OPTIMIZATION MODELS FOR FOREIGN EXCHANGE RATE HEDGING USING CURRENCY OPTIONS

by

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ABSTRACT

Currency options provide a convenient means for multinational corporations to hedge future revenues and liabilities against exchange rate risk. While, ideally, companies would prefer to hedge their entire exposure at a single favorable strike price (exchange rate), it is frequently too expensive to do so. Moreover, a fully hedged position is usually unnecessary to adequately eliminate downside risk. Instead, the optimal strategy lies somewhere between the two extremes of a fully hedged and an unhedged position. In general, the optimal hedging strategy will be comprised of a package of currency options of various strike prices, each covering a portion of the exposure. We refer to the problem of deciding which strike prices to buy the options at, and the percentage of the exposure that each covers, as the options hedging problem.

This thesis explores three sets of mathematical models to evaluate the options hedging problem. The first set of models uses a probabilistic approach to the problem and describes how a currency options package changes the shape of the efficient market forecast of exchange rates to create a new distribution of what we call the effective exchange rate. The second set of models relaxes the notion of market efficiency and describes how best to take advantage of differences between the market's forecast and one's own. Finally, the third set of models explores how one can apply linear and stochastic programming methodologies to the problem. In this final section, we extend the scope of the decision process by allowing multi-period, multi-objective decision criteria. Additionally, applying mathematical programming to the problem allows us to account for constraints that may be specific to each company, such as budgetary requirements and cash flow needs.

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CHAPTER ONE

INTRODUCTION

1.1 A RATIONALE FOR THE FINANCIAL HEDGING OF FOREIGN CASH FLOW

Multinational corporations are constantly exposed to foreign exchange rate risk. A U.S. exporter of pharmaceuticals to W. Germany and Japan, for example, would see its profits decline in U.S. dollar value if the dollar strengthened in relation to the yen and the mark. Similarly, a U.S. importer of French wine would see its profits shrink if the dollar weakened in relation to the French franc, since the dollar cost of French wine would rise.

There are many ways to insure against changes in foreign exchange rates. For example, in the event of a strengthening dollar, the U.S. exporter could:

1) Raise prices of products for sale in the foreign market
2) Build plants in the foreign country
3) Abandon the foreign market
4) Use financial hedging

The first three alternatives may be broadly characterized as operating hedges of foreign exchange, since they involve adjustments to the operations of the firm. Operating hedges may be difficult to implement in practice and even if carried out, could have serious long-term repercussions. Raising prices on exports can cause substantial losses of market share, especially for products in competitive markets. Building production facilities abroad is expensive and risky, since doing so effectively locks a firm into the foreign market. It also replaces one form of risk -- currency risk -- with another -- sovereign
risk. Finally, abandoning a market is usually a last resort for most firms and has obvious drawbacks.

A viable alternative to these sorts of operating hedges of foreign exchange, is the fourth alternative listed above -- financial hedging. Financial hedging involves the purchase of financial instruments, such as currency forward contracts or currency options, and gives corporations a means of locking in exchange rates on future cash flows without affecting a firm's operating strategy. Though financial hedging comes at a cost, a wisely implemented hedging strategy can provide the cheapest and most effective defense against exchange rate volatility.

1.2 WHEN IS FINANCIAL HEDGING WORTH THE COST?

The two most common forms of financial hedging of exchange rates involve the purchase of currency forward contracts and currency options. Below, we shall define exactly how these instruments work and how they are priced. Before we do, there is one important point to make: These instruments are traded on extremely liquid markets. The economist would say that if we believe the efficient market hypothesis, then buying a currency forward or option is a zero NPV (net present value) transaction. That is, on average there can be no gain to buying forwards or options.

Fortunately, a number of factors do make these financial instruments worthwhile investments despite the zero-NPV claim. First; corporations may be risk averse and seek to change the distribution of returns implied by the market. Thus while the market may predict some finite probability that exchange rates will fall below a
certain point, a company may be willing to pay to avoid the possibility. Moreover, a company may have a different forecast of exchange rates than the market. Although it may not wish to speculate against the market, it may wish to hedge based on its own predictions.

One instance when financial hedging is not sufficient is if the firm expects a permanent adjustment in exchange rates. If the yen were to fall permanently against the dollar, a financial hedge could only "fix" the problem in the short-term. Thereafter, financial hedging costs would be too expensive, and the company would need to look to one of the forms of operating hedging discussed above -- raising prices, building local production facilities, or abandoning the market.

Thus, financial hedging is most useful in volatile markets, where large fluctuations are common but not expected to be permanent.

1.3 CHOOSING AN "OPTIMAL" HEDGING STRATEGY WITH CURRENCY OPTIONS

This thesis describes how to hedge foreign exchange with one type of financial instrument -- currency options. Chapter 2 reviews some basic concepts in foreign exchange and currency options, including the roles of the spot and forward markets and the pricing of currency options. The remaining chapters present different mathematical models describing the predicament of the multinational corporation attempting to insure future foreign cash flows. As we shall see, because an "optimal" hedging strategy means different things to different people, there is no simple solution to this problem.
CHAPTER TWO
CURRENCY OPTIONS AND THE BASICS OF FOREIGN EXCHANGE

Two markets dominate foreign exchange: the spot market and the forward market. Below, we describe these markets from the perspective of a dollar-based corporation -- i.e. one which is exchanging dollars for foreign currency (FC), or FC for dollars.

2.1 THE SPOT MARKET

The spot price of a FC is simply the exchange rate -- the amount of FC it takes to buy (or sell) a dollar today. For most currencies (with the exception of the British Pound), prices are quoted in FC per dollar. The spot markets for the most common foreign currencies are some of the most liquid in the world.

2.2 THE FORWARD MARKET

A forward contract is an obligation to buy (or sell) foreign currency for dollars at a specific date in the future. The forward market is also extremely liquid. Forward rates are easily calculated from the spot rate and Euro-interest rates on the dollar and the FC via a theory known as interest rate arbitrage. The interest-rate arbitrage formula is:

\[ s^*(e^{FC_i^*T}) = F^*(e^{US_i^*T}) \]

where \( S \) is the spot rate (in FC per dollar), \( F \) is the forward rate, \( FC_i \) is the interest rate on Euro-FC deposits, and \( US_i \) is the interest rate
on Euro-dollar deposits for any period of time, $T$. The idea is that if you convert a dollar into foreign currency today and invest the proceeds in Euro-FC bonds, you should earn the same amount of FC as you would if you invested in Euro-dollar bonds and entered into a forward contract, trading dollars for FC at the end of $T$, (see Weisweller [1984]).

2.3 THE BASICS OF CURRENCY OPTIONS

The disadvantage of a forward contract is that there is an obligation to exchange. Let's say we have entered into a forward contract to trade 100 million yen for dollars at the rate of 125 yen/dollar a year from now. If the dollar weakens to 100 yen/dollar at the end of the year, we will have lost:

\[
100,000,000/100 - 100,000,000/125 = 200,000.
\]

The obligation to exchange inherent in a forward contract can lead to large losses if there is a drastic change in exchange rates. Many corporations would choose to avoid this risk if possible.

Currency options work similarly to forward contracts, but do not require the holder to make the exchange. Also, unlike forward contracts, the holder can set the rate at which the exchange occurs, when he/she purchases the option. Thus, the holder of a currency option has the right, but not the obligation, to buy (or sell) FC for dollars at a stated exchange rate (called the strike price), on or before a certain date.
Currency options are extremely confusing. To begin with, different countries have different definitions for the same instrument:

- In the U.S., a put option on German marks gives the holder the right to sell marks for dollars.
- In West Germany, the same option is referred to as a call option on U.S. dollars-- the right to buy dollars for marks.

Thus we must be careful in our definitions. From here on, we will take the U.S. perspective:

- A put option gives the holder the right to sell FC for dollars.
- A call option gives the holder the right to buy FC for dollars.

Another technical aside has to do with the difference between American and European options.

- An American option can be exercised at any time up to expiration.
- A European option can only be exercised at expiration.

American options are generally worth more than European options, since changes in interest rates may make it worthwhile to exercise early and guarantee interest income. For the sake of simplicity, we will focus exclusively on European options in this paper.

Currency options are sold both on public markets, such as the Philadelphia Exchange, and over-the-counter (OTC) by banks and other financial institutions. The way that these contracts are structured, the
option premium, (or cost of the option), is paid up front. The next section describes how to calculate the option premium.

2.4 THE PRICING OF EUROPEAN CURRENCY OPTIONS

Basic options pricing theory is readily applied to currency options. Black and Scholes [1973] developed most of the theory used to price currency options, while Feiger and Jacquillat [1979] filled in some of the gaps. The pricing formulas which follow are presented in a paper by Biger and Hull [1983] and are based on the others' research.

The theory relies on three important assumptions which are discussed in the papers:

1) The exchange rate, $S$, follows a Geometric Brownian Motion.
2) The foreign exchange market operates continuously with no transaction costs or taxes.
3) Options are priced such that the NPV of all options is zero.

The following market parameters are needed to price an option:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Represents</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>Spot price</td>
<td>The current exchange rate in FC per dollar</td>
</tr>
<tr>
<td>$US_i$</td>
<td>U.S. Interest</td>
<td>The interest rate on Euro-dollar deposits for period $T$.</td>
</tr>
<tr>
<td>$FC_i$</td>
<td>FC Interest</td>
<td>The interest rate on Euro-FC deposits for period $T$.</td>
</tr>
<tr>
<td>$V$</td>
<td>Volatility</td>
<td>The volatility of the exchange rate, measured as the instantaneous standard deviation of the spot rate (in dollars per FC) as a percentage of the current spot price.</td>
</tr>
</tbody>
</table>
The call option premium is calculated as follows:

\[
c = \frac{e^{-FC_{-i^*T}}}{S} \cdot N\left\{ \ln \left( \frac{X}{S} \right) + \left[ US_i - FC_i + \frac{V^2}{2} \right] T \right\} \sqrt{V T} - \frac{e^{-US_{-1^*T}}}{X} \cdot N\left\{ \ln \left( \frac{X}{S} \right) + \left[ US_i - FC_i - \frac{V^2}{2} \right] T \right\} \sqrt{V T},
\]

where \(c\) is the call premium (in dollars), \(X\) is the strike price written on the option (in FC per dollar), \(T\) is the time to maturity (in years), and \(N\) is the cumulative distribution function for the zero-mean, unit variance Gaussian probability density function:

\[
N(y) = \int_{-\infty}^{y} \frac{-x^2}{2\pi} e^{\frac{-x^2}{2}} dx.
\]

The put option premium can be derived from the call option:

\[
p = c + \frac{e^{-US_{-i^*T}}}{X} - \frac{e^{-FC_{-i^*T}}}{S},
\]

where \(p\) is the premium on the put option (in dollars).

The formulas imply some interesting properties of currency options. Below, we list some of the key parameters in the model and the effect each has on the option premium:
The interaction of the two interest rate parameters, US_i and FC_i, and the time to maturity, T, with the option prices is more complicated. This is because there are two opposing forces. On the one hand, for both put and call options, there is a time value to the option -- the longer the time to maturity, the more likely that the option will expire "in the money." This result, which is true for any type of option, tends to increase the option's value (premium) with increasing T.

On the other hand, we must also consider the effect of foregone interest in the foreign and U.S. currencies. Let's assume that we wish to exchange FC for dollars in the future with a FC put option. Furthermore, assume that the U.S. interest rate is much higher than the foreign one. Then the larger T is, the longer we delay the exchange, and the longer we have to wait before we can start earning the U.S. interest rate. In order to eliminate an arbitrage opportunity, the premiums on the put options in this scenario tend to decrease with T. The foregone interest effect works similarly for call options. To see how, merely state the exchange from the foreigner's perspective: View the FC call options as dollar put options.

Thus, the two forces which determine how changes in FC_i, US_i and T affect the premiums are: 1) The time value of the pure option; and 2) the effect of foregone interest. Without specific market
parameters it is difficult to make any generalizations about which of the two is the dominant factor.

Another important property to recognize about currency options is that all of the parameters used in the pricing formulas except for the volatility (V) are predetermined by other markets. Thus, for options purchased over-the-counter, banks will typically quote both the option premium and the volatility used in the pricing formulas.

Volatilities can be estimated in many different ways, and therefore different banks will quote different volatilities. Moreover, the same bank will use different volatilities for options of different duration. Volatility is typically quoted at a higher rate for options of longer duration. For example, in mid-March, 1989, one bank quoted the following volatilities on Yen currency options:

<table>
<thead>
<tr>
<th>Duration</th>
<th>3 mo.</th>
<th>6 mo.</th>
<th>1 yr.</th>
<th>2 yr.</th>
<th>3 yr.</th>
<th>4 yr.</th>
<th>5 yr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility(%)</td>
<td>8.9%</td>
<td>9.1%</td>
<td>9.4%</td>
<td>10.5%</td>
<td>11%</td>
<td>12%</td>
<td>12.5%</td>
</tr>
</tbody>
</table>

**2.5 HEDGING WITH CURRENCY OPTIONS: THE IMPORTER AND THE EXPORTER**

Now that we have described currency options and how to price them, we return to the central problem at hand: How can multinational corporations effectively use currency options to hedge exchange rate risk? We will focus on two types of potential users -- importers and exporters.
2.5.1 The Exporter Uses Options to Hedge Foreign Revenues

A U.S. exporter can hedge future cash inflows denominated in FC by buying FC put options. The put options insure a minimum amount of dollar revenues.

For example, let's say a company expects DM100 million one year from now. Suppose the current spot rate is 1.9 DM/$ and that the company wishes to guarantee a nominal exchange rate of 2.10 DM per dollar or better. To do this, the company would buy, "one year DM 2.10 puts on DM 100 million."

At the end of the year, if the spot rate went below 2.10 (say to 1.80 DM per $), then the options would expire out of the money and the company would not exercise them. The effective exchange rate on the DM 100 million would be the spot rate -- 1.80. On the other hand, if the spot rate moved above 2.1 (say to 2.2), then the company would exercise and the effective exchange rate on the DM 100 million would be 2.1.

2.5.2 The Importer Uses Options to Hedge Foreign Liabilities

A U.S. importer can hedge future cash outflows denominated in FC by buying FC call options. The currency risk problem of the U.S. importer is completely analogous to that of the exporter. For example, instead of looking to guarantee a strongest dollar scenario, the importer would try to guarantee a weakest dollar scenario. He focuses on the dollar value of his foreign currency obligations.
2.5.3 Other Uses of Currency Options

There are of course many other uses of foreign currency options. Feiger and Jacquillat describe the use of currency options for corporations making bids on foreign business. The option allows the bidder to lock in an exchange rate for a bid without committing to the actual exchange, (as it would need to with a forward contract).

Some multinationals will combine different options to produce hedging strategies. For example, and exporter who expects DM 100MM may want to guarantee a "strongest dollar" scenario of DM 2.1 per dollar by buying put options costing $1MM. If he wishes to reduce his cost, he can also sell call options on marks at DM1.7 for, say $300,000. In the end, he has guaranteed an exchange rate between 1.7 and 2.1 for $700,000.

There are a multitude of other so-called "hybrid strategies" that a company can tailor to its hedging needs. However, this paper will focus exclusively on the case of the U.S. exporter hedging foreign revenues with put options. By limiting the problem in this way, we simplify the modeling without sacrificing generality. It is our intention to lay the ground work for the general problem, not to cover all types of strategies possible. We shall therefore focus on ways to think about optimality rather than trying to span all of the different strategies imaginable.

As for focussing exclusively on the U.S. exporter, remember that the importer's problem is completely analogous to the exporter's. Therefore, the importer's optimal strategy is simply the mirror image of the exporter's optimal strategy.
2.7 MAJOR ASSUMPTIONS OF THE HEDGING PROBLEM

We shall analyze the problem making the following assumptions:

- The company will hedge cash flows up to five years out.
- The company will use only European put options.
- The company will hold all options until maturity.

Note that there are two aspects of this hedging strategy that differentiate it from traditional techniques. First of all, the company will hedge cash flows, not balance sheet items. Traditionally, the treasury department of a multinational focuses on hedging balance sheet exposure, managing the values of foreign currency deposits, receivables and payables, as they appear on financial statements. While the traditional approach is useful from an accounting perspective, it does little to help preserve the economic value of future cash flows. In fact, at times, a balance sheet hedge can counteract cash flow hedging, (see Lessard [1986]).

Second, the company will hedge cash flows over a far longer time period than is typically done. Traditional balance sheet hedging is done with options and forward contracts of relatively short duration -- usually a year or less. We plan to hedge foreign currency over multi-year periods, including periods of up to five years.

Because there are no publicly traded options with durations over a year, this will mean buying options over-the-counter (OTC). In practice, the long-term hedge period will vary from company to company, depending on the attitude of the firm toward exchange rate
risk and the volatility in the foreign exchange and euro-interest rate markets.

For simplicity, the models presented in this paper make one additional assumption: Each year's cash flow from every country will be evaluated independently. That is, a DM-denominated cash flow expected in 1992 will be considered separately from a Yen-denominated cash flow in 1992, and from DM-denominated cash flows in 1991 and 1993.

There are some limitations to this approach, since it forces the company to absorb two types of risk internally:

- **Cross-Currency Risk**: In general, currencies do not move independently from one another, and thus a truly optimal strategy would need to consider cross-currency effects. By considering each exposure separately in our models, we force the company to bear all cross-currency risk. In most cases, the risk will be minimal in dollar terms.

- **Year-to-Year Risk**: The models look at each year separately. Thus, cash flows expected in 1993 are hedged independently from cash flow expected in 1994. If the company's objectives involve some interaction of cash flows from separate years (e.g. maximize net present value of all cash flows), then the models are insufficient.

2.8 THE REMAINING CHAPTERS: THREE MODELS OF OPTIONS HEDGING

In the remaining chapters of this thesis, we introduce three alternative techniques of approaching the options hedging problem.

In Chapter 3, we look at a model that assumes that the market is efficient -- i.e. that the market forecast of exchange rates is the best one available. Using this approach, we show how one can use options
to reshape the distribution of returns from foreign exchange. In Chapter 4, we relax the efficient market assumption, and look at a way to value options given that our forecast differs from the market's.

Finally, in Chapter 5, we take a mathematical programming approach to the options hedging problem by discretizing the forecast and viewing the situation as a multi-period, multi-objective optimization problem. Techniques used in this last section include linear and multi-period stochastic programming with recourse.
CHAPTER THREE

USING CURRENCY OPTIONS IN AN EFFICIENT MARKET:

HOW OPTIONS RESHAPE THE MARKET FORECAST
OF EXCHANGE RATES

If we accept the efficient market hypothesis, then we must believe that there are no bargains in currency options. Specifically, the NPV of all currency options must be zero, and therefore, the present value of an option's expected payoff must equal its cost.

Does this mean that there is no point for a company to purchase options in order to hedge cash flows?

We don't think so -- not if the company is interested in the distribution of its returns from foreign exchange, rather than just the average return. The efficient market assumption only states that, on average, the net present value of an investment in currency options is zero. It says nothing about the shape of the distribution of possible returns.

We would argue that companies generally are not interested in averages. Rather, they are interested in actual results. If a strengthening dollar wipes out 25% of an exporter's foreign cash flow one year, it is no consolation that, according to the market, the outcome was a freak occurrence -- that on average the firm would have done better. By buying currency options, the firm could have averted the large losses or at least minimized the damage.

Firms need a better way of describing what does matter to them in terms of exchange rate risk. Only then can they assess the "value" of options in hedging foreign cash flows. In this chapter, we start with
the assumption that markets are efficient -- that options are fairly priced. We find that behind options' pricing, there exists an implied market forecast of exchange rates. We will show that this forecast is described by a lognormally distributed random variable. By focussing on the entire probability distribution rather than the expected value (the forward rate), we can gain a great deal of insight into the risks we face in an unhedged position.

We go on to show how a hedged position, in which we have covered all or part of our exposure with currency options struck at various prices, changes the shape of the distribution of returns. As we shall see, if any of the options expires in the money, it will change the effective exchange rate, and therefore, the entire distribution of returns.

Once we have described the way that options affect our exposure, we turn to the question of how to choose a package of options, only now incorporating the effects that each package has on risk and return.

3.1 THE MARKET FORECAST OF EXCHANGE RATES

In Chapter 2, we introduced the options pricing formulas for call and put options on foreign currency. We reproduce the equations here with a slightly different notation, and measuring the exchange rate in dollars per FC rather than FC per dollar:
\[
\begin{align*}
    c &= e^{-FC_i \cdot T} S_0 \times N\left\{ \frac{\ln\left(\frac{S_0}{X}\right) + \left[\text{US}_i - FC_i + (V^2/2)\right] T}{V\sqrt{T}} \right\} \\
    - e^{-US_i \cdot T} X \times N\left\{ \frac{\ln\left(\frac{S_0}{X}\right) + \left[\text{US}_i - FC_i - (V^2/2)\right] T}{V\sqrt{T}} \right\} \\
    p &= c + X e^{-US_i \cdot T} - S_0 e^{-FC_i \cdot T}
\end{align*}
\]

where \(c\) and \(p\) are the call and put premiums (in dollars); \(X\) is the strike price written on the option (in dollars per FC); \(S_0\) is the current spot price of the FC (in dollars per FC); \(T\) is the time to maturity (in years); \(N\) is the cumulative distribution function for the zero-mean, unit variance Gaussian distribution; and \(V\) is the instantaneous standard deviation of the return on a unit of foreign currency.

As we mentioned earlier, Black and Scholes derived these formulas by making two critical assumptions. The first is simply the efficient market hypothesis: The expected present value of any option must be zero. Thus, if we know the market forecast of the spot rate at time \(T\), \(f_{ST}(S)\), it must be true that:

\[
\begin{align*}
    c &= e^{-US_i \cdot T} \int_{S=X}^{S=X} (S - X) f_{ST}(S) \, dS \\
    p &= e^{-US_i \cdot T} \int_{S=0}^{S=X} (X - S) f_{ST}(S) \, dS
\end{align*}
\]

These two formulas just say that the cost of a call (put) option is equal to the present value of the expected future payoff.
Black and Scholes' second assumption -- that the spot rate, $S$, follows a Geometric Brownian Motion with instantaneous standard deviation $V$ -- implies that the distribution, $f_{S_T}(S)$, is lognormally distributed, (see Black [1973]). By the definition of lognormality, we know that the natural logarithm of $S_T$ must be normally distributed -- that is, $Y=\ln(S_T)$ is normally distributed. Furthermore, if we define $b$ to be the mean of $Y$ and $\sigma^2$ to be the variance, (so that $Y \sim N(b,\sigma^2)$), then the density function for $S_T$ is defined by:

$$f_{S_T}(S) = \frac{1}{\sqrt{2\pi} \sigma} \exp\{-(\ln(s) - b)^2/2\sigma^2\} \quad s \geq 0$$

0 otherwise

where,

$$E(S) = e^{b+0.5\sigma^2}$$

$$\sigma_s^2 = e^{2b + \sigma^2} (e^{\sigma^2} - 1)$$

To find the values of the parameters $\sigma^2$ and $b$, we use the fact that, by the definition of Brownian Motion, the variance of $\ln(S_T)$ must equal $\sigma^2$. Also, we know that the forward rate on the FC at time $T$, $F_T$, (in dollars per FC), must be an unbiased estimator of the spot rate at time $T$. Therefore,

$$E(S_T) = F_T,$$

where,

$$F_T = S_0 e^{(uS_1 - iFC_1)T}$$
It follows then that,

\[ b = \ln(S_0) + (US_1 - FC_1) \cdot T - 0.5 \cdot V^2 \]

We shall use the symbol \( \Lambda(b, \sigma^2) \) to denote lognormality, where \( b \) and \( \sigma^2 \) are the mean and variance of the gaussian distribution defined by the natural log of the lognormal random variable. Our result then is:

\[ S_T \sim \Lambda(\ln(S_0) + (US_1 - FC_1) \cdot T - 0.5 \cdot V^2, V^2) \]

We now have a complete description of the forecast of the spot rate implied by the options pricing formulas. And, because essentially all banks and financial institutions quote prices that are consistent with these pricing formulas, \( f_{ST}(S) \) is also the forecast implied by market prices.

An Example of a Market Forecast of Exchange Rates

To illustrate the derivation from the last section, we consider the market forecast of German Mark spot rates one year hence, given the following market parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>So</td>
<td>$0.56/\text{mark}$</td>
</tr>
<tr>
<td>US(_1)</td>
<td>10%</td>
</tr>
<tr>
<td>FC(_1)</td>
<td>7%</td>
</tr>
<tr>
<td>V</td>
<td>10%</td>
</tr>
</tbody>
</table>
The implied market forecast of the spot rate a year from now is:

$$S_1 \sim \Lambda [\ln(0.56) + (0.10 - 0.07) \cdot 1 - 0.5 \cdot (0.10)^2, (0.10)^2]$$

$$\sim \Lambda [-0.6148, 0.01]$$

This forecast is plotted in Figure 3.1.

![Figure 3.1](image)

3.2 USING CURRENCY OPTIONS TO CREATE AN EFFECTIVE EXCHANGE RATE

Now that we have determined the general form of the market forecast of exchange rates, we can begin to analyze how currency options affect exchange rate risk. To do so, we will need to make one important assumption regarding the nature of our foreign currency exposure:

- We assume that the exposure is nominally fixed, and that we know its exact amount.
This means that if our exposure is specified in real cash flows, we will need to convert them to nominal cash flows. For illustrative purposes, we shall further specify the problem, concentrating exclusively on using put options to hedge foreign currency inflows. Specifically, we assume that the company of interest needs to hedge 100 million German marks expected one year hence and that the market parameters are as in Section 3.1.

The essence of the hedging problem now boils down to two questions:

- What strike prices should the put options have?
- What percentages of the exposure should each be written on?

Although, ideally, the company would like to cover its entire exposure at a single high strike price, budget constraints will generally limit both the strike prices and the coverage.

We define the effective exchange rate, $S_E$, to be the total dollars received (from both options and exchanging operating flow) on the exercise date divided by the total operating flow (in FC). For example, if we cover 50% of our DM100 million exposure with put options struck at 50¢, and the spot rate moves to 49¢, our option expires in the money: The effective exchange rate at time $T$ is then:

$$
\frac{50,000,000 \times 0.5 + 50,000,000 \times 0.49}{100,000,000} = 49.5\text{¢ per DM}
$$

(Note: In this chapter, we do not include the option's cost in the effective exchange rate). In general, the effective exchange rate, $S_E$, is
a linear function of the strike prices of options purchased, of the
coverage rates on those options, and of the market forecast of
exchange rates, $S_T$. Moreover, because $S_T$ is a random variable, so too
is $S_E$. In the next few sections, we show that $f_{S_E}(S)$ is, in fact,
piecewise lognormal, where each lognormal region corresponds to an
option.

Note also that there is one aspect, common to all feasible
distributions for the effective exchange rate, which makes it trivial to
calculate the expected value of the effective exchange rate:

$$E(S_E) = E(S_T) + \exp(U_{S_i}T) \times \text{(cost of all puts)}.$$ 

This simply states that, on average, there are no bargains. The
expected effective exchange rate is the forward rate minus the future
value of the cost of the options package. The reader should keep this
in mind as we analyze the piecewise nature of the distribution.
3.2.1 Covering 100% of the Exposure with a Single Put Option

The simplest case to conceive of, other than no hedge at all, is when the entire exposure is hedged with a single option. For example, we might purchase a put option with a strike price of $X per mark, on the entire DM100 million. In this case, writing $S_E$ as a function of $S_T$ is simple. If $S_T \geq X$, the put option expires out of the money, and $S_E = S_T$. If $S_T < X$, the option expires in the money, and $S_E = X$. The density function $f_{S_E}(S)$ thus equals $f_{S_T}(S)$ for $S > X$; and is an impulse at $S = X$ with area:

$$\int_{S=0}^{X} f_{S_T}(S) \, dS$$

The distribution for $f_{S_E}(S)$, using parameters from our DM example is depicted in Figure 3.2, for $X = 53¢$. 

![Figure 3.2](image)
3.2.2 Covering Part of the Exposure with a Single Put Option

Next, we consider the more interesting case of partial coverage. If we purchase a put option with a strike price of \( X \), covering only part of the DM100 million exposure -- say \( f \% \) (where \( 0 < f < 1 \)), the distribution becomes more complicated. If \( S_T > X \), then the option expires out of the money and, just as before, \( S_E = S_T \). However, if \( S_T \leq X \), the option expires in the money. Unlike the 100% coverage case, however, only part of the exposure can be exchanged for $X. The remainder must be exchanged at the real rate, \( S_T \). Therefore, in this region, the effective exchange rate will be a weighted average of \( X \) and the real exchange rate:

\[
S_E = \begin{cases} 
S_T & \text{if } S_T > X \text{ (Region 1)} \\
(1-f)S_T + fX & \text{if } S_T \leq X \text{ (Region 2)} 
\end{cases}
\]

To obtain the distribution \( f_{S_E}(S) \), we use the law of conditional probability:

\[
f_{S_E}(S) = f_{S_E}(S \mid S_T> X) \cdot Pr(S_T> X) + f_{S_E}(S \mid S_T \leq X) \cdot Pr(S_T \leq X)
\]

The first term, corresponding to Region 1, is no problem to evaluate. Since \( S_E = S_T \) in this region, it is simply the original distribution cut off at \( S_E = X \). However, to evaluate the second term, corresponding to Region 2, where the option expires in the money and \( S_E = (1-f)S_T + fX \), we first need to explain the effect of a linear transformation of a lognormal random variable.
The Effect of a Linear Transformation of a Lognormal R.V.

In general, if a random variable $X$ is lognormally distributed, where $f_X(x) \sim \Lambda(b, \sigma^2)$, then the variable $Z$ defined by $Z = dX + g$ is a "shifted" lognormal random variable. To see why, consider $Z$ where $g=0$. In this case, we know that:

$$\ln(Z) = \ln(d \cdot X) = \ln(d) + \ln(X) = \Lambda(\ln(d) + b, \sigma^2).$$

From this, we can infer that if $g=0$, $Z$ is also lognormal with $Z \sim \Lambda(\ln(d)+b,\sigma^2)$. Now, in considering the general case, we realize that the effect of a non-zero $g$ is simply to shift the distribution. For example, if $g=1$, the distribution will be lognormal, starting at $Z=1$ rather than $Z=0$.

We adapt our notation for the lognormal distribution to include the shift property by adding a third "shift" term -- $g$. If $X \sim \Lambda(b,\sigma^2,g)$, then $X$ is lognormally distributed as before, except that it is shifted to the right by the constant $g$.

Returning to the options problem -- we need to find $f_{SE}(S \mid S \leq X)$, where $S_E = (1-f) \cdot S_T + f \cdot X$ and $S_T$ is lognormally distributed. Using the derivation just discussed, it is clear that:

$$f_{SE}(S \mid S \leq X) \sim \Lambda(\ln(1-f) + b, \sigma^2, f \cdot X) / \Pr(S_T < X) \quad \text{for } f \cdot X < S_E < X,$$

(where we have divided by $\Pr(S_T < X)$ to rescale the density function so that all probability sums to one. The factor will disappear when we add the joint probabilities). Thus, to obtain $f_{SE}(S)$ over all of $S_E$, we simply add the two joint distributions. Figure 3.3 depicts $f_{SE}(S)$ for our German mark example with $X = 53\text{¢}$ per mark and $f=40\%$. (This corresponds to buying a put option struck at 53¢ per mark on DM 40 million of our DM 100 million exposure). The cost of this hedge, calculated using the options pricing formula, would be approximately $220,000:
Figure 3.3
To get an idea of the magnitude of the change of exchange rates realized by buying the 53€ option, we have tabulated the cumulative probabilities for both the real and effective exchange rates:

<table>
<thead>
<tr>
<th>S</th>
<th>Unhedged Pr(S₁&lt; S)</th>
<th>Hedged Pr(Sₑ&lt; S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.47</td>
<td>2.3%</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.48</td>
<td>3.7%</td>
<td>0.6%</td>
</tr>
<tr>
<td>0.49</td>
<td>5.6%</td>
<td>1.6%</td>
</tr>
<tr>
<td>0.50</td>
<td>8.3%</td>
<td>3.7%</td>
</tr>
<tr>
<td>0.51</td>
<td>11.8%</td>
<td>7.3%</td>
</tr>
<tr>
<td>0.52</td>
<td>16.1%</td>
<td>13.1%</td>
</tr>
<tr>
<td>0.53</td>
<td>21.2%</td>
<td>21.2%</td>
</tr>
</tbody>
</table>
A few properties, apparent in this example, are generally true:

- There is no overlap on \( f_{SE}(S) \) of the two contributing regions
- All subregions are lognormally distributed
- The lowest possible effective exchange rate is \( f^*X \).

### 3.2.3 Covering the Exposure with a Variety of Put Options

The next logical extension is to allow hedging with a number of put options at various coverage rates. Consider the general case, in which we purchase \( N \) put options, where the strike prices are \( X_1 < X_2 < X_3 < \ldots < X_N \), and the coverage rate on each is given as \( f_1, f_2, f_3, \ldots, f_N \), respectively, (where \( \Sigma f_i = 1 \)).

The strike prices, \( X_i \), divide the event space of \( S_T \) into \( N+1 \) regions. Let's begin with Region 1: If the real exchange rate lands in Region 1, \( (S < X_1) \), all of the options will expire in the money, and the effective exchange rate will be a weighted average:

\[
S_{E | S < X_1} = \sum_{i=1}^{N} \left[ f_i X_i + (1-f_i) S_T \right] \quad \text{for} \quad \sum_{i=1}^{N} f_i X_i < S_E < \sum_{i=1}^{N} \left[ f_i X_i + (1-f_i) X_1 \right]
\]
In general, if the real exchange rate lands in Region K, (so that $X_{K-1} < S_T < X_K$), options with strike prices of $X_K$ or higher will expire in the money, while those with strike prices of $X_{K-1}$ or lower will expire out of the money and can be ignored. The equation for the effective exchange rate thus becomes the following weighted average:

$$S_E | X_{K-1} < S_T < X_K = \sum_{i=K}^{N} [f_i X_i + (1-f_i) S_T]$$

for $S_T > \sum_{i=K}^{N} [f_i X_i + (1-f_i) X_{K-1}]$ and $S_T < \sum_{i=K}^{N} [f_i X_i + (1-f_i) X_K]$

For each region, we see that the values of $S_E$ conditional on $S_T$ are linear combinations of the real exchange rate $S_T$. Therefore, as before, we can determine lognormal distributions which correspond to each region and connect pieces of the lognormals to obtain $f_{S_E}(S)$ (via conditional probability). Thus, Region K corresponds to the lognormal:

$$f_{S_E}(S | X_{K-1} < S_T < X_K) \sim \mathcal{N}(\sum_{i=K}^{N} (1-f_i) X_i, \sqrt{\sum_{i=K}^{N} f_i X_i})$$

for $S_T > \sum_{i=K}^{N} [f_i X_i + (1-f_i) X_{K-1}]$ and $S_T < \sum_{i=K}^{N} [f_i X_i + (1-f_i) X_K]$

= 0 otherwise.

The K+1 regions on $S_T$ will map to K+1 contiguous, non-overlapping regions of $S_E$. Note that the bounds on $S_E$ are determined by plugging in the corresponding bounds on $S_T$ into the linear equation. Therefore, the minimum value of $S_E$ corresponds to $\Sigma(f_i X_i)$ and occurs if the FC is completely worthless at time T.
To illustrate the effect of a multi-option hedge, we again consider the example of German mark hedging. This time we shall use a combination of three different put options:

\[
\begin{array}{c|c}
X & f \\
0.51 & 25\% \\
0.525 & 15\% \\
0.545 & 10\% \\
\end{array}
\]

The cost of this options package is $222,000 to hedge the DM 100 million exposure, (derived via the options pricing formula). This is only slightly more expensive than the single option strategy discussed earlier and therefore will be interesting for comparison.

As we noted above, a three option package will map to four regions on the event space of the PDF, \( f_{S_1}(S) \). Below, we have tabulated the linear mappings of the event space of \( S_1 \) onto \( S_E \). We also include the lognormal PDFs and their bounds from which the piecewise lognormal PDF, \( f_{S_E}(S) \), will be formulated:

\[
\begin{align*}
\text{Region 1: } & 0 < S_1 < 0.51 & \text{Region 2: } & 0.51 < S_1 < 0.525 \\
\Sigma(f_i) &= 0.5 & \Sigma(f_i) &= 0.25 \\
\Sigma(f_iX_i) &= 0.2608 & \Sigma(f_iX_i) &= 0.1333 \\
S_E &= 0.2608 + 0.5*(S_1) & S_E &= 0.1333 + 0.75*(S_1) \\
f_{S_E}(S) &\sim \Lambda(ln(0.5)+b, V^2, 0.2608) & f_{S_E}(S) &\sim \Lambda(ln(0.75)+b, V^2, 0.1333) \\
\text{Bounds: } & 0.2608 < S_E < 0.5158 & \text{Bounds: } & 0.5158 < S_E < 0.5270
\end{align*}
\]
Region 3: $0.525 < S_1 < 0.545$

Region 4: $0.545 < S_1<^*$

$\Sigma(f_i) = 0.1$

$\Sigma(f_iX_i) = 0.0545$

$S_E = 0.0545 + 0.9*(S_1)$

$E(f) = 0.0$

$E(f) = 0.0$

$S_E = S_1$

$f_{S_E}(S) \sim \Lambda(\ln(0.9) + b, V^2, 0.0545)$

$f_{S_E}(S) \sim \Lambda(b, V^2, 0)$

Bounds: $0.5270 < S_E < 0.5450$

Bounds: $0.545 < S_E < ^*$

$t^* t = \ln(0.56) + (0.10 - 0.07)*1 - 0.5*(0.10)^2 = -0.6148$

$V^2 = (0.10^2) = 0.01$

To obtain the PDF, $f_{S_E}(S)$, we simply connect the PDFs for each region.

The bounds on each region are non-overlapping and contiguous on $S_E$. In Figure 3.4, we have plotted $f_{S_E}(S)$ for this case. Notice that, despite the fact that this package of options costs about the same as the one-option package, its shape is quite different. So too are the cumulative probabilities, which we list below:

<table>
<thead>
<tr>
<th>S</th>
<th>Pr(S_1&lt;S)</th>
<th>Pr(S_E&lt;S)</th>
<th>Pr(S_E&lt;S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.47</td>
<td>2.3%</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.48</td>
<td>3.7%</td>
<td>0.0%</td>
<td>0.6%</td>
</tr>
<tr>
<td>0.49</td>
<td>5.6%</td>
<td>1.2%</td>
<td>1.6%</td>
</tr>
<tr>
<td>0.50</td>
<td>8.3%</td>
<td>3.4%</td>
<td>3.7%</td>
</tr>
<tr>
<td>0.51</td>
<td>11.8%</td>
<td>7.9%</td>
<td>7.3%</td>
</tr>
<tr>
<td>0.52</td>
<td>16.1%</td>
<td>14.1%</td>
<td>13.1%</td>
</tr>
<tr>
<td>0.53</td>
<td>21.2%</td>
<td>20.3%</td>
<td>21.2%</td>
</tr>
<tr>
<td>0.54</td>
<td>27.0%</td>
<td>26.6%</td>
<td>27%</td>
</tr>
<tr>
<td>0.54</td>
<td>33.4%</td>
<td>33.4%</td>
<td>33.4%</td>
</tr>
</tbody>
</table>
3.3 CHOOSING AN OPTIONS PACKAGE

Now that we know how options change the distribution of effective exchange rates, we turn to the question of which package of options to purchase.

The most natural way to incorporate the probabilistic analysis of the last section into hedging strategy is to think in terms of paying to get rid of unwanted probability density. The U.S. exporter hedging FC inflows buys put options in order to squeeze probability density out of the left tail of the distribution of the effective exchange rate. Similarly, the U.S. importer hedging FC outflows buys call options in order to squeeze probability density out of the right tail of the distribution.

One way to measure the relative performance of different options packages is to choose a point on the event space of $S_E$ and compare cumulative probabilities and costs of each package. For example, in the situation described in the last section where we are hedging German marks, we can choose the 49¢ per mark point as the critical juncture.
With no hedging, the cumulative probability of landing below 49¢ is 5.6%; with the 1-option package, it is 1.6%; while, with the 3-option package, it is 1.2%. Although the two packages cost the same (about $220,000), the 3-option package is more effective at avoiding an effective exchange rate of 49¢ or lower, and would be preferred.

Another approach is to select a set of points on the effective exchange rate distribution and require that the cumulative probabilities at each be below a certain threshold. Thus, in our example, we might require that the cumulative probabilities of being below 50¢ and 53¢ be below 4% and 13.5%, respectively. In this case, the 1-option package would be preferred to the 3-option package, since the latter has a 14.1% probability of landing below 53¢ per mark.

Procedures such as these are extremely useful. For one thing, they coincide nicely with issues that managers can immediately understand and act upon. Moreover, because market forecasts of effective and real exchange rates are continuous random variables, it is very natural to think in terms of manipulating the probabilities rather than the manipulating expected exchange rates.

The techniques that I suggest do not show how to choose the optimal options package -- only how to compare different packages. This is a valuable first step. The next logical step, of course, is to devise optimization schemes which exploit the properties of the derived distribution and sample the entire (infinite) set of possible options packages. For example, two interesting questions to ask are:

* Given that we will not pay more than $C_{\text{max}}$, which package will minimize the probability of facing an effective exchange rate below $S_{\text{juncture}}$?
• What is the least we need to pay to ensure that the probability of an effective exchange rate below \( S_{juncture} \) is less than \( P \)?

Unfortunately, unlike the simpler discrete models of later chapters, the continuous distributions derived in this chapter do not lend themselves well to conventional mathematical programming analyses.
CHAPTER FOUR

CHOOSING CURRENCY OPTIONS

IN A NON-EFFICIENT MARKET

In this chapter, we relax the notion that markets are efficient and that all options are worth the same in expected value terms. Markets can break down for a variety of reasons. Some currency option markets simply do not have the volume to be efficient. In other cases, government intervention may destroy the free movement of currencies, and make the assumption of Brownian Motion untenable.

Whatever the reason, it may be that our best forecast of future spot rates differs from the market's. In this case, one can differentiate options on the basis of expected values. In this chapter, we briefly describe how to find the strike price of the option with highest expected value per dollar of premium. We call this measure the utility of an option. And it follows that the option with the highest utility is the one that is the most valuable in terms of maximizing NPV.

4.1 THE UTILITY OF AN OPTION

We now assume that we have our own forecast of the future spot rate, defined by $f_{\text{own}}(S)$. This forecast may or may not be lognormal. But it definitely differs from the market forecast, $f_{\text{mkt}}(S)$. In particular:
In this chapter, as before, we focus mostly on put options. The analysis for call options is completely analogous. We define the utility of an option of strike price \( X \) to be its expected payoff minus the future value of its cost, all divided by its cost, (where the cost of the option is computed via the options pricing formulas, using the market forecast).

\[
U(X) = \frac{E(\text{payoff}) - FV(COST(X))}{\text{COST}(X)}
\]

Using this definition, it is clear that the option whose strike price yields the highest utility will have the highest expected payoff per dollar spent and therefore be the optimal investment. We call the strike price of this option \( X_{\text{Umax}} \).

4.2 **CALCULATING \( X_{\text{Umax}} \) FOR LOGNORMAL FORECASTS WITH VOLATILITY=\( V_{\text{OWN}} \)**

If we agree with Black-Scholes' assumption that the spot rate follows a Brownian motion, but disagree with the market's pricing on
currency options, then the point of disagreement must be the volatility quote, $V$. This is because the other market parameters -- interest rates and the current spot rate -- are fixed and available on other (extremely liquid) markets.

The Brownian Motion assumption guarantees that our own forecast of the spot rate must be lognormal, similar to the market’s. In fact, to calculate our forecast of the expected payoff, all we need to do is substitute our own estimate of volatility, $V_{own}$, into the options pricing formula. Thus, the formula for utility simplifies to:

$$U(X) = \frac{[P(V_{own}) - P(V_{mkt})]*\exp(US_{i}*T)]}{P(V_{mkt})} \quad \text{for put options}$$

$$U(X) = \frac{[C(V_{own}) - C(V_{mkt})]*\exp(US_{i}*T)]}{C(V_{mkt})} \quad \text{for call options}$$

where $P(V)$ and $C(V)$ are simply the prices of a put option and a call option, calculated using the options pricing formulas, evaluated at volatility $V$.

Below, we show an example of evaluating utilities for specific market parameters. Though, we could not prove the generalizations we make, we have experimented enough to feel confident that they are true.

<table>
<thead>
<tr>
<th>Market Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot Rate</td>
<td>1.87 DM/$ (=0.5348 $/DM)</td>
</tr>
<tr>
<td>US$_i$</td>
<td>10.19%</td>
</tr>
<tr>
<td>FC$_i$</td>
<td>7.00%</td>
</tr>
<tr>
<td>Time (Yrs.)</td>
<td>1</td>
</tr>
<tr>
<td>Volatility</td>
<td>11.30%</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>10%</td>
</tr>
</tbody>
</table>
4.2.1 If the Market is Underpricing Options: $V_{own} > V_{mkt}$

For put options, where the market volatility is lower than our own forecasted volatility, (i.e. we feel that options are underpriced), we found that the utility is a decreasing function of the strike price. For example, using our own forecast of $V_{own} = 13\%$, we get:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(V=11.3%)$</th>
<th>$P(V=13%)$</th>
<th>$U(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.0000</td>
<td>0.0001</td>
<td>1.97</td>
</tr>
<tr>
<td>0.42</td>
<td>0.0001</td>
<td>0.0004</td>
<td>1.74</td>
</tr>
<tr>
<td>0.44</td>
<td>0.0004</td>
<td>0.0010</td>
<td>1.55</td>
</tr>
<tr>
<td>0.46</td>
<td>0.0011</td>
<td>0.0022</td>
<td>0.98</td>
</tr>
<tr>
<td>0.48</td>
<td>0.0027</td>
<td>0.0043</td>
<td>0.66</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0056</td>
<td>0.0079</td>
<td>0.46</td>
</tr>
<tr>
<td>0.52</td>
<td>0.0103</td>
<td>0.0132</td>
<td>0.31</td>
</tr>
<tr>
<td>0.54</td>
<td>0.0172</td>
<td>0.0205</td>
<td>0.21</td>
</tr>
</tbody>
</table>

For call options, we get just the opposite result -- utility is an increasing function of the strike price. For example, using our own forecast of $V_{own} = 13\%$, we get:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$C(V=11.3%)$</th>
<th>$C(V=13%)$</th>
<th>$U(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.1374</td>
<td>0.1375</td>
<td>0.00</td>
</tr>
<tr>
<td>0.42</td>
<td>0.1195</td>
<td>0.1197</td>
<td>0.00</td>
</tr>
<tr>
<td>0.44</td>
<td>0.1017</td>
<td>0.1022</td>
<td>0.01</td>
</tr>
<tr>
<td>0.46</td>
<td>0.0843</td>
<td>0.0854</td>
<td>0.01</td>
</tr>
<tr>
<td>0.48</td>
<td>0.0679</td>
<td>0.0695</td>
<td>0.03</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0527</td>
<td>0.0550</td>
<td>0.05</td>
</tr>
<tr>
<td>0.52</td>
<td>0.0393</td>
<td>0.0422</td>
<td>0.08</td>
</tr>
<tr>
<td>0.54</td>
<td>0.0281</td>
<td>0.0314</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Thus, in general, given that we feel that the market is overpricing currency options (by quoting a low $V$), the farther the option is out-of-the-money, the higher its utility.
4.2.2 If the Market is Overpricing Options: \textit{Vown}<\textit{Vmkt}

Similarly, we can show that if the market is overestimating the volatility, (therefore overpricing options), then the further the option is in-the-money, the higher (less negative) its utility. (Just substitute \textit{Vmkt} for \textit{Vown}, and vice versa above). Note also, though, that if the market is overpricing options in this way, all investments are negative NPV transactions. In this case, we are better off \textit{writing} currency options than we are buying them.

4.3 \textbf{CALCULATING \textit{UXMAX} FOR ARBITRARY FORECASTS}

If we allow an arbitrary forecast of the spot rate, \textit{f_{S_{own}}}(S), then we must, in general, compute the integrals in the equation given for \textit{U(X)} is Section 4.1. Below, we show derivations of \textit{UXMAX} for a square (uniformly distributed) forecast and a triangular distribution. (Note: In these examples, we have assumed the market parameters from Section 4.2, and specified \textit{S} and \textit{X} in units of FC per dollar, not $/FC).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.1.png}
\caption{Market Prices of Put Options}
\end{figure}
Example 1: Square Distribution

Forecast: \( p_{s}(s_0) = 1.25 \) if \( 1.4 < s_0 < 2.2 \):

\[ = 0 \text{ otherwise} \]

Maximum Utility: \( X.\text{UMAX} = 2.04 \)

(Figures 4.2, 4.3, and 4.4)
Example 2: Triangular Distribution

Forecast: \( p_s(s_o) = 6.25s_o - 8.75 \) if \( 1.4 < s_o < 1.8 \)
\[ = -6.25s_o + 13.5 \] if \( 1.8 < s_o < 2.2 \)
\[ = 0 \] otherwise

Maximum Utility: \( X.UMAX = 1.9 \)

(Figures 4.5, 4.6, and 4.7)
4.4 SUMMARY

If our forecast does not coincide with the market's, it may be worthwhile to purchase options based on the expected value they provide. That is, the expected value of some options will be higher than that of others -- and we should look for the options which yield the most value. The utility of an option measures the expected value in such a way that it is easy to choose the option that yields the highest average payoff per dollar spent.

However, note that in cases where \( U(X) < 0 \) for all \( X \), there are no options that yield positive NPV -- all options are overpriced. The profitable alternative in this case is to write currency options rather than buy them. If one must buy them though, the best alternative is to buy the option with the least negative utility.
CHAPTER FIVE
A MATHEMATICAL PROGRAMMING APPROACH
TO FOREIGN EXCHANGE HEDGING WITH CURRENCY OPTIONS

This chapter describes models for analyzing the options hedging problem which rely on discrete, user-supplied forecasts of exchange rates. In contrast to the efficient market approach of Chapter 3, the approach here allows us to take advantage of a variety of mathematical programming methods, such as linear programming and stochastic programming with recourse. Using these techniques, we extend the scope of the decision process by allowing multi-period, multi-objective decision criteria and by specifically accounting for constraints such as budgetary requirements and cash flow needs.

Note that the models presented in this chapter are independent from those described in Chapters 3 and 4. One technical detail to note is that whereas the exchange rates were stated in dollars per FC in Chapters 3 and 4, here we define them in FC per dollar.

5.1 One-Period Models

Our first formulations are one-period models. Managers can purchase options just once, at the outset. Performance is evaluated at the end, according to the return on their options and the dollar value of their operating cash flows.

A one-period, single-objective model has some deficiencies that a multi-period, multi-objective model would not have. First; it allows the firm only one chance to buy options. In practice, the company will
have many opportunities. For instance, a manager who plans to receive foreign cash flows two years out may wish to wait one year to see how economic conditions develop before buying put options. Multi-period models are needed to provide the alternative to wait.

Second, firms generally have a number of simultaneous, conflicting objectives in financial hedging. The firm wishes to maximize cash flows, but it should also try to minimize up-front costs. Managers might also try to minimize the costs of guaranteeing a "worst-case" exchange rate. To satisfy many goals simultaneously, our models should use some form of multi-objective optimization.

The models in this section skirt these two issues for the time being. Instead, they present a condensed version of the options hedging problem which, nonetheless, will be useful to build upon to create the multi-period, multi-objective models.

5.1.1 The Discrete One-Period Forecast of Exchange Rates

The discrete forecast of exchange rates is simply a probability mass function. We must choose a number of scenarios for the spot rate at time $T$ along with associated probabilities for each scenario. For example, one might use the following forecast for the spot rate of German marks one year hence:
Our convention will be to use $S_1$, $S_2$, and $S_3$ to denote the spot rates at the end of Year 1 for Scenarios 1, 2, and 3. $P_1$, $P_2$, and $P_3$ denote the probabilities that the scenarios occur.

One way to generate such a distribution is to poll a group of economic forecasters and to incorporate their opinions into a discrete probability distribution. Another way is to base the discrete forecast on the market forecast described in Chapter 3 via some sampling technique. (see Hiller [1986]).

5.1.2 Definition of Key Parameters and Variables

Below, we list and describe the key parameters and variables used in the one-period models:

1. $C_{X}$ -- The cost (or premium) of an option of strike price $X$ (FC per dollar) derived via the options pricing formula.

2. $OCF_i$ -- The Dollar Value of an Operating Cash Flow given Scenario $i$ occurs. Thus, $OCF_i = 1/S_i$. 
(3) \( E(OCF) \) -- The Expected Value of 1 unit in operating cash flow, calculated as:

\[
\sum_{i=1}^{N} p_i \cdot OCF_i
\]

where \( N \) is the total number of scenarios.

(4) \( HCFXi \) -- The future value of an option on 1 unit of FC struck at \( X \) units of FC per dollar given Scenario \( i \). There are two cash flows to consider for each option:

- The option's payoff, if any; and
- The option's premium (or cost).

We define the cash flow of an option to be the future value of the difference of these two flows (payoff minus premium). Note that in the future value calculation, the option's premium must be inflated to adjust for the time value of money.

As an example, consider a 1 mark put option, struck at DM1.7, expiring in one year. Assume that market parameters imply a premium of 3¢ per DM covered, that we are using the forecast in Figure 5.1, and that the discount rate is 10%. The following payoffs result:

- **Scenario 1** (Spot rate = 1.5): The option is not exercised, and the payoff is 0. The future value of the option's premium is:
\[ 0 - 3\cdot(1.1) = -3.3\cdot. \]

Therefore, the FV of the option, given Scenario 1, is -3.3¢.

- **Scenario 2** (Spot rate = 1.8): The option is exercised, and the dollar payoff is:

\[ \$\left(\frac{1}{1.7}\right) - \$\left(\frac{1}{1.8}\right) = 3.27\cdot. \]

Since the FV of the premium is still -3.3¢, the FV of the option, given Scenario 2, is -0.03¢.

- **Scenario 3** (Spot rate = 2.0): The option is exercised and the dollar payoff is:

\[ \$\left(\frac{1}{1.7}\right) - \$\left(\frac{1}{2.0}\right) = 8.82\cdot. \]

Therefore, the FV of the option, given Scenario 3, is:

\[ 8.82 - 3.3 = 5.52\cdot. \]

(5) \(E(HCF_X)\) -- The expected future value of all cash flows from an option struck a \(X\) units of FC per dollar. It is defined as:

\[ \sum_{i=1}^{N} p_i \cdot HCF_{X_i} \]
For the case depicted above,

\[ E(HCF_{1.7}) = -3.3\times(25\%) -0.03\times(40\%) + 5.52\times(35\%) = 1.10\times. \]

(6) \( X_X \) -- The percentage of operating cash flow that we cover with put options struck at \( X \). Thus if, for example, we are covering DM100 million in operating cash inflow 1 year from now and our optimal solution reads:

\[
\begin{align*}
X_{190} &= 0.5 \\
X_{200} &= 0.25 \\
\text{all other } X_X &= 0,
\end{align*}
\]

then we should purchase the following options:

- A one year DM1.90 put on DM 50 million
- A one year DM2.00 put on DM 25 million.

Note that the \( X_X \)'s are the only variables in the problem.

(7) \( CF \) -- Expected Total Cash Flow -- defined as total dollars realized per unit of foreign currency exposure. This includes three items:

- Future value of dollar outflow from purchasing put options.
- Future value of expected payoff from put options, (if any).
- Future value of expected dollars realized from operating cash.

The formula for \( CF \) is:

\[ CF = E(OCF) + \sum_{\text{all strike prices}} E(HCF_X)X_X \]
Another way to think about CF is as the mean "effective" exchange rate, including the costs, measured in dollars per unit of foreign currency.

5.1.3 The Discretization of the Domain of Strike Prices

In order to use conventional optimization techniques such as linear programming and stochastic programming, all our equations describing objectives and constraints must be stated in linear form. This creates a dilemma if we hope to optimize over the entire range of strike prices X, because the options pricing formula is clearly nonlinear. The alternative, which we chose, is to select a finite sampled set of strike prices and optimize over that set. Because the pricing formula is well-behaved, such a strategy is likely to lead to a near-optimal solution.

Choosing a range of strike prices to consider is relatively straightforward and depends on the forecasts and on the maximum amount that the company can willingly spend on the hedge. The most expensive put options are those which have the lowest strike prices (measured in FC per dollar). A good lower bound is the forward rate on the foreign currency, since options with lower strike prices would be too expensive for most companies. The upper bound is the highest forecast of exchange rates, since put options with higher strike prices can never expire "in the money."

Once we have decided upon a range, we simply sample a number of discrete strike prices within it. The more dense the sampling, the more "optimal" the solution. Fortunately, since the options pricing
function is well-behaved, we need not be concerned about large discontinuities in the underlying pricing and expected value functions.

For each strike price that we sample, we can then calculate each option's cash flow (per unit of FC covered) under the different scenarios -- HCFX, and its expected value (per unit of FC) covered over all the scenarios -- E(HCF_X), as illustrated in the last section.

5.1.4 FORMULATION 1: MAXIMIZING EXPECTED CASH-FLOW SUBJECT TO COSTS

The most intuitive way to approach the options hedging problem is to think in terms of maximizing the total expected value of all cash flows -- both options flows and operating flows, subject to cost and coverage constraints:

\[
\begin{align*}
\text{Max} & \quad \text{CF} \\
\text{subject to} & \\
\sum_x C_x & \leq \text{max}\_\text{cost} \\
\sum_x X_x & \leq 1 \\
X_x & \geq 0
\end{align*}
\]

Constraint (1) limits the total dollars spent on options. The parameter MAX\_COST represents the maximum amount of dollars available (for purchasing options) per unit of FC-denominated operating cash flow. For example, if we can spend up to $1 million to hedge a £50 million cash flow expected a year from now, then MAX\_COST equals 2¢ (per £). This notion of a spending limit is consistent with
the budget limitations that most treasurers face in their spending on financial hedging.

Constraint (2) limits the total coverage: The firm can only cover its operating cash flow -- no more. Thus, as an example, if the firm expects ¥100 million in operating cash flow next year, it can only buy put options covering up to ¥100 million. This constraint is optional. In some cases, it may make sense to cover greater than 100% of an exposure with a cheap option.

This formulation is best used when we would like to take advantage of discrepancies between our forecast and the market's. The notion of utility that we introduced in Chapter 4 plays an important role in choosing the optimal solution. In general, to maximize the expected cash flow (CF) subject to a cost constraint, the optimal strategy is to choose the option with the highest utility, and buy as much of it as we can afford.

Example

Exhibit 1 shows the results of a hedge of German marks using Formulation 1, where the model's parameters are:

<table>
<thead>
<tr>
<th>Spot Rate</th>
<th>1.87</th>
<th>SCENARIOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>US_i</td>
<td>10.19%</td>
<td>EXCHANGE RATE PROB</td>
</tr>
<tr>
<td>FC_i</td>
<td>7.00%</td>
<td>1.8</td>
</tr>
<tr>
<td>Time (Yrs.)</td>
<td>1</td>
<td>2.07</td>
</tr>
<tr>
<td>Volatility</td>
<td>11.30%</td>
<td>1.53</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>10%</td>
<td>0.3333</td>
</tr>
<tr>
<td>MAXCOST</td>
<td>0.5$</td>
<td>0.3334</td>
</tr>
</tbody>
</table>
The optimal solution is to cover approximately 49% of the exposure with put options struck at 1.93 marks per dollar at a cost of 0.5¢ per mark of operating cash flow. \(\text{CF}\), in this case, equals 0.564.

5.1.5 **Formulation 2: Minimize Costs Subject to Achieving Target Cash Flows**

Now, we take the opposite perspective. Instead of maximizing the expected cash flow, we minimize the total cost subject to achieving some set of target cash flows in each scenario. Unlike the approach taken in Formulation 1, this approach does not rely on the fact that we try to outsmart an inefficient market. The formulation is as follows:

\[
\begin{align*}
\text{Min} & \quad \sum_x [C_x X_x] \\
\text{subject to} & \quad \sum_x X_x \leq 1 \quad (1) \\
& \quad \sum_x [HCFXH^*_X_X] + OCF1 \geq \text{TARGETH} \quad (2) \\
& \quad \sum_x [HCFXM^*_X_X] + OCF2 \geq \text{TARGETM} \quad (3) \\
& \quad \sum_x [HCFXL^*_X_X] + OCF3 \geq \text{TARGETL} \quad (4) \\
& \quad X_X \geq 0
\end{align*}
\]

Exhibit 2 shows the results of a hedge of German marks using

**Formulation 2**, where the model's parameters are:

<table>
<thead>
<tr>
<th>Spot Rate</th>
<th>1.87</th>
</tr>
</thead>
<tbody>
<tr>
<td>US_i</td>
<td>10.19%</td>
</tr>
<tr>
<td>FC_i</td>
<td>7.00%</td>
</tr>
<tr>
<td>Time (Yrs.)</td>
<td>1</td>
</tr>
<tr>
<td>Volatility</td>
<td>11.30%</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>10%</td>
</tr>
</tbody>
</table>

**SCENARIOS**

<table>
<thead>
<tr>
<th>EXCHANGE RATE</th>
<th>PROB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>0.3333</td>
</tr>
<tr>
<td>2.07</td>
<td>0.3333</td>
</tr>
<tr>
<td>1.53</td>
<td>0.3334</td>
</tr>
</tbody>
</table>

TARGET1=0.53 (=DM 1.887)  
TARGET2=0.50 (=DM 2.00)  
TARGET3=0.60 (=DM 1.667)
The optimal solution for this case is again to use the option that has the highest utility -- a put option struck at 1.93 marks per dollar. Now, we need to cover about 65% of our exposure at a cost of 0.7¢ per mark of operating cash flow.

5.2 Two-Period Stochastic Programming Models

In this section, we extend the models introduced in the last section to include multi-period decisions. To facilitate this, we introduce a method known as stochastic programming (SP) with recourse, which allows the decision-maker to incorporate future decisions as well as current decisions into his/her strategy. Thus, for example, if we wish to hedge cash flows two years out, the SP formulation will not only answer the question, "What options do we buy today?"; but also, "What options do we buy a year from now?"

5.2.1 THE MECHANICS OF STOCHASTIC PROGRAMMING

A number of texts describe stochastic programming in some detail. For an excellent discussion of the topic, see Wagner [1969], pp. 640-700. In this section, we shall briefly summarize the fundamentals of stochastic programming. In the next section, we show how to apply the technique to hedging with currency options.

As we argued earlier, traditional mathematical programming formulations, such as linear programming (LP) and utility maximization
formulations, force decision makers to commit all their resources at
the outset -- and only at the outset. In reality, decision makers often
have many sequential opportunities to commit resources. Though
perhaps at a cost, they can delay some of their decisions, and in fact
might be forced to delay others.

Stochastic programming uses a probabilistic model of the future to
allow decisions to be made later in the process. To see how, consider
the limitations of a typical one-period LP:

\[
\begin{align*}
\text{max} & \quad c_i^*(x_i) \\
\text{st} & \quad Ax < b \\
& \quad x_i > 0
\end{align*}
\]

In the LP formulation, all the coefficients -- those of the objective
function \(c_i\), and those of the constraints \(A\) -- are assumed to be
known, deterministic parameters. In terms of the options hedging
problem, the assumption is that we know all the costs and expected
final cash flows for all options at the start.

In many situations, though, the value of a coefficient will not be
known until some later date. In the hedging problem, we will not
know the prices of one-year put options purchased one year from now
until the date of purchase. And because we cannot predicts the prices,
we cannot assign deterministic coefficients, \(c_i\) and \(A\), to include in the
LP.

Stochastic programming takes a different approach. It recognizes
that even if we do not know the exact values of all coefficients at the
start, we can build a forecast for the ones we don't know. Then, as
coefficients take on values at the forecasted interim dates, the SP
solution can tell us what values to assign to the corresponding
variables. Thus, for our example, we can forecast the prices of 1-year options one year from now and incorporate the forecast into the formulation. Then, the SP solution will tell us not only which 2-year options to buy today, but also, which options to buy one year from now, for each of the possible scenarios in our forecast.

To see exactly how we modify the LP to create an SP, consider, now, a two-period model -- one having a single interim date. The SP formulation starts by separating variables into two categories: Those we can assign today (current decisions); and those that we will assign at the interim date, after one of the scenarios occurs (future decisions).

The basic strategy in SP is: To do the best we can today, given our view of the future, and given that we'll do the best we can in the future. Thus, the SP objective function maximizes the expected value of the original LP objective, across all the scenarios. Meanwhile, the original LP constraints now need to be satisfied for all interim scenarios. Thus, each scenario will map to a separate set of constraints, similar to the original LP constraints. The only difference will be in the interim coefficients. Below, we show mathematically what we just described in words:

A Stochastic Programming Formulation with Recourse

\[
\begin{align*}
\max & \sum_{i=1}^{N} c_i x_i + \sum_{j=1}^{M} \sum_{k=1}^{Q} p_k \cdot (d_{jk} y_{jk}) \\
\text{st} & \sum_{i=1}^{N} A x_i + \sum_{j=1}^{M} B_k y_{jk} \leq b & k=1,2,\ldots,Q \\
x_i \geq 0
\end{align*}
\]
N is the number of decisions \( (x_i) \) that can be determined at the initial stage; \( M \) is the number of decisions that must be determined at the interim period \( (y_{jk}) \); \( Q \) is the number of interim scenarios; \( d_{jk} \) and \( B_k \) are the values of \( d_i \) and \( B \) respectively given that scenario \( k \) occurs; and \( P_j \) is the probability that scenario \( j \) occurs.

The optimal solution to the stochastic formulation will not only tell us the values of the variables that we assign today -- \( x_i \) for \( i=1,2,3,\ldots,N \) -- but also the values that we assign to variables at the interim date, given each of the scenarios. If Scenario 1 occurs \( (k=1) \), then we will select \( y_{1j} \) \( (j=1,2,3,\ldots,M) \) at the interim date; if Scenario 2 occurs \( (k=2) \), we will select \( y_{2j} \) \( (j=1,2,3,\ldots,M) \) at the interim date; and so on.

5.2.2 The Two-Period Forecast

In order to use stochastic programming for the options hedging problem, we need to build probabilistic forecasts of a number of market parameters at interim dates. (Note that this is in addition to the forecast of the spot rate on the final date -- the date that we expect operating cash flow). The interim period data we need to predict option premiums include forecasts of:

- The Spot Rate \( (S_i) \)
- The Foreign and Domestic Interest Rates \( (US_i, \text{ and } FC_i) \)
- The Market Volatility \( (V) \)

In this chapter, we will consider only two-period hedging models -- i.e. models with just one interim date, such as a two year hedge with
the opportunity of buying options at the end of Year 1. Specifically, we shall focus on two-year hedging of German marks, with an opportunity to buy options midway through. There will be three interim scenarios for the end of Year 1, labelled H (strong dollar), M (medium-strength dollar), and L (weak dollar). When we run the stochastic programming model, the optimal solution will tell us which options to buy today, as well as which options to buy in Year 2 given each of the three interim scenarios.

For example, consider the following forecast: The interim date forecast of market parameters is as follows:

<table>
<thead>
<tr>
<th>Scenario:</th>
<th>Today</th>
<th>H</th>
<th>M</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>----</td>
<td>33%</td>
<td>33%</td>
<td>33%</td>
</tr>
<tr>
<td>Spot Rate</td>
<td>1.87</td>
<td>1.95</td>
<td>1.8</td>
<td>1.7</td>
</tr>
<tr>
<td>US_i</td>
<td>10%</td>
<td>11%</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>FC_i</td>
<td>7%</td>
<td>7%</td>
<td>7%</td>
<td>8%</td>
</tr>
<tr>
<td>Volatility</td>
<td>12%†</td>
<td>11.5%</td>
<td>11.0%</td>
<td>11.5%</td>
</tr>
</tbody>
</table>

†Today's volatility is for 2-year options.

Using these parameters, we can calculate the premiums (costs) of options purchased one year hence for each of the scenarios.

To calculate the expected cash flows for the options, we will need forecasts of the spot rate two years hence given each of the interim scenarios. In other words, what will the path of the spot rate be on the exercise date, conditional on the interim scenario?

So, for example, given that Scenario H occurs at the end of Year 1, we need to specify three separate "sub-scenarios" for the spot rate at the end of Year 2: HH (continued strong dollar), HM (medium
dollar), and HL (weaker dollar). Below, we specify parameters and probabilities for nine scenarios occurring at the end of Year 2:

<table>
<thead>
<tr>
<th>Scenario H: Sub-Scenarios</th>
<th>Spot Rate</th>
<th>Conditional Probability</th>
<th>Total Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>2.20</td>
<td>33%</td>
<td>11.1%</td>
</tr>
<tr>
<td>HM</td>
<td>2.05</td>
<td>33%</td>
<td>11.1%</td>
</tr>
<tr>
<td>HL</td>
<td>1.75</td>
<td>33%</td>
<td>11.1%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario: M Sub-Scenarios</th>
<th>Spot Rate</th>
<th>Conditional Probability</th>
<th>Total Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>MH</td>
<td>2.10</td>
<td>33%</td>
<td>11.1%</td>
</tr>
<tr>
<td>MM</td>
<td>1.70</td>
<td>33%</td>
<td>11.1%</td>
</tr>
<tr>
<td>ML</td>
<td>1.19</td>
<td>33%</td>
<td>11.1%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario L: Sub-Scenarios</th>
<th>Spot Rate</th>
<th>Conditional Probability</th>
<th>Total Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>LH</td>
<td>1.95</td>
<td>33%</td>
<td>11.1%</td>
</tr>
<tr>
<td>LM</td>
<td>1.60</td>
<td>33%</td>
<td>11.1%</td>
</tr>
<tr>
<td>LL</td>
<td>1.12</td>
<td>33%</td>
<td>11.1%</td>
</tr>
</tbody>
</table>
In Figure 5.2, below, we depict a schematic of the two-period forecast.

![Diagram of the two-period forecast]

**Figure 5.2**
The extension to multi-period models is relatively straightforward. For each extra period, simply add a forecast at a new interim date. Then in the SP formulation, extend the expected value of the objective function over the additional scenarios, and add extra sets of constraints to handle the new sub-scenarios.

5.2.3 Definition of Key Parameters and Variables

(1) C_X, CH_X, CM_X, CL_X -- These parameters describe the present value of the cost of buying an option struck at X units of FC per dollar. C_X is for options bought at time 0; CH_X, CM_X, and CL_X are for options bought at the interim date given Scenarios H, M, and L, respectively.

(2) E(HCF_X) -- The expected future value of cash flows received from an option struck at X, purchased at time 0. This is simply the expected payoff (over all subscenarios) minus the future value of the premium:

\[ E(HCF_X) = \sum_{i = H, M, L} \sum_{j = H, M, L} \max(0, \frac{1}{X} - \frac{1}{S_{ij}}) \cdot P_{ij} - C_X \cdot PF_2 \]

where \( S_{ij} \) is the spot rate at the end of the two periods for Scenario \( ij \) (e.g. HH); \( P_{ij} \) is the probability that Scenario \( ij \) occurs; and \( PF_2 \) is the future value multiplicand for two periods, (e.g. for 2 years at a 10% discount rate, \( PF_2 = (1.1)^2 = 1.21 \)).
(3) \(E(HCFH_X), E(HCFM_X), E(HCFL_X)\) -- The expected future value of cash flows received from an option struck at \(X\), purchased at the interim date, given Scenarios H, M, and L, respectively. Again -- this is simply the expected payoff (over all subscenarios) minus the future value of the premium (now inflated over only one period). Thus, for example, \(E(HCFL_X)\) would be:

\[
E(HCFL_X) = E(\text{Cash Flow of } XL_X \text{ over } LH, LM, LL)
= \sum_{ij = LH,LM,LL} \max(0, \frac{1}{X} - \frac{1}{S_{ij}}) \text{Prob}(ij|L) - CL_X \cdot PF_1
\]

where \text{Prob}(ij|L)\) are the probabilities that sub-scenarios LH, LM, and LL occur at the final date given that L has occurred at the interim date; and PF_1 is the FV factor over one period. Expected cash flows for options purchased under scenarios H and M are calculated similarly.

(4) \(X_X, XH_X, XM_X, XL_X\) -- These terms are variables and represent the percentages of our operating cash flow (exposure) that we cover with options struck at \(X\) units of FC per dollar. \(X_X\) is for options bought at time 0; \(XH_X, XM_X,\) and \(XL_X\) are for options bought at the interim date, given Scenarios H, M, and L, respectively.

(5) \(CF\) -- The expected (future) value of all options and operating cash flows:
The first term represents cash flows from options purchased at time 0; the next three terms represent cash flows from options purchased at the interim date, given H, M, and L; the last term represents the expected operating cash flow.

(6) FLOW\text{ij} -- The total cash flow (in future value terms) actually received in Scenario \text{ij} (where \text{ij=HH,HM,HL,MH,MM,ML,LH,LM,LL}). The general formula for FLOW\text{ij} is:

\[ \text{FLOW}_{ij} = \sum_{X} \left[ \max(0, \frac{1}{X} - \frac{1}{S_{ij}}) \cdot C_X \cdot PF_2 \right] X_X + \sum_{X} \left[ \max(0, \frac{1}{X} - \frac{1}{S_{ij}}) \cdot C_i \cdot PF_1 \right] X_i_X + \frac{1}{S_{ij}} \]

The first term represents the cash flows from options purchased at time 0; the second set represents cash flow from options purchased at the interim date; and the last term represents the cash flow from one unit of FC operating flow.
(7) **COSTH, COSTM, COSTL** -- The present value of the total cost, given some strategy of X_X, XH_X, XM_X, and XL_X, and given Scenario H, M, and L, respectively, occurs. Thus, for example, COSTM is the cost given Scenario M occurs, and is defined as:

\[
\text{COSTM} = \sum_{X} C_X * X_X + CM_X * XM_X
\]

5.2.4 **Formulation 3: Maximization of Total Cash Flow Subject to a Maximum Cost**

The two-period SP formulation for maximization of cash flow is completely analogous to the LP formulation presented in Section 5.1.4: We are trying to maximize total cash flow -- both from options and operating cash flows -- and are only limited by budget and coverage constraints. Specifically, no matter which interim scenario occurs, we cannot spend more than a specified amount on options (MAXCOST), and we cannot cover more than 100% of our operating flow. The formulation for this is as follows:

Max CF

\[
\begin{align*}
\text{COSTH} & \leq \text{MAXCOST} & (1) \\
\text{COSTM} & \leq \text{MAXCOST} & (2) \\
\text{COSTL} & \leq \text{MAXCOST} & (3) \\
\sum_{X} X_X + XH_X & \leq 1 & (4)
\end{align*}
\]
In Exhibit 3, we show a formulation of this type, along with the optimal solution, using the market parameters and forecast of Section 5.2.2 above. In this case, if we allow a maximum cost of 2¢ per mark of operating flow, we get the following strategy:

<table>
<thead>
<tr>
<th>Option</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>XH196</td>
<td>44%</td>
</tr>
<tr>
<td>XM192</td>
<td>61%</td>
</tr>
<tr>
<td>XM194</td>
<td>39%</td>
</tr>
<tr>
<td>XL182</td>
<td>72%</td>
</tr>
</tbody>
</table>

Note that we purchase all of our options in the second year. Also, note that we would not, in practice, buy options struck at 1.92 and 1.94 (as in Scenario M), since they are not significantly different. Rather, we would cover fully with one or the other. The split is merely a result of the discretization of the problem.

The costs of this strategy are:

<table>
<thead>
<tr>
<th>Cost</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>COSTH</td>
<td>0.5¢</td>
</tr>
<tr>
<td>COSTM</td>
<td>0.5¢</td>
</tr>
<tr>
<td>COSTL</td>
<td>0.5¢</td>
</tr>
<tr>
<td>AVGC</td>
<td>0.5¢</td>
</tr>
</tbody>
</table>

5.2.5 Formulation 4: Maximizing Expected Cash Flow Subject to a Maximum Average Cost

Another way to formulate a cost constraint is to set a maximum average cost rather than a maximum cost for each scenarios. Some
managers may feel that they can compensate for especially bad times by spending more on options, as long as they spend less when times are good. Thus, rather than requiring that they never exceed a specified maximum budget, they would rather make sure that they limit their average spending level. Thus the formulation for this is:

Max CF

s.t.

\[ P_H \cdot \text{COSTH} + P_M \cdot \text{COSTM} + P_L \cdot \text{COSTL} \leq \text{MAX_AVGC} \quad (1) \]

\[ \sum_{X} \text{X}_X + \text{XH}_X \leq 1 \quad (2) \]

\[ \sum_{X} \text{XM}_X \leq 1 \quad (3) \]

\[ \sum_{X} \text{XL}_X \leq 1 \quad (4) \]

\[ \text{X}_X, \text{XL}_X, \text{XM}_X \geq 0 \]

Note that MAX_AVGC is the maximum average cost, and is a parameter set before the model is run.

In Exhibit 4, we show an example formulation, again using the parameters from 5.2.2. Using a maximum average cost of 2.00, the optimal strategy and costs are as follows:
### Option Coverage

<table>
<thead>
<tr>
<th>Option</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>XH186</td>
<td>34%</td>
</tr>
<tr>
<td>XH188</td>
<td>66%</td>
</tr>
<tr>
<td>XM184</td>
<td>100%</td>
</tr>
<tr>
<td>XL176</td>
<td>100%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cost</th>
<th>AVGC</th>
</tr>
</thead>
<tbody>
<tr>
<td>COSTH</td>
<td>4.15¢</td>
</tr>
<tr>
<td>COSTM</td>
<td>0.89¢</td>
</tr>
<tr>
<td>COSTL</td>
<td>0.97¢</td>
</tr>
<tr>
<td>AVGC</td>
<td>2.00¢</td>
</tr>
</tbody>
</table>

#### 5.2.6 Formulation 5: Minimizing the Average Cost Subject to Achieving Target Cash Flows

This formulation is an extension of the model in 5.1.6. Now, we have nine endpoint scenarios to attach targets to:

\[
\text{Max AVGC}
\]

\[
\text{s.t.}
\]

\[
P_H \cdot \text{COSTH} + P_M \cdot \text{COSTM} + P_L \cdot \text{COSTL} = \text{AVGC} \quad (1)
\]

\[
\sum_{X} X_{X+XH_X} \leq 1 \quad (2)
\]

\[
\sum_{X} X_{X+XM_X} \leq 1 \quad (3)
\]

\[
\sum_{X} X_{X+XL_X} \leq 1 \quad (4)
\]

\[
\text{FLOWij} \geq \text{TARGETij} \quad \text{for ij=HH,HM,HL,MH,MM, ML,LH,LM,LL}
\]

\[
X_{X}, X_{LX}, X_{MX} \geq 0
\]

The parameters, TARGETij, are preselected target cash flows that we require in Scenarios ij.
Once again, we illustrate this technique using the market parameters and forecast from 5.2.2. We also set the following targets:

- FLOWHH >= 0.509
- FLOWHM >= 0.509
- FLOWHL >= 0.547
- FLOWMH >= 0.53
- FLOWMM >= 0.575
- FLOWML >= 0.827
- FLOWLH >= 0.554
- FLOWLM >= 0.611
- FLOWLL >= 0.879

For this formulation, the optimal strategy is as follows:

<table>
<thead>
<tr>
<th>Option</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>XH186</td>
<td>10%</td>
</tr>
<tr>
<td>XH188</td>
<td>90%</td>
</tr>
<tr>
<td>XM184</td>
<td>78%</td>
</tr>
<tr>
<td>XM186</td>
<td>22%</td>
</tr>
<tr>
<td>XL176</td>
<td>89%</td>
</tr>
<tr>
<td>XL178</td>
<td>11%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cost</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>COSTH</td>
<td>1.94¢</td>
</tr>
<tr>
<td>COSTM</td>
<td>1.01¢</td>
</tr>
<tr>
<td>COSTL</td>
<td>1.12¢</td>
</tr>
<tr>
<td>AVGC</td>
<td>1.35¢</td>
</tr>
</tbody>
</table>

5.3 A MULTI-OBJECTIVE STOCHASTIC PROGRAMMING MODEL

In the last section, we were concerned with the trade-offs of cost and return. Though for each cost constraint we can come up with an optimal cash flow, it is somewhat arbitrary what level to choose for each constraint. In this section, we introduce a method of multi-objective stochastic programming, in which we address more directly the trade-offs inherent in the options hedging problem.
5.3.1 Pricing Out Constraints With Shadow Prices

The shadow price of a constraint on a linear (or stochastic) program is defined as the amount by which the objective function changes if we change the right-hand-side of the constraint by 1. Thus, for example, the shadow price of a \( \leq \) constraint for a max problem is the amount by which the optimal solution increases if we increase the right-hand side by 1. [Winston p.224]. In Formulation 3, we interpret the shadow price on the COSTH constraint (Constraint 1) as the amount of additional expected total cash flow per unit of FC that we'll receive by increasing MAX_COST by 1.

Similarly, the shadow price of a \( \geq \) constraint for a min problem is the amount by which the optimal solution decreases if we decrease the right-hand side by 1. Thus, in Formulation 5, we interpret the shadow price on the FLOWHH constraint as amount that the average cost will decrease if we decrease TARGETHH by 1.

The shadow prices on constraints provide very useful information about the inherent trade-offs we are making by setting our strategy to a specific optimal solution. For example, in Exhibit 5, one of the binding constraints is Constraint 33 \( \Rightarrow \) FLOWHM\(\geq\)0.509. In the optimal solution, we can read the shadow (dual) price from the solution: SHADOWHM=30.19. Therefore, we could pay 0.3019\(\$\) less if we reduced the rhs of Constraint 33 to 0.499. The new Constraint 33 would read \( \Rightarrow \) FLOWHM\(\geq\)0.499.

Sometimes, we may be willing to make such a tradeoff -- relieving constraints to reduce costs or increase expected cash flows. In fact, we may decide, for example, that as long as we save more than .28\(\$\) for every penny FLOWHM lies below 0.509, (even less than the 0.3019\(\$\)
now), we will be willing to reduce the rhs of the constraint. If this is the case, we would reformulate the problem by pricing out Constraint 33 in the objective function so that the new one would read:

\[
\text{MIN AVGC - 28*FLOWHM}
\]

And we would eliminate Constraint 33, entirely.

Below, we generalize Formulations 5 and 6, by showing how one would price out multiple cost and target constraints. Note that in practice, the best way to apply this procedure is to run the single objective models to get an idea on the range of sensible shadow prices first, before attempting the multi-objective formulation.

5.3.2 Formulation 6: Trading off Expected Cash Flow with Average Cost

\[
\text{Max CF - SHADOW*AVGC}
\]

\[
\text{s.t.} \\
PH*COSTH + PM*COSTM + PL*COSTL - AVGC = 0 \quad (1)
\]

\[
\sum_{X} X_{X} + X_{H_{X}} \leq 1 \quad (2)
\]

\[
\sum_{X} X_{X} + X_{M_{X}} \leq 1 \quad (3)
\]

\[
\sum_{X} X_{X} + X_{L_{X}} \leq 1 \quad (4)
\]

\[X_{X}, X_{L_{X}}, X_{M_{X}} \geq 0\]
(Note: Constraint (1) above is used for convenience in defining AVGC and is not really a meaningful "constraint.") We price out the average cost by subtracting the shadow term in the max problem, because we want to penalize increases in cost.

5.3.3 Formulation 7: Trading off Target Cash Flows with Average Cost

\[
\text{MIN AVGC - SHADOWHH*FLOWHH - SHADOWHM*FLOWMH - etc.}
\]

\[
\text{s.t.}
\]

\[
P_{H}^{COSTH} + P_{M}^{COSTM} + P_{L}^{COSTL} - AVGC = 0 \quad (1)
\]

\[
\sum_{X}X_{-X} + X_{H-X} \leq 1 \quad (2)
\]

\[
\sum_{X}X_{-X} + X_{M-X} \leq 1 \quad (3)
\]

\[
\sum_{X}X_{-X} + X_{L-X} \leq 1 \quad (4)
\]

\[
X_{-X}, X_{L-X}, X_{M-X} \geq 0
\]

We price out the FLOWij terms by subtracting them in the min problem, because we want to reward increases in cash flows.

5.4 SUMMARY

In this chapter, we introduced a number of techniques for simplifying the options hedging problem. We assumed that the user of these models can generate his/her own discrete forecast of exchange rates one or more periods into the future, and wants a simple system for quantifying the benefits of different options packages, as well as a method for choosing the optimal package based on these criteria.
The purpose of the chapter was not to come up with insightful generalizations about how optimality is affected by specific changes in the forecast. We leave that to finance theorists. Rather, we attempted to set up a framework for normative decision models. The useful concepts to take away from these analyses include --

- How discretization of the event space of strike prices makes it possible to use mathematical programming.

- How mathematical programming constructs can be used to model real-world constraints such as budgetary requirements and target cash flow needs.

- How to measure various key parameters in an optimal solution, such as FLOW_i,j, the flow at each payoff scenario, and CF, the total expected cash flow from a hedging strategy.

- How to set up a multi-period stochastic programming model with recourse, to model the purchases of options in the future as well as the present.

- How to formulate the problem with multiple objectives by pricing out constraints in the objective function.
CHAPTER 6
CONCLUSION

The three sets of models presented in this thesis take very
different approaches to the options hedging problem. In Chapter 3,
we started with the assumption that markets are efficient and
presented a framework for evaluating how options change the risks we
face and the distributions of returns we can expect. There were no
optimization techniques presented along with this framework. Rather,
our objective was to define, in a probabilistic way, a convenient and
economically rational way of describing what an options hedge does
(and does not do) for us. What we found was that options can be used
to slice the event space of the effective exchange rate into sections of
lognormally distributed random variables derived from the original
lognormal market forecast. By choosing options of various strike
prices and coverages, we can tailor the distribution of returns to our
own specific needs.

In Chapter 4, we analyzed currency options in the face of an
inefficient market forecast. Here we focused directly on one way of
measuring optimality -- the utility of an option, which is simply its
expected value divided by its cost. We found that, in terms of
generating the most cash flow per dollar spent, the option with the
greatest utility was optimal. We also analyzed two special cases: Which
option to buy when our forecast is the same as the market's, but with a
higher volatility; and which option to buy when our forecast is the same
as the market's, but with a lower volatility. We found that when the
market is underpricing options in this way, (i.e. the volatility quote is too low), one should purchase the least expensive options. When the market is overpricing them in this way, (volatility is too high), one should buy the most expensive options.

Finally, in Chapter 5, we looked at a few ways of formulating the problem using mathematical programming constructs. Techniques such as stochastic programming and multi-objective optimization allowed us to represent the decision process over multiple periods and in the face of specific constraints, such as cash flow needs and budgets.

Future research in this area hopefully would lead to a synthesis of these three approaches into a single unified theory. While each theory has its merits, none of them alone is able to capture all aspects of the decision process. For instance, the probabilistic approach of Chapter 3 nicely depicts the circumstances surrounding the decision and the effect of a given options package on returns, but it fails to provide a means of normative decision making. Nor does it provide a way of evaluating multi-period, multi-objective decisions. For instance: How do we represent the alternative of buying options 1 year hence on the event space of the effective exchange rate?

The utility theory in Chapter 4 presents a narrow view of optimization, involving only a single objective -- maximizing expected cash flow per dollar spent -- with no constraints. Moreover, while utility alone may be important to the speculator, the hedger needs to look more closely at the risks associated with his/her total exposure, and also must consider constraints specific to the organization.

Finally, in Chapter 5, although we were able to incorporate specific objectives and constraints into the decision process while formulating
a multi-period model, further work is needed to develop a cohesive
 technique of linking the the discrete forecast to the market's as we did
 in Chapter 3. By converting the continuous probability distribution
describing the forecast of exchange rates into a discrete model, we
perhaps oversimplified the problem, and, in the process, lost a great
deal of the probabilistic insight which is so valuable when evaluating
risk and return.

Perhaps the unified theory of options hedging will come by
developing some combination of the three approaches. For example, it
would be extremely interesting to see a multi-period, multi-objective
mathematical model which manipulated probabilities rather than
scenarios of cash flows and costs. Thus, we would apply the
optimization techniques of Chapter 5 to the derived distribution
formulas of Chapter 3. For this, we would probably need to eliminate
the discretization of exchange rate scenarios and of strike prices
entirely.

For the time being, though, we feel that the three approaches
together will provide enough insight into the problem to be
operationally useful -- even if each approach must be considered
separately. There are a large number of multinational corporations
who potentially could use currency options to hedge foreign exchange
risk. For them, we hope that the notions of an effective exchange rate,
the utility of an option, and multi-period, multi-objective optimization
will be relevant criteria in their decisions of which options package to
buy.
EXHIBIT 1: 1-PERIOD MAX CF ST COSTS

<table>
<thead>
<tr>
<th>Spot Rate</th>
<th>1.87</th>
</tr>
</thead>
<tbody>
<tr>
<td>US_i</td>
<td>10.19%</td>
</tr>
<tr>
<td>FC_i</td>
<td>7.00%</td>
</tr>
<tr>
<td>Time (Yrs.)</td>
<td>1</td>
</tr>
<tr>
<td>Volatility</td>
<td>11.30%</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>10%</td>
</tr>
<tr>
<td>MAXCOST</td>
<td>0.5c</td>
</tr>
</tbody>
</table>

Subject to:

2) \( CF = 0.00238 \times X_{1807} - 0.00196 \times X_{1814} - 0.00157 \times X_{1821} - 0.00122 \times X_{1829} \)

3) \( 0.02347 \times X_{1807} + 0.02242 \times X_{1814} + 0.02139 \times X_{1821} + 0.02041 \times X_{1829} \)

4) \( X_{1807} + X_{1814} + X_{1821} + X_{1829} + X_{1836} + X_{1843} + X_{1851} + X_{1858} + X_{1866} + X_{1873} + X_{1881} + X_{1888} + X_{1896} + X_{1903} + X_{1911} + X_{1919} + X_{1926} + X_{1934} + X_{1942} + X_{1949} + X_{1957} + X_{1966} + X_{1973} + X_{1981} + X_{1989} + X_{1997} + X_{2005} + X_{2013} + X_{2021} - COST = 0 \)
Exhibit 1 -continued-

\[ + X_{2078} + X_{2086} + X_{2095} + X_{2103} + X_{2111} + X_{2120} + X_{2128} + X_{2137} + X_{2145} \]
\[ + X_{2154} + X_{2163} + X_{2171} + X_{2180} + X_{2189} + X_{2197} \leq 1 \]

5) \( \text{COST} \leq 0.005 \)

6) \[-0.0248 X_{1807} - 0.02582 X_{1814} - 0.02466 X_{1821} - 0.02353 X_{1829} - 0.02246 X_{1836} - 0.0214 X_{1843} - 0.02038 X_{1851} - 0.01939 X_{1858} + 0.01844 X_{1866} - 0.01752 X_{1873} - 0.01664 X_{1881} - 0.01579 X_{1888} - 0.01496 X_{1896} - 0.01418 X_{1903} - 0.01342 X_{1911} - 0.01269 X_{1919} - 0.01199 X_{1926} - 0.01132 X_{1934} - 0.01068 X_{1942} - 0.01007 X_{1949} - 0.00948 X_{1957} - 0.00893 X_{1965} - 0.00839 X_{1973} - 0.00788 X_{1981} - 0.0074 X_{1989} + 0.00694 X_{1997} + 0.0065 X_{2005} - 0.00609 X_{2013} - 0.0057 X_{2021} - 0.00532 X_{2029} - 0.00497 X_{2037} - 0.00464 X_{2045} - 0.00432 X_{2053} - 0.00402 X_{2061} - 0.00374 X_{2070} - 0.00348 X_{2078} - 0.00323 X_{2086} - 0.003 X_{2095} - 0.00278 X_{2103} - 0.00257 X_{2111} - 0.00238 X_{2120} - 0.0022 X_{2128} - 0.00203 X_{2137} - 0.00187 X_{2145} - 0.00173 X_{2154} - 0.00159 X_{2163} - 0.00146 X_{2171} + 0.00134 X_{2180} - 0.00123 X_{2189} + 0.00113 X_{2197} - 0.00104 X_{2197} - \text{FLOW1} = -0.5534 \]

7) \[ 0.04449 X_{1807} + 0.04345 X_{1814} + 0.04238 X_{1821} + 0.04127 X_{1829} + 0.04014 X_{1836} + 0.03899 X_{1843} + 0.03782 X_{1851} + 0.03662 X_{1858} + 0.03539 X_{1866} + 0.03414 X_{1873} + 0.03287 X_{1881} + 0.03157 X_{1888} + 0.03025 X_{1896} + 0.02891 X_{1903} + 0.02754 X_{1911} + 0.02616 X_{1919} + 0.02475 X_{1926} + 0.02332 X_{1934} + 0.02187 X_{1942} + 0.02041 X_{1949} + 0.01892 X_{1957} + 0.01742 X_{1965} + 0.0159 X_{1973} + 0.01436 X_{1981} + 0.01281 X_{1989} + 0.01124 X_{1997} + 0.00966 X_{2005} + 0.00807 X_{2013} + 0.00646 X_{2021} + 0.00484 X_{2029} + 0.00321 X_{2037} + 0.00157 X_{2045} - 0.00008 X_{2053} - 0.00174 X_{2061} + 0.00031 X_{2070} + 0.00323 X_{2078} - 0.003 X_{2086} + 0.00278 X_{2095} - 0.00257 X_{2103} - 0.00238 X_{2111} - 0.0022 X_{2120} - 0.00203 X_{2128} - 0.00187 X_{2137} - 0.00173 X_{2145} - 0.00159 X_{2154} - 0.00146 X_{2163} + 0.00134 X_{2171} - 0.00123 X_{2180} - 0.00113 X_{2189} - 0.00104 X_{2197} - \text{FLOW2} = -0.48309 \]

8) \[-0.02582 X_{1807} - 0.02466 X_{1814} - 0.02353 X_{1821} - 0.02246 X_{1829} - 0.0214 X_{1836} - 0.02038 X_{1843} - 0.01939 X_{1851} - 0.01844 X_{1858} - 0.01752 X_{1866} - 0.01664 X_{1873} - 0.01579 X_{1881} - 0.01496 X_{1888} - 0.01418 X_{1896} - 0.01342 X_{1903} - 0.01269 X_{1911} - 0.01199 X_{1919} - 0.01132 X_{1926} - 0.01068 X_{1934} - 0.01007 X_{1942} - 0.00948 X_{1949} - 0.00893 X_{1957} - 0.00839 X_{1965} - 0.00788 X_{1973} - 0.0074 X_{1981} - 0.00694 X_{1989} + 0.0065 X_{1997} + 0.00609 X_{2005} + 0.0057 X_{2013} - 0.00532 X_{2021} - 0.00497 X_{2029} - 0.00464 X_{2037} + 0.00432 X_{2045} + 0.00402 X_{2053} - 0.00374 X_{2061} + 0.00348 X_{2070} - 0.00323 X_{2078} - 0.003 X_{2086} - 0.00278 X_{2095} + 0.00257 X_{2103} - 0.00238 X_{2111} - 0.0022 X_{2120} + 0.00203 X_{2128} + 0.00187 X_{2137} + 0.00173 X_{2145} - 0.00159 X_{2154} + 0.00146 X_{2163} - 0.00134 X_{2171} + 0.00123 X_{2180} + 0.000104 X_{2197} - \text{FLOW3} = -0.48309 \]
- 0.00113 X2189 - 0.00104 X2197 - FLOW3 = - 0.65359

OBJECTIVE FUNCTION VALUE
1) .563712500

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### Exhibit 2: 1-Period Min Cost ST Targets

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<td><strong>US_i</strong></td>
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<td><strong>FC_i</strong></td>
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<td><strong>Time (Yrs.)</strong></td>
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<td><strong>TARGET3</strong></td>
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**MIN COST**

**SUBJECT TO**

2) - CF - 0.00238 X1807 - 0.00196 X1814 - 0.00157 X1821 - 0.00122 X1829

- 0.00089 X1836 - 0.00059 X1843 + 0.00033 X1851 - 0.00009 X1858
  + 0.00011 X1866 + 0.00029 X1873 + 0.00043 X1881 + 0.00055 X1888
  + 0.00063 X1896 + 0.00069 X1903 + 0.00072 X1911 + 0.00072 X1919
  + 0.00007 X1926 + 0.00065 X1934 + 0.00058 X1942 + 0.00048 X1949
  + 0.00036 X1957 + 0.00021 X1965 + 0.00004 X1973 - 0.00015 X1981
- 0.00036 X1989 - 0.00059 X1997 - 0.00084 X2005 - 0.01111 X2013
- 0.0014 X2021 - 0.0017 X2029 - 0.00202 X2037 - 0.00236 X2045
- 0.00271 X2053 - 0.00308 X2061 - 0.00346 X2070 - 0.00323 X2078
- 0.003 X2086 - 0.00278 X2095 - 0.00257 X2103 - 0.00238 X2111
- 0.0022 X2120 - 0.00203 X2128 - 0.00188 X2137 - 0.00173 X2145
- 0.00159 X2154 - 0.00146 X2163 - 0.00135 X2171 - 0.00124 X2180
- 0.00113 X2189 - 0.00104 X2197 = - 0.56337

3) 0.02347 X1807 + 0.02242 X1814 + 0.02139 X1821 + 0.02041 X1829

+ 0.01946 X1836 + 0.01853 X1843 + 0.01763 X1851 + 0.01677 X1858
+ 0.01593 X1866 + 0.01513 X1873 + 0.01435 X1881 + 0.0136 X1888
+ 0.01289 X1896 + 0.0122 X1903 + 0.01154 X1911 + 0.0109 X1919
+ 0.01029 X1926 + 0.00971 X1934 + 0.00915 X1942 + 0.00862 X1949
+ 0.00811 X1957 + 0.00763 X1965 + 0.00717 X1973 + 0.00673 X1981
+ 0.00631 X1989 + 0.00591 X1997 + 0.00554 X2005 + 0.00518 X2013
+ 0.00484 X2021 + 0.00452 X2029 + 0.00422 X2037 + 0.00393 X2045
+ 0.00366 X2053 + 0.0034 X2061 + 0.00316 X2070 + 0.00294 X2078
+ 0.00273 X2086 + 0.00253 X2095 + 0.00234 X2103 + 0.00217 X2111
+ 0.002 X2120 + 0.00185 X2128 + 0.0017 X2137 + 0.00157 X2145
+ 0.00145 X2154 + 0.00133 X2163 + 0.00122 X2171 + 0.00112 X2180
+ 0.00103 X2189 + 0.00094 X2197 - COST = 0

4) X1807 + X1814 + X1821 + X1829 + X1836 + X1843 + X1851 + X1858
   + X1866 + X1873 + X1881 + X1888 + X1896 + X1903 + X1911 + X1919 + X1926
Exhibit 2 -continued-

- 0.00113 X2189 - 0.00104 X2197 - FLOW3 = - 0.59418

8)  FLOW1 >=  0.49
9)  FLOW2 >=  0.45
10) FLOW3 >=  0.58
END

OBJECTIVE FUNCTION VALUE
1)  0.450254510E-02

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EXHIBIT 3: 2-PERIOD MAX CF ST MAXIMUM COST

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<td>HL</td>
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<td>HL</td>
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<th>HH</th>
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<tr>
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<tr>
<td>HL</td>
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<td>33%</td>
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MAXCOST = 0.5

MAX CF
SUBJECT TO
2) - CF + 0.11109 FLOWHH + 0.11109 FLOWHM + 0.11112 FLOWHL + 0.33333 FLOWL + 0.33333 FLOWM + 0.33333 FLOWH = 0
3) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
4) COSTL - MAXCOST <= 0
5) COSTM - MAXCOST <= 0
6) COSTH - MAXCOST <= 0
7) MAXCOST = 2
8) - CF + 0.11109 FLOWHH + 0.11109 FLOWHM + 0.11112 FLOWHL + 0.33333 FLOWL + 0.33333 FLOWM + 0.33333 FLOWH = 0
9) COSTL - MAXCOST <= 0
10) COSTM - MAXCOST <= 0
11) COSTH - MAXCOST <= 0
12) MAXCOST = 2
13) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
14) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
15) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
16) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
17) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0

MAX CF
SUBJECT TO
2) - CF + 0.11109 FLOWHH + 0.11109 FLOWHM + 0.11112 FLOWHL + 0.33333 FLOWL + 0.33333 FLOWM + 0.33333 FLOWH = 0
3) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
4) COSTL - MAXCOST <= 0
5) COSTM - MAXCOST <= 0
6) COSTH - MAXCOST <= 0
7) MAXCOST = 2
8) - CF + 0.11109 FLOWHH + 0.11109 FLOWHM + 0.11112 FLOWHL + 0.33333 FLOWL + 0.33333 FLOWM + 0.33333 FLOWH = 0
9) COSTL - MAXCOST <= 0
10) COSTM - MAXCOST <= 0
11) COSTH - MAXCOST <= 0
12) MAXCOST = 2
13) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
14) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
15) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
16) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0
17) - CF + 0.11112 FLOWHH + 0.11112 FLOWHM + 0.11116 FLOWHL + 0.11116 FLOWLM + 0.11116 FLOWLL = 0

X176
+ 2.88559 X178 + 2.6332 X178 + 2.39849 X182 + 2.18143 X184 + 1.98109 X186 + 1.79651 X188 + 1.62678 X190 + 1.47098 X192 + 1.32823 X194 + 1.19767 X196 + 1.07847 X198 + 0.96983 X200 + 0.87098 X202 + 0.78118 X204 + 0.69975 X206 + 0.62603 X208 + 1.78627 XL170 + 1.54227 XL172 + 1.32552 XL174 + 1.13404 XL176 + 0.96583 XL178 + 0.81887 XL180 + 0.69117 XL182 + 0.58081 XL184 + 0.48595 XL186 + 0.40482 XL188 + 0.3358 XL190 + 0.27739 XL192 + 0.22819 XL194 + 0.18695 XL196 + 0.15256 XL198 + 0.12401 XL200 + 0.10041 XL202 + 0.081 XL204 + 0.0651 XL206 + 0.05213 XL208 = 0
18) - COSTM + 4.1044 X170 + 3.76728 X172 + 3.45183 X174 + 3.15846 X176
   + 2.88559 X178 + 2.6332 X180 + 2.39849 X182 + 2.18143 X184
   + 1.98109 X186 + 1.79651 X188 + 1.62678 X190 + 1.47098 X192
   + 1.32823 X194 + 1.13767 X196 + 1.07847 X198 + 0.96983 X200
   + 0.87098 X202 + 0.78118 X204 + 0.69975 X206 + 0.62603 X208
   + 2.80544 XM170 + 2.46885 XM172 + 2.16205 XM174 + 1.88611 XM176
   + 1.63706 XM178 + 1.41432 XM180 + 1.21623 XM182 + 1.04104 XM184
   + 0.88698 XM186 + 0.75226 XM188 + 0.63509 XM190 + 0.53376 XM192
   + 0.44659 XM194 + 0.372 XM196 + 0.30852 XM198 + 0.25478 XM200
   + 0.2095 XM202 + 0.17155 XM204 + 0.13989 XM206 + 0.11362 XM208 = 0

19) - COSTH + 4.1044 X170 + 3.76728 X172 + 3.45183 X174 + 3.15846 X176
   + 2.88559 X178 + 2.6332 X180 + 2.39849 X182 + 2.18143 X184
   + 1.98109 X186 + 1.79651 X188 + 1.62678 X190 + 1.47098 X192
   + 1.32823 X194 + 1.13767 X196 + 1.07847 X198 + 0.96983 X200
   + 0.87098 X202 + 0.78118 X204 + 0.69975 X206 + 0.62603 X208
   + 2.6332 X180 + 2.39849 X182 + 2.18143 X184
   + 1.79651 X188 + 1.62678 X190 + 1.47098 X192
   + 1.13767 X196 + 1.07847 X198 + 0.96983 X200
   + 0.87098 X202 + 0.78118 X204 + 0.69975 X206 + 0.62603 X208
   + 4.57159 XM172 + 4.1506 XM174 + 3.75576 XM176
   + 3.04365 XM180 + 2.7256 XM182 + 2.43212 XM184
   + 1.9152 XM188 + 1.6914 XM190 + 1.48768 XM192
   + 1.13845 XM196 + 0.99052 XM198 + 0.85872 XM200
   + 0.74181 XM202 + 0.63855 XM204 + 0.54774 XM206 + 0.46821 XM208 = 0

20) - FLOWHH + 0.08403 X170 + 0.08127 X172 + 0.0784 X174 + 0.07542 X176
   + 0.07234 X178 + 0.06915 X180 + 0.06588 X182 + 0.06254 X184
   + 0.05912 X186 + 0.05563 X188 + 0.05209 X190 + 0.04849 X192
   + 0.04485 X194 + 0.04117 X196 + 0.03746 X198 + 0.03372 X200
   + 0.02997 X202 + 0.0262 X204 + 0.02242 X206 + 0.01845 X208
   + 0.07296 XH170 + 0.07153 XH172 + 0.06994 XH174 + 0.06819 XH176
   + 0.06627 XH178 + 0.06418 XH180 + 0.06193 XH182 + 0.0595 XH184
   + 0.05692 XH186 + 0.0542 XH188 + 0.0513 XH190 + 0.04829 XH192
   + 0.04514 XH194 + 0.04188 XH196 + 0.03852 XH198 + 0.03506 XH200
   + 0.03153 XH202 + 0.02792 XH204 + 0.02426 XH206 + 0.02056 XH208
   = - 0.45455

21) - FLOWHM + 0.05077 X170 + 0.04801 X172 + 0.04514 X174 + 0.04216 X176
   + 0.03908 X178 + 0.03589 X180 + 0.03262 X182 + 0.02928 X184
   + 0.02586 X186 + 0.02237 X188 + 0.01883 X190 + 0.01523 X192
   + 0.01159 X194 + 0.00791 X196 + 0.0042 X198 + 0.00046 X200
   - 0.00329 X202 - 0.00706 X204 - 0.00847 X206 - 0.00757 X208
   + 0.0397 XH170 + 0.03827 XH172 + 0.03669 XH174 + 0.03493 XH176
   + 0.03301 XH178 + 0.03092 XH180 + 0.02867 XH182 + 0.02624 XH184
   + 0.02366 XH186 + 0.02094 XH188 + 0.01804 XH190 + 0.01503 XH192
   + 0.01188 XH194 + 0.00862 XH196 + 0.00526 XH198 + 0.0018 XH200
Exhibit 3 -continued-
- 0.00173 XH202 - 0.00534 XH204 - 0.00663 XH206 - 0.00567 XH208
= - 0.4878

22) - FLOWHL - 0.03286 X170 - 0.03562 X172 - 0.03848 X174 - 0.03822
X176
- 0.03492 X178 - 0.03186 X180 - 0.02902 X182 - 0.0264 X184
- 0.02397 X186 - 0.02174 X188 - 0.01968 X190 - 0.0178 X192
- 0.01607 X194 - 0.01449 X196 - 0.01305 X198 - 0.01173 X200
- 0.01054 X202 - 0.00945 X204 - 0.00847 X206 - 0.00757 X208
- 0.04392 XH170 - 0.04535 XH172 - 0.04694 XH174 - 0.04544 XH176
- 0.04098 XH178 - 0.03683 XH180 - 0.03298 XH182 - 0.02943 XH184
- 0.02617 XH186 - 0.02317 XH188 - 0.02047 XH190 - 0.018 XH192
- 0.01578 XH194 - 0.01378 XH196 - 0.01199 XH198 - 0.01039 XH200
- 0.00898 XH202 - 0.00773 XH204 - 0.00663 XH206 - 0.00567 XH208
= - 0.57143

23) - FLOWMH + 0.06238 X170 + 0.05962 X172 + 0.05676 X174 + 0.05377
X176
+ 0.05069 X178 + 0.0475 X180 + 0.04424 X182 + 0.04089 X184
+ 0.03747 X186 + 0.03399 X188 + 0.03044 X190 + 0.02684 X192
+ 0.0232 X194 + 0.01952 X196 + 0.01581 X198 + 0.01207 X200
+ 0.00832 X202 + 0.00455 X204 + 0.00078 X206 - 0.003 X208 + 0.0781
XM170
+ 0.07533 XM172 + 0.07236 XM174 + 0.06917 XM176 + 0.0658 XM178
+ 0.06225 XM180 + 0.05854 XM182 + 0.0549 XM184 + 0.05071 XM186
+ 0.04662 XM188 + 0.04244 XM190 + 0.03818 XM192 + 0.03387 XM194
+ 0.02951 XM196 + 0.02513 XM198 + 0.02073 XM200 + 0.01632 XM202
+ 0.01193 XM204 + 0.00755 XM206 + 0.0032 XM208 = - 0.47619

24) - FLOWMM - 0.04966 X170 - 0.04558 X172 - 0.04177 X174 - 0.03822
X176
- 0.03492 X178 - 0.03186 X180 - 0.02902 X182 - 0.0264 X184
- 0.02397 X186 - 0.02174 X188 - 0.01968 X190 - 0.0178 X192
- 0.01607 X194 - 0.01449 X196 - 0.01305 X198 - 0.01173 X200
- 0.01054 X202 - 0.00945 X204 - 0.00847 X206 - 0.00757 X208
- 0.03395 XM170 - 0.02987 XM172 - 0.02616 XM174 - 0.02282 XM176
= - 0.58824

25) - FLOWML - 0.04966 X170 - 0.04558 X172 - 0.04177 X174 - 0.03822
X176
- 0.03492 X178 - 0.03186 X180 - 0.02902 X182 - 0.0264 X184
- 0.02397 X186 - 0.02174 X188 - 0.01968 X190 - 0.0178 X192
- 0.01607 X194 - 0.01449 X196 - 0.01305 X198 - 0.01173 X200
- 0.01054 X202 - 0.00945 X204 - 0.00847 X206 - 0.00757 X208
- 0.03395 XM170 - 0.02987 XM172 - 0.02616 XM174 - 0.02282 XM176
Exhibit 3 -continued-

- 0.01981 XM178 - 0.01711 XM180 - 0.01472 XM182 - 0.0126 XM184
- 0.01073 XM186 - 0.0091 XM188 - 0.00768 XM190 - 0.00646 XM192
- 0.0054 XM194 - 0.0045 XM196 - 0.00373 XM198 - 0.00308 XM200
- 0.00253 XM202 - 0.00208 XM204 - 0.00169 XM206 - 0.00137 XM208
= - 0.84034

26) - FLOWLH + 0.02575 X170 + 0.02299 X172 + 0.02013 X174 + 0.01714 X176+ 0.01406 X178 + 0.01087 X180 + 0.00761 X182 + 0.00426 X184+ 0.00084 X186 - 0.00264 X188 - 0.00619 X190 - 0.00979 X192
- 0.01343 X194 - 0.01449 X196 - 0.01305 X198 - 0.01173 X200
- 0.01054 X202 - 0.00945 X204 - 0.00847 X206 - 0.00757 X208 + 0.0538 XL170 + 0.04991 XL172 + 0.04585 XL174 + 0.04164 XL176 + 0.03729 XL178 + 0.03283 XL180 + 0.02827 XL182 + 0.02363 XL184 + 0.01893 XL186 + 0.0142 XL188 + 0.00943 XL190 + 0.00466 XL192 - 0.00012 XL194 - 0.00226 XL196 - 0.00185 XL198 - 0.0015 XL200 - 0.00121 XL202 - 0.00098 XL204 - 0.00079 XL206 - 0.00063 XL208
= - 0.51282

27) - FLOWLM - 0.04966 X170 - 0.04558 X172 - 0.04177 X174 - 0.03822 X176 - 0.03492 X178 - 0.03186 X180 - 0.02902 X182 - 0.0264 X184 - 0.02397 X186 - 0.02174 X188 - 0.01968 X190 - 0.0178 X192 - 0.01607 X194 - 0.01449 X196 - 0.01305 X198 - 0.01173 X200 - 0.01054 X202 - 0.00945 X204 - 0.00847 X206 - 0.00757 X208 - 0.02161 XL170 - 0.01866 XL172 - 0.01604 XL174 - 0.01372 XL176 - 0.01169 XL178 - 0.00991 XL180 - 0.00836 XL182 - 0.00703 XL184 - 0.00558 XL186 - 0.0049 XL188 - 0.00406 XL190 - 0.00336 XL192 - 0.00276 XL194 - 0.00226 XL196 - 0.00185 XL198 - 0.0015 XL200 - 0.00121 XL202 - 0.00098 XL204 - 0.00079 XL206 - 0.00063 XL208
= - 0.625

28) - FLOWLL - 0.04966 X170 - 0.04558 X172 - 0.04177 X174 - 0.03822 X176 - 0.03492 X178 - 0.03186 X180 - 0.02902 X182 - 0.0264 X184 - 0.02397 X186 - 0.02174 X188 - 0.01968 X190 - 0.0178 X192 - 0.01607 X194 - 0.01449 X196 - 0.01305 X198 - 0.01173 X200 - 0.01054 X202 - 0.00945 X204 - 0.00847 X206 - 0.00757 X208 - 0.02161 XL170 - 0.01866 XL172 - 0.01604 XL174 - 0.01372 XL176 - 0.01169 XL178 - 0.00991 XL180 - 0.00836 XL182 - 0.00703 XL184 - 0.00558 XL186 - 0.0049 XL188 - 0.00406 XL190 - 0.00336 XL192 - 0.00276 XL194 - 0.00226 XL196 - 0.00185 XL198 - 0.0015 XL200 - 0.00121 XL202 - 0.00098 XL204 - 0.00079 XL206 - 0.00063 XL208
= - 0.89286

29) X170 + X172 + X174 + X176 + X180 + X182 + X184 + X186+ X188 + X190 + X192 + X194 + X196 + X198 + X200 + X202 + X204 + X206 + X208 + XL170 + XL172 + XL174 + XL176 + XL178 + XL180 + XL182 + XL184+ XL186 + XL188 + XL190 + XL192 + XL194 + XL196 + XL198 + XL200 + XL202+ XL204 + XL206 + XL208 <= 1
Exhibit 3 -continued-

30) \( X_{170} + X_{172} + X_{174} + X_{176} + X_{178} + X_{180} + X_{182} + X_{184} + X_{186} + X_{188} + X_{190} + X_{192} + X_{194} + X_{196} + X_{198} + X_{200} + X_{202} + X_{204} + X_{206} + X_{208} + X_{M170} + X_{M172} + X_{M174} + X_{M176} + X_{M178} + X_{M180} + X_{M182} + X_{M184} + X_{M186} + X_{M188} + X_{M190} + X_{M192} + X_{M194} + X_{M196} + X_{M198} + X_{M200} + X_{M202} + X_{M204} + X_{M206} + X_{M208} \leq 1 \)

31) \( X_{170} + X_{172} + X_{174} + X_{176} + X_{178} + X_{180} + X_{182} + X_{184} + X_{186} + X_{188} + X_{190} + X_{192} + X_{194} + X_{196} + X_{198} + X_{200} + X_{202} + X_{204} + X_{206} + X_{208} + X_{H170} + X_{H172} + X_{H174} + X_{H176} + X_{H178} + X_{H180} + X_{H182} + X_{H184} + X_{H186} + X_{H188} + X_{H190} + X_{H192} + X_{H194} + X_{H196} + X_{H198} + X_{H200} + X_{H202} + X_{H204} + X_{H206} + X_{H208} \leq 1 \)

END

OBJECTIVE FUNCTION VALUE

1) .610925300

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EXHIBIT 4: 2-PERIOD MAX CF ST MAXIMUM AVERAGE COST
SEE EXHIBIT 3 FOR MARKET PARAMETERS

FORMULATION: SAME AS EXHIBIT 3, EXCEPT SUBSTITUTE FOR CONSTRAINT 7:

7) AVGC <= 2

OBJECTIVE FUNCTION VALUE

1) .617458220

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EXHIBIT 5: 2-PERIOD MIN AVERAGE COST ST TARGETS
SEE EXHIBIT 3 FOR MARKET PARAMETERS

FORMULATION: SAME AS EXHIBIT 3, EXCEPT:

CHANGE OBJECTIVE FUNCTION TO:

• MIN AVGC

• DELETE CONSTRAINT 7;

• ADD THE FOLLOWING CONSTRAINTS:

32) FLOWHH >= 0.509
33) FLOWHM >= 0.509
34) FLOWHL >= 0.547
35) FLOWMH >= 0.53
36) FLOWMM >= 0.575
37) FLOWML >= 0.827
38) FLOWLH >= 0.554
39) FLOWLM >= 0.611
40) FLOWLL >= 0.879

OBJECTIVE FUNCTION VALUE

1) 1.35399710

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