Higher Canonical Asymptotics of Kähler-Einstein Metrics on Quasi-Projective Manifolds

by

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Abstract

In this thesis, we derive the asymptotic expansion of the Kähler-Einstein metrics on certain quasi-projective varieties, which can be compactified by adding a divisor with simple normal crossings. The weighted Cheng-Yau Hölder spaces and the log-filtrations based on the bounded geometry are introduced to characterize the asymptotics. We first develop the analysis of the Monge-Ampère operators on these weighted spaces. We construct a family of linear elliptic operators which can be viewed as certain conjugacies of the specially linearized Monge-Ampère operators. We derive a theorem of Fredholm alternative for such elliptic operators by the Schauder theory and Yau's generalized maximum principle. Together these results derive the isomorphism theorems of the Monge-Ampère operators, which imply that the Monge-Ampère operators preserve the log-filtration of the Cheng-Yau Hölder ring.

Next, by choosing a canonical metric on the submanifold, we construct an initial Kähler metric on the quasi-projective manifold such that the unique solution of the Monge-Ampère equation belongs to the weighted -1 Cheng-Yau Hölder ring. Moreover, we generalize the Fefferman's operator to act on the volume forms and obtain an iteration formula. Finally, with the aid of the isomorphism theorems and the iteration formula we derive the desired asymptotics from the initial metric. Furthermore, we prove that the obtained asymptotics is canonical in the sense that it is independent of the extensions of the canonical metric on the submanifold.

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Contents

Chapter 1

Introduction

On a complex manifold M of dimension n, a *volume form* Ψ is a smooth positive (n,n) form. In a local coordinate neighborhood U_α with holomorphic coordinates $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$,

$$\Psi = \xi_\alpha \prod_{j=1}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_\alpha^j \wedge d\bar{z}_\alpha^j \right).$$

where ξ_α is a positive C^∞ function. Then the *Ricci form* $\operatorname{Ric}\Psi$ associated to Ψ is the real $(1,1)$ form given locally by

$$\operatorname{Ric}\Psi = dd^c \log \xi_\alpha.$$

It follows that $\operatorname{Ric}\Psi$ is globally defined on M and that $\operatorname{Ric}\Psi = c_1(K_M)$, where K_M is the *canonical bundle.*

We define the *generalized Fefferman operator* J by

$$J(\Psi) = \frac{(\operatorname{Ric}\Psi)^n}{\Psi},$$

where Ψ is a volume form. Let $\Psi_\varphi = e^\varphi \Psi$. Then we have the following iteration formula

$$J(\Psi_\varphi) = M_{\operatorname{Ric}\Psi}(\varphi) J(\Psi), \tag{1.1}$$

where

$$M_{\mathrm{Ric}\,\Psi}(\varphi) = \frac{(\mathrm{Ric}\,\Psi + dd^c\varphi)^n}{(\mathrm{Ric}\,\Psi)^n} e^{-\varphi},$$

which is the Monge-Ampère operator associated to $\mathrm{Ric}\,\Psi$. It follows from the definition that $J(\Psi_\varphi) = 1$ if and only if $M_{\mathrm{Ric}\,\Psi}(\varphi) = 1/J(\Psi)$.

Let M be a complex manifold. A divisor D on M is said to have *normal crossings* if locally D is given by an equation

$$z_1 \cdots z_k = 0$$

where $(z_1, \ldots, z_n)$ are local holomorphic coordinates on M. Moreover, if each irreducible component of D is smooth, then we shall say that D has *simple normal crossings.*

Let $\overline{M}$ be a compact complex manifold and D be a divisor in $\overline{M}$ with simple normal crossings. We impose the positivity condition

$$K_{\overline{M}} + [D] > 0. \tag{1.2}$$

Then a theorem of Carlson-Griffiths [4] assures that there exists a volume form Ω on $M \equiv \overline{M} \setminus D$ such that $\mathrm{Ric}\,\Omega > 0$ on M, and $(M, \mathrm{Ric}\,\Omega)$ is a complete Kähler manifold with negative Ricci curvature.

This theorem provides an initial metric $\omega \equiv \mathrm{Ric}\,\Omega$, which can be deformed to the canonical Kähler-Einstein metric on M. In fact, this was first addressed by Yau [39], and later on by Cheng-Yau [8], R. Kobayashi [21], H. Tsuji [34], Tian-Yau [33] and Bando [1]. It follows that M possesses a unique complete Kähler-Einstein metric of constant negative Ricci curvature -1.

In this thesis, we derive the asymptotic expansion of the Kähler-Einstein metrics near a simple normal crossing divisor D. In order to characterize the asymptotics, we introduce the weighted Cheng-Yau Hölder rings and the associated filtrations

based on the bounded geometry. We first develop the analysis of the Monge-Ampère operators on these weighted spaces. We derive certain isomorphism theorems for the Monge-Ampère operators and their linearizations. These results imply that the Monge-Ampère operators and the Laplacians preserve the log-filtration of the Cheng-Yau Hölder ring up to infinite weight.

Second, we construct certain initial volume form which can be used to approximate the canonical volume form in a nice way. This construction depends on the existence of a canonical metric on the smooth subvariety. Third, by the initial volume form and the iteration formula (1.1) we can derive a formal asymptotics of the Kähler-Einstein volume form; then, by the isomorphism theorems we prove that the formal asymptotics is indeed the real asymptotics. Furthermore, we prove that the obtained asymptotics is canonical in the sense that it is independent of the extensions of the canonical metric on the subvariety.

The plan of this thesis consists in starting from deriving the asymptotics in the special case to make it easier for the reader to comprehend what is going on before treating the more involved simple normal crossing case. The whole machinery deriving in the smooth case is generalized in a less trivial way to the difficulties arise from the higher codimensional situations. Also, the proof of canonicity requires some further development of the analysis on the bounded geometry.

In Chapter 2, we establish the basic setting, the bounded geometry and the weighted Cheng-Yau Hölder spaces, which are essential for our geometric analysis. We first recall the notions of local quasi-coordinate map, bounded geometry, and the Cheng-Yau Hölder spaces in the first section. Then in Section 2.2 we introduce the weighted Cheng-Yau Hölder rings and the associated filtrations, which will be used to characterize the asymptotics. Furthermore, we define the differential forms over the weighted rings, which is convenient for the later analysis.

In Chapter 3 we derive the asymptotics in the case of complement of a smooth divisor D. Section 3.1 establishes the isomorphism theorems for both the Monge-

Ampère operator M_ω and its linearizations. To do this, we first linearize M_ω as certain negative Laplacian Δ_u by fixing the Cheng-Yau solution u. Then we construct a family of linear elliptic operators which can be viewed as certain conjugacies of $\Delta_u - 1$. By the Schauder theory and Yau's generalized maximum principle we derive a theorem of Fredholm alternative for such elliptic operators, which, in turn, imply the isomorphism theorems.

In section 3.2, the Cheng-Yau solution u of the Monge-Ampère equation

$$M_\omega(u) = J(\Omega)^{-1} \tag{1.3}$$

can be characterized by the weighted -1 Hölder ring $\mathcal{R}_1(M)$. This is achieved by a careful construction of the initial metric, which depends on a canonical metric on the smooth divisor D. This characterization result, followed immediately from the isomorphism theorem for M_ω, is independent of the extension of the canonical metric. In section 3.3, a formal asymptotics is derived by the perturbation method based on the iteration formula (1.1). The crucial part is to prove that this formal asymptotics is the real asymptotics, which is achieved by the isomorphism theorems derived in Section 3.1

In Chapter 4, The whole machinery used in the previous chapter are generalized to derive the asymptotics near a divisor with simple normal crossings. More precisely, we obtain the asymptotics of Kähler-Einstein volume form near the complete intersections of the irreducible components of the divisor. In Section 4.1 we establish the isomorphism theorems on the homogeneous weighted Cheng-Yau Hölder spaces with respect to an index subset I, which is essential in the proof of asymptotics in the simple normal crossing case. Moreover, our isomorphism theorems are formulated and proved in a much general form, which, we believe, should have interests of their own.

The construction of initial volume forms in Section 4.2 is less trivial than that of the smooth case. In fact, we study the construction from the viewpoint of "moduli

spaces"; namely, we consider $\mathcal{M}_I$ and $\mathcal{M}_{I,\mathcal{H}_I}$, the families of the initial volume forms which satisfy the adjunction formula, and the formula plus certain metric restriction condition, respectively. The latter condition is useful in characterizing the canonicity of the asymptotics. We essentially present two ways to extend the canonical metric on the higher codimensional subvariety: One can be done by direct calculations, while the other way is to use the metric extension theorem (see Theorem 7), which should also have interests of its own.

Finally, together the results in the previous sections and the iteration formula, Section 4.3 derives the canonical asymptotics near the complete intersection. Namely, given $\Omega \in \mathcal{M}_I$, the Kähler-Einstein volume form $\Omega_{K-E} = e^u\Omega$ on M has the following canonical asymptotic expansion near the complete intersection D_I:

$$\Omega_{K-E} \sim \left(1 + \sum_{|r|=1}^{\infty} \frac{\phi_r}{\sigma^r}\right)\Omega, \tag{1.4}$$

where $\phi_r \in \mathcal{R}(M)$ for each $r \in (\mathbb{Z}_+^q \times 0)$. The asymptotics is canonical in the the following sense: Let $\Omega \in \mathcal{M}_{I,\mathcal{H}_I}$ for a given $\mathcal{H}_I$. If (1.4) is derived in terms of another $\Omega' \in \mathcal{M}_{I,\mathcal{H}_I}$, and coefficients $\{\phi'_r\}_{r\in\mathbb{Z}_+^q\times 0}$. Then

$$\begin{aligned} &u' - u \in \mathcal{R}_{I,\infty}(M) \\ &\phi'_r - \phi_r \in \mathcal{R}_{I,\infty}(M), \qquad \forall\, r \in \mathbb{Z}_+^q \times 0. \end{aligned}$$

Our work may be viewed as the counterpart of Fefferman [14] and Lee-Melrose's asymptotics [23] on the pseudoconvex domain in $\mathbb{C}^n$. The background of their work refers to Fefferman's papers [13], [14], [15], [2], and Cheng-Yau [7]. See also J. Bland [3], R. Graham [17], K. Hirachi [20] and the references therein for the further development of the asymptotic geometry in the pseudoconvex domains and Cauchy-Riemann manifolds.

In the special case of a smooth divisor, certain initial results and applications were obtained by Schumacher [31] and [32]. His main result is equivalent to $u \in \mathcal{R}_r(M)$

with $0 < r \leq 1$ undetermined, which is, however, less precise. Indeed the accurate weight, $r = 1$, is crucial for deriving the asymptotics, as seen in section 3.3. Also, his method, repeating the process of continuity method, does not give information on the higher order terms of the asymptotics.

Our work settles the case of simple normal crossing divisor. The approach was motivated by [23] and [22]. Indeed, we obtained a formal asymptotic expansion in terms of the log-filtration even before noticing [31] and [32]. The theorem of Fredholm alternative derived in Section 3.1 enable us to prove the isomorphism theorems on Cheng-Yau Hölder spaces up to infinite weight. These, in turn, imply that the formal asymptotics is the real asymptotics, which is furthermore canonical.

Our asymptotics near the simple normal crossings divisor can be viewed as a higher dimensional generalization of Nevanlinna's classic result on $\mathbb{P}^1 \setminus \{p_1, \ldots, p_\mu \mid \mu \geq 3\}$ (see [28, p. 249–250].), which played a fundamental role in the second main theorem of Nevanlinna theory. Hence, it is natural to expect that our work could have applications to modern Nevanlinna theory, which, in turn, has applications to analytic geometry (see, for example, [4], [9], and [19]).

Chapter 2

Bounded Geometry

In this chapter, we will establish our basic setting, the bounded geometry and the Cheng-Yau Hölder spaces. These notions were introduced in [7], [8] and [33]. We will recall the definitions in Section 2.1. In Section 2.2 we consider the initial volume form and the associated initial metric, which can be deformed to the Kähler-Einstein metric. Furthermore, to characterize the asymptotics, we introduce the weighted Ching-Yau Hölder rings and the associated filtrations, which will be used extensively in the next chapters.

2.1 Quasi-coordinate map and bounded geometry

Let X be an n-dimensional complex manifold. Recall that the notion of *quasi-coordinate* is given as follows.

Definition 1. *Let $V \subset \mathbb{C}^n$ be an open set. A holomorphic map $\phi : V \to X^n$ is called a quasi-coordinate map if $rank_p(d\phi) = n$ for every $p \in V$. In this case, (V, ϕ) is called a local quasi-coordinate of X.*

Next, the *bounded geometry* is defined below in terms of a system of local quasi-coordinates.

Definition 2. *Given a complete Kähler manifold (X, ω). We say that (X, ω) has bounded geometry of order $m + \mu$, where $m \in \mathbb{Z}_+$, and $\mu \in [0, 1)$, if there exists a*

system of quasi-coordinates $\mathcal{V} = \{(V_\eta, \phi_\eta)\}$ *such that*

1. $X = \bigcup_\eta \phi_\eta(V_\eta)$, *and each* $x \in X$ *is centered at some* V_η;

2. *For each* η, $1/2 \leq$ *radius of* $V_\eta \leq 1$;

3. *There exist constant* C *and* $\mathcal{A}_m$ *such that, for each* η, *if we write*

$$\phi_\eta^*(\omega) \equiv \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{\eta,i\bar{j}} dz^i \wedge d\bar{z}^j,$$

then

$$0 < C^{-1}(\delta_{ij}) \leq (g_{\eta,i\bar{j}}) \leq C(\delta_{ij}),$$
$$\left\| g_{\eta,i\bar{j}} \right\|_{C^{m,\mu}(V_\eta)} \equiv \sup_{|p|+|q| \leq m} \left\| \frac{\partial^{|p|+|q|}}{\partial v^p \partial \bar{v}^q} g_{\eta,i\bar{j}} \right\|_{C^\mu(V_\eta)} \leq \mathcal{A}_m.$$

Based on the bounded geometry, the Cheng-Yau's Hölder spaces are defined as follows:

Definition 3. *Fix a quasi-coordinate system* $\mathcal{V} = \{(V_\eta, \phi_\eta)\}$ *with (1), (2), and (3) stated in Definition 2. For* $k \in \mathbb{Z}_+$, $\alpha \in [0, 1)$, *define the norm* $\|\cdot\|_{k,\alpha}$ *on* $C^\infty(X)$ *by*

$$\|u\|_{k,\alpha} = \sup_{V_\alpha \in \mathcal{V}} \left\{ \|\phi_\eta^*(u)\|_{C^{k,\alpha}(V_\eta)} \right\}.$$

Define

$$C^{k,\alpha}(X) = \text{the completion of } \{u \in C^\infty(X)\,;\ \|u\|_{k,\alpha} < +\infty\} \text{ with respect to } \|\cdot\|_{k,\alpha}.$$

Define the *Cheng-Yau Hölder ring* by

$$\mathcal{R}(X) = \bigcap_{k \geq 0,\, 0 < \alpha < 1} C^{k,\alpha}(X).$$

2.2 Weighted Cheng-Yau Hölder rings and the associated filtrations

Let $\overline{M}$ be a compact complex manifold and $D = \sum_{i=1}^{p} D_i$ be a simple normal divisor, where the irreducible components D_i are smooth and intersect transversely. Denote $M = \overline{M} \setminus D$. Suppose $K_{\overline{M}} + [D] > 0$. Let $s_i \in H^0(\overline{M}, \mathcal{O}([D_i]))$ be the holomorphic section defining D_i. Then by the theorem of Carlson and Griffiths, there exists a C^∞ volume form V on $\overline{M}$ such that

$$\Omega \equiv \frac{V}{\prod_{i=1}^{p} |s_i|^2 (\log |s_i|^2)^2} \tag{2.1}$$

is a volume on M satisfying the following properties:

(i) $\operatorname{Ric}\Omega > 0$ on M, and $(M, \operatorname{Ric}\Omega)$ is a complete Kähler manifold with finite volume;

(ii) there is a positive constant C such that

$$C^{-1} < J(\Omega) < C \qquad \text{on } \overline{M}. \tag{2.2}$$

Such a volume form Ω is called an *initial volume form.* Denote

$$\begin{aligned} \omega &= \operatorname{Ric}\Omega \\ &= \omega_K + 2\sum_{i=1}^{p} \sigma_i^{-1}\omega_{c_i} + 2\sum_{i=1}^{p} \sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i, \end{aligned}$$

where

$$\begin{aligned} \omega_K &= \operatorname{Ric}\left(\frac{V}{\prod_{i=1}^{p} |s_i|^2}\right), \\ \omega_{c_i} &= -dd^c\sigma_i. \end{aligned}$$

It is well known that (M, ω) has a system of local quasi-coordinates (V_η, ϕ_η) with bounded geometry of order infinity. (See, for example, [8], [33] or [21].)

Denote by $\mathbb{R}_+$ and $\mathbb{Z}_+$ the sets of nonnegative real numbers and nonnegative integers, respectively; denote $\mathbb{R}_+^m \equiv (\mathbb{R}_+)^m$ and $\mathbb{Z}_+^m \equiv (\mathbb{Z}_+)^m$ for each $m \in \mathbb{N}$. For $r = (r_1, \ldots, r_p) \in \mathbb{R}_+^p$ with $|r| \equiv r_1 + \cdots + r_p$, denote by

$$\sigma^{-r} = \sigma_1^{-r_1} \cdots \sigma_p^{-r_p}.$$

Then the weighted Cheng-Yau Hölder spaces $\sigma^{-r}C^{k,\alpha}(M)$ are the Banach spaces defined as usual. For each $l \in \mathbb{N}$, the *homogeneous* weighted space $\sum_{|r|=l} \sigma^{-r}C^{k,\alpha}(M)$ is defined as a normed linear subspace of $C^{k,\alpha}(M)$. Similarly, let

$$\mathcal{R}_r(M) = \sigma^{-r}\mathcal{R}(M), \qquad \forall\, r \in \mathbb{R}_+^p.$$

Moreover, denote by $\mathcal{R}^1(M)$ the $\mathcal{R}(M)$–module of differential 1–forms on M; namely, for each 1–form $\varphi \in \mathcal{R}^1(M)$, there exist constants $\mathcal{A}_{\varphi,k}$, $k \in \mathbb{Z}_+$, such that if, for each local quasi-coordinate (V_η, ϕ_η),

$$\phi_\eta^*(\varphi) = f_i dv^i + g_{\bar{j}} d\bar{v}^j,$$

then

$$\|f_i\|_{C^{k,\alpha}(V_\eta)} \leq \mathcal{A}_{\varphi,k}, \quad \|g_{\bar{j}}\|_{C^{k,\alpha}(V_\eta)} \leq \mathcal{A}_{\varphi,k}, \quad \forall\, \alpha \in (0,1).$$

Note that $\mathcal{R}^1(M)$ is a subset of $A^1(M)$, the set of smooth 1-forms on M. Let

$$\mathcal{R}^m(M) = \wedge^k \mathcal{R}^1(M), \qquad \forall\, m \in \mathbb{N}.$$

Similarly, denote by $\mathcal{R}^{p,q}(M)$ the module of (p,q)-forms on M over the ring $\mathcal{R}(M)$. In particular, it follows from the definition that $\omega \in \mathcal{R}^{1,1}(M)$. Furthermore, a (p,q)-form φ on M belongs to $R^{p,q}(M)$ if and only if

$$\Lambda_\omega(\varphi) \equiv \frac{\omega^{n-p-q} \wedge \varphi \wedge \overline{\varphi}}{\omega^n} \in \mathcal{R}(M). \tag{2.3}$$

We are interested in the asymptotic behavior of the Kähler-Einstein metric near the (nonempty) complete intersections,

$$D_I \equiv D_{k_1} \cap \cdots \cap D_{k_i},$$

where the index set

$$I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, p\}$$

satisfies that

$$D_j \cap D_I = \emptyset, \qquad \forall\, j \notin I.$$

It is convenient to denote by $\mathcal{I}$ all such index subsets. By the definition of simple normal crossing, each D_I, $I \in \mathcal{I}$, is a smooth subvariety of codimension $|I|$, $1 \leq |I| \leq n$, but is not necessary connected. Write

$$D_I = \sum_{\nu} D_I^{\nu}$$

where each D_I^{ν} is a connected component. So, more precisely, we are interested in the asymptotics near each connected component D_I^{ν}.

To characterize the asymptotics, we introduce the *weighted Cheng-Yau Hölder rings* associated to D_I, $I = \{i_1, \ldots, i_k\} \in \mathcal{I}$, as below:

$$\mathcal{R}_{I,t}(M) = \tau_I^t \mathcal{R}(M), \qquad \forall\, t \in \mathbb{R}_+.$$
$$\mathcal{R}_{I,\infty}(M) = \bigcap_{t \geq 0} \mathcal{R}_{I,t}(M),$$

where the weight function

$$\tau_I \equiv \Big(\sum_{i \in I} \sigma_i^{-2}\Big)^{1/2}.$$

They are all ideals of $\mathcal{R}(M)$. Moreover, it follows from the definition that, for each $l \in \mathbb{N}$,

$$\begin{aligned}\mathcal{R}_{I,l}(M) &= \Big(\sum_{i\in I}\sigma_i^{-1}\Big)^l \mathcal{R}(M)\\ &= \sum_{|r_I|=l}\sigma^{-r_I}\mathcal{R}(M),\end{aligned}$$

where

$$\sigma^{-r_I} = \prod_{i\in I}\sigma_i^{-r_i}, \quad r_I = (r_{i_1},\dots,r_{i_k}) \in \mathbb{Z}_+^{|I|}.$$

Furthermore, we can also define the differential forms over the weighted Cheng-Yau Hölder rings as follows:

$$\begin{aligned}&\mathcal{R}^m_{I,k}(M) \equiv \tau_I^k\mathcal{R}^m(M), \qquad \mathcal{R}^{p,q}_{I,k}(M) \equiv \tau_I^k\mathcal{R}^{p,q}(M), \quad \forall\, m,k,p,q \in \mathbb{Z}_+,\\ &\mathcal{R}^m_{I,\infty}(M) \equiv \bigcap_{k\geq 0}\mathcal{R}^m_{I,k}(M), \quad \mathcal{R}^{p,q}_{I,\infty}(M) \equiv \bigcap_{k\geq 0}\mathcal{R}^{p,q}_{I,k}(M).\end{aligned}$$

By the definitions and (2.3) we have

$$\Lambda_\omega(\mathcal{R}^m_{I,k}(M)) \subset \mathcal{R}_{I,k}(M), \quad \Lambda_\omega(\mathcal{R}^{p,q}_{I,k}(M)) \subset \mathcal{R}_{I,k}(M), \quad \forall\, m,k,p,q \in \mathbb{Z}_+.$$

The similar results also hold for the Cheng-Yau Hölder rings with infinite weight.

Any sequence of non-decreasing numbers $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ give rise to a filtration of $\mathcal{R}(M)$:

$$\mathcal{R}_{I,t_1}(M) \supset \mathcal{R}_{I,t_2}(M) \supset \mathcal{R}_{I,t_3}(M) \supset \cdots.$$

Now we consider the *log-filtration* $\{\mathcal{R}_{I,k}(M)\}_{k\in\mathbb{Z}_+}$ of $\mathcal{R}(M)$. Define the *graded Cheng-Yau Hölder Ring* $\mathcal{R}_I^G(M)$ associated to the log-filtration by

$$\mathcal{R}_I^G(M) = \Bigg\{ u \in \mathcal{R}(M) \;\Bigg|\; \begin{array}{l} \text{there exist a multiple sequence} \\ \{\psi_r \mid r \in \mathbb{Z}_+^p, r_i = 0 \text{ if } i \notin I\}, \\ \text{where not all } \psi_r \text{ are zero, such that } u \sim \displaystyle\sum_{|r|=0}^{\infty} \psi_r \sigma^{-r}. \end{array} \Bigg\}.$$

Let

$$\mathcal{R}_{I,k}^G(M) = \mathcal{R}_I^G \bigcap \mathcal{R}_{I,k}(M), \qquad \forall\, k \in \mathbb{N}.$$

In the special case that D is *smooth*, we have

$$\mathcal{R}_{I,t}(M) = \mathcal{R}_t(M) = \sigma^{-t}\mathcal{R}(M), \quad \forall\, t \in \mathbb{R}_+,$$

where

$$\sigma = \log |s|^2,$$

in which $s \in H^0(\overline{M}, \mathcal{O}[D])$ defines D. In this case, $\mathcal{R}_k(M)_{k\in\mathbb{Z}_+}$ gives rise to the log-filtration of $\mathcal{R}(M)$. Similarly, we have the graded Cheng-Yau Hölder rings $\mathcal{R}_k^G(M)$ for each $k \in \mathbb{Z}_+$. We will first derive the asymptotics in this special case, which is the content of the next chapter.

Chapter 3

Smooth Divisor Case

In this chapter, we will derive the asymptotic expansion of Kähler-Einstein metric in the case of complement of a smooth divisor. The approach consists of three parts: The first part is to establish the isomorphism theorems for both the Monge-Ampère operator and its linearizations. Second, construct certain initial volume form which can approximate the Kähler-Einstein volume form in a nice way. Finally, by the initial volume form and the iteration formula we derive the asymptotics, which is proved to be the real asymptotics by the isomorphism theorems .

Indeed, in the following section the isomorphism theorems are stated and proved in a much general form, which, we believe, should have interests of their own.

3.1 Isomorphism theorems

Let $\overline{M}$ be a compact complex manifold and D be a smooth divisor. Denote $M = \overline{M} \setminus D$. Suppose $K_{\overline{M}} + [D] > 0$. Let $s \in H^0(\overline{M}, \mathcal{O}([D]))$ is a holomorphic section defining D. Then there exists a C^∞ volume form V on $\overline{M}$ such that

$$\Omega = \frac{V}{|s|^2 \log^2 |s|^2} \tag{3.1}$$

is an initial volume form on M. (See (i) and (ii) in Chapter 2.) Denote $\omega = \mathrm{Ric}\,(\Omega)$, and $\sigma = \log|s|^2$. Then

$$\omega = \omega_K + 2\sigma^{-1}\omega_c + 2\sigma^{-2}d\sigma \wedge d^c\sigma,$$

in which

$$\omega_K = \mathrm{Ric}\,(V/|s|^2) > 0,$$
$$\omega_c = -dd^c\sigma \in c_1([D]).$$

Let $C^{k,\alpha}(M)$, $k \geq 0$ and $\alpha \in (0,1)$, be the Cheng-Yau Hölder spaces formed by the local quasi-coordinates. Then we have the first isomorphism theorem as follows:

Theorem 1. *Fix an arbitrary $r \in \mathbb{R}_+$, $k \geq 0$ and $\alpha \in (0,1)$. For each $F \in \sigma^{-r}C^{k,\alpha}(M)$, let $u \in C^{k+2,\alpha}(M)$ be the unique Cheng-Yau solution of*

$$M_\omega(u) \equiv \frac{(\omega + dd^c u)^n}{\omega^n} e^{-u} = e^F, \tag{3.2}$$

$$\frac{1}{C}\omega \leq \omega + dd^c u \leq C\omega, \qquad C > 0. \tag{3.3}$$

Then $u \in \sigma^{-r}C^{k+2,\alpha}(M)$.

Proof of Theorem 1. *(3.2) implies that*

$$u + F = \log \frac{(\omega + dd^c u)^n}{\omega^n}.$$

Let $\omega_t = \omega + tdd^c u$. Then it follows that

$$\begin{aligned} u + F &= \int_0^1 \left[\frac{d}{dt}\log\left(\frac{\omega_t^n}{\omega^n}\right)\right] dt \\ &= \int_0^1 \left(\frac{n\omega_t^{n-1} \wedge dd^c u}{\omega_t^n}\right) dt; \end{aligned}$$

i.e., we can view the Monge-Ampère equation (3.2) as the following "linear" equation:

$$(\Delta_u - 1)u = F,$$

where

$$\Delta_u(v) \equiv \int_0^1 \left(\frac{n\omega_t^{n-1} \wedge dd^c v}{\omega_t^n} \right) dt, \qquad \forall\, v \in C^2(M). \tag{3.4}$$

Therefore, it suffices to show that

$$\Delta_u - 1 : \sigma^{-r} C^{k+2,\alpha}(M) \to \sigma^{-r} C^{k,\alpha}(M) \tag{3.5}$$

is an isomorphism for all $r \geq 0$, $k \geq 0$ *and* $\alpha \in (0,1)$.

Observe that (3.3) implies

$$[t/C + (1-t)]\,\omega \leq \omega_t \leq [Ct + (1-t)]\,\omega, \qquad \forall\, t \in [0,1], \tag{3.6}$$

which assures that $\Delta_u - 1$ *is uniformly elliptic in each local quasi-coordinate. It follows from Yau's generalized maximum principle and Schauder theory that*

$$\Delta_u - 1 : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$$

is an isomorphism for all $k \geq 0$ *and* $\alpha \in (0,1)$. *Now we want to construct a linear operator*

$$L_{u,r} : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M), \qquad \forall\, k \geq 0, \alpha \in (0,1), \tag{3.7}$$

which is uniformly elliptic in each local quasi-coordinate chart, and such that the following diagram commutes:

$$\begin{tikzcd}
C^{k+2,\alpha}(M) \arrow[r, "\cdot\sigma^{-r}", "\approx"'] \arrow[d, "L_{u,r}"'] & \sigma^{-r} C^{k+2,\alpha}(M) \arrow[r, "i"] & C^{k+2,\alpha}(M) \arrow[d, "\Delta_u - 1", "\approx"'] \\
C^{k,\alpha}(M) \arrow[r, "\cdot\sigma^{-r}", "\approx"'] & \sigma^{-r} C^{k,\alpha}(M) \arrow[r, "i"] & C^{k,\alpha}(M)
\end{tikzcd} \tag{3.8}$$

(In this paper the map i stands for the inclusion unless indicate)

Assume that (3.8) is true and that $L_{u,r}$ *is an isomorphism; then the proof is*

finished since (3.8) will give rise to the following commutative diagram:

$$\begin{array}{ccc} C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\cdot\sigma^{-r}} & \sigma^{-r}C^{k+2,\alpha}(M) \\ {\scriptstyle L_{u,r}}\downarrow\approx & & \downarrow{\scriptstyle \Delta_u-1} \\ C^{k,\alpha}(M) & \xrightarrow[\approx]{\cdot\sigma^{-r}} & \sigma^{-r}C^{k,\alpha}(M) \end{array} \tag{3.9}$$

Therefore, it remains to construct an isomorphism $L_{u,r}$ such that (3.8) holds. Now for each $v \in C^{k,\alpha}(M)$, $k \geq 0$ and $\alpha \in (0,1)$, we define

$$\begin{aligned} L_{u,r}(v) &\equiv \sigma^r(\Delta_u - 1)(\sigma^{-r}v) \\ &= \Delta_u v - 2r\sigma^{-1}H_u(\sigma, v) + c_{u,r} \cdot v, \end{aligned}$$

where

$$H_u(f,g) \equiv \int_0^1 \left(\frac{n\omega_t^{n-1} \wedge df \wedge d^c g}{\omega_t^n}\right) dt, \tag{3.10}$$

and

$$\begin{aligned} c_{u,r} &\equiv \sigma^r \Delta_u(\sigma^{-r}) - 1, \\ &= r\sigma^{-1}\Delta_u(-\sigma) + r(r+1)\sigma^{-2}H_u(\sigma,\sigma) - 1. \end{aligned}$$

It is easy to show $\sigma^{-1}H_u(\sigma,\cdot)$ and $c_{u,r}$ are well defined in each local quasi-coordinate chart $(V_\eta, \{v^1, \ldots, v^n\})$, $\eta \in (0,1)$. Indeed, it suffices to check that $\sigma^{-1}d\sigma$ and $\sigma^{-2}d\sigma \wedge d^c\sigma$ are well defined. For a coordinate neighborhood $(U, \{z^1, \ldots, z^n\})$ of $p \in D$ with $U \supset V_\eta$, assume that $D \cap U = \{z^1 = 0\}$ and that

$$|s|^2 = |z^1|^2 e^u, \qquad u \in C^\infty(U).$$

Then by definition one has

$$\begin{aligned} z^1 &= e^{\frac{1+\eta}{1-\eta}\frac{v^1+1}{v^1-1}} \\ z^j &= v^j, \qquad j = 2, \ldots, n, \end{aligned}$$

for all $0 \leq |v^1| < R < 1$ and $0 \leq |v^j| \leq 1$, $j = 2, \ldots, n$, where $R \in (1/2, 1)$ is a fixed

number independent of η. These imply that

$$\frac{dz^1}{z^1 \log |z^1|^2} = \frac{\bar{v}^1 - 1}{v^1 - 1} \cdot \frac{dv^1}{1 - |v^1|^2},$$

and so

$$\frac{dz^1 \wedge d\bar{z}^1}{|z^1|^2 \log^2 |z^1|^2} = \frac{dv^1 \wedge d\bar{v}^1}{(1 - |v^1|^2)^2},$$

for all $0 \leq |v^1| \leq R < 1$. Therefore, we have showed that $L_{u,r}$ is uniformly elliptic in each local quasi-coordinate chart and satisfies (3.8).

Next to show that $L_{u,r}$ is an isomorphism. First, observe that there exists a constant $K_r > 0$ such that

$$\sup c_{u,r} < K_r.$$

Indeed, one can do the estimate more precisely as follows: On the one hand there exists a constant $\Lambda > 0$ such that

$$-\Lambda \omega_K < \omega_c < \Lambda \omega_K.$$

We can choose the norm $|\cdot|$ of s sufficiently small such that

$$1 + \frac{2\Lambda}{\sigma} > \frac{1}{2}.$$

Then

$$\begin{aligned}
\Delta_u(\sigma) &= \int_0^1 \frac{n\omega_t^{n-1} \wedge (-\omega_c)}{\omega_t^n}\, dt \\
&\leq \Lambda \int_0^1 \frac{n\omega_t^{n-1} \wedge \omega_K}{\omega_t^n}\, dt \\
&\leq \int_0^1 \frac{dt}{t/C + (1-t)} \cdot \frac{n\omega^{n-1} \wedge \omega_K}{\omega^n}\, dt \\
&\leq 2\Lambda n C_1, \qquad C_1 \equiv \frac{C \ln C}{C - 1} > 0.
\end{aligned} \tag{3.11}$$

Hence,

$$r(-\sigma^{-1})\Delta_u(\sigma) \leq r\frac{2\Lambda}{-\sigma} n C_1 \leq nr\frac{C_1}{2}.$$

On the other hand, (3.6) implies that

$$\frac{n\omega_t^{n-1} \wedge 2\sigma^{-2} d\sigma \wedge d^c\sigma}{\omega_t^n} \leq \frac{1}{t/C + (1-t)} \cdot \frac{n\omega^{n-1} \wedge 2\sigma^{-2} d\sigma \wedge d^c\sigma}{\omega^n}.$$

Write

$$\frac{n\omega^{n-1} \wedge 2\sigma^{-2} d\sigma \wedge d^c\sigma}{\omega^n} = \frac{1}{1+f_b},$$

in which

$$f_b \equiv \frac{(\omega_K + 2\sigma^{-1}\omega_c)^n}{n(\omega_K + 2\sigma^{-1}\omega_c)^{n-1} \wedge 2\sigma^{-2} d\sigma \wedge d^c\sigma} > 0. \tag{3.12}$$

Hence,

$$\begin{aligned} 2\sigma^{-2} H_u(\sigma, \sigma) &\leq \frac{1}{1+f_b} \int_0^1 \frac{dt}{t/C + (1-t)} \\ &\leq \frac{C \ln C}{C-1} \equiv C_1. \end{aligned}$$

Let

$$K_r = \frac{r(r+1)}{2} C_1 + \frac{nr}{2} C_1 > 0;$$

therefore, we have

$$\sup_M c_{u,r} \leq K_r - 1. \tag{3.13}$$

Second, we have the following two lemmas:

Lemma 3.1. *$L_K \equiv L_{u,r} - K_r : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ is an isomorphism for all $k \geq 0$ and $\alpha \in (0,1)$.*

Proof of Lemma 3.1. *This proof is similar to Cheng-Yau's in ([7]). The uniqueness of L_K follows immediately from Yau's generalized maximum principle ([7, p.516]). The existence can be done as follows: Since (M, ω) is a complete manifold, we can choose a sequence of relatively compact domains $\{B_j\}_{j=1}^{\infty}$ to exhaust M. The standard Schauder theory (see, for example, Gilbarg-Trudinger[16, p.107].) implies the*

following Direchlet problem:

$$L_K v = f, \qquad \text{on } B_j;$$
$$v = 0, \qquad \text{on } \partial B_j.$$

has a unique solution $v_j \in C^{k+2,\alpha}(\overline{B_j})$. *It follows from (3.13) and the usual maximum principle that*

$$\sup_{\overline{B_j}} |v_j| \leq \sup_M |f|, \qquad \forall j \in \mathbb{N}.$$

Then the standard interior Schauder estimates (see, for example, [16, p.93].) applied to the local quasi-coordinates show that a subsequence of $\{v_j\}$ *converges to* $v \in C^{k+2,\alpha}(M)$, *which satisfies*

$$L_K v = f \qquad \text{on } M,$$
$$\|v\|_{k+2,\alpha} \leq \|f\|_{k,\alpha}.$$

This proves lemma 3.1.

Lemma 3.2. *Let* $L_{u,r}$ *defined as in (3.7), then either*

1. *the homogeneous problem*

$$L_{u,r} v = 0$$

   has nontrivial solutions, which form a finite dimensional subspace of $C^{k+2,\alpha}(M)$; *or,*

2. *the inhomogeneous problem*

$$L_{u,r} v = f$$

   has a unique $C^{k+2,\alpha}(M)$ *solution for all* $f \in C^{k,\alpha}(M)$.

For a proof of Lemma 3.2 we are going to make use of the standard Fredholm alternative for compact linear mapping:

Theorem (Fredholm alternative for compact linear mapping). *Let* V *be a normed linear space and* $T : V \to V$ *be a compact linear mapping. Then either*

1. $\ker(I-T) \neq \{0\}$, *and* $\dim\ker(I-T) < \infty$; *or*

2. $I-T : V \to V$ *is a linear isomorphism.*

See, for example, [16, p.76], for a proof.

Proof of Lemma 3.2. *By definition*

$$L_{u,r} = L_K + K_r.$$

It follows from Lemma 1 that L_K *has a bounded inverse*

$$L_K^{-1} : C^{k,\alpha}(M) \to C^{k+2,\alpha}(M).$$

Now for any $f \in C^{k,\alpha}(M)$, $L_{u,r}v = f$ *is equivalent to*

$$v + K_r L_K^{-1} v = L_K^{-1} f. \tag{3.14}$$

We claim that $L_K^{-1} : C^{k,\alpha}(M) \to C^{k,\alpha}(M)$ *is a bounded compact linear operator. Indeed, this follows from the Ascoli-Arzelà theorem and the following commutative diagram:*

$$\begin{tikzcd}
C^{k,\alpha}(M) \arrow[r, "L_K^{-1}", "\approx"'] \arrow[ddr, "L_K^{-1}"'] & C^{k+2,\alpha}(M) \arrow[d, "i"] \\
& C^{k+2}(M) \arrow[d, "i"] \\
& C^{k,\alpha}(M)
\end{tikzcd}$$

Therefore, by the above theorem of Fredholm alternative, either

$$v + K_r L_K^{-1} v = 0 \tag{3.15}$$

has nontrivial solutions which form a finite dimensional subspace of $C^{k,\alpha}(M)$; *or, for*

each $\tilde{f} \in C^{k,\alpha}(M)$, there exists a unique $v \in C^{k,\alpha}(M)$ such that

$$v + K_r L_K^{-1} v = \tilde{f}. \tag{3.16}$$

Note that (3.15) implies that $v = -K_r L_K^{-1} v \in C^{k+2,\alpha}(M)$, and so

$$\ker L_{u,r} = \ker(I + K_r L_K^{-1}).$$

If $\ker L_{u,r} = \{0\}$, then for each $f \in C^{k,\alpha}(M)$, $L_K^{-1} f \in C^{k+2,\alpha}(M) \subset C^{k,\alpha}(M)$, there is a unique $v \in C^{k,\alpha}(M)$ satisfies (3.14), which in turn implies that $v \in C^{k+2,\alpha}(M)$ and that $L_{u,r} v = f$. This finishes the proof of Lemma 2.

Finally, note that (3.8) implies $\ker L_{u,r} = \{0\}$. This together with Lemma 2 show that $L_{u,r}$ is an isomorphism. Therefore, the proof of Theorem 1 is completed.

□

Similarly we also have the following isomorphism theorem of linear version.

Theorem 2. *Fix an arbitrary $r \in \mathbb{R}_+$. Let $\omega_u \equiv \omega + dd^c u$ satisfy $\frac{1}{C}\omega \leq \omega_u \leq C\omega$ for some $C > 0$, where $u \in \mathcal{R}(M)$. We have the following commutative diagram:*

$$\begin{array}{ccc} \sigma^{-r} C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\Delta_{\omega_u} - 1} & \sigma^{-r} C^{k,\alpha}(M) \\ \downarrow i & & \downarrow i \\ C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\Delta_{\omega_u} - 1} & C^{k,\alpha}(M), \end{array}$$

where Δ_{ω_u} is the negative Laplacian with respect to the metric ω_u, and $\approx$ stands for the homeomorphism of the Banach spaces.

The proof is similar but easier. So we omit it here.

3.2 Characterization of Cheng-Yau solution

Recall the weighted Cheng-Yau Hölder rings $\mathcal{R}_r(M)$, $r \geq 0$, introduced in Chapter 2:

$$\mathcal{R}(M) = \bigcap_{k\geq 0, 0<\alpha<1} C^{k,\alpha}(M)$$

is the Cheng-Yau Hölder ring. For each $r \in \mathbb{R}_+$,

$$\mathcal{R}_r(M) = \sigma^{-r}\mathcal{R}(M)$$

is defined to be the weighted $-r$ Hölder ring, which is an ideal in $\mathcal{R}(M)$. Then any sequence of non-decreasing numbers $\{r_j\}_{j=1}^{\infty} \subset \mathbb{R}_+$ give rise to a filtration of $\mathcal{R}(M)$:

$$\mathcal{R}_{r_1}(M) \supset \mathcal{R}_{r_2}(M) \supset \mathcal{R}_{r_3}(M) \supset \cdots .$$

Now before carrying out the asymptotic expansion, we give a precise characterization of the Cheng-Yau solution by aid of theorem 1.

Theorem 3. *If $D \subset \overline{M}$ is a smooth divisor, then one can choose a canonical metric on D such that the Cheng-Yau solution of*

$$M_\omega(u) = J(\Omega)^{-1}, \tag{3.17}$$

$$\frac{1}{C}\omega \leq \omega_u \leq C\omega, \qquad C > 0. \tag{3.18}$$

is in $\mathcal{R}_1(M)$.

Proof of Theorem 3. *It suffices to construct an initial volume form Ω on M such that*

$$\log J(\Omega) \in \mathcal{R}_1(M).$$

Given any volume $\widetilde{V}$ on $\overline{M}$ and a metric h on the line bundle $[D]$, $\frac{\widetilde{V}}{h}$ give rise to a

metric on $K_{\overline{M}} + [D]$. Without loss of generality, suppose the curvature form

$$\widetilde{\omega} \equiv \mathrm{Ric}\,(\frac{\widetilde{V}}{h}) > 0.$$

Then by the adjunction formula the pull back

$$c_1(K_D) \ni i^*(\widetilde{\omega}) > 0.$$

It follows from Yau's solution of Calabi conjecture ([38]) that there exists a function φ_D on D unique up to a constant such that

$$\omega_{K-E,D} \equiv i^*(\widetilde{\omega}) + dd^c\varphi_D$$

is the Kähler-Einstein metric on D with the Einstein constant -1. Now extend φ_D to $\overline{M}$ by setting

$$\varphi \equiv \chi\left(\frac{|s|^2}{\delta}\right)\varphi_D, \qquad \delta > 0,$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a cut-off function with $\chi \equiv 1$ on $[-1,1]$ and vanishes outside $[-2,2]$, such that

$$\widetilde{\omega} + dd^c\varphi > 0$$

on $\overline{M}$. Now by adding φ a constant we get that

$$\frac{(e^{\varphi}\widetilde{\gamma}/h)|_{\{z^1=0\}}}{\det(g_{D,i\bar{j}})} = \frac{1}{2n!}, \tag{3.19}$$

where we write

$$\widetilde{V} \equiv \widetilde{\gamma}\prod_{j=1}^{n}\left(\frac{\sqrt{-1}}{2\pi}dz^j \wedge d\bar{z}^j\right).$$

and

$$\omega_{K-E,D} \equiv \sum_{2\le i,j\le n} g_{D,i\bar{j}}\left(\frac{\sqrt{-1}}{2\pi}dz^i \wedge d\bar{z}^j\right).$$

Let $V = e^{\varphi}\widetilde{V}$. Then we define the initial volume form Ω as in (3.1).

Recall that

$$\omega = \omega_K + 2\sigma^{-1}\omega_c + 2\sigma^{-2}d\sigma \wedge d^c\sigma$$

Then

$$\omega^n = (1 + \sum_{k=1}^{n-1} \sigma^{-k} f_k)(1 + f_b) \cdot n\omega_K^{n-1} \wedge \sigma^{-2} 2d\sigma \wedge d^c\sigma,$$

where

$$f_k = \frac{2^k \binom{n-1}{k} \omega_K^{n-k-1} \wedge \omega_c^k \wedge 2\sigma^{-2} d\sigma \wedge d^c\sigma}{\omega_K^{n-1} \wedge 2\sigma^{-2} d\sigma \wedge d^c\sigma}, \quad k = 1, \ldots, n-1.$$

and f_b is defined in (3.12). Then

$$\begin{aligned} J(\Omega) &\equiv \frac{(\mathrm{Ric}\,\Omega)^n}{\Omega} \\ &= \frac{\omega^n |s|^2 \log^2 |s|^2}{V} \\ &= f_0(1 + \sum_{k=1}^{n-1} \sigma^{-k} f_k)(1 + f_b), \end{aligned}$$

where

$$f_0 = \frac{2n\omega_K^{n-1} \wedge |s|^2 d\log|s|^2 \wedge d^c \log|s|^2}{V} > 0.$$

So it suffices to show that

$$f_0 - 1 \in \mathcal{R}_r(M), \qquad \forall r \geq 1. \tag{3.20}$$

In local coordinate $\{U, (z^1, \ldots, z^n)\}$, set $D \cap U = \{z^1 = 0\}$. Write

$$\begin{aligned} |s|^2 &= |z^1|^2 e^u, \qquad u \in C^\infty(U); \\ V &= \gamma \prod_{j=1}^n \frac{\sqrt{-1}}{2\pi} dz^j \wedge d\bar{z}^j, \quad \gamma = \tilde{\gamma} e^\varphi > 0. \\ \omega_K &= \sum R_{i\bar{j}} \frac{\sqrt{-1}}{2\pi} dz^i \wedge d\bar{z}^j. \end{aligned}$$

Then straightforward computations show that

$$f_0 = \frac{2n!}{\gamma e^{-u}} (R^{1\bar{1}} + z^1 R^{\bar{1}i} u_i + \bar{z}^1 R^{1\bar{j}} u_{\bar{j}} + |z^1|^2 R^{i\bar{j}} u_i u_{\bar{j}}),$$

where $R^{i\bar{j}}R_{k\bar{j}} = \delta_{ij}\det(R_{pq})$ *and* $u_k \equiv \frac{\partial u}{\partial z^k}$. *Denote*

$$H = \gamma e^{-u}.$$

It follows that

$$\begin{aligned} R^{1\bar{1}} &= \det\left(\left(\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log H\right)_{i,j\geq 2}\right). \\ &= \det\left(\left(\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log H|_{\{z^1=0\}}\right)_{i,j\geq 2}\right) + z^1 B + \bar{z}^1\overline{B} + \mathcal{O}(|z^1|^2). \end{aligned}$$

where $B \in C^\infty(U)$. *Therefore, to prove (3.20) it suffices to prove that for each* $r \geq 1$, *the* $C^{k,\alpha}(U)$ *norm of*

$$(\log|z^1|^2)^r \cdot \left(\frac{2n!\det\left(\left(\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log H|_{\{z^1=0\}}\right)_{i,j\geq 2}\right)}{H|_{\{z^1=0\}}} - 1\right)$$

is uniformly bounded. This, however, follows immediately from the above construction (3.19). We have also shown that the conclusion $u \in \mathcal{R}_1(M)$ *is independent of the extension of* $\omega^{n-1}_{K-E,D}$ *to* $\overline{M}$. □

Remark 1. *The reader is refer to Schumacher [31] for another way to construct the initial volume form.*

3.3 Asymptotic expansions

Recall that the *graded Cheng-Yau Hölder ring* $\mathcal{R}^G(M)$ associated to the log-filtration $\{\mathcal{R}_j(M)\}_{j=0}^\infty$ is given by

$$\begin{aligned} \mathcal{R}^G(M) = \{u \in \mathcal{R}(M) \mid &\text{ there exist a sequence } \{\psi_j\}_{j=0}^\infty, \text{ where} \\ &\text{not all } \psi_j \text{ are zero, such that for each } N \in \mathbb{N}, \\ &u - \sum_{j=0}^N \psi_j \sigma^{-j} \in \mathcal{R}_{N+1}(M). \} \end{aligned} \tag{3.21}$$

Note that $\sigma^{-\alpha} \in \mathcal{R}^G(M)$, but $|s|^\beta$, $|s|^\lambda(\log|s|^2)^\mu \notin \mathcal{R}^G(M)$, where α, β, λ and $\mu \in \mathbb{R}_+$. Let

$$\mathcal{R}_j^G(M) = \mathcal{R}^G(M) \cap \mathcal{R}_j(M), \qquad j \in \mathbb{N}.$$

Denote $L_j = L_{0,j}$ in (3.7), i.e.,

$$L_j(v) = \Delta_\omega(v) - 2j\sigma^{-1}H_\omega(\sigma, v) + c_j v, \quad j \in \mathbb{N}, \tag{3.22}$$

where Δ_ω is the negative Laplacian with respect to the metric ω,

$$H_\omega(f,g) = \frac{n\omega^{n-1} \wedge df \wedge d^c g}{\omega^n}, \quad f, g \in C^1(M);$$
$$c_j = j\sigma^{-1}\Delta_\omega(-\sigma) + j(j+1)\sigma^{-2}H_\omega(\sigma,\sigma) - 1.$$

Now we derive the desired asymptotic expansion in the following theorem.

Theorem 4. *With the assumptions in theorem 3, the Cheng-Yau solution u of* (3.17) *is in $\mathcal{R}_1^G(M)$. More precisely, there exists a sequence $\{\psi_j\}_{j=1}^\infty \subset \mathcal{R}(M)$ such that for any $N \in \mathbb{N}$,*

$$u - \sum_{j=1}^{N} \psi_j \sigma^{-j} \in \mathcal{R}_{N+1}(M),$$

where each ψ_j satisfies

$$L_j(\psi_j) = F_j, \qquad j \in \mathbb{N}, \tag{3.23}$$

in which

$$F_1 = -\frac{2n(n-1)\omega_k^{n-2} \wedge \omega_c \wedge \frac{2d\sigma \wedge d^c\sigma}{\sigma^2}}{V}$$

and $F_j \in \mathcal{R}(M)$, $j \geq 2$, are given by induction.

Remark 2. *The coefficients ψ_j of the asymptotics are required to satisfy the Elliptic linear second order PDE (3.23), in contrast to the ODE in the case of pseudoconvex domain (see [23]).*

Proof of Theorem 4. *As in the proof of theorem 3,*

$$\begin{aligned} J(\Omega) &= f_0(1+\sum_{k=1}^{n-1}\sigma^{-k}f_k)(1+f_b) \\ &= 1-\sum_{k=1}^{n-1}F_k\sigma^{-k}+F_b, \end{aligned}$$

in which $F_k = -f_0 f_k$, $1 \le k \le n-1$, *and*

$$F_b = f_0 - 1 + f_b f_0(1+\sum_{k=1}^{n-1}\sigma^{-k}f_k) \in \bigcap_{r\ge 0}\mathcal{R}_r(M).$$

Let

$$u_1 \equiv \psi_1\sigma^{-1},$$

where $\psi_1 \in \mathcal{R}(M)$ *solves* $L_1(\psi_1) = F_1$. *Thus*

$$(\Delta_\omega - 1)(u_1) = F_1\sigma^{-1}.$$

Want to show

$$u - u_1 \in \mathcal{R}_2(M). \tag{3.24}$$

Let $h_1 = u_1 - u$. *By theorem 3* $u \in \mathcal{R}_1(M)$, *so* $h_1 \in \mathcal{R}_1(M)$. *Observe that by construction*

$$\begin{aligned} \mathcal{R}_2(M) &\ni J(e^{u_1}\Omega) - J(e^u\Omega) \\ &= M_{\omega_u}(h_1) - 1 \\ &= (1+\Delta_{\omega_u}h_1 + G_2^u(h_2) + \cdots + G_n^u(h_1))e^{-h_1} - 1, \end{aligned}$$

where

$$G_i^u(h_1) = \frac{\binom{n}{i}\omega_u^{n-i}\wedge(dd^c h_1)^i}{\omega_u^n} \in \mathcal{R}_i(M), \quad i = 2,\ldots,n.$$

Hence,

$$(\Delta_{\omega_u} - 1)h_1 \in \mathcal{R}_2(M).$$

Therefore, it follows from Theorem 2 that $h_1 \in \mathcal{R}_2(M)$. This proves (3.24).

Now assume that by induction we have

$$u_{N-1} = \sum_{i=1}^{N-1} \psi_i \sigma^{-i}$$

such that

$$u - u_{N-1} \in \mathcal{R}_N(M). \tag{3.25}$$

$$J(e^{u_{N-1}}\Omega) = 1 - F_N \sigma^{-N}, \quad F_N \in \mathcal{R}(M) \tag{3.26}$$

Then there exist a $\psi_N \in \mathcal{R}(M)$ such that $L_N \psi_N = F_N$. Let

$$u_N = u_{N-1} + \psi_N \sigma^{-N}.$$

Thus we have

$$J(e^{u_N}\Omega) = 1 + F_{N+1}\sigma^{-(N+1)}$$

for some $F_{N+1} \in \mathcal{R}(M)$. Let $h_N = u_N - u$. Then $h_N \in \mathcal{R}_N(M)$. Furthermore,

$$\begin{aligned}
\mathcal{R}_{N+1}(M) &\ni J(e^{u_N}\Omega) - J(e^u\Omega) \\
&= M_{\omega_u}(h_N) - 1 \\
&= (1 + \Delta_{\omega_u} h_N + G_2^u(h_N) + \cdots + G_n^u(h_N))e^{-h_N} - 1,
\end{aligned}$$

which implies that

$$(\Delta_{\omega_u} - 1)h_N \in \mathcal{R}_{N+1}(M),$$

Hence, $h_N \in \mathcal{R}_{N+1}(M)$ by Theorem 2. This completes the induction.

□

Chapter 4

Simple Normal Crossing Case

The whole machinery, especially the weighted Ching-Yau Hölder rings and the associated filtrations, the isomorphism theorems, and the extension of hermitian metric with positive curvature form, are generalized in this chapter to derive the asymptotics near a divisor with simple normal crossings. Together these results with the iteration formula, we obtain the asymptotics of Kähler-Einstein volume form near the complete intersections of the irreducible components of the divisor. Furthermore, the obtained asymptotics is canonical in the sense that it is unique up to (negative) infinite weight once the metrics of the restricted normal bundles are fixed.

4.1 Isomorphism theorems

We first generalize Theorem 1, Theorem 2, and Theorem 4 to the case that D has simple normal crossing. In this case, let $D = \sum_{i=1}^{p} D_i$, where the irreducible components D_i are smooth and intersect transversely. Let $s_i \in H^0(\overline{M}, \mathcal{O}[D_i])$ define D_i, and denote by $\sigma_i = \log |s_i|^2$. Choose a volume form V on $\overline{M}$ such that

$$\Omega \equiv \frac{V}{\prod_{i=1}^{p} |s_i|^2 (\log |s_i|^2)^2} \tag{4.1}$$

is the initial volume form on M satisfying the properties (i) and (ii) in Chapter 2. Recall that

$$\begin{aligned}\omega &= \operatorname{Ric}\Omega \\ &= \omega_K + 2\sum_{i=1}^{p}\sigma_i^{-1}\omega_{c_i} + 2\sum_{i=1}^{p}\sigma_i^{-2}d\sigma_i\wedge d^c\sigma_i,\end{aligned}$$

where

$$\begin{aligned}\omega_K &= \operatorname{Ric}\left(\frac{V}{\prod_{i=1}^{p}|s_i|^2}\right), \\ \omega_{c_i} &= -dd^c\sigma_i.\end{aligned}$$

For any subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, p\}$, denote by

$$\sigma^{-r_I} = \sigma_{i_1}^{-r_{i_1}} \cdots \sigma_{i_k}^{-r_{i_k}}, \qquad \forall r_I = (r_{i_1}, \ldots, r_{i_k}) \in \mathbb{R}_+^{|I|}.$$

Denote $\sigma^{-r} = \sigma^{-r_I}$ if $I = \{1, \ldots, p\}$. Note that we implicitly assume $r_I \in \mathbb{Z}_+^{|I|}$ when we write $|r_I| = l \in \mathbb{Z}_+$ in the follows.

The weighted Cheng-Yau Hölder spaces $\sigma^{-r}C^{k,\alpha}(M)$ are the Banach spaces defined as usual. For each $l \in \mathbb{N}$, $\sum_{|r_I|=l}\sigma^{-r_I}C^{k,\alpha}(M)$ is defined to be the Cheng-Yau Hölder space with *homogeneous* weight $-l$ associated to the index set I. Recall that

$$\begin{aligned}\mathcal{R}_{I,l}(M) &= \tau_I^l\mathcal{R}(M) \\ &= \sum_{|r_I|=l}\sigma^{-r_I}\mathcal{R}(M), \quad l \in \mathbb{N}.\end{aligned}$$

in which

$$\tau_I \equiv \Big(\sum_{i\in I}\sigma_i^{-2}\Big)^{1/2}.$$

Now we first state and prove the following isomorphism theorem of the simple normal crossing version.

Theorem 5. *Fix an arbitrary $r \in \mathbb{R}_+^p$, $k \geq 0$, and $\alpha \in (0,1)$. Let*

$$\omega_\varphi \equiv \omega + dd^c\varphi, \qquad \varphi \in \mathcal{R}(M),$$

satisfies $\omega/C_1 < \omega_\varphi < C_1\omega$, $C_1 > 0$. *For each* $F \in \sigma^{-r}C^{k,\alpha}(M)$, *let* $u \in C^{k+2,\alpha}(M)$ *be the unique Cheng-Yau solution of*

$$M_{\omega_\varphi}(u) \equiv \frac{(\omega_\varphi + dd^c u)^n}{\omega_\varphi} e^{-u} = e^F,$$
$$\frac{1}{C}\omega_\varphi \le \omega_\varphi + dd^c u \le C\omega_\varphi, \qquad C > 0.$$

Then $u \in \sigma^{-r}C^{k+2,\alpha}(M)$. *Furthermore, for any subset* $I \subset \{1, \ldots, p\}$, *if*

$$F \in \sum_{|r_I|=l,\, r_I \in \mathbb{Z}_+^{|I|}} \sigma^{-r_I} C^{k,\alpha}(M), \qquad l \in \mathbb{N},$$

then

$$u \in \sum_{|r_I|=l,\, r_I \in \mathbb{Z}_+^{|I|}} \sigma^{-r_I} C^{k+2,\alpha}(M).$$

Proof of Theorem 5. *First, observe that*

$$\sigma^r d\sigma^{-r} = -\sum_{i=1}^{p} \frac{r_i}{\sigma_i} d\sigma_i,$$

and

$$\sigma^r dd^c(\sigma^{-r}) = \sum_{i=1}^{p} \left[-\frac{r_i}{\sigma_i} dd^c\sigma_i + \frac{r_i(r_i+1)}{\sigma_i^2} d\sigma_i \wedge d^c\sigma_i \right.$$
$$\left. + \sum_{j \ne i} \frac{r_i r_j}{\sigma_i \sigma_j} d\sigma_j \wedge d^c\sigma_i \right]$$

Similar to the proof of Theorem 1, we define

$$L_{u,r}(v) \equiv \sigma^r(\Delta_u - 1)(\sigma^{-r}v)$$
$$= \Delta_u v - 2\sum_{i=1}^{p} r_i \sigma_i^{-1} H_u(\sigma_i, v) + c_{u,r}\, v,$$

where Δ_u *and* H_u *are defined by the same form of (3.4) and (3.10) with* ω_t *replaced*

by $\omega_{\varphi,t} \equiv \omega_\varphi + tdd^c u$, *respectively; and*

$$\begin{aligned} c_{u,r} &\equiv \sigma^r \Delta_u(\sigma^{-r}) - 1, \\ &= \sum_{i=1}^{p} \left[\frac{r_i}{\sigma_i} \Delta_u(-\sigma_i) + \frac{r_i(r_i+1)}{\sigma_i^2} H_u(\sigma_i, \sigma_i) \right. \\ &\qquad \left. + \sum_{j \neq i} \frac{r_i r_j}{\sigma_i \sigma_j} H_u(\sigma_j, \sigma_i) \right] - 1. \end{aligned}$$

Similarly, Δ_u, $\sigma_i^{-1} H_u(\sigma_i, \cdot)$, *and* $c_{u,r}$ *are well defined in each local quasi-coordinate chart* $(V_\eta, \{v^1, \ldots, v^n\})$, $\eta = (\eta^1, \ldots, \eta^p) \in (0,1)^p$. *These follows from the well-definedness of* $\sigma_i^{-1} d\sigma_i$ *for* $i = 1, \ldots, p$: *Without loss of generality, assume that locally* $D \cap U = \{z^1 \cdots z^k = 0\}$ *and that*

$$|s_j|^2 = |z^j|^2 e^{u_j}, \qquad u_j \in C^\infty(U),\ j = 1, \ldots, k.$$

Then for a $V_\eta \subset U$,

$$\begin{aligned} z^i &= e^{\frac{1+\eta^i}{1-\eta^i} \frac{v^i+1}{v^i-1}}, \quad & i = 1, \ldots, k, \\ z^j &= v^j, & j = k+1, \ldots, n, \end{aligned}$$

for all $0 \leq |v^i| < R < 1$, $i = 1, \ldots, k$, *and* $0 \leq |v^j| \leq 1$, $j = k+1, \ldots, n$, *where* $R \in (1/2, 1)$ *is a fixed number independent of* η. *These imply that*

$$\frac{dz^i}{z^i \log |z^i|^2} = \frac{\bar{v}^i - 1}{v^i - 1} \cdot \frac{dv^i}{1 - |v^i|^2},$$

for all $0 \leq |v^i| \leq R < 1$, $i = 1, \ldots, k$. *Therefore, we have showed that* $L_{u,r}$ *is uniformly elliptic in each local quasi-coordinate chart and satisfies (3.8).*

Next to derive the following estimate

$$\sup_M c_{u,r} \leq K_r - 1, \tag{4.2}$$

where

$$K_r = \frac{1}{2} n C_2 \sum_{i=1}^{p} r_i + 2nC_2 \sum_{i=1}^{p} r_i(2r_i + 1) > 0$$

in which

$$C_2 = \frac{C_1 C \ln C}{(C-1)}. \tag{4.3}$$

There exists a constant $\Lambda > 0$ *such that*

$$-\Lambda\omega_K < \omega_{c_i} < \Lambda\omega_K.$$

Also, we can choose the norm $|\cdot|$ *of* s *sufficiently small such that*

$$1 + \Lambda \sum_{i=1}^{p} \frac{2}{\sigma_i} > \frac{1}{2}.$$

Note that for all $t \in [0,1]$,

$$[t/C + (1-t)]\omega/C_1 \leq \omega_t = \omega_\varphi + tdd^c u \leq C_1[Ct + (1-t)]\omega,$$

Then it follows from the same estimate in (3.11) that

$$\sum_{i=1}^{p} r_i \frac{\Delta_u(\sigma_i)}{-\sigma_i} \leq \frac{n}{2}|r|C_2, \tag{4.4}$$

where $C_2 > 0$ *is given by (4.3). Moreover, we have*

$$\sum_{i\neq j}\left(\frac{r_i d\sigma_i}{\sigma_i} \wedge \frac{r_j d^c\sigma_j}{\sigma_j}\right) \leq \sum_{k=1}^{p} \frac{r_k^2 d\sigma_k \wedge d^c\sigma_k}{\sigma_k^2},$$

and hence,

$$\sum_{j\neq i} \frac{r_i r_j}{\sigma_i\sigma_j} H_u(\sigma_j, \sigma_i) \leq \frac{r_i^2}{\sigma_i^2} H_u(\sigma_i, \sigma_i) \tag{4.5}$$

It remains to control $\sum r_i(r_i+1)\sigma_i^{-2} H_u(\sigma_i, \sigma_i)$*: Fix an arbitrary* $i = 1, \ldots, p$,

$$\frac{n\omega_t^{n-1} \wedge 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i}{\omega_t^n} \leq \frac{C_1}{t/C + (1-t)} \cdot \frac{n\omega^{n-1} \wedge 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i}{\omega^n}.$$

Write

$$\omega^n = (2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i) \wedge \omega^{n-1}$$
$$+ (\omega_K + 2\sum_{i=1}^{p} \sigma_i^{-1}\omega_{c_i} + 2\sum_{j\neq i} \sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i) \wedge \omega^{n-1}.$$

Then

$$\frac{\omega^{n-1} \wedge 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i}{\omega^n} = \frac{1}{1+f_{bi}},$$

in which

$$f_{bi} \equiv \frac{(\omega_K + 2\sum_{i=1}^{p} \sigma_i^{-1}\omega_{c_i} + 2\sum_{j\neq i} \sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i) \wedge \omega^{n-1}}{(2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i) \wedge \omega^{n-1}} > 0.$$

Hence,

$$\sum_{i=1}^{p} r_i(r_i+1)\sigma_i^{-2} H_u(\sigma_i, \sigma_i) \leq nC_2 \sum_{i=1}^{p} r_i(r_i+1). \tag{4.6}$$

Therefore, (4.2) follows from (4.4), (4.5) and (4.6).

Then, by going through the same process in the proof of Theorem 1 we show that $L_{u,r} : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ *is a linear isomorphism. This proves the first part of the theorem.*

For the second part, we write

$$F = \sum_{|r_I|=l} F_{r_I}, \qquad F_{r_I} \in \sigma^{-r_I} C^{k,\alpha}(M).$$

Denote by $u \in C^{k+2,\alpha}(M)$ *the corresponding Cheng-Yau solution for* F*. Each* $r_I \in \mathbb{Z}_+^{|I|}$ *can be viewed as an element in* $\mathbb{R}_+^p$ *via the natural embedding* $I \hookrightarrow \{1, \ldots, p\}$*; hence, by the argument above we know that*

$$\Delta_u - 1 : \sigma^{-r_I} C^{k+2,\alpha}(M) \to \sigma^{-r_I} C^{k,\alpha}(M)$$

is a linear isomorphism for each $r_I \in \mathbb{Z}_+^{|I|}$*. Then there exists a* $u_{r_I} \in \sigma^{-r_I} C^{k+2,\alpha}(M)$

such that

$$(\Delta_u - 1)(u_{r_I}) = F_{r_I}, \qquad \forall\, |r_I| = l.$$

Hence, by linearity,

$$(\Delta_u - 1)(\sum_{|r_I|=l} u_{r_I}) = F.$$

Therefore, it follows from the uniqueness of $(\Delta_u - 1)$ *in* $C^{k+2,\alpha}(M)$ *that*

$$u = \sum_{|r_I|=l} u_{r_I}.$$

This proves the second part. □

Theorem 5 also has a similar linear version, which can be stated in the following:

Theorem 6. *Let* $\omega_u \equiv \omega + dd^c u$ *satisfy* $\frac{1}{C}\omega \leq \omega_u \leq C\omega$ *for some* $C > 0$, *where* $u \in \mathcal{R}(M)$. *For each* $r \in \mathbb{R}^p_+$, *we have the following commutative diagram:*

$$\begin{array}{ccc} \sigma^{-r}C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\Delta_{\varphi,u}-1} & \sigma^{-r}C^{k,\alpha}(M) \\ \downarrow i & & \downarrow i \\ C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\Delta_{\varphi,u}-1} & C^{k,\alpha}(M), \end{array}$$

where Δ_{ω_u} *is the negative Laplacian with respect to the metric* ω_u, *and* $\approx$ *stands for the linear isomorphism of the Banach spaces. Consequently, for any* $I \subset \{1, \ldots, p\}$, *let* $r_I \in \mathbb{Z}^{|I|}_+$, $l \in \mathbb{N}$, *we have*

$$\begin{array}{ccc} \sum_{|r_I|=l} \sigma^{-r_I}C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\Delta_{\omega_u}-1} & \sum_{|r_I|=l} \sigma^{-r_I}C^{k,\alpha}(M) \\ \downarrow i & & \downarrow i \\ C^{k+2,\alpha}(M) & \xrightarrow[\approx]{\Delta_{\omega_u}-1} & C^{k,\alpha}(M), \end{array}$$

4.2 Canonical initial volume forms and the metric extension theorem

Based on these two isomorphism theorems, we can derive the asymptotics of the Kähler-Einstein metric near each connected component of the nonempty complete intersection D_I, where $I \in \mathcal{I}$ ($\mathcal{I}$ is defined in Section 2.2.) Without loss of generality, assume that the index set $I = \{1, \ldots, q\}$. Let

$$D_I = D_{\{1,\ldots,q\}} = \sum_{\nu} D_I^{\nu}$$

where each D_I^{ν} is a connected component.

Observe that the following adjunction formula

$$(K_{\overline{M}} + [D])|_{D_I^{\nu}} = K_{D_I^{\nu}} \tag{4.7}$$

holds for each component D_I^{ν}. Then, it follows from the positivity condition (1.2) and Yau's solution of Calabi conjecture that there exists a unique Kähler-Einstein metric $\omega_{K-E,\nu}$ of Ricci curvature -1 on D_I^{ν}, when $|I| = q < n$. In the case $q = n$, each D_I^{ν} is just a point in $\overline{M}$; we set $\omega_{K-E,\nu} = 1$.

Let

$$\mathcal{H}_I = \{h_{I,i} \mid i \in I = \{1, \ldots, q\}\},$$

where each $h_{I,i}$ is an arbitrary given metric on the restriction of the normal bundle $N_{D_i}|_{D_I} = [D_i]|_{D_I}$. Since $D_j \cap D_I = \emptyset$ for $j \notin I$, we have

$$\left(\prod_{j=q+1}^{p} \sigma_i^2\right)^{-1} \in C^{\infty}(D_I).$$

In order to derive an asymptotics near D_I with certain canonicity, we introduce the family, $\mathcal{M}_{I,\mathcal{H}_I}$, of initial volume forms

$$\Omega = \frac{V}{\prod_{i=1}^{p} |s_i|^2 (\log |s_i|^2)^2}$$

satisfy the following constraints:

1. For each connected component D_I^ν,

$$\left.\frac{\gamma}{\prod_{i\in I} h_i \prod_{j\notin I} |s_j|^2\sigma_j^2}\right|_{D_I^\nu} = \begin{cases} 2^q n! \det(g_{i\bar{j},\nu}), & \text{if } 1 \le |I| < n; \\ 2^n n!, & \text{if } |I| = n. \end{cases}$$

2. Each $h_i|_{D_I} = h_{I,i}$, for any $i \in I = \{1, \ldots, q\}$,

where locally D_I^ν is given by $\{z^1 \cdots z^q = 0\}$, and denote

$$V = \gamma \prod_{j=1}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz^j \wedge d\bar{z}^j \right),$$
$$\omega_{K-E,\nu} = \sum_{q+1\le i,j\le n} g_{i\bar{j},\nu} \left(\frac{\sqrt{-1}}{2\pi} dz^i \wedge d\bar{z}^j \right),$$

and h_i is the metric on $[D_i]$, i.e., $|s_i|^2 = h_i|z^i|^2$, $i = 1, \ldots, q$. Also, we denote by $\mathcal{M}_I$ the family of initial volume forms satisfies the condition (1) only. Before actually constructing such a family, we make the following remark:

Remark 3. *The initial volume form together with the condition (1) only enable us to obtain an asymptotics whose coefficients are all given by the solutions of certain second order linear elliptic PDE (see the proof of Theorem 8), similar to Theorem 4. The additional condition (2), however, will assure that the obtained asymptotics, with respect to $\mathcal{R}^G(M)$, is canonical in the sense of the following two propositions:*

Proposition 4.1. *Let Ω, $\Omega' \in \mathcal{M}_{I,\mathcal{H}_I}$, and $\omega = \mathrm{Ric}\,(\Omega)$, $\omega' = \mathrm{Ric}\,(\Omega')$. Let u and u' be the Cheng-Yau solution of*

$$M_\omega(u) = J(\Omega)^{-1}, \quad \frac{1}{C}\omega < \omega + dd^c u < C\omega, \qquad C > 0;$$
$$M_{\omega'}(u') = J(\Omega')^{-1}, \quad \frac{1}{C'}\omega' < \omega' + dd^c u < C'\omega', \quad C' > 0,$$

respectively. Then $u - u' \in \mathcal{R}_{I,\infty}(M)$.

Proof of Proposition 4.1. *By definition we have*

$$J(e^u\Omega) = 1 = J(e^{u'}\Omega').$$

By the uniqueness of Kähler-Einstein volume form on M,

$$e^u\Omega = e^{u'}\Omega' = \Omega_{K-E,M}.$$

Observe that

$$\frac{\Omega'}{\Omega} = G\prod_{i=1}^{q}\left(\frac{\log|s_i|^2}{\log|s_i|'^2}\right)^2$$

where

$$G = \left(\frac{V'}{V}\prod_{i=1}^{p}\frac{|s_i|^2}{|s_i|'^2}\prod_{j=q+1}^{p}\frac{\log|s_i|^2}{\log|s_i|'^2}\right) \in \mathcal{R}(M).$$

Also, G is smooth near each D_I^ν; furthermore, by condition (1) *we have*

$$G|_{D_I^\nu} = 1$$

for each ν. Hence, $G \in 1 + \mathcal{R}_{I,\infty}(M)$. On the other hand,

$$\log|s_i|^2 = \sigma_i = \sigma_i' + \log\beta_i = \sigma_i'(1 + \sigma_i'^{-1}\log\beta_i),$$

where $\beta_i = |s_i|^2/|s_i|'^2 = h_i/h_i' \in C^\infty(\overline{M})$. Since $\beta_i|_{D_I} = 1$, we have $\sigma_i'^{-1}\log\beta_i \in \mathcal{R}_{I,\infty}(M)$. Therefore,

$$u - u' = \log\frac{\Omega'}{\Omega} \in \mathcal{R}_{I,\infty}(M).$$

□

Our asymptotics will be derived by the following linear operator L_r associated with ω:

$$L_r(v) \equiv \Delta_\omega v - 2\sum_{i=1}^{q} r_i\sigma_i^{-1}H(\sigma_i, v) + c_r v = -f_r, \tag{4.8}$$

where Δ_ω is the negative Laplacian with respect to ω, and

$$
\begin{aligned}
H(v,w) &= \frac{n\omega^{n-1}\wedge dw\wedge d^c v}{\omega^n},\\
c_r &= \sum_{i=1}^{q}\Bigg[\frac{r_i}{\sigma_i}\Delta_\omega(-\sigma_i)+\frac{r_i(r_i+1)}{\sigma_i^2}H(\sigma_i,\sigma_i)\\
&\qquad +\sum_{j\neq i}\frac{r_ir_j}{\sigma_i\sigma_j}H(\sigma_j,\sigma_i)\Bigg]-1.
\end{aligned}
$$

Proposition 4.2. *Let Ω, $\Omega'\in\mathcal{M}_{I,\mathcal{H}_I}$, and $\omega=\mathrm{Ric}\,(\Omega)$, $\omega'=\mathrm{Ric}\,(\Omega')$. Consider the the following two equations:*

$$
\begin{aligned}
L_r v &= f,\\
L_r' v' &= f',
\end{aligned}
$$

where $v,v'\in\mathcal{R}(M)$. If $f-f'\in\mathcal{R}_{I,\infty}(M)$, then $v-v'\in\mathcal{R}_{I,\infty}(M)$.

To prove this proposition, we first observe the following isomorphic property of the operator L_r:

Lemma 4.1. *Let $v\in C^{k+2,\alpha}(M)$ satisfy $L_r(v)=f$. If*

$$f\in\tau_I^m C^{k,\alpha}(M),\qquad m\in\mathbb{N},$$

then

$$v\in\tau_I^m C^{k+2,\alpha}(M).$$

Consequently, if $f\in\mathcal{R}_{I,\infty}(M)$, then $v\in\mathcal{R}_{I,\infty}(M)$.

Proof of Lemma 4.1. *Recall that*

$$
\begin{aligned}
\tau_I &= \Big(\sum_{i\in I}\sigma_i^{-2}\Big)^{1/2}\\
&= \sum_{i\in I}\Big(\frac{\sigma_i}{\tau_I}\Big)\sigma_i\in\sum_{i\in I}\sigma_i\mathcal{R}(M).
\end{aligned}
$$

Then

$$\tau_I^m C^{l,\beta}(M) = \sum_{|r_I|=m} \sigma^{-r_I} C^{l,\beta}(M), \qquad \forall\, l \in \mathbb{Z}_+, \beta \in (0,1).$$

Let $f = \sum_{|r_I|=m} \sigma^{-r_I} f_{r_I}$, $f_{r_I} \in C^{k,\alpha}(M)$. *Then*

$$(\Delta_\omega - 1)(v\sigma^{-r}) = \sum_{|r_I|=m} \sigma^{-r}\sigma^{-r_I} f_I.$$

On the other hand, it follows from Theorem 6 that for each r_I, *there exists a* $v_{r_I} \in C^{k+2,\alpha}(M)$ *such that*

$$(\Delta_\omega - 1)(v_{r_I}\sigma^{r_I}\sigma^{-r}) = f_{r_I}\sigma^{-r_I}\sigma^{-r}.$$

Thus, by the uniqueness of $\Delta_\omega - 1$, *we have*

$$v = \sum_{|r_I|=m} \sigma^{-r_I} v_{r_I} \in \tau_I^m C^{k+2,\alpha}(M).$$

□

Next, we derive the following lemma:

Lemma 4.2. *Let* Ω, $\Omega' \in \mathcal{M}_{I,\mathcal{H}_I}$, *and* L_r *and* L_r' *be the corresponding linear operators with respect to* Ω *and* Ω' *respectively. Then*

$$(L_r - L_r')(v) \in \mathcal{R}_{I,\infty}(M), \qquad \forall\, v \in \mathcal{R}(M).$$

Proof of Lemma 4.2. *We want to show that*

$$\begin{aligned} (\Delta_\omega - \Delta_{\omega'})v \in \mathcal{R}_{I,\infty}(M), &\quad \forall\, v \in \mathcal{R}(M); \\ \sigma_i'^{-1}H(\sigma_i', v) - \sigma_i^{-1}H(\sigma_i, v) \in \mathcal{R}_{I,\infty}(M), &\quad \forall\, v \in \mathcal{R}(M); \\ c_r - c_r' \in \mathcal{R}_{I,\infty}(M). & \end{aligned}$$

It suffices to check the following:

$$d\mathcal{R}^m_{I,l}(M) \subset \mathcal{R}^{m+1}_{I,l}(M), \quad \forall\, l,\, m \geq 0, \tag{4.9}$$

$$\omega^k - \omega'^k \in \mathcal{R}^{k,k}_{I,\infty}(M), \quad \forall\, k \geq 1, \tag{4.10}$$

$$\sigma_i^{-1} d\sigma_i - \sigma_i'^{-1} d\sigma_i' \in \mathcal{R}^1_{I,\infty}(M). \tag{4.11}$$

Recall that

$$\mathcal{R}^m_{I,l}(M) = \tau_I^l \mathcal{R}^m(M) = \sum_{|r_I|=l} \sigma^{-r_I} \mathcal{R}^m(M).$$

For $\varphi \in \mathcal{R}^m(M)$, *we have*

$$\sigma^{r_I} d(\sigma^{-r_I}\varphi) = d\varphi + \sum_{i\in I} r_i \frac{d\sigma_i}{-\sigma_i} \wedge \varphi \in \mathcal{R}^{m+1}(M).$$

This proves (4.9). Similarly, we have

$$d^c \mathcal{R}^m_{I,l}(M) \subset \mathcal{R}^{m+1}_{I,l}(M). \tag{4.12}$$

Next, it follows from (4.9), (4.12), and the proof of Proposition 4.1 that

$$\omega - \omega' = dd^c \log \frac{\Omega}{\Omega'} \in \mathcal{R}^{1,1}_{I,\infty}(M).$$

Since $\omega \in \mathcal{R}^{1,1}(M)$, *we get*

$$\omega^k - \omega'^k \in \sum_{i=1}^k \omega^i \wedge \left(\mathcal{R}^{1,1}_{I,\infty}(M)\right)^{k-i} \subset \mathcal{R}^{k,k}_{I,\infty}(M).$$

Finally, as in the proof of Proposition 4.1,

$$\frac{\sigma_i}{\sigma_i'} \in 1 + \mathcal{R}_{I,\infty}(M);$$

therefore, by (4.9),

$$\sigma_i^{-1}d\sigma_i - \sigma_i'^{-1}d\sigma_i' = d\log\frac{\sigma_i}{\sigma_i'} \in \mathcal{R}^1_{I,\infty}(M).$$

This completes the proof. □

Proof of Proposition 4.2. *By the assumption and Lemma 4.2 we have*

$$L_r(v - v') = (f - f') - (L_r - L_r')v' \in \mathcal{R}_{I,\infty}(M).$$

Hence, it follows from Lemma 4.1 that $v - v' \in \mathcal{R}_{I,\infty}(M)$. □

Now we construct a family $\mathcal{M}_{I,\mathcal{H}_I}$ with certain canonicity on the $\mathcal{H}_I$: Fix a Kähler metric $\omega_{\overline{M}}$ on $\overline{M}$. Write

$$\omega_{\overline{M}}^n = \widetilde{\gamma}\prod_{j=1}^n\left(\frac{\sqrt{-1}}{2\pi}dz^j \wedge d\bar{z}^j\right)$$
$$(\iota_i^*\omega_{\overline{M}})^{n-1} = \widetilde{\gamma}_i\prod_{j\neq i}\left(\frac{\sqrt{-1}}{2\pi}dz^j \wedge d\bar{z}^j\right),$$

where $\iota_i : D_i \hookrightarrow \overline{M}$ is the inclusion. Then it follows from the usual adjunction formula that $\overline{M}$ induces a metric h_{D_i} on the normal bundle $N_{D_i} = [D_i]|_{D_i}$:

$$h_{D_i} = \frac{\widetilde{\gamma}|_{D_i}}{\widetilde{\gamma}_i}, \qquad i = 1, \ldots, q.$$

Now set

$$h_{I,i} = \epsilon_i h_{D_i}|_{D_I}, \qquad i = 1, \ldots, q.$$

where each ϵ_i is a small constant. Let

$$\mathcal{H}_I = \{h_{I,i} \mid i \in I\}$$

This implies that $\mathcal{H}_I$ is canonical in terms of $\omega_{\overline{M}}$ up to a constant. Extend each $h_{I,i}$

to a metric on $[D_i] \to \overline{M}$, which is denoted by h_i. Note that

$$[dd^c \log \frac{\widetilde{\gamma}}{h_1 \cdots h_p}] = c_1(K_{\overline{M}} + [D]). \tag{4.13}$$

We claim that there exists a function $\rho \in C^\infty(\overline{M})$ such that

$$\left.\frac{e^\rho \widetilde{\gamma}}{\prod_{i=1}^q h_i \prod_{j=q+1}^p |s_j|^2 \sigma_j^2}\right|_{D_I^\nu} = \begin{cases} 2^q n! \det(g_{i\bar{j},\nu}), & \text{if } 1 \le q < n; \\ 2^n n!, & \text{if } q = n. \end{cases} \tag{4.14}$$

$$dd^c \log \frac{e^\rho \widetilde{\gamma}}{h_1 \cdots h_p} > 0, \qquad \text{on } \overline{M}. \tag{4.15}$$

Indeed, (4.13) and (1.2) imply that there exists a $\widetilde{\rho} \in C^\infty(\overline{M})$ such that

$$dd^c \log \frac{e^{\widetilde{\rho}} \widetilde{\gamma}}{h_1 \cdots h_p} > 0.$$

Notice that

$$\log \big(\prod_{j \notin I} \sigma_j^{-2} \big) \in C^\infty(D_I).$$

If $q < n$, then by (4.7) there exists a $\varphi_{I,\nu} \in C^\infty(D_I^\nu)$ such that

$$dd^c \log \left.\frac{e^{\widetilde{\rho}} \widetilde{\gamma}}{h_1 \cdots h_p}\right|_{D_I^\nu} + dd^c \log \big(\prod_{j \notin I} \sigma_j^{-2} \big)\Big|_{D_I^\nu} + dd^c \varphi_{I,\nu} = \omega_{K-E,\nu}.$$

then, by adding a constant to $\varphi_{I,\nu}$ we get

$$\left.\frac{e^{\widetilde{\rho}} \widetilde{\gamma}}{\prod_{i=1}^q h_i \prod_{j=q+1}^p |s_j|^2 \sigma_j^2}\right|_{D_I^\nu} \cdot e^{\varphi_{I,\nu}} = 2^q n! \det(g_{i\bar{j},\nu}).$$

As for $q = n$, each D_I^ν is a point in $\overline{M}$. let $\varphi_{I,\nu}$ be the constant such that

$$\left.\frac{e^{\widetilde{\rho}} \widetilde{\gamma}}{\prod_{i=1}^q h_i \prod_{j=q+1}^p |s_j|^2 \sigma_j^2}\right|_{D_I^\nu} \cdot e^{\varphi_{I,\nu}} = 2^q n!.$$

Let $\chi \in C^\infty(\mathbb{R})$ be the cut-off function such that $\chi = 1$ on $(-\infty, 1]$ and $\chi = 0$ on

$[2, +\infty)$. Denote by

$$\varphi = \log\left[\exp\left(\chi(\sum_{i\in I}|s_i|^2/\delta)\varphi_I\right) + C(\sum_{i\in I}|s_i|^2)^3\right],$$

where $\varphi_I \in C^\infty(D_I)$ is defined by $\varphi_I = \varphi_{I,\nu}$ on each D_I^ν, $\delta > 0$ is a small number, and $C = C(\delta) > 1$ is a sufficient large number. It follows from direct calculations, or the following general theorem on metric extension, that

$$\rho \equiv \widetilde{\rho} + \varphi$$

satisfies both (4.14) and (4.15). Therefore, let $V = e^\rho \omega_{\widetilde{M}}^n$, and Ω be the volume form given by (4.1); then $\Omega \in \mathcal{M}_{I,\mathcal{H}_I}$ by the above construction.

As mentioned above, we can finish the construction of $\mathcal{M}_{I,\mathcal{H}_I}$ by direct calculations without using the following theorem. However, since the theorem on metric extensions may have interests of its own, we include it here.

Theorem 7. *Let X be an n-dimensional compact complex manifold, and L an ample line bundle over X. Let $E \subset X$ be a subvariety of complete intersection, i.e.,*

$$E = \bigcap_{i=1}^{k} D_i, \qquad 1 \le k \le n,$$

where $D_i \subset X$ are smooth irreducible hypersurfaces that meet transversally at each point of intersection. Let h_E be a smooth metric on $L|_E$ with positive curvature form on E. Then h_E can be extended to a smooth metric h on L with positive curvature form on X.

A similar result, which works for codimension 1, i.e., $E = D_1$ as above, was obtained by Schumacher (see Theorem 4 in [31]). His extended metric, however, is not globally smooth, since it contains a term as $|e|^{2/m}$, where e is the holomorphic section defining the divisor E (see [31, p.634].). Our theorem has generalized it to any higher codimension, and our proof below will be simpler. We need the following

two lemmas, in which Lemma 4.3 was learned from [31].

Lemma 4.3. *Let H_1 and H_2 be two metrics on a line bundle over a complex manifold. Let*

$$H = \frac{1}{\frac{1}{H_1} + \frac{1}{H_2}}$$

Denoted by Θ_j and Θ the curvature form of H_j and H, respectively. Then

$$\Theta \geq \frac{H}{H_1}\Theta_1 + \frac{H}{H_2}\Theta_2.$$

Proof of Lemma 4.3. *Let*

$$\mathcal{F}(f) = f dd^c \log f,$$

for any positive function f. Then one verifies immediately that for any two positive functions f and g the following holds:

$$\mathcal{F}(f+g) - \mathcal{F}(f) - \mathcal{F}(g) = \frac{g^3}{f(f+g)} d\left(\frac{f}{g}\right) \wedge d^c\left(\frac{f}{g}\right) \geq 0$$

We finish the proof by setting $f = 1/H_1$ and $g = 1/H_2$. □

Lemma 4.4. *Assume that there exists an open neighborhood U of E and a smooth metric h_U on $L|_U$ such that $h_U|_E = h_E$, and the curvature form of h_U is positive on U. Then the metric h_E can be extended to a metric h on L with positive curvature form on X.*

Proof of Lemma 4.4. *Since L is ample, by definition there exists a metric h_p on L with positive curvature form on X. On the other hand, extend h_U (with some shrinking of U, if necessary) to a smooth metric h_1 of L over X , without any curvature assumptions on $X \setminus U$. Let*

$$\frac{1}{h} = \frac{1}{h_1} + C\frac{1}{h_p}\left(\sum_{i=1}^{k} |s_i|^2\right)^3,$$

where $C > 0$ is a constant and each $s_i \in H^0(X, \mathcal{O}([D]))$ defines D_i. We claim that h is the desired extension metric.

Let $U' \subset U$ be a relatively compact open neighborhood of E. On $X \setminus U'$, applying Lemma 4.3, we can choose $C = C(U') > 0$ sufficiently large such that the curvature form of h is positive on $X \setminus U'$. On $U \setminus E$, by the assumption of h_U, applying Lemma 4.3 again we know that the curvature form of h is positive on $U \setminus E$. Finally, by direct calculation we have that the curvature forms of h and h_1 coincides at every point of the subvariety E. This completes the proof.

□

Proof of Theorem 7. *By Lemma 4.4 it suffices to extend h_E to a metric whose curvature form is positive on a neighborhood of E. It follows from positivity of L that there exists a metric h_p with positive curvature form Θ_p on X. Note that $[\Theta_p|_E] = c_1(L|_E)$. Hence, there is a $\varphi_E \in C^\infty(E)$ such that*

$$\frac{e^{\varphi_E}}{h_p|_E} = \frac{1}{h_E}.$$

Let

$$\varphi = \chi\Big(\sum_{i=1}^{k} |s_i|^2/\delta\Big)\varphi_E,$$

where $\chi \in C^\infty(\mathbb{R})$ is a cut-off function with value 1 on $(-\infty, 1)$ and value 0 on $(2, +\infty)$, and $\delta > 0$ is a small constant. Now let

$$\frac{1}{h_1} = \frac{e^{\varphi}}{h_p}.$$

Then h_1 is a smooth metric of L whose curvature form is positive on $\{\sum_{i=1}^{k} |s_i|^2 < \delta\}$. Hence, by Lemma 4.4 the metric

$$\frac{1}{h} = \frac{1}{h_p}\left(e^{\varphi} + C\Big(\sum_{i=1}^{k} |s_i|^2\Big)^3\right)$$

is the desired extension metric on X.

□

4.3 Higher canonical asymptotics of Kähler-Einstein metrics

Finally, we will state and prove the asymptotics theorem in the case of complement of a simple normal crossing divisor.

Theorem 8. *Suppose $I = \{1, \ldots, q\} \in \mathcal{I}$. Let $\Omega \in \mathcal{M}_I$. Then the Cheng-Yau solution u of*

$$M_\omega(u) = J(\Omega)^{-1}, \tag{4.16}$$

$$\frac{1}{C}\omega \leq \omega_u \leq C\omega, \qquad C > 0, \tag{4.17}$$

is in $\mathcal{R}^G_{I,1}(M)$. More precisely, $u \in \mathcal{R}_{I,1}(M)$; furthermore, there is a multiple sequence $\{\psi_r\}_{r \in (\mathbb{Z}^q_+ \times 0)} \subset \mathcal{R}(M)$ such that for any $N \in \mathbb{N}$,

$$u - \sum_{|r|=1}^{N} \psi_r \sigma^{-r} \in \mathcal{R}_{I,N+1}(M). \tag{4.18}$$

Moreover, assume $\Omega \in \mathcal{M}_{I,\mathcal{H}_I}$ for a given $\mathcal{H}_I$; if we start from another $\Omega' \in \mathcal{M}_{I,\mathcal{H}_I}$ and get

$$u' - \sum_{|r|=1}^{N} \psi'_r \sigma'^{-r} \in \mathcal{R}_{I,N+1}(M), \qquad \forall N \in \mathbb{N}$$

then

$$u' - u \in \mathcal{R}_{I,\infty}(M); \tag{4.19}$$

$$\psi'_r - \psi_r \in \mathcal{R}_{I,\infty}(M), \qquad \forall r \in \mathbb{Z}^q_+ \times 0. \tag{4.20}$$

Consequently, the Kähler-Einstein volume form $\Omega_{K-E} = e^u \Omega$ on M has the following canonical asymptotic expansion near the smooth subvariety D_I:

$$\Omega_{K-E} \sim \left(1 + \sum_{|r|=1}^{\infty} \frac{\phi_r}{\sigma^r}\right) \Omega, \tag{4.21}$$

where $\phi_r \in \mathcal{R}(M)$ for each $r \in (\mathbb{Z}_+^q \times 0)$. The asymptotics is canonical in the the following sense: If (4.21) is derived in terms of another $\Omega' \in \mathcal{M}_{I,\mathcal{H}_I}$, and coefficients $\{\phi'_r\}_{r\in\mathbb{Z}_+^q\times 0}$. Then

$$\phi'_r - \phi_r \in \mathcal{R}_{I,\infty}(M), \qquad \forall\, r \in \mathbb{Z}_+^q \times 0.$$

Proof of Theorem 8. *For simplicity, we identify the index set $\mathbb{Z}_+^q \times 0$ with $\mathbb{Z}_+^q$; namely, in the following proof, $r = r_I = (r_1, \ldots, r_q) \in \mathbb{Z}_+^q$ and $\sigma^{-r} = \sigma^{-r_I} = \sigma_1^{-r_1} \cdots \sigma_q^{-r_q}$. Write*

$$\begin{aligned} J(\Omega) &= \frac{\omega^n}{\Omega} \\ &= f_0(1 + \sum_{|r|=1}^{n-q} \sigma^{-r} f_r)(1 + f_b), \end{aligned}$$

where

$$\begin{aligned} f_0 &= \frac{2^q n!\, \omega_{K,I}^{n-q} \wedge \prod_{i=1}^q |s_i|^2 d\sigma_i \wedge d^c\sigma_i}{(n-q)!\, V}, \\ f_b &= \frac{\sum_{j=0}^{q-1} \binom{n}{j} (\omega_{K,I} + \sum_{i=1}^q 2\sigma_i^{-1}\omega_{c_i})^{n-j} \wedge (\sum_{i=1}^q 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i)^j}{q! \binom{n}{q} (\omega_{K,I} + \sum_{i=1}^q 2\sigma_i^{-1}\omega_{c_i})^{n-q} \wedge \prod_{i=1}^q 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i}, \\ f_r &= \frac{|r|! 2^{|r|} \binom{n-q}{|r|} \omega_{K,I}^{n-q-|r|} \wedge \prod_{i=1}^q \omega_{c_i}^{r_i} \wedge \prod_{i=1}^q 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i}{r! \binom{n}{q} \omega_{K,I}^{n-q} \wedge \prod_{i=1}^q 2\sigma_i^{-2} d\sigma_i \wedge d^c\sigma_i}, \end{aligned}$$

in which $1 \le |r| \le n-q$, and

$$\omega_{K,I} = \mathrm{Ric}\left(\frac{V}{\prod_{i=1}^p |s_i|^2 \prod_{j\notin I} \sigma_j^2}\right) = \omega_K - 2dd^c \log\Big(\prod_{j=q+1}^p \sigma_j\Big).$$

By definition we know that $f_r \in C^\infty(\overline{M}) \subset \mathcal{R}(M)$, $f_b \in \mathcal{R}_{I,\infty}(M)$, $f_b > 0$ on M, and that $f_0 \in \mathcal{R}(M)$, is smooth near D_I, and $f_0 > 0$ on $\overline{M}$. Since $\Omega \in \mathcal{M}_I$, by condition (1) we have

$$f_0\big|_{D_I} = 1.$$

Hence, $f_0 - 1 \in \mathcal{R}_{I,\infty}(M)$. *Moreover,*

$$\sum_{|r|=k} \sigma^{-r} f_r \in \mathcal{R}_{I,k}(M), \qquad k \geq 1.$$

These imply that

$$\log J(\Omega) \in \mathcal{R}_{I,1}(M).$$

Then, applying Theorem 5 yields that the Cheng-Yau solution

$$u \in \mathcal{R}_{I,1}(M).$$

Next to derive (4.18): For each $r \in \mathbb{Z}_+^q$ *and* $|r| = 1$, *let* $\psi_r \in \mathcal{R}(M)$ *be the unique solution of*

$$L_r(\psi_r) \equiv \Delta_\omega \psi_r - 2\sum_{i=1}^{q} r_i \sigma_i^{-1} H(\sigma_i, \psi_r) + c_r \psi_r = -f_r.$$

Denote by

$$u_1 = \sum_{|r|=1} \psi_r \sigma^{-r}.$$

Then we have

$$(\Delta_\omega - 1)(u_1) = -\sum_{|r|=1} f_r \sigma^{-r}.$$

Also,

$$\begin{aligned} J(e^{u_1}\Omega) &= M_\omega(u_1) J(\Omega) \\ &= \big(1 + \Delta_\omega u_1 + G_2(u_1) + \cdots + G_n(u_1)\big) e^{-u_1} J(\Omega) \\ &\in 1 + \sum_{|r|=2} \mathcal{R}_r(M) \subset 1 + \mathcal{R}_{I,2}(M), \end{aligned}$$

in which

$$G_i(v) \equiv \frac{\binom{n}{i} \omega^{n-i} \wedge (dd^c v)^i}{\omega}, \quad \forall\, v \in \mathcal{R}(M),\ i \geq 2. \tag{4.22}$$

By the same argument in the proof of Theorem 4, and applying Theorem 6, we have

that

$$u - \sum_{|r|=1} \psi_r \sigma^{-r} \in \mathcal{R}_{I,2}(M).$$

Then, similar to the proof of Theorem 4, by induction and Theorem 6, we prove that there exists a sequence $\{\psi_r\}_{r\in\mathbb{Z}_+^q}$ *such that for each* $N \in \mathbb{N}$,

$$J\left(\exp(\sum_{|r|=1}^{N} \psi_r \sigma^{-r}) \cdot \Omega\right) - 1 \in \sum_{|r|=N+1} \mathcal{R}_r(M) \subset \mathcal{R}_{I,N+1}(M),$$

$$u - \sum_{|r|=1}^{N} \psi_r \sigma^{-r} \in \mathcal{R}_{I,N+1}(M).$$

Therefore, this completes the proof of (4.18).

As for the canonicity, (4.19) follows from Proposition 4.1. It remains to prove (4.20): In fact, by construction

$$f_r' - f_r \in \mathcal{R}_{I,\infty}(M), \qquad \forall\, 1 \le |r| \le n-q. \tag{4.23}$$

Furthermore, it follows from (4.9) and (4.10), in the proof of Lemma 4.2, that

$$G_i'(v') - G_i(v) \in \mathcal{R}_{I,\infty}(M), \quad \forall\, v' - v \in \mathcal{R}_{I,\infty}(M), \tag{4.24}$$

where $G_i'(M)$ *is given by (4.22) with* ω' *replaced by* ω. *Then (4.20) follows immediately from (4.23), Proposition 4.2, (4.24), and induction.*

□

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