
$$

$$
= (1 - \lambda) \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n(t_m)e^{-\xi\delta t} \cos(n\pi\gamma)
$$

$$
+ (1 - \lambda) 2 \sum_{n=1}^{\infty} e^{-\xi\delta t} \left[ j_1(t_m) + \cos(n\pi) j_2(t_m) \right] \cos(n\pi\gamma)
$$

$$
+ \sqrt{2} \int_{t_m}^{t_{m+\delta t}} \sum_{n=1}^{\infty} n\pi e^{-\xi(t_{m+\delta t}-s)} f_n(s) \cos(n\pi\gamma) ds
$$

$$
- \lambda \sqrt{2} \sum_{n=1}^{\infty} n\pi \left( \frac{1 - e^{-\xi\delta t}}{\xi} \right) f_n(t_m) \cos(n\pi\gamma) + \lambda \beta
$$

(1.87)

The error is

$$
\epsilon_x(t_{m+1}) = \epsilon_x, m+1 = (1 - \lambda) \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n(t_m)e^{-\xi\delta t} \cos(n\pi\gamma)
$$

$$
+ (1 - \lambda) 2 \sum_{n=1}^{\infty} e^{-\xi\delta t} \left[ j_1(t_m) + \cos(n\pi) j_2(t_m) \right] \cos(n\pi\gamma)
$$

$$
+ \sqrt{2} \int_{t_m}^{t_{m+\delta t}} \sum_{n=1}^{\infty} n\pi e^{-\xi(t_{m+\delta t}-s)} f_n(s) \cos(n\pi\gamma) ds
$$

$$
- \lambda \sqrt{2} \sum_{n=1}^{\infty} n\pi \left( \frac{1 - e^{-\xi\delta t}}{\xi} \right) f_n(t_m) \cos(n\pi\gamma) + (\lambda - 1) \beta
$$

(1.88)

(1.89)

(1.90)
but we know

\[ \epsilon_x(t_m^+ - t_m^-) = \epsilon_{x,m} = \sqrt{2} \sum_{n=1}^{\infty} n\pi a_n(t_m) \cos(n\pi \gamma) \]

\[ + 2 \sum_{n=1}^{\infty} [j_1(t_m) + \cos(n\pi)j_2(t_m)] \cos(n\pi) \]

so

\[ (1 - \lambda)e^{-\xi T}\epsilon_{m,n} + \sqrt{2} \int_0^T n\pi e^{-\xi(T-u)} f_n(u + mT)du 
\]

\[ -\sqrt{2}\lambda n\pi \left(1 - e^{-\xi T}\right) \int_0^T f_n(mT) + (\lambda - 1)\beta_n = \epsilon_{m+1,n} \]  

(1.92)

Taking z-transform we get

\[ (1 - \lambda)e^{-\xi T}e(z) + \sqrt{2}2^z \left[ \int_0^T n\pi e^{-\xi(T-u)} f_n(u + mT)du - \lambda n\pi Bf_n(mT) + (\lambda - 1)\beta_n \right] 
\]

\[ = z(e(z) - e(0)) \]  

(1.93)

\[ B = \left(1 - e^{-\xi T}\right) = \text{Const.} \]  

(1.94)

Which is in general in the same form of the stability theorem for the controller 1. Adding \( k \) integrator to the error we get the following stability criteria

\[ \lim_{m \to \infty} \epsilon_{m,n} = \lim_{z \to 1} \frac{\sqrt{2}(z - 1)^{k+1}}{[z - 1]^k(z + A) + \lambda n\pi Bf_n(mT) + (\lambda - 1)\beta_n} \]

\[ 2^z \left[ \int_0^T n\pi e^{-\xi(T-u)} f_n(u + mT)du 
\]

\[ -\lambda n\pi Bf_n(mT) + (\lambda - 1)\beta_n \]  

(1.95)

**Corollary 7. Rate of convergence**

Equation (1.76) can be written in a simpler format

\[ \epsilon_{m+1} = A\epsilon_m + G_m \]  

(1.96)
The general solution of this equation is

$$\epsilon_m = A^m \epsilon_0 + \sum_{j=0}^{m-1} A^j G_{m-j}$$  \hspace{1cm} (1.97)

where $A$ is a measure of the rate of convergence. It means that the number of iterations which are needed to damp initial condition to a desired ratio is depend on the amount of $A$. Similar to the condition that we already derived using z-transform, if $|A| > 1$ the controller is unstable.

Now we assume $G_m < \bar{M} < \infty$ and $|A| < 1$, we get

$$|\epsilon_m| = |A^m \epsilon_0 + \sum_{j=0}^{m-1} A^j G_{m-j}|$$

$$\leq |\epsilon_0| + \sum_{j=0}^{m-1} |A|^j \bar{M}$$

$$\leq |\epsilon_0| + \frac{\bar{M}}{1 - |A|}$$  \hspace{1cm} (1.99)

if $G_m$ is bounded we can conclude that the final error must be bounded. If $|A| = 1$ then the condition for boundedness of error is changed to $\sum_{j=0}^{\infty} G_m < \infty$

1.6.3 Proof of Theorem 3

Expanding Taylor series of the function $g(t) = f(t + t_m)$ around zero we have

$$g(t) = b_0 + b_1 t + b_2 t^2 + O(t^3)$$  \hspace{1cm} (1.101)

Inserting this series to the equation (??) up to second term we get

$$\sum_{n=1}^{\infty} \left( \frac{1 - e^{-\xi_\delta t}}{\xi} b_0 - \left\{ b_0 \frac{1 - e^{-\xi_\delta t}}{\xi} + b_1 \frac{\delta t \xi + e^{-\xi_\delta t} - 1}{\xi^2} + \ldots \right\} \right) \sin(n\pi\gamma)$$

$$= \epsilon(t_m) = \sum_{n=1}^{\infty} \epsilon_{m,n} \sin(n\pi\gamma)$$  \hspace{1cm} (1.102)

which gives
\[-b_1 \delta t \xi + e^{-\xi \delta t} - 1 = \epsilon_m \]
\[-b_1 \delta t \xi + e^{-\xi \delta t} - 1 = \epsilon_{m+1} \]

\(b^*\)’s are coefficient of Taylor expansion of the function \(g\) at the next time step, i.e., \(t = \delta t\). For the control to reduce the error in the second time step we should have

\[ \left| \frac{\epsilon_{m+1}}{\epsilon_m} \right| < 1 \]

but for small enough \(\delta t\)

\[ b_1^* = b_1 + 2b_2 \delta t + O(\delta t^3) \]

so the equation reduces to

\[ \left| \frac{b_1 + 2b_2 \delta t}{b_1} \right| < 1 \]

\[ \Rightarrow \quad 0 < \delta t < \frac{b_1}{b_2} \]

\(\text{or}\)

\[ 0 < \delta t < \frac{2g'}{g''} \]

\(\text{Example}\)

for \(f(t) = e^{-t}\) we get \(0 < \delta t < 2\) for all times.
Chapter 2

Classical and Fuzzy Approaches to Separation Control in General Compressible Unsteady Flows

2.1 Introduction

The idea behind developing a feedback control law for imposing the separation is from skin friction measurement on the wall and its relation to the location of the separation. From a mathematical point of view, as mentioned in Haller 2004 ([10]), the separation profile acts as an unstable manifold of the flow: collects and ejects materials from the vicinity of separation point. Therefore, we expect that the direction of flow velocity on the left and right side of the separation point is different. From a physical point of view, the point of separation is the point where two very-close-to-the-wall streamlines impinge. Opposite velocity directions in the left and right hand sides of separation point give a good criteria to design a model independent controller. In this section we explain the structure of a classical controller for different separation criteria, i.e., steady and unsteady, and show their capabilities by real flow simulation.
2.1.1 Control of Necessary Condition for Steady Separation

Necessary condition for steady separation requires vanishing \( u_y \) at the desired separation point. To impose this condition at a specified point \((x, y) = (\gamma, 0)\) we consider our control strategy: while flow is steaming in one direction the sign of \( u_y \) does not change. Therefore to get a zero \( u_y \), wall jets should blow against the background flow. We conclude for steady flow we need one blower (wall jet) to blow against the background flow (see Figure 3-1 without right wall jet). We measure \( u_y \) only at the intended point of separation and try to vanish it by adjusting the strength of wall jets. The control law is (Controller-1)

\[
J(t) = C_1 u_y(\gamma, 0, t) + C_2 \int_0^t u_y(\gamma, 0, \tau) d\tau
\]  

(2.1)

It is to be noted that by steady separation, in this context, we mean that the background flow does not change with time and the criteria is the steady criteria. The control objective is to impose a separation at a specific point and this can achieve only during a transient unsteady period. We assume the background flow behaves like a force input to a system of order one, so to vanish the steady state error we need at least one integrator.

To show this controller by real flow simulation, we took Figure 3-1 as our numerical simulation setup. We turn off the right wall jet outlet and set the angle of blowing to \( \arctg(0.1) \). We set the desired separation point to \((x, y) = (\gamma, 0) = (0.5, 0)\) and the coefficients of the controller \( C_1 = 1/500, C_2 = 1/1000 \). By changing this coefficient different transient behavior would be observed. Background flow speed is \( V_{bg} = 1 \text{m/s} \), the fluid is air \((\rho = 1.225, \mu = 1.775 \times 10^{-5})\) and the Reynolds number is of order \( 10^3 \). Time sampling is \( \delta t = 1 \text{sec} \), although the controller do the job very well for a wide range of Reynolds number and time sampling. Figure 3-15 shows the response of flow during applying this controller. As it is expected, the wall jet grows to a steady state value. The steady state error of \( u_y \) is zero that shows modeling flow field like an order one system and step input is correct and the integral of \( u_y \) is not zero but bounded. For the case that there is an uncertainty about the direction of background flow or for slowly varying background flow (and one may want to apply steady criteria), you may run both wall jets in Figure 3-1 and use two independent form of above controller to impose the necessary condition of steady separation.
Because this controller only satisfies the necessary condition for flow separation there is no guarantee to have a separation at that point. For example the necessary condition for separation is already satisfied by a flow at rest while it does not show any separation and this control does not do any attempt to change the situation. To ensure occurring separation we propose using the second controller.

2.1.2 Control of Sufficient Condition for Steady Separation

Sufficient condition for flow separation includes vanishing $u_y$ on the wall and simultaneously a nonzero $u_{xy}$. Use of both right and left wall jets showed in Figure 3-1 is unavoidable because we are trying to control two objectives. As we don’t use any model for flow, deriving a coupled controller (MIMO-Multi Input Multi Output) by using traditional control approaches is almost impossible. To overcome this fact we set two independent controller for two wall jets by a wise choosing of coefficients. It is not a general recipe for a MIMO system. For example to control an inverted pendulum, both angle and position must be controlled simultaneously. In the pendulum case two independent controller can not control both objectives. In separation control the second condition can be achieved after getting the first condition, means we can apply a higher coefficient to first impose zero skin friction at the point and then gradually impose the intended nonzero strength of separation at that point. The form of two controllers are (Controller -2)

\[
J_{\text{right}}(t) = C_1u_y(\gamma, 0, t) + C_2 \int_0^t u_y(\gamma, 0, t) \, dt \\
J_{\text{left}}(t) = C_3e(t) + C_4 \int_0^t e(t) \, dt \\
e(t) = u_{xy}(\gamma, 0, t) - \beta.
\]

where $\beta$ is the desired separation strength at the point of separation. To apply this controller we need to measure one more quantity. This quantity can be directly $u_{xy}(\gamma, 0, t)$ or $u_y(\gamma + \delta x, 0, t)$. Not every $\beta$ can be attained by this controller. Usually $\beta$ is a very big number and at the point of separation during transient period easily changes its number. Based on this fact and to enhance the stability of the controller, we set the coefficient of the $u_{xy}$ in equation 2.3 to zero.
For simulation we consider an interesting extreme case: background flow is at rest. Theoretically Controller-1 does not do anything since its motivation is only the error at the point $(x, y) = (\gamma, 0)$ which is already zero, but in practice disturbances in the flow and measurement cause it to act. It is obvious that one blower can not make a stationary separation. We show by controller-2 how easy and exact we can get a separation. The setup is the same as example of controller-1 and background flow velocity is set to zero. Controller-2 is applied to right (equation 2.2) and left (equation 2.3) wall jets, the coefficient of the first term is as previous example and the coefficient of integral term in the second controller is set to $10^{-5}$. Time sampling is 1 second. The results are shown in Figure (3-16). It is shown that the derivative of wall shear stress at the intended separation point (strength of the separation) begins from zero and under the effect of two blower grows. Two controllers eventually suppress the skin friction at the point of separation while keeping the strength of separation at the desired value $\beta = -1000$. Variation of $J_1 = J_{left}$ and $J_2 = J_{right}$ are quite smooth. For higher coefficients an overshoot can be observed but system response is faster. Figure (3-17) shows the pathlines initiated from the wall after the system reaches to the steady state: there is a separation. We will discuss later how can we develop a coupled controller, instead of two independent controller using Fuzzy logic.

2.1.3 Control of Necessary Condition for Unsteady Separation

Necessary condition for unsteady separation in incompressible flow is a milder criteria that requires the integral of $\int_0^t u_y(\gamma, 0, t)dt$ to remain bounded during time. To get this condition we only need one wall jet. Following the same reasoning in Control-1 we get (Controller-3)

$$J(t) = C_1 \int_0^t u_y(\gamma, 0, t)dt + C_2 \int_0^t \int_0^\tau u_y(\gamma, 0, \tau)d\tau dt$$  \hspace{1cm} (2.5)

2.1.4 Control of sufficient Condition for Unsteady Separation

Sufficient condition for unsteady separation in incompressible flow requires two conditions: $\int_0^t u_y(\gamma, 0, t)dt$ remains bounded during time and $\int_0^t u_{xy}(\gamma, 0, t)dt$ grows unbounded. For first condition we use control-3 law and to impose the second condition we use a similar controller that we used in control-2. By forcing $u_{xy}$ to be a specific non-zero constant we force its integral
to grow unbounded. The final formula for this controller is (Controller-4)

\[
J_{\text{right}}(t) = C_1 \int_0^t u_y(\gamma, 0, t) dt + C_2 \int_0^t \int_0^t u_y(\gamma, 0, \tau) d\tau dt
\]

\[
J_{\text{left}}(t) = C_3 e(t) + C_4 \int_0^t e(t) dt
\]

\[
e(t) = u_{xy}(\gamma, 0, t) - \beta.
\]

For periodic flow this formula becomes

\[
J(t) = C_1 \int_t^{t-T} u_y(\gamma, 0, t) dt + C_2 \int_t^{t-T} \int_t^{t-T} u_y(\gamma, 0, \tau) d\tau dt
\]

\[
J_{\text{left}}(t) = C_3 e(t) + C_4 \int_0^t e(t) dt \quad \text{Controller - 4}
\]

\[
e(t) = \int_t^{t-T} u_{xy}(\gamma, 0, t) - \beta.
\]

2.1.5 Limitations of this Controller

Are these controllers work for all velocities and time sampling? The answer is NO. If the time sampling is big enough, the system gets the current state of the flow very late and it sets the strength of the wall jets during another big time sampling. Now if the coefficients of the controller are very small, the controller can work stably but very slow and if the coefficients are very big, controller easily becomes unstable which is the case for any other discrete control systems. But what if one chooses a smaller time sampling? We re-simulated the example of controller-1 by choosing a time sampling interval 10 times smaller than that example. The flow response becomes oscillatory. Phase difference is appeared when one compares time history of wall jet and skin friction variation in time (Figure(3-18)). This phase difference is because of inertial terms in Navier-Stokes equation. We name it “time delay” in fluid flow.

2.2 Time Delay in Flows

How can we quantify time delay in our system? We defined time delay as the time needed for information to reach to the point of separation from the outlet of wall jet. Physically, many parameters like compressibility, viscosity and velocities in flow may affect this time. For our
purpose, that is for incompressible, viscous and laminar flow, based on our physical intuition we choose viscosity ($\mu$), a reference length ($L$), density ($\rho$), a reference velocity ($V$) as parameters that may affect time delay ($t$) in the system. Using dimensional analysis (any reference book in fluid mechanics, see for example Schlichting 2000 [18]) we can find two dimensionless groups

$$\Pi_1 = \frac{Vt}{L}, \quad \Pi_2 = \frac{\rho V L}{\mu}. \quad (2.12)$$

The second dimensionless group is the famous Reynolds number and we name the first one 'dimensionless time'. Dimensional analysis also predict that the relation between involved parameters in a problem must be in the form of relation between dimensionless quantities. It means

$$\frac{Vt}{L} = f\left(\frac{\rho V L}{\mu}\right); \quad (2.13)$$

dimensionless time is a function of Reynolds number. Physically speaking by this time we are seeking for a measure of allowable time step in our controller. In other words this formula defines a time constant for this setup ($L/V$). The higher the time constant the higher the time needed for information to reach to the intended point. Later, using this definition we categorize the flow to two short time delay and long time delay setups. By short time delay systems we mean those regimes in which transferring time is shorter than the time constant of the controller (like sampling time). Therefore the controller does not feel the effect of the time delay. While in long time delay systems, after a control signal applies to the flow, it takes a considerable time to affect the intended point.

The basic idea for measuring a numeric relation between these two dimensionless quantities is to set a periodic wall jet stream and to measure $u_y$ along the wall. The graph of $u_y$ with respect to time has a phase difference with the wall jet oscillating velocity. We define this phase difference (converted to time) the time delay of the flow at this point and with this Reynolds number. The setup is shown in Figure 3-19. All measurements are done after a transient period (see Figure 3-20). We used Gambit for mesh generation and Fluent 6.1 for simulation. The fluid is air and measurements have been transferred to Techplot for high precision curve fitting and computing the phase difference. Although the phase difference shows a continuous change while moving from exit point of wall jets, the amplitude of $u_y$ does not change continuously,
probably because of the local small bubbles or disturbances and instabilities resulting from transient in wall jet flow (see Gogineni and Shih [9]).

The final results are shown in Figure 3-21. The background flow in as same order of magnitude as the wall jet flow (but is not constant in all simulations). Reynolds number is measured by mean velocity of wall jet flow and length along the wall. Figure 3-21 shows for a very small Reynolds number the time delay of the system tends to zero. It means the information from wall jet exit point reaches to the point of separation in a very short time. This behavior resembles solid media and of course is expected in very low Reynolds number regimes in fluid mechanics. For moderate Reynolds number, which is the range of our interest, the behavior is almost independent of Reynolds number and dimensionless time is equal to one. As the simulations have been done for different frequencies and different background flow, it seems that the only dominant term in determining the time delay in our setup is the mean velocity of the wall jet and the length along the wall. This behavior is similar to solid balls kicked individually along straight path without interacting with each other. The dimensionless number \( T^* = \frac{t_d V}{L} \) in which \( t_d \) is the minimum time constant of the controller and \( V \) is the jet exit speed and \( L \) is the distance of the jet from the desired separation point is the basis criteria for validating no-delay assumption. The higher this number, the lower time delay.

Now we are at the point to categorize the setup into two main groups: short time delay and long time delay. The system is short time delay if \( T^* > 1 \). In this case we can safely use the controllers in section 2.1. The main idea is to adjust the sampling time to get a suitable \( T^* \) number. Physically, short time delay setup means that after changing the wall jet output, controller software waits until the desired separation point would be affected by this change, after this time, it measures the error again and set new values for wall jet outputs.

### 2.3 Fuzzy Control

Fuzzy Logic is a model free estimator to approximate a system via linguistic input-output associations. Fuzzy Logic is a departure from classical two-valued sets and logic, that uses qualitative linguistic (e.g. high, big, hot) system variables and a continuous range of truth values in the interval \([0,1]\), rather than strict binary (True or False) decisions and assignments.
The motivation for fuzzy control comes from the fact that the world is nonlinear and linear models often are a good 'local' approximation. Although, by the way, linear control usually is sufficient (today most of the controllers in the world industrial projects are PID controllers), increasing functionality and complexity, rapid production change, higher precision and wider operation requires more sophisticated controller approach. Nonlinearity in control system may comes from the nonlinear system or in linear systems by putting constraint on inputs, states and/or outputs. Fuzzy Logic, introduced by Lotfi Zadeh (Zadeh 1965 [24]) is one kind of nonlinear mapping. Neural Networks and Wavelets are two other examples of nonlinear mapping. Modern fuzzy mathematics seems to become a standard tool of modeling systems with nonprobabilistic uncertainties. Mathematically, ordinary or crisp sets are the sets $A = \{x | x \in A\}$ with characteristic function

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise.} \end{cases}$$

The set operations are intersection, union, complement and subset. In fuzzy sets a membership function is

$$\mu_A : x \to [0, 1]$$

$$A = \{(x, \mu_A(x)) | x \in U\}$$

$\mu_A(x)$ expresses to what degree “$x$ is $A$”. Fuzzy logic is another way of looking at the world; continuous thinking. For example in Figure 3-22 a specific temperature can be to some degree ‘cold’ and to some degree ‘ok’. The type of membership in Figure 3-22 is triangular-trapezoidal function, the other common choices for shape of membership function is Gaussian and Singleton.

Set operations in fuzzy logic are

- intersection (AND): $A \cap B$, $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
- union (OR): $A \cup B$, $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
- complement (NOT): $A'$, $\mu_A'(x) = 1 - \mu_A(x)$.

Figure 3-23 shows example of these operations. (there are different definition for operations. One may consider AND as product or T-norm and OR as bounded sum or T-conorm of the elements.)
Fuzzy logic is a generalization of ordinary boolean logic. In fuzzy logic, propositions have truth values between 0 and 1 (see Figure 3-23.) AND, OR and NOT connects simple proposition into compound propositions. A typical fuzzy inference is written in the general form of “If ((fuzzy proposition)) Then ((fuzzy proposition)) ”. In fuzzy control two main types of inference is used: Mamdani Inference Systems and Takagi-Sugeno Inference System. The general rule in Mamdani inference system is “IF x IS A THEN u IS B” where x is the controller input and u is the controller output and A and B are fuzzy sets. To calculate the controller output we follow these steps:

i. input fuzzy set evaluation, for example if the point of separation is at $x = -10$ it is OK to degree 0.6 and Left to degree 0.4. Note that in general these two sets are not complement, i.e., the summation of these two numbers can be greater or lesser than unity.

ii. calculation of degree of fulfillment of each rule. For example in MIMO (Multi Input Multi Output) systems like combustion chamber control, one rule is 'If the pressure is high and temperature is very low then do action number one.' Now if pressure is high to degree 0.5 and temperature is very low to degree 0.65 then the degree of fulfillment of this rule is minimum of these two number and so 0.5. This procedure repeats for all rules in the system.

iii. calculation of fuzzy output of each rule. Based on the degree of each rule, output of that rule can be computed.

iv. aggregation of fuzzy outputs.

v. defuzzification. In defuzzification process we assign to each fuzzy set a number based of the defuzzifier rule. Common choice is center of gravity of the output graphs.

in summary, Fuzzy control is a user friendly way of designing low-order nonlinear controller. It can be easily extended to multi input multi output systems and allows explicit representation of process control knowledge. Fuzzy control has the capability of identification and adaption. On the other hand, it is computationally expensive and it is very difficult to predict its behavior or prove its stability.

2.3.1 Separation Control by Fuzzy Logic

If the time delay in the system is high enough, applying the simple classical controller can not impose separation correctly and furthermore the system response may easily diverge. We
discussed earlier that choosing a big sampling time results in a late convergence and the danger to further instability. Assume a moment at which the error is very high, the controller sets the actuator to a very high strength and waits for the next signal that will be available at the next sampling time. But the next sampling time will come after a long while. During this long period, the high strength jet, pushes the separation point over the set point. When the next signal is read, it is again high error with opposite sign and this story goes on. Oscillation around the set point is another issue in using classical controller. Suppose that the separation point is located at the desired set point. Any small perturbation causes controller to produce actuation to suppress the disturbance. Because of the time delay, this actuation reaches to the intended separation point after a long time, while during this period controller has continued to actuate. It is the initiation of oscillation about a set point. The amplitude of these oscillations is related to the time delay; the higher amplitude, the bigger amplitude. Figure 3-25.a,b and c show result of applying classical controller to a long time delay setup. The oscillation around the final values are not because of the order of the system and will not suppress during time. You can not avoid them by changing the coefficient of the controller, they happen because of the time delay in the fluid. Classical controller gains a zero mean error, but with non-zero instantaneous error (i.e., there is a steady state periodic error in system, the amplitude and frequency depends on the time delay, sampling time and control parameters). The Fuzzy controller for a long delay system leaves a steady state error (which can be set to be small enough) but with no oscillatory motion. For most application the fuzzy remedy is much better, because oscillations are sources of strange behaviors and instability.

Since viscous flow dissipates energy, the flow can be approximated by a second order dissipative system. For this system a combination of a proportional term and an integrator term in the feedback rout can suppress the error (adding further a differentiator can modify the transient behavior). To design a fuzzy controller we use the integration idea, i.e., our fuzzy controller based on the measured data, determines how much should be the increment or decrement in the actuator strength in the next time step. Designing the membership functions is mostly based on experience. The first case that we are interested in is the necessary condition for steady background flow. This control objective as we discussed in section 2.1 needs one wall jet in opposite direction of background flow. The controller is a SISO (Single Input Single
Output) system. We design input and output membership functions to suppress $u_y$ at the desired point. Our experiences showed that when $u_y = \pm 20 s^{-1}$ the steady separation point is close to intended separation point by the error less than 5%. Therefore the objective of the fuzzy control is to keep $u_y$ in this range. Figure 3-24 shows the input and output membership functions. The abbreviations are S, M and H prefixes which stand for Small, Medium and High, P, N, A and S suffixes that stand for Positive error, Negative error, Add to input and Subtract from the input respectively. EQ and Z stand for Equilibrium and Zero. The fuzzy rules, as the number of functions in input and output are the same, are easily one by one correspondence between membership functions, e.g., when the input is in EQ (equilibrium) region, output is corresponds to Z region. In a same manner, SP (small positive error) corresponds with SA (small add to input) and so on. There are many freedom to design membership functions to achieve the desired behavior. By contracting the input area, the response will become more sharp and as a result more oscillations (overshoot) before final damping will be occurred. Designing fuzzy membership function depends on time sampling and flow specification, but when a fuzzy controller is designed, it is robust enough for a relatively wide range of variation of system parameters.

To design a controller for satisfying sufficient condition of steady separation, the control problem becomes a MIMO (Multi Input Multi Output) problem. Using fuzzy approach we can combine two objectives and derive a coupled controller that tries to satisfy both conditions simultaneously. We need two separate input membership functions and two separate output membership function. For the first couple of input-outputs, we reuse the previous membership functions (see Figure 3-24), then we design the second pair of membership functions in accordance with the first couple (see Figure 3-24). In Fuzzy Control literature it is usual to speak about the surface of the controller which is the surface of the outputs versus inputs. These surfaces, that for our problem are two three-dimensional surfaces, are shown in Figure 3-27. Although the number of membership functions is proportional to the number of inputs and outputs, the number of fuzzy rules change dramatically. Tables 2.1 and 2.2 show the fuzzy rules for our two input-two output fuzzy controller.

To satisfy necessary condition of unsteady separation we just need to change the feedback of the fuzzy controller from $u_y(\gamma, 0, t)$ to $\int_0^t u_y(\gamma, 0, t)dt$. Although the basis of designing fuzzy
controller is still the same, the membership functions should changes in a consistent manner. To satisfy sufficient condition of unsteady separation, we use a feedback for controlling \( \int_{0}^{t} u_y(\gamma, 0, t)dt \) and another feedback to impose a nonzero \( u_{xy} \).

In developing all control algorithm we use one of the wall jets (the one that blows against the background flow) as the controller for the point of separation and the second wall jet (that blows in the same direction of background flow) for controlling the strength of separation. We can change the role of these two wall jets by measuring the background flow direction upstream and/or downstream. By applying this idea we can use this controller for those unsteady background flows that change their directions.

2.4 Conclusion

Control of the point of separation is a subject of interest in many fluid mechanics problems; control of the drag and lift in airplanes, control of the mixing which is highly affected by the point of separation and control and stall prevention in turbine blades are only a few examples. The nonlinear and complex nature of Navier-Stokes equation as the main and most accurate
descriptor of fluid motion turn the flow control into one of the most exciting control problems in the past decades. The complication in the definition of the point of separation - still a matter of controversy - is another feature that makes separation control challenging. In this section we used the necessary and sufficient conditions for steady separation (Prandtl 1904) and unsteady separation (Haller 2004 [10]) and a descriptive idea to control the point of separation by feedback. We aimed to use measurement at only one point (intended point of separation), as opposed to chapter I in which we use measurements along the wall. We showed that by classical use of feedback signal and by integrating the error in the feedback route we can suppress error in a wide range of flow conditions. We then went further and showed that inertia terms in Navier-Stokes equation cause a delay in transferring information between two points and must be considered in designing a stable controller. We defined a delay time criterion for control setup and showed that for short time-delay setup, the classical controller works very good. For long time-delay systems we propose to use Fuzzy Control. Although Fuzzy Logic is a well known and widely used control procedure in many industrial applications, so far it has not been used in flow control. We began with a brief introduction on Fuzzy Control. Then we designed fuzzy controllers for controlling the point of separation for different objective functions. In each case numerical simulations are presented to verify the utility of the controller. This work can also be extended for compressible flow but it needs a new definition for time delay. For short time delay systems, the classical controller still works. For long time delay compressible flows our fuzzy controller needs to be modified.
Bibliography


Chapter 3

FIGURES
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(a) 

(b) 

(c) 

(d)
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