On ℓ^∞ to ℓ^∞ Performance of Slowly Varying Systems

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Abstract

In this paper we present a result on the ℓ^∞ to ℓ^∞ performance of slowly time varying systems. In particular we show that the performance of such systems cannot be much worse than that of the frozen-time systems which are time invariant. We also demonstrate via an example that optimal frozen-time design may in general result to far from optimal ℓ^∞ to ℓ^∞ performance eventhough the time variations of the system can be arbitrarily slow.

Keywords: frozen-time design, slowly time-varying, optimal, worst case, discrete-time, coprime.

1 Introduction

The problem of controlling slowly time-varying systems arises in many applications. The main paradigm is in gain-scheduling where the plant is time-varying and at successive points in time a controller is designed to satisfy certain stability and performance specifications based on the "frozen-time" system which is time invariant (LTI). Therefore, the resulting controller is itself time-varying. However, it is expected that if the rate of time variation is small enough then the frozen-time properties carry on to the overall time-varying system. In other words, it is expected that the stability of "frozen-time" designs will guarantee stability of the global time-varying system and also that the performance of the global system cannot be considerably worse than that of the frozen designs. As a matter of fact, these expectations have not only been confirmed in practice but also in theory by the work of several researchers in this area. We refer to the work of [1] and [2] on stability of time varying systems; for performance results in addition to stability, we refer to the work in [3, 4, 5, 6]. In particular, stability as well as performance results for the global time varying system in terms of the frozen-time properties of $\mathcal{H}^\infty$ interpolants were given in [3]. Moreover, in their subsequent papers [4, 5, 6] the authors established a novel framework for frozen-time analysis based on the notion of a normed local-global algebra. Using this framework the authors in [4, 5, 6] obtained stability and performance results of the global system in relation to the local frozen time properties. In their approach exponentially weighted norms were used which could capture both ℓ^2 and ℓ^∞ input-output behavior. Analysis using the notion of frozen-time system as in [3] can also be found in [7] where the authors produced ℓ^∞ stability results analogous to the ones in [3, 4]. Their analysis however did not involve the use of exponentially weighted norms as in the general formal framework of [4, 5], but rather, it was carried out using only standard ℓ^∞ induced norms. Hence the ℓ^∞ stability results were obtained in a direct manner. Finally, work related to stability and robustness of gain-scheduled finite dimensional systems can be found in [8, 9].

In this paper we take exactly the same point of view as in [7] to continue their work that was centered only at the stability issue, and extend it to capture the performance part in a bounded input to bounded output (i.e. ℓ^∞ to ℓ^∞) sense; this is done in a direct fashion focusing only to the ℓ^∞ behavior and thus using only ℓ^∞ induced norms. We use exactly the same setup as in [7] that allows infinite-dimensional, stable or unstable plants and controllers. Hence, the need of a fixed degree is not apparent. The main result of this paper is given

for single-input single-output (SISO) discrete slowly varying systems. It states that the ℓ^∞ to ℓ^∞ performance of the global time varying system cannot be much worse than the worst frozen-time ℓ^∞ to ℓ^∞ performance given that the rates of variation of the plant and the controller are sufficiently small. Our main result is in parallel with the results in [3, 6] which were given for stable plants, and, is connected to the coprime factorization analysis for unstable systems in the double-algebra framework performed in [4]. Moreover, given the continuity properties of the optimal ℓ^1 design established in [10] it follows that, under certain conditons, optimal ℓ^1 [11, 12] frozen-time design can yield an upper bound on the ℓ^∞ to ℓ^∞ performance of the global system.

2 Preliminary Definitions and Problem Statement

In this paper the following notation is used:

ℓ_m^∞ : The space of real $m \times 1$ vectors u each of whose components is a magnitude bounded real sequence $(u_i(k))_{k=0}^\infty$. The norm is defined as:

$$\|u\|_{\ell_m^\infty} = \max_i(\sup_k |u_i(k)|)$$

$\ell_m^{\infty,e}$: The space of real $m \times 1$ vector valued sequences.

$\mathcal{L}_{TV}^{m\times n}$: The space of all linear bounded and causal maps from ℓ_n^∞ to ℓ_m^∞. We refer to these operators as stable.

$\mathcal{L}_{TI}^{m\times n}$: The subspace of $\mathcal{L}_{TV}^{m\times n}$ consisting of the maps that commute with the shift operator (i.e. the time invariant maps).

Π_m^k : The kth-truncation operator on $\ell_m^{\infty,e}$ defined as:

$$\Pi_m^k : \{u(0), u(1), \ldots\} \longrightarrow \{u(0), \ldots, u(k), 0, 0, \ldots\}$$

Note: To avoid proliferation of notation we will drop the m and n which indicate dimensions in the above definitions since we mostly use SISO systems. Also, subscripts on the norms that indicate the spaces are usually dropped whenever it is clear from the context.

Let T be a SISO operator in $\mathcal{L}_{TI}$ with transform representation

$$\hat{T}(\lambda) = \sum_{i=0}^{\infty} T(i)\lambda^i.$$

Then, its $\mathcal{L}_{TI}$ norm is given as

$$\|T\| = \sum_{i=0}^{\infty} |T(i)|.$$

Further, we give the following definition

Definition 2.1 *The Integral Time Absolute Error ITAE associated with T is defined as*

$$ITAE(T) = \sum_{k=0}^{\infty} k|T(k)|.$$

If $T^{(1)}$ is the LTI operator associated with the derivative $\hat{T}^{(1)}(\lambda) = \frac{d\hat{T}(\lambda)}{d\lambda}$ then it follows that

$$\text{ITAE}(T) = \left\|T^{(1)}\right\|.$$

Given a sequence of SISO LTI operators $\{A_t\}_{t=0}^{\infty}$ where each A_t is a map from $\ell^{\infty,e}$ to $\ell^{\infty,e}$ we can generate a time varying operator A as $(Ay)(t) = (A_t y)(t),\ t = 0, 1, \ldots$.

Definition 2.2 *The operator A is called slowly time-varying if there is a constant $\gamma > 0$ such that*

$$\|A_t - A_\tau\| \leq \gamma|t - \tau| \quad \forall t, \tau.$$

This is denoted by $A_t \in STV(\gamma)$.

If $A_t \in \mathcal{L}_{TI}$ for all t and also the $\mathcal{L}_{TI}$ norm is bounded uniformly in t then $A \in \mathcal{L}_{TV}$ and $\|A\| = \sup_t \|\Pi^t A_t\|$.

The problem we want to analyze is the stability and performance of the feedback system in Figure 1 where P is a SISO slowly varying plant and C is a controller obtained by "frozen-time" control. Specifically, the plant P is defined as $P = A^{-1}B$ where A, B are slowly varying operators in $\mathcal{L}_{TV}$ associated with the sequences $\{A_t\}, \{B_t\},\ t = 0, 1, 2, \ldots$ of LTI stable SISO operators respectively. We assume that A^{-1} exists as an operator on $\ell^{\infty,e}$ which is equivalent to $A_t(0) \neq 0\ \forall t$. Hence, the plant model is

$$y(t) = (Pu)(t) = (A^{-1}Bu)(t), \quad t = 0, 1, 2, \ldots$$

or, equivalently,

$$(A_t y)(t) = (B_t u)(t), \quad t = 0, 1, 2, \ldots .$$

We refer to the LTI system $P_t = A_t^{-1}B_t$ as the "frozen-time" plant. We should point out that the not all time-varying systems can be represented as the product $A^{-1}B$ with A, $B \in \mathcal{L}_{TV}$. However, such representation includes most systems of interest such as systems which are

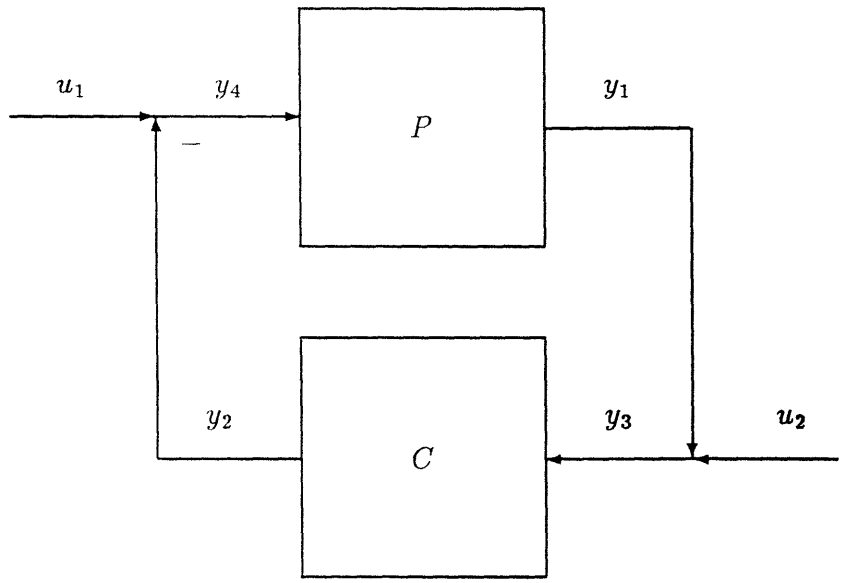

Figure 1: Feedback Loop of the Pair $(P,\ C)$

stabilizable by time-varying controllers [7]. Furthermore, in applications of adaptive control such systems appear naturally.

The controller is given as $C = L^{-1}M$ where L, M are associated with the sequences $\{L_t\}, \{M_t\}$, $t = 0, 1, 2, \ldots$ of LTI stable SISO operators i.e., $(Ly)(t) = (L_t y)(t)$ and $(My)(t) = (M_t y)(t)$. The following definition of stability is now given

Definition 2.3 *The closed loop system of Figure 1 is stable if the map* $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *is in* $\mathcal{L}_{TV}$.

Let L_t, M_t be such that the LTI controller defined as $C_t = L_t^{-1} M_t$ stabilizes the frozen time plant P_t. The controller operates as

$$(Cy)(t) = (L^{-1}My)(t), \quad t = 0, 1, 2, \ldots$$

or, equivalently,

$$(L_t u)(t) = (M_t y)(t), \quad t = 0, 1, 2, \ldots .$$

The question we want to answer is under what conditions the feedback loop is stable and, if so, what is the relation between the performance of the frozen-time pair (P_t, C_t) and the actual time varying feedback pair (P, C). This is done in the following section.

3 Main Result

In [7] an input-output point of view was taken to prove that, under the assumption of sufficiently small rate of variation, stability of the frozen-time feedback pair (P_t, C_t) implies stability of the pair (P, C). Yet, the performance part of the problem was not investigated. As already indicated in the introduction, ℓ^∞ performance results based on a general formal framework are available in [4, 6] for stable plants and strongly parallel the results of Theorem 3.1 below. In our approach however, we take the exact same point of view as in [7], which is more direct and suited to ℓ^∞ behavior, and extend the results of [7] to capture the performance issue. In particular, for the system in Figure 1 define the stable LTI operator for each $t = 0, 1, 2, \ldots$ from the following (Bezout) identity

$$G_t = L_t A_t + M_t B_t.$$

Since C_t stabilizes P_t then $H_t = G_t^{-1} \in \mathcal{L}_{TI}$. Now let S^{ij} represent the map from u_j to y_i in the system of Figure 1 and S_t^{ij} the (LTI) map from u_j to y_i for the frozen system (P_t, C_t). The following theorem which is an extension of Theorem 1 in [7] supplies the answer to our problem.

Theorem 3.1 *Assume the following:*

1. *The operators defining the plant P are slowly time-varying with rates γ_A, γ_B i.e., $A_t \in STV(\gamma_A), B_t \in STV(\gamma_B)$.*

2. *The operators defining the controller C are slowly time-varying with rates γ_L, γ_M i.e., $L_t \in STV(\gamma_L), M_t \in STV(\gamma_M)$.*

3. *The $\mathcal{L}_{TI}$ norms and the ITAE of the operators A_t, B_t, L_t, M_t are uniformly bounded in t; this of course means that $A, B, L, M, \in \mathcal{L}_{TV}$*

4. *The $\mathcal{L}_{TI}$ norms and the ITAE of the operator $H_t = G_t^{-1}$ are uniformly bounded in t.*

Then, for a given $\epsilon > 0$, there exists a nonzero constant γ such that, if $\gamma_A, \gamma_B, \gamma_L, \gamma_M \leq \gamma$, the closed loop system is internally stable and

$$(1 - \epsilon) \left\| S^{ij} \right\| \leq \sup_t \left\| S_t^{ij} \right\| + \epsilon.$$

Proof see Appendix ∎

The above theorem indicates that if the rates of variation of the plant and the controller

are sufficiently small then, modulo the uniform boundness assumptions 3 and 4, frozen time control would not only provide stability but also the resulting performance cannot be much worse than the worst frozen time design. In [10] it was shown under certain assumptions of existence and uniqueness that the ℓ^1 design methodology produces optimal frozen-time LTI controllers for the frozen-time plant that possess the slow variation property given that the plant is slowly varying. Hence, in these cases, one can obtain an upper bound on the achievable $\|S^{ij}\|$ by evaluating the expression $\sup_t \left\|S_t^{ij}\right\|$ obtained by frozen ℓ^1 optimal designs. Of course, the rate of variation should be sufficiently slow.

It is important to stress at this point that this theorem is useful as an analysis tool i.e., for a given controller. For this result to be useful as a synthesis tool, the design methodology should necessarily be continuous in the plant parameters, as well as the plant's parameters should vary slowly enough such that the conditions of the theorem are met. The later can be accomplished if the variation rate of the plant's parameter goes to zero. Otherwise, it is quite difficult to find apriori bounds on the plant's variation rate that guarantees stability.

Remark

A natural question that arises in the case where the plant P is slowly time varying is whether optimal frozen-time design at each time t will result in an optimal or near-optimal design (depending on the rate of variation) for the time-varying system. Although it is tempting to conjecture that, if the rate of variation is sufficiently small then the optimal performance cannot be far from the performance provided by optimal frozen-time control at each time t, the following example shows that this might not be true in general: Consider the plant $P \in \mathcal{L}_{TV}$ defined by the sequences $\{A_t\}, \{B_t\}$ where $\hat{A}_t(\lambda) = 1\,\forall t$, $\hat{B}_t(\lambda) = 2\lambda + 1$ for $t = 0, 1$ and $\hat{B}_t(\lambda) = 2\lambda + (1 + \gamma t)$ for $2 \leq t \leq T = [1/\gamma + 1]$, $\hat{B}_t(\lambda) = 2\lambda + (1 + \gamma T)$ for $t > T$, with $\gamma > 0$. The resulting lower triangular, infinite, matrix representation of P is

$$P = \begin{pmatrix} 1 & & & & & & & \\ 2 & 1 & & & & & & \\ & 2 & 1+\gamma & & & & & \\ & & 2 & 1+2\gamma & & & & \\ & & & \ddots & \ddots & & & \\ & & & & 2 & 1+T\gamma & & \\ & & & & & 2 & 1+T\gamma & \\ & & & & & & \ddots & \ddots \end{pmatrix}$$

Clearly, for this P we have $\|B_t - B_\tau\| \leq \gamma|t - \tau|\ \forall t, \tau$. Suppose we are interested in min-

imizing the $\mathcal{L}_{TV}$ norm of the sensitivity map $S = (1 + PC)^{-1}$. Then as it is well known [13, 14, 15] the optimization problem transforms to

$$\inf_{Q \in \mathcal{L}_{TV}} \|1 - PQ\| .$$

For this particular P we have that $P^{-1} \in \mathcal{L}_{TV}$ since $P_t = B_t$ is eventually ($t > T$) stably invertible. To view this, let P^{-1} be represented by the lower diagonal structure

$$P^{-1} = \begin{pmatrix} q(0,0) & 0 & \cdots \\ q(1,0) & q(1,1) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} .$$

Then the following recursion holds:

$$q(i,j) = -2q(i,i)q(i-1,j), \; i > 0, \; j = 0, 1, \ldots, i-1$$

with $q(0,0) = 1$, $q(i,i) = 1/(1 + (i-1)\gamma)$ for $i = 1, \ldots, T$ and $q(i,i) = 1/(1 + T\gamma)$ for $i \geq T$. Note that for $i \geq T$ we have that $|q(i,i)| = |q(T,T)| < 1$. Therefore, for any $k = 1, 2, \ldots$ we have

$$\begin{aligned} \sum_{j=0}^{T+k} |q(T+k,j)| &\leq (\max_{0 \leq j \leq T-1} |q(T-1,j)|)|q(T,T)|^k + \textstyle\sum_{j=1}^{k} |q(T,T)|^j \\ &\leq c_1 |q(T,T)| + \tfrac{1}{1-|q(T,T)|}. \end{aligned}$$

This evidently shows that $P^{-1} \in \mathcal{L}_{TV}$. Hence, by choosing $Q_o = P^{-1}$ we can make $\|1 - PQ_o\| = 0$ for any γ. On the other hand, using ℓ^1 optimal frozen time design yields [11, 12] $S_{t=0} = 1, S_{t=1} = 1, \ldots, S_{t=T-1} = 1, S_{t=T} = 0, S_{t=T+1} = 0, \ldots$. The reason for $S_{t=0}, \ldots S_{t=T-1} \neq 0$ is of course the unstable zero of $P_{t=0}, \ldots, P_{t=T-1}$ at $\lambda = (1 + t\gamma)/2$ for all γ. Moreover, the resulting frozen time based controller will yield a performance $\|S\| \geq 1$ for any $\gamma > 0$ no matter how small, since the system will behave exactly as the frozen LTI one for $t = 0, 1$.

In effect, the previous example illustrates a case of discontinuity of the ℓ^1-optimal design. Such a case however, is excluded in the analysis of [10] since, in this treatment, unstable zeros are not allowed to exit the unit disk. In fact, this example can also serve to illustrate discontinuity of $\mathcal{H}^\infty$ interpolants [6]. This discontinuity is attributed [16] to the discontinuity of the inner factors in the Nehari formulation of the optimal $\mathcal{H}^\infty$ problem. Nevertheless, in the absence of such discontinuities, it was shown in [17] that the frozen-time optimal $\mathcal{H}^\infty$ design yields arbitrarily close (depending on the rate of variation) to optimal ℓ^2 to ℓ^2 global behavior. Such a property has not yet been established for optimal ℓ^1 interpolation and is an issue of future research.

4 Conclusions

In this paper we presented a ℓ^∞ to ℓ^∞ performance result in the case of SISO discrete slowly time varying systems. We showed that the performance of a slowly varying system cannot be much worse than that of the frozen time systems. Moreover, we demonstrated that optimal ℓ^∞ to ℓ^∞ frozen-time design may yield far from optimal ℓ^∞ to ℓ^∞ performance in the overall system.

5 APPENDIX

Proof of Theorem 3.1: The proof of the stability part is given in [7]. Here we repeat in brief the main steps because we will use them to prove the claim for the performance. The closed loop equations for the system in Figure 1 are as follows:

$$(A_t y_1)(t) = (B_t(u_1 - y_2))(t)$$

$$(L_t y_2)(t) = (M_t(u_2 + y_1))(t)$$

$$A_t L_t + M_t B_t = G_t$$

By adding subtracting and grouping terms we finally arrive [7] at

$$\begin{pmatrix} G+X & Y \\ -Z & G+W \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} LB & -BM \\ MB & AM \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{CL}$$

where G is the operator in $\mathcal{L}_{TV}$ associated with the family $\{G_t\}$ and X, Y, Z, W are "perturbation" operators which are due to the time variation of the system P. As indicated in lemmas 1 and 2 in [7] these operators have $\mathcal{L}_{TV}$ norm bounded by the term $\gamma \times$constant where $\gamma = \max(\gamma_A, \gamma_B, \gamma_L, \gamma_M)$ and the constant depends on the uniform bounds of assumption 3 of the Theorem 3.1; i.e., there are constants $c_X, c_Y, c_Z, c_W > 0$ such that

$$\|X\| \leq \gamma c_X, \quad \|Y\| \leq \gamma c_Y, \quad \|Z\| \leq \gamma c_Z, \quad \|W\| \leq \gamma c_W$$

Now, from the first equation in (CL) we have

$$G y_1 + X y_1 + Y y_2 = v$$

where $v = LBu_1 - BMu_2$. If we fix some t, G_t is a LTI operator; adding and subtracting this operator in the above operator equation we obtain

$$G_t y_1 + (G - G_t) y_1 + X y_1 + Y y_2 = v$$

or since $H_t = G_t^{-1}$ we obtain

$$y_1 + H_t(G - G_t) y_1 + H_t X y_1 + H_t Y y_2 = H_t v.$$

Evaluating this operator equation at time t we obtain

$$y_1(t) + (H_t(G - G_t) y_1)(t) + (H_t X y_1)(t) + (H_t Y y_2)(t) = (H_t v)(t).$$

Define the operator H as $(Hz)(\tau) = (H_\tau z)(\tau)$, $\tau = 0, 1, 2, \ldots$. Also define the operator R as $(Ry_1)(\tau) = (H_\tau(G - G_\tau)y_1)(\tau)$, $\tau = 0, 1, 2, \ldots$. Rewriting the above equation in operator form we have

$$y_1 + Ry_1 + HXy_1 + HYy_2 = Hv.$$

Similarly working with the second equation, letting $w = MBu_1 + AMu_2$ and putting both equations together in operator form we get

$$(I + F)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Hv \\ Hw \end{pmatrix}$$

where

$$F = \begin{pmatrix} R + HX & HY \\ -HZ & R + HW \end{pmatrix}.$$

Note that from the uniform bound assumption on H_τ it follows that $H \in \mathcal{L}_{TV}$ and therefore the norms of the operators HX, HZ, HY, HW can be bounded by $\gamma\times$constant. Also, utilizing the fact that the ITAE of H_τ is uniformly bounded it is shown in [7] that the norm of R is bounded in the same way i.e., $\|R\| \leq \gamma\times$constant. The stability of the loop then, follows from the small gain theorem for sufficiently small γ.

We now come to the performance part. We will prove our claim for the maps S^{12}, S^{22}; the proof for any other map is completely analogous. Let $u_1 = 0$ and let $\|u_2\| \leq 1$. Then from the system equations we get

$$y_1(t) = -(H_t BMu_2)(t) - (H_t Xy_1)(t) - (H_t Yy_2)(t) - (H_t(G - G_t)y_1)(t)$$

Consider now the frozen LTI feedback system at time t i.e., (P_t, C_t) subjected to the same input u_2 and let y_{1t} denote the output that corresponds to y_1 in the time varying loop (P, C). Then evaluating y_{1t} at t we have

$$y_{1t}(t) = -(H_t B_t M_t u_2)(t).$$

Subtracting the above two equations we obtain

$$y_{1t}(t) - y_1(t) = (H_t(BM - B_t M_t)u_2)(t) + (H_t Xy_1)(t) + (H_t Yy_2)(t) + (H_t(G - G_t)y_1)(t).$$

The idea here is to bound $|(H_t(BM - B_t M_t)u_2)(t)|$ by $\gamma\times$constant. For this purpose define the operator $K \in \mathcal{L}_{TV}$ as

$$(Kz)(\tau) = (B_\tau M_\tau z)(\tau) \quad \tau = 0, 1, 2, \ldots$$

then

$$(H_t(BM - B_tM_t)u_2)(t) = (H_t(BM - K)u_2)(t) + (H_t(K - B_tM_t)u_2)(t).$$

By lemma 1 in [7] and the fact that H_t has norm uniformly bounded it follows that

$$|(H_t(BM - K)u_2)(t)| \leq \gamma c_1$$

with c_1 a positive constant. For the term $(H_t(K - B_tM_t)u_2)(t)$ we have the following:

$$\begin{aligned} \|B_\tau M_\tau - B_tM_t\| &\leq \|B_\tau\| \, \|M_\tau - M_t\| + \|M_t\| \, \|B_t - B_\tau\| \\ &\leq \|B_\tau\| \, \gamma_M |t - \tau| + \|M_t\| \, \gamma_B |t - \tau| \\ &\leq \gamma c_2 |t - \tau|. \end{aligned}$$

Hence, if $z(\tau) = ((K - B_tM_t)u_2)(\tau)$, then $|z(\tau)| \leq \gamma c_2 |t - \tau|, \quad \tau = 0, 1, 2, \ldots$ with $c_2 > 0$. But then from the fact that H_t has bounded (uniformly in t) ITAE it follows as in theorem 1 of [7] that

$$\begin{aligned} |(H_t(K - B_tM_t)u_2)(t)| &= |\sum_{\tau=0}^{\tau=t} h_t(t - \tau)z(\tau)| \\ &\leq \gamma c_2 \sum_{\tau=0}^{\tau=t} |h_t(\tau)|\tau \\ &\leq \gamma c_3, \quad c_3 > 0. \end{aligned}$$

Now, looking at the rest of the terms and since $\|u_2\| \leq 1$ we have $|(H_tXy_1)(t)| \leq \gamma c_4 \|S^{12}\|$, $|(H_tYy_2)(t)| \leq \gamma c_5 \|S^{22}\|$ and $|(H_t(G - G_t)y_1)(t)| \leq \gamma c_6 \|S^{12}\|$ so putting everything together it follows that there are $c, c_{12}, c_{22} > 0$ such that

$$|y_1(t) - y_{1t}(t)| \leq \gamma c + \gamma c_{12} \left\|S^{12}\right\| + \gamma c_{22} \left\|S^{22}\right\|$$

or since $\|u_2\| \leq 1$ then $|y_{1t}(t)| \leq \|S_t^{12}\|$ and therefore

$$\sup_t |y_1(t)| \leq \sup_t \left\|S_t^{12}\right\| + \gamma c + \gamma c_{12} \left\|S^{12}\right\| + \gamma c_{22} \left\|S^{22}\right\|$$

and since u_2 is arbitrary

$$\left\|S^{12}\right\| \leq \sup_t \left\|S_t^{12}\right\| + \gamma c + \gamma c_{12} \left\|S^{12}\right\| + \gamma c_{22} \left\|S^{22}\right\|.$$

Similarly working for $\|S^{22}\|$ we get

$$\left\|S^{22}\right\| \leq \sup_t \left\|S_t^{22}\right\| + \gamma k + \gamma k_{22} \left\|S^{22}\right\| + \gamma k_{12} \left\|S^{12}\right\|.$$

Now noting that $\|H_t\|$ is uniformly bounded then $\sup_t \|S_t^{12}\|, \sup_t \|S_t^{12}\| < \infty$ and hence by assuming γ sufficiently small the proof of the theorem is complete. ∎

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