On Synthesizing Robust Decentralized Controllers *

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Abstract

Many advances have been made in the area of multidimension, centralized control in recent years. These technical advances have led to increased application in ever widening domains of complexity and scale. Unfortunately, the control of many large scale systems still present prohibitive costs in terms of instrumenting up centralized control solutions. Decentralized control continues to be a viable alternative when face with issues of complexity, limitations on computation and restrictions on implementation. However, a methodology for the systematic synthesis of robust decentralized controllers

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still remains a difficulty. This paper addresses the issue of placing the decentralized control problem in a μ-framework. The difficulties in developing a robust, concurrent decentralized design methodology is detailed. An alternative sequential design approach which leads to a D-K like alogorithm for the design of robust decentralized controllers is presented.

1 Introduction

The last decade has witness a phenomenal growth in computing power. This accessible computing capability has been exploited by new control methodologies which rely on it to develop sophisticated controllers capable of robustness in the face of model and disturbance uncertainty. These technical advances have led to increased application in ever widening domains of complexity and scale. Unfortunately, the control of many large scale systems still present prohibitive costs in terms of instrumenting up centralized control solutions. Issues of complexity, limitations on computation, ease of implementation and physical dimension continue to play a significant role in forcing the control engineer to place structural constraints on the feedback controller eventually used to control large scale systems.

Decentralized controllers still continue to be the most readily relied on method for structurally constrained controllers useful for large scale system control. Recently, a new generalized control paradigm has been developed which accounts for model uncertainty and provides for robust performance in the design of centralized controllers [1]. Linking the design of decentralized controllers to this paradigm would provide the distinct guarantees of synthesizing robust decentralized controllers. An important distinction to bear in mind is that the framework under which controllers are generated can make a considerable difference. For instance, designing decentralized controllers which provide for robust stability and good nominal performance does not in general guarantee that the overall system will achieve robust performance [2]. For these reasons it becomes important to provide a formulation of the decentralized control problem which can take advantage of the μ-synthesis framework

which currently supports the synthesis of robust centralized controllers.

This paper provides such a connection. In section 2 a recent parameterization for the set of all stabilizing decentralized controllers is introduced [3], [4]. From this set of controllers a subset is extracted which will prove to be useful in the development of robust decentralized controllers. Section 3 presents the necessary essentials for the development of a μ-framework. Section 4 places the decentralized control problem in the μ-framework and develops a decentralized, stable factor, $M(\cdot)$ operator. The problems in developing a concurrent algorithm for generating simultaneously the design parameters for robust controllers is detailed. And in section 5 a methodology for the development of sequential design of robust decentralized controllers is presented.

1.1   Notation

H	principle ideal domain
$U \subset H$	is the group of units of H
G	is the ring of fractions associated with H
$m(H)$	set of matrices with elements in H
$m(G)$	set of matrices with elements in G
$m(0)$	set of matrices whose elements are 0
$|F|$	determinant of F
unimodular	$F \in m(H)$ is unimodular iff $|F| \in U$
$\|\cdot\|$	refers to the H_∞ norm of enclosed operator

2 Parameterizations

A recent parameterization of the set of stabilizing decentralized controllers, [3], [4], will be presented in this section. From this parameterization, in section 2.1, a subset of stable decentralized controllers will be extracted. This subset will be found to be useful for the design of robust decentralized controllers in section 5. The parameterization to be given will be developed for the two channel case in order to allow the notation to be manageable. The definition for the two channel partition of a plant $P \in m(G)$ is as follows.

Definition 1 (Two Channel Partition) *For a plant $P \in m(G^{p \times q})$ with left coprime factorization, (l.c.f.), $(\tilde{D}_d, \tilde{N}_d)$ and right coprime factorization, (r.c.f), (N_d, D_d) the following represents a two channel partition of a plant where the input channel dimensions are q_1 and q_2 with $q = q_1 + q_2$ and the output channel dimensions are p_1 and p_2 with $p = p_1 + p_2$*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{1}$$

$$D_d = \begin{bmatrix} D_{d_{11}} & D_{d_{12}} \\ D_{d_{21}} & D_{d_{22}} \end{bmatrix} \quad N_d = \begin{bmatrix} N_{d_{11}} & N_{d_{12}} \\ N_{d_{21}} & N_{d_{22}} \end{bmatrix}$$

$$\tilde{D}_d = \begin{bmatrix} \tilde{D}_{d_{11}} & \tilde{D}_{d_{12}} \\ \tilde{D}_{d_{21}} & \tilde{D}_{d_{22}} \end{bmatrix} \quad \tilde{N}_d = \begin{bmatrix} \tilde{N}_{d_{11}} & \tilde{N}_{d_{12}} \\ \tilde{N}_{d_{21}} & \tilde{N}_{d_{22}} \end{bmatrix} \tag{2}$$

where $P_{11} \in m(G^{p_1 \times q_1})$, $P_{22} \in m(G^{p_2 \times q_2})$, $D_{d_{11}} \in m(H^{q_1 \times q_1})$, $D_{d_{22}} \in m(H^{q_2 \times q_2})$,

$\tilde{D}_{d_{11}} \in m(H^{p_1 \times p_1})$, $\tilde{D}_{d_{22}} \in m(H^{p_2 \times p_2})$, $N_{d_{11}}, \tilde{N}_{d_{11}} \in m(H^{p_1 \times q_1})$, $N_{d_{22}}, \tilde{N}_{d_{22}} \in m(H^{p_2 \times q_2})$ and

the other blocks have conforming dimensions.

A necessary and sufficient condition on the plant $P \in m(G)$ in order to be stabilizable by a decentralized controller is that the partitioned plant must satisfy the following identity.

$$\begin{bmatrix} \tilde{V}_1 & 0 & \tilde{U}_1 & 0 \\ 0 & \tilde{V}_2 & 0 & \tilde{U}_2 \\ -\tilde{N}_{d_{11}} & -\tilde{N}_{d_{12}} & \tilde{D}_{d_{11}} & \tilde{D}_{d_{12}} \\ -\tilde{N}_{d_{21}} & -\tilde{N}_{d_{22}} & \tilde{D}_{d_{21}} & \tilde{D}_{d_{22}} \end{bmatrix} \begin{bmatrix} D_{d_{11}} & D_{d_{12}} & -U_1 & 0 \\ D_{d_{21}} & D_{d_{22}} & 0 & -U_2 \\ N_{d_{11}} & N_{d_{12}} & V_1 & 0 \\ N_{d_{21}} & N_{d_{22}} & 0 & V_2 \end{bmatrix} = I \qquad (3)$$

For some $\tilde{V}_{bd} = blkdiag[\tilde{V}_1, \tilde{V}_2]$, $\tilde{U}_{bd} = blkdiag[\tilde{U}_1, \tilde{U}_2]$, $V_{bd} = blkdiag[V_1, V_2]$ and $U_{bd} = blkdiag[U_1, U_2]$ where $\tilde{V}_i \in m(H^{q_i \times q_i})$, $V_i \in m(H^{p_i \times p_i})$, $\tilde{U}_i, U_i \in m(H^{q_i \times p_i})$. Equation (3) is referred to as the decentralized doubly coprime Bezout identity, (DDCBI), and implicit in this identity is the result that any plant which satisfies eq. (3) contains no unstable fixed modes [5].

Now that the decentralized doubly coprime Bezout identity (DDCBI), eq. (3), has been defined the parameterization of all stabilizing decentralized controllers can be stated.

Theorem 1 (Parameterization of Decentralized Controllers) *For a plant $P \in m(G)$ which satisfies the partition of definition 1 and for which a DDCBI as in eq. (3) can be established, the parameterized set of stabilizing decentralized controllers is given by the following left and right coprime expansions. Expansion of the parameterization into left coprime*

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parameterized factors is as follows

$$C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{C_1}^{-1} \tilde{N}_{C_1} & 0 \\ 0 & \tilde{D}_{C_2}^{-1} \tilde{N}_{C_2} \end{bmatrix} \tag{4}$$

$$\tilde{D}_{C_1}^{-1} \tilde{N}_{C_1} = (Q_{11} \tilde{V}_1 - Q_1 \tilde{N}_{d_{11}})^{-1} (Q_{11} \tilde{U}_1 + Q_1 \tilde{D}_{d_{11}}), \quad |Q_{11} \tilde{V}_1 - Q_1 \tilde{N}_{d_{11}}| \neq 0 \tag{5}$$

$$\tilde{D}_{C_2}^{-1} \tilde{N}_{C_2} = (Q_{22} \tilde{V}_2 - Q_2 \tilde{N}_{d_{22}})^{-1} (Q_{22} \tilde{U}_2 + Q_2 \tilde{D}_{d_{22}}), \quad |Q_{22} \tilde{V}_2 - Q_2 \tilde{N}_{d_{22}}| \neq 0 \tag{6}$$

where the individual parameters must be selected such that the following operator is unimodular

$$Q_u = \begin{bmatrix} Q_{11} & Q_1 W_{12} \\ Q_2 W_{21} & Q_{22} \end{bmatrix} \tag{7}$$

and where W_{12} and W_{21} are composed of stable factors from the partitioned plant as follows

$$\begin{aligned} W_{12} &= -\tilde{N}_{d_{11}} D_{d_{12}} + \tilde{D}_{d_{11}} N_{d_{12}} = \tilde{N}_{d_{12}} D_{d_{22}} - \tilde{D}_{d_{12}} N_{d_{22}} \\ W_{21} &= -\tilde{N}_{d_{22}} D_{d_{21}} + \tilde{D}_{d_{22}} N_{d_{21}} = \tilde{N}_{d_{21}} D_{d_{11}} - \tilde{D}_{d_{21}} N_{d_{11}} \end{aligned} \tag{8}$$

Expansion of the parameterization into right coprime parameterized factors is as follows

$$C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} N_{C_1} D_{C_1}^{-1} & 0 \\ 0 & N_{C_2} D_{C_2}^{-1} \end{bmatrix} \tag{9}$$

$$N_{C_1} D_{C_1}^{-1} = (U_1 \hat{Q}_{11} + D_{d_{11}} \hat{Q}_1)(V_1 \hat{Q}_{11} - N_{d_{11}} \hat{Q}_1)^{-1}, \quad |V_1 \hat{Q}_{11} - N_{d_{11}} \hat{Q}_1| \neq 0 \tag{10}$$

$$N_{C_2} D_{C_2}^{-1} = (U_2 \hat{Q}_{22} + D_{d_{22}} \hat{Q}_2)(V_2 \hat{Q}_{22} - N_{d_{22}} \hat{Q}_2)^{-1}, \quad |V_2 \hat{Q}_{22} - N_{d_{22}} \hat{Q}_2| \neq 0 \qquad (11)$$

where the individual parameters must be selected such that the following operator is unimodular

$$\hat{Q}_u = \begin{bmatrix} \hat{Q}_{11} & W_{12} \hat{Q}_2 \\ W_{21} \hat{Q}_1 & \hat{Q}_{22} \end{bmatrix} \qquad (12)$$

with the individual parameters being members of the following stable matrix rings

$$Q_{11} \in m(H^{q_1 \times q_1}), \quad Q_{22} \in m(H^{q_2 \times q_2}), \quad \hat{Q}_{11} \in m(H^{p_1 \times p_1})$$
$$\hat{Q}_{22} \in m(H^{p_2 \times p_2}), \quad Q_1, \hat{Q}_1 \in m(H^{q_1 \times p_1}), \quad Q_2, \hat{Q}_2 \in m(H^{q_2 \times p_2}) \qquad (13)$$

Proof of theorem 1 is available in [3], [4]. From the controllers of theorem 1 a specialized subset of decentralized controllers will be extracted which will find use in a iterative scheme for the development of robust decentralized controllers in section 4. In order to develop this specialized set of decentralized controllers, properties concerning auxiliary Bezout identities must first be presented.

2.1 Reliance on Auxiliary Bezout Identities

The proof of theorem 1 is dependent on the use of auxiliary doubly coprime Bezout identities (ADCBI) which follow directly from the decentralized doubly coprime Bezout identity (DD-CBI), eq. (3). These auxiliary doubly coprime Bezout identities are given in the following corollary.

Corollary 1 (Auxiliary Doubly Coprime Bezout Identities) *The stable factors which satisfy the DDCBI (eq. (3)) also satisfy the following auxiliary doubly coprime Bezout identities (ADCBI)*

$$
\begin{bmatrix} \tilde{V}_1 & \tilde{U}_1 \\ -\tilde{N}_{d_{11}} & \tilde{D}_{d_{11}} \end{bmatrix} \begin{bmatrix} D_{d_{11}} & -U_1 \\ N_{d_{11}} & V_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
$$
$$
\begin{bmatrix} \tilde{V}_2 & \tilde{U}_2 \\ -\tilde{N}_{d_{22}} & \tilde{D}_{d_{22}} \end{bmatrix} \begin{bmatrix} D_{d_{22}} & -U_2 \\ N_{d_{22}} & V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{14}
$$

These auxiliary identities indicate that not only does the overall compensator, expressed by say the stable factors $U_{bd} = blkdiag[U_1, U_2]$ and $V_{bd} = blkdiag[V_1, V_2]$, stabilize the plant P as indicated by DDCBI, eq. (3), but the individual subcompensators by satisfying the ADCBI of corollary 1 stabilize fictitious plant operators formed from the main diagonal (see eq. (2)) of the decentralized stable plant factors, (i.e. $(N_{d_{11}}, D_{d_{11}})$ and $(N_{d_{22}}, D_{d_{22}})$). Note, that if the plant was decoupled it would be immediately obvious that the above auxiliary doubly coprime Bezout identities would be satisfied. This follows since the aforementioned fictitious plant operators would no longer be fictitious. They would correspond to the stable factors associated with the individual plant operators P_{11} and P_{22} of the decoupled plant. And the individual subcontrollers would be the respective stabilizing controllers for P_{11} and P_{22}. It is less obvious that the auxiliary doubly coprime identities should hold for a plant with coupling, but when the plant stable factors are placed in a form which satisfies eq. (3), the auxiliary properties, eq. (14), can be shown to be true. Proof of corollary 1 is as follows.

Proof The following three equations are immediately available from eq. (3).

$$\begin{bmatrix} \tilde{V}_1 & \tilde{U}_1 \end{bmatrix} \begin{bmatrix} D_{d_{11}} \\ N_{d_{11}} \end{bmatrix} = I$$

$$\begin{bmatrix} -\tilde{N}_{d_{11}} & \tilde{D}_{d_{11}} \end{bmatrix} \begin{bmatrix} -U_1 \\ V_1 \end{bmatrix} = I$$

$$\begin{bmatrix} \tilde{V}_1 & \tilde{U}_1 \end{bmatrix} \begin{bmatrix} -U_1 \\ V_1 \end{bmatrix} = 0 \tag{15}$$

Also directly available from eq. (3) is the following relation

$$\tilde{N}_{d_{12}} U_2 + \tilde{D}_{d_{12}} V_2 = 0 \tag{16}$$

Operating on the left by $V_2^{-1} N_{d_{21}}$ gives

$$\tilde{N}_{d_{12}} U_2 V_2^{-1} N_{d_{21}} + \tilde{D}_{d_{12}} N_{d_{21}} = 0 \tag{17}$$

Using the relation $C_2 = \tilde{V}_2^{-1} \tilde{U}_2 = U_2 V_2^{-1}$ we obtain

$$\tilde{N}_{d_{12}} \tilde{V}_2^{-1} \tilde{U}_2 N_{d_{21}} + \tilde{D}_{d_{12}} N_{d_{21}} = 0 \tag{18}$$

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Applying the following relation (which is also from the DDCBI, eq. (3))

$$\tilde{V}_2 D_{d_{21}} + \tilde{U}_2 N_{d_{21}} = 0 \implies \tilde{U}_2 N_{d_{21}} = -\tilde{V}_2 D_{d_{21}} \tag{19}$$

to eq. (18) yields

$$-\tilde{N}_{d_{12}} D_{d_{21}} + \tilde{D}_{d_{12}} N_{d_{21}} = 0 \tag{20}$$

From eq. (3) we have that

$$-\tilde{N}_{d_{11}} D_{d_{11}} + \tilde{D}_{d_{11}} N_{d_{11}} - \tilde{N}_{d_{12}} D_{d_{21}} + \tilde{D}_{d_{12}} N_{d_{21}} = 0 \tag{21}$$

By application of eq. (20) to eq. (21) we obtain

$$\begin{bmatrix} -\tilde{N}_{d_{11}} & \tilde{D}_{d_{11}} \end{bmatrix} \begin{bmatrix} D_{d_{11}} \\ N_{d_{11}} \end{bmatrix} = 0 \tag{22}$$

Combining eq. (22) with eq. (15) gives the following

$$\begin{bmatrix} \tilde{V}_1 & \tilde{U}_1 \\ -\tilde{N}_{d_{11}} & \tilde{D}_{d_{11}} \end{bmatrix} \begin{bmatrix} D_{d_{11}} & -U_1 \\ N_{d_{11}} & V_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

□

The proof for the other auxiliary Bezout identity in corollary 1 is completely analogous.

Left DIP	Right DIP
$W_{12}\tilde{V}_2 = \tilde{N}_{d_{12}}$	$U_1 W_{12} = -D_{d_{12}}$
$W_{21}\tilde{V}_1 = \tilde{N}_{d_{21}}$	$U_2 W_{21} = -D_{d_{21}}$
$W_{12}\tilde{U}_2 = -\tilde{D}_{d_{12}}$	$V_1 W_{12} = N_{d_{12}}$
$W_{21}\tilde{U}_1 = -\tilde{D}_{d_{21}}$	$V_2 W_{21} = N_{d_{21}}$

Table 1: Decentralized Interaction Properties (DIP)

As mentioned earlier the ADCBI are used in the proofs of theorem 1. In section 2.2 they will be used in the parameterization of a special class of decentralized compensators. In addition the ADCBI are used in establishing a set of relations between the interaction terms, W_{12} and W_{21}, and the decentralized stable plant factors (see eq.(2)). These relations are used in the necessary and sufficient parts of the proofs for theorem 1 and will be used in section 4 to simplify stable factor terms. Although, the relationships were never given explicitly in reference [4] an analysis of the proofs given in that reference indicate that a number of algebraic relationships in the proofs relied on these properties being true. For completeness these properties are collected in table 1 and will be referred to as the decentralized interaction properties (DIP). These properties are derived by applying the definitions of W_{12}, W_{21}, (see eq. (8)) and the ADCBI (see corollary 1).

2.2 Decentralized Controllers Which Always Satisfy ADCBI

This section is devoted to characterizing a subclass of stabilizing decentralized controllers which are useful in autonomous design methods and in design methods based on iteration.

These controllers will be used in section 5. The subclass of controllers is defined by imposing a unimodular restriction on the parameters used in the decentralized parameterization of theorem 1. The parameters affected by this restriction are given in the following definition.

Definition 2 (Unimodular Parameter Restriction (UPR)) *For the set of parameters satisfying the decentralized parameterization, (theorem 1), unimodular parameter restriction (UPR) refers to constraining the parameters, Q_{11}, Q_{22}, $\hat{Q}_{11}$, and $\hat{Q}_{22}$ to being unimodular.*

An important relationship between the parameters established in [4] is the following.

$$Q_1 \hat{Q}_{11} = Q_{11} \hat{Q}_1 \tag{23}$$

$$Q_2 \hat{Q}_{22} = Q_{22} \hat{Q}_2 \tag{24}$$

For the case involving UPR these relationships become

$$Q_{11}^{-1} Q_1 = \hat{Q}_1 \hat{Q}_{11}^{-1} \quad \text{where} \quad Q_{11}^{-1} Q_1, \; \hat{Q}_1 \hat{Q}_{11}^{-1} \in m(H) \tag{25}$$

$$Q_{22}^{-1} Q_2 = \hat{Q}_2 \hat{Q}_{22}^{-1} \quad \text{where} \quad Q_{22}^{-1} Q_2, \; \hat{Q}_2 \hat{Q}_{22}^{-1} \in m(H) \tag{26}$$

The following theorem 2 shows that the UPR leads to a set of subcontrollers which always satisfies an ADCBI.

Theorem 2 (Subcontrollers Which Always Satisfy ADCBI) *Given the decentralized parameterization, (theorem 1), selecting a subset of the parameters to satisfy UPR, (definition 2), results in subcontrollers which satisfy a parameterized ADCBI, (corollary 1).*

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Proof Starting with the following ADCBI, for $i = 1, 2$.

$$\begin{bmatrix} \tilde{V}_i & \tilde{U}_i \\ -\tilde{N}_{d_{ii}} & \tilde{D}_{d_{ii}} \end{bmatrix} \begin{bmatrix} D_{d_{ii}} & -U_i \\ N_{d_{ii}} & V_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{27}$$

Operating on the left by $\quad \begin{bmatrix} Q_{ii} & Q_i \\ 0 & I \end{bmatrix} \quad$ and on the right by $\quad \begin{bmatrix} I & -\hat{Q}_i \\ 0 & \hat{Q}_{ii} \end{bmatrix} \tag{28}$

yields the following

$$\begin{bmatrix} Q_{ii}\tilde{V}_i - Q_i\tilde{N}_{d_{ii}} & Q_{ii}\tilde{U}_i + Q_i\tilde{D}_{d_{ii}} \\ -\tilde{N}_{d_{ii}} & \tilde{D}_{d_{ii}} \end{bmatrix} \begin{bmatrix} D_{d_{ii}} & -(D_{d_{ii}}\hat{Q}_i + U_i\hat{Q}_{ii}) \\ N_{d_{ii}} & V_i\hat{Q}_{ii} - N_{d_{ii}}\hat{Q}_i \end{bmatrix} =$$

$$\begin{bmatrix} Q_{ii} & Q_i\hat{Q}_{ii} - Q_{ii}\hat{Q}_i \\ 0 & \hat{Q}_{ii} \end{bmatrix} \tag{29}$$

Equations (23)-(24) imply

$$Q_i\hat{Q}_{ii} - Q_{ii}\hat{Q}_i = 0 \tag{30}$$

Since Q_{ii} and $\hat{Q}_{ii}$ are unimodular by UPR, the operator $blkdiag[Q_{ii}, \hat{Q}_{ii}]$ is unimodular.

Operating on eq. (29) from the right by the stable inverse of $blkdiag[Q_{ii}, \hat{Q}_{ii}]$ yields

$$\begin{bmatrix} Q_{ii}\tilde{V}_i - Q_i\tilde{N}_{d_{ii}} & Q_{ii}\tilde{U}_i + Q_i\tilde{D}_{d_{ii}} \\ -\tilde{N}_{d_{ii}} & \tilde{D}_{d_{ii}} \end{bmatrix} \begin{bmatrix} D_{d_{ii}}Q_{ii}^{-1} & -(D_{d_{ii}}\hat{Q}_i + U_i\hat{Q}_{ii})\hat{Q}_{ii}^{-1} \\ N_{d_{ii}}Q_{ii}^{-1} & (V_i\hat{Q}_{ii} - N_{d_{ii}}\hat{Q}_i)\hat{Q}_{ii}^{-1} \end{bmatrix} = I$$

which is a parameterized version of ADCBI for both $i = 1, 2$.

□

The following theorem gives the subclass of decentralized controllers which always satisfy ADCBI and result from applying the unimodular parameter restriction.

Theorem 3 (Unimodular Parameter Restricted Controllers (UPRC)) *Given the form of stabilizing decentralized controllers, (theorem 1), applying the UPR, (definition 2), leads to the following subset of stabilizing decentralized controllers.*

Expansion of the parameterization into left coprime parameterized factors is as follows

$$C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{C_1}^{-1}\tilde{N}_{C_1} & 0 \\ 0 & \tilde{D}_{C_2}^{-1}\tilde{N}_{C_2} \end{bmatrix} \tag{31}$$

$$\tilde{D}_{C_1}^{-1}\tilde{N}_{C_1} = (\tilde{V}_1 - \bar{Q}_1\tilde{N}_{d_{11}})^{-1}(\tilde{U}_1 + \bar{Q}_1\tilde{D}_{d_{11}}), \quad |\tilde{V}_1 - \bar{Q}_1\tilde{N}_{d_{11}}| \neq 0 \tag{32}$$

$$\tilde{D}_{C_2}^{-1}\tilde{N}_{C_2} = (\tilde{V}_2 - \bar{Q}_2\tilde{N}_{d_{22}})^{-1}(\tilde{U}_2 + \bar{Q}_2\tilde{D}_{d_{22}}), \quad |\tilde{V}_2 - \bar{Q}_2\tilde{N}_{d_{22}}| \neq 0 \tag{33}$$

where the individual parameters must be selected such that the following operator is unimodular

$$\begin{bmatrix} I & \bar{Q}_1 W_{12} \\ \bar{Q}_2 W_{21} & I \end{bmatrix} \tag{34}$$

Expansion of the parameterization into right coprime parameterized factors is as follows

$$C_d = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} N_{C_1}D_{C_1}^{-1} & 0 \\ 0 & N_{C_2}D_{C_2}^{-1} \end{bmatrix} \tag{35}$$

$$N_{C_1} D_{C_1}^{-1} = (U_1 + D_{d_{11}} \bar{Q}_1)(V_1 - N_{d_{11}} \bar{Q}_1)^{-1}, \quad |V_1 - N_{d_{11}} \bar{Q}_1| \neq 0 \qquad (36)$$

$$N_{C_2} D_{C_2}^{-1} = (U_2 + D_{d_{22}} \bar{Q}_2)(V_2 - N_{d_{22}} \bar{Q}_2)^{-1}, \quad |V_2 - N_{d_{22}} \bar{Q}_2| \neq 0 \qquad (37)$$

where the individual parameters must be selected such that the following operator is unimodular

$$\begin{bmatrix} I & W_{12} \bar{Q}_2 \\ W_{21} \bar{Q}_1 & I \end{bmatrix} \qquad (38)$$

Where $\bar{Q}_1 \in m(H^{q_1 \times p_1})$ and $\bar{Q}_2 \in m(H^{q_2 \times p_2})$

Proof   The form of the left coprime parameterized subcompensators of eq. (5) and eq. (6) can be rewritten

$$
\begin{aligned}
C_i &= (Q_{ii} \tilde{V}_i - Q_i \tilde{N}_{d_{ii}})^{-1}(Q_{ii} \tilde{U}_i + Q_i \tilde{D}_{d_{ii}}) \qquad i = 1, 2 \\
&= (Q_{ii}(\tilde{V}_i - Q_{ii}^{-1} Q_i \tilde{N}_{d_{ii}}))^{-1}(Q_{ii} \tilde{U}_i + Q_i \tilde{D}_{d_{ii}}) \\
&= (\tilde{V}_i - Q_{ii}^{-1} Q_i \tilde{N}_{d_{ii}})^{-1}(\tilde{U}_i + Q_{ii}^{-1} Q_i \tilde{D}_{d_{ii}}) \\
&= (\tilde{V}_i - \bar{Q}_i \tilde{N}_{d_{ii}})^{-1}(\tilde{U}_i + \bar{Q}_i \tilde{D}_{d_{ii}}) \qquad (39)
\end{aligned}
$$

where $\bar{Q}_i = Q_{ii}^{-1} Q_i \in m(H)$, for $i = 1, 2$, since Q_{ii} is constrained to be unimodular. The form of eq. (39) is UPRC, (see eq. (32-33)), the unimodular operator constraint, (eq. (34)), is obtained as follows. By rewriting the unimodular operator of eq. (7) the following is obtained

$$Q_u = \begin{bmatrix} Q_{11} & Q_1 W_{12} \\ Q_2 W_{21} & Q_{22} \end{bmatrix}$$

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$$= \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \begin{bmatrix} I & Q_{11}^{-1}Q_1 W_{12} \\ Q_{22}^{-1}Q_2 W_{21} & I \end{bmatrix} \qquad (40)$$

Since Q_{11} and Q_{22} are unimodular, $blkdiag[Q_{11}, Q_{22}]$ is also unimodular. Hence the above unimodular constraint becomes

$$Q_u \quad \text{unimodular} \quad \Longleftrightarrow \quad \begin{bmatrix} I & \bar{Q}_1 W_{12} \\ \bar{Q}_2 W_{21} & I \end{bmatrix} \quad \text{unimodular} \qquad (41)$$

$\square$

The proof for the right coprime UPRC, (see eq. (36)-(38)), is completely analogous.

3   Essentials of Robust Stability/Performance

In this section the essential tools needed for using the μ-framework will be defined. The source for this material and many more of the details is available from [1], [2]. Placing the decentralized problem in this framework will then be demonstrated with the use of a specific example in section 4.

A preliminary element needed for a robust framework is a means to incorporate model uncertainties and modeling errors associated with the nominal plant model. Within the generalized robustness framework uncertainty is modeled via a norm bounded perturbation and a scalar weight. The weight is usually restricted to be a unit, $W_i \in U$, which for continuous time definitions implies that the weight is usually restricted to be a real-rational

transfer function which is stable, minimum phase, proper and has an inverse which is also stable, minimum phase and proper [6]. The uncertainty modeling perturbations in this section will be restricted to $\Delta_i \in m(H)$ which for continuous time systems implies LTI, stable. In addition the perturbations will be constrained to satisfy a unity norm bound, $||\Delta_i|| \leq 1$. Given this perturbation form of modeling uncertainty, the following are examples of plant sets and their associated modeling assumptions.

Plants modeled using multiplicative input uncertainty: $P \in \{P_n(I + W_I \Delta_I)\}$

Plants modeled using multiplicative output uncertainty: $P \in \{(I + W_O \Delta_I)P_n\}$

Plants modeled using additive uncertainty: $P \in \{P_n + W_A \Delta_A\}$

Figure 1 represents the generalized control problem formulation where the nominal plant, G, is modified to effectively include a set of plants formed by the perturbation approach to modeling uncertainty. Viewing the generalized plant G as being partitioned in the following manner

$$\begin{bmatrix} \begin{bmatrix} b \\ z \end{bmatrix} \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a \\ w \end{bmatrix} \\ u \end{bmatrix} \tag{42}$$

allows formulating the set of systems to be controlled in the following way

$$\{F_u(G, \Delta_u) : \Delta_u \in m(H), ||\Delta_u|| \leq 1\} \tag{43}$$

where Δ_u is the perturbation used to model uncertainty in the plant and $F_u(\cdot)$ is an upper
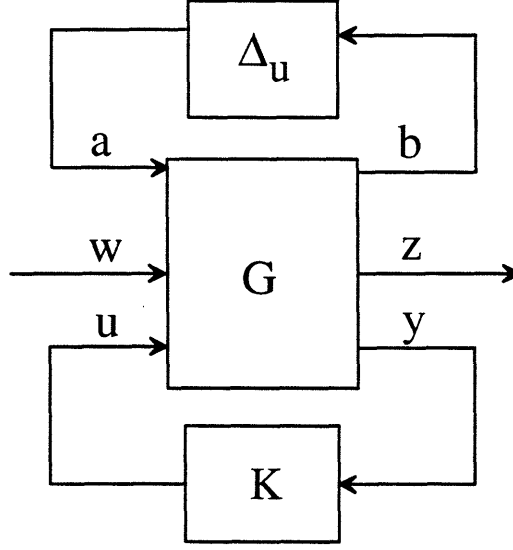
Figure 1: Block diagram of the formulation for the general control problem

linear fractional transformation (LFT) given by

$$F_u(G, \Delta_u) = G_{22} + G_{21}\Delta_u(I - G_{11}\Delta_u)^{-1}G_{12} \tag{44}$$

An important LFT which can be extracted from the general formulation using the 2×2 partition of the general plant G, eq. (42), is

$$M(G, K) := F_l(G, K) \tag{45}$$

Where $F_l(\cdot)$ is a lower LFT given by

$$F_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{46}$$
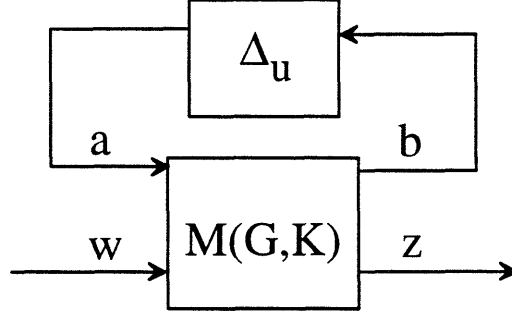
Figure 2: M-system With Uncertainty Perturbation Loop Closed

The $M(\cdot)$ designation is commonly used for this LFT in the literature and this particular

operator will be the one use in analysis tests to determine whether the system is meeting

the desired robust stability and performance under closed loop control. Figure 2 illustrates

the general control formulation in terms of the $M(\cdot)$ operator. The following LFT

$$F_u(M(G, K), \Delta_u) = M_{22} + (M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12} \qquad (47)$$

represents the nominal performance operator perturbed by the model uncertainty. If there is

no model uncertainty, (i.e. $||\Delta_u|| = 0$), eq. (47) reduces to the original nominal performance

operator, T_{zw}. Or in other words: $M_{22} = T_{zw}$. Other elements of the $M(\cdot)$ operator yield

the following facts [1].

Nominal performance is satisfied if and only if $||M_{22}|| < 1$

System is robustly stable if and only if $||M_{11}|| < 1$

Finally, since the nominal performance objective is $||T_{zw}|| < 1$, the robust performance

objective is to try and maintain this performance in the face of the uncertainty perturbation, or specifically the LFT of eq. (47) should be less than one for all unity norm bound perturbations. This is expressed by the following equation.

$$||F_u(M(G,K), \Delta_u)|| < 1 \quad \text{for all} \quad ||\Delta_u|| \leq 1 \tag{48}$$

Equation (48) represents our optimization objective. In terms of analysis however the norm bounded LFT of eq. (48) represents a difficulty due to the dependence on the uncertainty perturbation, Δ_u. What is needed is an analysis tool which operates on an expression independent of Δ_u but indicates when the objective of eq. (48) is satisfied.

Such a tool exists, it is referred to as the structured singular value. To provide a definition useful for the robust control problems we desire to solve, the perturbation structure used in the general control formulation must be augmented. Figure 3 shows how connecting a fictitious perturbation, (denoted Δ_p for performance perturbation) between the performance output, z, and the performance input, w, produces overall a closed loop consisting of a structured perturbation operator, Δ, and the $M(\cdot)$ operator which represents the "known" closed loop system. By "known" closed loop system we mean that $M(\cdot)$ has the performance and uncertainty weighting functions reflected into it, contains the nominal performance operator, the nominal plant operator and other operators resulting from the uncertainty structure. The structured perturbation, Δ, is an element of the following set
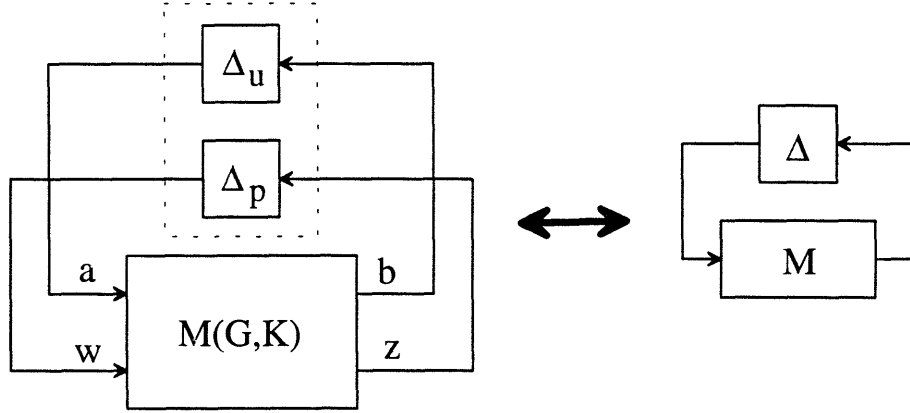
Figure 3: M-system With Uncertainty and Performance Perturbation Loops Closed

$$\Delta = \left\{ \begin{bmatrix} \Delta_u & 0 \\ \\ 0 & \Delta_p \end{bmatrix} \mid \Delta_u \in \Delta_u, \Delta_p \in m(H^{n_w \times n_z}) \right\} \tag{49}$$

Placing the generalized control problem in the form of a $M(\cdot)$-Δ closed loop will allow applying a metric on the $M(\cdot)$ operator to assess whether the desired robustness properties under closed loop control have been achieved. The following definition is a operator equivalent definition for the structured singular value.

Definition 3 (Structured Singular Value) *For* $\Delta \in \Delta$ *and* $M \in m(H)$ *the Structured Singular Value is a map from the matrix ring,* $m(H)$, *of stable operators to the positive reals and is defined as*

$$\mu_\Delta(M) = \left[\inf_\Delta \{ \|\Delta\| \mid (I - M\Delta) \quad \text{is no longer unimodular} \} \right]^{-1} \tag{50}$$

If for every $\Delta \in \Delta$, $(I - M\Delta)$ *is unimodular, then* $\mu_\Delta(M) := 0$.

Using definition 3 the following robustness theorem is obtained.

Theorem 4 (Robust Stability/Performance Test) *The generalized control system, figure 1, is stable and satisfies the perturbed performance objective of eq. (48) for all $\Delta_u \in \Delta_u$ iff $M(G,K)$ is an element of $m(H)$ and the following condition holds*

$$\mu_\Delta(M(G,K)) < 1 \tag{51}$$

Due to a "maximum-modulus-like" theorem associated with linear fractional transformations, [7], μ-robustness tests for continuous time systems reduces to one dimensional searches along the $j\omega$ axis. The robustness theorem for this continuous time case then becomes

Theorem 5 *Robust Stability/Performance is guaranteed iff*

$$\max_\omega \mu_\Delta(M(G,K)(j\omega)) < 1 \tag{52}$$

Note the structured singular value used in thm. 5 is defined in terms of complex matrices at each frequency. This definition is commonly given as

Definition 4 (μ_Δ in terms of complex matrices) *For $M \in \mathbf{C}^{n\times n}$*

$$\mu_\Delta(M) = [\min\{\bar{\sigma}(\Delta) \mid \Delta \in \Delta, det(I - M\Delta) = 0\}]^{-1} \tag{53}$$

If for every $\Delta \in \Delta$, $(I - M\Delta)$ nonsingular, then $\mu_\Delta(M) := 0$.

The set, Δ, for definition 4 is defined as, $\Delta \in \mathbf{C}^{n \times n}$ with

$$\Delta = \{blkdiag[\delta_1 I_{r_1}, \cdots, \delta_s I_{r_s}, \Delta_1, \cdots, \Delta_f] \mid \delta_i \in \mathbf{C}$$

$$\Delta_j \in \mathbf{C}^{m_j \times m_j}, \quad 1 \leq i \leq s, 1 \leq j \leq f\} \tag{54}$$

The formulation of the structured singular value in terms of complex matrices plays an important role in numerical computations. In section 5 a convex upper bound calculation for $\mu_\Delta(\cdot)$ will be given. This upper bound also plays an additional role in the synthesis of robust controllers and this also will be discussed in section 5.

4   Placing the Decentralized Problem in the μ-Framework

Given the background provided in section 3, the decentralized control problem can now be placed in the μ-framework through the use of a specific example. Figure 4 is a representative robust control problem. Model uncertainty is given in the form of output multiplicative uncertainty perturbation indicated by the scalar weight W_u and the uncertainty perturbation Δ_u. The performance operator will effectively be a input sensitivity transfer function matrix scaled by W_p. In order to develop the M operator to be used for robust analysis, as indicated in section 3, a fictitious unity norm bound performance perturbation, Δ_p, is included in the control setup of figure 4. M is a map of the perturbation outputs to the perturbation inputs
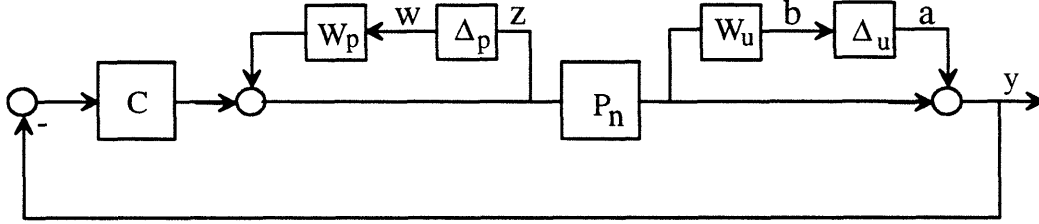
Figure 4: Robust Control Problem

and is defined as

$$
M : \begin{bmatrix} a \\ w \end{bmatrix} \longrightarrow \begin{bmatrix} b \\ z \end{bmatrix} \tag{55}
$$

For LTI systems, the elements of the 2×2 M operator can be found by breaking the loops associated with the perturbations and finding the transfer matrices, T_{ba}, T_{bw}, T_{za}, and T_{zw} which result from the four input/output combinations of the uncertainty and performance perturbations, Δ_p and Δ_u. These individual transfer matrices, which are elements of M, take on the following values.

$$
\begin{aligned}
M_{11} &= T_{ba} = -W_u P_n (I - C P_n)^{-1} C \\
M_{12} &= T_{bw} = W_u P_n (I + C P_n)^{-1} W_p \\
M_{21} &= T_{za} = -(I + C P_n)^{-1} C \\
M_{22} &= T_{zw} = (I + C P_n)^{-1} W_p
\end{aligned} \tag{56}
$$

Which implies the M operator has the following form.

$$M = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -P_n(I - CP_n)^{-1}C & P_n(I + CP_n)^{-1} \\ -(I + CP_n)^{-1}C & (I + CP_n)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix} \tag{57}$$

The M operator can be written in terms of stable coprime factors of the nominal plant, P_n. To prove this the following equivalence between a sensitivity transfer matrix and its stable factor form must be observed [8].

$$(I + CP_n)^{-1} = D(\tilde{V} - Q\tilde{N}) \tag{58}$$

Application of eq. (58) to the elements of the M operator, eq. (57), along with parameterized stable factor form of C, [8], and stable factors of P_n, (i.e. $P_n = \tilde{D}^{-1}\tilde{N} = ND^{-1}$), yield the following stable factor form for M.

$$M = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N(\tilde{U} + Q\tilde{D}) & N(\tilde{V} - Q\tilde{N}) \\ -D(\tilde{U} + Q\tilde{D}) & D(\tilde{V} - Q\tilde{N}) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix} \tag{59}$$

The M operator associated with the decentralized problem is obtained by assuming the nominal plant, $P_n \in m(G)$, satisfies the two channel partition of eq. (1) without inducing any unstable fixed modes. A decentralized doubly coprime Bezout identity (DDCBI) of the form found in eq. (3) then exists for the nominal plant, P_n. The expression for the operator M, eq. (59), can then be rewritten in terms of the decentralized stable factors satisfying the

DDCBI.

$$M = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d & N_d \\ -D_d & D_d \end{bmatrix} \begin{bmatrix} (\tilde{U}_{bd} + Q\tilde{D}_d) & 0 \\ 0 & (\tilde{V}_{bd} - Q\tilde{N}_d) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix} \quad (60)$$

Substituting in the decentralizing parameters from eq. (5)-(7) yields the following expressions for $(\tilde{U}_{bd} + Q\tilde{D}_d)$ and $(\tilde{V}_{bd} - Q\tilde{N}_d)$.

$$(\tilde{U}_{bd} + Q\tilde{D}_d) = Q_u^{-1} \begin{bmatrix} Q_{11}\tilde{U}_1 + Q_1\tilde{D}_{d_{11}} & Q_1[W_{12}\tilde{U}_2 + \tilde{D}_{d_{12}}] \\ Q_2[W_{21}\tilde{U}_1 + \tilde{D}_{d_{21}}] & Q_{22}\tilde{U}_2 + Q_2\tilde{D}_{d_{12}} \end{bmatrix} \quad (61)$$

$$(\tilde{V}_{bd} - Q\tilde{N}_d) = Q_u^{-1} \begin{bmatrix} Q_{11}\tilde{V}_1 + Q_1\tilde{N}_{d_{11}} & Q_1[W_{12}\tilde{V}_2 - \tilde{N}_{d_{12}}] \\ Q_2[W_{21}\tilde{V}_1 - \tilde{N}_{d_{21}}] & Q_{22}\tilde{V}_2 + Q_2\tilde{N}_{d_{12}} \end{bmatrix} \quad (62)$$

Application of left decentralized interaction properties from table 1 results in the following simplifications for eqs. (61)-(62).

$$(\tilde{U}_{bd} + Q\tilde{D}_d) = Q_u^{-1} \begin{bmatrix} Q_{11}\tilde{U}_1 + Q_1\tilde{D}_{d_{11}} & 0 \\ 0 & Q_{22}\tilde{U}_2 + Q_2\tilde{D}_{d_{12}} \end{bmatrix} \quad (63)$$

$$(\tilde{V}_{bd} - Q\tilde{N}_d) = Q_u^{-1} \begin{bmatrix} Q_{11}\tilde{V}_1 + Q_1\tilde{N}_{d_{11}} & 0 \\ 0 & Q_{22}\tilde{V}_2 + Q_2\tilde{N}_{d_{12}} \end{bmatrix} \quad (64)$$

Hence the M operator for the decentralized version of the robust problem in figure 4 takes the following form.

$$M = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d Q_u^{-1} & N_d Q_u^{-1} \\ -D_d Q_u^{-1} & D_d Q_u^{-1} \end{bmatrix} \begin{bmatrix} T_{d_1} & 0 \\ 0 & T_{d_2} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix} \qquad (65)$$

Where

$$T_{d_1} = \begin{bmatrix} Q_{11}\tilde{U}_1 + Q_1\tilde{D}_{d_{11}} & 0 \\ 0 & Q_{22}\tilde{U}_2 + Q_2\tilde{D}_{d_{12}} \end{bmatrix}$$

$$T_{d_2} = \begin{bmatrix} Q_{11}\tilde{V}_1 + Q_1\tilde{N}_{d_{11}} & 0 \\ 0 & Q_{22}\tilde{V}_2 + Q_2\tilde{N}_{d_{12}} \end{bmatrix} \qquad (66)$$

As will be shown in the following section 5 the M operator is used in a standard H_∞ formulation for the synthesis of robust controllers. For centralized controller problems there exists a solution methodology [9], but for the decentralized form of the M operator given in eq. (65) a difficulty exists. The inverse of the unimodular constraint, Q_u^{-1}, effects every element of the M operator (see eq. (65)). The formulation of a convex, concurrent solution to generate simultaneously the design parameters Q_{11}, Q_1, Q_{22}, Q_2 is hindered by the presence of the Q_u^{-1} term associated with each element of M. To see this more clearly consider the nominal performance operator, M_{22}. For the centralized case this operator takes the form

$$M_{22} = D_d(\tilde{V} - Q\tilde{N}_d)W_p \qquad (67)$$

This equation clearly takes the affine form, $T_1 - T_2QT_3$, where

$$T_1 = D_d\tilde{V}W_p \qquad T_2 = D_d \qquad T_3 = \tilde{N}_dW_p \tag{68}$$

The centralized nominal performance problem

$$\inf_Q ||T_1 - T_2QT_3|| \tag{69}$$

is solvable for $Q \in m(H)$. However, the M_{22} operator for the decentralized problem, using the unimodular parameter restriction of section 2.2, is of the form

$$M_{22} = D_dQ_u^{-1} \begin{bmatrix} \tilde{V}_1 + Q_1\tilde{N}_{d_{11}} & 0 \\ 0 & \tilde{V}_2 + Q_2\tilde{N}_{d_{12}} \end{bmatrix} W_p \tag{70}$$

where the unimodular constraint takes the form

$$Q_u = \begin{bmatrix} I & Q_1W_{12} \\ Q_2W_{21} & I \end{bmatrix} \tag{71}$$

Now the elements in the bracketed center term of eq. (70) take on an affine structure, but because the Q_u shares similar terms (namely, Q_1 and Q_2) the overall M_{22} term is not convex with respect to the design parameters Q_1 and Q_2 and hence convex solution algorithms for

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the following type of concurrent design problem

$$\inf_{Q_1,Q_2} ||M_{22}|| \tag{72}$$

are not available. This is the same difficulty associated with concurrent design problems for decentralized M operator, eq. (65). In the next section a iterative strategy will be introduced which will restore a convexity property for the parameter searches and allow the problem formulation to remain in the μ-framework, thereby providing for the synthesis of robust stability/robust performance decentralized controllers.

5 D-K Methodology for Sequential Design of Decentralized Controllers

In this section a methodology for the sequential design of decentralized controllers is developed. It is adapted from a centralized synthesis technique for the design of robust controllers known as the D-K synthesis technique. The D-K method is developed using the structured singular analysis tools outlined in section 3. Before developing the method for decentralized controllers some essential elements of the D-K method for centralized systems must be presented.

The μ-synthesis methods result from an upper bound developed to compute $\mu_\Delta(\cdot)$. The following notation will be used for norm-bounded subsets of the perturbation set Δ given in

eq. (54).

$$\mathbf{B}_\Delta = \{\Delta \in \Delta \mid \bar{\sigma}(\Delta) \le 1\} \tag{73}$$

The following subset of $\mathbf{C}^{n \times n}$ will shortly be shown to be rather useful.

$$\mathbf{D} = \Big\{ blkdiag[D_1, \cdots, D_s, d_{s+1}I_{m_1}, \cdots, d_{s+f}I_{m_f}] \mid$$

$$D_i \in \mathbf{C}^{r_j \times r_j}, D_i = D_i^* > 0, d_{s+j} > 0 \Big\} \tag{74}$$

Where for any $\Delta \in \Delta$, and $D \in \mathbf{D}$, $D\Delta = \Delta D$. From these definitions it can be shown, [2], that the following are tight upper and lower bounds for the computation of $\mu_\Delta(\cdot)$.

$$\max_{\Delta \in \mathbf{B}_\Delta} \rho(\Delta M) = \mu_\Delta(M) \le \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{75}$$

The upper and lower bounds of $\mu_\Delta(\cdot)$ allow it to be numerically tractable and the upper bound has convex properties which make it computationally attractive (see [2] for details).

The synthesis method relies on developing from the upper bound, frequency domain scales denoted $D(s)$. This is accomplished as follows. From the μ robust stability/performance test of eq. (52) the following synthesis equation can be formulated.

$$\min_K \max_\omega \mu_\Delta[M(G, K)(j\omega)] \tag{76}$$

This equation formulates the following objective, find the controller, from the set of all stabilizing controllers, which minimizes the peak value $\mu_\Delta(M(G, K))$. Where $M(G, K)$ represents

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the closed loop system transfer matrices of the general control problem. Equation (76) can be approximated using the $\mu_\Delta(\cdot)$ upper bound as follows

$$\min_K \max_\omega \min_{D_\omega \in \mathbf{D}} \bar\sigma[D_\omega M(G,K)(j\omega)D_\omega^{-1}] \tag{77}$$

where D_w is chosen from the set of scalings, $\mathbf{D}$, independently at every ω. From these D_w scalings frequency domain scalings $\hat{D}(s)$ can be constructed. These scalings are usually restricted to real-rational, stable, minimum-phase transfer functions and the optimization becomes

$$\min_K \min_{\hat{D}(s) \in \mathbf{D}} ||\hat{D}M(G,K)(j\omega)\hat{D}|| \tag{78}$$

We are now in a position to describe the D-K synthesis. The robust controllers are synthesized under D-K method by performing a number of iterations where alternately the $\hat{D}(s)$ scales or the compensator $K(s)$ are held fixed. Holding the $\hat{D}(s)$ scales fixed it is readily established, [2], that the following equation

$$\min_K ||\hat{D}M(G,K)(j\omega)\hat{D}|| \tag{79}$$

is equivalent to

$$\min_K ||M(G_D,K)|| \tag{80}$$

Where the frequency scales $\hat{D}(s)$ and $\hat{D}(s)^{-1}$ are absorbed directly into the generalized plant, G. The form of eq. (80) is in a standard H_∞ formulation for which a solution algorithm

exists, [9].

Holding the compensator, $K(s)$, fixed, the following upper bound calculation of $\mu_\Delta(\cdot)$ is performed.

$$\min_{D_\omega \in D} \bar{\sigma}[D_\omega M(G,K)(j\omega)D_\omega^{-1}] \tag{81}$$

From the set of D_w found at each discrete frequency point evaluated, a set of $\hat{D}(s)$ scale transfer functions are constructed. Reflecting these $\hat{D}(s)$ back into the generalized plant is the mechanism by which the H_∞ minimization is forced to focus its efforts over specific frequency ranges to try an lower the peak value of the $\mu_\Delta(\cdot)$ for the closed loop generalized system. Iterating back and forth between the steps of fixing the $\hat{D}(s)$ scales and the compensator, $K(s)$, comprises the D-K methodology. Although, as indicated in [1], the D-K method does not converge to a global minimum, it has proven quite successful in practice for synthesizing robust controllers [2].

Now adapting the D-K method to synthesizing robust decentralized controllers is accomplished as follows. Having placed the decentralized control problem in the μ-framework (see section 4), the decentralized problem is positioned to develop a set of $\hat{D}(s)$ scalings in an identical fashion to the centralized case by holding the decentralized compensator fixed during the D step of the D-K iteration.

The difficulty resides in the step where the $\hat{D}(s)$ scales are held fixed and a decentralized compensator is sought out to satisfy eq. (80). The way this can be resolved is to impose the unimodular parameter restriction, definition 2, section 2.2. This then reduces the number of design parameters to be found for each subcompensator to one. The design parameter of

the subcompensator is individually found by holding the other subcompensator's parameters fixed. After a new design parameter is found, the decentralized doubly coprime Bezout identity, DDCBI, is recomputed so that the new parameterized subcompensator becomes the factorized subcompensator for the newly adjusted DDCBI. After this step, the design parameter for the second subcompensator is sought out, while holding the first subcompensator fixed. In order for this algorithm to be effective two issues must be resolved.

1. If at each step the resulting $M(\cdot)$ operator can be shown to be convex in the individual design parameter sought, the problem can then be reduced to a solvable algorithm using convex methods.

2. Iterating between the controllers must reduce the over all optimization problem in a monotonic decreasing fashion.

Both properties will be demonstrated for this sequential design method for decentralized controllers.

5.1   Convexity of the $M(\cdot)$ Operator

To demonstrate the resulting convexity of the $M(\cdot)$ in terms of the single design parameter when sequentially designing subcontrollers we will work with the $M(\cdot)$ operator developed for the two channel decentralized control problem in section 3. The M operator for the

decentralized control problem, eq. (65), can be rewritten as

$$
M = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d & N_d \\ -D_d & D_d \end{bmatrix} \begin{bmatrix} Q_u^{-1} T_{d_1} & 0 \\ 0 & Q_u^{-1} T_{d_2} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix} \tag{82}
$$

where

$$
\begin{aligned}
Q_u^{-1} T_{d_1} &= \begin{bmatrix} Q_{11} & Q_1 W_{12} \\ Q_2 W_{21} & Q_{22} \end{bmatrix}^{-1} \begin{bmatrix} Q_{11}\tilde{U}_1 + Q_1\tilde{D}_{d_{11}} & 0 \\ 0 & Q_{22}\tilde{U}_2 + Q_2\tilde{D}_{d_{12}} \end{bmatrix} \\
Q_u^{-1} T_{d_2} &= \begin{bmatrix} Q_{11} & Q_1 W_{12} \\ Q_2 W_{21} & Q_{22} \end{bmatrix}^{-1} \begin{bmatrix} Q_{11}\tilde{V}_1 + Q_1\tilde{N}_{d_{11}} & 0 \\ 0 & Q_{22}\tilde{V}_2 + Q_2\tilde{N}_{d_{12}} \end{bmatrix}
\end{aligned} \tag{83}
$$

For the case of finding subcontroller one, C_1, impose the unimodular restriction on the parameters Q_{11}, Q_{22} and set $||Q_2|| = 0$. Equation (83) becomes

$$
\begin{aligned}
Q_u^{-1} T_{d_1} &= \begin{bmatrix} I & -Q_1 W_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{U}_1 + Q_1\tilde{D}_{d_{11}} & 0 \\ 0 & \tilde{U}_2 \end{bmatrix} \\
&= \begin{bmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{U}_2 \end{bmatrix} + \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} \tilde{D}_{d_{11}} & -W_{12}\tilde{U}_2 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
Q_u^{-1} T_{d_2} &= \begin{bmatrix} I & -Q_1 W_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{V}_1 + Q_1\tilde{N}_{d_{11}} & 0 \\ 0 & \tilde{V}_2 \end{bmatrix} \\
&= \begin{bmatrix} \tilde{V}_1 & 0 \\ 0 & \tilde{V}_2 \end{bmatrix} + \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} \tilde{N}_{d_{11}} & -W_{12}\tilde{V}_2 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{85}
$$

The middle term of eq. (82) can now be written

$$
\begin{bmatrix} Q_u^{-1} T_{d_1} & 0 \\ 0 & Q_u^{-1} T_{d_2} \end{bmatrix} = \begin{bmatrix} U_{bd} & 0 \\ 0 & V_{bd} \end{bmatrix} + \begin{bmatrix} Q_{1_d} & 0 \\ 0 & Q_{1_d} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \tag{86}
$$

with

$$
U_{bd} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \qquad V_{bd} = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \qquad Q_{1_d} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}
$$

$$
S_1 = \begin{bmatrix} \tilde{D}_{d_{11}} & -W_{12}\tilde{U}_2 \\ 0 & 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} \tilde{N}_{d_{11}} & -W_{12}\tilde{V}_2 \\ 0 & 0 \end{bmatrix} \tag{87}
$$

Given the form of eq. (86) the $M(\cdot)$ operator becomes

$$
M = T_1 + T_2 \hat{Q}_1 T_3 \tag{88}
$$

where the expressions T_1, T_2 and T_3 take on the values

$$
T_1 = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d & N_d \\ -D_d & D_d \end{bmatrix} \begin{bmatrix} U_{bd} & 0 \\ 0 & V_{bd} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix}
$$

$$
T_2 = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d & N_d \\ -D_d & D_d \end{bmatrix}
$$

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$$T_3 = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix}$$

$$\hat{Q}_1 = \begin{bmatrix} Q_{1_d} & 0 \\ 0 & Q_{1_d} \end{bmatrix} = blkdiag[Q_1, Q_1, Q_1, Q_1] \tag{89}$$

The $M(\cdot)$ operator is convex in the design parameter Q_1. The proof of this is as follows.

Proof M is affine in $\hat{Q}_1$ (see eq. (88)) and $\hat{Q}_1$ is convex in the parameter Q_1 which implies M is convex in the design parameter Q_1.

$\square$

This means that the optimization problem eq. (80) is solvable using convex algorithmic methods, [10].

Assume the optimization problem eq. (80) for the individual subcontroller parameterized by Q_1 is solved and the selected parameter is Q_1^*. Using the stable factors of the new subcontroller, C_1, the decentralized doubly coprime Bezout identity, eq. (3), can be readjusted such that the fixed stable factors of DDCBI associated with C_1 correspond to these newly found subcontroller factors. This will preserve the stable factor structure of $M(\cdot)$ operator, eq. (82), with the old stable factors replaced by the appropriate new factors from the adjusted DDCBI. This will then allow a design iteration for subcontroller two, C_2, by once again enforcing a unimodular parameter restriction for parameters Q_{11}, Q_{22} and setting $||Q_1|| = 0$ to obtain a $M(\cdot)$ operator which is convex in Q_2.

Following a similar method as used with controller one, C_1, the $M(\cdot)$ in terms of Q_2 has

the following form. The middle term of eq. (82) will become

$$
\begin{bmatrix} Q_u^{-1} T_{d_1} & 0 \\ 0 & Q_u^{-1} T_{d_2} \end{bmatrix} = \begin{bmatrix} U_{bd}^{(1)} & 0 \\ 0 & V_{bd}^{(1)} \end{bmatrix} + \begin{bmatrix} Q_{2_d} & 0 \\ 0 & Q_{2_d} \end{bmatrix} \begin{bmatrix} \hat{S}_1 & 0 \\ 0 & \hat{S}_2 \end{bmatrix} \tag{90}
$$

with

$$
U_{bd}^{(1)} = \begin{bmatrix} U_1^{(1)} & 0 \\ 0 & U_2 \end{bmatrix} \qquad V_{bd}^{(1)} = \begin{bmatrix} V_1^{(1)} & 0 \\ 0 & V_2 \end{bmatrix} \qquad Q_{2_d} = \begin{bmatrix} Q_2 & 0 \\ 0 & Q_2 \end{bmatrix}
$$

$$
\hat{S}_1 = \begin{bmatrix} 0 & 0 \\ -W_{21} \tilde{V}_1^{(1)} & \tilde{N}_{d_{12}}^{(1)} \end{bmatrix} \qquad \hat{S}_2 = \begin{bmatrix} 0 & 0 \\ -W_{21} \tilde{V}_1^{(1)} & \tilde{N}_{d_{12}}^{(1)} \end{bmatrix} \tag{91}
$$

Where the superscript, $(\cdot)^{(1)}$, refers to the new stable factors resulting from the first iteration which designed a new controller for the first channel. The $M(\cdot)$ operator will once again have the form

$$
M^{(1)} = T_1 + T_2 \hat{Q}_2 T_3 \tag{92}
$$

where the expressions T_1, T_2 and T_3 take on the values

$$
T_1 = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d^{(1)} & N_d^{(1)} \\ -D_d^{(1)} & D_d^{(1)} \end{bmatrix} \begin{bmatrix} U_{bd}^{(1)} & 0 \\ 0 & V_{bd}^{(1)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix}
$$

$$
T_2 = \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -N_d^{(1)} & N_d^{(1)} \\ -D_d^{(1)} & D_d^{(1)} \end{bmatrix}
$$

$$T_3 = \begin{bmatrix} \hat{S}_1 & 0 \\ 0 & \hat{S}_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_p \end{bmatrix}$$

$$\hat{Q}_2 = \begin{bmatrix} Q_{2_d} & 0 \\ 0 & Q_{2_d} \end{bmatrix} = blkdiag[Q_2, Q_2, Q_2, Q_2] \qquad (93)$$

Due to the similar form of $M(\cdot)^{(1)}$ in eq. (92) to eq. (88), $M(\cdot)^{(1)}$ is convex in the subcontroller parameter Q_2.

5.2 Monotonic Decreasing Property of Iterative Subcontroller Design

In order for the iteration between subcontrollers to be useful, the overall optimization problem should decrease in a monotonic fashion. The optimization problem in terms of $M(\cdot)$ is given by eq. (80). For the sequential design of subcontrollers, eq. (80) is rewritten in the following manner

$$\min_{Q_i} \ ||M^{(j)}(Q_1, Q_2)|| \quad \text{for} \quad i = 1 \text{ or } 2, \quad j = 0, 1, 2, \cdots \qquad (94)$$

where only one design parameter is being sought during a given minimization. In other words if the minimization is over the entire set of $Q_1 \in m(H)$, then $||Q_2|| = 0$ and vice-versa. The superscript, j, is an iteration index to keep track of what iteration is currently proceeding. The alternating between controller parameters Q_1 and Q_2 has the desirable effect

of monotonically decreasing the H_∞ norm bound of the $M(\cdot)$ operator. To see this consider the following, before any iteration takes place, the DDCBI has assigned stable factors for a stabilizing compensator (i.e. $[\tilde{V}_{bd}, \tilde{U}_{bd}]$ and $[U_{bd}, V_{bd}]$). Using these factors the value of the $M(\cdot)$ before any iteration is

$$||M^{(0)}(0,0)|| = \delta \tag{95}$$

The first iteration optimization problem is

$$\min_{Q_1} ||M^{(0)}(Q_1,0)|| \tag{96}$$

Since the above is convex in Q_1, we'll assume Q_1^* is the minimum of eq. (96). Define

$$||M^{(0)}(Q_1^*,0)|| =: \delta_0 \tag{97}$$

By definition we have that

$$\delta_0 \leq \delta \tag{98}$$

The subcontroller one obtained from Q_1^* is absorbed back in to a newly adjusted DDCBI and we obtain the following

$$||M^{(1)}(0,0)|| = ||M^{(0)}(Q_1^*,0)|| \tag{99}$$

Now on this next iteration we are looking to solve

$$\min_{Q_2} \ ||M^{(1)}(0, Q_2)|| \tag{100}$$

Once again since the above is convex in Q_2, we'll assume Q_2^* is the minimum of eq. (100). Define

$$||M^{(1)}(0, Q_2^*)|| =: \delta_1 \tag{101}$$

By definition we have that

$$\delta_1 \leq \delta_0 \tag{102}$$

Continued iteration proceeds in a similar fashion and hence we have established the monotonic decreasing property for iterating between the subcontrollers.

6   Conclusions

This paper has covered a number of issues concerning the development of decentralized controllers for robust performance. Section 3 provided the necessary structures and definitions for developing controllers under the μ-framework. Methods of modeling uncertainty, the generalized control problem, the application of LFT's, and finally definitions for the structured singular value and the robust stability/performance tests where provided. Section 4 demonstrated how the decentralized control problem could be placed in the μ-framework. Use of decentralizing interaction properties from section 2.1 helped to simplify the decen-

tralized stable factor formulation of the $M(\cdot)$ operator. An examination of the difficulties in developing a concurrent method for generating design parameters for the decentralized controllers in the robust framework was also provided. Finally, section 5 provided a methodology for developing sequentially, robust decentralized controllers in the μ-framework. As illustrated in this paper development of decentralized controllers under this framework provides the benefit of specifically trying to synthesize decentralized controllers which satisfies the structured singular value robustness tests for the closed loop system. The net result is decentralized control with robust stability and robust performance properties.

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