Nonperturbative QCD Calculations

by

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Abstract

The research described in this thesis is an exact transformation of the Yang-Mills quantum chromodynamics (QCD) Lagrangian into a form that is suitable for nonperturbative calculations. The conventional Yang-Mills Lagrangian has proven to be an excellent basis for perturbative calculations, but in nonperturbative calculations it is difficult to separate gauge problems from physical properties. To mitigate this problem, I develop a new equivalent Lagrangian that is not only expressed completely in terms of the field strengths of the gauge field but is also manifestly Lorentz and gauge invariant.

The new Lagrangian is quadratic in derivatives, with non-linear local couplings, thus it is ideally suited for a numerical calculation.

The field-strength Lagrangian is of such a form that it is possible to do a straightforward numerical stationary path expansion and find the fundamental QCD properties. This thesis examines several approximations analytically, investigating different ways to utilize the new Lagrangian.

Advisor: Professor Arthur K. Kerman, Ph.D.

Acknowledgments

Throughout my long time at MIT, my advisor Arthur has been surprisingly willing to discuss everything from the outrageous to the mundane. Usually helpful, and always inspiring, he has taught me how to wrest nuggets of wisdom from the onslaught of complication confronted in field theory.

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My wife Beth has miraculously and happily tolerated my half-decade long delving into the nature of nature. Staunchly supporting me while I enjoy the pursuit.

Last but not least, Nature who has generated such a wonderful spectrum of phenomena, so simple and yet so complicated.

Dedication

To the pursuit of happiness

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The Yang-Mills Action in Terms of Field Strengths

Summary

Our objective in this thesis is to eliminate the gauge field completely from the Yang-Mills Lagrangian, and replace it with something that we can regard as the color electric and magnetic field strengths. The reasons for doing this are plentiful:

First, gauge invariance tends to make the properties of an expression opaque: All states used must be invariant under gauge transformations. The vector potential A has a complicated transformation rule under gauge transformations, in that it not only rotates, but also picks up a gradient, whereas the field strengths only rotate.

Second, semi-classical calculations require well-defined stationary paths that correspond to the actual physical field configurations, with smooth behavior around the stationary-path solutions. As we shall see, we can make our full quantum-mechanical field-strength transition amplitude functional integral be a smooth, well-behaved function that falls off in all directions around the stationary path.

The field-strength formulation problem was dealt with in Goldstone and Jackiw (1978) by writing the temporal gauge Hamiltonian in terms of E-field strengths, with the gauge field A being the conjugate momentum. The next step was recognizing that in su(2) the E-field can be written as a product of gauge-invariant variables and space and color rotation matrixes. Then the gauge invariance could be explicitly separated.

More similar to our approach is that of Halpern (1977) which was a transformation of the Euclidean Yang-Mills action into a field strength action, relying heavily on analyticity to define values to manifestly divergent integrals. Following Halpern, the Euclidean action has been transformed into a complex field strength action (e.g., Schaden et al. 1990) and analyzed.

Without resorting to Euclidean formalism we show that the Minkowski vector-potential action is equivalent to a real field-strength action. This makes it possible to utilize all the tools of non-perturbative calculations on the field-strength action, in particular we can identify the well behaved classical field-strength Hamiltonian that will yield the QCD-vacuum solutions.

We introduce effective field-strengths that are Gaussian averages around the exact field-strengths, thus including the local gauge-field phase space density in the field strength measure. Secondly, we transform our action completely into field strengths, and then consider gauge fixing. The second point is the one that generates most of the interesting results. Since in a field strength formulation gauge invariance is a completely local symmetry, not including any derivatives, the gauge fixing can in a natural way be done independently at each space-time point.

Without transforming into gauge-invariant variables à la Goldstone and Jackiw, it is then possible to write down the free-gauge classical Lagrangian, and a simple gauge condition for quantizing around that solution. Namely, the classical solution determines the gauge. It is worth noting that the simple gauge-fixing condition will come directly from imposing Gauss' law exactly.

The main result is the field strength transition amplitude that we generate, which can be written in a manifestly Lorentz invariant form as

$$W = \int D\sqrt{\mathrm{f}^{abc} F^c}\, \exp\left(\frac{i}{\hbar}\int d^3x dt \frac{1}{4}F^2 + \frac{1}{2g}(\partial\cdot F^a - g\mathbf{J}^a)(\mathrm{f}^{abc}F^c)^{-1}(\partial\cdot F^b - g\mathbf{J}^b)\right).$$

Where F is the conventional field strength tensor. This expression is also gauge invariant, as the gauge invariance condition is $\partial\cdot(\partial\cdot F^{a} + g\mathbf{J}^{a}) = 0$. In this expression the gauge has not been fixed.

The details of the calculation are more transparent if we separate out the time and space components of F explicitly. Also, to do practical calculations and to be able to extract the dynamics of the system it is necessary to work in a non-covariant form, where the action can be written as,

$$L = -\frac{1}{2}E^2 + \frac{1}{2}B^2 + \frac{1}{2}\mathbf{j}_i^a (\mathrm{f}^{abc}\varepsilon_{ijk}B_k^c)^{-1}\mathbf{j}_j^b - \frac{1}{2}(\mathbf{q}^a + \mathbf{j}_i^g Z_i^{ga})Y_j^{ab}(\mathrm{f}^{bcd}\varepsilon_{jkl}B_l^d)^{-1}Y_k^{ec}(\mathbf{q}^e + \mathbf{j}_i^f Z_i^{ef})$$

with

$$\mathbf{j}_i^a = \varepsilon_{ijk}\partial_j B_k^a - \dot{E}_i^a + \mathbf{J}_i^a,$$

$$\mathbf{q}^a = -\partial_i E_i^a + \mathbf{Q}^a,$$

$$Z_i^{ab} = E_j^d \mathrm{f}^{acd}(\mathrm{f}^{cbe}\varepsilon_{jik}B_k^e)^{-1} \text{ and}$$

$$Y_i^{ab} = E_i^d \mathrm{f}^{acd}(E_j^f \mathrm{f}^{cef}(\mathrm{f}^{egh}\varepsilon_{jkl}B_l^h)^{-1}E_k^i \mathrm{f}^{gbi})^{-1}.$$

We denote external currents and charges by $\mathbf{J}$ and $\mathbf{Q}$. We have here also defined a deficit current $\mathbf{j}$ and a deficit charge $\mathbf{q}$, such that Maxwell's equations correspond to $\mathbf{j} = 0$ and $\mathbf{q} = 0$. The only derivatives that occur in our Lagrangian are in these terms. There is a vacuum impedance Z and admittance Y, defined in terms of the local field strengths, that in a natural way relate charges and currents.

Only the electric field has a time derivative acting on it, so the magnetic field should be regarded as a constraint. All our kinetic terms are coupled trough the matrix $(\mathrm{f}^{abc}\varepsilon_{ijk}B_k^c)^{-1}$, this has the effect of acting as a centrifugal barrier forcing the eigenvalues of $\mathrm{f}^{abc}\varepsilon_{ijk}B_k^c$ away from zero. We thus acquire a magnetic field that is not only nonzero but also has a complicated internal structure.

Noticing that ZY is a projector, we see how the last term in the Lagrangian exactly cancels the quadratic contribution for one third of the $\dot{E}$-modes, making their time evolution completely determined by charge conservation.

In the Lorentz invariant form we saw that the dynamical variables were $\sqrt{\mathrm{f}^{abc} F^{c}}$, we find that using $\sqrt{\mathrm{f}^{abc} B^{c}}$ and E as our dynamical variables, we acquire a gauge constraint by demanding that the Jacobian determinant for the transformation should exactly cancel the Fadeev-Popov determinant.

The constraint is that the local color matrix $\sqrt{\left| E_i^c \mathrm{f}^{adc} (\mathrm{f}^{def} \varepsilon_{ijk} B_k^f)^{-1} E_j^g \mathrm{f}^{ebg} \right|}$ should map a colored field χ onto a diagonal representation. The important observations here are that a local rotation invariant object can be constructed—making the local character of gauge invariance manifest, and that we can derive a gauge fixed theory without ghosts.

Derivation of Field Strength Action

The Stationary Phase Approximation

Our whole discussion is based on the observation that the classical equations of motion are identical to the stationary path in the stationary phase approximation (SPA) . We therefore try to make the SPA as accurate as possible, thus making the classical limit meaningful. As we shall see, the same Lagrangian can be written in terms of different dynamical variables, all corresponding to the same physics, but with very different classical limits.

We will also see how the issue of scale generation in QCD is intimately connected with symmetry breaking and the balancing of the classical and quantum energy contributions. In fact we will demonstrate a general principle that symmetry breaking generates logarithmically large classical and quantum contributions to the energy, and that only their *sum* is finite. This is important to notice for any semi-classical calculations, as one would otherwise erroneously be led to work on the wrong scale for the fields.

Even though we are not actually using the SPA for transforming our Lagrangian, we need to become familiar with the properties of it, so as to be able to see which transformations are good.

The SPA is usually done as follows: In an integral involving an exponential, we find the stationary point(s) of the function in the exponent, and then make a power series expansion of the exponent around that point. If the expansion is good then the constant and quadratic terms yield a good estimate of the integral. For imaginary exponents we need to expand around all stationary points, whereas for real exponents we only expand around local maxima of the exponent.

Formally the expansion can be regarded as an asymptotic expansion in the following manner: Consider $\int dx \exp(\theta f(x))$ and expand for large θ, then the leading order term is

the quadratic SPA expansion. We can always choose $f(x)$ such that the point of interest is $\theta = 1$. The classical limit is often referred to as taking the limit of $\hbar$ going to zero, but even to the extent that such a limit is meaningful, the leading order quantum contributions persist.

For example:

The integral $\int_{-\infty}^{\infty} dx \exp(\theta(-\frac{x^4}{4} + \frac{x^2}{2}))$, *has maxima of the exponent of the integrand of magnitude* $\frac{\theta}{4}$ *at* $x = \pm 1$, *and second derivative of* $\theta \frac{\partial^2}{\partial x^2}(-\frac{x^4}{4} + \frac{x^2}{2}) = -3\theta x^2 + \theta$, *which evaluates to* -2θ *at the maxima, yielding the SPA estimate of the integral at* $\theta = 1$ *to be* $2\int_{-\infty}^{\infty} dy \exp(\frac{1}{4} - y^2) = 2\sqrt[4]{e\pi^2}$. *Compared to the exact value for this integral, this estimate is* 16% *too large.*

Now let's consider the somewhat trickier situation of the exponent containing a logarithm, such as:

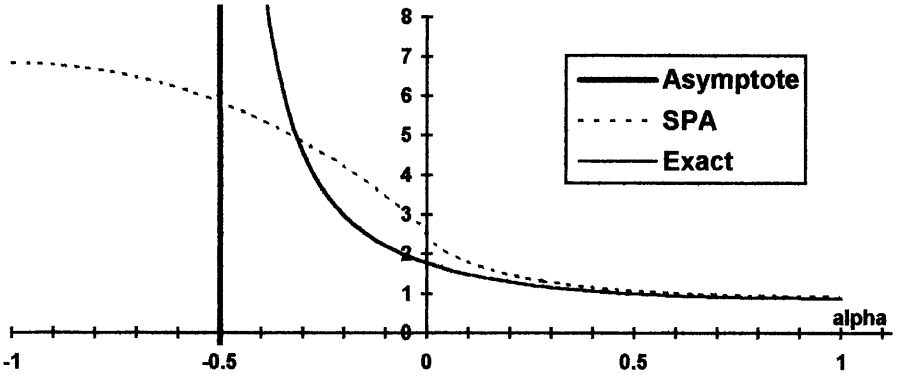
$\int_{-\infty}^{\infty} dx \exp(\alpha \log x^2 - x^2)$ *where we are interested in the regime of* α *near* $-\frac{1}{2}$. *The exact solution is* $\Gamma(\alpha + ½)$.

If we introduce an overall factor θ *in the exponent* $\int_{-\infty}^{\infty} dx \exp(\theta(\alpha \log x^2 - x^2))$ *and try to do the SPA in* θ, *then we get the stationary point condition*

$0 = \frac{\partial}{\partial x}(\alpha \log x^2 - x^2) = \frac{2\alpha}{x} - 2x$. *The stationary points are* $x = \pm\sqrt{\alpha}$, *imaginary for* $\alpha < 0$. *Leaving us with an SPA integral of*

$2\exp(\theta(\alpha \log \alpha - \alpha)) \int_{-\infty}^{\infty} dx \exp(-2\theta x^2) = 2\alpha^{\theta\alpha} e^{-\theta\alpha} \sqrt{\frac{\pi}{2\theta}}$, *which we evaluate at* $\theta = 1$ *to get* $\alpha^{\alpha} e^{-\alpha} \sqrt{2\pi}$.

In the graph below we compare the exact value of the integral with the absolute value of the SPA estimate.



The approximation is reasonably good for positive values for α, but completely fails to account for the singularity as α approaches -½.

We see here how we have to be a little bit careful when expanding our functions, especially if there is a term in the exponent that has a singularity, for it might dominate the integral but it neither has a stationary point nor is it possible to expand around the singularity.

When we look at a transition amplitude integral the easiest way to get an idea of the physics of the system is to consider the classical limit, which is the same as the stationary path, but only if the zero order approximation is good in the SPA does the classical limit have any physical meaning. Thus it is of fundamental importance to make the system such that the SPA is good.

Conditionally Convergent Integrals

In our calculations we deal with conditionally convergent integrals, that is

$\lim_{b\to\infty}\int_{-b}^{b} dx|f(x)| \to \infty$ but there exists a finite A such $\left|\int_{-b}^{b} dx f(x)\right| < A \;\forall\; b$. For such an integral to be defined it is not sufficient to specify the integration space, one must also specify an

integration-measure limit process. We define the value of the integral as the limit of the measured integral as the measure approaches one everywhere,

$$\int dx f(x) \underset{\text{def}}{=} \lim_{m(x) \to 1} \int dx f(x) m(x).$$

In particular we will choose $m(x) = e^{-\varepsilon \frac{x^2}{2}}$, with the regulated limit $\varepsilon \to 0^+$, so that $\int dx f(x)$ can be regarded as an analytical continuation of $\int dx f(x) m(x)$. If $\int dx f(x) m(x)$ is a Cauchy convergent function of ε in the limit $\varepsilon \to 0^+$, then this defines the value of the integral $\int dx f(x)$.

To be absolutely correct in our regulation of our functional integrals on $\mathbf{A}^a_\mu(x,t)$ we should define a positive definite matrix field $\varepsilon^{ab}_{\mu\nu}(x,t)$, so that we regulate each conditionally convergent integral independently at each point in space and in time. We will take $\varepsilon^{ab}_{\mu\nu}(x,t)$ to be block diagonal in space and time components, writing it as $\varepsilon^{ab}_{\mu\nu}(x,t) = \varepsilon^{ab}_{ij}(x,t) + \eta^{ab}_{00}(x,t)$. As the space-time dependence, and the space and color structures of ε and η are irrelevant in all points of our calculation, we will suppress all indication thereof throughout our calculations, and simply remember at each point in our calculations that ε and η are properly, sufficiently small positive-definite arbitrary functions.

The Transition Amplitude

The starting point for our discussion is the Yang-Mills transition amplitude: $W = N\int D[A] e^{i/\hbar \int dt d^3x L}$, where L is the standard Yang-Mills Lagrangian (Yang and Mills 1954):

$L = -\frac{1}{4} \mathrm{F}_{\mu\nu}\mathrm{F}_{\mu\nu}$, where $\mathrm{F}^a_{\mu\nu} = \frac{1}{g}\left[D_\mu, D_\nu\right] = \partial_\mu \mathbf{A}^a_\nu - \partial_\nu \mathbf{A}^a_\mu + g\mathrm{f}^{abc}\mathbf{A}^b_\mu \mathbf{A}^c_\nu$, and $D^{ab}_\mu = \delta^{ab}\partial_\mu + g\mathrm{f}^{abc}\mathbf{A}^c_\mu$.

The f^{abc}'s are of course the structure constants of su(n). We have chosen the **A**'s to be *i* times the $n^2 - 1$ members of the special unitary *n* by *n* matrix group, so that all expressions are manifestly real. In particular f^{abc} is real and antisymmetric, making $f^{abc}\mathbf{A}^c_\mu$ and $\delta^{ab}\partial_\mu$ anti-hermitian and real. To get the covariant derivative commutator we use:

$$\frac{1}{g}\left[D_\mu, D_\nu\right]^{ab} = \frac{1}{g}(\delta^{ac}\partial_\mu + gf^{acd}\mathbf{A}^d_\mu)(\delta^{cb}\partial_\nu + gf^{cbe}\mathbf{A}^e_\nu)$$
$$-\frac{1}{g}(\delta^{ac}\partial_\nu + gf^{acd}\mathbf{A}^d_\nu)(\delta^{cb}\partial_\mu + gf^{cbe}\mathbf{A}^e_\mu)$$
$$= \partial_\mu f^{abe}\mathbf{A}^e_\nu - \partial_\nu f^{abe}\mathbf{A}^e_\mu + gf^{acd}\mathbf{A}^d_\mu f^{cbe}\mathbf{A}^e_\nu - gf^{acd}\mathbf{A}^d_\nu f^{cbe}\mathbf{A}^e_\mu.$$

Noting the antisymmetry of the adjoint matrix, and using the closure property of the su(n) group, the matrix must be proportional to the structure constants, using the general relationships

$f^{acd}f^{bcd} = \kappa\delta^{ab}$ and $f^{ade}f^{bef}f^{cfd} = \frac{\kappa}{2}f^{abc}$ to write $F^c_{\mu\nu}f^{abc} = \frac{1}{g}\left[D_\mu, D_\nu\right]^{ab}$ thus,

$$F^c_{\mu\nu} = \frac{1}{g\kappa}f^{abc}\left[D_\mu, D_\nu\right]^{ab} = \partial_\mu\mathbf{A}^c_\nu - \partial_\nu\mathbf{A}^c_\mu + \frac{g}{2}(f^{cde}\mathbf{A}^d_\mu\mathbf{A}^e_\nu - f^{cde}\mathbf{A}^d_\nu\mathbf{A}^e_\mu)$$
$$= \partial_\mu\mathbf{A}^c_\nu - \partial_\nu\mathbf{A}^c_\mu + gf^{cde}\mathbf{A}^d_\mu\mathbf{A}^e_\nu.$$

We are working in a Minkowski space with a (1,-1,-1,-1) Lorentz metric.

We will drop infinite numerical constants in our functional integrals whenever this causes no confusion.

By making the substitution $g\mathbf{A} \to \mathbf{A}$ we can absorb all factors of *g* in the definitions of our variables, with an overall factor of $\frac{1}{\hbar g^2}$ multiplying the whole exponent. Just as we can work in units where $\hbar = 1$ and simply reinsert the factor of $\hbar$ at the end of the calculation, we can choose units where $\frac{1}{\hbar g^2} = 1$ and reinsert the factor at the end of the calculation.

There is nothing profound about this, we are simply avoiding having to pull along a constant factor in front of all our equations. No physical aspect is necessarily associated with this scale, yielding,

$$W = N\int D[A]e^{i\int dtd^3x\frac{-1}{4}(\partial_\mu \mathbf{A}^a_\nu - \partial_\nu \mathbf{A}^a_\mu + \mathrm{f}^{abc}\mathbf{A}^b_\mu \mathbf{A}^c_\nu)^2}.$$

To make it easier to interpret our results we also introduce external charged current and charge densities $\mathbf{J}^a_i$ and $\mathbf{Q}^a$.

Separating the spatial and temporal components of $F_{\mu\nu}$, we can write L as,

$$L = \frac{1}{2}(\partial_i \mathbf{A}^a_0 - \partial_0 \mathbf{A}^a_i + \mathrm{f}^{abc}\mathbf{A}^b_i \mathbf{A}^c_0)^2 - \frac{1}{4}(\partial_i \mathbf{A}^a_j - \partial_j \mathbf{A}^a_i + \mathrm{f}^{abc}\mathbf{A}^b_i \mathbf{A}^c_j)^2 + \mathbf{Q}^a \mathbf{A}^a_0 - \mathbf{J}^a_i \mathbf{A}^a_i .$$

Before we do any manipulations of the functional integral we will regulate the integrals in $\mathbf{A}$ to assure convergence of all integrals, by adding a quadratic term in $\mathbf{A}_i$ and one in $\mathbf{A}_0$. This regularization scheme ensures that our integration is over a uniform distribution of $\mathbf{A}$ space. This integration space is the same as that to which the Fadeev-Popov (1967) scheme makes gauge-fixed integrals equivalent. It is wise to remember that this is a choice, and it corresponds to regarding the full gauge-field vector potential as the dynamical variable that is being quantized. If we were to separate the gauge freedom from the dynamical degrees of freedom and then only consider dynamical excitations, we would in fact be considering a different dynamical system, which may or may not have the same physical content. At present there is no test of contradicting predictions for different gauges.

$$L = \frac{1}{2}(\partial_i \mathbf{A}^a_0 - \partial_0 \mathbf{A}^a_i + \mathrm{f}^{abc}\mathbf{A}^b_i \mathbf{A}^c_0)^2 - \frac{1}{2}(\partial_i \mathbf{A}^a_j - \partial_j \mathbf{A}^a_i + \mathrm{f}^{abc}\mathbf{A}^b_i \mathbf{A}^c_j)^2$$
$$+\mathbf{Q}^a \mathbf{A}^a_0 - \mathbf{J}^a_i \mathbf{A}^a_i + \frac{i\varepsilon}{2}\mathbf{A}^a_i \mathbf{A}^a_i + \frac{i\eta}{2}\mathbf{A}^a_0 \mathbf{A}^a_0$$

After we have performed all integrals in $\mathbf{A}$ we can take ε and η to zero independently, thus assuring that all integrals have been handled consistently.

Field Strengths

Introducing a new pair of color vector fields E_i^a and B_i^a, and using the identity

$\lim_{\varepsilon\to 0} N\int D[X]e^{i\frac{1}{2}XX+iXY-\frac{\varepsilon}{2}XX} = N'e^{-i\frac{1}{2}YY}$, we get:

$W = N\int D[E]D[B]D[A]e^{i\int dtd^3xL_{EB}}$, where

$$L_{EB} = \frac{-1+i\varepsilon_E}{2}E_i^aE_i^a + \frac{1+i\varepsilon_B}{2}B_i^aB_i^a + E_i^a(\partial_iA_0^a - \partial_0A_i^a + f^{abc}A_i^bA_0^c)$$
$$-B_i^a\tfrac{1}{2}\varepsilon_{ijk}(\partial_jA_k^a - \partial_kA_j^a + f^{abc}A_j^bA_k^c) + Q^aA_0^a - J_i^aA_i^a + \frac{i\varepsilon}{2}A_i^aA_i^a + \frac{i\eta}{2}A_0^aA_0^a.$$

Which we also can regard as multiplying the functional integral by the constant

$$\int D[E]e^{-\frac{i}{2}(E_i^a-\partial_iA_0^a+\partial_0A_i^a-f^{abc}A_i^bA_0^c)^2-\frac{\varepsilon_E}{2}E^2}$$
$$\cdot\int D[B]e^{\frac{i}{2}(B_k^a-\frac{1}{2}\varepsilon_{ijk}(\partial_iA_j^a-\partial_jA_i^a+f^{abc}A_i^bA_j^c))^2-\frac{\varepsilon_B}{2}B^2},$$

inside the **A** integrals. We now have four independent regulating parameters ensuring that all our integrals are well defined and convergent.

In this expression it is crucial to notice that we have introduced *E* and *B* fields that are the expectation values of the corresponding expressions in **A**, since the actual functions we multiplied by inside the double **A** and *E* integral were $e^{-\frac{i}{2}(E^a-\partial_iA_0^a+\partial_0A_i^a-f^{abc}A_i^bA_0^c)^2}$, which is a function of *E* and the particular combination of gauge fields that is usually defined as the color electric field. Furthermore, we notice that the particular function of *E* and **A** that we have chosen is a Gaussian distribution around the point where *E* equals the electric field, so *E* corresponds to an expectation value of the electric field. In the same way, *B* corresponds to an expectation value of the magnetic field.

Our goal now is to completely eliminate the **A** field from our expression. If this can be done we will have a Lagrangian in terms of *E* and *B* fields that is completely equivalent to

the conventional gauge-field Lagrangian. The big advantage of such a Lagrangian is that the E and B fields have physical meaning. A field configuration that is a pure gauge always corresponds to vanishing E and B fields, by virtue of their only rotating under gauge transformations. They vanish if the $\mathbf{A}$ field vanishes.

Integrability

Our Lagrangian is a quadratic expression in the spatial components of $\mathbf{A}$. To the extent that we can complete the square without encountering vanishing eigenvalues, can we perform the integrals on them.

As we are performing a functional integral we will have to be a little bit careful in our limiting processes. A well defined method of performing functional integrals is to place all our fields on a grid and then do a conventional multi-dimensional integral over all the field values at all points, after performing all the integrals we take the limit of vanishing grid spacing (Feynman 1948). To see if the integrals on $\mathbf{A}_i$ are well behaved, we need to consider only the quadratic terms in $\mathbf{A}_i$ in the limit of vanishing regulators, it is trivial to see that everything is well defined for finite regulators, but only if the integrals are well defined in the limit as we remove the regulators can we be sure that our results will be independent of how we remove the regulators. Thus we set $\varepsilon = 0$.

Conveniently, the coupling of pairs of $\mathbf{A}_i$ is diagonal in space and is represented by $-\frac{i}{2}\mathbf{A}_j^b(x)(B_i^a(x)\varepsilon_{ijk}\mathrm{f}^{abc})_{jk}^{bc}\delta(x-y)\mathbf{A}_k^c(y)$. This means that we can perform the integral independently at each point in space, and the integral is well behaved everywhere, except where $(B_i^a(x)\varepsilon_{ijk}\mathrm{f}^{abc})_{jk}^{bc}$ has a vanishing eigenvalue, which will occur for some particular values of B at any point x.

Noticing that $(B_i^a\varepsilon_{ijk}\mathrm{f}^{abc})_{jk}^{bc}$ is a matrix that has a zero eigenvalue only on a region without compact support in B, we can complete the square in $\mathbf{A}_i$. We stress that we are inte-

grating over B at each point x, and it is the *conventional* integral over B that has the property that there is a lower-dimensional sub-space, for which the integrand is not defined. By compactness and the definition of conventional integrals, we know that the value of an integral is independent of the value of the integrand on a subspace of the integration volume of measure zero.

It is a general property of integrals of a phase, that singularities have vanishing contributions. This follows from Cauchy-Schwarz inequality $\left|\int f\right| \leq \int |f|$. As $|f| = 1$ everywhere, the contribution of a singularity in the phase is bounded by the volume of the smallest subspace containing the singularity. So as long as $(B_i^a \varepsilon_{ijk} f^{abc})_{jk}^{bc}$ is invertible almost everywhere we can without effect restrict our integration space to exclude all singular points.

To complete our proof of integrability over $\mathbf{A}_i^a$ we need to show that the matrix in B does not have a vanishing eigenvalue on a compact region in B. This can easily be seen numerically by diagonalizing the matrix for some B value where all eigenvalues are well away from zero. By analyticity of the determinant of a finite-dimensional matrix, it is then invertible almost everywhere. For definiteness we note that the signature for the su(2) matrix $\sqrt{2}\delta_i^a \varepsilon_{ijk} f^{abc}$ is $(2,1^3,-1^5)$. It is easily verifiable for some su(3) matrix as well, though we will omit an explicit calculation here as it is non-informative.

Group Properties of $B_i^a \varepsilon_{ijk} f^{abc}$

We can see why the matrix $B_i^a \varepsilon_{ijk} f^{abc}$ lacks vanishing eigenvalues for small n and a non-degenerate B through some simple arguments. If the matrix has a zero eigenvalue, then it should be possible to find a nontrivial vector X_i^a that, when acted on by our matrix using a non-degenerate B, will be mapped to zero. The only way this can happen is if the B components orthogonal to X in space commute with X in color. Now choose a component X_0^a of X, and by non-degeneracy of B there will be two color vector components of B that should

commute with X_0^a. As we can choose any X, this reduces the problem to finding the smallest group su(n) such that for any pair of group members B_1^a and B_2^a, there is at least one other group member X^a that commutes with both.

Since we can find a coordinate system where B_1^a is diagonal, we will denote this component by B_z to indicate its diagonal status.

The second B component B_2^a can be separated into two parts; that which commutes with B_z, and that which does not commute with B_z. The commuting part can be diagonalized into a component B_Y orthogonal to B_z, and another, ignorable part proportional to B_z. Together B_Y and B_z define an su(3) subspace of our group. The non-commuting part of B_2^a is by genericity both inside and outside the su(3) subspace, ensuring that an su(4) subspace of X does not commute. For now let us assume that we are working in an su(4) or smaller space.

Completing the Square

Having demonstrated that the terms quadratic in $\mathbf{A}_i$ are non-zero almost everywhere, we can now complete the square in $\mathbf{A}_i$.

To make our equations more legible we will introduce the following notation: $(\partial\times B)_i = \varepsilon_{ijk}\partial_j B_k$, $(S\cdot B)_{ij} = \varepsilon_{ijk}B_k$, $(\Lambda\cdot B)^{ab} = \mathrm{f}^{abc}B^c$, $(S\cdot\Lambda\cdot B)^{ab}_{ij} = \varepsilon_{ijk}\mathrm{f}^{abc}B^c_k$ and $\tilde{E} = \mathrm{f}^{abc}E^c$, where S is simply a spin one matrix and Λ is the su(n) analog of a spin one matrix. This makes the Lagrangian look as follows:

$$
\begin{aligned}
L_{EB} &= \frac{-1+i\varepsilon_E}{2}E\cdot E + \frac{1+i\varepsilon_B}{2}B\cdot B - (\partial\cdot E^a - \mathbf{Q}^a)\mathbf{A}_0^a + \frac{i\eta}{2}\mathbf{A}_0^a\mathbf{A}_0^a \\
&-\tfrac{1}{2}[\mathbf{A}_i^a + (\partial\times B - \dot{E} + \mathbf{J} + \mathbf{A}_0\tilde{E})^c_k (S\cdot\Lambda\cdot B - i\varepsilon)^{-1\,ca}_{ki}]^a_i \\
&\cdot(S\cdot\Lambda\cdot B - i\varepsilon)^{ab}_{ij}[\mathbf{A}_j^b + (\partial\times B - \dot{E} + \mathbf{J} + \mathbf{A}_0\tilde{E})^c_k (S\cdot\Lambda\cdot B - i\varepsilon)^{-1\,cb}_{kj}]^b_j \\
&+\tfrac{1}{2}[\partial\times B - \dot{E} + \mathbf{J} + \mathbf{A}_0\tilde{E}]^a_i (S\cdot\Lambda\cdot B - i\varepsilon)^{-1\,ab}_{ij}[\partial\times B - \dot{E} + \mathbf{J} + \mathbf{A}_0\tilde{E}]^b_j,
\end{aligned}
$$

since $XMX + 2XY = (X + YM^{-1})M(X + YM^{-1}) - YM^{-1}Y$. Wherever there is any issue of order of contractions in products, an index indicating the proper contraction has been added.

Integration over $\mathbf{A_i}$

Shifting $\mathbf{A_i}$ by the appropriate amount in the completed square, we get:

$$L_{EB} = \frac{-1+i\varepsilon_E}{2} E \cdot E + \frac{1+i\varepsilon_B}{2} B \cdot B - (\partial \cdot E^a - \mathbf{Q}^a)\mathbf{A}_0^a + \frac{i\eta}{2}\mathbf{A}_0^a\mathbf{A}_0^a$$
$$-\tfrac{1}{2}\mathbf{A}(S \cdot \Lambda \cdot B - i\varepsilon)\mathbf{A}$$
$$+\tfrac{1}{2}[\partial \times B - \dot{E} + \mathbf{J} + \mathbf{A}_0\tilde{E}]_i^a (S \cdot \Lambda \cdot B - i\varepsilon)^{-1}{}_{ij}^{ab} [\partial \times B - \dot{E} + \mathbf{J} + \mathbf{A}_0\tilde{E}]_j^b .$$

It is worth noting that the matrix in between the $\mathbf{A_i}$'s is traceless, having both positive and negative eigenvalues, so only because we are integrating $\prod_{x,t} e^{-\frac{i}{2}\mathbf{A}(S\cdot\Lambda\cdot B)\mathbf{A}}$ with the all-important factor of i in the exponent is there any meaning to the integral. If we had been working with the partition function we would still have had the factor of i in front of the expression, as we would have been forced to shift the original E and B integrals by an imaginary rather than a real amount. In fact the whole calculation is identical for the partition function, apart from that the terms in E^2 and B^2 would have a pre-factor of $-\frac{1}{2}$ instead of an imaginary number and a regulator.

The integral over the $\mathbf{A_i}$'s generates the term $\underset{x,t}{\text{Det}} \frac{1}{\sqrt[4]{(S \cdot \Lambda \cdot B)^2 + \varepsilon^2}}$. As we take the determinant over all of space we get an amplitude that only depends on the absolute values of the $S \cdot \Lambda \cdot B$ eigenvalues. To see better how this term compares with the rest of our Lagrangian, we will write it as a term in the exponent.

Determinants over Space-Time

Let's digress for a moment and consider the meaning of a determinant over space-time. The determinant of a function is simply the product of all its eigenvalues. In particular, if the function is diagonal in space and time, it is simply the product over space-time of the local determinants. To be able to properly define this product we need to count the number of space-time points. The usual way to do this is to introduce a cutoff scale Λ, such that the unit space-time hypercube contains Λ^4 independent points.

The second relevant issue here is the dimension of our functional. A functional integral is only definable up to an overall constant factor, which follows from the fact that we integrate over an indeterminate number of variables, each of which carries a dimension. Thus, the overall factor must carry a factor of the inverse dimension for each integration point, just to keep the dimension of the integral independent of the number of points in our integral. To keep this independence manifest, we will choose to explicitly pull out from the norm a constant factor of the opposite dimension to that of the field integrated over. The magnitude of this factor is of course arbitrary, but it is useful for keeping track of dimensions. For simplicity we will use the cutoff scale rather than introducing an arbitrary scale.

More properly, the integral over A_i then yields: $\underset{x,t}{\mathrm{Det}} \frac{\Lambda}{\sqrt[4]{(S\cdot\Lambda\cdot B)^2+\varepsilon^2}}$.

Our definitions are now clear enough that we can use the old formula without further ado, $\mathrm{Det}(X) = e^{Tr(\log(X))}$, and write: $e^{-\int dtd^3x\frac{\Lambda^4}{4}Tr\log(((S\cdot\Lambda\cdot B)^2+\varepsilon^2)/\Lambda^4)}$.

Upon absorption of the constant term and re-exponentiation, this can be written as a new term in the Lagrangian: $\frac{i\Lambda^4}{4} Tr \ln(S\cdot\Lambda\cdot B/\Lambda^2)^2$. After completing the square in $\mathbf{A}_0$ the Lagrangian is now:

$$L_{EB} = \frac{-1+i\varepsilon_E}{2} E\cdot E + \frac{1+i\varepsilon_B}{2} B\cdot B + \frac{i\Lambda^4}{4} Tr\ln([(S\cdot\Lambda\cdot B)^2+\varepsilon^2]/\Lambda^4)$$
$$+\tfrac{1}{2}[\partial\times B-\dot{E}+\mathbf{J}]^a_i (S\cdot\Lambda\cdot B-i\varepsilon)^{-1\,ab}_{ij}[\partial\times B-\dot{E}+\mathbf{J}]^b_j$$
$$+\tfrac{1}{2}[\mathbf{A}^a_0+\{-\partial\cdot E^c+\mathbf{Q}^c+(\partial\times B-\dot{E}+\mathbf{J})^d_k (S\cdot\Lambda\cdot B-i\varepsilon)^{-1\,de}_{kl}\tilde{E}^{ec}_l\}M^{-1\,ca}]^a M^{ab}$$
$$\cdot[\mathbf{A}^b_0+\{-\partial\cdot E^c+\mathbf{Q}^c+(\partial\times B-\dot{E}+\mathbf{J})^d_k (S\cdot\Lambda\cdot B-i\varepsilon)^{-1\,de}_{kl}\tilde{E}^{ec}_l\}M^{-1\,cb}]^b$$
$$-\tfrac{1}{2}[-\partial\cdot E^a+\mathbf{Q}^a+(\partial\times B-\dot{E}+\mathbf{J})^d_k (S\cdot\Lambda\cdot B-i\varepsilon)^{-1\,de}_{kl}\tilde{E}^{ea}_l]M^{-1\,ab}$$
$$\cdot[-\partial\cdot E^b+\mathbf{Q}^b+(\partial\times B-\dot{E}+\mathbf{J})^d_k (S\cdot\Lambda\cdot B-i\varepsilon)^{-1\,de}_{kl}\tilde{E}^{eb}_l],$$

with $M^{ab} = -\tilde{E}^{ac}_i (S\cdot\Lambda\cdot B-i\varepsilon)^{-1\,cd}_{ij}\tilde{E}^{db}_j + i\eta$. For clarity we have added indices on some expressions. The minus sign in front of the definition for M is because $\tilde{E}$ is antisymmetric, so we need to transpose one of the $\tilde{E}$ to $-\tilde{E}$ to make the symmetry of M manifest. We see that the $\tilde{E}^{ab}_i$'s perform a rotation and rescaling of the $(S\cdot\Lambda\cdot B-i\varepsilon)^{-1}$ matrix into a smaller subspace. Thus the signs of the eigenvalues are not changed, and we only need to look at the imaginary part of $(S\cdot\Lambda\cdot B-i\varepsilon)^{-1}$ to verify the convergence of the $\mathbf{A}_0$ integrals with respect to the regulating parameter ε : $\mathrm{Im}(S\cdot\Lambda\cdot B-i\varepsilon)^{-1} = \varepsilon((S\cdot\Lambda\cdot B)^2+\varepsilon^2)^{-1} > 0$.

For the general ε of a positive symmetric matrix we get

$$\mathrm{Im}(S\cdot\Lambda\cdot B-i\varepsilon)^{-1} = [(S\cdot\Lambda\cdot B+i\varepsilon)(S\cdot\Lambda\cdot B+i\varepsilon)^*]^{-1}\varepsilon > 0.$$

Integration over $\mathbf{A}_0$

We note that as the ε convergence factor for $\mathbf{A}_i$ also aids in inducing convergence for the $\mathbf{A}_0$ integrals, there is no issue of order of limits for ε and η to assure convergence. The integration on $\mathbf{A}_i$ raised our dependence on $\mathbf{A}_0$ from linear to quadratic. As in the case of

$\mathbf{A}_i$, we need to verify the lack of vanishing eigenvalues in M. The matrix M can quite easily be shown to lack vanishing eigenvalues: Since $S \cdot \Lambda \cdot B$ is nonvanishing, we see that an eigenvector with a zero eigenvalue must commute with all three components of E, but we already showed that the requirement of commutation with two components brought us up to su(5). By the non-degeneracy argument for E, we do not need to worry about the possibility of an E whose different spatial components accidentally make two non-zero components in $S \cdot \Lambda \cdot B$ cancel.

Just as we earlier did for the integral over $\mathbf{A}_i$, we can shift $\mathbf{A}_0$ by the appropriate amount and perform the integral, yielding an effective Lagrangian in terms of the E and B fields:

$$
\begin{aligned}
L_{EB} &= \frac{-1+i\varepsilon_E}{2} E \cdot E + \frac{1+i\varepsilon_B}{2} B \cdot B \\
&+ \tfrac{i\Lambda^4}{4} Tr \ln([(S \cdot \Lambda \cdot B)^2 + \varepsilon^2] / \Lambda^4) + \tfrac{i\Lambda^4}{4} Tr \ln(M^* M / \Lambda^4) \\
&+ \tfrac{1}{2} [\partial \times B - \dot{E} + \mathbf{J}]^a_i (S \cdot \Lambda \cdot B - i\varepsilon)^{-1\,ab}_{ij} [\partial \times B - \dot{E} + \mathbf{J}]^b_j \\
&- \tfrac{1}{2} [-\partial \cdot E^a + \mathbf{Q}^a + (\partial \times B - \dot{E} + \mathbf{J})^d_k (S \cdot \Lambda \cdot B - i\varepsilon)^{-1\,dc}_{kl} \tilde{E}^{ca}_l] M^{-1\,ab} \\
&\cdot [-\partial \cdot E^b + \mathbf{Q}^b + (\partial \times B - \dot{E} + \mathbf{J})^d_k (S \cdot \Lambda \cdot B - i\varepsilon)^{-1\,dc}_{kl} \tilde{E}^{eb}_l]^b .
\end{aligned}
$$

Since we have now completed the integrals in $\mathbf{A}_0$ and $\mathbf{A}_i$ we should complete the removal of these variables by taking the limits $\varepsilon \to 0$ and $\nu \to 0$. Conveniently we already regulated the large E and B limits, and as we shall see, the integrals do not have any fundamental singularities. So we can simply set $\varepsilon = 0$ and $\nu = 0$.

It has been claimed that the potential in QCD should be of the form $B^2 \log B^2/\mu^2$ on dimensional grounds, but we see here how the expression in B is dimensionally just as reasonable. The argument inside the logarithm here is a matrix rather than a scalar, making the potential sensitive to the internal relative directions of B's components. So we see how dimensional arguments do not necessarily yield the right form.

Lagrangian with Mixed Signs in the Kinetic Term

Example:

The Lagrangian we acquired above does not have a definite sign for the kinetic term, so before we go on, let's digress for a moment and consider a very simple similar Lagrangian to familiarize ourselves with the properties thereof.

Let us consider the following Lagrangian, $L = \frac{\dot{x}^2 - \dot{y}^2}{4B} + \frac{B^2}{2} - V(x,y)$. *The conjugate momenta are* $\Pi_x = \frac{\dot{x}}{2B}$ *and* $\Pi_y = \frac{-\dot{y}}{2B}$. *The Hamiltonian is* $H = (\Pi_x^2 - \Pi_y^2)B - \frac{B^2}{2} + V(x,y)$. *Solving the constraint yields* $B = (\Pi_x^2 - \Pi_y^2)$, *which can be reinserted into the Hamiltonian, for a completely positive definite kinetic energy of* $\frac{1}{2}(\Pi_x^2 - \Pi_y^2)^2$. *The reason this works is that the constraint determines the sign of the total kinetic energy in such a way that it always remains positive.*

Finding the Natural Coordinate System in *B*

The logarithm of determinants of various matrices that show up in our Lagrangian make the stationary phase approximation a very poor method of evaluating our integral. We will therefore attempt to find the natural coordinate system for our functional integrals.

We will first try to find a change of coordinates in the B integral that will absorb the factor of $1/\sqrt{|\mathrm{Det}(S \cdot \Lambda \cdot B)|}$. It is possible to do this change independently at each point in space, as the factor we are trying to absorb is a completely local expression in B. We remind ourselves that in a functional integral we always first integrate over all the fields independently at each space-time point, and only after these integrals are completed do we take the limit of vanishing grid-spacing. So our discussion here is only concerning the conventional integral over B at some point x.

Noting that dimensionally we need a variable that is the square root of B, we will simply try to use the obvious candidate: $(S\cdot\Lambda\cdot t)^2 = S\cdot\Lambda\cdot B$, where we will of course need to take our path in t into the complex plane. Since $S\cdot\Lambda\cdot B$ is diagonalizable, we can choose t such that diagonalizing $S\cdot\Lambda\cdot B$ automatically diagonalizes $S\cdot\Lambda\cdot t$. We introduce yet another piece of notation, a star to denote a set of floating indices: $(S\cdot\Lambda\cdot *)^{ab}_{ij} = \varepsilon_{ijk} f^{abc}$.

We then get $(S\cdot\Lambda\cdot t)^2 = S\cdot\Lambda\cdot B$, which implies

$(S\cdot\Lambda\cdot dt)(S\cdot\Lambda\cdot t) + (S\cdot\Lambda\cdot t)(S\cdot\Lambda\cdot dt) = (S\cdot\Lambda\cdot dB)$.

Using: $Tr((S\cdot\Lambda\cdot *)(S\cdot\Lambda\cdot dB)) = dB$, $Tr((S\cdot A)(S\cdot B)(S\cdot C)) = \frac{1}{2}A(S\cdot C)B$ and

$Tr((\Lambda\cdot A)(\Lambda\cdot B)(\Lambda\cdot C)) = \frac{1}{2}A(\Lambda\cdot C)B$, we get:

$Tr((S\cdot\Lambda\cdot *)((S\cdot\Lambda\cdot dt)(S\cdot\Lambda\cdot t) + (S\cdot\Lambda\cdot t)(S\cdot\Lambda\cdot dt))) = dB$, or

$*(S\cdot\Lambda\cdot t)dt + dt(S\cdot\Lambda\cdot t)* = 4dB$, which is equivalent to

$(S\cdot\Lambda\cdot t)dt = 2dB$, thus $dt = \dfrac{2}{\sqrt{S\cdot\Lambda\cdot B}}dB$. The constant factor of 2 is of course completely irrelevant in a functional integral. This expression has a Jacobian that exactly matches the factor we were trying to absorb, apart from a phase that is i to the power of the number of negative eigenvalues in $S\cdot\Lambda\cdot B$. Which is the proper factor to make our integral real. For each factor of i, there is a corresponding component in t that is imaginary. Combining the two and formally writing $(i)dt = d\sqrt{B}$, we get:

$$\int \frac{D[E]D[\sqrt{B}]}{\sqrt{|\mathrm{Det}(M)|}}\, e^{i\int d^3x dt(-\frac{1}{2}E^2+\frac{1}{2}B^2+\frac{1}{2}[\partial\times B-\dot{E}+\mathbf{J}](S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}]}$$

$$-\tfrac{1}{2}[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}]M^{-1}$$

$$\cdot[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}],$$

$$M = -\tilde{E}(S\cdot\Lambda\cdot B)^{-1}\tilde{E}.$$

Removing the Determinant of M

Just as the factor $1/\sqrt{|\mathrm{Det}(S\cdot\Lambda\cdot B)|}$ above could be absorbed into the B integrals by virtue of being a local function of our fields, we note that the $1/\sqrt{|\mathrm{Det}(M)|}$ factor is only a function of the local E and B fields. Thus we are entitled to simply consider local integral transformations in the conventional E and B integrals at each point to see if there is a way of absorbing this factor.

Recall the definition of M as $M = -\tilde{E}(S\cdot\Lambda\cdot B)^{-1}\tilde{E}$, and note that M is a matrix in color only. The natural way to generate a determinant of a color matrix, is by converting an integral over color, but our integrals in E and B do not easily separate out an integral over color. Further, we know that all gauge transformations can be generated by a colored scalar field, $U(x)=e^{\Lambda\cdot\chi(x)}$. As we have not at any point violated gauge invariance it is clear that we have implicit in our integrals over E and B an integral over gauge transformations, further the action does not depend on this integral, so it is natural to consider an explicit integral over a colored scalar field χ^a.

We replace the factor $1/\sqrt{|\mathrm{Det}(M)|}$ by an integral over a new field χ^a:

$\int d\chi^a \delta(\sqrt{|M|}^{ab}\chi^b - U^a) = 1/2\sqrt{|\mathrm{Det}(M(x))|}$ at each point x, with U^a an arbitrary non-vanishing color field. In this expression we have made use of the $|M|$ notation to denote the positive square root of the square of M; all eigenvalues of M are replaced by their absolute value. Since the delta function is even we can choose any sign signature for $\sqrt{|M|}$ inside the delta function.

Hence the functional relation $N\int D[\chi^a]\delta(\sqrt{|M|}^{ab}\chi^b - U^a) = 1/\sqrt{|\mathrm{Det}(M)|}$.

This leaves us with a completely regular functional integral. This integral contains an arbitrary fixed field $U^a(x)$ and is an integral over $\chi^a(x)$, which we can choose to rotate under gauge transformations, as it is integrated over.

We have now removed the determinant-of-M dependence, but at the cost of adding a new dynamical field degree of freedom, as well as an arbitrary fixed field. The expression looks like a gauge fixing term, and we shall adjust it further to make its gauge properties manifest.

Integration Order in Functional Integrals

In the following we will need to consider a certain amount of changes in integration order as our objective is to explicitly factor out an integral over gauge transformations. Obviously a functional integral over a field X can never be factored out unless we interchange the limit of vanishing gridspacing and the integration over X at each point. Explicitly:

$\int D[X]D[Y]f[X,Y] = \lim_{n\to\infty} N_n \int \prod_{i\le n} dX_i dY_i f(X_0,..,X_n,Y_0,..,Y_n)$, interchange of limit and integration yields,

$$\lim_{n\to\infty} N_n \int \prod_{i\le n} dX_i \lim_{m\to\infty} N_m \int \prod_{j\le m} dY_j f(X_0,..,X_n,Y_0,..,Y_m) = \int D[X]\int D[Y]f[X,Y].$$

Such an interchange is wrought with hazards, but we only need a less general case:

If the functional f above is separable in X and Y then we are allowed to separate the integrals over X and Y thus, $\int D[X]D[Y]f[X]g[Y] = \int D[X]f[X]\int D[Y]g[Y]$.

Proof:

$$\int D[X]D[Y]f[X]g[Y] = \lim_{n\to\infty} N_n \int \prod_{i\le n} dX_i dY_i f(X_0,..,X_n)g(Y_0,..,Y_n)$$

$$= \lim_{n\to\infty} N_n \int dX_i f(X_0,..,X_n) N_n' \int dY_i g(Y_0,..,Y_n)$$

$$= \lim_{n\to\infty} N_n \int dX_i f(X_0,..,X_n) \lim_{m\to\infty} N_m \int dY_i g(Y_0,..,Y_n)$$

$$= \int D[X]f[X] \int D[Y]g[Y].$$

Where we only needed to assume that the right hand side was well defined, thus we are allowed to separate separable functional integrals, provided that our result is well defined. Specifically if neither of the separated integrals is well defined then it is still possible that their product integral is. Conversely, if either of the separated integrals is well defined, then the product of the integrals equals the integral of the product.

Removing the Integral on χ

As long as we have symmetries of the action, such that we can span the whole target space of $\sqrt{|M|}\chi$ inside the delta function by means of applying the symmetries, we will not need to perform any χ integrals after calculating the field-strength integrals. The argument is simple; if the symmetries apply to the action, $\sqrt{|M|}$, and χ, then it will also apply to $\sqrt{|M|}\chi$.

At this point it looks as though we have removed all ill-behaved functions from our integral, and separated out an integral over gauge transformations, however $\sqrt{|M|}\chi$ rotates under gauge transformations, thus it has n-1 gauge invariant components. An su(n) matrix can always be diagonalized by a suitable gauge transformation, but the gauge transformation will not affect the eigenvalues of the matrix, so there are n-1 invariants (number of eigen-

values, less one for tracelessness). Conversely there are n-1 gauge degrees of freedom left, as any su(n) gauge transformation matrix that is diagonal in the coordinate system where $\sqrt{|M|}\chi$ is diagonal will commute with the gauge-fixed field.

The same phenomenon exists in all adjoint-representation gauge-fixing schemes, only by using the fundamental representation can we fully fix the gauge in a straightforward way. Recall that our constraint was, $\delta(\sqrt{|M|}^{ab}\chi^b - U^a)$. From our discussion of gauge-fixing in the adjoint representation we see that this expression is really n-1 physical constraints, and $n^2 - n$ gauge constraints. These are all constraints on χ^a, but as we wish to regard the integral over χ^a as an integral over gauge transformations we will remove the constraints on physical degrees of freedom by choosing a suitable superposition of U^a's.

Choose an orthonormal set of n-1 commuting su(n) matrices $\hat{U}^a_\alpha$, for different points in space we may choose different sets. Consider the superposition $U^a = \int d^{n-1}u_\alpha\, u_\alpha \hat{U}^a_\alpha$, then we can perform the integrals on u_α as,

$$\int d\chi^a \int d^{n-1}u_\alpha \delta(\sqrt{|M|}^{ab}\chi^b - u_\alpha \hat{U}^a_\alpha)$$
$$= \int d\chi^a \delta^{\hat{U}}_\perp(\sqrt{|M|}^{ab}\chi^b)\frac{1}{\left|\hat{U}_1\right|\cdots\left|\hat{U}_{n-1}\right|} = \int d\chi^a \delta^{\hat{U}}_\perp(\sqrt{|M|}^{ab}\chi^b).$$

Where we have denoted the product delta function over the color space orthogonal to the $\hat{U}$ directions by $\delta^{\hat{U}}_\perp$. Our delta-function is now explicitly an adjoint-representation gauge-fixing integral, it fixes $\sqrt{|M|}\chi$ to be in the $\hat{U}$ basis-space and leaves the residual gauge freedom completely unperturbed. Our integrand is now separated into two completely disjoint components, the action which is gauge invariant by construction, and a delta function that only depends on gauge transformations.

Here the residual gauge freedom is manifestly obvious. As we will be expanding around a nontrivial field configuration in order to do the quantum mechanics, we will simply note that the residual gauge freedom generates a set of n-1 "0"-modes which we can explicitly

remove from our expansion. There is also an $n!$ component discrete global gauge symmetry, corresponding to all possible rearrangements of the eigenvalues of the n eigenvalues of the gauge fixed matrix. We see how a local gauge fixing is much more explicit than a non-local expression.

We recapitulate: Our integral over χ can be written as the integral over the signature of χ and all gauge-transformed versions thereof, but the integrand is independent of the signature and there is a delta function of the gauge-transformed object in the integrand, thus we can identify the integral over χ with an integral over gauge transformations.

This demonstrates that we are allowed to do the functional integrals over χ, after we have performed the functional integrals in E and B. But as the integral on χ only corresponds to a gauge transformation, we do not need to perform it, except so as to make gauge invariance manifest.

Fadeev-Popov Comparison

At this point we can identify our variables with the variables in the Fadeev-Popov (1967) construction, which told us to isolate an integral over gauge transformations, with a delta function inside it, and then recognizing that nothing in our transition amplitude depended on the gauge, we're entitled to drop the integral—evaluate it for some particular value of the gauge fixed parameter.

Fadeev and Popov's original construction was to state that the gauge fixed object should be chosen and fixed, such as $\partial \cdot A = 0$. In subsequent implementations this was modified by noting that exponential superpositions of different gauges can be useful, such as the Feynman gauges, $\int D[x] e^{ikx^2/2} \delta(\partial \cdot A - x) \Rightarrow L = L_0 + \frac{k}{2}(\partial \cdot A)^2$. Enabling us to add any function of the gauge fixed quantity to the Lagrangian, conversely we are entitled to add $k\chi M\chi$ to our Lagrangian, making χ act as a physical field.

Let us now write out the full transition amplitude integral, recognizing that the integral over χ is simply the gauge fixing component,

$$\int D[\chi]\int D[E]D[\sqrt{B}]\,\delta_{\perp}^{\hat{U}}(\sqrt{|M|}\chi)\,e^{i\int d^3x dt\; -\frac{1}{2}E^2+\frac{1}{2}B^2+\frac{1}{2}[\partial\times B-\dot{E}+\mathbf{J}](S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}]}$$
$$-\tfrac{1}{2}[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}]M^{-1}$$
$$\cdot[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}].$$

This expression looks superficially as though the expression $\delta_{\perp}^{\hat{U}}(\sqrt{|M|}\chi)$ inside the field-strength integral is both χ dependent and E and B dependent, but this is in fact not the case.

Suppose we integrated over E and B for some fixed χ, then the support of the integral is the subspace where $\sqrt{|M|}\chi$ is in the $\hat{U}_\alpha$ space. If we then were to consider a different χ, call it χ', $\sqrt{|M|}\chi'$ would then no longer be in $\hat{U}_\alpha$. But there is always a gauge transformation that will rotate $\sqrt{|M|}\chi'$ into $\hat{U}_\alpha$. All our integration measures are gauge invariant, and the integrand, apart from $\delta_{\perp}^{\hat{U}}(\sqrt{|M|}\chi)$ is also gauge invariant, thus gauge transforming our integrals such that $\sqrt{|M|}\chi'$ rotates into $\hat{U}_\alpha$ will not change the value of the integrals. So properly speaking $\delta_{\perp}^{\hat{U}}(\sqrt{|M|}\chi)$ depends on the gauge transformation part of the E and B integrals and the rest of the integrand depends only on the gauge invariant part of the fields. This proves that $\delta_{\perp}^{\hat{U}}(\sqrt{|M|}\chi)$ only picks out a particular gauge.

Gauge Invariance

We have been using the fact that our action is gauge invariant by construction, but as the derivatives used in the action are not covariant, this gauge invariance is not manifest. It is an essential property of the action to be gauge invariant, and it is a good verification of our calculations to verify the gauge invariance.

We recognize that there is a large amount of freedom in the way $\mathbf{J}$ transforms under a gauge transformation. Conventionally when working with the vector potential $\mathbf{J}$ is chosen to transform such that $D\cdot\mathbf{J}=0$, in order to preserve gauge invariance. It would be more

appealing to have a transformation rule for **J** that does not involve the gauge fields, as we wish to consider the reaction of the gauge fields from a given external current. We will therefore consider a **J** that only rotates under gauge transformations. Gauge invariance will then yield the additional constraint $\partial\cdot \mathbf{J}^a = 0$, which tells us that any charged fields must have a Lagrangian that conserves color charge. Further for a fixed external charge we see that there is an intrinsic frame corresponding to $\partial\cdot \mathbf{J}^a = 0$, as this condition is not gauge invariant, and in fact an external charge $\mathbf{J}^a$ that can not be gauge transformed into satisfying the constraint is disallowed.

Let's consider an infinitesimal gauge transformation $\Lambda\cdot\omega$ on the action

$$\int d^3x dt(-\tfrac{1}{2}E^2+\tfrac{1}{2}B^2+\tfrac{1}{2}[\partial\times B-\dot{E}+\mathbf{J}](S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}]$$
$$-\tfrac{1}{2}[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}]M^{-1}$$
$$\cdot[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}].$$

We only need to consider terms linear in ω, and as all color indices are summed over, only terms that contain derivatives of ω need to be considered, all other components already are manifestly invariant. Using,
$\partial\cdot E\to\partial_i\omega\tilde{E}_i$, $\dot{E}\to\dot{\omega}\tilde{E}$ and $\partial\times B\to\partial\omega(S\cdot\Lambda\cdot B)$ we get,

$$\delta_\omega=\int d^3x dt\,[\partial\omega(S\cdot\Lambda\cdot B)-\dot{\omega}\tilde{E}](S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}]$$
$$-[-\partial_i\omega\tilde{E}_i+(\partial\omega(S\cdot\Lambda\cdot B)-\dot{\omega}\tilde{E})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}]M^{-1}$$
$$\cdot[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}].$$

Collecting terms in $\dot{\omega}$ we get,

$$\delta_{\dot{\omega}}=\int d^3x dt-\dot{\omega}\tilde{E}(S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}]$$
$$-\dot{\omega}MM^{-1}[-\partial\cdot E+\mathbf{Q}]+\dot{\omega}MM^{-1}\tilde{E}(S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}],$$

using the antisymmetry of $\tilde{E}$, $M=-\tilde{E}(S\cdot\Lambda\cdot B)^{-1}\tilde{E}$ and the invertibility of M. Canceling terms, we are left with, $\delta_{\dot{\omega}}=\int d^3x dt\dot{\omega}(\partial\cdot E-\mathbf{Q})$.

Collecting terms in $\partial\omega$ we get,

$$\delta_{\partial\omega} = \int d^3x dt \partial\omega(\partial\times \boldsymbol{B} - \dot{\boldsymbol{E}} + \mathbf{J}).$$

Adding the two contributions, and doing extensive integrations by parts,

$$\delta_{\omega} = \int d^3x dt\, \dot{\omega}\partial\cdot \boldsymbol{E} - \partial\omega\dot{\boldsymbol{E}} + \partial\omega\partial\times \boldsymbol{B} - \dot{\omega}\mathbf{Q} + \partial\omega\mathbf{J}$$
$$= \int d^3x dt\, \dot{\omega}\partial\cdot \boldsymbol{E} - \dot{\omega}\partial\cdot \boldsymbol{E} - \omega\partial\cdot \partial\times \boldsymbol{B} + \omega(\dot{\mathbf{Q}} - \partial\cdot \mathbf{J})$$
$$= \int d^3x dt\omega(\dot{\mathbf{Q}} - \partial\cdot \mathbf{J}).$$

This proves that our action is gauge invariant, as long as all external currents are conserved. Incidentally, the integrations by parts necessary in order to get this result, leave us with the following contribution the surface integrals,

$$\delta_{\omega} = \int d^3x \left.\right|_{t=-\infty}^{t=+\infty} (-E\partial\omega - \omega\mathbf{Q}) + \oint d^2s \int dt(\dot{\omega}E - \partial\omega\times \boldsymbol{B} + \omega\mathbf{J}).$$

As our action is gauge invariant, we have verified our calculations up to this point. Incidentally we have found that even though we integrated over all vector-potential configurations—even those corresponding to different topological charges, the possibility of instantons in this formulation has not been excluded nor confirmed, if the surface integrals had vanished, then there would have been no possibility for topological charge. Even though we here only considered infinitesimal gauge transformations, we can get the large gauge transformation contributions by splitting a large gauge transformation into pieces and integrating.

We could at this point argue that as the motivation for the original Yang-Mills Lagrangian was that it was gauge invariant and of dimension four, then our field-strength Lagrangian would in fact have been an equally good choice. Our Lagrangian is also gauge invariant and of dimension four, but does not contain any unphysical degrees of freedom. Curiously we see that any local expression in the field strengths can be added to the Lagrangian without spoiling our invariance.

Gauge Fixing

The beauty of the way that the gauge is fixed is that for the classical Lagrangian there is no gauge fixing whatsoever. The gauge-fixed expression contains a variable that is not in the action, so from a classical point of view no gauge fixing is necessary. Without any assumptions or approximations we have proven, at least for some choice of dynamical variables, that gauge fixing is necessary for the quantum theory, but not for the classical formulation. As the integral over χ is just a summation over different gauges of the same expression we get the same integrals for E and B if we gauge rotate their integrals, such that $\sqrt{|M|}\chi$ is just gauge rotated.

It is well worth the effort to dwell a bit on the meaning of the expression

$$\int D[\chi]\int D[E]D[\sqrt{B}]\,\delta_{\perp}^{\hat{U}}(\sqrt{|M|}\chi)\,e^{i\int d^3xdt\ -\frac{1}{2}E^2+\frac{1}{2}B^2+\frac{1}{2}[\partial\times B-\dot{E}+\mathbf{J}](S\cdot\Lambda\cdot B)^{-1}[\partial\times B-\dot{E}+\mathbf{J}]}$$
$$-\tfrac{1}{2}[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}]M^{-1}$$
$$\cdot[-\partial\cdot E+\mathbf{Q}+(\partial\times B-\dot{E}+\mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}].$$

The Lagrangian we get must preserve all the symmetries of the original problem, since all that we are doing is a change of variables. Thus the complete gauge independence must be preserved. The straightforward way of doing calculations from a Lagrangian is to find the classical solutions and then consider small fluctuations around those solutions. If our Lagrangian has a symmetry, then the fluctuations in the directions along that symmetry will be arbitrarily large, which was the problem encountered by Feynman (1962). The solution to this problem, as formalized by Fadeev and Popov, was to explicitly factor out the gauge symmetry of the problem, which in that case forced them to introduce a determinant of a matrix as they separated out the gauge transformations.

In our case, as we integrated over $\mathbf{A}_0$, which corresponds to solving Gauss' law, we got a determinant as a result of our integral. This determinant told us that there is a unique choice of gauge separation for which this determinant will be completely absorbed.

It is interesting to note that the reason the gauge fixing determinant popped out was that we solved Gauss' law exactly.

Comparing this with the situation when we work directly with the gauge-potential fields, where we either manifestly violate covariance of our system or are forced to introduce "ghost" fields, with the exact choice completely up to the user, we see how the field strength formulation much more explicitly exposes the underlying structure. In the field strength formulation we get a particular form of gauge separation thrust upon us with no real choice available about how to do it.

In the Fadeev-Popov scheme we need to find an object that transforms under gauge transformations in such a way that fixing the object determines the gauge, without constraining the physical configuration.

Here we can read off the object that we get $\delta_{\perp}^{\hat{v}}(\sqrt{\left|\tilde{E}(S\cdot\Lambda\cdot B)^{-1}\tilde{E}\right|}\chi)$, and look at its properties . It is a local object, which is what one would expect from dealing with fields that rotate under gauge transformations. And it is rotationally invariant, but not manifestly Lorentz invariant, which is to be expected, considering that we separated time and space components into E and B respectively. It is also worth noting that the form of the object is such that there is no possible regime where the kinetic term in the action decouples from the gauge fixing expression, since their functional dependence is virtually identical.

Solution Method

Above we separated out the integral over gauge transformations and were left with functional integrals over the field strengths, and two arbitrary fixed constant fields $\hat{U}_\alpha$ and χ. We can now define a method for calculating the stationary path expansion:

First we find the stationary path in E and B, which corresponds to solving the classical equations of motion.

Second we choose $\hat{U}_\alpha$ and the signature of $\sqrt{|M|}$ arbitrarily, which determines χ from our E and B solution.

Third we expand in E and $\sqrt{S \cdot \Lambda \cdot B}$ to second order around the path found above, as the action is stationary in B there will not be a problem expanding in the square root thereof.

Fourth we constrain our expansion to the subspace where $\sqrt{|M|}\chi$ is in the space spanned by $\hat{U}_\alpha$, this satisfies the gauge-fixing constraint. We further constrain our expansion to be orthogonal to any gauge transformation that is generated by $\hat{U}_\alpha$, this constraint corresponds to removing the residual gauge freedom that our gauge fixing determinant did not handle. We do not need to remove these modes, so much as to recognize their existence, as otherwise numerical errors would tend to drive their eigenvalues negative, generating fictitious instabilities. This means that all modes corresponding to gauge transformations have been removed. We can do this complete gauge fixing because the residual gauge-freedom modes are easily identifiable in a system where all gauge dependence is local.

As we solve for the classical equations of motion it is clear that we are dealing with a highly degenerate theory, but all the "zero"- modes corresponding to gauge invariance and (the presumably broken) translation invariance are easily identifiable, thus we will only need to do an expansion in the space of physical modes.

Properties of the Classical Lagrangian

In the following we will refer to the stationary phase solutions as the solutions to the classical Lagrangian, not because they are solutions of the classical Yang-Mills equations, but because they are the leading-order, average field configurations for the full quantum mechanical problem.

Now we are in the situation of having a smooth function in the exponent and no factors outside the exponential, so we would expect the stationary phase approximation to be good, the stationary phase paths being the classical equations of motion.

The constraint in χ implies that our Euler-Lagrange equations are not homogeneous. We can, if we so desire, choose to solve the constrained system, but it is usually easier to ignore constraints. The unconstrained system is just the special case of vanishing Lagrange multipliers. After finding the classical solution, and finding the χ gauge that the solution corresponds to, we will want to calculate the perturbation expansion around this solution, so as to get the full quantum mechanical transition amplitude. In the expansion around the classical solution we have to remember to impose the χ constraint on $\boldsymbol{M}$ as well as using $\sqrt{S\cdot\Lambda\cdot B}$ as the dynamical variable, instead of just $\boldsymbol{B}$.

Now we are ready to deal with the classical Lagrangian, which is:

$$
\begin{aligned}
&L_{cl} = \frac{-1}{2}E^2 + \frac{1}{2}B^2 + \tfrac{1}{2}[\partial\times B - \dot{E} + \mathbf{J}](S\cdot\Lambda\cdot B)^{-1}[\partial\times B - \dot{E} + \mathbf{J}] \\
&-\tfrac{1}{2}[-\partial\cdot E + \mathbf{Q} + (\partial\times B - \dot{E} + \mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}] \\
&\cdot M^{-1}[-\partial\cdot E + \mathbf{Q} + (\partial\times B - \dot{E} + \mathbf{J})(S\cdot\Lambda\cdot B)^{-1}\tilde{E}], \\
&M = -\tilde{E}(S\cdot\Lambda\cdot B)^{-1}\tilde{E}.
\end{aligned}
$$

Where we have set $\varepsilon_E = \varepsilon_B = 0$ as they are only part of the definition of the transition amplitude integrals and not part of the Lagrangian.

Notation

To analyze this it will be convenient to define some shorthand notation, namely, the matrix form of the magnetic field $\vec{\vec{B}} = S \cdot \Lambda \cdot B$, the impedance $Z = \tilde{E} \cdot \vec{\vec{B}}^{-1}$, and the admittance $Y = \tilde{E}(\tilde{E} \cdot \vec{\vec{B}}^{-1} \cdot \tilde{E})^{-1}$. These definitions will make our equations very transparent, in particular the magnetic field tends to occur in the matrix form $S \cdot \Lambda \cdot B$ throughout our equations. The impedance and admittance are somewhat peculiar objects, in that they are vectors in space, but matrixes in color, just as our previously defined $\tilde{E}$. All our equations turn out to be a color vector on each end, with a set of color matrices in between, thus an implied contraction over color indices does not lead to any ambiguities. The spatial indices are substantially more irregular so we will use a $\cdot$ (dot) product to indicate contraction between the spatial indices.

Looking at this Lagrangian we see how the external current and charge densities are always incorporated in the same way, thus it seems reasonable to define induced current and charge densities as, $\mathbf{j} = \partial \times B - \dot{E} + \mathbf{J}$, and $\mathbf{q} = -\partial \cdot E + \mathbf{Q}$. We call these the induced current and charge, as they are identical to the nonabelian terms in the charge and current conservation equations. We see how setting $\mathbf{j}$ and $\mathbf{q}$ to zero yields the conservation laws of Maxwell's equations.

Results

The Lagrangian can in our new variables be written as,

$$L_{cl} = \frac{-1 + i\varepsilon_E}{2} E^2 + \frac{1 + i\varepsilon_B}{2} B^2 + \frac{1}{2} \mathbf{j} \cdot (\vec{\vec{B}}^{-1} - \vec{\vec{B}}^{-1} \cdot YZ) \cdot \mathbf{j} - \frac{1}{2} \mathbf{q} Y \cdot \vec{\vec{B}}^{-1} \cdot Y\mathbf{q} - \mathbf{q} Y \cdot \vec{\vec{B}}^{-1} \cdot \mathbf{j}.$$

We need to consider the meanings of the admittance and impedance tensors as defined in this expression. They are the connections between charge and current, unlike electromagnetism where the connection is between the gradient of the charge and the current. Further the nonabelian nature of QCD makes them be functions of the local E and B fields.

The charge is a spatial scalar whereas the current is a vector, thus the admittance and impedance have one spatial and two color indices. The product of Y and Z can be constructed in two ways: as a color matrix $Y \cdot Z$—in which case it is equal to the identity, or as a color and space matrix YZ—in which case it is trivial to see that it acts as a projector $YZ = Y(Z \cdot Y)Z = (YZ) \cdot (YZ) = (YZ)^2$.

We will need a few more properties of these variables, as follows:

$$ZY \cdot \vec{B}^{-1} = [\vec{B}^{-1} \cdot \tilde{E}][(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}\tilde{E}] \cdot [\vec{B}^{-1}] = [\vec{B}^{-1}] \cdot [\tilde{E}(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}][\tilde{E} \cdot \vec{B}^{-1}] = \vec{B}^{-1} \cdot YZ,$$

$$\begin{aligned} ZY \cdot \vec{B}^{-1} \cdot YZ &= [\vec{B}^{-1} \cdot \tilde{E}][(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}\tilde{E}] \cdot [\vec{B}^{-1}] \cdot [\tilde{E}(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}][\tilde{E} \cdot \vec{B}^{-1}] \\ &= [\vec{B}^{-1} \cdot \tilde{E}][(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E}][(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}\tilde{E}] \cdot [\vec{B}^{-1}] = ZY \cdot \vec{B}^{-1} \end{aligned}$$

The interpretation of our Lagrangian is now obvious. The currents couple through a factor $\vec{B}^{-1}$, and charges are first converted to their corresponding currents by means of the impedance and then couple as currents. An alternate way of writing the Lagrangian is:

$$L_{cl} = \frac{-1+i\varepsilon}{2}E^2 + \frac{1+i\varepsilon}{2}B^2 + \frac{1}{2}\mathbf{j} \cdot \vec{B}^{-1} \cdot \mathbf{j} - \frac{1}{2}(\mathbf{q} + \mathbf{j} \cdot Z)Y \cdot \vec{B}^{-1} \cdot Y(\mathbf{q} + Z \cdot \mathbf{j}).$$

As this is the final result of the reformulation of the transition amplitude in terms of field strengths, we will write out the whole expression here for reference:

$$\mathbf{W}_{\text{Yang-Mills}} = N\int D[E]D[\sqrt{B}]D[\chi]\delta(\sqrt{|M|}\chi - U)$$
$$\cdot \; e^{\frac{i}{\hbar g^2}\int dxdt\left(\frac{-1+i\varepsilon}{2}E^2 + \frac{1+i\varepsilon}{2}B^2 + \frac{1}{2}\mathbf{j} \cdot \vec{B}^{-1} \cdot \mathbf{j} - \frac{1}{2}(\mathbf{q} + \mathbf{j} \cdot Z)Y \cdot \vec{B}^{-1} \cdot Y(\mathbf{q} + \mathbf{j} \cdot Z)\right)}$$

where

$M = -\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E}$, $\mathbf{j} = \partial \times B - \dot{E} + \mathbf{J}$, $\mathbf{q} = -\partial \cdot E + \mathbf{Q}$, $Y = \tilde{E}(\tilde{E} \cdot \vec{B}^{-1} \cdot \tilde{E})^{-1}$ and $Z = \tilde{E} \cdot \vec{B}^{-1}$.

In this form it is perhaps even clearer what happens to the charge-current interaction. The $\mathbf{j} \cdot Z$ term is a projection of the current into a charge. The projected current is then added to the charge to yield a new total induced charge.

We can say a bit more on the nature of the current projection, by noting that the charge is converted to a current through the admittance, and then couples to itself in the same way as the current couples to itself. The current is projected to a charge through the impedance, thus the part of the current that couples to the charge is simply that part that could have been generated by a charge.

Lorentz Invariance

Our Lagrangian is not in a form that makes the Lorentz invariance manifest, so we will now try to re-express it in terms of the *F*-tensor.

We note the identity $-\frac{1}{2}E^2 + \frac{1}{2}B^2 = \frac{1}{4}F^2$ for the first two terms in our Lagrangian. The induced current and charge can be written as $\mathbf{j}_i^a = -\partial_\mu F_{\mu i}^a + \mathbf{J}_i^a$ and $\mathbf{q}^a = -\partial_\mu F_{\mu 0}^a + \mathbf{Q}^a$, so that we can define an induced four-current $\mathbf{j}_\mu^a = -\partial_\nu F_{\nu\mu}^a + \mathbf{J}_\mu^a$. Using a (0,i) matrix notation, we can write our Lagrangian as,

$$L = \frac{1}{4}F^2 + \frac{1}{2}\mathbf{j}_\mu^a \begin{pmatrix} -Y \cdot \vec{B}^{-1} \cdot Y & Y \cdot \vec{B}^{-1} \\ \vec{B}^{-1} \cdot Y & \vec{B}^{-1} - ZY\vec{B}^{-1} \end{pmatrix} \mathbf{j}_\nu^b.$$

Now we claim that the $\mathbf{j} \bullet \mathbf{j}$ matrix that occurs in this Lagrangian is simply the matrix $(F^{ab}_{\mu\nu})^{-1}$. To prove our contention we multiply our matrix by $F^{ab}_{\mu\nu}$,

$$\begin{pmatrix} -Y\cdot\vec{B}^{-1}\cdot Y & Y\cdot\vec{B}^{-1} \\ \vec{B}^{-1}\cdot Y & \vec{B}^{-1} - ZY\cdot\vec{B}^{-1} \end{pmatrix}\begin{pmatrix} 0 & \tilde{E} \\ \tilde{E} & \vec{B} \end{pmatrix}$$
$$= \begin{pmatrix} Y\cdot\vec{B}^{-1}\cdot\tilde{E} & -Y\cdot\vec{B}^{-1}\cdot Y\tilde{E} + Y\cdot\vec{B}^{-1}\cdot\vec{B} \\ \vec{B}^{-1}\tilde{E} - ZY\cdot\vec{B}^{-1}\cdot\tilde{E} & \vec{B}^{-1}\cdot Y\tilde{E} + \vec{B}^{-1}\vec{B} - ZY\cdot\vec{B}^{-1}\cdot\vec{B} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Where we have made repeated use of the definitions of $Y = \tilde{E}(\tilde{E}\cdot\vec{B}^{-1}\cdot\tilde{E})^{-1}$ and $Z = \tilde{E}\cdot\vec{B}^{-1}$. In this calculation we also note the vagueness with respect to the sign of expressions in our notation.

Our Lagrangian can now be written in the manifestly Lorentz-invariant form

$$L = \frac{1}{4}F^2 + \frac{1}{2}(\partial_\mu F^a_{\mu\nu} - \mathbf{J}^a_\nu)F^{-1\,ab}_{\nu\alpha}(\partial_\beta F^b_{\beta\alpha} - \mathbf{J}^b_\alpha).$$

This expression is, of course, gauge invariant as is easily seen using $F^{ab}_{\mu\nu} = \Lambda^{ab}_c \cdot F^c_{\mu\nu}$ and $F^a_{\mu\nu} = -F^a_{\nu\mu}$. We have,

$\delta L = \partial_\mu\omega^c\Lambda^{da}_c F^d_{\mu\nu}F^{-1\,ab}_{\nu\alpha}(\partial_\beta F^b_{\beta\alpha} - \mathbf{J}^b_\alpha) = \partial_\alpha\omega^a(\partial_\beta F^a_{\beta\alpha} - \mathbf{J}^a_\alpha)$, integrated by parts to

$\delta L = -\omega^a(\partial_\alpha\partial_\beta F^a_{\beta\alpha} - \partial\cdot\mathbf{J}^a) = \omega^a\partial\cdot\mathbf{J}^a$.

Let us consider the determinants that we earlier absorbed in a change of integration variables in B and a gauge fixing integral. We note that the determinants $\mathrm{Det}(-\tilde{E}(S\cdot\Lambda\cdot B)^{-1}\tilde{E})$ and $\mathrm{Det}(S\cdot\Lambda\cdot B)$ together are over the same number of degrees of freedom as $\mathrm{Det}(\Lambda\cdot F)$. In matrix notation we write

$$F = \begin{pmatrix} 0 & \tilde{E} \\ \tilde{E} & \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{E} \\ \tilde{E} & \vec{B} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \vec{B}^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{E}\vec{B}^{-1} \\ \tilde{E} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \vec{B} \end{pmatrix}.$$ By the chain rule of de-

terminants we then have $\mathrm{Det}(F) = \mathrm{Det}(-\tilde{E}\ddot{B}^{-1}\tilde{E})\mathrm{Det}(\ddot{B})$. Consequentially we can absorb all the determinants by the variable substitution $\frac{dF}{Det(2\sqrt{F})} = d\sqrt{F}$.

Finally we can write the manifestly gauge and Lorentz invariant transition amplitude as:

$$W = \int D\sqrt{F}\exp(\frac{i}{\hbar}\int d^3xdt\frac{1}{4}F^2 + \frac{1}{2g}(\partial\cdot F - g\mathbf{J})F^{-1}(\partial\cdot F - g\mathbf{J}))$$

We need to comment a bit on this expression:

Fundamentally we expect the dynamics to be in terms of dimension one objects, as the field strength is dimension two our integrals are over the square root of the field strength.

To get dynamical bosons we need a term that is quadratic in derivatives, as such a term must contain two fields and two derivatives, it would be of dimension six hence it must be divided by the field strength to get the right dimension.

The simplest potential that one can write is F^2, and by the fact that this is the highest power of F we are assured that our integrals are convergent.

This transition amplitude can therefore be regarded as the simplest possible result. Incidentally if we take the naïve $g \to 0$ limit, then we recover the free Maxwell's equations.

Lorentz Invariant Derivation

From this result we also see that we can derive this action in a manifestly Lorentz invariant manner.

The Transition amplitude is

$$W = \int D[A]\exp(-\tfrac{i}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \Lambda A_\nu)^2 - iA_\mu \mathbf{J}_\mu).$$

Introducing a field F, we can write this as

$$W = \int D[A]D[F]\exp(\tfrac{i}{4}F^2 - \tfrac{i}{2}F_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \Lambda A_\nu) - iA_\mu \mathbf{J}_\mu).$$

Integrating by parts, we get

$$W = \int D[A]D[F]\exp(\tfrac{i}{4}F^2 + \tfrac{i}{2}[\partial_\mu(F_{\mu\nu} - F_{\nu\mu}) - 2\mathbf{J}_\nu]A_\nu - \tfrac{i}{2}A(\Lambda \cdot F)A).$$

If we now restrict F to be antisymmetric in space, by integrating away all other components of F, then F is writeable as $\begin{pmatrix} 0 & E \\ -E & S \cdot B \end{pmatrix}$.

Completing the square, which we can only do because the previous calculation was done by explicitly proving that $\Lambda \cdot F$ is a matrix that does not have any vanishing eigenvalues.

The separation of $\Lambda \cdot F$ into $\begin{pmatrix} 1 & 0 \\ 0 & \vec{B} \end{pmatrix}$ and $\begin{pmatrix} 0 & \tilde{E}\vec{B}^{-1} \\ \tilde{E} & 1 \end{pmatrix}$, proves the regularity. Hence,

$$W = \int D[A]D[F]\exp(\tfrac{i}{4}F^2 + i[\partial \cdot F - \mathbf{J}](\Lambda \cdot F)^{-1}[\partial \cdot F - \mathbf{J}]$$
$$-\tfrac{i}{2}(A - [\partial \cdot F - \mathbf{J}](\Lambda \cdot F)^{-1})(\Lambda \cdot F)(A - [\partial \cdot F - \mathbf{J}](\Lambda \cdot F)^{-1}))$$

Integrating on A,

$$W = \int D[F]\frac{1}{\sqrt{\mathrm{Det}(\Lambda \cdot F)}}\exp(\tfrac{i}{4}F^2 + \tfrac{i}{2}[\partial \cdot F - \mathbf{J}](\Lambda \cdot F)^{-1}[\partial \cdot F - \mathbf{J}]).$$

Substituting the integration variable by $\sqrt{\Lambda \cdot F}$, recovers the earlier expression

$$W = \int D\sqrt{F}\exp(\frac{i}{g^2\hbar}\int d^3x dt \frac{1}{4}F^2 + \frac{1}{2}(\partial \cdot F - \mathbf{J})F^{-1}(\partial \cdot F - \mathbf{J})).$$

Concluding the Lorentz-invariant derivation.

Analysis of Properties

The Classical Hamiltonian

In order to get a greater understanding of our classical Lagrangian, we will derive a Hamiltonian from it. As we go from a Lagrangian to a Hamiltonian, we have to be careful about how we handle regulating factors, but as we are here only considering the classical Hamiltonian it is sufficient to set the regulators to zero.

After we have a classical solution we need to reinsert it into the Lagrangian in order to derive the proper quantum-mechanical Hamiltonian.

Let's start by finding the conjugate momentum to $\dot{E}$, recalling $\mathbf{j} = \partial\times B - \dot{E} + \mathbf{J}$.

$\Pi = \frac{\delta L}{\delta \dot{E}}$ yields the equation $\Pi = -\vec{B}^{-1}\cdot\mathbf{j} + ZY\cdot\vec{B}^{-1}\cdot Y(\mathbf{q} + Z\cdot\mathbf{j})$. We immediately see that Π only depends on the transverse modes of $\dot{E}$.

It is important to recognize this lack of dependence on the longitudinal modes, as we will show by the following little example:

> $L = \frac{\varepsilon}{2}\dot{x}^2 + \dot{x}f(x)$, *hence* $P = \varepsilon\dot{x} + f(x)$, *and* $H = \frac{1}{2\varepsilon}(P - f(x))^2$. *In the limit* $\varepsilon \to 0$, *we get the same ground-state solution as the direct insertion of* $\varepsilon = 0$, *but we get a whole spectrum of divergent energy states from including the possibility of a nonzero* ε. *For any approximate method, these infinities would be a major inconvenience.*

Using $H = \Pi\dot{E} - L$, and the general relation:

$L = \frac{1}{2}(\dot{x} + a)m(\dot{x} + a) + \frac{1}{2}(\dot{x} + b)n(\dot{x} + b)$ yields $p = m(\dot{x} + a) + n(\dot{x} + b)$,

$H = \frac{1}{2}(p - ma - nb)(m + n)^{-1'}(p - ma - nb) + \frac{1}{2}ama + \frac{1}{2}bnb$, where the prime on the inverse denotes that the inverse is only taken over the nonzero eigenvalues, and any vanishing eigenvalues are taken to have inverse zero. This is simply a reminder that in

$p = (m+n)\dot{x} + ma + nb$, p is not dynamical if the coefficient in $\dot{x}$ vanishes so that the vectors bracketing $(m+n)^{-1}$ are in fact identically zero for the vanishing eigenvalues.

We get,

$$H = \frac{1}{2}E^2 - \frac{1}{2}B^2 - \tfrac{1}{2}(\partial\times \boldsymbol{B}+\mathbf{J})\cdot\vec{\boldsymbol{B}}^{-1}\cdot(\partial\times \boldsymbol{B}+\mathbf{J})$$
$$+\tfrac{1}{2}[\mathbf{q}+(\partial\times \boldsymbol{B}+\mathbf{J})\cdot Z]Y\cdot\vec{\boldsymbol{B}}^{-1}\cdot Y[\mathbf{q}+Z\cdot(\partial\times \boldsymbol{B}+\mathbf{J})]$$
$$+\tfrac{1}{2}[\Pi+\vec{\boldsymbol{B}}^{-1}\cdot(\partial\times \boldsymbol{B}+\mathbf{J})-ZY\cdot\vec{\boldsymbol{B}}^{-1}\cdot Y(\mathbf{q}+Z\cdot(\partial\times \boldsymbol{B}+\mathbf{J}))]$$
$$\cdot(\vec{\boldsymbol{B}}^{-1}-ZY\cdot\vec{\boldsymbol{B}}^{-1}\cdot YZ)^{-1'}\cdot[\Pi+\vec{\boldsymbol{B}}^{-1}\cdot(\partial\times \boldsymbol{B}+\mathbf{J})-ZY\cdot\vec{\boldsymbol{B}}^{-1}\cdot Y(\mathbf{q}+Z\cdot(\partial\times \boldsymbol{B}+\mathbf{J}))],$$

where we have denoted a restricted inverse by -1', as the matrix has a large number (⅓) of vanishing eigenvalues. The vectors multiplying the vanishing-eigenvalues of the matrix are the new Gauss' law, and are by the definition of Π supposed to be exactly zero. To get the correct Hamiltonian it is necessary that $0(0^{-1'})0 = 0$, which is most easily implemented by taking the inverse in the subspace where the matrix has non-zero eigenvalues and preserving the vanishing eigenvalues.

Collecting terms, this expression can be written as

$$H = \tfrac{1}{2}\Pi\cdot(\vec{\boldsymbol{B}}^{-1}-ZY\cdot\vec{\boldsymbol{B}}^{-1}\cdot YZ)^{-1'}\cdot\Pi+\Pi\cdot(1-ZY)\cdot(\partial\times \boldsymbol{B}+\mathbf{J})+\tfrac{1}{2}E^2-\tfrac{1}{2}B^2$$
$$+\tfrac{1}{2}[\mathbf{q}+(\partial\times \boldsymbol{B}+\mathbf{J})\cdot Z]Y\cdot\vec{\boldsymbol{B}}^{-1}\cdot Y[\mathbf{q}+Z\cdot(\partial\times \boldsymbol{B}+\mathbf{J})],$$

where we have made extensive use of the $Z, Y, \vec{\boldsymbol{B}}^{-1}$ symmetry properties. We proved previously that the ZY projector commutes with $\vec{\boldsymbol{B}}^{-1}$, hence
$(\vec{\boldsymbol{B}}^{-1}-ZY\cdot\vec{\boldsymbol{B}}^{-1}\cdot YZ)^{-1'} = (1-YZ)\cdot\vec{\boldsymbol{B}}$, by $(\vec{\boldsymbol{B}}^{-1}-ZY\vec{\boldsymbol{B}}^{-1}YZ)(1-YZ)\vec{\boldsymbol{B}} = 1-ZY$. This simply follows from $\vec{\boldsymbol{B}}$ being block diagonal in $(ZY, 1-ZY)$-space.

Stability

Let us now consider the case of no external currents, then we can prove that our Hamiltonian has positive definite energy, even though our kinetic term has mixed signs. The proof is based on the observation that since there is no kinetic term in B, the variation of the Hamiltonian with respect to any change in B should vanish. In particular, if we uniformly rescale B we see the following relationships.

$B(x) = \alpha B_0(x)$ implies the following α dependence,

$\int d^3x H(x) = \alpha \int d^3x (H_0 - \frac{1}{2} E_0^2 + \frac{1}{2} B_0^2) + \int d^3x \frac{1}{2} E_0^2 - \alpha^2 \int d^3x \frac{1}{2} B_0^2$, where sub-index 0 denotes the value of the variable as evaluated at $\alpha = 1$. Setting the variation with respect to α to zero yields $\int d^3x (H_0 - \frac{1}{2} E_0^2 + \frac{1}{2} B_0^2) = \alpha \int d^3x B_0^2$. Evaluating at $\alpha = 1$ yields the result: $\int d^3x H(x) = \int d^3x \frac{1}{2} (E^2 + B^2)$.

We see how the total energy conforms to the conventional result, but this does not mean that the local energy density obeys the same relationship.

As we have proven that the total energy is positive definite, and as the Hamiltonian is a finite function for some choice of E and B, we have proven that there is a finite minimum for the Hamiltonian density.

To get a better understanding of the dynamics of this Hamiltonian we will consider the simplified problem of solving the Lagrangian without the "coulomb" term,
$-\frac{1}{2}(\mathbf{q} + \mathbf{j} \cdot Z) Y \cdot \vec{B}^{-1} \cdot Y(\mathbf{q} + Z \cdot \mathbf{j})$. The Lagrangian is then
$L'_{cl} = -\frac{1}{2} E^2 + \frac{1}{2} B^2 + \frac{1}{2} \mathbf{j} \vec{B}^{-1} \mathbf{j}$.

The Hamiltonian becomes
$H' = \frac{1}{2} \Pi \cdot \vec{B} \cdot \Pi + \Pi \cdot (\partial \times B + \mathbf{J}) + \frac{1}{2} E^2 - \frac{1}{2} B^2$ with $\Pi = -\vec{B}^{-1} \cdot \mathbf{j}$.

This expression is simple enough that we can explicitly eliminate B:

$\frac{\delta H'}{\delta B} = \frac{1}{2}\Pi(S\Lambda^*)\Pi + \partial\times\Pi - B = 0$, hence

$H' = \frac{1}{2}(\frac{1}{2}\Pi(S\Lambda^*)\Pi + \partial\times\Pi)^2 + \Pi\cdot\mathbf{J} + \frac{1}{2}E^2$.

Recognizing that Π is conjugate to E we see that this is in fact the usual Yang-Mills Hamiltonian for the $\mathbf{A}_0 = 0$ gauge. From this we conclude that the intuitive interpretation of E as the conjugate momentum to $\mathbf{A}$ is indeed appropriate, and that the "coulomb" term is indeed the charge conservation term.

Further we see that the choice of B that we had to make was the conventional chromo-magnetic field strength, allowing us to interpret B as the magnetic field.

Symmetries

It is interesting to see what the symmetry properties of our fields are. From the terms in E^2 and B^2 we see that the Lagrangian must be even under transformations if it preserves that symmetry, $CL = PL = TL = +L$. All other terms in the Lagrangian can be written in the form $V\cdot(S\cdot\Lambda\cdot B)^{-1}\cdot V$, as the Lagrangian is even then so must B be, $CB = PB = TB = +B$.

By definition $C\mathbf{Q} = -\mathbf{Q}$ and $C\mathbf{J} = -\mathbf{J}$, but $\mathbf{J}$ occurs only in the combination $\partial\times B - \dot{E} + \mathbf{J}$, which therefore must be an eigenstate of charge conjugation, however B is even and $\mathbf{J}$ is odd so C is not a symmetry of the Lagrangian.

CP can be made a good symmetry if $P\mathbf{J} = +\mathbf{J}$ and $CPE = -E$ as $CP\,\partial\times B = -\partial\times B$, from the combination $-\partial\cdot E + \mathbf{Q} + (\partial\times B - \dot{E} + \mathbf{J})\cdot B^{-1}\cdot\tilde{E}$, we get $CP(-\partial\cdot E + \mathbf{Q}) = +(-\partial\cdot E + \mathbf{Q})$. Hence $P\mathbf{Q} = +\mathbf{Q}$.

T being a symmetry requires $TE = -E$ from $T\,\partial\times B = +\partial\times B$, and consequentially $T\mathbf{J} = +\mathbf{J}$ and $T\mathbf{Q} = -\mathbf{Q}$.

We see how our Lagrangian is an eigenfunction of CP and of T, if we transform our fields appropriately, but that it cannot be made an eigenfunction of C or P separately. This

does not mean that the vacuum necessarily violates these symmetries, but rather that charge-conjugation is a somewhat ill-defined concept. Consider a colored vector $\mathbf{J}_i^a$, then we can make the gauge and rotational invariant object $\mathbf{J}_i^a\mathbf{J}_i^a + x\varepsilon_{ijk}f^{abc}\mathbf{J}_i^a\mathbf{J}_j^b\mathbf{J}_k^c$, which is not an eigenstate of parity nor of charge conjugation, but only of their product.

The Static Stationary Solution

As all results depend on the solutions of the stationary paths, which are defined by the classical equations of motion, we will now try to find such solutions. To find the ground state we assume that the Hamiltonian coordinates and momenta are time independent. Completing the square for the Hamiltonian, we get,

$$H = \tfrac{1}{2}[\Pi + (\partial\times B + \mathbf{J})\cdot\ddot{B}^{-1}]\cdot(1-YZ)\cdot\ddot{B}\cdot(1-ZY)\cdot[\Pi + (\partial\times B + \mathbf{J})\cdot\ddot{B}^{-1}]$$
$$-\tfrac{1}{2}[\partial\times B + \mathbf{J}]\cdot(1-ZY)\cdot\ddot{B}^{-1}\cdot[\partial\times B + \mathbf{J}] + \tfrac{1}{2}E^2 - \tfrac{1}{2}B^2$$
$$+\tfrac{1}{2}[\mathbf{q} + (\partial\times B + \mathbf{J})\cdot Z]Y\cdot\ddot{B}^{-1}\cdot Y[\mathbf{q} + (\partial\times B + \mathbf{J})\cdot Z].$$

With the static Π solution being obviously $\Pi\cdot(1-YZ) = -(\partial\times B + \mathbf{J})\cdot\ddot{B}^{-1}\cdot(1-YZ)$. As $\ddot{B}$ is block diagonal in the YZ projection splitting, we can define a new sign-changed B by $\overline{B} = -\ddot{B} + 2YZ\ddot{B}$. leaving us with the following potential,

$$V = \tfrac{1}{2}[\mathbf{q}Y + (\partial\times B + \mathbf{J})]\cdot\overline{B}^{-1}\cdot[\mathbf{q}Y + (\partial\times B + \mathbf{J})] + \tfrac{1}{2}E^2 - \tfrac{1}{2}B^2.$$

For a numerical calculation this would be the natural static problem to solve, we can however do some further analytical work on this expression, reinsert $\mathbf{q} = -\partial\cdot E + \mathbf{Q}$, and define the magnitude fields of E and B as $E = e\hat{E}$ and $B = b\hat{B}$. For a potential of

$$V = \frac{1}{2b}[(-\partial\cdot(e\hat{E}) + \mathbf{Q})\frac{b}{e}\hat{Y} + \partial\times(b\hat{B}) + \mathbf{J}]\cdot\overline{\hat{B}}^{-1}$$
$$\cdot[(-\partial\cdot(e\hat{E}) + \mathbf{Q})\frac{b}{e}\hat{Y} + \partial\times(b\hat{B}) + \mathbf{J}] + \tfrac{1}{2}e^2 - \tfrac{1}{2}b^2.$$

We must still remember that the potential should be stationarized in B but minimized in E. From our definitions $Y = \tilde{E}(\tilde{E}\cdot\ddot{B}^{-1}\cdot\tilde{E})^{-1}$ and $Z = \tilde{E}\cdot\ddot{B}^{-1}$, we see that $\hat{Y} = \tilde{\hat{E}}(\tilde{\hat{E}}\cdot\ddot{\hat{B}}^{-1}\cdot\tilde{\hat{E}})^{-1}$.

A Uniform Static Solution

To get some idea of the solutions to the static problem, and to illustrate some details we now try to solve for the minimum of the potential. Whenever we run into trouble we will make further simplifying assumptions without strict justification.

The problem we will try to solve is the field configuration for an infinite capacitor, thus no external currents, and only external surface charges.

Further assume that the magnitude of the *B*-field is very smooth in space, so that we can ignore any derivatives of its magnitude. This assumption really means that we expect the *B*-field to change directions much more rapidly than it changes magnitude. Then we get,

$$V \cong \frac{b}{2}[(-\partial\cdot(e\hat{E}) + \mathbf{Q})\frac{1}{e}\hat{Y} + \partial\times\hat{B}]\cdot\bar{\hat{B}}^{-1}\cdot[(-\partial\cdot(e\hat{E}) + \mathbf{Q})\frac{1}{e}\hat{Y} + \partial\times\hat{B}] + \tfrac{1}{2}e^2 - \tfrac{1}{2}b^2.$$

With these restrictions we can explicitly solve for *b*, to make the potential manifestly positive definite.

In the interior region we set **Q** to zero, and we take $\partial\cdot\hat{E} = 0$ corresponding to the assumption that the *E*-field does not transform the color of charges, to get,

$V \cong \frac{1}{8}[(-\partial\ln e\cdot\hat{E}\hat{Y} + \partial\times\hat{B})\cdot\bar{\hat{B}}^{-1}\cdot(-\partial\ln e\cdot\hat{E}\hat{Y} + \partial\times\hat{B})]^2 + \frac{1}{2}e^2$ for the interior region. We now see that in the weak *e* limit the natural coordinate for *e* is $e = ae^{-\mu(x)}$ with the potential $V \cong \frac{1}{8}[(\partial\mu\cdot\hat{E}\hat{Y} + \partial\times\hat{B})\cdot\bar{\hat{B}}^{-1}\cdot(\partial\mu\cdot\hat{E}\hat{Y} + \partial\times\hat{B})]^2$. Uniformity makes it reasonable to assume that there exists a matrix *T* such that $T\hat{B} = \partial\times\hat{B}$, leading us to μ being linear in space for the uniform solution. The linear μ is of course the usual Yukawa decay.

The interesting aspects of this solution is that it is consistent with our potential to get solutions of exponentially decaying electric fields away from charges, and that the magnetic field stays at approximately constant magnitude everywhere in space. We note that as the magnetic field couples to the relative rate of change of the electric field, even though the

electric field strength decays exponentially fast, the associated potential energy does not decay. We see how the static potential contains a candidate mechanism for generating the conventional linear potential.

It is an important caveat in this analysis that we have not in any way demonstrated that the ground state corresponds to the static solution, in fact it would be a curious coincidence if this were the case. We simply noted that the static solution stationarized the Hamiltonian, but the *B*-matrix that couples the quadratic term in the momentum is not positive definite and only if it had been a fixed positive semi-definite matrix could we have known that the point of vanishing kinetic term is the minimum. To find the true classical minimum, we must, alas, consider arbitrarily complicated time-dependent field configurations.

Euclidean-Space Formulation

Let us go back and consider the changes that occur in these calculations if we start out with the Euclidean action instead of the Minkowski action. As we have already gone through the whole calculation in detail for the Minkowski case, we will introduce some shorthand notation as follows:

$e(\mathbf{A}) = \partial \mathbf{A}_0 - \dot{\mathbf{A}} + \mathbf{A}\Lambda\mathbf{A}_0$, $b(\mathbf{A}) = \partial S\mathbf{A} + \frac{1}{2}\mathbf{A}S\Lambda\mathbf{A}$

Considering the case of no external charges, the Euclidean partition function is,

$$\int D[\mathbf{A}]\, e^{-\int d^3x dt\, \frac{1}{2}e^2 + \frac{1}{2}b^2}.$$

To do our integral transformations we need to make the exponent linear in e and b using, $\int D[X]e^{-\frac{1}{2}XX-iXY} = N'e^{-\frac{1}{2}YY}$. The only way we can make the quadratic term in the transforming exponent negative is by performing a shift by an imaginary amount, this yields the transformed partition function,

$\int D[E]\int D[B]\int D[\mathbf{A}]e^{-\int dxdt\frac{1}{2}E^2+\frac{1}{2}B^2-iEe+ibB}$. Which we can compare with the Minkowski transition amplitude,

$\int D[E]\int D[B]\int D[\mathbf{A}]_e i\int d^3xdt\frac{-1+i\varepsilon_E}{2}E^2+\frac{1+i\varepsilon_E}{2}B^2+Ee-Bb$. None of the steps we performed in the elimination of the vector potential, and related determinants depended on the coefficients in front of E^2 and B^2, only on their convergence. From this we conclude that the Euclidean partition function is identical to the Minkowski amplitude, except for the quadratic terms in the fields that transform as $\frac{-i-\varepsilon_E}{2}E^2+\frac{i-\varepsilon_B}{2}B^2\rightarrow-\frac{1}{2}E^2-\frac{1}{2}B^2$.

The intrinsic meaning of this change is that the factors of i in our exponent are irremovable, preventing any numerical simplification by rotating the vector potential **A** into imaginary time formalism.

The Partition Function

For many numerical methods, such as Monte Carlo calculations, it is necessary to somehow change the factor of i in front of the action to -1, so that the magnitude of the action for a particular path indicates the size of the contribution to the full functional integral. For such a transformation to be successful it is necessary to get to a form where the new action is bounded from below.

We proved that our Hamiltonian is indeed bounded from below, so there is a starting point for a Monte Carlo calculation, but the Hamiltonian is only bounded from below if we solve the B field explicitly.

From the manifestly Lorentz invariant formulation we have the action

$$S = \int d^3x dt \frac{1}{4} F^2 + \frac{1}{2}(\partial \cdot F - \mathbf{J}) F^{-1} (\partial \cdot F - \mathbf{J})).$$

To keep the action manifestly real, we must choose an analytic continuation of F such that F^{-1} is positive definite. But, the essential property for preserving gauge invariance in this expression was the antisymmetry of $F^{ab}_{\mu\nu}$ under interchange of space indices, hence there is no real F^{-1} that is positive while preserving gauge invariance. We must therefore work in the E and B representation so that we can keep our components manifestly real.

What we would like to do, is to find a transformation that would make our kinetic term $\frac{1}{2}[\partial \times B - \dot{E} + \mathbf{J}] \cdot (\ddot{B}^{-1} - ZY \cdot \ddot{B}^{-1} \cdot YZ) \cdot [\partial \times B - \dot{E} + \mathbf{J}]$ be manifestly positive. Remembering that ZY is a projector, the obvious way of ensuring positivity is to make $\ddot{B}^{-1} = (S \cdot \Lambda \cdot B)^{-1}$ positive, as the projection cannot change the signs of the eigenvalues.

When we removed the factor $1 / \sqrt{|\det(S \cdot \Lambda \cdot B)|}$ from the transition amplitude, we were forced to introduce the new dynamical variables t by $(S \cdot \Lambda \cdot t)(S \cdot \Lambda \cdot t) = (S \cdot \Lambda \cdot B)$. At that point, we needed to create complex integration paths for the t-variables, so that their square would have the proper sign structure. Now we see that the analytical continuation of all the t's to the real axis is exactly the transformation needed to make $(S \cdot \Lambda \cdot B)^{-1}$ positive. We will denote the rotated B with $\mathbf{B}$.

To verify that we do not violate gauge invariance we recalculate the relevant part of the gauge-transformation properties of the Lagrangian:

$$\partial\times B \to \partial\omega(S\cdot\Lambda\cdot B)$$
$$\Rightarrow \partial\cdot(\Lambda^{a}(S\cdot\Lambda\cdot t)^{2})_{bb} \to 2(\Lambda^{a}_{bc}(S_{ij}\cdot\Lambda_{cd}\cdot t)(S_{jk}\cdot\Lambda_{db}\cdot(\partial_{i}\omega_{e}\Lambda_{ef}t_{f}))$$
$$= \partial_{i}\omega_{e}\Lambda^{a}_{bc}(S_{ij}\cdot t_{b})(S_{jk}\cdot\Lambda^{c}_{ef}t_{f})$$
$$= \partial_{i}\omega(S_{ij}\cdot\Lambda\cdot t)(S_{jk}\cdot\Lambda\cdot t) = \partial\omega(S\cdot\Lambda\cdot\mathbf{B}).$$

The rest of the gauge invariance calculation is the same as before.

The second issue in converting to Euclidean coordinates is to make the Euclidean action manifestly positive and real. Ignoring external charges we need to find the complex rotations that make the rotation $t = e^{i\alpha}\tau$, with the Wick rotation $\alpha = 0 \to \pi/2$ well behaved. The action then becomes,

$$\exp\{ie^{i\alpha}\int d^{3}xd\tau\frac{-1}{2}E^{2}+\frac{1}{2}\mathbf{B}^{2}+\frac{1}{2}(\partial\times\mathbf{B}-e^{-i\alpha}E'+\mathbf{J})\cdot\tilde{\mathbf{B}}^{-1}\cdot(\partial\times\mathbf{B}-e^{-i\alpha}E'+\mathbf{J})$$
$$-\frac{1}{2}[\mathbf{Q}-\partial\cdot E+(\partial\times\mathbf{B}-e^{-i\alpha}E'+\mathbf{J})\cdot Z]Y\cdot\tilde{\mathbf{B}}^{-1}\cdot Y[\mathbf{Q}-\partial\cdot E+Z\cdot(\partial\times\mathbf{B}-e^{-i\alpha}E'+\mathbf{J})]\}.$$

Where the sign for the term in E^{2} is wrong, and the terms $\partial\times\mathbf{B}-e^{-i\alpha}E'$ mix a real and an imaginary quantity. Both of these problems can be solved if we rotate E by $E = e^{i\alpha}\mathbf{E}$. Which makes $Y = e^{-i\alpha}\mathbf{Y}$ and $Z = e^{i\alpha}\mathbf{Z}$, to match we need to rotate the external charge by $\mathbf{Q} = e^{-i\alpha}\mathbf{Q}_{E}$.

Leaving us with an almost positive definite Euclidean Lagrangian,

$$L_{E} = \frac{1}{2}\mathbf{E}^{2}+\frac{1}{2}\mathbf{B}^{2}+\frac{1}{2}(\partial\times\mathbf{B}-\mathbf{E}'+\mathbf{J})\cdot\tilde{\mathbf{B}}^{-1}\cdot(\partial\times\mathbf{B}-\mathbf{E}'+\mathbf{J})$$
$$-\frac{1}{2}[-\mathbf{Q}_{E}-\partial\cdot\mathbf{E}+(\partial\times\mathbf{B}-\mathbf{E}'+\mathbf{J})\cdot\mathbf{Z}]\mathbf{Y}\cdot\tilde{\mathbf{B}}^{-1}\cdot\mathbf{Y}[-\mathbf{Q}_{E}-\partial\cdot\mathbf{E}+\mathbf{Z}\cdot(\partial\times\mathbf{B}-\mathbf{E}'+\mathbf{J})],$$

with $\mathbf{Y} = \tilde{\mathbf{E}}(\tilde{\mathbf{E}}\cdot\tilde{\mathbf{B}}^{-1}\cdot\tilde{\mathbf{E}})^{-1}$ and $\mathbf{Z} = \tilde{\mathbf{E}}\cdot\tilde{\mathbf{B}}^{-1}$.

We are still left with an unbounded term from the divergence of **E**, the reason for this is that only two thirds of the modes in $\mathbf{E}'$ are dynamic, the remainder are constrained by $\partial\cdot\mathbf{E}$.

If we are willing to consider some extra assumptions about external charges, then the Lagrangian can be made manifestly positive definite. We recognize that there is an arbitrary external charge in the action, simply postulating $-\partial\cdot\mathbf{E} = \mathbf{Q}_E$ automatically makes the Lagrangian positive definite. To preserve gauge invariance our external charge must obey $\dot{\mathbf{Q}} - \partial\cdot\mathbf{J} = \mathbf{0}$, which becomes $\mathbf{Q}'_E + \partial\cdot\mathbf{J} = \mathbf{0}$ in Euclidean space. Eliminating $\mathbf{Q}'_E$ we get $\partial\cdot(-\mathbf{E}' + \mathbf{J}) = \mathbf{0}$. Postulating that the external charges are irrotational, we get the following Lagrangian,

$L_E = \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{B}^2 + \frac{1}{2}(\partial\times\mathbf{B} - P_t\mathbf{E}')\cdot(\vec{\mathbf{B}}^{-1} - \mathbf{ZY}\cdot\vec{\mathbf{B}}^{-1}\cdot\mathbf{YZ})\cdot(\partial\times\mathbf{B} - P_t\mathbf{E}')$, where we have introduced the conventional transverse mode projector $P_{t\,ij} = \delta_{ij} - \partial_i\frac{1}{\partial^2}\partial_j$.

In this representation we can interpret our postulates on the external currents as the assumption that the quark background is a perfect dielectric medium—completely shielding all charges but not coupling to any currents.

The Wick rotation is straightforward, but there is no easy way of showing that the changing of roots in $\sqrt{(S\cdot\Lambda\cdot B)^2}$ will be safe.

Induced Kinetic Term

As many of our complications stem from the fact that only some of the E-field excitations have a kinetic term, it might be useful to find a transformation that induces such a term on all of them. The treatment will not be completely rigorous, as the transformations raise some thorny issues regarding gauge invariance.

Recall our Hamiltonian,

$$H = \tfrac{1}{2}\Pi\cdot(\ddot{B}^{-1} - ZY\cdot\ddot{B}^{-1}\cdot YZ)^{-1'}\cdot\Pi + \Pi\cdot(1-YZ)\cdot(\partial\times B + \mathbf{J}) + \tfrac{1}{2}E^2 - \tfrac{1}{2}B^2$$
$$+\tfrac{1}{2}[\mathbf{q} + (\partial\times B + \mathbf{J})\cdot Z]Y\cdot\ddot{B}^{-1}\cdot Y[\mathbf{q} + Z\cdot(\partial\times B + \mathbf{J})]$$

and the definition of Π as $\Pi = -\ddot{B}^{-1}\cdot\mathbf{j} + ZY\cdot\ddot{B}^{-1}\cdot Y(\mathbf{q} + Z\cdot\mathbf{j})$. In this expression there is no dependence on the part of Π represented by $ZY\cdot\Pi = ZY\cdot\ddot{B}^{-1}\cdot Y\mathbf{q}$, but there is a corresponding $\mathbf{q}$ dependence. Ignoring quantization issues, we write

$\mathbf{q}Y\cdot\ddot{B}^{-1}\cdot Y\mathbf{q} = \mathbf{q}Y\cdot ZY\cdot\ddot{B}^{-1}\cdot\ddot{B}\cdot\ddot{B}^{-1}\cdot YZ\cdot Y\mathbf{q} = \Pi\cdot YZ\cdot\ddot{B}\cdot ZY\cdot\Pi$ and

$(\partial\times B + \mathbf{J})\cdot ZY\cdot\ddot{B}^{-1}\cdot Y\mathbf{q} = (\partial\times B + \mathbf{J})\cdot ZY\cdot\Pi$.

Our Hamiltonian is now

$$H = \tfrac{1}{2}\Pi\cdot[(\ddot{B}^{-1} - YZ\cdot\ddot{B}^{-1}\cdot ZY)^{-1'} + ZY\cdot\ddot{B}\cdot YZ]\cdot\Pi + \Pi\cdot(\partial\times B + \mathbf{J}) + \tfrac{1}{2}E^2 - \tfrac{1}{2}B^2$$
$$+\tfrac{1}{2}(\partial\times B + \mathbf{J})\cdot ZY\cdot\ddot{B}^{-1}\cdot YZ\cdot(\partial\times B + \mathbf{J}).$$

After eliminating projectors we get,

$$H = \tfrac{1}{2}\Pi\cdot\ddot{B}\cdot\Pi + \Pi\cdot(\partial\times B + \mathbf{J}) + \tfrac{1}{2}E^2 - \tfrac{1}{2}B^2 + \tfrac{1}{2}(\partial\times B + \mathbf{J})\cdot ZY\cdot\ddot{B}^{-1}\cdot YZ\cdot(\partial\times B + \mathbf{J}).$$

The external charge is completely gone in this expression, so it is clear that we have lost some dynamics. Because of the linear term in the momentum Π, there is still no obvious complex continuation that will render the Hamiltonian manifestly positive definite.

On Scale Invariance

Our Lagrangian has a scale invariance, in the sense that if we have a solution $\overline{E}(x,t)$, $\overline{B}(x,t)$ that satisfies the Euler-Lagrange equations, then there is a family of solutions $E = \alpha^2\overline{E}(\alpha x,\alpha t)$, $B = \alpha^2\overline{B}(\alpha x,\alpha t)$ such that the equations are still satisfied. This is easily seen by realizing that the substitution of α simply multiplies the Lagrangian density by α^4 and expands the space-time dependence by α.

The main consequence of this scale invariance is that it is not possible to talk about a small or large B-field limit, since we do not have anything to relate the magnitude to. If we

wanted to consider the properties of our system for vanishing fields, we would still have to define the relative magnitudes and directions of the fields to make the Lagrangian defined, but then by scale invariance there is no point in taking the limit of vanishing fields.

The scale of the system will be introduced as we integrate over the quantum fluctuations, since that integral depends on the cutoff scale. We note that the absorption of the coupling constant in the variable redefinition that we initially did, will be impractical for calculating the quantum fluctuations.

Quantum Mechanics

To get the leading-order quantum contributions we only need to expand to second order around the stationary path, as we outlined earlier in the description of the solution method.

Let us consider the Hamiltonian for a stationary path, by our previous proof of the energy being positive there must exist a minimum-energy solution. If we expand to second order, then we must get a purely quadratic expression in δE, $\delta\Pi$ and δB. From being a classical minimum the expression must be positive semi-definite. So we simply integrate away all non-dynamic degrees of freedom and diagonalize the remainder to find out the complete mass spectrum of the theory. The total quantum contribution to the energy is then $H_q = -\sum_n m_n^4 \log(\Lambda^2/m_n^2)$, where all the m_n's are proportional to the scale. As all the terms are negative and the classical contribution was positive this proves that the solution we get will spontaneously transmutate the bare coupling constant to a symmetry-breaking scale.

From a practical point of view it would be easier to reinsert the Π_{cl}, E_{cl} and B_{cl} into the Lagrangian and eliminate the nonphysical modes before generating a new "free" quantum-mechanical Hamiltonian.

On Scale Generation and Quantum Mechanics

In general we can see how the quantum fluctuations generate a scale. After canceling all higher-order (point-like) divergences, the leading-order quantum fluctuations will contribute a term of the form $H_q = -\mu^4 h_q \log(\Lambda^2/\mu^2) + \mu^4 q$, where μ is the scale of the classical fields, q and h_q are cutoff independent numerical factors from adding all quantum contributions. h_q is scale independent whereas q has bounded derivatives with respect to the scale. We have chosen the sign in front of h_q such that a free massive theory has $h_q > 0$ and that the quadratic expansion around a stable minimum has $h_q > 0$. The classical contributions are $H_{cl} = \frac{1}{g^2(\Lambda)}\mu^4 h_{cl}$, from scale invariance, with h_{cl} strictly positive. Adding the contributions, and minimizing the sum with respect to the scale, we get:

$H = H_q + H_{cl} = \mu^4(\frac{h_{cl}}{g^2(\Lambda)} - h_q \log(\Lambda^2/\mu^2) + q)$, hence

$\frac{\partial H}{\partial \mu^2} = 2\mu^2(\frac{h_{cl}}{g^2(\Lambda)} - h_q(\log(\Lambda^2/\mu^2) - \frac{1}{2}) + q + \frac{\mu^2}{2}q'_{\mu^2}) = 0$, which has two solutions,

$\mu^2 = 0$ and $\mu^2 = \frac{\Lambda^2}{\exp(\frac{h_{cl}}{g^2(\Lambda)h_q} + \frac{1}{2} - \frac{1}{h_q}(q + \frac{\mu^2}{2}q'_{\mu^2}))}$. For quantum contributions where $h_q < 0$, only the first solution is valid, and corresponds to the simple vacuum of all fields vanishing. For quantum contributions where $h_q > 0$, only the second solution is a minimum, implying that the leading order term in the renormalized coupling constant must be

$g_0^2(\Lambda) = \frac{h_{cl}}{h_q(\log(\Lambda^2/\mu_r^2) + \frac{h_{cl}}{h_q g_r^2(\mu_r^2)})}$, where we have introduced an arbitrary renormaliza-

tion scale μ_r. Reinserting the renormalized coupling we get

$$\mu^2 = \frac{\mu_r^2}{\exp(\frac{h_{cl}}{h_q g_r^2(\mu_r^2)} + \frac{1}{2} - \frac{1}{h_q}(q + \frac{\mu^2}{2} q'_{\mu^2}))}.$$

It should be noted that re-inserting the generated scale into the expression for the energy will cancel the leading-logarithm contributions to the energy, so that for a finite vacuum energy density it is sufficient that the classical contribution is not more than logarithmically divergent. Conversely for a finite classical contribution, the quantum contribution must be massless—implying that all modes must be massless.

We note that the explicit elimination of the coupling constant from the action, led to an overall factor of $\frac{1}{g^2(\Lambda)}$ in front of the action, and consequentially in front of the energy. Our minimization of the energy with respect to the scale showed that the total energy density can be finite and non-zero only if the classical contribution is logarithmically divergent.

As a corollary we note that if the classical contribution vanishes, as in QED, then the only solution when minimizing the energy with respect to the scale is $\mu^2 = 0$.

Whenever we do a semi-classical calculation for a process, we must remember the crucial cancellation of the always logarithmically-divergent classical and quantum contributions. Suppose we have a field theory with two different classical stationary paths, but a single coupling constant. Only if the ratio between the classical and quantum contributions for the two paths is identical (up to a logarithmically small quantity), can both of these paths contribute to the vacuum.

Some Comments on Larger Symmetry Groups

The fundamental property of B that we used in our previous derivation was that there is in general no gauge transformation that leaves an adjoint-color spatial vector invariant. In

particular, given two colored objects, no third mutually commuting colored object can be constructed.

In the case of larger symmetry groups it is however possible to find such an object, for definiteness let us consider the smallest such group su(5). We can now define a new gauge covariant color projector by $P_Z(S\cdot\Lambda\cdot B)=0$ and letting Z denote the subspace where $S\cdot\Lambda\cdot B=0$.

For the subspace in $\mathbf{A}$ orthogonal to Z we can complete the square as before. Letting a prime denote the subspace orthogonal to Z and subscript z denote components in the Z-space we can write the Lagrangian as,

$$L_{EB}=-\tfrac{1}{2}E\cdot E+\tfrac{1}{2}B\cdot B-\partial\cdot E^{a}\mathbf{A}_0^{a}-\tfrac{1}{2}\mathbf{A}'(S\cdot\Lambda\cdot B)'\mathbf{A}'+(\partial\times B-\dot{E}+\mathbf{A}_0\tilde{E})_z\,\mathbf{A}_z$$
$$+\tfrac{1}{2}[\partial\times B-\dot{E}+\mathbf{A}_0\tilde{E}]'(S\cdot\Lambda\cdot B)'^{-1}[\partial\times B-\dot{E}+\mathbf{A}_0\tilde{E}]'.$$

Where we have shifted only the Z-orthogonal part of $\mathbf{A}$-spatial. The crucial point here is that the color matrix $\tilde{E}\cdot(S\cdot\Lambda\cdot B)'^{-1}\cdot\tilde{E}$ will lack vanishing eigenvalues for the generic E. Thus we can perform our shift in coordinates in $\mathbf{A}_0$, but now there's an extra term in $\mathbf{A}_z$. The shift in integration path for $\mathbf{A}_0$ brings the $\mathbf{A}_z$ dependence up from linear to quadratic, so that we can integrate over $\mathbf{A}_z$.

Before integrating over $\mathbf{A}_z$ we have,

$$L_{EB}=\frac{-1}{2}E^2+\frac{1}{2}B^2+\tfrac{1}{2}(\partial\times B-\dot{E})\cdot\ddot{B}^{-1'}\cdot(\partial\times B-\dot{E})$$
$$+(\partial\times B-\dot{E})_z\cdot\mathbf{A}_z$$
$$-\tfrac{1}{2}[-\partial\cdot E+\tilde{E}_z\mathbf{A}_z+(\partial\times B-\dot{E})\cdot\ddot{B}^{-1}\cdot\tilde{E}]\cdot M'^{-1}$$
$$\cdot[-\partial\cdot E+\tilde{E}_z\cdot\mathbf{A}_z+(\partial\times B-\dot{E})\cdot\ddot{B}^{-1}\cdot\tilde{E}],$$
with $M'=-\tilde{E}'\ddot{B}'^{-1}\tilde{E}'$.

Completing the square and shifting $\mathbf{A}_z$ we get,

$$L_{EB} = \frac{-1}{2}E^2 + \frac{1}{2}B^2 + \tfrac{1}{2}(\partial\times B - \dot{E})\cdot \ddot{B}^{-1\prime}\cdot(\partial\times B - \dot{E})$$
$$-\tfrac{1}{2}[-\partial\cdot E + (\partial\times B - \dot{E})\cdot\ddot{B}^{-1}\cdot\tilde{E}]M'^{-1}[-\partial\cdot E + (\partial\times B - \dot{E})\cdot\ddot{B}^{-1}\cdot\tilde{E}]$$
$$+\tfrac{1}{2}[(-\partial\cdot E + (\partial\times B - \dot{E})\cdot\ddot{B}^{-1}\cdot\tilde{E})\tilde{E}_z - (\partial\times B - \dot{E})_z]\cdot(\tilde{E}_z M'^{-1}\tilde{E}_z)_z^{-1}$$
$$\cdot[(-\partial\cdot E + (\partial\times B - \dot{E})\cdot\ddot{B}^{-1}\cdot\tilde{E})\tilde{E}_z - (\partial\times B - \dot{E})_z].$$

So we see that the method flows through nicely, and that the effect is that the "mass" of the modes which used to be $S\cdot\Lambda\cdot B$, got replaced by $\tilde{E}_z(\tilde{E}'\ddot{B}'^{-1}\tilde{E}')^{-1}\tilde{E}_z$ for the vanishing eigenvectors of $S\cdot\Lambda\cdot B$. The gist of this is that our transformations were not particularly dependent on the non-singularity of $(S\cdot\Lambda\cdot B)^{-1}$, but that the essential property was that we could reduce the gauge-potential dependence to quadratic and then by successive integrations of the quadratic parts we could bring all gauge vector-potential parts to a form that could be integrated away.

The only essential properties we used were that all **A**-components were either quadratic or coupled to other components that were quadratic or likewise coupled, and that after bringing our dependence to quadratic, we must be able, through integration by parts, to remove all derivatives acting on our **A**-fields. By the time we reach approximately su(7) the scheme will generate a residual constraint $(\partial\times B - \dot{E})_{z,\tilde{E}_\perp} = 0$ on components that are simultaneously *zero* eigenvectors of $S\cdot\Lambda\cdot B$ and vanish upon contraction with $\tilde{E}$.

In particular this means that the manifestly Lorentz invariant formulation is unchanged in form as we go from su(3) to su(5), since the success of making the **A** dependence quadratic is equivalent to proving the invertibility of F.

Conclusions

We have seen how it is quite straightforward to transform the Yang-Mills Lagrangian into a field-strength Lagrangian, and how all the issues of gauge invariance resolve themselves in a simple way, which was the primary goal of this transformation. We have also been able to observe the natural form of the dynamical variables of the system and how they separate themselves into dynamics and constraints.

The price we had to pay in order to get to this form was replacing the four-component A-field with the pair of three-component E and B-fields. The number of dynamical components stayed at two, which means that we doubled the number of constraints in the system.

The remaining issue is, of course, how useful is this formalism for practical calculations? The strength of the formalism is that we have taken care of gauge invariance in a way that fully implements the phase space density around physical configurations, and arrived at a non-singular Lagrangian. We therefore have good reasons to believe that the *zero* order stationary path, or classical solution, will contain much of the physics of the system. Our Lagrangian is quite complex, but all the algebraic complexity is located in inversions of local matrices, so a direct numerical solution on a lattice should not pose any severe problems. Nor would the first quantum corrections to the solution.

The leading-order quantum contributions are essential to the physics of the system, since the vacuum energy density can only be made finite by summing the divergent classical and quantum contributions. The quantum corrections are trivial, in the sense that we need only consider the "free" theory around the classical solution, so the quantum mass matrix can be explicitly diagonalized, yielding all the masses, symmetries and physical eigenmodes.

The attempt to convert the field-strength Lagrangian to a Euclidean formulation was less successful, only by introducing an ad hoc background charge can the Euclidean action be made real and positive. The origin of this problem is that the dynamical modes need to be

treated differently from the local degrees of freedom, but there is no obvious way of separating the two algebraically.

The great virtue of the reformulation is that all the complexity that occurs in the field strength Lagrangian is in terms of local expressions. Gauge invariance is imposed by a local constraint, which makes it very easy to find the particular gauge transformation that will transform a field configuration to the equivalent gauge fixed configuration.

Perhaps the most satisfying formal result is that the Fadeev-Popov gauge-fixing determinant was generated by our transformation into field strengths, so that there is in fact a natural preferred gauge where no ghost fields need be introduced.

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