Recovering the Moments of a Function From Its Radon-Transform Projections: Necessary and Sufficient Conditions *

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Abstract

The question we wish to address in this paper is the following: To what extent does a limited set of noise-free Radon-Transform projections of a function \( f(z, y) \) determine this function? This question has been dealt with in the mathematical literature to some extent. Noteworthy are results due to Volcic [4, 9], Fishburn et al [10], Falconer [3], Gardner [5, 11], Kuba [8] and other references contained therein. These results almost exclusively deal with the case when the function \( f(z, y) \) is an indicator function over its domain of definition \( \mathcal{O} \). There has also been some effort in the physics and engineering literature to answer this question. Some noteworthy examples are [17, 18]. In this paper, we will prove that one may uniquely recover the first \( p \) geometric moments [13] of a bounded, positive function \( f(z, y) \), with compact support, from a fixed number \( p \) of Radon-Transform [6] projections. We further show that one can not uniquely recover any higher order moments of \( f(z, y) \) from such limited information. The importance of this result lies in the fact that it directly shows to what extent a limited number of projections of a function determine the function. This, in essence, is a precise notion of the geometric complexity that a limited set of projections can support. In applications of this result to tomographic reconstruction problems such as in Medical Imaging [1], our result shows, in a quantitative way, how well one can theoretically expect to reconstruct an object being imaged in the absence of noise. From a more abstract viewpoint, it sheds some light on the reconstructability of certain elementary binary objects from a limited number of tomographic projections.

1 Background and Notation

The Radon-Transform of a function \( f(z, y) \) defined over a compact domain of the plane \( \mathcal{O} \) is defined by

\[
g(t, \theta) \equiv \int_{\mathbb{R}^2} f(z, y) \delta(t - \omega \cdot [z, y]^T) \, dz \, dy.
\]

For every fixed \( t \) and \( \theta \), \( g(t, \theta) \) is simply the line-integral of \( f \) over \( \mathcal{O} \) in the direction \( \omega = [\cos(\theta), \sin(\theta)]^T \), where \( \delta(t - \lfloor \cos(\theta), \sin(\theta) \rfloor \cdot [z, y]^T) \) is a delta function on a line at angle \( \theta + (\pi/2) \) from the \( z \)-axis, and distance \( t \) from the origin.

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Not all functions \( g(t, \theta) \), however, are Radon Transforms of some \( f(z, y) \). Several well-known mathematical properties of the Radon transform known as the consistency relations specify valid 2-D Radon transforms. The Radon transform of some function \( f(z, y) \) is constrained to lie in a particular functional subspace of the space of all real-valued functions \( g(t, \theta) : \mathcal{R} \times S^1 \to \mathcal{R} \), where \( S^1 \) is the unit circle. This subspace is characterized by the fact that \( g \) must be an even function of \( t \), and that certain coefficients of the Fourier expansion of \( g \) must be zero \([6, 7]\).

Let \( \omega = [\cos(\theta), \sin(\theta)] \) denote a unit direction vector, and \( x = [z, y] \) a vector in \( \mathcal{R}^2 \). Using the definition of the Radon Transform we can write

\[
\int_{-\infty}^{\infty} g(t, \omega) t^k dt = \int_{-\infty}^{\infty} t^k \int_{x \in \mathcal{R}^2} f(x) \delta(t - \omega \cdot x) dx dt = \int_{x \in \mathcal{R}^2} f(x)(\omega \cdot x)^k dx
\]

(2)

The above identity clearly holds in higher dimensions as well. For our purposes, however, we shall only deal with \( \mathcal{R}^2 \). Assuming that the support, \( O \), of \( f(z, y) \) is contained within the unit disk, we can write the identity (2) as

\[
H^{(k)}(\theta) \equiv \int_{-1}^{1} g(t, \theta) t^k dt = \sum_{j=0}^{k} \binom{k}{j} \cos^{k-j}(\theta) \sin^j(\theta) \mu_{k-j, j}
\]

(3)

where the right hand side is obtained by expanding the term \( (\omega \cdot x)^k = (\cos(\theta)z + \sin(\theta)y)^k \) according to the binomial theorem, and where \( \mu_{k-j, j} \) are the geometric moments of \( f(z, y) \) defined as follows.

\[
\mu_{p,q} = \int_{[x,y] \in \mathcal{O}} f(z, y) x^p y^q dz dy
\]

(4)

The identity (3) was apparently first discovered by I.M. Gelfand and M.I. Graev in 1961 \([19]\). Note that the left hand side of (3) is simply the \( k \)th order moment of the projection function \( g(t, \theta) \) for a fixed angle \( \theta \), denoted by \( H^{(k)}(\theta) \). Defining the vector of \( k \)th order moments of \( f(z, y) \) as

\[
\mu^{(k)} = [\mu_{k,0}, \mu_{k-1,1}, \cdots, \mu_{0,k}]^T
\]

(5)

we have that

\[
H^{(k)}(\theta) = D^{(k)}(\theta) \mu^{(k)}
\]

(6)

\[
D^{(k)}(\theta) = [\gamma_{k,0} \cos^k(\theta), \gamma_{k,1} \cos^{k-1}(\theta) \sin(\theta), \cdots, \gamma_{k,k-1} \cos(\theta) \sin^{k-1}(\theta), \gamma_{k,k} \sin^k(\theta)]
\]

(7)

\[
\gamma_{k,j} = \binom{k}{j}
\]

(8)

so that \( D^{(k)}(\theta) \) is a \( 1 \times (k + 1) \) matrix. Thus moments of order \( k \) of the projection only depend on moments of order \( k \) of \( f(z, y) \). This observation will form the basis for the result that will be established in the next section.

2 The Result

Here we will show our main result, that given a fixed number \( p \) of noise free integral projections of a real valued function of 2 variables \( f(z, y) \), we may uniquely recover the first \( p \) geometric moments of \( f(z, y) \) but cannot uniquely recover any higher order moments.

The function of interest, \( f(z, y) \), is completely and uniquely defined by the complete set of its \( (k \)th order) geometric moments \([12]\), \( \mu^{(k)} \), for all \( k \geq 0 \). In particular, we define the vector of geometric moments of \( f(z, y) \) up to order \( N \) by \( \mathcal{M}_N = [\mu^{(0)^T}, \mu^{(1)^T}, \cdots, \mu^{(N)^T}]^T \). We assume that we are given (noise-free) integral projections of \( f(z, y) \) at a fixed number of angles \( \theta_i \). Essentially we are given "cuts" of the Radon transform \( g(t, \theta_i) \) of \( f(z, y) \) at a finite number of \( \theta \). Note that each projection \( g(t, \theta_i) \) itself is uniquely and completely
defined by the complete set of its geometric moments \( H^{(k)}(\theta_i) \) for all \( k \geq 0 \). In the previous section we showed that \( H^{(k)}(\theta) \) and \( \mu^{(k)} \) are related as follows:

\[
H^{(k)}(\theta) = D^{(k)}(\theta) \mu^{(k)}
\]

so that \( D^{(k)}(\theta) \) is a \( 1 \times (k+1) \) matrix. In particular, for a given projection at angle \( \theta \), we have the following

\[
\begin{bmatrix}
H^{(0)}(\theta) \\
H^{(1)}(\theta) \\
\vdots \\
H^{(N)}(\theta)
\end{bmatrix}
= 
\begin{bmatrix}
D^{(0)}(\theta) & 0 & \cdots & 0 \\
0 & D^{(1)}(\theta) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D^{(N)}(\theta)
\end{bmatrix}
\begin{bmatrix}
\mu^{(0)} \\
\mu^{(1)} \\
\vdots \\
\mu^{(N)}
\end{bmatrix}
\]

(9)

where the obvious association is made in the last equation.

Now consider the problem where we observe a number \( p \) of (noise free) projections \( g(t, \theta_i), i = 1, \ldots, p \), and wish to uniquely recover as much of \( f(z, y) \) as possible. We look at this problem in the following way: we observe \( H^{(k)}(\theta_i) \) \( \forall k, i = 1, \ldots, p \) and wish to uniquely recover as many of the \( \mu^{(j)} \) as possible. Note that we treat moments of a given order, \( j \), as a unit, e.g. even though \( \mu^{(1)} \) consists of two numbers, we consider it determined only if both are uniquely determined. Our main result is the following:

**Theorem 1 (\( p \) Moments From \( p \) Projections)** Given \( p \) (line integral) projections of \( f(z, y) \) at \( p \) different angles \( \theta_i \) in \([0, \pi)\), one can uniquely determine the first \( p \) geometric moment vectors \( \mu^{(j)}, 0 \leq j < p \) of \( f(z, y) \). Further this can be done using only the first \( p \) order geometric moments \( H^{(k)}(\theta), 0 \leq k < p \) of the projections. Conversely, moments of \( f(z, y) \) of higher order cannot be uniquely determined from \( p \) projections.

**Proof** Consider (11) and suppose that we stack up our observations at the different angles to obtain:

\[
\begin{bmatrix}
H_N(\theta_1) \\
H_N(\theta_2) \\
\vdots \\
H_N(\theta_p)
\end{bmatrix}
= 
\begin{bmatrix}
D_N(\theta_1) \\
D_N(\theta_2) \\
\vdots \\
D_N(\theta_p)
\end{bmatrix}
\begin{bmatrix}
\mu^{(1)} \\
\mu^{(2)} \\
\vdots \\
\mu^{(p)}
\end{bmatrix}
\]

(12)

so that the matrix \( D_N \) relating \( \mathcal{M}_N \) and the projections is \( pN \times \frac{(N+1)(N+2)}{2} \). Clearly to be able to determine \( \mathcal{M}_N \) uniquely we must have full column rank of \( D_N \). Now we may rearrange the rows of (12), grouping together the moments of the same order from all projections, without changing the column rank of \( D_N \).
This operation yields the following equivalent equation:

\[
\begin{bmatrix}
H^{(0)}(\theta_1) \\
H^{(0)}(\theta_2) \\
\vdots \\
H^{(0)}(\theta_p)
\end{bmatrix}
\begin{bmatrix}
D^{(0)}(\theta_1) \\
D^{(0)}(\theta_2) \\
\vdots \\
D^{(0)}(\theta_p)
\end{bmatrix}
\begin{bmatrix}
\mu^{(0)} \\
\mu^{(1)} \\
\vdots \\
\mu^{(N)}
\end{bmatrix}
\]

\[
\begin{bmatrix}
H^{(1)}(\theta_1) \\
H^{(1)}(\theta_2) \\
\vdots \\
H^{(1)}(\theta_p)
\end{bmatrix}
\begin{bmatrix}
D^{(1)}(\theta_1) \\
D^{(1)}(\theta_2) \\
\vdots \\
D^{(1)}(\theta_p)
\end{bmatrix}
\begin{bmatrix}
\mu^{(0)} \\
\mu^{(1)} \\
\vdots \\
\mu^{(N)}
\end{bmatrix}
\]

\[
\begin{bmatrix}
H^{(N)}(\theta_1) \\
H^{(N)}(\theta_2) \\
\vdots \\
H^{(N)}(\theta_p)
\end{bmatrix}
\begin{bmatrix}
D^{(N)}(\theta_1) \\
D^{(N)}(\theta_2) \\
\vdots \\
D^{(N)}(\theta_p)
\end{bmatrix}
\begin{bmatrix}
\mu^{(0)} \\
\mu^{(1)} \\
\vdots \\
\mu^{(N)}
\end{bmatrix}
\]

(14)

where \(\bar{H}^{(k)}_p\) is the collection of moments of order \(k\) in each projection and \(\bar{D}^{(k)}_p\) is the matrix relating moments of order \(k\) of the object to moments of order \(k\) in each of the \(p\) projections. Now since the overall matrix is block diagonal the problem of determining each \(\mu^{(k)}\) decouples so that \(\bar{H}^{(k)}_p = \bar{D}^{(k)}_p \mu^{(k)}\). In particular, we can determine \(\mu^{(k)}\) uniquely if and only if the corresponding matrix block:

\[
\bar{D}^{(k)}_p
\]

has full column rank. Let us now examine the conditions when \(\bar{D}^{(k)}_p\) will have full column rank.

Substituting for \(D^{(k)}(\theta_i)\) from (7) we find that \(\bar{D}^{(k)}_p\) is of the form:

\[
\bar{D}^{(k)}_p = \begin{bmatrix}
\gamma_{k,0} \cos^k(\theta_1) & \gamma_{k,1} \cos^{k-1}(\theta_1) \sin(\theta_1) & \cdots & \gamma_{k,k-1} \cos(\theta_1) \sin^{k-1}(\theta_1) & \gamma_{k,k} \sin^k(\theta_1) \\
\gamma_{k,0} \cos^k(\theta_2) & \gamma_{k,1} \cos^{k-1}(\theta_2) \sin(\theta_2) & \cdots & \gamma_{k,k-1} \cos(\theta_2) \sin^{k-1}(\theta_2) & \gamma_{k,k} \sin^k(\theta_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma_{k,0} \cos^k(\theta_p) & \gamma_{k,1} \cos^{k-1}(\theta_p) \sin(\theta_p) & \cdots & \gamma_{k,k-1} \cos(\theta_p) \sin^{k-1}(\theta_p) & \gamma_{k,k} \sin^k(\theta_p)
\end{bmatrix}
\]

(17)

and the matrix \(\bar{D}^{(k)}_p\) is \(p \times (k + 1)\). Now \(\bar{D}^{(k)}_p\) will have full column rank if and only if its columns are independent. Note that we must have \(p > k\). The columns will be independent if and only if there is no set of \(\alpha_i\) (not all zero) such that:

\[
\alpha_0 \cos^k(\theta_1) + \alpha_1 \cos^{k-1}(\theta_1) \sin(\theta_1) + \cdots + \alpha_{k-1} \cos(\theta_1) \sin^{k-1}(\theta_1) + \alpha_k \sin^k(\theta_1) = 0 \quad \forall \theta_1, \ 1 \leq i \leq p
\]

(18)

In particular, this will be true (when \(p > k\)) if the homogeneous trigonometric polynomial of order \(k\) defined by (18) has at most \(k\) roots \(\theta\) in \([0, \pi]\). This motivates the following lemma, which we prove in Appendix A.
Lemma 1 (Roots of Homogeneous Trigonometric Polynomial of order k) A homogeneous trigonometric polynomial of order k of the following form:

\[ a_0 \cos^k(\theta) + a_1 \cos^{k-1}(\theta) \sin(\theta) + \ldots + a_{k-1} \cos(\theta) \sin^{k-1}(\theta) + a_k \sin^k(\theta) \]  

vanishes for at most k distinct values of \( \theta \) in \([0, \pi)\).

Given this lemma we see that \( \bar{D}_p^{(k)} \) will have full column rank if and only if \( p > k \), i.e. the number of projections \( p \) greater than the order \( k \) of the moment vector \( \mu^{(k)} \) that we are interested in! In particular, we can achieve this full column rank if \( p = k + 1 \), so we use no more than the first \( p \) moments of the projection. Also note that this implies that if \( \bar{D}_n^{(n)} \) is of full column rank for some \( n \) then so is \( \bar{D}_p^{(k)} \) for all \( k < n \).

Now the preceding arguments essentially show the first two statements of the theorem. Given \( p \) projections the matrices \( \bar{D}_p^{(k)} \) will have full column rank for \( 0 \leq k < p \). Thus we can uniquely find the moments \( \mu^{(k)} \) from the corresponding \( \bar{H}_p^{(k)} \) and \( \bar{D}_p^{(k)} \). In particular, note that due to the block diagonal structure of (15), using higher order moments of the projections is of no help in determining a given moment \( \mu^{(k)} \).

3 Conclusion

We have shown that to uniquely specify the first \( p \) moments of a function \( f(x, y) \) one needs exactly \( p \) integral projections at distinct angles in the interval \([0, \pi)\), and that this number is both necessary and sufficient. The importance of this result lies in the fact that it directly shows to what extent a limited number of projections of a function determine the function. From a practical standpoint this is quite important. In applications of the above result to tomographic reconstruction problems such as in Medical Imaging [1], our result shows, in a quantitative way, how well one can theoretically expect to reconstruct an object being imaged in the absence of noise. From a more abstract viewpoint, it sheds some light on the reconstructability of certain elementary binary objects from a limited number of tomographic projections. For instance, it is known [2, 12] that any binary ellipse is uniquely determined from 3 projections at distinct angles. It is also known that the moment sets \( \{\mu^{(0)}, \mu^{(1)}, \mu^{(2)}\} \) uniquely determine a binary ellipse in the plane [15]. Our result shows essentially that 3 projections suffice to uniquely determine the set of numbers \( \{\mu^{(0)}, \mu^{(1)}, \mu^{(2)}\} \) hence showing that 3 projections suffice to uniquely determine any binary ellipse, thereby providing an alternate proof.

A somewhat more subtle instance of the usefulness of our result occurs in the reconstruction of binary polygonal objects in the plane. A fascinating theorem due to Davis [16, 14] states that a triangle in the plane in uniquely determined by its moments of up to order 3. i.e.\( \{\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \mu^{(3)}\} \). Furthermore, Davis has, in essence, provided an explicit algorithm for reconstructing the triangle from this set of numbers. Our result would imply that exactly 4 projections are sufficient to determine this set of moments. Hence, together with the work of Davis, our result provides a closed form solution to the problem of reconstructing a triangular region in the plane from only 4 tomographic projections in the absence of noise.
Proof: Let $p(\theta)$ denote the homogeneous polynomial in question. i.e.

$$p(\theta) = \alpha_0 \cos^k(\theta) + \alpha_1 \cos^{k-1}(\theta) \sin(\theta) + \cdots + \alpha_{k-1} \cos(\theta) \sin^{k-1}(\theta) + \alpha_k \sin^k(\theta).$$

(20)

- **CASE I:** Assume that $p(\pi/2) \neq 0$. Then we can write $p(\theta)$ as

$$p(\theta) = \cos^k(\theta)q(\theta).$$

(21)

where $q(\theta)$ has no roots at $\theta = \pi/2$ and

$$q(\theta) = \alpha_0 + \alpha_1 \tan(\theta) + \cdots + \alpha_{k-1} \tan^{k-1}(\theta) + \alpha_k \tan^k(\theta).$$

(22)

Letting $u = \tan(\theta)$ we observe that the right hand side of (22) is simply a polynomial of order $k$ in $u$. By the Fundamental Theorem of Algebra [20], this polynomial has at most $k$ real roots. This is to say that there exist at most $k$ values $u_i \in \mathbb{R}$ such that $q(\tan^{-1}(u_i)) = 0$. Given this, we have that the roots of $q(\theta)$ are

$$\theta_i = \tan^{-1}(u_i).$$

(23)

We know that the function $\tan^{-1}$ is one-to-one over the interval $[0, \pi)$. Since $q(\pi/2) \neq 0$ by assumption, it follows that there exist at most $k$ angles $\theta_i \in [0, \pi)$ for which $q(\theta_i) = 0$.

- **CASE II:** Let $r_0(\theta) = p(\theta)$, and define the functions $r_i(\theta)$ for $1 \leq i \leq k$ as follows.

$$r_i(\theta) = \begin{cases} 
\frac{r_{i-1}(\theta)}{\cos(\theta)} & \text{if } r_{i-1}(\pi/2) = 0 \\
1 & \text{if } r_{i-1}(\pi/2) \neq 0
\end{cases}$$

(24)

If $p(\pi/2) = 0$, then we have

$$p(\pi/2) = \alpha_k \sin^k(\pi/2) = \alpha_k = 0.$$ 

(25)

Therefore we have that

$$p(\theta) = \cos(\theta)(\alpha_0 \cos^{k-1}(\theta) + \cdots + \alpha_{k-1} \sin^{k-1}(\theta)) = \cos(\theta)r_1(\theta)$$

(26)

If $r_1(\theta)$ does not vanish at $\pi/2$, Case I shows that it has at most $k - 1$ roots in $[0, \pi)$, which together with $\cos(\theta) = 0$ give at most $k$ roots for $p(\theta)$ in $[0, \pi)$.

From the definition of $r_1(\theta)$, it is clear that

$$r_1(\theta) = \begin{cases} 
\alpha_0 \cos^{k-1}(\theta) + \cdots + \alpha_{k-1} \sin^{k-1}(\theta) & \text{if } r_{i-1}(\pi/2) = 0 \\
1 & \text{if } r_{i-1}(\pi/2) \neq 0
\end{cases}$$

(27)

Now suppose that $r_i(\pi/2) = 0$ for $i = 0, 1, \cdots, n - 1$ and $r_n(\pi/2) \neq 0$, where $1 \leq n \leq k$. Again, from the definition of $r_i(\theta)$ it follows that

$$p(\theta) = \cos^n(\theta)r_n(\theta).$$

(28)

From Case I, $r_n(\theta)$ has at most $k - n$ roots in $[0, \pi)$, which along with $\cos^n(\theta) = 0$ give at most $k$ roots for $p(\theta)$ in $[0, \pi)$. 

$\square$
References


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