Topology of Hybrid Systems*

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Abstract

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Finding some anomalies in completing this loop, we discuss a new view of hybrid systems that may broach them and is more in line with traditional control systems. In fact, the most widely used fuzzy control system is related to this new view and does not possess these anomalies. Indeed, we show that fuzzy control leads to continuous maps (from measurements to controls) and that all such continuous maps may be implemented via fuzzy control.

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Finding some anomalies in completing this loop, we discuss a new view of hybrid systems that may broach them and is more in line with traditional control systems. In fact, the most widely used fuzzy control system is related to this new view and does not possess these anomalies. Indeed, we show that fuzzy control leads to continuous maps (from measurements to controls) and that all such continuous maps may be implemented via fuzzy control.

1 Introduction

In traditional feedback control systems—continuous-time, discrete-time, and sampled-data—the maps from output measurements to control inputs are continuous (in the usual metric-based topologies). When dealing with hybrid systems, however, one immediately runs into problems with continuity using the "usual" topologies. Whereby we begin ...

In this paper, we discuss some results relating to the topology of hybrid (mixed continuous and finite dynamics) systems. We begin with a model of a hybrid system as shown in Figure 1. We are interested in maps from the continuous plant's output or measurement space into a finite set of symbols. We call these AD maps. We are also interested in the map from this symbol space into the control or input space of the continuous plant (DA map). In many control applications, both the measurement and control spaces are (connected) metric spaces. Therefore, we will keep our discussion germane to such assumptions.

The paper is organized as follows: In the next section, we discuss AD maps. First, we illuminate the general issues. Then, we examine at length an AD

map proposed in [3]. We verify that the map is indeed continuous, developing enough technical lemmas to easily add the fact that the symbol space topology they constructed is the same as the quotient topology induced by their AD map.

In Section 3, we discuss what happens if we try to impose continuity from the measurement to the control spaces. We first illuminate why this is unreasonable given the fact that the measurement and control spaces are normally connected metric spaces. We then impose a new topology on the control space that gives rise to continuous maps.

In Section 4, we introduce a new view of hybrid systems. This view allows us to meaningfully discuss continuity of maps from the measurement to control spaces without introducing new topologies. We also show that the most widely used fuzzy logic control structure is related to this form and that it indeed is a continuous map from measurements to controls. It is further demonstrated that these fuzzy logic controllers are dense in the set of such continuous functions.

We end with a short summary and conclusions. The Appendix collects the proofs of parts of a technical lemma.

2 Continuous AD Maps

2.1 General Discussion

In this section, we will discuss continuity of maps from the measurement space of the continuous plant into the finite symbol space. Such continuity is desirable when implementing control loops, since we want, roughly, small changes in measurement to lead to small changes in control action.

The basic problem we have in going from the continuum, M, into a finite set of symbols, I, is that I usually comes equipped with the discrete topology and the only continuous maps from M to I in this case are constant (since M is connected and any subset of I with more than one point is not). Therefore, we must search for topologies on I which are not the discrete topology. At first, we may be disheartened by the fact that this also precludes all Hausdorff, and even T_1 topologies, from consideration. However, the topologies associated with (finite) observations

are naturally T_0 [4]. Fortunately, there do exist T_0 topologies other than the discrete topology on any finite set of more than one point:

Example 1 *Suppose X is a finite set having $n > 1$ elements. There exists a T_0 topology on X that is not the discrete topology (and hence neither T_1 nor Hausdorff).*

Proof Take as a basis the following subsets of X: $\emptyset, \{x_1\}, \ldots, \{x_{n-1}\}, X$. $\square$

Using this idea, a way of getting around the problem above is to append the symbol space, I, with a single new symbol, $\bot$. Then, we place the following topology on $I' = I \cup \{\bot\}$: $2^I, I \cup \{\bot\}$, where 2^I is the power set of I. This topology on I' makes it homeomorphic to X in the proof above (when X and I' have the same number of elements). Therefore, it is T_0, but not T_1. Now, we can create continuous maps from a continuum, M, into I' as follows: Let A_i be N mutually disjoint open sets *not* covering M. Let $I = \{1, \ldots, N\}$, and define $f(A_i) = i$ and $f(M - \bigcup_i A_i) = \bot$. We claim f is continuous. It is enough to check the basis elements of the topology on I', which are the singleton sets of elements of I plus the set I' itself. We have $f^{-1}(i) = A_i$, open, for each $i \in I$. Further, $f^{-1}(I') = M$, which is open.

Another topology which works is the following: $\emptyset, \{U \cup \{\bot\} \mid U \in 2^I\}$, with the A_i *closed* instead of open (see Section 3 for a use of a topology like this). There are presumably many other choices one can make. Below we examine at length one espoused in [3].

2.2 AD Map of Nerode-Kohn

2.2.1 Definition:
The AD map is a map from the measurement space, M, into a finite set of symbols, I. Nerode and Kohn [3] create a continuous AD map as follows:

1. First, take any finite open cover of the measurement space: $M = \bigcup_{i=1}^{n} A_i$, where the A_i are open in the given topology of M.

2. Next, find the so-called *small topology*, $\mathcal{T}_M$, generated by the subbasis A_i. This topology is finite (as we will argue below) and its open sets can be enumerated, say, as $B_1, \ldots, B_p$.

3. Next, find all the non-empty *join irreducibles* in the collection of the B_i (that is, all non-empty sets B_j such that if $B_j = B_k \cup B_l$, then either $B_j = B_k$ or $B_j = B_l$). Again, there are a finite number of such join irreducibles, which we will denote $C_1, \ldots, C_N$.

4. Let the set of symbols be $I = \{1, \ldots, N\}$. Further, define the function $AD(m) = i$ if C_i is the smallest open set containing m.

5. Create a topology, $\mathcal{T}_I$, on I as follows. For each $i \in I$, declare $D_i = \{j \mid C_j \subset C_i\}$ to be open. Let $\mathcal{T}_I$ be the topology generated by the D_i.

Here is a simple example of the construction:

Example 2 *Let our measurement space be $M = [0,3]$ and the open cover of this measurement space be*

$$A_1 = [0,2), \qquad A_2 = (1,3]$$

The small topology generated by this subbasis can be enumerated as follows: $B_1 = \emptyset$, $B_2 = (1,2)$, $B_3 = [0,2)$, $B_4 = (1,3]$, $B_5 = [0,3]$. Next, we find the non-empty join irreducibles:

$$C_1 = [0,2), \qquad C_2 = (1,3], \qquad C_3 = (1,2)$$

Thus, we let our set of symbols be $I = \{1,2,3\}$ and define the function AD as follows:

$$AD(m) = \begin{cases} 1, & m \in [0,1] \\ 2, & m \in [2,3] \\ 3, & m \in (1,2) \end{cases}$$

The open sets D_i are found to be

$$D_1 = \{1,3\}, \qquad D_2 = \{2,3\}, \qquad D_3 = \{3\}$$

and the resulting topology on I, $\mathcal{T}_I$ is

$$\emptyset, \qquad \{3\}, \qquad \{1,3\}, \qquad \{2,3\}, \qquad \{1,2,3\}$$

One can readily check that $\mathcal{T}_I$ is T_0 and that AD is continuous. $\square$

2.2.2 Filter Interpretation:
Here, we give an intuitive interpretation of the Nerode-Kohn approach to hybrid systems as described in [3] (herein, *N-K*) in terms of bandpass filters. Our discussion covers both AD and DA maps.

The starting point of the *N-K* approach is an assumption that one can only realistically distinguish points up to knowing the open sets in which they are contained.[1] Thus, one takes small topologies on the measurement (a.k.a. plant output) and control (a.k.a. plant input) spaces. The open sets in these topologies correspond to events that are distinguishable and achievable, respectively. For example, they represent measurement error or actuator error (or equivalence classes that are adequate for the task at hand).

A good way to think of the open sets in the small topology is as notch filters. On the input side, we can pass our measurements through these filters. The level of information that we glean is, Did it go through the filter or not? Now, the total information from our sensors is summarized in the string of

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[1] However, the theory developed from this principle is destined to contradict itself. In particular, we have seen that closed sets may be distinguished (these arise from the partition of the measurement space into symbol pre-images, the so-called "essential parts.") More provocatively, we can distinguish single points in the measurement space. Consider as a representative example zero in $[-1, 1]$. Then the open sets [-1,1], (0,1], and [-1,0) give us information to exactly deduce $x = 0$.

Yes/No answers.[2] (Of course, we also implicitly have the filters themselves, that link these binary symbols with real regions of measurement space.) By taking the intersection of all filters which had a Yes answer, we obtain the join irreducible from which the measurement came. The input symbols of the finite automaton are simply "names" given to join irreducibles. By also taking into account the No answers, we obtain a partition of the measurement space into what N-K call "essential parts."

Likewise, on the output side one constructs the join irreducibles. The output symbols of the finite automata are exactly "names" given to these join irreducibles. Now, the finite automata controller is simply a map from input symbols to output symbols (modulated by its internal state). To fix ideas, let's say that the output symbol corresponds to join irreducible K_j.

Again, we can think of the control space small topology as a set of notch filters. Here, we imagine some broadband source signal (which is not exactly flat) which we use to produce our control in the following way: Instead of choosing a single output from the named join irreducible deliberately (normal AD conversion), we simply construct one in the correct equivalence class. We do this by using as a control signal the signal that results from passing our broadband source through each of the filters (open sets) which intersect to form the join irreducible K_j.

It is also interesting to note that N-K seem to have adopted the idea (cf. Appendix II of [3]) that the finite automaton and small topologies are used to construct approximations to maps from the measurement to control spaces, the approximation (of a continuous control law) necessarily approaching that law as the cover becomes finer.

2.2.3 Verification of Continuity: One of the results of [3] is the fact that their AD map is continuous from $(M, \mathcal{T}_M)$ to $(I, \mathcal{T}_I)$. Namely, they give (without proof) the following proposition, whose proof *we* provide for completeness:

Proposition 3 ([3]) $AD : M \mapsto I$ *is continuous.*

We will need several technical lemmas first, which will also be used to prove later results:

Lemma 4 *The non-empty join irreducibles of the topology generated by the (subbasis) A_i are exactly those sets which can be written as*

$$U_m = \bigcap_{j \in J(m)} A_j$$

where $J(m)$ is the set of all j such that $m \in A_j$.

Proof Pick $m \in M$ arbitrarily. Then U_m is non-empty since it contains m. Next, suppose that U_m is not join irreducible, so that it can be written as $U_m =$

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[2]That is, we can basically do peak detection (now allowing nonideal filters). Biological auditory and olfactory systems may work like this [1].

$A \cup B$ where $A \neq U_m$ and $B \neq U_m$. Then we must have that $m \in A$ or $m \in B$, or both. Without loss of generality, assume $m \in A$. But, since A is an element of the topology generated by the A_i, it must be of the form $A = \bigcup_{i \in I} \bigcap_{j \in J_i} A_j$, where $J_i \subset \{1, \ldots, n\}$, for each $i \in I$, that is, an arbitrary union of finite intersections of elements of the subbasis. Since $m \in A$, it must be in at least one of the sets in the union, say, the kth: $m \in \bigcap_{j \in J_k} A_j$. However, this means $m \in A_j$ for each $j \in J_k$. By definition, $J_k \subset J(m)$, so that $U_m \subset A$. But, since $U_m = A \cup B$, we also have $U_m \supset A$. So that $A = U_m$, a contradiction. $\square$

Thus, AD is a well-defined function, with $AD(m) = i$ where C_i is the smallest join irreducible containing m. In fact, C_i equals the U_m defined in the lemma. Now, it is easy to see that the set of non-empty join irreducibles is finite: there are at most $2^n - 1$ distinct non-empty sets that can be written in this manner. Thus, the topology generated by the A_i has less than 2^{2^n} elements (since each element is a union of basis sets).

Lemma 5 *1. The non-empty join irreducibles C_i form a basis of the topology, $\mathcal{T}_M$, which they generate.*

2. The sets D_i are a basis for the topology, $\mathcal{T}_I$, which they generate.

3. AD is surjective.

4. If f is surjective, $f(f^{-1}(X)) = X$.

5. $C_j = AD^{-1}(D_j)$

Proof The detailed proofs appear in the Appendix. Items 1 through 3 are straightforward. (For those with a knowledge of lattice theory, the C_i and D_i are lower closures in their respective lattices and give rise to the (dual) Alexandrov topologies thereupon [4].) Item 4 is in [2, p. 20]. Item 5 is almost immediate in the $\supset$-direction and follows with the help of Lemma 4 in the $\subset$-direction. $\square$

Now, we are ready to prove the proposition:

Proof (of Prop. 3) Lemma 5 says the D_j are a basis and that $AD^{-1}(D_j) = C_j$, which is open in M. $\square$

2.2.4 $\mathcal{T}_I$ is the Quotient Topology: Next, we want to show that the AD topology of Nerode and Kohn, $\mathcal{T}_I$, is exactly the quotient topology of their AD map. This is accomplished by proving that $\mathcal{T}_I$ is both coarser and finer than the quotient topology. The following is well-known [2, p. 143]: *Let X be a space; let A be a set; let $p : X \mapsto A$ be a surjective map. Then the quotient topology on A induced by p is the finest (largest) topology relative to which p is continuous.* Since the AD map is continuous in $\mathcal{T}_I$ and surjective, we trivially have: *$\mathcal{T}_I$ is coarser than the quotient topology, $\mathcal{T}_Q$, corresponding to AD.* Now, it remains to show that *$\mathcal{T}_I$ is finer than $\mathcal{T}_Q$.*

Proof Suppose J is open in $\mathcal{T}_Q$. Then $AD^{-1}(J)$ is open in $\mathcal{T}_M$. Finally, it can be written as $AD^{-1}(J) = \bigcup_{\beta \in B} C_\beta$, where B is some subset of $\{1, \ldots, N\}$, since

the C_β are a basis for $\mathcal{T}_M$. We want to show that $J \in \mathcal{T}_I$. But note that since AD is surjective

$$J = AD(AD^{-1}(J)) = AD\left(\bigcup_{\beta \in B} C_\beta\right) = \bigcup_{\beta \in B} AD(C_\beta)$$

Now, we have from Lemma 5 that $C_\beta = AD^{-1}(D_\beta)$. Since AD is surjective, this implies $AD(C_\beta) = AD(AD^{-1}(D_\beta)) = D_\beta$ So that $J = \bigcup_{\beta \in B} D_\beta$, which is open in $\mathcal{T}_I$, being a union of basis elements. □

Summarizing, we have shown

Theorem 6 *The AD topology of Nerode and Kohn, $\mathcal{T}_I$, is exactly the quotient topology of their AD map.*

We have also gotten something else along the way. In the last proof we showed that $AD(C_\beta) = D_\beta$. From Lemma 5, we have $AD^{-1}(D_\beta) = C_\beta$ and that D_β and C_β are bases for $\mathcal{T}_I$ and $\mathcal{T}_M$, resp. Thus, AD is a homeomorphism between the topological spaces $(M, \mathcal{T}_M)$ and $(I, \mathcal{T}_I)$. (This homeomorphism was also noted without proof in [3].)

3 Completing the Loop

3.1 Problems Completing the Loop

In this section, we discuss problems which arise when considering continuous mappings from the measurement to control spaces (see Figure 1). Specifically, we have

Remark 7 *If M is connected and C is T_1, the only continuous maps from M to a finite subset of C (i.e., $f(M) = \{c_1, \ldots, c_N\}$, $c_1, \ldots, c_N \in C$) are constant maps.*

Proof First, constant maps are always continuous, and their image is a single point of C, hence finite. Next, suppose for contradiction that f is a non-constant continuous map from M into C and the image $f(M) = \{c_1, \ldots, c_N\}$, where the c_i are distinct points in C for some finite N greater than or equal to two. Since C is T_1, we can construct open sets $U_{i,j}$ for $i \neq j$ such that $U_{i,j}$ contains c_i but not c_j. Thus, there is an open set about c_1 not containing $c_2, \ldots, c_N$, viz., $U = \bigcap_{j=2}^N U_{1,j}$. Also, we can construct an open set which contains each $c_2, \ldots, c_N$ yet does not contain c_1: $V = \bigcup_{i=2}^N U_{i,1}$. Therefore, $f(M) = U \cup V$ is not connected. □

3.2 Topologies Completing the Loop

In the previous subsection, we saw that, under mild assumptions, there are no non-constant continuous maps from the measurement to control spaces. In this subsection, we wish to give a topology on the (augmented) control space which allows us to construct a non-constant continuous map.

We make no assumptions on M and C (except those implicit in the definition of f below). Suppose that the topology on C is $\mathcal{T}$. Then we let

$C' = C \cup \{\perp\}$, that is, we append a single element, $\perp$, to C. Next, we define a topology, $\mathcal{T}'$, on C' as follows: $\mathcal{T}' = \emptyset, \{U \cup \{\perp\} \mid U \in \mathcal{T}\}$. Suppose we wish to have image points $c_1, \ldots, c_N$ in C. Let $f^{-1}(c_i) = K_i$ be disjoint closed sets not covering M. Let $f(M - \bigcup_{i=1}^N K_i) = \perp$. Then

Remark 8 *f is continuous.*

Proof $f^{-1}(\emptyset) = \emptyset$, which is open. Now, suppose U' is any non-empty open set of C'. Then $U' = U \cup \{\perp\}$, where U is open in C. Therefore,

$$\begin{aligned} f^{-1}(U') &= f^{-1}(U) \cup f^{-1}(\{\perp\}) \\ &= \left(\bigcup_{j \in J} K_j\right) \cup \left(M - \bigcup_{i \in I} K_i\right) \\ &= M - \bigcup_{i \in I-J} K_i \end{aligned}$$

(where J is the set of indices j for which $c_j \in U$, and $I = \{1, \ldots, N\}$) which is open since its complement is closed; the formula is well-defined if $J \neq I$. If $J = I$, then $f^{-1}(U') = M$, which is open. □

4 A New View of Hybrid Systems

We wish to propose a new view of hybrid systems as shown in Figure 2. The difference between this and the previous prototypical hybrid system is that there is feedback on the signal level. This feedback modulates the symbols coming down from the higher level. Alternatively, one can view the symbols as specifying one of several controllers whose output is to be the control signal.

The most widely used fuzzy control scheme is related to this model in the sense that it achieves continuous maps—despite a finite number of so-called fuzzy rules—by utilizing the continuous measurement information. We discuss this in more detail below.

4.1 Why the New View?

Before, we had a natural fan-in of sensory information from the signal to symbol levels. This models abstraction and reduction. In our new view, we also have an analogous, natural fan-out of control commands from the symbol to signal level that was not present before. Basically, we are saying that the finite description of the plant's dynamics as seen from automaton's point of view is not an exact aggregation of the plant's dynamics. Therefore, one should utilize the continuous information present at the lower level as well as the discrete decision made above in order to choose a control input for the lower level. There is no need to arbitrarily pick a member from the set of controls (fixed for normal AD conversion, always arbitrary in the Nerode-Kohn view). Instead, the set is given by the automaton, while the member of that set is chosen using information from the lower level. Thus, the aggregated and continuous dynamics are related, but the first is not a substitute for the latter.

If it were, the plant could have been modeled directly as a finite automaton.

4.2 Fuzzy Control Systems

4.2.1 The Control Scheme:

A fuzzy control scheme is given by the commuting diagram of Figure 3, where F denotes fuzzification, G the inference map of the fuzzy rule base, and D defuzzification. Here, the fuzzy controller has M rules of the form

RULE$_i$: IF x is A_i, THEN y is B_i, $i = 1, \ldots, M$

and $\mathcal{F}^M$ is a cross product of the space of fuzzy sets on Y. The most widely used inference rule computes $\mu_{B'_i}(z) = \min\{\mu_{A_i}(x), \mu_{B_i}(z)\}$, for all $z \in Y$, and defuzzifies using the centroid: $y = (\sum_{i=1}^{M} \overline{y}^i \ \mu_{B'_i}(\overline{y}^i))/(\sum_{i=1}^{M} \mu_{B'_i}(\overline{y}^i))$, where $\overline{y}^i$ equals the centroid of $\mu_{B'_i}$: $\overline{y}^i = (\int_Y z \ \mu_{B'_i}(z)dz)/(\int_Y \mu_{B'_i}(z)dz)$.

The finite rule base is related to the finite symbols of our hybrid model. For instance, the rules which fire are akin to the filters which passed data in our discussion of the Nerode-Kohn approach. However, here one utilizes the underlying continuous information (represented in the continuous membership functions for the measurement space) in order to construct the precise control output. Thus, it fits into our new view.

4.2.2 Producing Continuous Maps:

We will deal with the prototypical case where X and Y are closed intervals in R (for specificity, $[a, b]$ and $[c, d]$, resp.). The case where X is a multi-interval in R^m is a straightforward extension. The case where y is a multi-interval in R^n then follows from considering each dimension componentwise. We claim that the induced map $g = D \circ G \circ F$ is continuous from X to Y. We will assume, for the proof, that the μ_{A_i} and μ_{B_i} are continuous on X and Y, resp. This is fairly typical (e.g., triangular functions).

Proof If the μ_{A_i} are continuous, then F is continuous. It is also easy to see that centroid defuzzification, D, is continuous. It remains to show that G is continuous. Well, a fuzzy inference rule gives rise to the following situation: $H_{\alpha, f}(y) = \min\{\alpha, f(y)\}$, where $H_{\alpha, f}$, f, and α are playing the role of fixed $\mu_{B'_i}$, μ_{B_i}, and $\mu_{A_i}(x)$ resp. Thus, by assumption, $f(y) \in C([c, d] \to [0, 1])$. Now, we need G to be continuous as a map from, componentwise, $[0, 1]$ to $C([c, d] \to [0, 1])$. But, if $|\alpha_1 - \alpha_2| < \epsilon$, then $\|H_{\alpha_1, f} - H_{\alpha_2, f}\| < \epsilon$ where $\| \cdot \|$ denotes the sup norm. $\square$

4.2.3 Approximating Continuous Maps:

Fuzzy control maps are also dense in the set of continuous functions from X to Y. It is enough to note that triangular functions, which are prevalent for descriptions of fuzzy membership sets, are so dense. To more easily see this, note that triangular functions can be combined to construct arbitrary piecewise linear functions.

5 Summary and Conclusions

In this paper, we discussed some of the topological issues that arise when differential equations and finite automata interact in hybrid systems. We concentrated on the maps from a continuum to a finite symbol space and back again that such systems possess. We illuminated the general difficulties with the usual topologies in allowing continuous AD maps, constructed several topologies which bypassed them, and examined at length one such topology from [3].

We showed that there are inherent limitations present when one desires continuous maps from continuum to continuum through a finite symbol space, *viz.*, one must equip the continua with new topologies. We constructed a control space topology allowing a continuous map which "completes the loop."

We ended with a new view of hybrid systems that may broach these problems. As an example, we showed that the most widely used fuzzy logic control structure is related to this new view and that it indeed is a continuous map from measurements to controls. We further demonstrated that these fuzzy logic controllers are dense in the set of such continuous functions.

The full power of a hybrid, hierarchical structure is unknown at this stage. The popularity of fuzzy control seems due in part to the fact that humans deal better with—creating, constructing, modifying—a finite set of rules than with the continuous maps that they represent. Perhaps it is this aspect of a hybrid system, as an interdependent, hierarchical decomposition with feedback at various levels of abstraction, that will inevitably lead to truly intelligent control.

Appendix

This appendix collects the full proofs for statements 1–3 and 5 in Lemma 5. They are listed as separate lemmas for convenience.

Lemma 5.1 *The non-empty join irreducibles C_i form a basis of the topology, $\mathcal{T}_M$, which they generate.*

Proof Each $m \in M$ is contained in such a set since the A_i are a cover of M. The intersection of two such sets that contain the point m is a superset of U_m. $\square$

Lemma 5.2 *The sets D_i are a basis for the topology, $\mathcal{T}_I$, which they generate.*

Proof Each $i \in I$ is contained in D_i since $C_i \subset C_i$, so there is a basis element containing each $i \in I$. If $i \in I$ belongs to the intersection of two basis elements, say D_{j_1} and D_{j_2}, then we need a basis element

D_{j_3} containing i such that $D_{j_3} \subset D_{j_1} \cap D_{j_2}$. But then $C_i \subset C_{j_1}$ and $C_i \subset C_{j_2}$. So that $C_i \subset C_{j_1} \cap C_{j_2}$. From this, we want to show that D_i is contained in $D_{j_1} \cap D_{j_2}$. But this is evident from the definition of D_i:

$$D_i = \{j \mid C_j \subset C_i\}$$

So, if $j \in D_i$, then $C_j \subset C_i \subset C_{j_1}$, so that $j \in D_{j_1}$. Likewise, $C_j \subset C_i \subset C_{j_2}$, so that $j \in D_{j_2}$. Therefore, $D_i \subset D_{j_1} \cap D_{j_2}$, is the required basis element. $\square$

Lemma 5.3 *AD is surjective.*

Proof Pick $i \in I$. By construction, there exists $m \in M$ such that C_i is the smallest non-empty join irreducible containing m. $AD(m) = i$. $\square$

Lemma 5.5 $C_j = AD^{-1}(D_j)$

Proof

1. $C_j \supset AD^{-1}(D_j)$.

$$
\begin{aligned}
AD^{-1}(D_j) &= AD^{-1}(\cup_{k \in D_j} k) \\
&= \bigcup_{k \in D_j} AD^{-1}(k) \subset \bigcup_{k \in D_j} C_k \subset C_j
\end{aligned}
$$

The last inequality follows from the fact that $C_k \subset C_j$ for all $k \in D_j$.

2. $C_j \subset AD^{-1}(D_j)$. Suppose $m \in C_j$. Then either C_j is the smallest non-empty join irreducible containing m, in which case we are done, or there is some other smallest non-empty join irreducible C_k containing m. We claim $C_k \subset C_j$, in which case $k \in D_j$ and $m \in AD^{-1}(D_j)$, which is the desired result.

Therefore, it remains to show that $C_k \subset C_j$. The smallest join irreducible containing m is (see Lemma 4) is $U_m = \bigcap_{j \in J(m)} A_j$ where $J(m)$ is the set of all j such that $m \in A_j$. However, C_j is also a join irreducible, so that it can be written $C_j = \bigcap_{j \in J} A_j$ for some $J \subset \{1, \ldots, n\}$. But C_j contains m, so that each of the A_j in the intersection must contain m. So that by definition $J \subset J(m)$, whence $C_k \equiv U_m \subset C_j$.

$\square$

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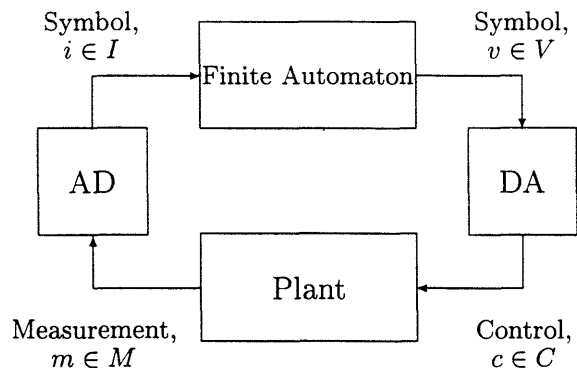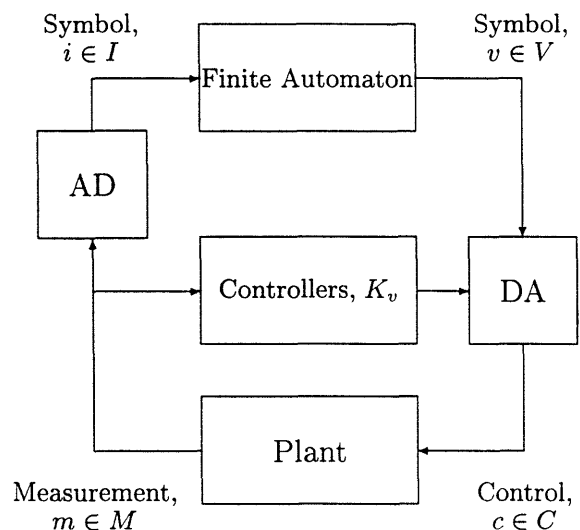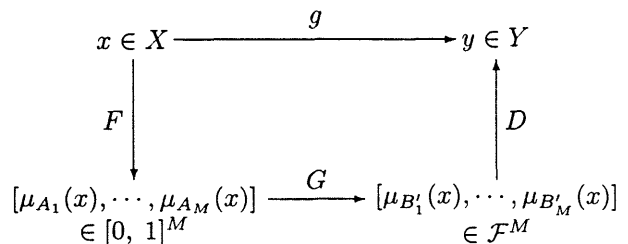
Figure 1: Prototypical Hybrid System

Figure 2: New Prototypical Hybrid System

Figure 3: Fuzzy Logic Controller