A QUARK MODEL FOR NUCLEAR MATTER

by

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ABSTRACT

Properties of nuclear matter are calculated with a quark model in the spirit of the MIT bag model.A many body wavefunction is written in terms of the quark degrees of freedom.With this wavefunction local correlations are built in the many quark system such that three quarks close to each other are dominantly in a color singlet nucleon state. The energy of the system is calculated as the sum of the contributions from, the kinetic energy of the quarks,a bag energy proportional to the volume of the system and the interaction of the quarks through gluon exchange treated in perturbation theory to lowest order in the strong coupling constant α_s.The energy of an uncorrelated fermi gas is calculated within the same approximation.At low densities,the system described by the correlated wavefunction has a lower energy per baryon compared to the fermi gas whereas the fermi gas gives a lower energy at high densities.This result suggests a transition from the correlated state of quark matter,i.e. nuclear matter,at low densities to a quark fermi gas at high densities.This calculation indicates that this transition can be expected to occur at a few times the normal nuclear density.

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CONTENTS

I.Introduction

With accumulating evidence indicating quantum chromodynamics to be the underlying theory of the strong interactions,considerable effort has been devoted to calculating the predictions of QCD using a wide variety of methods and comparing with experiment whenever possible[1].The asymptotic freedom property of QCD makes it possible to use perturbation theory in calculating processes involving momentum transfers large compared to the QCD scale parameter[2] $\Lambda \approx$ .05-.15 Gev.Our knowledge of the low energy behavior of QCD,however,is rather limited.The strong coupling phenomenon demands genuinely new methods for calculating with a quantum field theory.Lattice gauge theories provide[3] us with the only numerical results starting from the original field theory.Although the results so far obtained are encouraging we are still not in a position to calculate physical observables like hadron masses with lattice gauge theories because of numerical limitations. The MIT bag model[4] offers a simple and physically motivated framework for calculating low energy bound states of quarks and gluons and it has been quite succesful in reproducing the low energy hadron spectra with a small number of adjustable,yet physically meaningful,parameters.With the QCD picture of hadrons made up of quarks and gluons gaining a firmer footing,there has been growing interest in determining the consequences of this hadron substruce in the realm of nuclear physics[5] and ultimately achieving a unified understanding of the strong interactions from the particle and nuclear physics perspectives. The parameters of the bag model are the bag constant B,the coefficient of the zero point energy Z_o,the strong coupling constant α_s and the strange quark mass m_s whenever strange hadrons are concerned. In this model,hadrons are bound states of quarks interacting

through exchange of gluons.All the complicated aspects of this problem,the strong coupling phenomenon,self interaction of gluons etc., are represented by associating a positive energy density B wherever the quark or gluon fields are nonvanishing.This procedure automatically ensures confinement,i.e. the observation that aggregates of quarks always occur in color singlet states, because any system carrying a net color charge would have an electric field extending over the whole space and hence would have an infinite energy.The zero point energy term is proportinal to Z_o/R,where R is the length scale associated with the problem.This energy is a consequence of doing field theory in a finite domain,i.e. "inside the bag".It is associated with the zero point energy of all the modes in the theory.Usually,this extensive energy contribution is discarded in any field theoretical calculation since the total volume over which the fields are nonvanishing does not change in any process.This volume,however, is a relevant variable in the bag model.(The 1/R behavior requires a more detailed calculation[4]).The strong coupling constant, $\alpha_s=g_s^2/4\pi$,is directly significant in the context of perturbation theory.The second,and lowest,order contributions to the interaction energy are proportional to α_s,with the interaction Hamiltonian being proportional to g_s.These contributions are customarily represented by the Feynman diagrams indicated in figure 1.The implicit motivation for a perturbation treatment is of course the expectation for a convergent series.The popularity of the method rests perhaps mostly on its very successful application to calculation of processes with quantum electrodynamics[6].In the case of QCD,however,the situation is disconcerting.As compared to $\alpha_{QED}\approx 1/137$ for QED,many calculations indicate that $\alpha_s\approx 1$ for QCD at low energies.Thus we have no reason to believe that the usual perturbation series in powers of α_s will be convergent.We

can,however, still utilize the corrections to physical quantities calculated as a power series around α=0 by using wavefunctions that possess nonperturbative features.The strong coupling phenomenon is not completely new.It emerges as a setback to the naive application of perturbation theory in the conventional nuclear many body problem in the same way.[7] Because of the strong short range repulsion in the nucleon nucleon interaction,any perturbation expansion starting from a wavefunction built as an uncorrelated product of single particle wavefunctions is bound to be divergent.The solution to this problem is offered by Brueckner theory[8] where one starts out with a correlated wavefunction that does not allow any pair of particles to get very close to each other.This wavefunction indeed is more similar to the true many nucleon wavefunction as we know that the interaction energy of the nucleons in a nucleus is not infinite.In other words Brueckner theory incorporates a partial resummation of a (divergent) perturbation series into the initial wavefunction and produces a convergent series for the calculation of physical quantities of interest,e.g. the energy,with this wavefunction.We can adopt the same philosophy in the case of calculating the low energy states of quarks with QCD,especially if we are willing to relax the rigor in the perturbation treatment.We know again that the true wavefunctions for quarks will suppress the configurations where the corresponding energy will be very high,e.g. two quarks separated by a large distance.We can then eliminate these configurations from our wavefunctions from the start and leave the strong coupling aspect of the problem aside. This is what the bag model does in essence,and whether we have been succesful in eliminating these undesirable components from our wavefunction remains yet to be seen for no one has yet calculated higher order corrections in the framework of the bag model.Doing

this with mathematical rigor as in the case of Brueckner theory still requries much more effort because the treatment of bound states in a field theory is more complicated than it is in nonrelativistic many body theory.The results of the bag model treatment are quite encouraging and indicate at least that we are proceeding in the right direction.

The extension of the bag model philosophy to the case of many nucleons is not obvious.Perhaps the most naive picture of a nucleus would be a collection of many bags standing for nucleons.This picture is not very promising,however,because the mere existence of nuclei is due to the interaction of their constituent nucleons,which in turn are nothing but quarks and gluons.If the nucleons do not overlap,they do not interact. Also the picture of nonoverlapping nucleons is not consistent with the experimental observation of the density of valence quarks in nuclear matter,$\rho \approx .5/fm^3$ and the value obtained by the bag model for the same density in a nucleon,$\rho \approx .7/fm^3$.There is not much volume in nuclear matter that is not covered by nucleons.We can circumvent this difficulty by letting the bags overlap.The dynamics of the surface of the bag,however,is determined by the boundary conditions and in that sense it is not a real dynamical variable.A nucleon nucleon interaction energy determined by the dynamics of the bag boundaries when the nucleons overlap,represents the energy difference in the overlapping and nonoverlapping configurations and would have cotributions from all the terms in the bag Hamiltonian including the volume and zero point energy terms and the kinetic energies of the quarks.This energy may represent the hard core in the N-N interaction but it is contrary to our intuition regarding the N-N interaction at larger distances where it has been customarily viewed from the point of meson exchange.Meson exchange translates into QCD as quark

and gluon exchange and we would want the interaction that produces nuclear binding to be due to quark and gluon exchange among nucleons.

Once we have overlapping wavefunctions for quarks in different nucleons ,the antisymmetrization of the many body wavefunction becomes important and one can not associate a quark with a definite nucleon.A nucleon is then just a very strong local correlation in the many quark wavefunction.The success of conventional nuclear physics approach of taking nuclei to be made out of nucleons reminds us that these correlations are still the most prominent feature of the many quark wavefunction.One way of accomplishing this is by building the many quark wavefunction as an antisymmetrized product of three quark wavefunctions representing nucleons at different places.This situation is similar to the case of electrons in a crystal.An electron does not belong to a particular atom yet the electron density is modulated by the presence of the crystal of nuclei,and in turn it is this modulated density that holds the crystal together.In the case of nuclear matter there is no crystal in the background and it is not the one body density that is modulated but instead the two body correlations are modified.

In this work we attempt to describe nuclear matter by constructing a many body wavefunction for quarks.We incorporate the features which we believe are important for the quark wavefunctions.We are after indicative order of magnitude results which can hopefully be refined later.

The organization of the remainder of this work is as follows.In section II we work with simple nonrelativistic models to illustrate the basic method.In section III we work on the more realistic systems after developing the necessary tools for the relativistic calculation.A

comparison of the quark wavefunctions used in this work with the bag model wavefunctions is presented in section IV. Section V is on the incorporation of massive strange quarks and finally a discussion of the results and conclusions are presented in section VI.

IIa.A Simple Model :

The ground state of a non-interacting fermion system is a fermi gas obtained by filling up the lowest plane wave states in accordance with the Pauli principle.When we introduce an interaction,the wave function will be modified to give the lowest possible expectation value of the Hamiltonian,kinetic plus the interaction energy,as demanded by the Rayleigh-Ritz[10] variational principle.From the studies of superconducting systems[11],we know that any attractive interaction among the particles near the fermi surface will result in the correlated BCS[12] ground state where the particles near the fermi surface are coupled pairwise to form Cooper pairs of zero total momentum and spin.Since the BCS ground state yields a lower energy,we expect the true ground state wave function to be more similar to the correlated wave function than to a fermi gas.In the problem we want attack,however,we would like the particles to be correlated in configuration space instead of momentum space and would not want this phenomenon to be restricted to only the particles close to the fermi surface.In other words,it is the quarks that are spatially close that we want to couple to a nucleon state and not quarks of opposite momenta whose wave functions extend over the whole system.As we indicated in the introduction,one can accomplish this by building the many body wave function by antisymmetrizing the product of few body wavefunctions coupled to the correct quantum numbers.[13]

To begin with,let us assume the two body interaction to be given as,

$$V_{12} = \alpha \frac{\vec{S}_1 \cdot \vec{S}_2}{r_{12}} \qquad (IIa.1)$$

If we had only two particles,this interaction would favor them to be coupled to a spin

singlet because by using,

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left[(\vec{S}_1+\vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2 \right] \qquad (IIa.2)$$

we have

$$\langle S=0|\vec{S}_1\cdot\vec{S}_2|S=0\rangle = \frac{-3}{4} \quad , \quad \langle S=1|\vec{S}_1\cdot\vec{S}_2|S=1\rangle = \frac{1}{4} \qquad (IIa.3)$$

so the ground state wave function would have the form,

$$\psi(1,2) = u(\vec{r}_1)u(\vec{r}_2)\left\{\frac{|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2}{\sqrt{2}}\right\} \qquad (IIa.4)$$

We expect the many body wave function to carry the property that the particles that are close to each other,or in the quantum mechanical language that have considerably overlapping wave functions,to be coupled to spin singlet states. To accomplish this we first label the two body wave functions $\Psi(1,2)$ by an index i that denotes the location of the pair.$\Psi_i(1,2)$ will be localized around the point r_i.The locations of the points r_i will be determined later.Thus we have,

$$\psi_i(1,2) = u_i(\vec{r}_1)u_i(\vec{r}_2)\left\{\frac{|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2}{\sqrt{2}}\right\} \qquad (IIa.5)$$

The many body wave function will be the antisymmetrized product of the two body wave functions Ψ_i.This can be most easily done in the second quantization formalism by defining the creation and annihilation operators, $a_i^\dagger$ and a_i (where α stands for $\alpha = \uparrow$ or $\downarrow$.),so that

$a_i^\dagger$ creates the single particle state $|u_i\rangle$ acting on the vacuum $|0\rangle$.

$$|u_{i\alpha}\rangle = a^\dagger_{i\alpha}|0\rangle \tag{IIa.6}$$

and the a's satisfy the anticommutation relations

$$\left\{a_{i\alpha}, a_{j\beta}\right\} = \left\{a^\dagger_{i\alpha}, a^\dagger_{j\beta}\right\} = 0 \quad , \quad \left\{a_{i\alpha}, a^\dagger_{j\beta}\right\} = \delta_{ij}\delta_{\alpha\beta} \tag{IIa.7}$$

We then define

$$S^\dagger_{i,S=0} = a^\dagger_{i\uparrow} a^\dagger_{i\downarrow} \tag{IIa.8}$$

which creates the state $|\Psi_i(1,2)\rangle$ acting on the vacuum. We note that the antisymmetrization aspect is automatically taken care of by the anticommutation relations (IIa.7) of the a's and we do not need to write

$$S^\dagger = \frac{1}{2}\left(a^\dagger_{i\uparrow}a^\dagger_{i\downarrow} - a^\dagger_{i\downarrow}a^\dagger_{i\uparrow}\right) = \frac{1}{2}\left(a^\dagger_{i\uparrow}a^\dagger_{i\downarrow} + a^\dagger_{i\uparrow}a^\dagger_{i\downarrow}\right) = a^\dagger_{i\uparrow}a^\dagger_{i\downarrow}$$

We finally construct the antisymmetrized product of the Ψ_i's, by applying all the S 's on the vacuum. Thus,

$$|\psi\rangle = \prod_{i,\alpha} S^\dagger_{i,\alpha}|0\rangle \tag{IIa.9}$$

is our correlated many body wave function.

We now turn to the spatial part of the Ψ_i's given by the u_i's. They will be single particle orbitals localized around the point $\vec{r}_i$. The most localized function around a point

would be a delta function, $\delta^3(\vec{r}-\vec{r}_i)$,or more precisely the square root of a δ function if we want it to be square integrable.This choice would be undesirable,however, as can be seen from the Fourier decomposition,

$$\delta^3(\vec{r}-\vec{r}_i)=\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot(\vec{r}-\vec{r}_i)}\,d^3\vec{k} \qquad (IIa.10)$$

it contains components with momenta up to infinity and,we would not want such a drastic localization of single particle orbits in space,either.We can control the degree of localization of the function u_i if we let the upper limit of the integral in (IIa.10) to be a variable k_F and define,

$$u_i(\vec{r}-\vec{r}_i)=\frac{1}{N}\int_{|\vec{k}|<k_F} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_i)}\,d^3\vec{k} \qquad (IIa.11)$$

where N is a normalization factor to be determined later.We can easily evaluate (IIa.11) to obtain,

$$u_i(\vec{r})=\frac{4\pi k_F^3}{N}\frac{j_1(k_F|\vec{r}-\vec{r}_i|)}{k_F|\vec{r}-\vec{r}_i|} \qquad (IIa.12)$$

where j stands for the spherical Bessel function.As one can see from the form of (IIa.12),as k_F is increased one obtains a more localized function until in the limit $k_F\to\infty$, u_i becomes a δ function at $\vec{r}_i$.

The anticommutation relations (IIa.7) demand the orthogonality relations

$$\langle u_{i\alpha}\,|\,u_{j\beta}\rangle=\delta_{ij}\delta_{\alpha\beta} \qquad (IIa.13)$$

which are essential for the development of the second quantized formalism. The orthogonality relation with respect to the spin indices α,β is obviously satisfied so we only need to satisfy,

$$\int u_i^*(\vec{r})\, u_j(\vec{r})\, d\tau = \frac{1}{N^2} \int_{\substack{|\vec{k}|<k_F \\ |\vec{k}'|<k_F}} d^3\vec{k}\, d^3\vec{k}'\, d^3r\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_i)}\, e^{i\vec{k}'\cdot(\vec{r}-\vec{r}_j)}$$

$$= \frac{1}{N^2} \int_{\substack{|\vec{k}|<k_F \\ |\vec{k}'|<k_F}} d^3\vec{k}\, d^3\vec{k}'\, (2\pi)^3\, \delta^3(\vec{k}-\vec{k}')\, e^{i\vec{k}\cdot\vec{r}_i - i\vec{k}'\cdot\vec{r}_j} = \frac{(2\pi)^3}{N^2} \int_{|k|<k_F} e^{i\vec{k}\cdot(\vec{r}_i-\vec{r}_j)} = \frac{(2\pi)^3}{N^2}\, 4\pi k_F^3\, \frac{j_1(k_F|\vec{r}_i-\vec{r}_j|)}{k_F|\vec{r}_i-\vec{r}_j|} \qquad (IIa.14)$$

We realize that it is not possible to satisfy $\langle u_{i\alpha} | u_{j\beta}\rangle = \delta_{ij}\delta_{\alpha\beta}$ with the form of u_i given by (IIa.12) except for a small number of points $\vec{r}_i$, unless we can come up with a uniform lattice where the distance between any two points is a root of the spherical Bessel function of first order! We can get around this difficulty in the following way. We keep the integrand in (IIa.11) the same but change the region of integration to a fermi cube from a fermi sphere so that,

$$u_i(\vec{r}) = \frac{1}{N} \int_{-k_F}^{k_F} dk_x\, dk_y\, dk_z\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_i)} = \frac{8}{N}\, \frac{\sin k_F(r_x - r_{ix})\, \sin k_F(r_y - r_{iy})\, \sin k_F(r_z - r_{iz})}{(r_x - r_{ix})(r_y - r_{iy})(r_z - r_{iz})} \qquad (IIa.15)$$

so the orthogonality relation becomes,

$$\int u_i^*(\vec{r})\, u_j(\vec{r})\, d\vec{r} = \frac{1}{N^2} \int_{-k_F}^{k_F} d\vec{k}\, d\vec{k}'\, d\vec{r}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_i)}\, e^{-i\vec{k}'\cdot(\vec{r}-\vec{r}_j)} = \frac{(2\pi)^3}{N^2} \int_{-k_F}^{k_F} d\vec{k}\, d\vec{k}'\, e^{i\vec{k}\cdot(\vec{r}_j-\vec{r}_i)}\, \delta^3(\vec{k}-\vec{k}')$$

$$= \frac{(2\pi)^3}{N^2} \left[\int_{-k_F}^{k_F} dk_x\, e^{ik_x(r_j-r_i)_x}\right] \left[\int_{-k_F}^{+k_F} dk_y\, e^{ik_y(r_j-r_i)_y}\right] \left[\int_{-k_F}^{+k_F} dk_z\, e^{ik_z(r_j-r_i)_z}\right]$$

$$= \frac{8(2\pi)^3}{N^2}\, \frac{\sin k_F(r_j-r_i)_x}{(r_j-r_i)_x}\, \frac{\sin k_F(r_j-r_i)_y}{(r_j-r_i)_y}\, \frac{\sin k_F(r_j-r_i)_z}{(r_j-r_i)_z}$$

$$= \frac{8(2\pi)^3}{N^2} \frac{\sin k_F(r_j-r_i)_x}{(r_j-r_i)_x} \frac{\sin k_F(r_j-r_i)_y}{(r_j-r_i)_y} \frac{\sin k_F(r_j-r_i)_z}{(r_j-r_i)_z} \qquad (IIa.16)$$

Thus,if we choose

$$N=(2\pi)^{3/2}(2k_F)^{3/2} \quad \text{and} \quad (\vec{r}_i-\vec{r}_j)_\alpha = \frac{n\pi}{k_F} \quad \alpha = x,y,z \qquad (IIa.17)$$

we can satisfy $\langle u_i | u_j \rangle = \delta_{ij}$.The last condition on the vectors $\vec{r}_i$ can easily be satisfied if the $\vec{r}_i$ are placed on a simple cubic lattice of spacing π/k_F.With this accomplished,we have an orthonormal basis of localized wave functions equivalent to the basis of plane waves of momenta inside the fermi cube.We will use this basis for constructing our many body wave functions in the manner described earlier.

We now can calculate the expectation value of the Hamiltonian,

$$H = T+V = \sum_{\alpha,\beta} t_{\alpha\beta} a^\dagger_\alpha a_\beta + \frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta,\gamma\delta}\, a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma \qquad (IIa.18)$$

written in the second quantized notation.Our wave function is,

$$|\psi\rangle = \prod_i S^\dagger_i |0\rangle = \prod_i a^\dagger_{i\uparrow} a^\dagger_{i\downarrow} |0\rangle \qquad (IIa.19)$$

It is constructed by putting two particles coupled to spin singlet in each localized orbit i.By virtue of the form of (IIa.19)$|\Psi\rangle$is a Slater determinant.Furthermore since each localized orbit is a linear combination of momentum eigenstates,this Slater determinant,which can be obtained from the determinant of plane waves by adding rows and columns,is equivalent to the fermi gas having a spin up and a spin down particle at each momentum state.[34] Thus

our correlated wave function describes nothing but a fermi gas!We will continue working with this wave function in the form (IIa.19) to prepare the way for the calculation to be carried out in section IIb where the coupled wave function will not be a determinant.The equivalence between the fermi gas and Ψ given in (IIa.19) also provides us with illustrative checks on the calculation of two body operators,most importantly the interaction energy which we now proceed to calculate.

The expectation value of the two body operator $\hat{V}$ in a determinantal wave function $|\Psi\rangle$is given by the usual expression,[33]

$$\langle\psi|\hat{V}|\psi\rangle=\frac{1}{2}\sum_{i\alpha,j\beta}\langle i\alpha,j\beta|V|i\alpha,j\beta\rangle-\langle i\alpha,j\beta|V|j\beta,i\alpha\rangle=\frac{1}{2}\sum v_{i\alpha j\beta,i\alpha j\beta}-v_{i\alpha j\beta,j\beta i\alpha} \quad (IIa.20)$$

where $(i\alpha),(j\beta)$ stand for the single particle states out of which Ψ is constructed.All we need to calculate is the direct and exchange terms $v_{ij,ij}$ and $v_{ij,ji}$ respectively.This is straightforward,

$$v_{i\alpha j\beta,i\alpha j\beta}=\langle\alpha\beta|\vec{s}_1\cdot\vec{s}_2|\alpha\beta\rangle\int d^3\vec{r}_1 d^3\vec{r}_2\, u^*(\vec{r}_1-\vec{r}_i)u^*(\vec{r}_2-\vec{r}_j)\,V(\vec{r}_1-\vec{r}_2)\,u(\vec{r}_1-\vec{r}_i)\,u(\vec{r}_2-\vec{r}_j) \quad (IIa.21)$$

We insert the expressions for the wave functions u given in (IIa.14) and let $\vec{r}_1-\vec{r}_i\to\vec{r}_1$,$\vec{r}_2-\vec{r}_j\to\vec{r}_2$ in the integrand.With these and,

$$V(\vec{r}_1-\vec{r}_2)=\frac{\alpha}{|\vec{r}_1-\vec{r}_2|}$$

we obtain

$$v_{i\alpha j\beta,i\alpha j\beta}=\frac{\langle\alpha\beta|\vec{s}_1\cdot\vec{s}_2|\alpha\beta\rangle}{k_F^6\pi^6}\int\frac{\sin^2k_Fr_{1x}\sin^2k_Fr_{1y}\sin^2k_Fr_{1z}\sin^2k_Fr_{2x}\sin^2k_Fr_{2y}\sin^2k_Fr_{2z}\,d^3\vec{r}_1\,d^3\vec{r}_2}{r_{1x}^2\,r_{1y}^2\,r_{1z}^2\,r_{2x}^2\,r_{2y}^2\,r_{2z}^2\,|\vec{r}_1-\vec{r}_2+\vec{r}_i-\vec{r}_j|} \quad (IIa.22)$$

with the same manipulations the exchange integral becomes,

$$V_{i_\alpha j_\beta, j_\beta i_\alpha} = \frac{\langle\alpha\beta|\vec{s}_1\cdot\vec{s}_2|\beta\alpha\rangle}{\pi^6 k_F^6} \int \frac{\sin k_F r_{1x}\sin k_F r_{1y}\sin k_F r_{1z}\sin k_F r_{2x}\sin k_F r_{2y}\sin k_F r_{2z}}{r_{1x}r_{1y}r_{1z}r_{2x}r_{2y}r_{2z}}$$

$$\times \frac{\sin k_F(r_{2x}+r_{jx}-r_{ix})\sin k_F(r_{2y}+r_{jy}-r_{iy})\sin k_F(r_{2z}+r_{jz}-r_{iz})\,\sin k_F(r_{1x}+r_{ix}-r_{jx})\sin k_F(r_{1y}+r_{iy}-r_{jy})\sin k_F(r_{1z}+r_{iz}-r_{jz})\,d^3r_1\,d^3r_2}{(r_{2x}+r_{jx}-r_{ix})(r_{2y}+r_{jy}-r_{iy})(r_{2z}+r_{jz}-r_{iz})(r_{1x}+r_{ix}-r_{jx})(r_{1y}+r_{iy}-r_{jy})(r_{1z}+r_{iz}-r_{jz})} \qquad (IIa.23)$$

Both of these integrals have a simple k_F dependence as can be seen by substituting $u_l = k_F r_l$

$$V^{i_\alpha j_\beta}_{\alpha\beta\,\alpha\beta} = \frac{\alpha k_F}{\pi^6}\langle\alpha\beta|\vec{s}_1\cdot\vec{s}_2|\alpha\beta\rangle \int \frac{\sin^2 u_{1x}\sin^2 u_{1y}\sin^2 u_{1z}\sin^2 u_{2x}\sin^2 u_{2y}\sin^2 u_{2z}\,d^3\vec{u}_1\,d^3u_2}{u_{1x}^2 u_{1y}^2 u_{1z}^2\; u_{2x}^2 u_{2y}^2 u_{2z}^2\,|\vec{u}_1-\vec{u}_2+k_F(\vec{r}_i-\vec{r}_j)|}$$

$$V^{i_\alpha j_\beta}_{\alpha\beta\,\beta\alpha} = \frac{\alpha k_F}{\pi^6}\langle\alpha\beta|\vec{s}_1\cdot\vec{s}_2|\beta\alpha\rangle \int \frac{d^3u_1 d^3u_2 \sin u_{1x}\sin(u_{1x}+k_F(r_{ix}-r_{jx}))\sin u_{1y}\sin(u_{1y}+k_F(r_{iy}-r_{jy}))\sin u_{1z}\sin(u_{1z}+k_F(r_{iz}-r_{jz}))}{u_{1x}^2 u_{1y}^2 u_{1z}^2\; u_{2x}^2 u_{2y}^2 u_{2z}^2\,|\vec{u}_1-\vec{u}_2+k_F(\vec{r}_i-\vec{r}_j)|}$$

$$\times \frac{\sin u_{2x}\sin(u_{2x}-k_F(r_{ix}-r_{jx}))\sin u_{2y}\sin(u_{2y}-k_F(r_{iy}-r_{jy}))\sin u_{2z}\sin(u_{2z}-k_F(r_{iz}-r_{jz}))}{u_{2x}(u_{2x}-k_F(r_{ix}-r_{jx}))u_{2y}(u_{2y}-k_F(r_{iy}-r_{jy}))u_{2z}(u_{2z}-k_F(r_{iz}-r_{jz}))} \qquad (IIa.24)$$

The dependence of the integrands on k_F is only apparent since $k_F(r_i-r_j)=n\pi$, with $n=(n_x,n_y,n_z)$, a integer vector. Thus the matrix elements v depend linearly on k_F. This is expected. Since the distances, e.g. the distance between two orbits or the size of the wave functions, scale as $1/k_F$, the matrix elements of an interaction that has a $1/r$ behavior scale as k_F.

To simplify the integrals further for numerical evaluation at the end we use the Fourier transform of

$$\frac{1}{|u_1-u_2+\vec{n}\pi|}$$

that comes from the factor v_{12}.

$$\frac{1}{|\vec{u}_1 - \vec{u}_2 + \vec{n}\pi|} = \int \frac{d^3\vec{q}\, e^{i\vec{q}\cdot(\vec{u}_1 - \vec{u}_2 + \vec{n}\pi)}}{2\pi^2 q^2} \qquad (IIa.25)$$

This yields,

$$V^{ij,ij}_{\alpha\beta,\alpha\beta} = \frac{\alpha k_F}{\pi^6}\langle\alpha\beta|\vec{S}_1\cdot\vec{S}_2|\alpha\beta\rangle \int \frac{d\vec{q}\, e^{i\vec{q}\cdot\vec{n}\pi}}{2\pi^2 q^2} \prod_\alpha \left| \int \frac{\sin^2 u_\alpha e^{iq_\alpha u_\alpha} du_\alpha}{u_\alpha^2} \right|^2$$

$$V^{ij,ji}_{\alpha\beta,\beta\alpha} = \frac{\alpha k_F}{\pi^6}\langle\alpha\beta|\vec{S}_1\cdot\vec{S}_2|\beta\alpha\rangle \int \frac{d\vec{q}\, e^{i\vec{q}\cdot\vec{n}\pi}}{2\pi^2 q^2} \left[\prod_\alpha \int \frac{\sin u_\alpha \sin(u_\alpha + n_\alpha\pi) e^{iq_\alpha u_\alpha} du_\alpha}{u_\alpha (u_\alpha + n_\alpha\pi)}\right.$$

$$\left. \times \int \frac{\sin v_\alpha \sin(v_\alpha - n_\alpha\pi) e^{-iq_\alpha v_\alpha} dv_\alpha}{v_\alpha (v_\alpha - n_\alpha\pi)}\right] \qquad (IIa.26)$$

where α=x,y,z runs over the components of a vector. We evaluate these expressions using,[14]

$$\int \frac{\sin^2 u\, e^{iq_\alpha u} du}{u^2} = \left(1 - \frac{|q_\alpha|}{2}\right) \qquad |q_\alpha| < 2$$

$$= 0 \qquad \text{otherwise}$$

and

$$\int \frac{\sin u \sin(u - n_\alpha\pi) e^{-iq_\alpha u}}{u(u - n_\alpha\pi)} = \frac{1}{2n_\alpha}\left[i\cos n_\alpha\pi \left(1 - e^{iq_\alpha n_\alpha \pi}\right) \text{sign}\{q_\alpha\}\right]$$

$$\text{for} \quad 0 \le |q_\alpha| < 2$$

$$= \frac{1}{4\eta_\alpha}\left[i \cos\eta_\alpha\pi (1 - e^{i q_\alpha \eta_\alpha \pi}) \operatorname{sign}\{q_\alpha\}\right] \quad \text{for } |q_\alpha| = 0$$

$$= 0 \qquad \text{otherwise}$$

Wih these we obtain,

$$V_{\substack{ij,ij \\ \alpha\beta\alpha\beta}} = \frac{\alpha k_F}{2\pi^2} \langle \alpha\beta | \vec{s}_1 \cdot \vec{s}_2 | \alpha\beta \rangle \int_{-2}^{+2}\int\int \frac{d\vec{q}\, e^{i\vec{q}\cdot\vec{n}\pi}}{q^2} \prod_\alpha \left(1 - \frac{|q_\alpha|}{2}\right)^2 \qquad (IIa.27)$$

and for the exchange term for i≠j,

$$V_{\substack{ij,ji \\ \alpha\beta,\beta\alpha}} = \frac{\alpha k_F}{2\pi^2} \langle \alpha\beta | \vec{s}_1 \cdot \vec{s}_2 | \beta\alpha \rangle \int_{-2}^{+2}\int\int \frac{d\vec{q}\, e^{i\vec{q}\cdot\vec{n}\pi}}{q^2} \prod_\alpha \left| \frac{\cos\eta_\alpha\pi (1 - e^{i q_\alpha \eta_\alpha \pi})}{2\eta_\alpha \pi} \right|^2 \qquad (IIa.28)$$

with $k_F(\vec{r}_i - \vec{r}_j) = \vec{n}\pi$

We can now calculate these matrix elements numerically using (IIa.27) and (IIa.28). The only dependence on i and j comes through the vector $\vec{n}$. As $|\vec{n}|$ increases the distance between the orbits increase and we expect the interaction matrix elements to become smaller. This comes about in (IIa.27) and (IIa.28) because of the more rapidly oscillating exponential integrated over a fixed domain. For the same reason it becomes harder to calculate these matrix elements numerically as $|\vec{n}|$ increases. To illustrate the behaviour of the $v_{ij,ji}$, its typical values are given for different values of n in Table 1. They indicate the expected behaviour as $|\vec{n}|$ increases. We also find that,

$$\langle\uparrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\downarrow\rangle=\frac{1}{\sqrt{2}}\left[\langle s=1,s_z=0|+\langle s=0|\right]\vec{s}_1\cdot\vec{s}_2\left[|s=0\rangle+|s=1,s_z=0\rangle\right]\frac{1}{\sqrt{2}}$$

$$=\frac{1}{2}\left[\frac{1}{4}-\frac{3}{4}\right]=\frac{-1}{4}$$

$$\langle\uparrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\uparrow\rangle=\frac{1}{\sqrt{2}}\left[\langle s=1,s_z=0|+\langle s=0|\right]\vec{s}_1\cdot\vec{s}_2\left[|s=1,s_z=0\rangle-|s=0\rangle\right]\frac{1}{\sqrt{2}}=\frac{1}{2}\left[\frac{1}{4}+\frac{3}{4}\right]=\frac{1}{2} \qquad (IIa.29)$$

and similarly,

$$\langle\uparrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\uparrow\rangle=\frac{1}{4} \qquad \langle\downarrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\downarrow\rangle=\frac{1}{4}$$

$$\langle\downarrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\uparrow\rangle=\frac{-1}{4} \qquad \langle\downarrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\downarrow\rangle=\frac{-1}{4}$$

If we calculate the expectation value of the total interaction energy with these expressions,we observe that it diverges as the total volume or the number of particles as would be expected.We instead calculate the interaction energy per particle,by adding the contributions coming from the interaction of the particles in a given orbit with all the rest of the particles and dividing this result by two,the number of particles in each orbit.This result will not depend on the particular orbit chosen because of the invariance of the infinite system by a translation vector belonging to the cubic lattice providing the sites for the different orbits.Thus we have

$$\langle\Psi|V|\Psi\rangle=\sum_{\substack{\text{orbits}\\ i}}\frac{\alpha k_F}{2\pi^2}\left[\left[\langle\uparrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\downarrow\rangle-\langle\uparrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\uparrow\rangle\right]v_{ii,ii}\right.$$

$$\left.-\sum_{\substack{\text{orbits}\\ j\neq i}}\left[\langle\uparrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\uparrow\rangle+\langle\uparrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\uparrow\rangle+\langle\downarrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\downarrow\rangle+\langle\downarrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\downarrow\rangle\right]v_{ij,ji}\right. \qquad (IIa.30)$$

where we have defined,

$$v^{\alpha\beta\gamma\delta}_{ijkl}=\langle\alpha\beta|\vec{s}_1\cdot\vec{s}_2|\gamma\delta\rangle\frac{\alpha k_F}{2\pi^2}v_{ij,kl} \qquad (IIa.31)$$

We note the absence of the direct term in the contribution coming from the interaction of the particles from different orbits that would be proportional to

$$\sum_i \sum_{j\neq i} v_{ij,ij}$$

because its coefficient

$$\left[\langle\uparrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\uparrow\rangle+\langle\uparrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\uparrow\downarrow\rangle+\langle\downarrow\downarrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\downarrow\rangle+\langle\downarrow\uparrow|\vec{s}_1\cdot\vec{s}_2|\downarrow\uparrow\rangle\right]=0$$

vanishes. Thus we see that, if the wavefunctions from different orbits did not overlap there wouldn't be any interaction between particles in different orbits. By evaluating the term in the brackets using the expression (IIa.29) for the spin matrix elements we obtain,

$$\langle\Psi|v|\Psi\rangle=\sum_i\left(\frac{-3}{4}\right)\left[v_{ii,ii}+\sum_{j\neq i}v_{ij,ji}\right]\frac{\alpha k_F}{2\pi^2} \qquad (IIa.32)$$

which yields an interaction energy per particle,

$$\frac{V}{A}=\frac{-3}{8}\left[v_{ii,ii}+\sum_{j\neq i}v_{ij,ji}\right]\frac{\alpha k_F}{2\pi^2} \qquad (IIa.33)$$

We then need to evaluate the infinite sum in the second term in principle. Because the numerical factors $v_{ij,ji}$ decrease rapidly as i and j become further apart as indicated in Table 1, we can approximate the infinite sum by the sum of contributions from nearest, next nearest neighbors and so on. We will check the reliability of this approximation later on by

calculating the same interaction energy by using the plane wave basis. We obtain then,

$$\frac{V}{A} \cong -.65 \alpha k_F \qquad (IIa.34)$$

To get the total energy we also calculate the expectation value of the kinetic energy. Since the kinetic energy operator is diagonal in the plane wave basis it is convenient to use the plane wave representation for this calculation. This yields,

$$\langle \psi | T | \psi \rangle = \sum_k T_k = \iiint_{-k_F}^{+k_F} \frac{2 d^3k \cdot V}{(2\pi)^3} \frac{k^2}{2m} = \frac{V k_F^5}{m \pi^3} \qquad (IIa.35)$$

The sum $\sum_k$ is replaced with the continuous integral in $\vec{k}$ over the fermi cube with the appropriate normalization factors so that,

$$\sum_{\vec{k}} 1 = \iiint_{-k_F}^{+k_F} \frac{2V d^3k}{(2\pi)^3} = 2 \frac{k_F^3}{\pi^3} V = \rho V$$

(Since the lattice spacing is π/k_F the density of particles is $2(k_F/\pi)^3$). So the kinetic energy per particle is,

$$\frac{T}{A} = \frac{k_F^5/m\pi^3}{2\left(\frac{k_F}{\pi}\right)^3} = \frac{k_F^2}{2m} \qquad (IIa.36)$$

We can compare this result with the average kinetic energy of a fermi gas filling a fermi sphere at the same density. To prevent confusion we will denote the fermi momentum of the fermi sphere by p_F. The density of a fermi sphere gas is,

$$\rho = \frac{g p_F^3}{6\pi^2}$$

where g is the degeneracy of each momentum state,with g=2 in our case. Equating the two densities yields,

$$2\left(\frac{k_F}{\pi}\right)^3 = \frac{2\cdot P_F^3}{6\pi^2} \quad \Rightarrow \quad k_F = \left(\frac{\pi}{6}\right)^{\frac{1}{3}} P_F \tag{IIa.37}$$

The total kinetic energy in the fermi sphere is,

$$\iiint_{-k_F}^{+k_F} \frac{2\, d^3k \cdot V}{(2\pi)^3} \cdot \frac{k^2}{2m} = \frac{P_F^5 V}{10m\pi^2} \tag{IIa.38}$$

which yields the average kinetic energy

$$\frac{T}{A} = \frac{P_F^5 V / 10m\pi^2}{P_F^3 / 3\pi^2} = \frac{3P_F^2}{10m} = \frac{3}{5}\left(\frac{\pi}{6}\right)^{2/3} \frac{k_F^2}{2m} \tag{IIa.39}$$

We see that the average kinetic energy in the fermi cube is higher by a factor of $\frac{5}{3}\left(\frac{6}{\pi}\right)^{2/3}$ .To allow local correlations in the wavefunction we have to use higher momentum components in the wavefunction which result in a higher expectation value of the kinetic energy.Adding the two contributions and an energy proportional to the volume of the system we obtain the total energy per particle.

$$\frac{\mathcal{E}}{A} = \frac{6\pi^2 B}{2P_F^3} + [-.65]\left(\frac{6}{\pi}\right)^{1/3} \alpha P_F + \frac{3}{10}\left(\frac{6}{\pi}\right)^{2/3} \frac{P_F^2}{m} \tag{IIa.40}$$

We will now calculate the interaction energy in the plane wave basis as well and compare our result for the fermi cube with the one above.Applying (IIa.20) to a fermi gas wave function yields,

$$\langle\psi|V|\psi\rangle=\frac{1}{2}\sum_{\substack{k_1s_1\\k_2s_2}}\langle k_1s_1k_2s_2|V|k_1s_1k_2s_2\rangle-\langle k_1s_1k_2s_2|V|k_2s_2k_1s_1\rangle$$

$$=\frac{1}{2}\left[\sum_{s_1s_2}\sum_{k_1k_2}\left\{\langle s_1s_2|\vec{S}_1\cdot\vec{S}_1|s_1s_2\rangle\langle k_1k_2|V|k_1k_2\rangle-\langle s_1s_2|\vec{S}_1\cdot\vec{S}_2|s_2s_1\rangle\langle k_1k_2|V|k_2k_1\rangle\right.\right. \tag{IIa.41}$$

The first,direct,term again vanishes because,

$$\sum_{s_1s_2}\langle s_1s_2|\vec{S}_1\cdot\vec{S}_1|s_1s_2\rangle=0$$

and we have

$$\sum_{s_1s_2}\langle s_1s_2|\vec{S}_1\cdot\vec{S}_2|s_2s_1\rangle=2\left\{\langle\uparrow\uparrow|\vec{S}_1\cdot\vec{S}_2|\uparrow\uparrow\rangle+\langle\uparrow\downarrow|\vec{S}_1\cdot\vec{S}_2|\downarrow\uparrow\rangle\right\}=2\left[\frac{1}{4}+\frac{1}{2}\right]=\frac{3}{2}$$

which yields

$$\langle\psi|V|\psi\rangle=-\frac{3}{4}\sum_{k_1k_2}\langle k_1k_2|V|k_2k_1\rangle \tag{IIa.42}$$

The properly normalized momentum eigenstates are

$$\langle r|k\rangle=\frac{1}{\sqrt{V}}e^{-i\vec{k}\cdot\vec{r}}$$

which, with

$$V_{12}=\alpha\frac{\vec{S}_1\cdot\vec{S}_2}{r_{12}}$$

give

$$\langle\Psi|V|\Psi\rangle=\frac{-3}{4}\frac{1}{V^2}\int_{|\vec{k}|<k_F}\frac{e^{-i\vec{k}_1\cdot\vec{r}_1-i\vec{k}_2\cdot\vec{r}_2+i\vec{k}_2\cdot\vec{r}_1+i\vec{k}_1\cdot\vec{r}_2}\,\alpha}{|r_1-r_2|}\frac{d^3\vec{r}_1\,d^3r_2\,d^3k_1\,d^3k_2\;V^2}{(2\pi)^6} \qquad (IIa.43)$$

Since we expect this expression to diverge as the volume V we concentrate on

$$\frac{\langle\Psi|V|\Psi\rangle}{V}=\frac{-3}{4}\frac{\alpha}{V}\frac{1}{(2\pi)^6}\int_{k<k_F}\frac{e^{i(\vec{k}_1-\vec{k}_2)\cdot(\vec{r}_2-\vec{r}_1)}}{|\vec{r}_1-\vec{r}_2|}d^3\vec{r}_1d^3\vec{r}_2d^3\vec{k}_1d^3\vec{k}_2 \qquad (IIa.44)$$

We make the substitutions $r_2-r_1\to r$ and $r_1\to R$ in this integral to obtain,

$$\frac{V}{V}=\frac{-3}{4}\frac{\alpha}{V}\frac{1}{(2\pi)^6}\int\frac{d^3r\,e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}}}{|\vec{r}|}d^3k_1d^3k_2\underbrace{\int d\vec{R}}_{V} \qquad (IIa.45)$$

The last integral which factorizes is equal to the volume V and it cancels the V in the denominator yielding,

$$\frac{V}{V}=\frac{-3}{4}\frac{\alpha}{(2\pi)^6}\int\frac{d^3r\,d^3k_1\,d^3k_2\,e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}}}{r} \qquad (IIa.46)$$

Using,

$$\frac{V}{V}=\int\frac{d^3r\,e^{i\vec{k}\cdot\vec{r}}}{r}=\frac{4\pi}{k^2}$$

this becomes,

$$\frac{V}{V}=\frac{-3\pi\alpha}{(2\pi)^6}\int_{k<k_F}\frac{d^3\vec{k}_1\,d^3\vec{k}_2}{|\vec{k}_1-\vec{k}_2|}\equiv\frac{-3\pi\alpha}{(2\pi)^6}I_F \qquad (IIa.47)$$

The only thing that changes when we go from a fermi sphere to a fermi cube is the integral I_f which we will calculate now first for the fermi sphere.

$$I_f = \int_{\substack{|k_1|<p_F \\ |k_2|<p_F}} \frac{d^3k_1\, d^3k_2}{|k_1 - k_2|^2} = \frac{p_F^4}{4\pi} \int_{\substack{|u_1|<1 \\ |u_2|<1}} \frac{d^3u_1 d^3u_2 e^{i\vec{u}_1\cdot\vec{r} - i\vec{u}_2\cdot\vec{r}}}{r}\, d^3r = \frac{k_F^4}{4\pi} \int \frac{d^3r}{r} \left| \int_{|u|<1} e^{\frac{iu\cdot r}{4\pi}} \right|^2 \qquad (IIa.48)$$

We evaluate

$$\int_{|u|<1} e^{i\vec{u}\cdot\vec{r}}\, d^3\vec{u} = 4\pi \int u^2 du \frac{\sin(ur)}{ur} = \frac{4\pi}{r} \int_0^1 u \sin ur\, du = \frac{4\pi}{r^3} \int_0^r z \sin z\, dz$$

$$= \frac{4\pi}{r^3}\left[-r\cos r + \sin r\right] = \frac{4\pi}{r} j_1(r) \qquad (IIa.49)$$

which gives,

$$I_f = \frac{p_F^4}{4\pi} \int \frac{d^3r}{r} \left| \frac{4\pi}{r} j_1(r) \right|^2 = 4\pi p_F^4 \int \frac{d^3r\, j_1^2(r)}{r^3} \qquad (IIa.50)$$

The spherical Bessel function j_1 is given in terms of the ordinary Bessel functions as,

$$j_1(z) = \sqrt{\frac{\pi/2}{z}}\, J_{3/2}(z) \qquad (IIa.51)$$

We evaluate the integral,[14]

$$\int_0^\infty J_\nu(at) J_\mu(at) t^{-\lambda}\, dt = \frac{a^{\lambda-1}\Gamma(\lambda)\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)}$$

for $(\mu+\nu+1) > \lambda > 0$

with $\mu=\nu=3/2$,a=1,$\lambda=2$ this yields,

$$I_f = (2\pi)^3 P_F^4 \left\{ \frac{\Gamma(2)\Gamma(1)}{4\,\Gamma(\frac{3}{2})\Gamma(3)\Gamma(\frac{3}{2})} \right\}$$

using the properties of the gamma function

$$\Gamma(3)=2!=2 \quad , \quad \Gamma(\tfrac{3}{2})=\tfrac{1}{2}\Gamma(\tfrac{1}{2})=\tfrac{1}{2}\sqrt{\pi} \quad , \quad \Gamma(1)=1$$

giving,

$$I_f = (2\pi)^4 P_F^4 \qquad (IIa.52)$$

Similarly for the fermi cube we evaluate,

$$\int_{|u_\alpha|<1} e^{i\vec{u}\cdot\vec{r}}\, d^3u = \int_{-1}^{+1} du_x e^{iu_x r_x} \int_{-1}^{+1} du_y e^{iu_y r_y} \int_{-1}^{+1} du_z e^{iu_z r_z} = 8\,\frac{\sin r_x}{r_x}\,\frac{\sin r_y}{r_y}\,\frac{\sin r_z}{r_z} \qquad (IIa.53)$$

which gives,

$$\tilde{I}_f = \frac{k_F^4}{4\pi} \int d^3r \left| \frac{8 \sin r_x \sin r_y \sin r_z}{r_x r_y r_z} \right|^2 = \frac{16 k_F^4}{\pi} \int \frac{d^3r}{r} \frac{\sin^2 r_x \sin^2 r_y \sin^2 r_z}{r_x^2 r_y^2 r_z^2} \qquad (IIa.54)$$

Fourier transforming the 1/r to separate the integrals we obtain,

$$\tilde{I} = \frac{16 k_F^4}{\pi} \int \frac{d^3q\, d^3r\, e^{i\vec{q}\cdot\vec{r}}}{2\pi^2 q^2} \frac{\sin^2 r_{1x}}{r_{2x}^2} \frac{\sin^2 r_{1y}}{r_{1y}^2} \frac{\sin^2 r_{1z}}{r_{1z}^2}$$

$$= \frac{8 k_F^4}{\pi^3} \int \frac{d^3\vec{q}}{q^2} \prod_\alpha \int_{-\infty}^{\infty} \frac{e^{iq_\alpha r_\alpha} \sin^2 r_\alpha\, dr_\alpha}{r_\alpha^2} \qquad (IIa.55)$$

We then evaluate the integral,[14]

$$\int_{-\infty}^{\infty} \frac{e^{i\vec{q}\cdot\vec{r}} \sin^2 r \, dr}{r^2} = \int_{-\infty}^{\infty} \frac{\cos q_\alpha r_\alpha \sin^2 r_\alpha \, dr}{r^2} = \left(1 - \frac{|q_\alpha|}{2}\right) \Theta\left(2 - |q_\alpha|\right) \pi$$

which gives,

$$\tilde{I}_f = \frac{8k_F^4}{\pi^3} \int \frac{d^3q}{q^2} \prod_\alpha \left(1 - \frac{|q_\alpha|}{2}\right) \Theta\left(2 - |q_\alpha|\right) = 8k_F^4 \int_{|q_\alpha|\geq 2} \frac{d^3q}{q^2} \prod_\alpha \left(1 - \frac{|q_\alpha|}{2}\right) \equiv 8k_F^4 K$$

The last integral denoted as K has to be evaluated numerically. We find $K \approx 10.71$ giving

$$\tilde{I}_f \cong 85.68 \, k_F^4 \qquad (IIa.56)$$

so that we have,

$$\frac{V}{\bar{V}} = \frac{-3\alpha\pi}{(2\pi)^6} (2\pi)^3 p_F^4 = \frac{-3\alpha p_F^4}{16\pi^3} \qquad (IIa.57)$$

for the fermi sphere and

$$\frac{V}{\bar{V}} = \frac{-42.84}{2\pi^2} \frac{3\alpha k_F^4}{16\pi^3} \qquad (IIa.58)$$

for the fermi cube. Since they are at the same density we have,

$$k_F = \left(\frac{\pi}{6}\right)^{1/3} p_F$$

so that at the same density,

$$\frac{\frac{V}{V}\Big)_{sphere}}{\frac{V}{V}\Big)_{cube}} = \frac{1}{\frac{42.84}{2\pi^2}\cdot\left(\frac{\pi}{6}\right)^{4/3}} \cong 1.09$$

The fermi sphere is more strongly bound than the fermi cube.This is expected. Since the interaction matrix element behaves as $1/|k_1-k_2|^2$ in momentum space,the more closely packed fermi sphere gives a slightly higher interaction energy.With these we have all the contributions to the energy and we obtain the energy per particle for both the fermi cube and the fermi sphere gases,

$$\frac{E}{N}\Big)_1 = \frac{3\pi^2 B}{p_F^3} + \frac{3}{10m}p_F^2 - \frac{9\alpha p_F}{16\pi} \qquad \{\text{Fermi-sphere}\} \qquad (IIa.59)$$

and

$$\frac{E}{N}\Big)_2 = \frac{3\pi^2 B}{p_F^3} + \frac{3}{10m}\left(\frac{6}{\pi}\right)^{2/3}p_F^2 - \frac{\alpha\left(\frac{6}{\pi}\right)^{1/3}(10.71)}{2\pi^2}p_F \qquad \{\text{Fermi-cube}\} \qquad (IIa.60)$$

comparing (IIa.60) with (IIa.40) we see that our approximation of the sum $\Sigma v_{ij,ji}$ is quite good.Figure 2 shows the $E/N)_2$ curve for different relative values of the dimensional parameters B,m and α.We now proceed to the calculation of the average energy of a more realistic nonrelativistic system.

IIb. A more realistic system :

We do not expect to be able to describe the dynamics of quarks in a nucleon,or in nuclear matter,by using nonrelativistic quantum mechanics.The reason is the large kinetic and interaction energies of the light quarks compared to their masses.Historically,nonrelativistic calculations of the low lying hadron spectra has been attempted many times[15].These calculations have also introduced a large variety of nonrelativistic quark models usually differing in how confinement is accomplished and in the interaction among the quarks.A nonrelativistic quark model similar to the bag model can be constructed in the following manner.We take two different types of quarks of mass m.These u and d quarks are fermions and carry an additional color charge customarily denoted by b,r and g.The total energy of the system includes the kinetic energy of the quarks,a "bag energy" proportional to the volume of the system and an interaction between the quarks in the form,

$$V_{12} = -\alpha \frac{\vec{s}_1 \cdot \vec{s}_2 \; \vec{\lambda}_1 \cdot \vec{\lambda}_2}{r_{12}} \qquad (IIb.1)$$

The Gell Mann matrices λ^i are the generators of the color group SU(3)[17].They act on the color wavefunctions and

$$\vec{\lambda}_1 \cdot \vec{\lambda}_2 = \sum_{a=1,8} \lambda_1^a \cdot \lambda_2^a$$

This form of the interaction is chosen because it can be shown that the color and spin

dependence of the lowest order color magnetic gluon exchange interaction roughly has the same form and this interaction predicts the spin 1/2 isospin 1/2 nucleon to be the lowest state of the baryon multiplet to which it belongs.It also produces the splitting of the other members of the multiplet in the right direction.

The states of 3 flavors of spin 1/2 quarks can be classified according to the representations of the group SU(6).For the nucleon the spin isospin part of the wave function is,

$$\Psi(1,2,3)=\frac{1}{\sqrt{18}}\Big[2(u\uparrow)(d\downarrow)(u\uparrow)+2(u\uparrow)(u\uparrow)(d\downarrow)+2(d\downarrow)(u\uparrow)(u\uparrow)$$

$$-(u\uparrow)(u\downarrow)(d\uparrow)-(u\downarrow)(u\uparrow)(d\uparrow)-(u\uparrow)(d\downarrow)(u\downarrow)-(u\downarrow)(d\downarrow)(u\uparrow)-(d\downarrow)(u\uparrow)(u\downarrow)-(d\downarrow)(u\downarrow)(u\uparrow)\Big] \qquad (IIb.2)$$

this is multiplied by the totally antisymmetric color singlet state of three quarks given by,

$$\Psi_{singlet}(1,2,3)=\frac{1}{\sqrt{6}}\Big[brg-bgr+rgb-rbg+gbr-grb\Big] \qquad (IIb.3)$$

The total wave function appears to have 6X9=54 terms,but since the color wave function exhausts all the permutations of three colors we can see that there will be identical terms,up to a permutation of the three particles forming a determinant,among the 54 terms.We find that one can write the total wave function as a sum of 9 Slater determinants in the following form

$$\Psi(1,2,3)=\frac{1}{\sqrt{18}}\Big[2(bu\uparrow)(ru\uparrow)(gd\downarrow)+2(bu\uparrow)(rd\downarrow)(gu\uparrow)+2(bd\downarrow)(ru\uparrow)(gu\uparrow)-(bu\uparrow)(ru\downarrow)(gd\uparrow)$$

$$-(bu\downarrow)(ru\uparrow)(gd\uparrow)-(bu\uparrow)(rd\uparrow)(gu\downarrow)-(bu\downarrow)(rd\uparrow)(gu\uparrow)-(bd\uparrow)(ru\uparrow)(gu\downarrow)-(bd\uparrow)(ru\downarrow)(gu\uparrow)\Big] \qquad (IIb.4)$$

We then define the creation operator,

$$S_i^\dagger = \frac{1}{\sqrt{18}}\Big[2a^\dagger_{b\uparrow}a^\dagger_{r\uparrow}a^\dagger_{g\downarrow}+2a^\dagger_{b\uparrow}a^\dagger_{r\downarrow}a^\dagger_{g\uparrow}+2a^\dagger_{b\downarrow}a^\dagger_{r\uparrow}a^\dagger_{g\uparrow}-a^\dagger_{b\uparrow}a^\dagger_{r\downarrow}a^\dagger_{g\downarrow}-a^\dagger_{b\downarrow}a^\dagger_{r\uparrow}a^\dagger_{g\downarrow}$$

$$-a^\dagger_{b\downarrow}a^\dagger_{r\downarrow}a^\dagger_{g\uparrow}-a^\dagger_{b\downarrow}a^\dagger_{r\uparrow}a^\dagger_{g\downarrow}-a^\dagger_{b\uparrow}a^\dagger_{r\downarrow}a^\dagger_{g\uparrow}-a^\dagger_{b\uparrow}a^\dagger_{r\uparrow}a^\dagger_{g\downarrow}\Big] \qquad (IIb.5)$$

$S_i^\dagger$ creates a nucleon state at the position $\vec{r}_i$ with radial wave functions $u_i(r)$. In addition to the matrix elements of the operator $\vec{s}_1\cdot\vec{s}_2$ between various spin states we need the similar matrix elements of the operator $\vec{\lambda}_1\cdot\vec{\lambda}_2$ which we calculate by straightforward algebra using the explicit forms of the λ matrices. We find

$$\langle aa|\vec{\lambda}_1\cdot\vec{\lambda}_2|aa\rangle=\frac{4}{3}$$

$$\langle ab|\vec{\lambda}_1\cdot\vec{\lambda}_2|ab\rangle=\frac{-2}{3}$$

$$\langle ab|\vec{\lambda}_1\cdot\vec{\lambda}_2|ba\rangle=2 \qquad (IIb.6)$$

where a and b stand for different colors among b, r and g.

To calculate the interaction energy per particle, we again calculate the interaction energy per orbit. To do this we need to calculate the expectation value of the operator $\vec{s}_1\cdot\vec{s}_2\vec{\lambda}_1\cdot\vec{\lambda}_2$ in the nucleon state (IIb.4). This can be done by using group theory and observing that $\vec{s}_1\cdot\vec{s}_2\vec{\lambda}_1\cdot\vec{\lambda}_2$ is a Casimir operator of the color-spin group SU(6),[19] or by straightforward but tedious evaluation using the matrix elements (IIa.30) and (IIb.6). In either case we

obtain,

$$\langle N | \vec{\lambda}_1 \cdot \vec{\lambda}_2 \vec{s}_1 \cdot \vec{s}_2 | N \rangle = 8 \qquad (IIb.7)$$

which yields the contribution to the interaction energy due to interactions of the particles in the same orbit.

$$\frac{V}{N} = \frac{-8\alpha k_F}{3} V_{ii,ii} \qquad (IIb.8)$$

The calculation of the contribution from the interactions between different orbits is calculated by direct evaluation of the matrix elements as described above.Since the two body operator V can act on only two creation operators to its right we need to consider only two orbits at once.The 6-particle wave functions created by two $\vec{S}^{\dagger}$s is a sum of 81 determinants.The diagonal contributions are straigtforward to evaluate.They are the expectation values of a two body operators in determinants which we evaluate by using (IIa.20).In doing this we carefully leave out the contributions that correspond to the interaction of particles in the same orbit to prevent double counting.There will not be any cross term contributions because there is no pair of determinants that differ by only two creation operators,each one coming from a different $\vec{S}^{\dagger}$.This is because specifying two of the creation operators in each of the triplets in an $S^{\dagger}$ automatically determines the remaining $\vec{c}^{\dagger}$.Thus there are only diagonal contributions.This calculation yields for the contribution from the interaction among different orbits,

$$\frac{V}{N} = \frac{-1.44}{3} \frac{\alpha k_F}{2\pi^2} \sum_{j \neq i} V_{ij,ji} \qquad (IIb.9)$$

The coefficient in front depends on the relative orientation of the spins and isospins of the two nucleons in consideration.The factor in (IIb.9) corresponds to spin up protons at each site.If we choose to populate each alternating site with a different nucleon,then the sum in (IIb.9) is replaced by two dif erent sums,one over the sites that can be reached from site i by travelling through an odd number of basic lattice vectors and the other sum over the remaining sites.The second sum is multiplied by .48 and the first is multiplied by a different factor depending on the relative orientations of the spin and isospin vectors at the two sites.These numerical factors are given in Table 1.2 .

With these results and the numerical factors $v_{ii,ii}$ and $\Sigma v_{ij,ji}$ evaluated earlier in section IIa we obtain the total interaction energy per particle with spin up protons,

$$\frac{V}{N} = \frac{-1.44}{3}\frac{\alpha k_F}{2\pi^2}\sum_{j\neq i} v_{ij,ji} = \frac{-1.86}{2\pi^2}\alpha k_F \qquad (IIb.10)$$

We now proceed to the calculation of the kinetic energy.As the correlated wave function is given by

$$|\Psi\rangle = \prod_i S_i^\dagger |0\rangle$$

with $S^\dagger$given by (IIb.5) we can see that Ψ is an infinite sum of Slater determinants.Since the kinetic energy is a single particle operator,it can act on only one creation operator to its right and we can consider the contributions from each orbit separately.Furthermore,since there are no determinants differing by only a single creation operator there will not be any cross term contributions to the kinetic energy.Thus the total contribution will be a sum of diagonal contributions corresponding to the expectation value of the kinetic energy in each

determinant.In addition to these simplifications since the kinetic energy operator is not sensitive to the color,spin and isospin quantum numbers of the particles this contribution depending only on the spatial wave functions will be the same for each diagonal term.We then can pick any single determinant in the infinite sum for Ψ and the expectation value of the kinetic energy is the same in this determinant as in the infinite sum.By the arguments given in section IIa this determinant is equivalent to an incompletely occupied fermi cube gas contructed by putting 3 particles in each momentum state. With the results of section IIa for average kinetic energy for a fermi cube gas we have the kinetic energy in the correlated wavefunction.

$$\frac{K.E.}{N} = \frac{k_F^2}{2m} \qquad (IIb.11)$$

We now calculate the energy of a fermi gas of quarks having three different colors.The wave function for system is a Slater determinant which can be written as,

$$|\psi\rangle = \prod_k S_k^\dagger |0\rangle \qquad (IIb.12)$$

where

$$S_k^\dagger = a^\dagger_{u\uparrow b} a^\dagger_{u\uparrow r} a^\dagger_{u\uparrow g} a^\dagger_{d\uparrow b} a^\dagger_{d\uparrow r} a^\dagger_{d\uparrow g} a^\dagger_{u\downarrow b} a^\dagger_{u\downarrow r} a^\dagger_{u\downarrow g} a^\dagger_{d\downarrow b} a^\dagger_{d\downarrow r} a^\dagger_{d\downarrow g} \qquad (IIb.13)$$

The average kinetic energy of this system is the same as that of a fermi sphere gas considered in section IIa,which is

$$\frac{K.E.}{N} = \frac{3}{5}\frac{p_F^2}{2m}$$

The calculation of the interaction energy is also the same as in IIa except instead of the factor

$$\sum_{s_1 s_2} \langle s_1 s_2 | \vec{s}_1 \cdot \vec{s}_2 | s_2 s_1 \rangle = \frac{3}{2}$$

we now have

$$\sum_{\substack{s_1 s_2 \\ c_1 c_2 \\ t_1 t_2}} \langle s_1 t_1 c_1 s_2 c_2 t_2 | \vec{s}_1 \cdot \vec{s}_2 \, \vec{\lambda}_1 \cdot \vec{\lambda}_2 | s_2 c_2 t_2 s_1 c_1 t_1 \rangle$$

coming from the exchange term in the expression for the interaction energy in (IIa.41). The coefficient of the direct term again vanishes. We now evaluate,

$$\sum_{\substack{s_1 s_2 \\ c_1 c_2 \\ t_1 t_2}} \langle s_1 c_1 t_1 s_2 c_2 t_2 | \vec{\lambda}_1 \cdot \vec{\lambda}_2 \, \vec{s}_1 \cdot \vec{s}_2 | s_2 c_2 t_2 s_1 c_1 t_1 \rangle = \sum_{t_1} \sum_{s_1 s_2} \langle s_1 s_2 | \vec{s}_1 \cdot \vec{s}_2 | s_2 s_1 \rangle \sum_{c_1 c_2} \langle c_1 c_2 | \vec{\lambda}_1 \cdot \vec{\lambda}_2 | c$$

We had,

$$\sum_{s_1 s_2} \langle s_1 s_2 | \vec{s}_1 \cdot \vec{s}_2 | s_2 s_1 \rangle = \frac{3}{2}$$

and now easily evaluate,

$$\sum_{c_1 c_2} \langle c_1 c_2 | \vec{\lambda}_1 \cdot \vec{\lambda}_2 | c_2 c_1 \rangle = 3 \left[2 \langle ab | \vec{\lambda}_1 \cdot \vec{\lambda}_2 | ba \rangle + \langle aa | \vec{\lambda}_1 \cdot \vec{\lambda}_2 | aa \rangle \right]$$

$$= 3 \left[2 \cdot 2 + \frac{4}{3} \right] = 16 \qquad \text{, with } a \neq b$$

(*IIb.14*)

With this we get the interaction energy per particle

$$\frac{V}{N} = \frac{12\alpha p_F}{\pi} \tag{IIb.15}$$

The density of the fermi gas is,

$$\rho = \frac{12 p_F^3}{6\pi^2} = \frac{2 p_F^3}{\pi^2} \tag{IIb.16}$$

and the density of the correlated system is,

$$\rho = \frac{3 k_F^3}{\pi^3} \tag{IIb.17}$$

equating these two expressions yield

$$k_F = \left(\frac{2\pi}{3}\right)^{1/3} p_F \tag{IIb.18}$$

Putting all the results together we write down the energy per particle for both systems in terms of p_F which really is acting as a parameter for the density of the system. We also include the "bag energy" contributions to get

$$\left.\frac{E}{N}\right)_{\substack{\text{Fermi}\\ \text{Gas}}} = \frac{\pi^2 B}{2 p_F^3} + \frac{3}{10 m} p_F^2 + \frac{12\alpha}{\pi} p_F \tag{IIb.19}$$

and

$$\left.\frac{E}{N}\right)_{\substack{\text{correlated}\\ \text{matter}}} = \frac{\pi^2 B}{2 p_F^3} + \frac{1}{2m}\left(\frac{2\pi}{3}\right)^{2/3} p_F^2 - \frac{1.81}{2\pi^2}\left(\frac{2\pi}{3}\right)^{1/3} \alpha p_F \tag{IIb.20}$$

We again obtain a higher kinetic energy per particle with the correlated wavefunction because we use higher momentum components than demanded by the Pauli principle.But the local correlations we build into the wave function pays off in the interaction energy.The interaction energies for the fermi gas and for the correlated systems have even different signs.Since the bag energy contribution is the same ,at lower densities and hence low p_F,the term linear in p_F will be the important one and the correlated wavefunction will have a lower energy per particle than the fermi gas.As the density increases, however,the kinetic energy term becomes more important and at high densities it is the fermi gas that gives a lower energy for the system. These two E/N curves should cross at the density corresponding to,

$$\frac{3p_F^\circ}{10m} + \frac{12\alpha}{\pi} = \frac{1}{2m}\left(\frac{2\pi}{3}\right)^{2/3} p_F^\circ - \frac{1.86}{2\pi^2}\left(\frac{2\pi}{3}\right)^{1/3}\alpha$$

$$\Rightarrow \quad p_F^\circ \cong 4.2\,\alpha m \qquad (IIb.21)$$

These two E/N curves are shown in Figures 3a and 3b for different relative values of the parameters m and α.

The simple color magnetic interaction is thus able to predict a locally correlated system at low densities and a fermi gas at high densities with a nonrelativistic system.This result was our target in doing this calculation and we have accomplished it for the presented nonrelativistic system.We would like it to carry over to a more satisfactory field theoretical treatment of the gluon exchange interaction.This will be done in the next section.

In closing the discussion of the nonrelativistic calculations we would like to note some

general features.The lattice introduced by demanding the orthogonality of the single particle wave functions is not a real lattice in the sense of solid state physics.We had already noted that there is no sign of the lattice structure in the one body densities.The system has a uniform density as in the case of a fermi gas.To see that the overall location of the lattice points in space is not a physical variable,we examine the effect of shifting all the lattice points by a vector $\vec{R}$.Then the u_i are modified according to (IIa.15)

$$u_i(r)=\frac{1}{N}\int d^3k\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_i)}$$

$$\longrightarrow u_{i,\vec{R}}(\vec{r})=\frac{1}{N}\int d^3\vec{k}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_i-\vec{R})}=\frac{1}{N}\int d^3\vec{K}\, e^{-ik\cdot(r-R_i)}\; e^{ikr_i}$$

The many body wave function Ψ is a Slater determinant constructed by using these single particle wave functions.We have factored out the e^{ikr_i} for the following reason.We can think of the original expression (IIa.15) for the $u_i(\mathrm{r})$ as a superposition of plane wave states e^{-ikr} with the corresponding coefficients e^{ikr_i} and the equivalence of the plane wave determinant and the determinant of u's follows provided that the orthonormality relationships are preserved.In the same way we can think of the expression for $u_{i,R}(\mathrm{r})$ as a superposition of plane waves $e^{ik(R-r)}$ with the corresponding coefficients e^{ikr_i}. It is trivial to show that the orthonormality relationships are unchanged with the new plane wave states so that the determinant of $u_{i,R}$'s is equivalent to the determinant of $e^{ik(R-r)}$'s.But $e^{ik(R-r)}$ is still a plane wave,in fact it is the original one multiplied by a phase factor e^{ikR}. Each term in the Slater determinant constructed with these functions gets multiplied by

the factor

$$\left[\prod_k e^{i\vec{k}\cdot\vec{R}}\right]^g$$

since each $\vec{k}$ occurs g times in each term,g being the degeneracy of each momentum state.Thus the overall many body determinant is multiplied by the factor

$$\exp\left[ig\left[\sum_k \vec{k}\right]\cdot\vec{R}\right]$$

If we want to describe a stationary system,we should have

$$\sum_k \vec{k} = 0$$

Thus,the overall wavefunction remains unchanged when the lattice is shifted by any vector $\vec{R}$!There is no signature of the lattice in the one particle observables,it is the two particle structure of the wave function, two body correlation functions etc.,that are modified in the correlated wave function with respect to the fermi gas.The resulting potential energy with the correlated wave function indicates that these modifications are in the right direction toward the true ground state of the system.

The crossing of the E/N curves for the fermi gas and correlated matter fulfils our physically motivated expectation of nuclear matter going to an asymptotically free fermi gas at high densities.We also observe,however,that this crossing occurs because of the different powers of the density entering in the kinetic and the potential energies so that

one of them is the dominant contribution at low densities and the other at high densities. The crossing of the curves may be meaningless if it occurs at a value of the density where $k_F \gg m$ and we should not be using the nonrelativistic expression for the kinetic energies of the particles.Since the crossing density depends on the values of the parameters B,α and m which we are not able to specify,we can not determine whether the results of this nonrelativistic calculation can indeed be an indication of what is happening in the real physical systems.We will see that the relativistic calculation of the energies of the two systems also predict the crossing of the E/N curves.In that case ,this effect is tied to the well established asymptotical freedom property of QCD rather than kinematics and provides a more convincing prediction of the real phenomenon.This will be done in the next section.

IIIa.Relativistic Calculations

To begin our relativistic treatment,we recite the basic features of the bag model.[20] The hadrons in the bag model are finite regions of space where quark and gluon fields are defined.Inside the bag the quark wave functions obey the Dirac equation,

$$(\not{p}-m)\psi=0 \quad \Rightarrow \quad (-i\vec{\gamma}\cdot\vec{\nabla}+m)=E\psi \tag{IIIa.1}$$

and the confining boundary condition,

$$(1-\vec{\gamma}\cdot\hat{r})\psi\Big|_{r=R}=0 \tag{IIIa.2}$$

For u and d quarks m=0.The 4X4 γ matrices are given in the standard representation as,

$$\vec{\gamma}=\gamma^0\vec{\alpha} \qquad \gamma^0=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \vec{\alpha}=\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \gamma^5=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{IIIa.3}$$

The solutions of (IIIa.1) can be written in the form,[19]

$$\psi_{\nu k\mu}(\vec{r})=\begin{pmatrix} g_{\nu k}(r)\,\chi^{\mu}_{k}(\hat{r}) \\ i\,f_{\nu k}(r)\,\chi^{\mu}_{-k}(\hat{r}) \end{pmatrix} \tag{IIIa.4}$$

with,

$$g_{\nu k}=\frac{N_{\nu k}}{R^{3/2}}\,j_{\ell}\left(x_{\nu k}\frac{r}{R}\right)$$

$$f_{\nu k}=\frac{N_{\nu k}}{R^{3/2}}\,\tau_{\nu k}\,j_{\bar{\ell}}\left(x_{\nu k}\frac{r}{R}\right) \tag{IIIa.5}$$

and

$$\gamma_{\nu\kappa} = \text{sign}\{\kappa\}\frac{x_{\nu\kappa}}{(\epsilon_{\nu\kappa}+\xi)} \;,\; \epsilon_{\nu\kappa} = E_{\nu\kappa}R = (x_{\nu\kappa}^2+\xi^2)^{1/2} \;,\; \xi = mR \qquad (IIIa.6)$$

The $X_\kappa^\mu(\mathbf{r})$ are the spin spherical harmonics.[21] ν,κ and μ are the radial,angular and azimuthal quantum numbers respectively.The total angular momentum j is determined by,

$$j = |\kappa| - \tfrac{1}{2}$$

$$\ell = j + \tfrac{1}{2}\,\text{sign}\{\kappa\}$$

$$\bar{\ell} = j - \tfrac{1}{2}\,\text{sig}\{\kappa\} \qquad (IIIa.7)$$

The normalization constants $N_{\mu\kappa}$ are chosen so that the wave functions $\Psi_{\nu\kappa\mu}$ are normalized

$$\int_0^R d^3\vec{r}\;\psi^+_{\nu\kappa\mu}(\vec{r})\,\psi_{\nu\kappa\mu}(\vec{r}) = 1$$

This yields,

$$\frac{1}{N_{\nu\kappa}^2} = \left[2\epsilon_{\nu\kappa}(\epsilon_{\nu\kappa}+\kappa)+\xi\right]\left[\frac{j_\ell(x_{\nu\kappa})^2}{x_{\nu\kappa}^2}\right] \qquad (IIIa.8$$

The dimensionless eigenvalues $x_{\nu\kappa}$ are determined by the boundary condition (IIIa.2) which can be written as,

$$j_\ell(x_{\nu\kappa}) + \gamma_{\nu\kappa}\, j_{\bar{\ell}}(x_{\nu\kappa}) = 0 \qquad (IIIa.9)$$

The $x_{\nu\kappa}$ for the case of a massless quark are given in table 2. Since we will only be considering the ground state of hadrons we will only be interested in the lowest energy mode among the $\Psi_{\nu\kappa\mu}$. For this special case we have,

$$j=\tfrac{1}{2}\ ,\ \ell=0\ ,\ \bar{\ell}=1\ ,\ \kappa=-1\ ,\ \nu=1\ ,\ \gamma_{\nu\kappa}=-1\ ,\ x_{1,-1}=2.0428\equiv x \qquad (IIIa.10)$$

$$\text{and}\quad \frac{1}{N}=.44$$

The two spin spherical harmonics X^{μ}_{κ} and $X^{\mu}_{-\kappa}$ are related by,

$$\chi^{\mu}_{-\kappa}(\hat{r})=-\hat{r}\cdot\vec{\sigma}\ \chi^{\mu}_{\kappa}$$

so that we have the lowest eigenmode of the bag model for massless quarks as,

$$\psi_{bag}(\vec{r})=\frac{1}{NR^{3/2}}\begin{pmatrix} \dfrac{\sin\frac{xr}{R}}{\left(\frac{xr}{R}\right)}\,\chi \\ \left[\dfrac{\sin\left(\frac{xr}{R}\right)}{\left(\frac{xr}{R}\right)^2}-\dfrac{\cos\frac{xr}{R}}{\left(\frac{xr}{R}\right)}\right](-i\vec{\sigma}\cdot\hat{r})\,\chi \end{pmatrix} \qquad (IIIa.11)$$

where X is an arbirary spinor.

Although the ground state of a hadron containing quarks does not contain any gluon modes, the intermediate states required for the perturbative evaluation of the gluon exchange energy do contain such modes. For completeness we also write down the gluon modes in the bag model. There are eight gluon modes and since we are neglecting the self interaction and the coupling of different gluon modes, they really act as eight independent

fields each one analogous to the usual electromagnetic field.Let $A_\mu^{a\lambda}(\mathbf{r})$ be the gluon wave-function of mode a and polarization λ.It satisfies the free field Helmholtz equation inside the bag,

$$(\nabla^2+k_\lambda^2)\,A_\mu^{a\lambda}(\vec{r})=0 \qquad (IIIa.12)$$

The eigenvalues k_λ are determined by the boundary conditions,

$$\left.\hat{r}\cdot\vec{E}_\lambda^a\right|_{r=R}=0$$
$$\left.\hat{r}\times\vec{B}_\lambda^a\right|_{r=R}=0 \qquad (IIIa.13)$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields corresponding to the solution $A_\mu^{a\lambda}(\mathbf{r})$.The different polarizations λ are classified as,scalar,transverse electric and magnetic,and longitudinal. The scalar polarized mode has the form,

$$A_\mu^{scalar}(\vec{r})=\left(\phi(r),\vec{0}\right)$$

and the remaining modes have a zero 4^{th} component.

The transverse electric and magnetic modes of the solutions of (IIIa.12) are,

$$\vec{A}^{mag}_{NJM}(\vec{r})=h^{mag}_{NJ}(r)\;\vec{Y}_{JJM}(\hat{r})$$
$$\vec{A}^{el}_{NJM}(\vec{r})=\left(\frac{J+1}{2J+1}\right)^{1/2}h^{el(r)}_{N,J-1}\vec{Y}_{J,J-1,M}(\hat{r})-\left(\frac{J}{2J+1}\right)^{1/2}h_{N,J+1}(r)\;\vec{Y}_{J,J+1,M}(\hat{r}) \qquad (IIIa.14)$$

where,

$$h^{mag}_{NJ}(r) = N^{mag}_{NJ}\, j_J(k^{mag}_{NJ} r) \,/\, R^{3/2}$$

$$h^{el}_{N,J\pm1}(r) = N^{el}_{NJ}\, j_{J\pm1}(k^{el}_{NJ} r) \,/\, R^{3/2} \qquad (IIIa.15)$$

and the $\vec{Y}_{JJM}$ are the vector spherical harmonics.[22] The boundary conditions (IIIa.13) then determine the dimensionless eigenvalues $\omega^{\lambda}_{NJ}=k^{\lambda}_{NJ}R$, through the equations

$$\frac{d}{dr}\left[r\, j_J(\omega^{mag}_{NJ}\frac{r}{R})\right]_{r=R} = 0$$

$$j_J(\omega^{el}_{NJ}) = 0 \qquad (IIIa.15)$$

The normalization constants N are given by

$$\frac{1}{(N^{mag})^2} = \frac{1}{2}\, j^2_J(\omega^{mag}_{NJ})\left[1 - \frac{J(J+1)}{\omega^{mag\,2}_{NJ}}\right]$$

$$\frac{1}{(N^{el}_{NJ})^2} = \frac{1}{2}\, J^2_{J+1}(\omega^{el}_{NJ}) \qquad (IIIa.16)$$

Similarly for the scalar and longitudinal polarizations we have the wave functions,

$$\phi_{NJM}(\vec{r}) = \phi_{NJ}(r)\, Y_{JM}(\hat{r}) = N^{SC}_{NJ}\, j_J(k^{SC}_{NJ} r)\, Y_{JM}(\hat{r})$$

$(IIIa.17)$

$$\vec{A}_{NJM}(\vec{r}) = N_{NJ}^{\ell_g}\left(\vec{\nabla} j_J(k_{NJ}^{\ell_g} r) Y_{JM}(\hat{r})\right) \Big/ i k_{NJ}^{\ell_g} R^{3/2} \qquad (IIIa.18)$$

These two modes have identical eigenvalues and normalization constants given by

$$\frac{d}{dr}\left[j_J(\omega_{NJ}^{sc,\ell_g})\right] = 0 \qquad (IIIa.19)$$

and

$$\frac{1}{N_{NJ}^2} = \frac{1}{2|\omega_{10}^{sc}|^2}\left(1 - j_0(2\omega_{10}^{sc})\right) \quad \text{for } N=1, J=0$$

$$\frac{1}{2} j^2(\omega_{NJ}^{sc})\left[1 - \frac{J(J+1)}{(\omega_{NJ}^{sc})^2}\right], \text{ otherwise}$$

Some of the eigenvalues ω^λ are listed in table 2. With this we have all the quark and gluon wavefunctions that enter the bag model calculations of the hadron spectra. We now proceed to the case of infinite quark matter and determine the appropriate quark and gluon modes for expanding our many body wave functions.

We first examine the solutions of the massless Dirac equation (IIa.1) (with m=0) without the boundary condition (IIIa.2). In doing this, it is advantageous to work in another representation to decouple the upper and lower components. This representation is related to the standard one given in (IIIa.3) through the unitary matrix

$$U = U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} \qquad (IIIa.20)$$

In this representation we have instead of (IIIa.3)

$$\vec{\alpha} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}$$

$$\vec{\gamma} = \gamma^0\vec{\alpha}\ , \qquad \gamma^0 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\gamma^5 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad (IIIa.21)$$

In this representation,the Dirac equation becomes,

$$\begin{pmatrix}\vec{\sigma}\cdot\vec{k} & 0\\ 0 & \vec{\sigma}\cdot\vec{k}\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \epsilon\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \qquad (IIIa.22)$$

which decouples into two separate equations for ϕ and χ in an obvious way.

$$(\vec{\sigma}\cdot\vec{k})\varphi = \epsilon\varphi$$
$$(\vec{\sigma}\cdot\vec{k})\chi = -\epsilon\chi \qquad (IIIa.23)$$

We take the z direction along the vector k so that

$$\vec{\sigma}\cdot\vec{k} = \sigma_z\ , \quad \vec{k} = \frac{\vec{p}}{|\vec{p}|}\ , \quad \epsilon = \frac{E}{|\vec{p}|}$$

and obtain the four possible linearly independent solutions to the equations (IIIa.23)

$$i)\quad \varphi = \begin{pmatrix}1\\ 0\end{pmatrix} \qquad \chi = 0 \qquad \epsilon = +1 \qquad (IIIa.24)$$

ii) $\varphi=\begin{pmatrix}0\\1\end{pmatrix}$ $\quad\chi=0\quad$ $\epsilon=-1$ (*IIIa*.25)

iii) $\varphi=0\quad\chi=\begin{pmatrix}1\\0\end{pmatrix}\quad\epsilon=-1$ (*IIIa*.26)

iv) $\varphi=0\quad\chi=\begin{pmatrix}0\\1\end{pmatrix}\quad\epsilon=+1$ (*IIIa*.27)

The corresponding solutions in the standard representation are obtained by transforming back with the matrix U in (IIIa.20).We obtain,

$$u_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad (IIIa.28)$$

$$u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \quad (IIIa.29)$$

$$u_3=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \quad (IIIa.30)$$

$$u_4=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \quad (IIIa.31)$$

So the wave functions in the standard representation are

$$\psi_i = e^{-ipz} u_i \qquad (IIIa.32)$$

with $p=|\vec{p}|=E$. To express these wave functions in another coordinate system where the z axis is not necessarily along k, we need to rotate the spinors $\binom{1}{0}$ and $\binom{0}{1}$. If the angles of $\vec{k}$ are θ and ϕ, to get from a spin up in the z direction to a spin up in the $\vec{k}$ direction we perform a rotation by θ along the y axis and another one by ϕ along the z axis. We then have,

$$\varphi_i = e^{-iJ_z\varphi} e^{-iJ_y\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (IIIa.33)$$

with

$$e^{-iJ_z\varphi} = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix} \quad \text{and} \quad e^{-iJ_y\theta} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \qquad (IIIa.34)$$

which yield

$$\varphi = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2} e^{i\frac{\varphi}{2}} \end{bmatrix} \qquad (IIIa.35)$$

and

$$\chi = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\frac{\theta}{2} e^{-i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2} e^{i\frac{\varphi}{2}} \end{bmatrix} \qquad (IIIa.36)$$

using these, the four Dirac spinor solutions become,

$$u_1 = \frac{1}{\sqrt{2}} e^{-i\varphi/2} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix} \qquad (IIIa.37)$$

$$u_2 = \frac{1}{\sqrt{2}} e^{i\varphi/2} \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\varphi} \\ \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} \qquad (IIIa.38)$$

$$u_3 = \frac{1}{\sqrt{2}} e^{-i\varphi/2} \begin{pmatrix} \cos\theta/2 \\ \sin\frac{\theta}{2} e^{i\varphi} \\ -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix} \qquad (IIIa.39)$$

$$u_4 = \frac{1}{\sqrt{2}} e^{i\varphi/2} \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\varphi} \\ \cos\theta/2 \\ \sin\theta/2\, e^{-i\varphi} \\ -\cos\theta/2 \end{pmatrix} \qquad (IIIa.40)$$

The phase factors in front are unimportant and can be set equal to unity. We finally classify these solutions according to their energies and chiralities

$$\begin{aligned} \psi_1 &= e^{-i\vec{k}\cdot\vec{r}}\, u_1(k) \qquad E = +|\vec{k}| \qquad \gamma_5\psi_1 = +\psi_1 \\ \psi_2 &= e^{-i\vec{k}\cdot\vec{r}}\, u_2(k) \qquad E = -|\vec{k}| \qquad \gamma_5\psi_2 = +\psi_2 \\ \psi_3 &= e^{-i\vec{k}\cdot\vec{r}}\, u_3(k) \qquad E = -|\vec{k}| \qquad \gamma_5\psi_3 = -\psi_3 \\ \psi_4 &= e^{-i\vec{k}\cdot\vec{r}}\, u_4(k) \qquad E = +|\vec{k}| \qquad \gamma_5\psi_4 = -\psi_4 \end{aligned} \qquad (IIIa.41)$$

We will only be using the positive energy and $\pm$ chirality states ψ_1 and ψ_4. Two other linear combinations of the positive energy solutions which prove to be useful are,

$$u_+ = \cos\frac{\theta}{2} u_1 - \sin\frac{\theta}{2} e^{i\varphi} u_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \cos\theta \\ \sin e^{i\varphi} \end{pmatrix} \qquad (IIIa.42)$$

$$u_- = \sin\frac{\theta}{2} u_1 e^{-i\varphi} + \cos\frac{\theta}{2} u_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \sin\frac{\theta}{2} e^{-i\varphi} \\ -\cos\theta \end{pmatrix} \qquad (IIIa.43)$$

The forms of the equations for u_+ and u_- are suggestive.They indicate that the upper component is in a definite spin up or down state.The lower components,however,are pointing in other directions depending on k.Since the squared amplitudes of the upper and lower components are equal for the massless solutions above,it would not be accurate to ascribe the spin of the upper components to the whole mode as might be done in the case of a low energy massive Dirac particle.

We now describe the gluon wave functions which are again solutions of the free field Helmholtz equation (IIIa.12) without the boundary conditions (IIIa.13).We can write these solutions as plane waves,

$$A^{\lambda a}_{\mu}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\; a^{\lambda a}(k)\,\epsilon^{\lambda}_{\mu}(k) \qquad (IIIa.44)$$

where the polarization vectors ϵ^{λ}_{μ} can be written in the reference frame with the z axis parallel to k as,

$$\underset{\text{scalar}}{\epsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}} \qquad \epsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \underset{\text{transverse}}{\epsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}} \qquad \underset{\text{longitudinal}}{\epsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}} \qquad (IIIa.45)$$

We will work in the Lorentz (or Landau) gauge which demands that the potential A_μ satisfy the gauge condition

$$\partial_\mu A^{\mu}_{\lambda} = 0 \quad \Rightarrow \quad k^{\mu}\,\epsilon^{\lambda}_{\mu}\,a^{\lambda} = 0 \qquad (IIIa.46)$$

which yields $a^0 = a^4$

We construct the localized wavefunctions the same way as in the nonrelativistic case

by superposing plane waves in a fermi cube with appropriate coefficients. We define

$$u_i^{(\alpha)}(\vec{x}) = \iiint_{-k_F}^{k_F} \frac{d^3\vec{k}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}_i)}}{N}\, u^{(\alpha)}(\vec{k}) \qquad , \alpha = 1,4 \text{ or } +,- \qquad (IIIa47.)$$

and try to satisfy the orthonormality relations with this form of the $u_i^{(\alpha)}(x)$. We have,

$$\int d^3\vec{r}\; u_i^{(\beta)}(\vec{r})^{+}\, u_j^{(\alpha)}(\vec{r}) = \int \frac{d^3\vec{r}\; d^3\vec{k}_1\, d^3\vec{k}_2\, e^{i\vec{k}_1\cdot(\vec{r}-\vec{r}_i)}\, e^{-i\vec{k}_2\cdot(\vec{r}-\vec{r}_j)}\, u^{(\beta)+}(k_1)\, u^{(\alpha)}(k_2)}{N^2}$$

$$= \int_{-k_F}^{k_F} \frac{d^3\vec{k}_1\, d^3\vec{k}_2\, e^{i(\vec{k}_1\vec{r}_j - \vec{k}_1\vec{r}_i)}\, (2\pi)^3\, \delta^3(\vec{k}_1-\vec{k}_2)\; u^{(\beta)+}(k_1)\, u^{(\alpha)}(k_2)}{N^2}$$

$$= (2\pi)^3 \int_{-k_F}^{+k_F} \frac{d^3k\; e^{-i\vec{k}\cdot(\vec{r}_i-\vec{r}_j)}}{N^2}\, \delta_{\alpha\beta} = \frac{(2\pi)^3\,(2k_F)^3\,\delta_{\alpha\beta}}{N^2} \prod_{\alpha} \frac{\sin k_F(r_j-r_i)_\alpha}{k_F(r_j-r_i)_\alpha} \qquad \alpha = x,y,z$$

$$= \delta_{\alpha\beta}\,\delta_{ij}\,\frac{(2\pi)^2\,(2k_F)^3}{N^2}$$

thus we see that by choosing

$$N^2 = (2\pi)^3\,(2k_F)^3$$

we can satisfy the ortonormality relations,

$$\langle u_i^{(\alpha)} \mid u_j^{(\beta)} \rangle = \delta_{\alpha\beta}\,\delta_{ij} \qquad (IIIa.48)$$

So we have the localized single particle orbital,

$$u_i^{(\alpha)}(r) = \int_{-k_F}^{k_F} \frac{d^3k\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_i)}\, u^{(\alpha)}(\vec{k})}{(2\pi)^{3/2}\ (2k_F)^{3/2}} \qquad (IIIa.49)$$

We now expand the quark and gluon field operators in terms of the normal modes we have developed.For the spin 1/2 quark fields we have,

$$\psi(x) = \sum_n u_n(x)\, c_n + v_n(x)\, b_n^\dagger \qquad (IIIa.50)$$

and

$$\bar{\psi}(x) = \sum_n \bar{u}(x)\, c_n^\dagger + \bar{v}_n(x)\, b_n \qquad (IIIa.51)$$

here $v_i^{(\alpha)}$ is the localized wavefunction of polarization α built by superposing the negative energy states u_2 and u_3 in (IIIa.38-39) .The operators $c_n^\dagger$ and c_n create and destroy quarks in states n.The b's do the same for antiquarks. (IIa.51) implicitly assumes that a Hamiltonian having the eigenmodes u_i exists and the sum over n goes over <u>all</u> the eigenmodes of this Hamiltonian.To preserve the equal time anticommutation relation

$$\left\{\psi(\vec{x}), \psi^\dagger(\vec{y})\right\}_t = \delta^3(\vec{x}-\vec{y}) \qquad (IIIa.52)$$

the operators c and b must have the anticommutation relations

$$\left\{c_m, c_n^\dagger\right\} = \delta_{mn} \qquad \left\{b_m, b_n^\dagger\right\} = \delta_{mn} \qquad (IIIa.53)$$

with all other anticommutators zero.Similarly we have the gluon field operators

$$A_\mu(x)=\int\frac{d^3k}{2k(2\pi)^3}\sum_{\lambda=0}^{3}\left[a^{(\lambda)}(k)\,\varepsilon_\mu^{(\lambda)}(k)e^{-ik\cdot x}+a^{(\lambda)\dagger}(k)\,\varepsilon_\mu^{(\lambda)*}(k)e^{ik\cdot x}\right] \qquad (IIIa.54)$$

here a (k) creates and a (k) destroys a gluon mode of momentum k.The polarization vectors ϵ_μ^λ are given by (IIIa.45).Again, to preserve the commutation relations[32]

$$\left[A_\mu(x),A_\nu(y)\right]=-ig_{\mu\nu}\,\Delta(x-y) \qquad (IIIa.55)$$

among the photon field operators the a's must have the commutation relations

$$\left[a^{(\lambda)}(k),a^{(\lambda')\dagger}(k')\right]=-g^{\lambda\lambda'}2k(2\pi)^3\,\delta^3(\vec{k}-\vec{k}') \qquad (IIIa.56)$$

With these developments we now go on to calculate the one gluon exchange energy in the next section.

IIIb.The Second Order Interaction Energy

The Gell Mann-Low theorem gives the energy shift of the ground state of a field theory due to a perturbation term in the Hamiltonian.[23] We will use Hubbard's modification of the Gell Mann-Low result.[24] In this form the theorem gives the energy shift as,

$$E-E_0=\langle\Phi_0|\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}i\delta(t_1)d^4x_1\,d^4x_2\cdots d^4x_n\,T\left[H_I(x_1)-H_I(x_n)\right]|\Phi_0\rangle_{connected} \quad (IIIb.1)$$

where E_o and E are the unperturbed and perturbed ground state energies, $|\Psi\rangle$ is the unperturbed ground state and $H_I(x)$ is the perturbing Hamiltonian. The subscript "connected" emphasizes that we should only include the connected diagrams among those generated by the time ordered product of operators in the infinite sum. For the coupling of quarks and gluons the perturbation Hamiltonian is,

$$H_I=-ig_s\bar{\psi}(x)\gamma^\mu t^a\psi(x)A^a_\mu(x) \quad (IIIb.2)$$

here $t^a=\lambda^a/2$ and g_s is the strong coupling constant. The lowest nonvanishing term in (IIIb.1) is the second order one, which is

$$\Delta E^{(2)}=\langle\Phi_0|-\tfrac{1}{2}\int_{-\infty}^{\infty}i\delta(t_1)\,d^4x_1\,d^4x_2\,T\left[H_I(x_1)H_I(x_2)\right]|\Phi_0\rangle \quad (IIIb.3)$$

inserting (IIIb.2) into (IIIb.3) gives,

$$\Delta E^{(2)}=+\tfrac{1}{2}ig_s^2\langle\Phi_0|\int_{-\infty}^{\infty}\delta(t_1)\,d^4x_1\,d^4x_2\,T\left[\bar{\psi}(x_1)\gamma^\mu t^a\psi(x_1)A^a_\mu(x_1)\bar{\psi}(x_2)\gamma^\nu t^b\psi(x_2)A^b_\nu(x_2)\right]|\Phi_0\rangle \quad (IIIb.4)$$

we use the eigenmode expansions (IIIa.51) and (IIIa.52) for the field operators Ψ and A

and keep only the contributions from the positive energy modes to obtain,

$$\Delta E^{(2)} = \tfrac{1}{2} i g_s^2 \langle \Psi_0 | \int_{-\infty}^{\infty} \delta(t_1)\, d^4x_1\, d^4x_2 \left[\left[\sum_{ijke} \bar{u}_i(x_1)\gamma^\mu t^a u_j(x_1) \bar{u}_k(x_2)\gamma^\nu t^b u_e(x_2) \right] \right.$$

$$\times \left[\theta(t_1 - t_2)\, c_i^\dagger c_j c_k^\dagger c_e \sum_{\substack{\lambda_1 \lambda_2 \\ p,q}} \left(a_{\lambda_1 p} A^{\lambda_1 a}_{p\mu}(\vec{r}_1) e^{-ip_{\lambda_1} t_1} + a^\dagger_{\lambda_1 p} A^{\lambda_1 a}_{p\mu}(r_1) e^{ip_{\lambda_1} t_1} \right) \left(a_{\lambda_2 q} A^{\lambda_2 b}_{q\nu}(r_2) e^{-iq_{\lambda_2} t_2} + a^\dagger_{\lambda_2 q} A^{\lambda_2 b}_{q\nu}(r_2) e^{iq_{\lambda_2} t_2} \right) \right.$$

$$\left. \left. + \theta(t_2 - t_1)\, c_k^\dagger c_e c_i^\dagger c_j \sum_{\substack{\lambda_1 \lambda_2 \\ p,q}} \left(a_{\lambda_2 q} A^{\lambda_2 b}_{q\nu}(r_2) e^{-iq_{\lambda_2} t_2} + a^\dagger_{\lambda_2 q} A^{\lambda_2 b}_{q\nu}(\vec{r}_2) e^{iq_{\lambda_2} t_2} \right) \left(a_{\lambda_1 p} A^{\lambda_1 a}_{p\mu}(r_1) e^{-ip_{\lambda_1} t_1} + a^\dagger_{\lambda_1 p} A^{\lambda_1 a}_{p\mu}(r_1) e^{ip_{\lambda_1} t_1} \right) \right] | \Psi_0 \rangle$$

(IIIb.5)

The contributions from the negative energy states are physical and important in principle,however,they are left out in the original bag model calculations with the expectation that their effects could be incorporated into the renormalized parameters B and Z_0.It is therefore only consistent to leave them out in this treatment when we are aiming to make a connection with the bag model.We have,

$$\Delta E^{(2)} = \tfrac{1}{2} i g_s^2 \int_{-\infty}^{\infty} d^3\vec{r}_1\, d^3\vec{r}_2\, dt_2 \left[\sum_{ijke} \bar{u}_i(\vec{r}_1)\gamma^\mu t^a u_j(\vec{r}_1) \bar{u}_k(\vec{r}_2)\gamma^\nu t^b u_e(\vec{r}_2) \right] e^{i(t_1\varepsilon_i - t_1\varepsilon_j + t_2\varepsilon_k - t_2\varepsilon_e)}$$

$$\times \left[\theta(-t_2) \sum_{\lambda_1 \lambda_2} \left[\frac{d^3p\, d^3q}{4pq(2\pi)^6} \langle \Psi_0 | c_i^\dagger c_j c_k^\dagger c_e | \Psi_0 \rangle \langle \Psi_0 | a^a_{\lambda_1 p} a^{b\dagger}_{\lambda_2 q} | \Psi_0 \rangle A^{\lambda_1 a}_{p\mu}(\vec{r}_1) A^{\lambda_2 b *}_{q\nu}(\vec{r}_2) e^{iq_{\lambda_2} t_2} \right. \right.$$

$$\left. \left. + \theta(t_2) \sum_{\lambda_1 \lambda_2} \left[\frac{d^3p\, d^3q}{4pq(2\pi)^6} \langle \Psi_0 | c_k^\dagger c_e c_i^\dagger c_j | \Psi_0 \rangle \langle \Psi_0 | a^b_{\lambda_2 q} a^{a\dagger}_{\lambda_1 p} | \Psi_0 \rangle A^{\lambda_2 b}_{q\nu}(\vec{r}_2) A^{\lambda_1 a}_{p\mu}(\vec{r}_1) e^{-iq_{\lambda_2} t_2} \right] \right]$$

(IIIb.6)

Evaluating,

$$\langle \gamma_0 | a^a_{\lambda_1 p} a^{+b}_{\lambda_2 p} | \gamma_0 \rangle = -g^{\lambda_1\lambda_2} \delta^{ab} (2p) (2\pi)^3 \delta^3(\vec{p}-\vec{q})$$

by using the commutation relation (IIIa.56),we get

$$\Delta E^{(2)} = \tfrac{1}{2} i g_s^2 \int_{-\infty}^{\infty} d^3\vec{r}_1 \, d^3\vec{r}_2 \sum_{ijkl} \left[\bar{u}_i(\vec{r}_1) \gamma^\mu t^a u_j(\vec{r}_1) \bar{u}_k(\vec{r}_2) \gamma^\nu t^b u_l(\vec{r}_2) \right]$$

$$\times \left[\langle \gamma_0 | c_i^\dagger c_j c_k^\dagger c_l | \gamma_0 \rangle \int \frac{d^3p \, d^3q}{4pq (2\pi)^6} \left[-g^{\lambda_1\lambda_2} (2p) (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \right] A^{\lambda_1 a}_{p\mu}(r_1) A^{\lambda_2 b *}_{q\nu}(r_2) \int_{-\infty}^{\infty} dt_2 \, e^{it_2(\varepsilon_k + q_{\lambda_2} - \varepsilon_l)} \theta(-\right.$$

$$\left. + \langle \gamma_0 | c_k^\dagger c_l c_i^\dagger c_j | \gamma_0 \rangle \int \frac{d^3p \, d^3q}{4pq (2\pi)^6} \left[-g^{\lambda_1\lambda_2} (2p) (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \right] A^{\lambda_1 a *}_{p\mu}(r_1) A^{\lambda_2 b}_{q\nu}(r_2) \int_{-\infty}^{\infty} dt_2 \, \theta(t_2) e^{it_2(\varepsilon_k - \varepsilon_l - iq} \right.$$

(IIIb.7)

We evaluate the integrals,

$$\int_{-\infty}^{\infty} dt_2 \, \theta(-t_2) e^{it_2(\varepsilon_k - \varepsilon_l + q_{\lambda_2})} = \frac{-i}{\varepsilon_k - \varepsilon_l + q_{\lambda_2}}$$

$$\int_{-\infty}^{\infty} dt_2 \, \theta(t_2) e^{it_2(\varepsilon_k - \varepsilon_l - q_{\lambda_2})} = \frac{i}{\varepsilon_k - \varepsilon_l - q_{\lambda_2}}$$

which when inserted in (IIIb.6) give

$$\Delta E^{(2)} = \frac{1}{2} g_s^2 \int_{-\infty}^{\infty} d^3r_1\, d^3r_2 \sum_{ijkl} \left[\bar{u}_i(r_1) \gamma^\nu t^a u_j(r_1)\, \bar{u}_k(r_2) \gamma^\nu t^a u_l(r_2) \right]$$

$$\times \left[\langle \psi_0 | c_i^\dagger c_j\, c_k^\dagger c_l | \psi_0 \rangle \sum_\lambda \left\{ \frac{d^3p_\lambda \left[-g^{\lambda\lambda} \right]}{2p_\lambda (2\pi)^3} \; \frac{A_{p\mu}^{\lambda a}(r_1)\, A_{p\nu}^{\lambda a *}(r_2)}{\varepsilon_k - \varepsilon_l + p_\lambda} \right. \right.$$

$$\left. \left. + \langle \psi_0 | c_k^\dagger c_l\, c_i^\dagger c_j | \psi_0 \rangle \sum_\lambda \left\{ \frac{d^3p_\lambda \left[-g^{\lambda\lambda} \right]}{2p_\lambda (2\pi)^3} \; \frac{A_{p\mu}^{\lambda a *}(r_1)\, A_{p\nu}^{\lambda a}(r_2)}{\varepsilon_l - \varepsilon_k + p_\lambda} \right. \right. \qquad (IIIb.8)$$

we now evaluate the matrix elements of the quark creation and destruction operators.Since,

$$c_i^\dagger c_i = n_i$$

is the number operator for orbit i we have

$$\langle \psi_0 | c_i^\dagger c_j\, c_k^\dagger c_l | \psi_0 \rangle = \delta_{ij}\delta_{kl} n_i n_k + \delta_{il}\delta_{jk}(1 - n_k + \delta_{ik})$$

$$\langle \psi_0 | c_k^\dagger c_l\, c_i^\dagger c_j | \psi_0 \rangle = \delta_{kl}\delta_{ij} n_k n_i + \delta_{il}\delta_{kj}(1 - n_i + \delta_{ik}) \qquad (IIIb.9)$$

with n_k=1 or 0 if the orbit is occupied in Ψ_o or not.Also for later use in calculating cross terms in the expectation value of the same operator in a coupled state given as a sum of Slater determinants, we evaluate,

$$\langle 0 | c_2 c_3\, c_i^\dagger c_j\, c_k^\dagger c_l\, c_4^\dagger c_5^\dagger | 0 \rangle = -\delta_{2i}\delta_{4j}\delta_{3k}\delta_{5l} + \delta_{3i}\delta_{4j}\delta_{2k}\delta_{5l} + \delta_{2i}\delta_{5j}\delta_{3k}\delta_{4l} \qquad (IIIb.10)$$
$$- \delta_{3i}\delta_{5j}\delta_{2k}\delta_{4l}$$

and also define,

$$\Theta_\lambda \equiv -g^{\lambda\lambda} \qquad (IIIb.11)$$

With these we have,

$$\Delta E^{(2)} = \tfrac{1}{2} g_s^2 \Bigg[\sum_\lambda \int \frac{d^3p}{2p(2\pi)^3} \frac{1}{\varepsilon_k - \varepsilon_\ell + p_\lambda} \Big[\delta_{ij}\delta_{k\ell} n_i n_k + \delta_{i\ell}\delta_{jk}(1 - n_j + \delta_{ij}) \Big]$$

$$\times \theta_\lambda \int d^3r_1\, \bar{u}_i(r_1)\gamma^\mu t^a u_j(r_1) A^{\lambda a}_{p\mu}(r_1) \int d^3r_2\, \bar{u}_k(r_2)\gamma^\nu t^a u_\ell(r_2) A^{\lambda a *}_{p\nu}(r_2)$$

$$- \sum_\lambda \int \frac{d^3p_\lambda}{2p(2\pi)^3} \frac{1}{\varepsilon_k - \varepsilon_\ell + p_\lambda} \Big[\delta_{k\ell}\delta_{ij} n_i n_k + \delta_{kj}\delta_{\ell i} n_k (1 - n_i + \delta_{ij}) \Big]$$

$$\times \theta_\lambda \int d^3r_1\, \bar{u}_i(r_1)\gamma^\mu t^a u_j(r_1) A^{\lambda a *}_{p\mu}(r_1) \int d^3r_2\, \bar{u}_k(r_2)\gamma^\nu t^a u_\ell(r_2) A^{\lambda a}_{p\nu}(r_2) \qquad (IIIb.12)$$

at this point we define,

$$T^{\lambda a}_p(i,j) \equiv \int d^3r\, \bar{u}_i(\vec{r})\gamma^\mu t^a u_j(\vec{r}) A^{\lambda a}_{p\mu}(\vec{r}) \qquad (IIIb.13)$$

so that

$$\int d^3r\, \bar{u}_k(r)\gamma^\nu t^a u_\ell(r) A^{\lambda a *}_{p\nu}(r) = T^{\lambda a \dagger}_p(\ell,k) \qquad (IIIb.14)$$

in terms of the T's the expression for the second order interaction energy becomes,

$$\Delta E^{(2)} = \tfrac{1}{2} g_s^2 \Bigg[\sum_\lambda \sum_{i,k} \int \frac{d^3p_\lambda}{2p_\lambda (2\pi)^3} \Bigg[\frac{n_i n_k \theta_\lambda T^{\lambda a}_p(i,i) T^{\lambda a \dagger}_p(k,k)}{p_\lambda}$$

$$- \frac{n_i n_k \theta_\lambda T^{\lambda a}_p(i,k) T^{a\lambda \dagger}_p(i,k)}{\varepsilon_k - \varepsilon_i + p_\lambda}$$

$$+ \frac{n_i n_k \theta_\lambda T^{\lambda a \dagger}_p(i,) T^{\lambda a}_p(k,k)}{p_\lambda}$$

$\Rightarrow$

$$+\frac{n_i n_k \Theta_\lambda T_p^{\lambda a}(k,i) T_p^{\lambda a\dagger}(k,i)}{\varepsilon_k-\varepsilon_i-p_\lambda}$$

$$+\left[\frac{n_i \Theta_\lambda T_p^{\lambda a}(i,k) T_p^{\lambda a\dagger}(i,k)}{\varepsilon_k-\varepsilon_i+p_\lambda}-\frac{n_k \Theta_\lambda T_p^{\lambda a}(k,i) T_p^{\lambda a\dagger}(k,i)}{\varepsilon_k-\varepsilon_i-p_\lambda}\right]\Bigg] \qquad (IIIb.15)$$

or collecting terms,

$$\Delta E^{(2)} = \Delta E^{(2)}_{\text{direct}} + \Delta E^{(2)}_{\text{exchange}} + \Delta E^{(2)}_{\text{self energy}} \qquad (IIIb.16)$$

with,

$$\Delta E^{(2)}_{\text{direct}} = -\tfrac{1}{2} g_s^2 \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \sum_{i,k} \frac{T_p^{\lambda a}(i,i) T_p^{\lambda a\dagger}(k,k) n_i n_k \Theta_\lambda}{p_\lambda^2} \qquad (IIIb.17)$$

$$\Delta E^{(2)}_{\text{exchange}} = -\tfrac{1}{2} g_s^2 \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \sum_{i,k} \frac{|T_p^{\lambda a}(i,k)|^2 n_i n_k \Theta_\lambda}{(\varepsilon_i-\varepsilon_k)^2-p_\lambda^2} \qquad (IIIb.18)$$

$$\Delta E^{(2)}_{\text{self}} = -\tfrac{1}{2} g_s^2 \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \sum_{i,k} \frac{\Theta_\lambda |T_p^{\lambda a}(i,k)|^2 n_i}{p_\lambda (\varepsilon_k-\varepsilon_i+p_\lambda)} \qquad (IIIb.19)$$

The direct and exchange energies are the counterparts of the nonrelativistic direct and exchange interactions.The direct term vanishes for a color singlet system because the integrand is really a product of two sums

$$\Delta E^{(2)}_{\text{direct}} = -\tfrac{1}{2} g_s^2 \sum_\lambda \int \frac{d^3p_\lambda}{(2\pi)^3 p_\lambda^2} \left[\sum_i T_p^{\lambda a}(i,i)\right]\left[\sum_k T_p^{\lambda a\dagger}(k,k)\right]$$

for a color singlet determinant each color occurs once in the sum i and hence the sum is proportional to $\mathrm{tr}(\lambda^a)=0$.The self energy sum is independent of the occupation of one of the interacting particles and so is a sum of single particle energies.For massless quarks in the

infinite system with no energy scale the self energy is zero.For the finite bag calculations the self energy is proportional to the energy scale 1/R.The exchange energy (IIIb.17) is the main quantity of interest in infinite matter and we will now calculate it with our correlated wave function. We construct our correlated wave function in the same manner as in the nonrelativistic system with colored quarks,so that

$$|\Psi_0\rangle = \prod_i S_i^\dagger |0\rangle$$

with S_i given by (IIb.5)The spin indices $\uparrow$ and $\downarrow$ are now replaced by the polarization indices of the Dirac spinors.The derivation of the second order interaction energy is the same as with a determinantal wave function except for evaluation of the matrix element

$$\langle\Psi_0| c_i^\dagger c_j c_k^\dagger c_\ell |\Psi_0\rangle = \langle 0| \cdots S_n \cdots c_i^\dagger c_j c_k^\dagger c_\ell \cdots S_m^\dagger \cdots |0\rangle$$

Since the operator $c_i c_j c_k c_l$ can only act on two different creation operators we need to consider only two cases;

i)both creation operators belong to the same orbital i.In this case

$$\langle\Psi_0| c_i^\dagger c_j c_k^\dagger c_\ell |\Psi_0\rangle = \langle 0| S_i c_i^\dagger c_j c_k^\dagger c_\ell S_i^\dagger |0\rangle$$

and there will be two sets of contributions.The first set comes from the diagonal terms where the same Slater determinant in S occurs both on the right and left of the operator $c_i c_j c_k c_l$.These contributions lead to the same form of the interaction energy as in (IIIb.16-

18)The second set of contributions will come from the cross terms where the determinants on different sides of $c_i c_j c_k c_l$ differ by only two creation operators. These contributions are evaluated with the aid of (IIIb.10).We will then have,

$$\Delta E^{(2)} = \left[\sum \text{diagonal terms} + \sum \text{cross terms}\right] \qquad (IIIb.20)$$

with

$$\sum \text{diagonal terms} = \sum_n \frac{\text{cof}(n)^2}{18} \sum_{\substack{i=1,3 \\ j=1,3}} \tfrac{1}{2} g_s^2 \sum_\lambda \int \frac{d^3p_\lambda}{(2\pi)^3} \sum_{i,j} \frac{|T_p^{\lambda\lambda}(\ ,n_j)|^2 \theta_\lambda}{(\epsilon_{n_i} - \epsilon_{n_j})^2 - p_\lambda^2} \qquad (IIIb.21)$$

$$\sum \text{cross terms} = \tfrac{1}{2} g_s^2 \sum_{\substack{m,n \\ m \neq n \\ P(ijkl)}} \frac{\text{cof}(m)\,\text{cof}(n)\,(-)^P}{18} \int \frac{d^3p_\lambda}{(2\pi)^3} \frac{\theta_\lambda T_j^{p\lambda}(i,j) T_j^{\dagger p\lambda}(l,k)}{p_\lambda^2} \qquad (IIIb.22)$$

Here,the sum on m,n go over the different terms in $S^\dagger$,cof(n) is the coefficient of the term n,the sum on P is over the different permutations of (ijkl) given by (IIIb.10) and $(-)^P$ is the sign associated with P in (IIIb.10).

ii)If (ijkl) belong to two different orbits then there are no cross terms as in the nonrelativistic case because there are no two terms,among the 6 particle determinants obtained by multiplying two $S^\dagger$'s,that differ by only two $a^\dagger$'s each one coming from a different S.We then have only determinantal contributions similar to (IIIb.21) except with the contributions corresponding to the interactions of the particles from the same orbit discarded to prevent double counting with the case i.With these the final result for the interaction

energy becomes,

$$\frac{\Delta E^{(2)}}{\text{per orbit}} = -\frac{1}{2}g^2\Bigg[\sum_n \sum_{\substack{i=1,3\\ j=1,3}} \frac{(\omega_f(n))^2}{18} \sum_\lambda \int \frac{d^3\vec{p}_\lambda}{(2\pi)^3} \frac{|T_p^{\lambda a}(i,j)|^2 \theta_\lambda}{(\varepsilon_{n_i} - \varepsilon_{n_j})^2 - p_\lambda^2}$$

$$+\frac{1}{2}\sum_{\substack{m,n\\ m\neq n}} \sum_{\substack{i=1,3\\ j=1,3}} \frac{(\omega_f(m))^2(\omega_f(n))^2}{18^2} \sum_\lambda \int \frac{d^3\vec{p}_\lambda}{(2\pi)^3} \frac{|T_p^{\lambda a}(m_i,n_j)|^2 \theta_\lambda}{(\varepsilon_{m_i} - \varepsilon_{n_j})^2 - p_\lambda^2}$$

$$+\sum_{P(ijkl)} \frac{(\omega_f(1))(\omega_f(1))(-)^P}{18} \sum_\lambda \int \frac{d^3p_\lambda}{(2\pi)^3} \frac{\theta_\lambda T_p^{\lambda a}(i,j)\, T_p^{\dagger\lambda a}(l,k)}{(\varepsilon_{m_i} - \varepsilon_{n_j})^2 - p_\lambda^2} \qquad (IIIb.23)$$

One important simplification occurs due to the fact that all the occupied single particle orbitals have the same energy for symmetry reasons.(They are nothing but the same wave function located at different points.)With this, the energy denominators in (IIIb.23) become independent of i and j. We now calculate the $T_\lambda^{pa}(i,j)$ which were defined in (IIIb.13) as,

$$T_p^{\lambda a}(i,j) = \int d^3\vec{r}\; A^\lambda_{p\mu}(r)\, \bar{u}_i(\vec{r})\, \gamma^\mu t^a\, u_j(\vec{r})$$

with,

$$A^\lambda_{k\mu}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\, \varepsilon^{(\lambda)}_\mu(\vec{k}) \qquad (IIIb.24)$$

in the frame where the z axis is parallel to k we have,

$$\varepsilon^{(\lambda)}_\mu = \delta^\lambda_\mu \qquad (IIIb.25)$$

and the orbitals u_i are given by (IIIa.49). We then have,

$$T_q^{a\lambda}(i,j) = \int \frac{d^3r\, d^3k\, d^3k'}{(2\pi)^3 (2k_F)^3} e^{i\vec{k}'\cdot(\vec{r}-\vec{r}_i) - i\vec{k}\cdot(\vec{r}-\vec{r}_j)} \bar{u}^{(\alpha_i)}(k') \gamma^\lambda u^{(\alpha_j)} e^{i\vec{q}\cdot\vec{r}} t^a_{c_i c_j}$$

$$= \frac{1}{(2k_F)^3} \iiint_{-k_F}^{k_F} d^3k\, e^{i(\vec{q}-\vec{k})\cdot\vec{r}_i + i\vec{k}\cdot\vec{r}_j} \bar{u}^{(\alpha_i)}(\vec{k}-\vec{q}) \gamma^\lambda u^{(\alpha_j)}(k) \Theta\left(k_F - |(\vec{k}-\vec{q})_d|\right) t^a_{c_i c_j}$$

$$= \frac{e^{i\vec{q}\cdot\vec{r}_i}}{8} \iiint_{-1}^{+1} d^3k\, e^{ik\cdot k_F(r_j - r_i)} \bar{u}^{(\alpha_i)}\left(\vec{k} - \frac{\vec{q}}{k_F}\right) \gamma^\lambda u^{(\alpha_j)}(\vec{k}) \Theta\left(k_F - |(\vec{k}-\vec{q})_d|\right) t^a_{c_i c_j}$$

$$= \frac{e^{i\vec{q}\cdot\vec{r}_i}}{8} \iiint_{-1}^{+1} d^3k\, e^{i\vec{k}\cdot\vec{n}_{ij}\pi} \bar{u}^{(\alpha_i)}(\vec{k} - g\vec{k}) \gamma^\lambda u^{(\alpha_j)}(\vec{k}) \Theta\left(1 - |(\vec{k}-\hat{k})_d|\right) t^a_{c_i c_j} \quad (IIIb.26)$$

with all the tools for the calculation of the interaction energy now developed we proceed to the calculation of the energy densities.

IIIc. The Energy Densities

We first start with the uncorrelated Fermi gas. In the completely occupied gas each momentum state is filled with 12 particles exhausting all the possible polarizaton, isospin and color states. (We are still considering only u and d quarks). For the moment we take the degeneracy to be g and set g=12 later. The kinetic energy is,

$$K.E. = \int_{|k|<p_F} \frac{g\, d^3k\, k}{(2\pi)^3} = \frac{g p_F^4}{8\pi^2} \qquad (IIIc.1)$$

the number density is

$$\rho = \int_{|k|<p_F} \frac{g\, d^3k}{(2\pi)^3} = \frac{g p_F^3}{6\pi^2} \qquad (IIIc.2)$$

(IIIc.1) with (IIIc.2) yields the energy per particle.

$$\frac{K.E.}{N} = \frac{3}{4} p_F \qquad (IIIc.3)$$

The interaction energy is calculated using (IIIb.18) with

$$|\psi_s\rangle = \prod_i S_k^\dagger |0\rangle$$

and

$$S_k^\dagger = C^\dagger_{k\alpha ub} C^\dagger_{k\alpha ur} C^\dagger_{k\alpha ug} C^\dagger_{k\alpha db} \cdots C^\dagger_{k\beta dg} \qquad (IIIc.4)$$

with k labelling the momentum states and α,β running over the possible polarizations.Since the total wave function is a Slater determinant this evaluation is straightforward.We then use the plane waves for the single particle wavefunctions to obtain the well known result,[25,26]

$$\frac{\Delta E^{(2)}}{V} = \frac{1}{8\pi^4} g_s^2 p_F^4 N_f \qquad (IIIc.5)$$

with N_f=2 the number of quark flavors.This with the density (IIIc.2) yields,

$$\frac{\Delta E^{(2)}}{N} = \frac{1}{8\pi^2} g_s^2 p_F \qquad (IIIc.6)$$

The bag energy contribution is,

$$\frac{\Delta E_{bag}}{N} = \frac{B}{\rho} = \frac{\pi^2 B}{2 p_F^3} \qquad (IIIc.7)$$

With these we have the expression for the total energy per particle of the Fermi gas

$$\left.\frac{E}{N}\right)_{FG} = \frac{\pi^2 B}{2 p_F^3} + \frac{3}{4}\left(1+\frac{g_s^2}{6\pi^2}\right) p_F \qquad (IIIc.8)$$

We now go on to the calculation of the corresponding contributions in the correlated system.The kinetic energy,a one body operator,again has the same expectation value in any of the determinants composing Ψ_o,as in Ψ_o itself.Furhermore these determinants are equivalent to an incompletely filled fermi cube gas wavefunction as in the nonrelativistic case.So the kinetic energy is

$$\frac{K.E.}{V} = \iiint_{-k_F}^{+k_F} \frac{d^3k \cdot 3 \cdot k}{(2\pi)^3} = \frac{3k_F^4}{(2\pi)^3} \iiint_{-1}^{+1} d^3K K \equiv \frac{3k_F^4}{(2\pi)^3} I \qquad (IIIc.9)$$

The integral I is evaluated numerically. We find,

$$I \simeq 7.686 \qquad (IIIc.10)$$

which yields,

$$\frac{K.E.}{V} = \frac{3 \cdot 7.686}{(2\pi)^3} k_f^4 \qquad (IIIc.11)$$

The number density is,

$$\rho = \frac{3 k_F^3}{\pi^3} \qquad (IIIc.12)$$

which with (IIIc.11) gives,

$$\frac{K.E.}{N} = .96\, k_F \qquad (IIIc.13)$$

We see again that in the correlated wavefunction the average kinetic energy is higher than it would be in a fermi sphere gas by more than a factor of 1.5 .

The interaction energy is calculated numerically as described in the preceding section. We obtain,

$$\frac{\Delta E}{N} = -.078 \cdot \frac{g_s^2}{8\pi^2} k_F \qquad (IIIc.14)$$

Since the correlated system has the same density, the bag energy contribution is the same as in the Fermi gas, It is given by (IIIc.7)

Gathering all the results we have the total energy per particle for the correlated system

$$\left.\frac{E}{N}\right)_{\substack{\text{correlated}\\ \text{matter}}} = \frac{\pi^3 B}{3k_F^3} + \left(.96 - .078\,\frac{g_s^2}{8\pi^2}\right)k_F \qquad (IIIc.15)$$

Demanding the correlated system to be at the same density as the fermi gas yields,

$$k_F = \left(\frac{2\pi}{3}\right)^{1/3} p_F$$

Substituting this in (IIIc.15) yields

$$\left.\frac{E}{N}\right)_{\substack{\text{correlated}\\ \text{matter}}} = \frac{\pi^2 B}{2p_F^3} + \left(1.23 - .1\,\frac{g_s^2}{8\pi^2}\right)p_F \qquad (IIIc.16)$$

We observe that,since the relativistic kinetic energy scales with the density the same way as the interaction energy,the only difference between the two E/N expressions (IIIc.8) and (IIIc.16) is in the coefficients of the linear term in p_F.The fermi gas has the lowest possible kinetic energy but has an unfavorable interaction energy and the situation is reversed in the case of the correlated system.At first sight ,this result seems to indicate that the fermi gas will lie above or below the correlated system depending on the coefficients of the linear term in p_F which in turn depends on the strong coupling constant g_s.In the discussion so far,however, we have left out an important property of QCD because we have left out all the complications introduced by the fact that QCD is a non abelian gauge theory.This property is asymptotic freedom[21] which tells that the renormalized quark gluon coupling constant g_s becomes smaller at higher momentum transfers and conversely it is large for low momentum transfer and large distance phenomena.In our problem the interaction energy is obtained

as a sum of diagrams involving a whole range of momenta.The overall scale for these momentum transfers,however, is set by the fermi momentum p_F.Thus at high densities or large p_F we have weak coupling and at low densities $\alpha_s=g_s^2/4\pi$ becomes large.When we examine the equations (IIIc.16) and (IIIc.8) in the light of this expected behavior of g_s^2 we see that at high densities the interaction terms will be negligible and the important contribution will come from the kinetic energy of the quarks and the fermi gas will be the energetically favorable state.On the contrary,at low densities g_s^2 will become large and the important term will be interaction contribution favoring the correlated system.We then expect the quark matter system to undergo a transition from the correlated,nuclear matter,state at low densities to a fermi gas at high densities.We note that this conclusion has been drawn with a minimal assumption on the behavior of the coupling constant as a function of the density.This assumption is theoretically predicted by QCD and has been experimentally verified whenever tested.If we want to examine the detailed features of this transition we need to parametrize the density dependence of the effective coupling in some plausible form.The asymptotic form of α_s is given by,[27]

$$\alpha_s = \frac{2\pi}{\frac{(33-2n_f)}{6}\ln\left(\frac{q^2}{\Lambda^2}\right) + \frac{(153-19n_f)}{(33-2n_f)}\log\left(\log\left(\frac{q^2}{\Lambda^2}\right)\right)} \qquad (IIIc.17)$$

where n_f is the effective number of the flavors of quarks and Λ_{QCD} is the QCD scale parameter,expected to be $\Lambda_{QCD}\approx$50-150 Mev.[29] We then parametrize our effective coupling constant as,

$$\alpha(k_F) = \frac{\alpha(\Lambda)\ln 2}{\ln\left(1+\frac{k_F^2}{\Lambda^2}\right)} \qquad (IIIc.18)$$

to be consistent with (IIIc.17) at high densities.The 1 in the logarithm is inserted to remove the unphysical singularity at p_F=A. We do not expect (IIIc.18) to provide an accurate description of the variation of the effective coupling constant for p_F A.In the region of interest to us p_F 2A.At very low densities the dominant contributions come from the bag energy which has a $1/p_F^3$ behavior and so the value of the interaction term which at low densities varies as $1/p_F$ is not important.With this parametrization of α we have the two E/N curves as

$$\frac{E}{N}\Big)_1 = \left[\frac{b}{k^3} + \left(\frac{3}{4} + \beta\frac{\alpha_s(\Lambda)}{\ln(1+k^2)}\right)k\right]\Lambda_{QCD} \qquad (IIIc.19)$$

$$\frac{E}{N}\Big)_2 = \left[\frac{b}{k^3} + \left(1.23 - .1\beta\frac{\alpha_s(\Lambda)}{\ln(1+k^2)}\right)\right]\Lambda_{QCD} \qquad (IIIc.20)$$

where we have defined

$$b = \frac{\pi^2}{2}\frac{B}{\Lambda_{QCD}^4} \qquad (IIIc.21)$$

$$\beta = \frac{\ln 2}{2\pi} \qquad (IIIc.22)$$

$$k = \frac{p_F}{\Lambda_{QCD}} \qquad (IIIc.23)$$

Once we have the values of B,α(A),and A we can calculate the energy per particle in each

phase.In its original version the bag model fit to hadron masses yielded,

$$B^{1/4} \cong 145 \text{ MeV} \quad , \quad \alpha_s (k_f \approx 300 \text{ MeV}) \cong 2.2$$

with these and using Λ_{QCD}=100 Mev,we obtain the E/A curves plotted in Figure 4.A detailed discussion of these curves with the other results will be given in the last chapter.

IV. The Overlap Of The Localized Wave Functions With Bag Model Wave Functions

The form of the single nucleon wave functions in infinite matter was forced by the technical constraint of the orthonormality of the basis of single particle wavefunctions. This constraint is necessary for a feasible many body calculation. We now would like to see how this wave function compares with the original bag model wave function (IIIa.11). To do this we calculate the overlap of the function $u_i(r)$ given by (IIIa.48) with the lowest mode of the bag model (IIIa.11). We define,

$$A=\int d^3\vec{r}\; u^{+}_{bag}(\vec{r})\, u_{local}(\vec{r}) \tag{IV.1}$$

The overlap of the N particle wave functions constructed with the two different sets is then A^N. Using (IIIa.11) and (IIIa.48) for the two wave functions we obtain

$$\int d^3\vec{r}\; u^{+}_{bag}(\vec{r})\, u_{l_0}(\vec{r})=\frac{1}{NR^{3/2}}\left[\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\left(\frac{k_F}{2}\right)^{3/2}\right]\iiint_{-1}^{+1} d^3k\left[e^{i\vec{k}\cdot k_F\vec{r}}\, d^3\vec{r}\right.$$

$$\left.\times\left[\frac{\sin\frac{\pi r}{R}}{\frac{\pi r}{R}}+\left[\frac{\sin\frac{\pi r}{R}}{\left(\frac{\pi r}{R}\right)^2}-\frac{\cos\frac{\pi r}{R}}{\frac{\pi r}{R}}\right]\right]\left[i\left(\hat{r}_z\cos\theta_k+\hat{r}_x\sin\theta_k e^{i\varphi_k}-i\hat{r}_y\sin\theta_k e^{i\varphi_k}\right)\right] \tag{IV.2}$$

by changing variables to u=xr/R in the r integral, we get

$$A=\frac{1}{NR^{3/2}}\left[\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\left(\frac{k_F}{2}\right)^{3/2}\right]\frac{R^3}{\pi^3}\iiint_{-1}^{+1} d^3k\left[d^3u\, e^{-i\frac{k_F R}{\pi}(\vec{u}\cdot\vec{k})}\left[\frac{\sin u}{u}+i\left[\frac{\sin u}{u^2}-\frac{\cos u}{u}\right]\right]u\left[\cos\theta_u\cos\theta_k+\sin\theta_k e^{i\varphi_k}\right.\right.$$
$$\left.\left.\times\sin\theta_u e^{-i\varphi_u}\right]\right]$$

$$=\frac{1}{4(2\pi)^{3/2}}\frac{(k_F R)^{3/2}}{N\pi^3}\iiint_{-1}^{+1} d^3k\left[d^3u\, e^{-i\frac{k_F R}{\pi}ku\cos\theta_u}\left[\frac{\sin u}{u}+i\left[\frac{\sin u}{u}-\cos u\right]\right]\left[\cos\theta_u\cos\theta_k+\sin\theta_k\sin\theta_u e^{i\varphi_k}e^{-i\varphi_u}\right]\right.$$

(IV.3)

The last term vanishes because of the integration over $e^{-i\phi}$.The term with the $\cos\theta_k$ is an odd integrand with respect to the k integration and it drops out leaving

$$A=\frac{1}{4(2\pi)^{3/2}}\frac{(k_FR)^{3/2}}{x^3N}\iiint_{-1}^{+1}d^3k\left[\frac{4\pi x}{k(k_FR)}\left[\frac{1}{2}\int_0^1\cos(u-\frac{k_FR}{x}u)-\cos(u+\frac{k_FR}{x}u)\right]\right]$$

$$=\frac{\pi}{(2\pi)^{3/2}}\frac{(k_FR)^{3/2}}{x^3N}\left[\frac{x}{2k_FR}\iiint_{-1}^{+1}\frac{d^3k}{k}\left[\frac{\sin(1-\frac{k_FR}{x})}{(1-\frac{k_FR}{x})}-\frac{\sin(x+k_FR/x)}{(1+\frac{k_FR}{x})}\right]\right.$$

$$=\frac{\pi}{Nx^3}\left(\frac{k_FR}{2\pi}\right)^{3/2}\left[\frac{x}{2k_FR}\left[\frac{\sin(x-k_FR)}{(1-\frac{k_FR}{x})}-\frac{\sin(x+k_FR)}{(1+\frac{k_FR}{x})}\right]\right]\iiint_{-1}^{+1}\frac{d^3k}{k} \qquad (IV.4)$$

putting $k_FR=y$ we have,

$$A=\frac{\pi}{(2\pi)^{3/2}}\frac{y^{3/2}}{Nx^3}\left[\frac{x}{2y}\left[\frac{\sin(x-y)}{(1-\frac{y}{x})}-\frac{\sin(x+y)}{(1+\frac{y}{x})}\right]\right]\tilde{I} \qquad (IV.5)$$

with

$$\tilde{I}=\iiint_{-1}^{+1}\frac{d^3k}{k}=9.49 \qquad (IV.6)$$

evaluated numerically.Inserting the numerical values of the constants appearing and with,

$$x=2.0428 \qquad N^{-1}=.44$$

we obtain

$$A=.01y^{3/2}\left[\frac{9.68}{y}\left[\frac{\sin(2.04-y)}{(1-\frac{y}{2.04})}-\frac{\sin(2.04+y)}{(1+\frac{y}{2.04})}\right]\right. \qquad (IV.7)$$

which has a maximum near $y=k_F R \approx = 2.62$ with $A \approx .37$ .With the average nuclear density of

$$\rho \cong .5 \text{ quarks}/\text{fm}^3 \tag{IV.8}$$

corresponding to

$$k_F \cong 1.75 \text{ fm}^{-1} \tag{IV.9}$$

the maximum overlap occurs at

$$R \cong 1.5 \text{ fm}$$

We also calculate the same overlap with the chiral localized wave function

$$u = \frac{1}{\sqrt{2}} \int_{-k_F}^{k_F} \frac{e^{-i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}(2k_F)^{3/2}} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \\ \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{pmatrix}$$

and obtain a maximum overlap of $A \approx .3$ .We thus observe that neither wave function has a large overlap with the exact bag model wave function. They are trial wave functions exhibiting the general features of the bag model wave functions as much as is allowed by calculational difficulties. We hope that more progress can be done in this respect in future calculations.

V.Incorporation Of Strange Quarks

The speculation that long lived states of quark matter having a number of strange quarks comparable to that of the light u and d quarks has led to many investigations on the properties of such systems with macroscopic strangeness.[30] The expectation of a low energy system with a large number of strange quarks stems from the simple observation of the one gluon exchange energy of a fermi gas of quarks of mass m,[31]

$$\frac{\Delta E}{N} = -\frac{g_s^2}{4\pi^2}\frac{m}{x^3}\left[x^4 - \frac{3}{2}\left[x(1+x^2)^{1/2} - \ln(x+(1+x^2)^{1/2})\right]^2\right] \qquad (V.1)$$

with $x=k_F/m$ plotted in Figure 5.We see that the gluon exchange provides a repulsive interaction for light quarks and an attractive interaction for heavy quarks in a fermi gas.It is concievable then that a favorable gluon exchange interaction could produce a long lived hyperstrange quark matter system which could decay to normal nuclear matter only by weak interactions. At high densities, with large fermi momenta(compared to the strange quark mass)for the light quarks,this strange matter could even turn out to be the true ground state.This is contrary to our intuition regarding strange systems with few strange quarks.The lowest observed strange baryon is the Λ particle with mass m_Λ=1116 Mev.We do not have any real information regarding a system of many Λ particles but we would expect this system to have a higher energy compared to a system of u and d quarks or nucleons having the same total baryon number.In this chapter we perform a calculation to check these intuitive forecasts regarding quark systems with a macroscopic number of strange quarks.

To do this we first contruct our localized wave functions with massive quarks.The four

linearly independent solutions of the free dirac equation for a particle of mass m,

$$(\not{p}-m)\psi=0 \tag{V.2}$$

can be written as,

$$u^{(\alpha)}(k)=\begin{pmatrix}\left(\frac{E+m}{2m}\right)^{1/2}\varphi^{(\alpha)}\\ \frac{\vec{\sigma}\cdot\vec{k}}{[2m(m+E)]^{1/2}}\varphi^{(\alpha)}\end{pmatrix} \tag{V.3}$$

and

$$v^{(\alpha)}(k)=\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{k}}{[2m(m+E)]^{1/2}}\varphi^{(\alpha)}\\ \left(\frac{E+m}{2m}\right)^{1/2}\varphi^{(\alpha)}\end{pmatrix} \tag{V.4}$$

with

$$E=p^0=\sqrt{p^2+m^2} \tag{V.5}$$

These solutions have the orthogonality relations,

$$\bar{u}^{(\alpha)}(k)\,u^{(\beta)}(k)=\bar{v}^{(\alpha)}(k)\,v^{(\beta)}(k)=\delta_{\alpha\beta}$$

$$\bar{u}^{(\alpha)}(k)\,\gamma^0\,u^{(\beta)}(k)=u^{\dagger(\alpha)}(k)\,u^{(\beta)}(k)=v^{\dagger(\alpha)}(k)\,v^{(\beta)}(k)=\frac{E}{m}\,\delta_{\alpha\beta} \tag{V.6}$$

We construct our localized wave functions in the same manner,

$$u_i^{(\alpha)}=\iiint_{-k_f}^{+k_f}\frac{d^3k\,e^{-i\vec{k}\cdot(\vec{r}-\vec{r}_i)}\,u^{(\alpha)}(k)}{(2\pi)^{3/2}\,(2k_f)^{3/2}\,\left(\frac{E}{m}\right)^{1/2}} \tag{V.7}$$

and verify the orthogonality relation,

$$\int d^3\vec{r}\, u_i^{(\alpha)\dagger}(\vec{r})\, u_j^{(\beta)}(\vec{r}) = \int \frac{d^3\vec{r}\, d^3\vec{k}\, d^3\vec{k}'\, e^{i(\vec{k}'\cdot\vec{r}_j - \vec{k}\cdot\vec{r}_i)}}{(2\pi)^3\,(2k_F)^3\,\left(\frac{EE'}{m^2}\right)^{1/2}}\, u^{(\alpha)\dagger}(k)\, u^{(\beta)}(k')$$

$$= \frac{\delta_{\alpha\beta}}{(2k_F)^3} \iiint_{-k_F}^{+k_F} e^{i\vec{k}\cdot(\vec{r}_j - \vec{r}_i)}\, d^3k$$

$$= \frac{\delta_{\alpha\beta}}{(2k_F)^3} \left[\prod_{\alpha} \frac{2\sin k_F(\vec{r}_j - \vec{r}_i)_\alpha}{(\vec{r}_j - \vec{r}_i)_\alpha} \right] = \delta_{\alpha\beta}\,\delta_{ij} \qquad (V.8)$$

Using these wave functions we calculate the T(i,j) that are needed for calculating the exchange energy by using (IIIb.20).For the case of two strange quarks occupying the orbits i and j we have

$$T_q^{\lambda a}(i,j) = \int d^3\vec{r} \int_{-k_F}^{+k_F} \frac{d^3\vec{k}_1\, d^3\vec{k}_2\, e^{i\vec{k}_1\cdot(\vec{r}-\vec{r}_i) - i\vec{k}_2\cdot(\vec{r}-\vec{r}_j)}}{(2\pi)^3 (2k_F)^3 (E_1 E_2/m^2)^{1/2}}\, \bar{u}^{(\alpha_i)}(k_1)\gamma^\lambda u^{(\alpha_j)}(k_2)\, e^{i\vec{q}\cdot\vec{r}}\, t^a_{c_i c_j}$$

$$= \int \frac{d^3k_1\, d^3k_2\, e^{i\vec{k}_2\cdot\vec{r}_j - i\vec{k}_1\cdot\vec{r}_i}}{(2\pi)^3\,(2k_F)^3\,(E_1 E_2/m^2)^{1/2}}\, \bar{u}^{(\alpha_i)}(k_1)\gamma^\lambda u^{(\alpha_j)}(k_2)\, \delta(\vec{k}_1 - \vec{k}_2 + \vec{q})\,(2\pi)^3\, t^a_{c_i c_j}$$

$$= \frac{m e^{i\vec{q}\cdot\vec{r}_i}}{(2k_F)^3} \iiint_{-k_F}^{+k_F} \frac{d^3k\, e^{i\vec{k}\cdot\vec{r}_j - i\vec{k}_1\cdot r_i}\; \bar{u}^{(\alpha_i)}(\vec{k}-\vec{q})\gamma^\lambda u^{(\alpha_j)}(\vec{k})\,\theta(k_F - |\vec{k}-\vec{q}|_\alpha|)}{((\vec{k}-\vec{q})^2 + m^2)^{1/4}\,(\vec{k}^2 + m^2)^{1/4}} \qquad (V.9)$$

by substituting $\kappa = \mathbf{k}/k_F$ in the integral we obtain

$$= \frac{m e^{i\vec{q}\cdot\vec{r}_i}}{8} \int_{-1}^{+1} \frac{d^3\kappa\, e^{i\vec{\kappa}\cdot\vec{n}_{ij}\pi}\; \bar{u}^{(\alpha_i)}(\vec{\kappa}k_F - \vec{q})\gamma^\lambda u^{(\alpha_j)}(\vec{\kappa}k_F)\,\theta(k_F - |(k_F\vec{\kappa} - \vec{q})_\alpha|)\, t^a_{c_i c_j}}{((k_F\vec{\kappa} - \vec{q})^2 + m^2)^{1/4}\,(\kappa^2 k_F^2 + m^2)^{1/4}} \qquad (V.10)$$

at this point we also substitute $\Delta = m/k_F = 1/x$. We then have

$$\left[\frac{E+m}{2m}\right]^{1/2} = \left[\frac{1+\sqrt{1+\frac{k^2}{\Delta^2}}}{2}\right]^{1/2}$$

$$\frac{\vec{\sigma}\cdot\vec{k}}{\left[2m\,[E+m]\right]^{1/2}} = \frac{\vec{\sigma}\cdot\vec{k}}{\left[2\Delta\left(\sqrt{k^2+\Delta^2}+\Delta\right)\right]^{1/2}} \qquad (V.11)$$

and

$$\left((k_F\vec{k}-\vec{q})^2+m^2\right)^{1/4} = \sqrt{m}\left[\frac{(\vec{k}-\vec{\zeta})^2}{\Delta^2}+1\right]^{1/4}$$

$$\left(k^2+m^2\right)^{1/4} = \sqrt{m}\left[\frac{k^2}{\Delta^2}+1\right]^{1/4} \qquad , \ \vec{\zeta} = \frac{\vec{q}}{|\vec{q}|} \qquad (V.12)$$

which altogether result in

$$T_q^{\lambda\lambda}(i,j) = \frac{e^{i\vec{q}\cdot\vec{r}_i}}{8}\iiint_{-1}^{+1} \frac{d^3k\, e^{i\vec{k}\cdot\vec{n}_{ij}\pi}\,\Theta(1-|\vec{k}-\vec{\zeta}|)}{\left(\frac{k^2}{\Delta^2}+1\right)^{1/4}\left(\frac{|\vec{k}-\vec{\zeta}|^2}{\Delta^2}+1\right)^{1/4}}\,\bar{u}^{(\alpha,i)}(\vec{k}-\vec{\zeta})\,\gamma^\lambda\,u^{(\alpha,j)}(\vec{k})\,t^\lambda_{c_i c_j} \qquad (V.13)$$

with

$$u^{(\alpha)}(\vec{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \left[\frac{k^2}{\Delta^2}+1\right]^{1/2}\varphi^{(\alpha)} \\ \frac{(\vec{\sigma}\cdot\vec{k})\,\varphi^{(\alpha)}}{\left[\Delta\left(\sqrt{\Delta^2+k^2}+\Delta\right)\right]^{1/2}} \end{pmatrix} \qquad (V.14)$$

Along with this, the correlated wave function becomes

$$|\Psi_0\rangle = \prod_i S_i^\dagger\,|0\rangle$$

with the creation operator $S_i^\dagger$ now given by,

$$S_i^\dagger = \frac{1}{6}\Big[2c^\dagger_{u\uparrow b}c^\dagger_{d\uparrow r}c^\dagger_{s\downarrow g} - 2c^\dagger_{u\uparrow b}c^\dagger_{d\uparrow g}c^\dagger_{s\downarrow r} + 2c^\dagger_{u\uparrow r}c^\dagger_{d\uparrow g}c^\dagger_{s\downarrow b} - 2c^\dagger_{u\uparrow r}c^\dagger_{d\uparrow b}c^\dagger_{s\downarrow g}$$

$$+ 2c^\dagger_{u\uparrow g}c^\dagger_{d\uparrow b}c^\dagger_{s\downarrow r} - 2c^\dagger_{u\uparrow g}c^\dagger_{d\uparrow r}c^\dagger_{s\downarrow b} - c^\dagger_{u\downarrow b}c^\dagger_{d\uparrow r}c^\dagger_{s\uparrow g} + c^\dagger_{u\downarrow b}c^\dagger_{d\uparrow g}c^\dagger_{s\uparrow r} - c^\dagger_{u\downarrow r}c^\dagger_{d\uparrow g}c^\dagger_{s\uparrow b} + c^\dagger_{u\downarrow r}c^\dagger_{d\uparrow b}c^\dagger_{s\uparrow g} - c^\dagger_{u\downarrow g}c^\dagger_{d\uparrow b}c^\dagger_{s\uparrow r}$$

$$+ c^\dagger_{u\downarrow g}c^\dagger_{d\uparrow r}c^\dagger_{s\uparrow b} - c^\dagger_{u\uparrow b}c^\dagger_{d\downarrow r}c^\dagger_{s\uparrow g} + c^\dagger_{u\uparrow b}c^\dagger_{d\downarrow g}c^\dagger_{s\uparrow r} - c^\dagger_{u\uparrow r}c^\dagger_{d\downarrow g}c^\dagger_{s\uparrow b} + c^\dagger_{u\uparrow r}c^\dagger_{d\downarrow b}c^\dagger_{s\uparrow g} - c^\dagger_{u\uparrow g}c^\dagger_{d\downarrow b}c^\dagger_{s\uparrow r} + c^\dagger_{u\uparrow g}c^\dagger_{d\downarrow r}c^\dagger_{s\uparrow b}\Big] \qquad (V.15)$$

derived from the spin isopin color wavefunction of the Λ particle.[16]

It is now straightforward to calculate the exchange energy in the same manner as was done in section IIIb and IIIc.The result has the form

$$\frac{\Delta E^{(2)}}{\text{per orbit}} = \frac{1}{2}g_s^2\Bigg[\sum_n \sum_{\substack{i=1,3\\ j=1,3}} \frac{(\cos f(n))^2}{36}\sum_\lambda \int \frac{d^3p_\lambda}{(2\pi)^3}\frac{|T_p^{\lambda\lambda}(i,j)|^2\,\vartheta_\lambda}{p_\lambda^2}$$

$$+\frac{1}{2}\sum_{\substack{m,n\\ m\neq n}}\sum_{\substack{i=1,3\\ j=1,3\\ i\neq j}} \frac{(\cos f(m))^2(\cos f(n))^2}{36^2}\sum_\lambda \int \frac{d^3p_\lambda}{(2\pi)^3}\frac{|T_p^{\lambda\lambda}(m_i,n_j)|^2\,\vartheta_\lambda}{p_\lambda^2}$$

$$+\sum_{\substack{\text{cross}\\ \text{terms}\\ P(ijk\ell)}} \frac{(\cos f(1))(\cos f(2))(-)^P}{36}\sum_\lambda \int \frac{d^3p_\lambda}{(2\pi)^3}\frac{T_p^{\lambda\lambda}(i,j)T_p^{\dagger\lambda\lambda}(\ell,k)}{p_\lambda^2}\Bigg] \qquad (V.16)$$

Here,the sum in m,n goes over the 18 terms in (V.15) and the sum over the cross terms goes over the pairs of determinants that differ by two orbits as described in IIIb.We note that T(i,j) vanishes unless i and j are of the same flavor.The end result is again obtained through a numerical evaluation of (V.16).The result for the gluon exchange contribution

for the energy per particle of Λ matter is calculated numerically for four different values of the parameter Δ.The result has the form,

$$\frac{\Delta E^{(2)}}{N} = \frac{1}{2} g_s^2 \gamma(\Delta) k_F \qquad (V.17)$$

the calculated values of $\gamma(\Delta)$ are given in Table 3.Putting this together with the kinetic energy contribution for strange quarks,

$$\frac{K.E.}{V} = \iiint_{-k_F}^{+k_F} \frac{d^3k}{(2\pi)^3} \sqrt{k^2+m^2} \equiv \frac{k_F^4}{(2\pi)^3} I(\Delta) \Rightarrow \frac{K.E}{N} = \frac{3}{4\pi} I(\Delta) k_F \qquad (V.18)$$

which yields an average kinetic energy,with I(Δ) defined in (V.18) ,and the bag energy contribution yields the energy per particle in correlated strange quark matter,

$$\frac{E}{N} = \frac{\pi^2 B}{2k_F^3} + \left(\frac{3}{4\pi} I(\Delta) + \frac{1}{2} g_s^2 \gamma(\Delta)\right) k_F \qquad (V.20)$$

The kinetic energy contribution in the fermi gas is

$$\frac{K.E.}{N} = \frac{6\pi^2}{p_F^3} \int_0^{p_F} \frac{d^3k}{(2\pi)^3} \sqrt{k^2+m^2} = \frac{3}{4\pi} p_F \int_0^1 d^3K \sqrt{k^2+\Delta^2}$$
$$= \frac{3p_F}{8} \Delta^4 \left[\frac{1}{\Delta}\sqrt{1+\frac{1}{\Delta^2}}\left(1+\frac{2}{\Delta^2}\right) + \ln\left(\frac{1}{\Delta}+\sqrt{1+\frac{1}{\Delta^2}}\right)\right] \equiv p_F f(\Delta) \qquad (V.21)$$

which combined with (V.1) and the bag contribution gives the energy per particle of the uncorrelated strange quark fermi gas consisting of equal numbers of u,d and s quarks.

$$\frac{E}{N} = \frac{\pi^2 B}{2k_F^3} + \frac{p_F}{3}\left[2\cdot\frac{3}{4}\left(1+\frac{g_s^2}{6\pi^2}\right) + f(\Delta) - \frac{g_s^2}{4\pi^2} \times \Delta^4 \left\{\frac{1}{\Delta^4} - \frac{3}{2}\left[\frac{1}{\Delta}\left(1+\frac{1}{\Delta^2}\right)^{1/2} - \ln\left(\frac{1}{\Delta}+\left(1+\frac{1}{\Delta^2}\right)^{1/2}\right)\right]\right] \qquad (V.22)$$

also define, $h(\Delta) \equiv 2\Delta^4\left\{\frac{1}{\Delta^4} - \frac{3}{2}\left[\frac{1}{\Delta}\left(1+\frac{1}{\Delta^2}\right)^{1/2} - \ln\left(\frac{1}{\Delta}+\left(1+\frac{1}{\Delta^2}\right)\right)^{1/2}\right]^2\right\}$

We again put these equations into dimensionless form by using the scale parameter Λ.By using the same parametrization of the running coupling constant as in the nuclear matter case,and substituting $k_F=\pi^{1/3}\ p_F$ obtained by equating the densities,we find

$$\left.\frac{E}{N}\right)_{\substack{\text{corr.}\\ \text{strange}\\ \text{matter}}} = \left(\frac{b}{\kappa^3}+\left[\left(\frac{3}{4\pi}I\left(\frac{m}{\kappa\Lambda}\right)+2\cdot\frac{3}{4}\right)/3+\frac{\beta\alpha_s(\Lambda)\gamma\left(\frac{m}{\kappa\Lambda}\right)}{e_m(1+\kappa^2)}\right]\pi^{1/3}\kappa\right)\Lambda_{QCD} \qquad (V.23)$$

$$\left.\frac{E}{N}\right)_{\substack{\text{strange}\\ \text{quark}\\ \text{gas}}} = \left(\frac{b}{\kappa^3}+\left[\left(\frac{1}{4}I\left(\frac{m}{\kappa\Lambda}\right)+2\cdot\frac{3}{4}\right)/3+\frac{\beta\alpha_s(\Lambda)}{e_m(1+\kappa^2)}\left[2-h\left(\frac{m}{\kappa\Lambda}\right)\right]/6\pi\right]\kappa\right)\Lambda_{QCD} \qquad (V.24)$$

with b,β and κ given by (IIIc.22-24).By using the values of the bag model parameters given in section IIIc we obtain the curves shown in Figure 5.

One aspect of the problem we have left out is the self energy of strange quarks in infinite matter which unlike the case of massless quarks need not be zero.We plan to include this effect in a future work.Here we only note that,the self energy enters to both systems in the same way and it would not alter the relative positions of the E/N curves which is the main variable of interest in this work.

A discussion of the main features of Figure 5 will be presented in the next section along with the discussion of the other results.

VI.Discussion Of The Results And Conclusion :

Our goal in this work has been singling out the important features of the quark wavefunctions in nuclear matter.To accomplish this goal we have modified uncorrelated fermi gas wave functions by building local and spin,isopin,color dependent correlations among the quarks.The fact that these correlated wave functions give a lower energy than a fermi gas does,indicates that these modifications are in the correct direction towards the true wave functions of the systems considered,at low densities.

Our initial relativistic treatment of nuclear matter as a correlated quark matter as presented in section III has many weaknesses which can hopefully be remedied in future calculations. The energy per particle curve given in figure 4a. lies well above a realistic nuclear matter equation of state.Although it has moved away from the fermi gas curve in the right direction the average energy near the minimum is still too high to claim that this wave function accurately describes nuclear matter.The main reason behind this is the kinetic energy of the quarks in the correlated wavefunction.To maintain the othogonality properties of our basis of single particle wavefunctions,we were forced to a form of the wave function that yields a higher average kinetic energy,by a factor of 1.64 than a fermi gas does. This considerably offsets the gain in interaction energy obtained by coupling the particles locally.This gain in interaction energy is quite insensitive to the detailed spatial behaviour of the single particle orbitals,e.g.whether they are build out of plane waves in a fermi cube or fermi sphere.We could ideally maintain the same gain in the interaction energy without giving up in kinetic energy,if we were able to make orhogonal orbits out of the plane waves in a sphere.If one could find the special lattice,satisfying

the condition that all distances between any of its two points is a root of the first order spherical Bessel function,as mentioned in section IIIa.,we could accomplish this goal.The energy per particle given by such a hypothetical wave function is plotted in figure 4b.with the same curves in 4a.Figure 4b.indicates that such a goal would be too ambitious because the energy per particle at the minimum is less than 300 Mev.We should,then, pay in kinetic energy for building local correlations but not as much as one does with the simplest technically motivated form of the single particle wave functions.We hope to improve our initial calculation in this respect in future calculations,and expect the resulting energy per particle curve to lie between the curves 2 and 3 in Figure 4b.

One remarkable feature of the energy per particle of the correlated system is that its minimum occurs at almost exactly the nuclear matter density.We have not adjusted any parameters to produce this result.In fact all the parameters are fitted to hadron spectra.This result plausibly is more than just a scaling effect , reflecting the fact that the valence quark densities in hadrons and nuclear matter are not very different,because the minimum of the uncorrelated system occurs at a different density.

The results of our calculation with strange quarks also indicate a similar transition from a correlated state at low density to a fermi gas at high density.We are in a much weaker position to make a comparison of these results with the real systems than in the nuclear matter case because there is no information available on the real systems except in the case of very low density where the ground state is a collection of separated Λ particles.We would like to obtain a better fit to the nuclear matter properties with our model before drawing quantitatively predictive conclusions regarding the behavior of correlated strange

quark matter systems.

In conclusion,we have succeded to isolate an important property of quark wave functions in low energy nuclear matter.This property is spin-color-isospin dependent,strong,local and short range correlations among quarks.We are optimistic that the technical difficulties encountered can be overcome in the future to obtain a more accurate description of nuclear matter with quark degrees of freedom.

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34.As a simple example consider the two particle determinants built out of the two different

orthogonal bases,

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) \\ \phi_2(x_1) & \phi_2(x_2) \end{vmatrix} \qquad \psi_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} \frac{1}{\sqrt{2}}\left(\phi_1(x_1) - \phi_2(x_1)\right) & \frac{1}{\sqrt{2}}\left(\phi_1(x_2) - \phi_2(x_2)\right) \\ \frac{1}{\sqrt{2}}\left(\phi_1(x_1) + \phi_2(x_1)\right) & \frac{1}{\sqrt{2}}\left(\phi_1(x_2) + \phi_2(x_2)\right) \end{vmatrix}$$

These two determinants are equal. This can be seen directly by evaluating the determinants or by realizing that either one can be obtained from the other by adding rows in the following way.

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) \\ \phi_2(x_1) & \phi_2(x_2) \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) \\ \left(\phi_1(x_1) + \phi_2(x_1)\right) & \left(\phi_1(x_2) + \phi_2(x_2)\right) \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \frac{1}{2}\left(\phi_1(x_1) - \phi_2(x_1)\right) & \frac{1}{2}\left(\phi_1(x_2) - \phi_2(x_2)\right) \\ \left(\phi_1(x_1) + \phi_2(x_1)\right) & \left(\phi_1(x_2) + \phi_2(x_2)\right) \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \frac{1}{\sqrt{2}}\left(\phi_1(x_1) - \phi_2(x_1)\right) & \frac{1}{\sqrt{2}}\left(\phi_1(x_2) - \phi_2(x_2)\right) \\ \frac{1}{\sqrt{2}}\left(\phi_1(x_1) + \phi_2(x_1)\right) & \frac{1}{\sqrt{2}}\left(\phi_1(x_2) + \phi_2(x_2)\right) \end{vmatrix} = \psi_2$$

Table Captions :

Table 1.1.The behavior of the magnitude of the exchange interaction $v_{ij,ji}$ between quarks in different orbits as a function of the separation between the orbits denoted by the integer vector $\vec{n}=(n_x, n_y, n_z)$.The actual distance scales with k_F as $\vec{r}_i$-$\vec{r}_j=\vec{n}_{ij}\pi/k_F$.

Table 1.2.The relative magnitudes of the interaction energy $v_{ij,ji}$ between different orbits that have three quarks coupled to the designated spin and isospin states.

Table 2.The dimensionless eigenvalues of the massles quark and gluon modes in the bag model obtained through the boundary conditions (IIIa.9), (IIIa.15) and (IIIa.19)

Table 3.Behavior of the coefficient of the interaction term in the energy per particle curve for the coupled strange matter system representing the system of Λ particles, as defined in (V.17), as a function of $\Delta = m/p_F$.

Table 1.1

$n_x=1$

		n_z		
	v	1	2	3
	1	.24	.16	.03
n_y	2	.16	.03	.007
	3	.03	.007	.002

$n_x=2$

		n_z		
	v	1	2	3
	1	.16	.03	.007
n_y	2	.03	.007	.0015
	3	.007	.0015	.0005

Table 1.2

lattice	$\langle v \rangle$
$p\uparrow\ p\uparrow$	-1.44
$p\uparrow\ p\downarrow$	-1.97
$p\uparrow\ n\uparrow$	-1.53
$p\uparrow\ n\downarrow$	-1.21

Table 2.

		ν		
	$x_{K\nu}$	1	2	3
	-1	2.04	5.40	8.58
K	1	3.81	7.30	10.16
	-2	3.20	6.76	10.00
	2	5.12	8.40	11.61

		N		
	ω_{NJ}^{scalar}	1	2	3
	1	0.	4.49	7.72
J	2	2.08	5.94	9.21
	3	3.34	7.29	10.61

		N		
	ω_{NJ}^{mag}	1	2	3
	1	2.74	6.12	9.32
J	2	3.87	7.44	10.71
	3	4.97	8.72	12.06

		N		
	ω_{NJ}^{el}	1	2	3
	1	4.49	7.72	10.90
J	2	5.76	9.09	12.32
	3	6.99	10.42	13.70

Table 3.

Δ	$\gamma(\Delta) \times 10^2$
.25	-.115
.5	-.49
1	-.51
2	-.67

Figure Captions :

Figure 1.The second order quark gluon diagrams.The solid lines denote the quark propogators and the dashed lines are for the gluon propogators both calculated with the appropriate quark and gluon modes used in expanding the many body wavefunctions. The diagram at the very top stands for the exchange term, the second for the direct term, and the last one denotes the quark self energy.

Figure 2a.)The energy per particle (IIa.54) for the Fermi cube gas without color. The values of the dimensionless parameters b and c are $b \approx c \approx 1$ with $b=3\pi^2 B/m^4$ and $c=.67\alpha$

Figure 2b.)The energy per particle (IIa.54) for the Fermi cube gas without color. The values of the dimensionless parameters b and c are such that $b \gg c$ with $b=3\pi^2 B/m^4$ and $c=.67\alpha$

Figure 2c.)The energy per particle (IIa.54) for the Fermi cube gas without color. The values of the dimensionless parameters b and c are such that $b \ll c$ with $b=3\pi^2 B/m^4$ and $c=.67\alpha$

Figure 3a.)The energy per particle of the correlated quark matter with three quarks coupled to a color singlet at each orbit (curve 1) and the uncorrelated quark fermi gas (curve 2),with $b \gg c$, $b=3\pi^2 B/m^4$ and $c=.67\alpha$

Figure 3b.)The energy per particle of the correlated quark matter with three quarks coupled to a color singlet at each orbit (curve 1) and the uncorrelated quark fermi gas (curve 2),with $b \ll c$, $b=3\pi^2 B/m^4$ and $c=.67\alpha$

Figure 4a.)The energy per particle of a relativistic fermi gas of masless u and d quarks (curve 1) and the correlated system with coupled quarks occupying the localized

orbitals obtained by superposing chirally symmetric states Ψ_1 and Ψ_4 given by (IIIa.37) and (IIIa.40)(curve 2).

Figure 4b.)The energy per particle of a relativistic fermi gas of masless u and d quarks (curve 1) and the correlated system with coupled quarks occupying the localized orbitals obtained by superposing chirally symmettric states (curve 2) with the additional curve 3 providing a lower bound as the energy of a hypothetical system possesing the average kinetic energy of a fermi gas and the average interaction energy of the correlated system.

Figure 5.)The energy per particle of a fermi gas of u,d and s quarks (curve 1) and a correlated system representing a collection of Λ (1116) particles. The strange quark mass is taken to be $m_s \approx 300$ Mev with the same bag model parameters $B^{1/4}=145$ Mev and $\alpha_s = 2.2.\Lambda_{QCD}$ is taken to be 100 Mev.

Figure 6.)The one gluon exchange energy of a quark fermi gas of mass m given by (V.1),as a function of the density,ρ. This interaction is attractive for large quark masses, compared to $\rho^{1/3}$, and repulsive for light quarks.

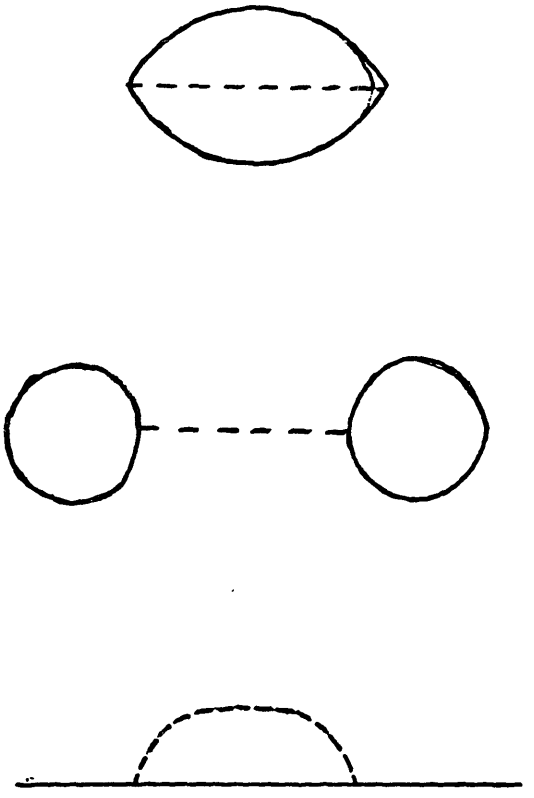

Fig. 1

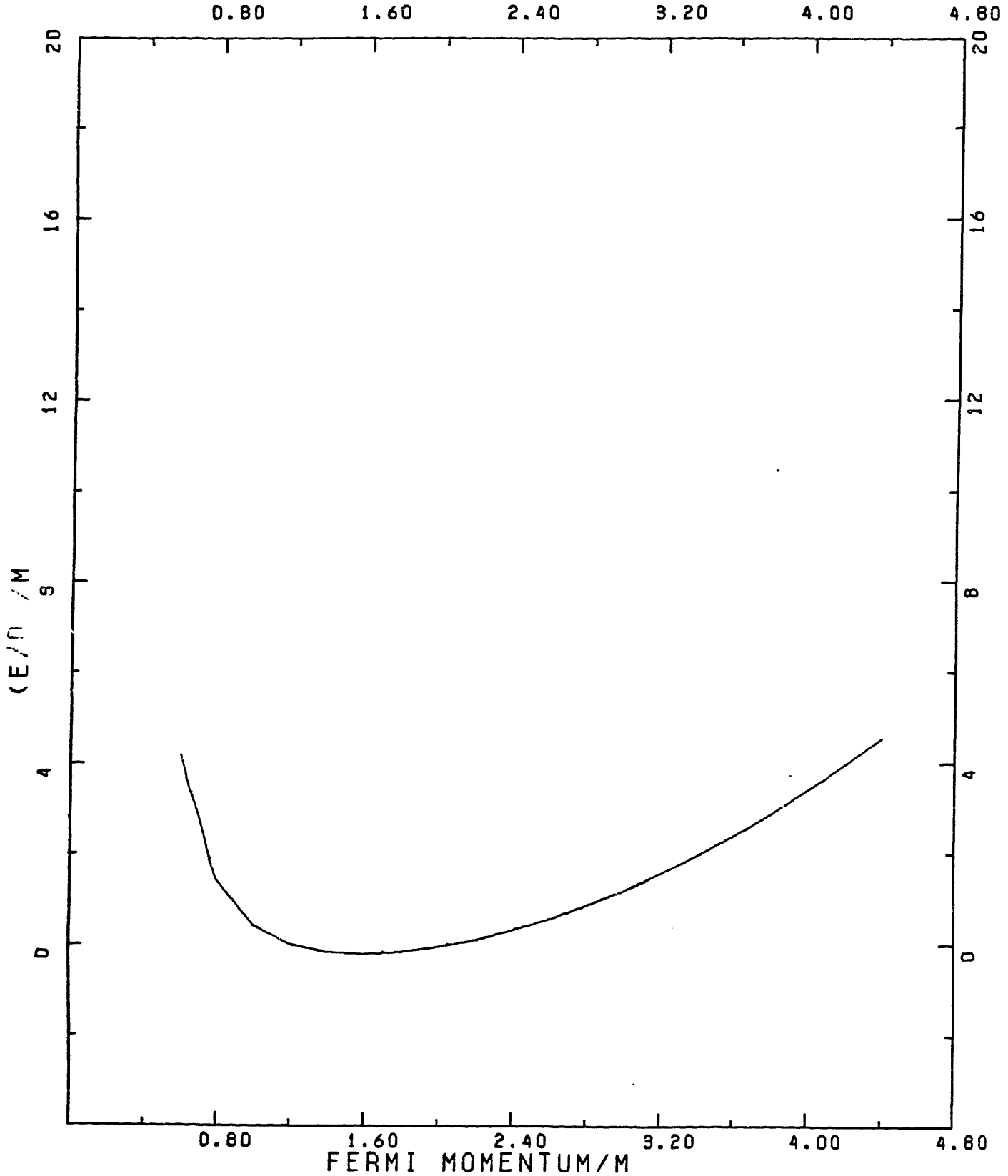

Fig 2a.

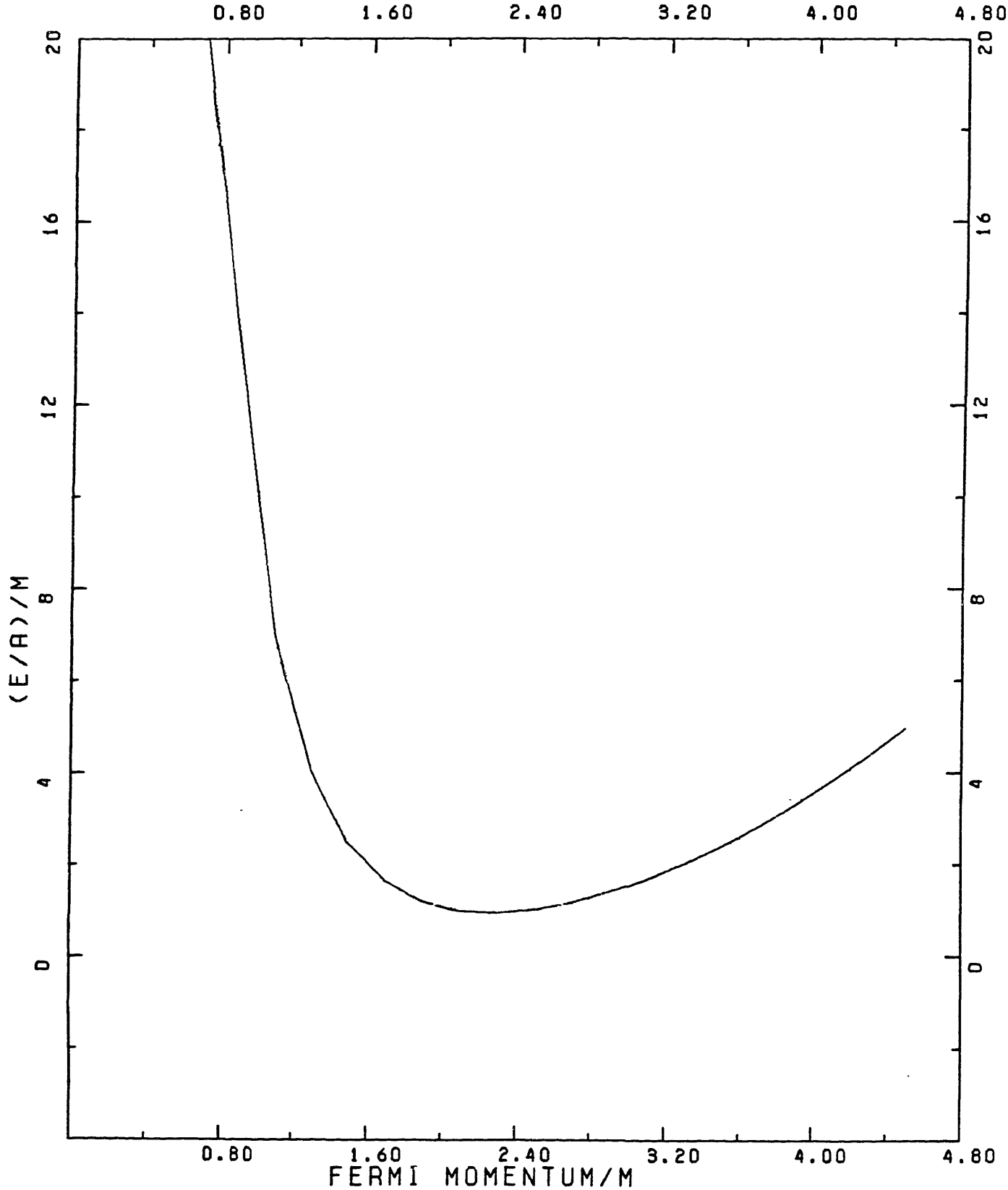

Fig 2b.

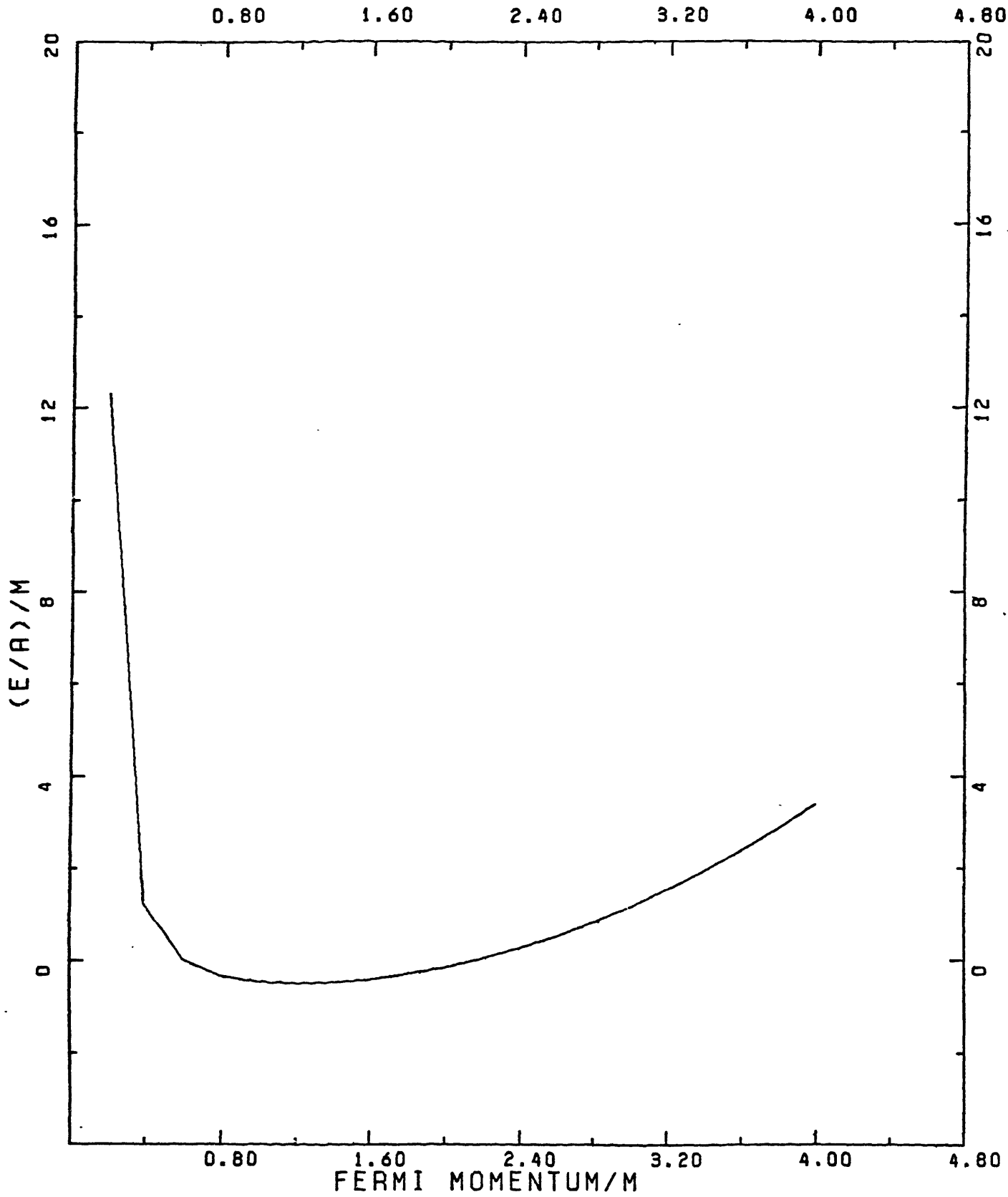

Fig 2c

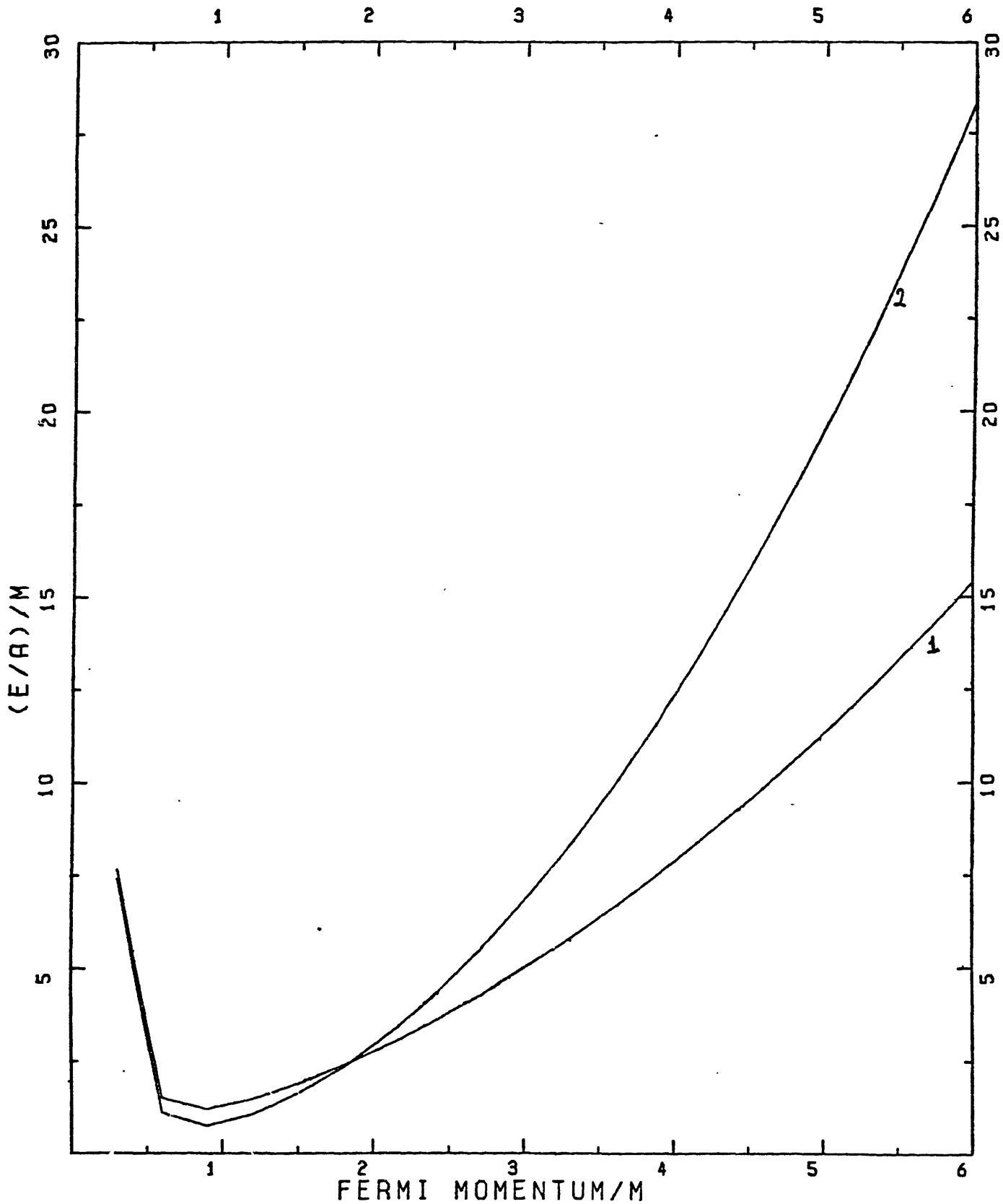

Fig 3a.

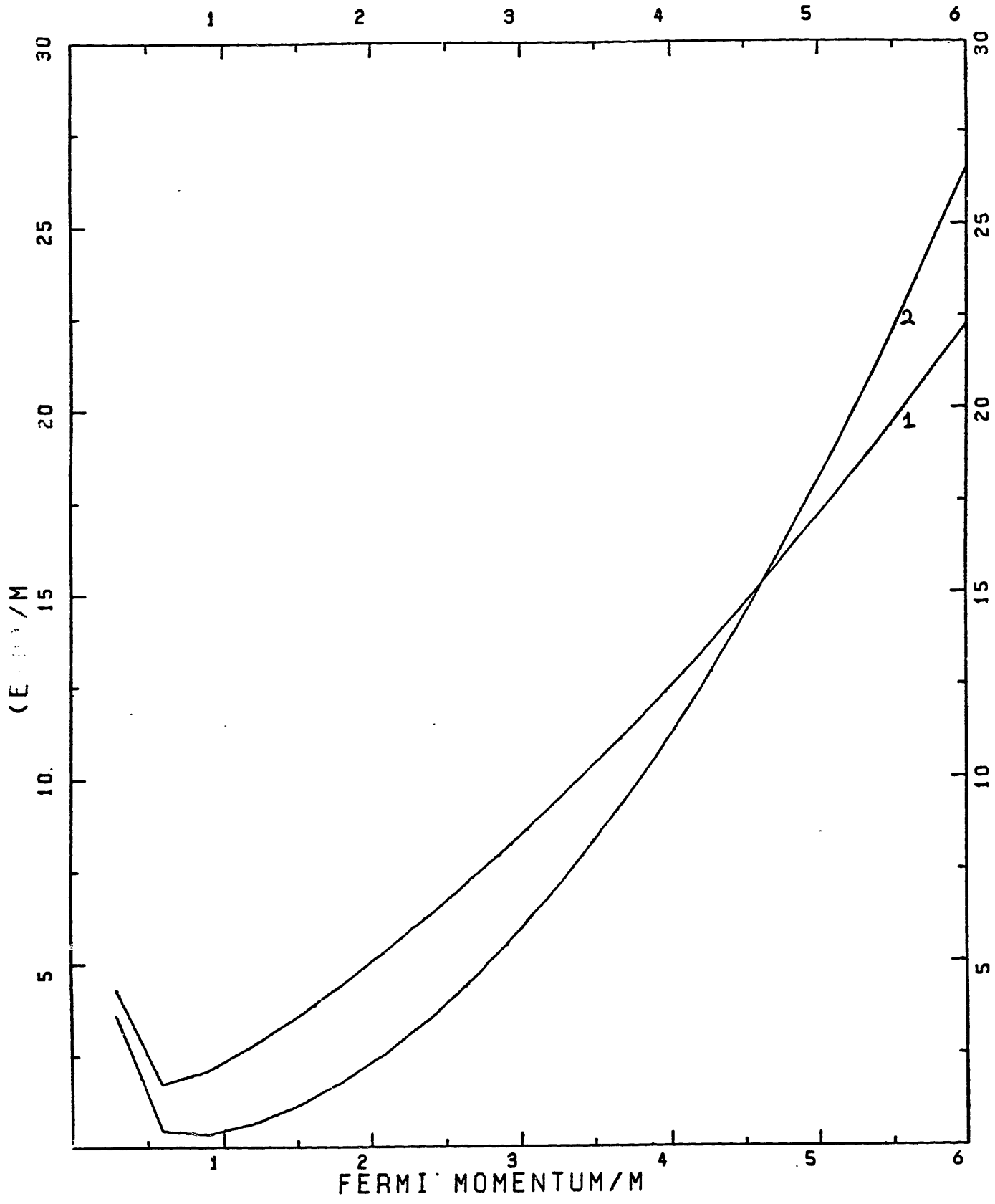

Fig 3b.

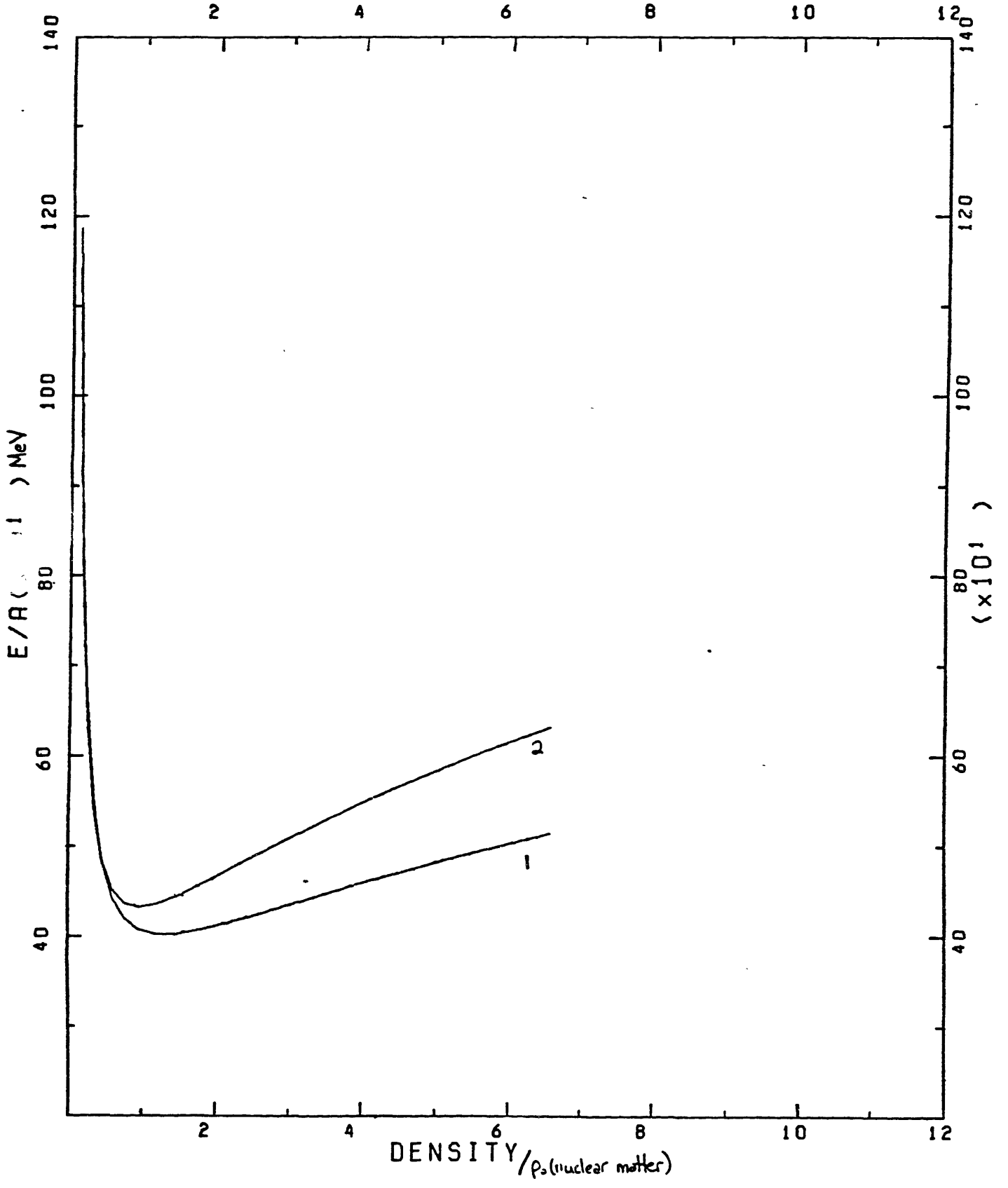
2
4
6
8
10
12
140
120
100
80
60
40
E/A( ) MeV
(×10¹)
2
1
DENSITY /ρ₀(nuclear matter)

Fig 4a.

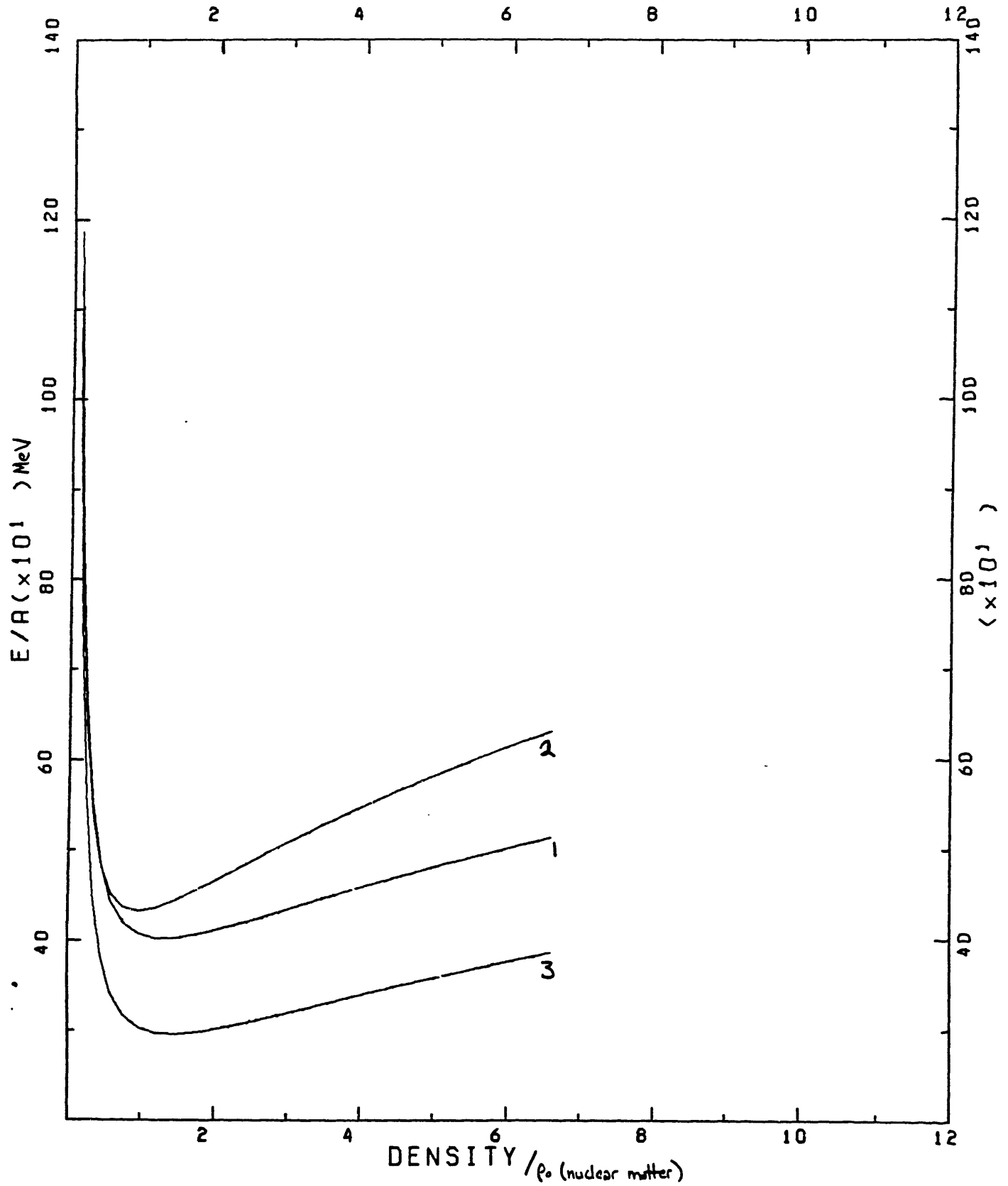
2
4
6
8
10
12
140
120
100
80
60
40
E/A(×10¹)MeV
BD (×10¹)
1
2
3
DENSITY /ρ₀ (nuclear matter)

Fig 4b.

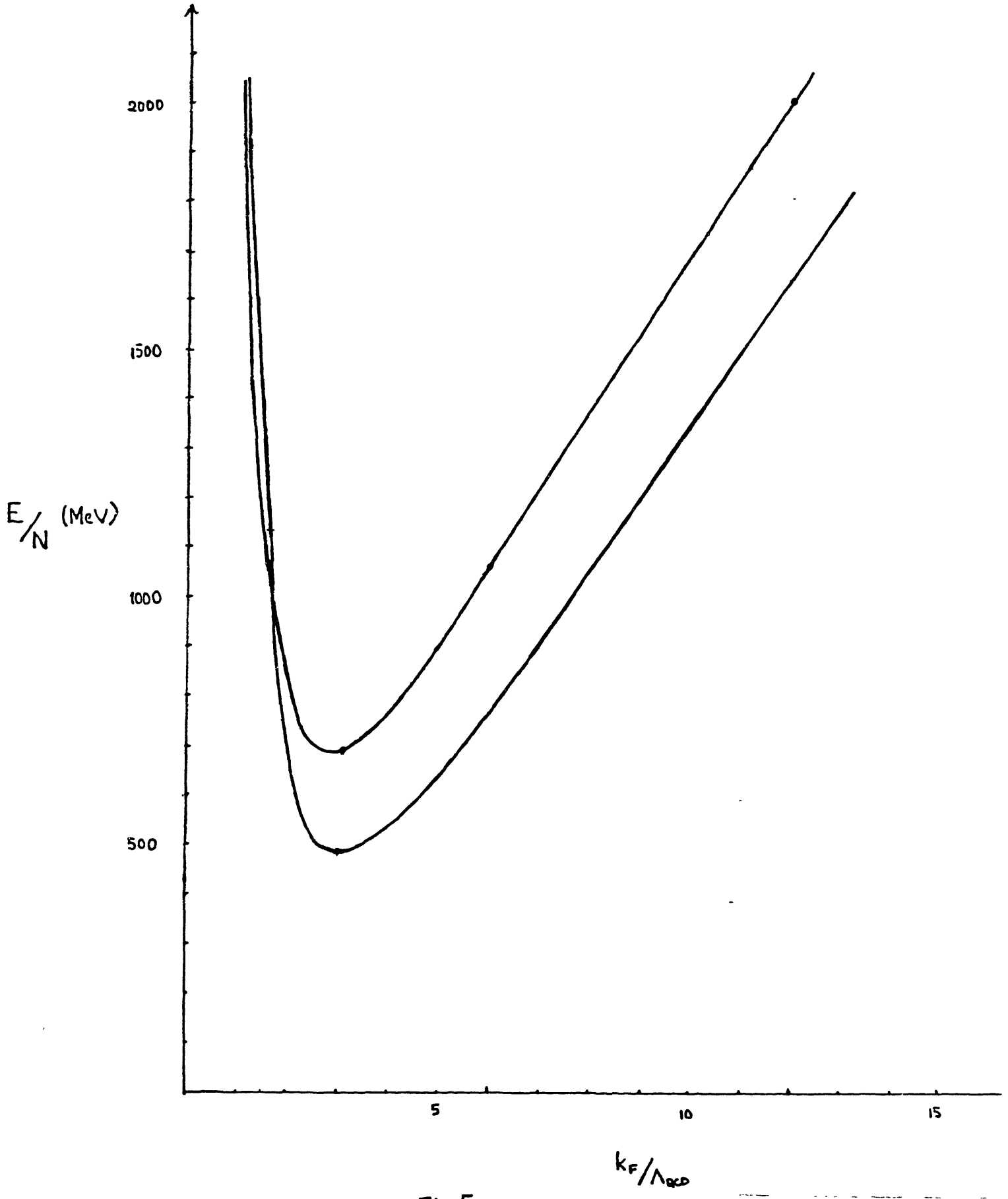

Fig. 5

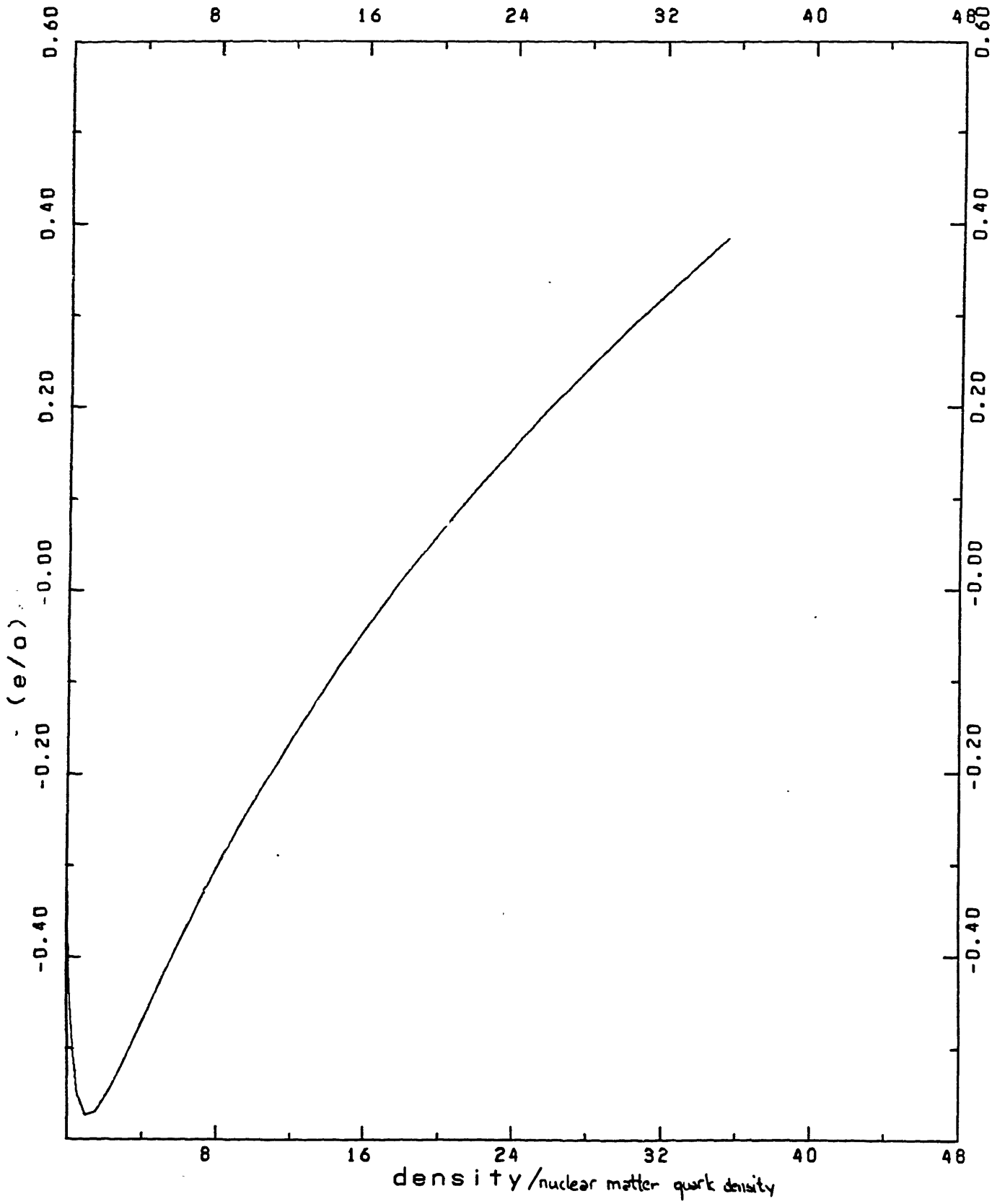
8
16
24
32
40
48
0.60
0.40
0.20
-0.00
-0.20
-0.40
- (e/a)
density/nuclear matter quark density

Fig.6

Biographical Note :

The author was born in Konya,Turkey, on September 14,1961.He was raised there until he left to attend the Ankara High School For Science from where he graduated in 1979.

In the fall of 1979, he began his undergraduate work at Bogazici University,in Istanbul,where he followed a superbly privileged program under the supervision of Prof.Alpar Sevgen.He spent the summer of 1981 at the Max Planck Institut Fur Kernphysik,in Heidelberg W.Germany.During this stay,which was generously made possible by Prof.Hans Weidenmueller,he started his first project in nuclear physics on investigating the microscopic justifications of the effective interactions used in mean field calculations.He graduated from Bogazici U. in June 1982.

He began graduate school at MIT in September 1982.During his first year,he worked on developing a model for describing dissipative collective nuclear motion.The following year was occupied mainly by a project on the statistical theory of nuclear reactions.Finally his last year was spent on developing a quark model for nuclear matter which is the subject of this thesis.

Through college and graduate school his time has been divided,apparently in comparable portions,between physics and sports with frequent leisurely activities in music.

He is planning to spend his time in the near future at the California Institute Of Technology.