The p–adic Local Langlands Conjecture

by

Christopher D. Malon

Bachelor of Science, University of Chicago, 2000
Master of Science, University of Chicago, 2000

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2005

Author ....
Department of Mathematics
July 21, 2005

Certified by ....
David A. Vogan, Jr.
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Pavel Etingof
Chairman, Department Committee on Graduate Students

The p–adic Local Langlands Conjecture

by

Christopher D. Malon

Submitted to the Department of Mathematics
on July 21, 2005, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

Let k be a p–adic field. Split reductive groups over k can be described up to k–isomorphism by a based root datum alone, but other groups, called rational forms of the split group, involve an action of the Galois group of k. The Galois action on the based root datum is shared by members of an *inner class* of k–groups, in which one k–isomorphism class is quasi–split. Other forms of the inner class can be called *pure* or *impure*, depending on the Galois action. Every form of an adjoint group is pure, but only the quasi–split forms of simply connected groups are pure.

A p–adic Local Langlands correspondence would assign an *L–packet*, consisting of finitely many admissible representations of a p–adic group, to each Langlands parameter. To identify particular representations, data extending a Langlands parameter is needed to make "completed Langlands parameters."

Data extending a Langlands parameter has been utilized by Lusztig and others to complete portions of a Langlands classification for pure forms of reductive p–adic groups, and in applications such as endoscopy and the trace formula, where an entire L–packet of representations contributes at once. We consider a candidate for completed Langlands parameters to classify representations of arbitrary rational forms, and use it to extend a classification of certain supercuspidal representations by DeBacker and Reeder to include the impure forms.

Thesis Supervisor: David A. Vogan, Jr.
Title: Professor of Mathematics

Acknowledgments

I wish to thank my mentors, colleagues, family, and friends for their unwavering support. This work was sponsored in part by an NSF Graduate Research Fellowship.

Contents

Chapter 1

L–groups and Langlands parameters

1.1 L–groups

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraically closed field $\overline{k}$. If $\mathbf{T} \subset \mathbf{B}$ are Cartan and Borel subgroups of $\mathbf{G}$, let X and $X^\vee$ denote the character and cocharacter groups of $\mathbf{T}$, and let Δ and $\Delta^\vee$ be the sets of simple roots and coroots for the action of the Lie algebra $\mathfrak{t}$ on $\mathfrak{b}$. Let Φ and $\Phi^\vee$ denote the full sets of roots and coroots. The sets Δ and $\Delta^\vee$ come with a canonical bijection $\delta : \Delta \to \Delta^\vee$. The quadruple $(X, \Delta, X^\vee, \Delta^\vee)$ is called a *based root datum*; it includes the identification of Δ with a subset of X (and $\Delta^\vee$ with a subset of $X^\vee$).

An isomorphism of based root data $(X, \Delta, X^\vee, \Delta^\vee)$ and $(X', \Delta', X'^\vee, \Delta'^\vee)$ is an isomorphism $X \to X'$ sending $\Delta \to \Delta'$, in such a way that the transpose isomorphism $X'^\vee \to X^\vee$ sends $\Delta'^\vee \to \Delta^\vee$, and that the two maps $\Delta \to \Delta'$ and $\Delta'^\vee \to \Delta^\vee$ respect the bijections δ and δ'.

Because $\overline{k}[\mathbf{T}] = \overline{k} \otimes_{\mathbb{Z}} X$, algebraic automorphisms of $\mathbf{T}$ are in natural correspondence with automorphisms of the abelian group X.

Suppose $\mathbf{T}' \subset \mathbf{B}'$ is another pair consisting of a Cartan and Borel subgroup in

$\mathbf{G}$. Then there is a unique element $g\mathbf{T} \in \mathbf{G}/\mathbf{T}$ such that $(g\mathbf{T}g^{-1}, g\mathbf{B}g^{-1}) = (\mathbf{T}', \mathbf{B}')$. Therefore, having fixed the identification $X \cong X^*(\mathbf{T})$, the pair $\mathbf{T}' \subset \mathbf{B}'$ determines an isomorphism between X and the character group $X^*(\mathbf{T}')$.

If $\sigma \in \mathrm{Aut}_{alg}(\mathbf{G})$ is an algebraic automorphism, then $\sigma\mathbf{T} \subset \sigma\mathbf{B}$ is another Cartan–Borel pair, determining an element $g\mathbf{T} \in \mathbf{G}/\mathbf{T}$ as above. We then have two maps:

$$\begin{aligned} \sigma^* &: X^*(\sigma\mathbf{T}) \to X^*(\mathbf{T}) \\ \mathrm{Int}(g)^* &: X^*(\sigma\mathbf{T}) \to X^*(\mathbf{T}) \end{aligned}$$

(here $\mathrm{Int}(g)$ is the inner automorphism $x \to gxg^{-1}$). The map $\sigma^* \circ \mathrm{Int}(g^{-1})^*$ induces an automorphism of $X^*(\mathbf{T})$ preserving the set Δ, whence an automorphism of the based root datum $(X, \Delta, X^\vee, \Delta^\vee)$. Thereby we get a map

$$\beta : \mathrm{Aut}_{alg}(\mathbf{G}) \to \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee)) \tag{1.1}$$

with kernel $\mathrm{Int}(\mathbf{G}) \cong \mathbf{G}_{ad} = \mathbf{G}/Z(\mathbf{G})$.

From here on, suppose that $\overline{k}$ is the algebraic closure of some p–adic field k. (By this, we mean that k is a finite extension of $\mathbb{Q}_p$.) Write $\Gamma = \Gamma_k = \Gamma_{\overline{k}/k}$ for the absolute Galois group of k. Let ϖ be a uniformizer of k.

We will use the term "group over k" to refer to a connected reductive algebraic group $\mathbf{G}$ over $\overline{k}$ equipped with an action

$$\sigma : \Gamma \to \mathrm{Aut}_{abs}(\mathbf{G}) \tag{1.2}$$

where Aut_{abs} refers to the group of (not necessarily algebraic) group automorphisms of $\mathbf{G}$. The action σ is required to have the property that if $f : \mathbf{G} \to \overline{k}$ is a regular function, then the function

$$x \to \gamma \cdot f(\sigma(\gamma^{-1}) \cdot x) \tag{1.3}$$

is regular. In this expression, the γ outside f acts via the natural Galois action on $\overline{k}$.

An isomorphism of groups over k is an isomorphism of algebraic groups that is equivariant for the action of Γ on each group. The various groups over k that are isomorphic to $\mathbf{G}$ over $\overline{k}$ are called rational forms of $\mathbf{G}$. Two rational forms are said to be equivalent if there is an algebraic automorphism of $\mathbf{G}$ that is equivariant for the corresponding Galois actions.

Fix a pinning for $\mathbf{G}$. Any $\kappa \in \mathrm{Hom}(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee)))$ determines a quasi–split form over k, as follows. Write κ_γ for the value of κ at $\gamma \in \Gamma$. Let $\omega \in X_*(\mathbf{T})$, and let $u_\alpha : \overline{k} \to \mathbf{G}$ be the homomorphism from the pinning associated to $\alpha \in \Delta$. The conditions that:

$$\begin{aligned} \gamma(\omega(k)) &= \kappa_\gamma(\omega)(\gamma k) && (1.4)\\ \gamma(u_\alpha(k)) &= u_{\kappa_\gamma(\alpha)}(\gamma k) && (1.5) \end{aligned}$$

(for all such ω and α, and $\gamma \in \Gamma$) characterize the action of Γ on elements of $\mathbf{G}$.

Fix any rational form on $\mathbf{G}$, writing γx for the action of $\gamma \in \Gamma$ on a point $x \in \mathbf{G}$. Suppose $\sigma : \Gamma \to \mathrm{Aut}_{abs}(\mathbf{G})$ is another rational form. Then $\gamma^{-1} \circ (\sigma(\gamma))$ is an algebraic automorphism of $\mathbf{G}$. Indeed, if f is a regular function on $\mathbf{G}$, then

$$y \to \gamma^{-1} f(\sigma(\gamma) y)$$

is regular. Applying this to the regular function

$$x \to \gamma f(\gamma^{-1} x)$$

we deduce that

$$y \to \gamma^{-1} \gamma f(\gamma^{-1} \sigma(\gamma) y)$$

is regular, as needed.

With this observation and the map (1.1), we may classify all rational forms of $\mathbf{G}$ over k in terms of a split form over k. Let $\sigma : \Gamma \to \mathrm{Aut}_{abs}(\mathbf{G})$ be a rational form and let $\tau : \Gamma \to \mathrm{Aut}_{abs}(\mathbf{G})$ be the split form preserving the pair $\mathbf{T} \subset \mathbf{B}$ and our fixed pinning. By the above paragraph, for each γ the element $\alpha(\gamma) = \sigma(\gamma)\tau(\gamma^{-1})$ is in $\mathrm{Aut}_{alg}(\mathbf{G})$. Let Γ act on $\mathrm{Aut}_{alg}(\mathbf{G})$ by sending

$$\phi \to \tau(\gamma) \circ \phi \circ \tau(\gamma^{-1}) \tag{1.6}$$

for each $\gamma \in \Gamma$. For this action, $\alpha \in Z^1(\Gamma, \mathrm{Aut}_{alg}(\mathbf{G}))$. The cocycles determined by two rational forms are cohomologous if and only if the rational forms are equivalent, and we obtain a bijection between rational forms and $Z^1(\Gamma, \mathrm{Aut}_{alg}\mathbf{G})$, and between their equivalence classes and $H^1(\Gamma, \mathrm{Aut}_{alg}(\mathbf{G}))$. Henceforth we typically will regard rational forms as cocycles.

Giving $\mathrm{Aut}((X, \Delta, X^\vee, \Delta))$ the trivial Galois action,

$$\begin{aligned} H^1(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee)) &= Z^1(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee))) \\ &= \mathrm{Hom}(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee))). \end{aligned}$$

The fibers of

$$\beta_* : Z^1(\Gamma, \mathrm{Aut}_{alg}(\mathbf{G})) \to Z^1(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee)) \tag{1.7}$$

are called *inner classes* of rational forms. Each inner class contains a unique equivalence class of quasi–split forms. If $\kappa \in Z^1(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee)))$, write τ_κ for the quasi–split form defined by equations (1.4)–(1.5). The fiber of β_* over κ is in bijection with $Z^1(\Gamma, \mathbf{G}_{ad})$, where the Galois action on $\mathbf{G}_{ad}$ is defined by equation (1.6), replacing τ by τ_κ.

If σ splits over a finite cyclic extension k' of k of degree m, and $\Gamma_{k'/k}$ is generated by an element $\gamma \in \Gamma$, then σ is determined by the value $\sigma(\gamma)$. Thus, the inner classes of rational forms splitting over k' are in correspondence with the automorphisms of the based root datum of order dividing m. For example, every quasi–split form that splits over $K = k^{unr}$ is determined, up to equivalence, by the action of a Frobenius element on the based root datum. Suppose $\kappa \in \mathrm{Hom}(\Gamma, \mathrm{Aut}((X, \Delta, X^\vee, \Delta^\vee)))$ describes this action, and τ_κ is the corresponding quasi–split form. The action of Γ on $\mathbf{G}_{ad}$ specified by τ_κ factors through $\Gamma_{K/k}$, so we obtain an inflation map $H^1(K/k, \mathbf{G}_{ad}) \to H^1(k, \mathbf{G}_{ad})$. In the inflation–restriction sequence

$$1 \to H^1(K/k, \mathbf{G}_{ad}) \to H^1(k, \mathbf{G}_{ad}) \to H^1(K, \mathbf{G}_{ad}) \tag{1.8}$$

we have $H^1(K, \mathbf{G}_{ad}) = 1$, by Steinberg's Theorem [14], applied to the connected k–group $\mathbf{G}_{ad}$. This exact sequence of pointed sets gives a bijection from $H^1(K/k, \mathbf{G}_{ad})$ to $H^1(k, \mathbf{G}_{ad})$. Thus, if a quasi–split form splits over K, so does every form in its inner class.

An inner class of rational forms determines an *L–group*, as follows. Let σ be a quasi–split rational form of a connected reductive k–group with based root datum $(X, \Delta, X^\vee, \Delta^\vee)$. The classification of reductive groups over an algebraically closed field associates a complex reductive group $\check{G}$ to the dual based root datum $(X^\vee, \Delta^\vee, X, \Delta)$. The group $\check{G}$ is unique up to inner isomorphism. Fix a Cartan–Borel pair $\check{T} \subset \check{B}$ in $\check{G}$. This choice determines a surjective homomorphism

$$\check{\beta} : \mathrm{Aut}_{alg}(\check{G}) \to \mathrm{Aut}((X^\vee, \Delta^\vee, X, \Delta) \tag{1.9}$$

as in (1.1).

Using the transpose $\beta^\top : \mathrm{Aut}_{alg}(\mathbf{G}) \to \mathrm{Aut}((X^\vee, \Delta^\vee, X, \Delta)$, our rational form $\sigma \in Z^1(\Gamma, \mathrm{Aut}_{alg}(\mathbf{G}))$ determines

$$\beta^\top_*(\sigma) \in \mathrm{Hom}(\Gamma, \mathrm{Aut}((X^\vee, \Delta^\vee, X, \Delta))).$$

A pinning $(\check{B}, \check{T}, \{\check{\omega}\}, \{u_{\check{\alpha}}\})$ for $\check{G}$ determines $\check{\sigma} \in \mathrm{Hom}(\Gamma, \mathrm{Aut}_{alg}(\check{G}))$ in the fiber of

$$\check{\beta}_* : \mathrm{Hom}(\Gamma, \mathrm{Aut}_{alg}(\check{G})) \to \mathrm{Hom}(\Gamma, \mathrm{Aut}(X^\vee, \Delta^\vee, X, \Delta))$$

over $\beta_*^\top(\sigma))$ via

$$\check{\sigma}(\omega(k)) = \beta_*^\top(\sigma)(\check{\omega})(k) \tag{1.10}$$

$$\check{\sigma}(u_\alpha(k)) = u_{\beta_*^\top(\sigma)(\check{\alpha})}(k) \tag{1.11}$$

The map $\check{\sigma}$ determines a semidirect product

$${}^L\mathbf{G} = \check{G} \rtimes \Gamma$$

which we call the *L–group* of G.

1.2 Langlands parameters

We denote by $\mathfrak{f} = \mathbb{F}_q$ the residue field of k, and by $\bar{\mathfrak{f}} = \overline{\mathbb{F}_q}$ its separable closure. The isomorphism $\Gamma_{\bar{\mathfrak{f}}/\mathfrak{f}} \cong \hat{\mathbb{Z}}$ sending Fr to 1 gives a natural embedding of $\mathbb{Z}$ in $\Gamma_{\bar{\mathfrak{f}}/\mathfrak{f}}$. Because $\mathbb{Z}$ is not a closed subgroup of $\hat{\mathbb{Z}}$, this subgroup does not correspond to any extension of $\mathfrak{f}$.

Let $\mathcal{W} = \mathcal{W}_k$ be the Weil group for k (see Tate [15]); it comes with an open subgroup $\mathcal{W}_{k'}$ for any finite Galois extension k'/k such that $\mathcal{W}_k/\mathcal{W}_{k'} \cong \Gamma_{k'/k}$, and the maximal abelian quotient of $\mathcal{W}_k$ is isomorphic to $k^\times$ under the Artin map of local class field theory. We may identify $\mathcal{W}_k$ as the inverse image of $\mathbb{Z}$ under the reduction map $\Gamma_k \to \Gamma_{\mathfrak{f}}$.

Let $K = k^{unr}$ be the maximal unramified extension of k in $\overline{k}$. The subgroup $\mathcal{I} = \Gamma_K$ of Γ is called the inertia group, and is contained in $\mathcal{W}$. It fits into an exact sequence

$$1 \to \mathcal{I} \to \mathcal{W} \to \mathbb{Z} \to 1.$$

via the reduction map $\Gamma \to \Gamma_{\mathfrak{f}}$.

Let k^t be the maximal tamely ramified extension of k in $\overline{k}$; it contains $K = k^{unr}$. The subgroup $\mathcal{I}^+ = \Gamma_{k^t}$ of $\mathcal{I}$ is called the *wild inertia group*; and the quotient $\mathcal{I}_t = \mathcal{I}/\mathcal{I}^+$ is called the *tame inertia group.* The quotient $\mathcal{I}_t$ is abelian, but $\mathcal{W}/\mathcal{I}^+$ is not. If Fr is a Frobenius element in $\Gamma_{\mathfrak{f}}$ and $w \in \mathcal{W}/\mathcal{I}^+$ with $w \to \mathrm{Fr}$, then for $\gamma \in \mathcal{I}_t$ we have

$$w\gamma w^{-1} = \gamma^q. \tag{1.12}$$

One way to see this is that $\mathcal{I}_t$ is a quotient of the Weil group $\mathcal{W}_K$ for $K = k^{unr}$ (where $\mathcal{W}_K = \mathcal{I}_K = \mathcal{I}$, the inertia group for k), and the local Artin map is equivariant for the natural action of Γ_k on $K^\times$ and its action on $\mathcal{W}_K$ by conjugation (see Tate [15], (W_2), page 3). In fact, the choice of w gives a splitting $\mathcal{W}/\mathcal{I}^+ \cong \mathbb{Z} \ltimes \mathcal{I}_t$ on which $\mathbb{Z}$ acts on $\mathcal{I}_t$ by $\gamma \to \gamma^q$.

The Weil–Deligne group is the semidirect product

$$\mathcal{W}' = \mathcal{W} \ltimes \mathbb{C} \tag{1.13}$$

where $w \in \mathcal{W}$ acts on $z \in \mathbb{C}$ by $wzw^{-1} = q^n z$ if $w \to n$ in the reduction map $\mathcal{W} \to \Gamma_{\mathfrak{f}}$.

Our Langlands parameters are homomorphisms $\phi : \mathcal{W}' \to {}^L\mathbf{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{W}' \arrow[rr, "\phi"] \arrow[dr] & & {}^L\mathbf{G} \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

commutes, and satisfying some additional restrictions. However, it will be more convenient to think of these homomorphisms in two parts:

1. The Galois group Γ acts on $\check{G}$ via $\gamma \cdot g = \gamma g \gamma^{-1}$, where the right hand side is

computed inside ${}^L\mathbf{G}$. Define $\alpha_\phi \in Z^1(\mathcal{W}, \check{G})$ for this action by

$$\alpha_\phi(\gamma) = \phi(\gamma)\gamma^{-1}. \tag{1.14}$$

2. The restriction of ϕ to $\mathbb{C}$ can be written as

$$\phi|_{\mathbb{C}}(z) = \exp(zN_\phi) \tag{1.15}$$

for some $N_\phi \in \check{\mathfrak{g}}$.

The structure of $\mathcal{W}'$ imposes a compatibility condition between these two parts α_ϕ and N_ϕ, which we will consider and utilize later.

Chapter 2

L–packets

Assume $\mathbf{G}$ to be quasi–split, and fix a pinning with Cartan and Borel subgroups $\mathbf{T} \subset \mathbf{B}$, with $\mathbf{T}$ a maximally k–split k–torus. Write $H^*(k', -)$ for Galois cohomology of a module for the absolute Galois group $H^*(\Gamma_{k'}, -)$ of a field k' containing k. If l/k' is a subextension over k, we write $H^*(l/k', -)$ for $H^*(\Gamma_{l/k'}, -)$ for a module under the relative Galois group. Finally, if σ is a generator of a topologically cyclic group, we may write $H^*(\sigma, -)$ for the Galois cohomology of its modules.

Within an inner class, we explained in Chapter 1 how rational forms are parameterized by elements of $Z^1(k, \mathbf{G}_{ad})$ where $\mathbf{G}_{ad}$ is viewed as a Γ_k–module via the action from equations (1.4)–(1.5) arising from the quasi–split form. One can also give $\mathbf{G}$ a Γ_k–module structure in the same way. Elements of $Z^1(k, \mathbf{G})$ are called *pure inner forms* of the given quasi–split form; when the inner class is not fixed, we speak of *pure rational forms.* Although pure rational forms determine rational forms via the natural map $Z^1(k, \mathbf{G}) \to Z^1(k, \mathbf{G}_{ad})$, this map is neither injective nor surjective on cohomology; already for $\mathbf{G} = \mathbf{SL_2}$ it is not surjective, and easy examples of non–injectivity arise in non–split isogeny classes ${}^2A_3$ and ${}^2D_4$.

In applications such as endoscopy, data pertaining to each pure rational form of an inner class arise simultaneously in typical constructions. Because these forms have the same L–group, Langlands parameters should classify the representations of each pure form of the inner class at once: A Langlands parameter conjecturally gives an L–packet consisting of finitely many pairs $(\pi_\sigma, \tau_\sigma)$, indexed by some set of "parameter extensions" σ, where π_σ is a representation of the rational form of $\mathbf{G}$ determined by τ_σ. (We will describe these extensions more precisely momentarily.) However, if one wishes to use the Langlands conjectures to obtain information about arbitrary rational forms, one needs a bigger parameterization of L–packets that include representations of all the forms of an inner class.

2.1 Langlands parameter extensions

In this section, we suppose we are given a k–group $\mathbf{G}$ with a pinning as above, defining an L–group ${}^L\mathbf{G}$. Let $\phi : \mathcal{W}' \to {}^L\mathbf{G}$ be a Langlands parameter.

The subgroups $[\mathbf{G}, \mathbf{G}]$ and $Z(\mathbf{G})$ are defined over k; let $\overline{\mathbf{G}} = \mathbf{G}/([\mathbf{G}, \mathbf{G}] \cap Z(\mathbf{G}))$. This group also is defined over k, and has the property that $[\overline{\mathbf{G}}, \overline{\mathbf{G}}] = \mathbf{G}_{ad}$. Write the based root datum for $\overline{\mathbf{G}}$ as $(\tilde{X}, \tilde{\Delta}, \tilde{X}^\vee, \tilde{\Delta}^\vee)$.

The isogeny $\mathbf{G} \to \overline{\mathbf{G}}$ induces maps of Γ–modules $\tilde{X} \to X$ and $X^\vee \to \tilde{X}^\vee$ (both injective) restricting to bijections $\tilde{\Delta} \to \Delta$ and $\Delta^\vee \to \tilde{\Delta}^\vee$. (Thus we also may write Δ in place of $\tilde{\Delta}$.) We wish to consider the lattices $\tilde{X}$ and $\tilde{X}^\vee$ in the same ambient spaces as X and $X^\vee$. The cocharacter group $X^\vee_{ad}$ for $\mathbf{G}_{ad}$ may be recognized as the lattice of integral coweights in the rational vector space $\mathbb{Z}\Delta^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, giving a natural embedding $X^\vee_{ad} \to X^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. Via this embedding, we may identify the lattices

$$\tilde{X}^\vee = X^\vee_{ad} + X^\vee \tag{2.1}$$

$$\tilde{X} = \{\chi \in X : \langle \chi, \omega \rangle \in \mathbb{Z} \text{ for all } \omega \in X^\vee_{ad}\} \tag{2.2}$$

because the right hand sides yield a based root datum isomorphic to that of $\overline{\mathbf{G}}$.

On the dual side, the L–group of $\overline{G}$ is a semidirect product ${}^L\overline{\mathbf{G}} = \tilde{G} \rtimes \Gamma$ where $\tilde{G}$ is a complex reductive algebraic group with the based root datum $(\tilde{X}^\vee, \tilde{\Delta}^\vee, \tilde{X}, \tilde{\Delta})$.

It gives a covering

$$1 \to K \to {}^L\overline{\mathbf{G}} \xrightarrow{\pi} {}^L\mathbf{G} \to 1 \tag{2.3}$$

with $K \subset \tilde{G}$.

We will need the following elementary identifications:

Proposition 2.1 *Let A be an algebraic subgroup of a complex torus, and let $H = \mathrm{Hom}_{alg}(A, \mathbb{C}^\times)$. Then:*

1. $\mathrm{Hom}_{alg}(\pi_0(A), \mathbb{C}^\times) = H_{tor}$, *the subgroup of torsion elements of H.*
2. *If $\Upsilon \subset \mathrm{Aut}_{alg}(A)$ is a finite subgroup, consider the transpose action of Υ on H and let $H(\Upsilon)$ be the group generated by ${}^\sigma h h^{-1}$ for $\sigma \in \Upsilon$. Then $\mathrm{ann}_H(A^\Upsilon) = H(\Upsilon)$, and $\mathrm{Hom}_{alg}(A^\Upsilon, \mathbb{C}^\times) = H/H(\Upsilon)$, the coinvariants of Υ in H.*

Define the groups:

$$\overline{X^\vee} = X^\vee/(\mathbb{Z}\Delta^\vee) = \mathrm{Hom}(Z(\check{G}), \mathbb{C}^\times) \qquad \overline{\tilde{X}^\vee} = \tilde{X}^\vee/(\mathbb{Z}\Delta^\vee) = \mathrm{Hom}(Z(\tilde{G}), \mathbb{C}^\times) \tag{2.4}$$

These are the cocenters of $\check{G}$ and $\tilde{G}$, respectively. We will write

$$\check{Z} = Z(\check{G}) \qquad \tilde{Z} = Z(\tilde{G}) \tag{2.5}$$

for the centers of these complex groups; because the center of Γ is trivial (*cf.* [15]), their Γ–invariants are respectively the centers $\check{Z}^\Gamma = Z({}^L\mathbf{G})$ and $\tilde{Z}^\Gamma = Z({}^L\overline{\mathbf{G}})$ of the L–groups to which they belong.

Write $X^\vee_{rad}$ for $X_*((Z(\mathbf{G}))^0)$. This lattice has the same $\mathbb{Q}$–span as $X_*((Z(\overline{\mathbf{G}}))^0)$, so we may naturally identify

$$(\tilde{X}^\vee \otimes \mathbb{Q})/(X^\vee_{rad} \otimes \mathbb{Q}) \cong X^\vee_{ad} \otimes \mathbb{Q}. \tag{2.6}$$

Under the quotient map $\tilde{X}^\vee \otimes \mathbb{Q} \to X^\vee_{ad} \otimes \mathbb{Q}$, the image of $\tilde{X}^\vee$ is contained in $X^\vee_{ad}$, because every element of $\tilde{X}^\vee$ pairs with all the roots as an integer. Thereby we get a map $\tilde{X}^\vee \to X^\vee_{ad}$.

The basic tool for interpreting $H^1(k, \mathbf{G})$ in terms of based root data is a theorem of Kottwitz:

Theorem 2.2 (Kottwitz, [6], [7]) *Let G be a quasi–split connected reductive group over k. Then there is a natural bijection*

$$\xi_G : \mathrm{Hom}(\pi_0(\check{Z}^\Gamma), \mathbb{C}^\times) \to H^1(k, \mathbf{G}) \tag{2.7}$$

in which the trivial homomorphism is sent to the base point of $H^1(k, \mathbf{G})$. For quasi–split structures on $\mathbf{G}$ and $\mathbf{G}_{ad}$ corresponding to the same automorphism of the Dynkin diagram, the diagram

$$\begin{tikzcd} \mathrm{Hom}(\pi_0(Z({}^L\mathbf{G})), \mathbb{C}^\times) \arrow[r, "\mathrm{Res}"] \arrow[d, "\xi_G"] & \mathrm{Hom}(\pi_0(Z({}^L\mathbf{G}_{ad})), \mathbb{C}^\times) \arrow[d, "\xi_{G_{ad}}"] \\ H^1(k, \mathbf{G}) \arrow[r] & H^1(k, \mathbf{G}_{ad}) \end{tikzcd} \tag{2.8}$$

is commutative.

The L–group ${}^L\mathbf{G}$ determines an inner class of rational forms; write $\mathbf{G}$ for the quasi–split form in this class with a pinning in which the Galois action is given by equations (1.4)–(1.5), and $\mathbf{G}_\tau$ for the inner form of $\mathbf{G}$ corresponding to the cocycle $\tau \in Z^1(k, \mathbf{G}_{ad})$. For each inner form $\mathbf{G}_\tau$, we want to classify the representations of the group $\mathbf{G}_\tau(k)$ of k–rational points.

Let ϕ be a Langlands parameter, and let $\check{G}^\phi$ denote the centralizer of its image in $\check{G}$. Let $\pi_0(\check{G}^\phi)$ be its component group. Write S^{pure}_ϕ for the set $\mathrm{Hom}(\pi_0(\check{G}^\phi), \mathbb{C}^\times)$. We call S^{pure}_ϕ the set of *pure extensions* of the Langlands parameter ϕ.

Through the natural map $\pi_0(Z({}^L\mathbf{G})) \to \pi_0(\check{G}^\phi)$, we get a restriction homomorphism

$$r^{pure} : S^{pure}_\phi \to \mathrm{Hom}(\pi_0(Z({}^L\mathbf{G})), \mathbb{C}^\times). \tag{2.9}$$

Conjecturally, if $\tau \in \mathrm{Hom}(\pi_0(Z({}^L\mathbf{G})), \mathbb{C}^\times)$, then the fiber of r^{pure} over τ is supposed to parameterize the set of representations of $\mathbf{G}_{\mathrm{Res} \circ \xi_G(\tau)}(k)$ associated to the Langlands parameter ϕ. For example, for groups with connected center, Lusztig has explicitly constructed unipotent representations for each pure extension of an unramified Langlands parameter. [9, 10]

We revise this setting by replacing S_ϕ^{pure} with a different set that maps naturally into $H^1(k, \mathbf{G}_{ad})$, in the hope of parameterizing representations of all the forms in our inner class. Recall the covering $\pi : {}^L\overline{\mathbf{G}} \to {}^L\mathbf{G}$ (equation (2.3)). Let $R_\phi = \pi^{-1}(\check{G}^\phi)$. We call the set of irreducible representations

$$S_\phi = \mathrm{Irr}(\pi_0(R_\phi)) \tag{2.10}$$

the set of *extensions* of the Langlands parameter ϕ.

When $\mathbf{G}$ is split, the Galois action on $\check{G}$ defined by ${}^L\mathbf{G}$ is trivial, so that the natural map $\tilde{Z}^\Gamma \to \check{Z}^\Gamma$ is a surjection. When $\mathbf{G}$ is assumed, in addition, to be semisimple, Theorem 2.2 shows that the mapping from pure rational forms to rational forms $H^1(k, \mathbf{G}) \to H^1(k, \mathbf{G}_{ad})$ is injective. This need not be the case for non–split inner classes.

2.2 Stable rational forms

Although there will be a natural map $S_\phi \to H^1(k, \mathbf{G}_{ad})$, the fibers of this map will turn out too big to use in our construction of Deligne–Lusztig representations below. To make a one to one correspondence, we will introduce a set of so–called *stable rational forms*, to be denoted $R(k, \mathbf{G})$, and factor the map $S_\phi \to H^1(k, \mathbf{G}_{ad})$ as

$$S_\phi \to R(k, \mathbf{G}) \to H^1(k, \mathbf{G}_{ad}). \tag{2.11}$$

The map $R(k, \mathbf{G}) \to H^1(k, \mathbf{G}_{ad})$ will be a surjection, independent of ϕ.

The action of Γ_k on the based root datum of $\mathbf{G}$ factors through a finite quotient, generated by finite–order automorphisms. The Kottwitz isomorphism (Theorem 2.2), with Proposition 2.1, has identified

$$H^1(k, \mathbf{G}_{ad}) \cong \mathrm{Hom}(\pi_0(Z({}^L\mathbf{G}_{ad}{}^0)^\Gamma), \mathbb{C}^\times) \cong \left(\overline{X}^\vee_{ad}/\overline{X}^\vee_{ad}(\Gamma)\right)_{tor}. \tag{2.12}$$

Through this equation and one like it for $\overline{\mathbf{G}}$, the map $\tilde{X}^\vee \to X^\vee_{ad}$ identifies $H^1(k, \overline{\mathbf{G}})$ as a cover of $H^1(k, \mathbf{G}_{ad})$. (Recall from equation (2.4) that $\overline{\tilde{X}^\vee}$ is the cocenter of $\tilde{G}$.) Our stable rational forms replace the coboundary relation in $H^1(k, \mathbf{G}_{ad})$ with one that reflects conjugacy in $\mathbf{G}$:

Definition 2.3 *Let* $\mathbf{G}$ *be a* k*–group, split over* K*. The set of* stable rational forms *is the set*

$$R(k, \mathbf{G}) = \mathrm{Hom}(\pi_0(\pi^{-1}(Z({}^L\mathbf{G}))), \mathbb{C}^\times). \tag{2.13}$$

For any Langlands parameter ϕ of $\mathbf{G}$, the group R_ϕ includes $\pi^{-1}(Z({}^L\mathbf{G})) = \pi^{-1}(Z(\check{G})^\Gamma)$ as a subgroup. Therefore there is a natural restriction map $S_\phi \to R(k, \mathbf{G})$.

Proposition 2.4 *The map* $R(k, \mathbf{G}) \to H^1(k, \mathbf{G}_{ad})$ *is surjective.*

Proof. Recall that we have constructed a map $\tilde{X}^\vee \to X^\vee_{ad}$, via equation (2.6). Proposition 2.1 gives a natural isomorphism

$$R(k, \mathbf{G}) \cong \left(\overline{\tilde{X}^\vee}/\overline{X^\vee}(\Gamma)\right)_{tor}. \tag{2.14}$$

Let ζ be the natural map $\overline{\tilde{X}^\vee} \to \overline{\tilde{X}^\vee}/\overline{X^\vee}(\Gamma)$. Put

$$R_{lift}(k, \mathbf{G}) := \{r \in \overline{\tilde{X}^\vee} : \zeta(r) \in R(k, \mathbf{G})\}. \tag{2.15}$$

Suppose $r \in R_{lift}(k, \mathbf{G}_{ad})$. If $m \in \mathbb{N}$ is such that $mr \in \overline{X^\vee}_{ad}(\Gamma)$, then there exists $m' \in \mathbb{N}$ such that $m'r \in \overline{X^\vee}(\Gamma)$. Therefore, the inclusion of $\tilde{X}^\vee$ in $X^\vee_{ad}$ sends r to an element of $R_{lift}(k, \mathbf{G})$. Thus, the natural map $R(k, \mathbf{G}) \to R(k, \mathbf{G}_{ad})$ is surjective. Via equation (2.12), the second set is simply $H^1(k, \mathbf{G}_{ad})$. $\square$

We will use stable rational forms in our construction in a way similar to that in which Adams, Barbasch, and Vogan use so–called rigid rational forms in the local Langlands correspondence for real groups [1]. Stable rational forms behave more simply than the set of strong rational forms envisioned by Vogan in [17], Problem 9.3—in particular, we have avoided introducing a pro–finite covering of $\check{G}$—and unlike the set of rigid rational forms Vogan introduces, $R(k, \mathbf{G})$ is a finite set even when the center of $\mathbf{G}$ is infinite.

We propose that stable rational forms be used in the local Langlands conjecture as follows. The complex dual group $\check{G}$ acts on the set of pairs (ϕ, ρ) where ϕ is a Langlands parameter and $\rho \in S_\phi$, with $g \in \check{G}$ sending a pair (ϕ, ρ) to $({}^g\phi, g \cdot \rho) = (\mathrm{Ad}(g) \circ \phi, \mathrm{Ad}(\tilde{g})^* \rho)$, where $\tilde{g} \in \tilde{G}$ is any preimage of g. (Since $\tilde{G} \to \check{G}$ is a central isogeny, any choice of $\tilde{g}$ yields the same element of $S_{\mathrm{Ad}(g) \circ \phi}$, and the map $S_\phi \to R(k, \mathbf{G})$ factors through this action on S_ϕ.)

Conjecture 2.5 *Suppose $\mathbf{G}$ is a quasi–split reductive k–group, split over K. Let $\tau \in Z^1(k, \mathbf{G}_{ad})$, and $\sigma \in R(k, \mathbf{G})$ mapping to $[\tau] \in H^1(k, \mathbf{G}_{ad})$. The irreducible admissible representations of the inner form $\mathbf{G}_\tau$ are in natural one–to–one correspondence with $\check{G}$–orbits of pairs (ϕ, ρ), where $\phi : \mathcal{W}' \to {}^L\mathbf{G}$ is a Langlands parameter, and $\rho \in S_\phi$ such that ρ maps to σ.*

A pure rational form $\tau^{pure} \in H^1(k, \mathbf{G})$ determines a stable rational form $\pi^*(\tau^{pure})$ in $R(k, \mathbf{G})$, via the map $\pi^* : \mathrm{Hom}(\pi_0(Z({}^L\mathbf{G})), \mathbb{C}^\times) \to \mathrm{Hom}(\pi_0(\pi^{-1}(Z({}^L\mathbf{G}))), \mathbb{C}^\times)$. It is clear that the fiber $S_\phi(\pi^*(\tau^{pure}))$ is in bijection with $S_\phi^{pure}(\tau^{pure})$.

Proposition 2.6 *If $\tau_1, \tau_2 \in R(k, \mathbf{G})$ lie over the same rational form $\tau \in H^1(k, \mathbf{G}_{ad})$, and ϕ is any Langlands parameter for $\mathbf{G}$, then the fibers $S_\phi(\tau_1)$ and $S_\phi(\tau_2)$ are in natural bijection.*

Proof. View τ_1 and τ_2 as elements of $\mathrm{Hom}(\pi^{-1}(Z({}^L\mathbf{G})), \mathbb{C}^\times)$. The map $R(k, \mathbf{G}) \to H^1(k, \mathbf{G}_{ad})$ is the restriction to $[\tilde{G}, \tilde{G}] \cap \pi^{-1}(Z({}^L\mathbf{G}))$. The product $\tau_2\tau_1^{-1}$ is trivial on this subgroup, and gives a homomorphism

$$\kappa : \pi^{-1}(Z({}^L\mathbf{G}))/\left([\tilde{G}, \tilde{G}] \cap \pi^{-1}(Z({}^L\mathbf{G}))\right) \to \mathbb{C}^\times. \tag{2.16}$$

The quotient on the left is isomorphic to $R_\phi/\left([\tilde{G}, \tilde{G}] \cap \pi^{-1}(Z({}^L\mathbf{G}))\right)$. Thus the map $\rho \to \rho\kappa$ gives a map $S_\phi \to S_\phi$ sending the fiber $S_\phi(\tau_1) \to S_\phi(\tau_2)$, as desired. □

In Chapter 3, we will verify that $\check{G}$–orbits of pairs (ϕ, ρ), where $\phi : \mathcal{W}' \to {}^L\mathbf{G}$ is a tame regular semisimple elliptic Langlands parameter, and $\rho \in S_\phi$ such that ρ maps to σ, are in natural one–to–one correspondence with the class of Deligne–Lusztig representations, giving evidence for our formulation of Conjecture 2.5.

2.3 Unramified groups

Now assume that $\mathbf{G}$ is a quasi–split k–group that splits over $K = k^{unr}$. In this situation, we may describe $R(k, \mathbf{G})$ in terms of group cohomology.

Let Fr be a generator for $\Gamma_{K/k}$. Through the quasi–split structure on $\mathbf{G}$, Fr acts on X and $X^\vee$ with finite order, through a finite–order automorphism we denote F. For the cyclic action on X, for example, we have $X(\langle F\rangle) = (1 - F)X$, and the group of coinvariants is $X/(1 - F)X$.

When R is a discrete group with a continuous action of an infinite topologically cyclic group $C = \langle\tau\rangle$ (for example, the unramified Galois group $\Gamma_{K/k} = \langle F\rangle$), we often will view cocycles in $Z^1(C, R)$ as elements in R:

Lemma 2.7 ([3], Section 2.1) *Let R be a discrete group with a continuous action of an infinite topologically cyclic group $C = \langle\tau\rangle$. Let*

$$A_R^n = \{g \in R : g \cdot \tau(g) \cdot \dots \cdot \tau^{n-1}(g) = 1\}. \tag{2.17}$$

Then evaluation at τ defines a bijection

$$Z^1(C,R) \to \cup_{n\geq 1} A_R^n \tag{2.18}$$

in which $\alpha, \beta \in Z^1(C,R)$ are cohomologous if and only if the corresponding elements $g, h \in R$ are τ–conjugate, i.e. *there exists $x \in R$ such that $h = xg\tau(x^{-1})$.*

When we take this viewpoint, if g is an element of R regarded as a cocycle, we will write $[g]$ for the cohomology class it represents.

Sometimes we have a further interpretation:

Lemma 2.8 ([3], Lemma 2.3.1) *If, furthermore, R is a finitely generated abelian group, then*

$$Z^1(C,R) = \{r \in R : mr \in (1-\tau)R \text{ for some } m \geq 1\}.$$

Under this interpretation, we have:

$$\begin{aligned} H^1(k,\mathbf{G}_{ad}) &= Z^1(K/k,\overline{X^\vee}_{ad}) &(2.19)\\ R_{lift}(k,\mathbf{G}) &= Z^1(K/k,\overline{\tilde{X}^\vee}) &(2.20)\\ R(k,\mathbf{G}) &= Z^1(K/k,\overline{\tilde{X}^\vee})/B^1(K/k,\overline{X^\vee}). &(2.21)\end{aligned}$$

This point of view will be useful in the next chapter.

Chapter 3

Deligne–Lusztig Representations

3.1 A Class of Depth Zero Supercuspidal Representations

Definition 3.1 *Let $\phi : \mathcal{W}' \to {}^L\mathbf{G}$ be a Langlands parameter, and say it determines $\alpha_\phi \in Z^1(\mathcal{W}, \check{G})$ and $N_\phi \in \check{\mathfrak{g}}$. We say that ϕ is* tame *if $\phi(\gamma) = \gamma$ for all $\gamma \in \mathcal{I}^+$. If ϕ is tame, we say that it is* regular semisimple *if the centralizer $Z_{\check{G}}(\alpha_\phi(\mathcal{I}))$ is a maximal torus of $\check{G}$. We say that ϕ is* elliptic *if the identity component of $\check{G}^\phi$ is contained in $\check{Z} = Z(\check{G})$.*

Recently, DeBacker and Reeder [3] have associated a depth zero supercuspidal representation of a pure rational form to every pure extension of a tame, regular semisimple, elliptic Langlands parameter.

The supercuspidal representations DeBacker and Reeder construct are precisely those compactly induced modulo center from unramified twists of the inflation of a Deligne–Lusztig representation $R_{S,\theta}$ to a parahoric subgroup P over k from its reductive quotient $\mathsf{P} = P/P_+$, where (S, θ) is a representative of any geometric conjugacy class of the finite group P such that S is a minisotropic maximal $\mathfrak{f}$–torus of P and $\theta \in \mathrm{Hom}(S(\mathfrak{f}), \mathbb{C}^\times)$ is in general position. We will call this class the class of representations of *Deligne–Lusztig type.* The Deligne–Lusztig representations are some depth zero, supercuspidal ([3], Lemma 4.5.1), irreducible representations of $\mathbf{G}_\tau(k)$. We will parameterize the same class of representations for non–pure inner forms.

3.2 Some Weyl groups

We begin by comparing the setup of [3] in the cases of $\mathbf{G}$, $\overline{\mathbf{G}}$, and $\mathbf{G}_{ad}$. The Frobenius element in $\Gamma_{K/k}$ acts on the group of K–points $G = \mathbf{G}(K)$ through the quasi–split structure by an action we denote F, and we may take the Cartan–Borel pair $\mathbf{T} \subset \mathbf{B}$ so that $\mathbf{T}$ and $\mathbf{B}$ are F–stable and $\mathbf{T}$ is K–split; write $T = \mathbf{T}(K)$. Write $\eta : G \to \overline{G}$ for the restriction of the covering map $\mathbf{G} \to \overline{\mathbf{G}}$; the map η need not be surjective. Let $\overline{T}$ be the maximal torus in $\overline{G}$ containing $\eta(T)$. Let $X_*(T) = \mathrm{Hom}(K^\times, T)$, and write $\mathcal{A}(G,T) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ for the apartment of T in G over K. Because T is K–split, we have a natural isomorphism $X^\vee \to X_*(T)$. The apartment is naturally embedded in the building $\mathcal{B}(G)$ (see Tits [16]), which carries natural actions by G and $\overline{G}$. Additionally, $N_G(T)$, $N_{\overline{G}}(T)$, W, and W_{ad} act on $X_*(T)$ and hence on $\mathcal{A}(G,T)$, preserving the simplicial structure. There is also an action by $\Gamma_{K/k}$ on both $\mathcal{A}(G,T)$ and $\mathcal{B}(G)$; again we denote the action of the Frobenius generator by F. The map η induces a bijection of affine spaces $\eta_* : \mathcal{A}(G,T) \to \mathcal{A}(\overline{G},\overline{T})$ with the same the simplicial structure. However, the orbits of $N_G(T)$ on $\mathcal{A}(G,T)$ may be smaller than the orbits of $N_{\overline{G}}(\overline{T})$ on $\mathcal{A}(\overline{G},\overline{T})$.

Let $\mathcal{O}_K$ be the ring of integers in K, and put $T^0 = T(\mathcal{O}_K)$, $T^0_{ad} = T_{ad}(\mathcal{O}_K)$, and $\overline{T}^0 = \overline{T}(\mathcal{O}_K)$. We have affine Weyl groups for $\mathbf{G}$, $\mathbf{G}_{ad}$, and $\overline{\mathbf{G}}$, which we may compute over K:

$$W = N_G(T)/T^0 \qquad W_{ad} = N_{G_{ad}}(T_{ad})/T^0_{ad} \qquad \tilde{W} = N_{\overline{G}}/\overline{T}^0 \tag{3.1}$$

The finite Weyl group $\overline{W} = N_{G_{ad}}(T_{ad})/T_{ad}$ appears as a quotient of each of these groups, via the maps $W \to \tilde{W} \to W_{ad}$ induced by the reductions $G \to \overline{G} \to G_{ad}$. If

$a, b \in \tilde{W}$ (or W or W_{ad}), write $a * b$ for $abF(a^{-1})$.

Let $C \subset \mathcal{A}(G,T)$ be an F–stable alcove, and let $\mathbf{o}$ be an F–stable hyperspecial vertex in its closure. We use the same notation for the images of $\mathbf{o}$ and C in $\mathcal{A}(\overline{G},\overline{T})$ and $\mathcal{A}(G_{ad}, T_{ad})$. Write $W^{\mathbf{o}}$ for the stabilizer of $\mathbf{o}$ in W; the natural inclusion maps induce isomorphisms with the stabilizer of $\mathbf{o}$ in W_{ad} or $\tilde{W}$, and these groups are canonically isomorphic to $\overline{W}$. Fix lifts $\dot{w} \in N_G(T)$ of each $\overline{w} \in W^{\mathbf{o}}$. Then $\eta(\dot{w})$ is a lift of $\overline{w}$ up to $N_{\overline{G}}(\overline{T})$.

Write W_C for the subgroup of W generated by reflections in the walls of C, and W^C for the stabilizer of C in W (*not* the pointwise stabilizer). Similarly, write $(W_{ad})_C$, W_{ad}^C, $\tilde{W}_C$, and $\tilde{W}^C$, but note that $W_C \cong (W_{ad})_C \cong \tilde{W}_C$. Thus we have decompositions:

$$W = X^\vee \rtimes W^{\mathbf{o}} \qquad W_{ad} = X_{ad}^\vee \rtimes W^{\mathbf{o}} \qquad \tilde{W} = \tilde{X}^\vee \rtimes W^{\mathbf{o}} \tag{3.2}$$

$$W = W_C \rtimes W^C \qquad W_{ad} = W_C \rtimes W_{ad}^C \qquad \tilde{W} = W_C \rtimes \tilde{W}^C \tag{3.3}$$

On the dual side, recall the Cartan–Borel pair $\check{T} \subset \check{B}$ in $\check{G}$ fixed when defining the semidirect product structure on the L–group ${}^L\mathbf{G}$. Let $\check{W} = N_{\check{G}}(\check{T})/\check{T}$. Via the transpose map, there is a canonical isomorphism $\overline{W} \to \check{W}$. Inside ${}^L\mathbf{G}$, Γ acts on $\check{G}$ by conjugation; write F for the automorphism by which the Frobenius generator acts. This action stabilizes $\check{T}$ and $\check{B}$ by assumption.

3.3 Construction

Let $\mathbf{G}$ be a quasi–split reductive k–group, split over K. Let ϕ be a tame regular semisimple elliptic Langlands parameter, and $\rho \in S_\phi$. Suppose ρ maps to $\sigma \in R(k, \mathbf{G})$, in the preimage of $[\tau] \in H^1(k, \mathbf{G}_{ad})$. We construct a Deligne–Lusztig representation $\pi = \pi_{(\phi,\rho)}$ as follows.

3.3.1 Computation of S_ϕ

First, we invoke DeBacker and Reeder's interpretation of a tame regular semisimple elliptic Langlands parameter ϕ. Recall from Chapter 1 the cocycle α_ϕ and the nilpotent element $N_\phi \in \check{\mathfrak{g}}$ attached to ϕ. Because $Z_{\check{G}}(\alpha_\phi(\mathcal{I}))$ is a torus, $N_\phi = 0$. Let $\mathrm{Fr} \in \mathcal{W}$ be a Frobenius element. The continuity of ϕ forces $\alpha_\phi|_{\mathcal{I}}$ to factor through a finite cyclic quotient. Let $n = \alpha_\phi(\mathrm{Fr})$. The semidirect product structure on $\mathcal{W}$ (see equation (1.12)) forces $n \in N_{\check{G}}(Z_{\check{G}}(\alpha_\phi(\mathcal{I})))$; by the assumption that ϕ is regular, this is the normalizer of a maximal torus in $\check{G}$. We say that ϕ is in *good position* if $\alpha_\phi(\mathcal{I}) \subset \check{T}$; any tame regular semisimple ϕ is conjugate to one in good position. Assume ϕ is in good position. Then n determines an element $\check{w} \in \check{W}$, and the centralizer $\check{G}^\phi = \check{T}^{\check{w}F}$. The assumption that ϕ is elliptic implies that the identity component of $\check{G}^\phi$ is contained in $Z(\check{G})$, so that $\check{w}$ is an elliptic element of $\check{W}$. The set of pure extensions S_ϕ^{pure} is the set of characters of $\pi_0(\check{G}^\phi) = \pi_0(\check{T}^{\check{w}F})$.

Under the transpose isomorphism $\check{W} \cong \overline{W}$ and the isomorphism $\overline{W} \cong W^{\mathbf{o}}$, let $\check{w}$ correspond to $w \in W^{\mathbf{o}}$. Since $X^\vee = \mathrm{Hom}(\check{T}, \mathbb{C}^\times)$, Proposition 2.1 yields $S_\phi^{pure} = \mathrm{Hom}(\pi_0(\check{T}^{\check{w}F}), \mathbb{C}^\times) = (X^\vee/(1-wF)X^\vee)_{tor}$. By Lemmas 2.7 and 2.8, we also have $S_\phi^{pure} = H^1(\langle wF\rangle, X^\vee)$. Observe that $Z^1(\langle wF\rangle, X^\vee)$ is the set of elements in $X^\vee$ whose image in $(X^\vee/(1-wF)X^\vee)$ has finite order. (In [3], this set of cocycles is denoted X_w.)

Using the set $R_\phi = \{t \in \tilde{T} : \pi(t) \in \check{T}^{wF}\}$, Proposition 2.1 similarly allows us to identify

$$S_\phi = \left(\tilde{X}^\vee/(1-wF)X^\vee\right)_{tor}. \tag{3.4}$$

Because $(1-wF)X^\vee$ has finite index in $(1-wF)\tilde{X}^\vee$, we may identify the right hand side as $Z^1(\langle wF\rangle, \tilde{X}^\vee)/B^1(\langle wF\rangle, X^\vee)$.

3.3.2 Computation of $R(k, \mathbf{G})$

Henceforth we will regard $X^\vee$ as a subgroup of W via the map $\lambda \to t_\lambda$, sending a cocharacter λ to translation by λ. Similarly, we will regard $\tilde{X}^\vee$ as a subgroup of $\tilde{W}$. Because the coroot lattice is $\tilde{X}^\vee \cap W_C = \mathbb{Z}\Delta^\vee$, we have $\overline{\tilde{X}^\vee} = \tilde{X}^\vee / \tilde{X}^\vee \cap W_C$. The splitting $\tilde{W} = W_C \rtimes \tilde{W}^C$ yields a homomorphism $\tilde{W} \to \tilde{W}^C$; because we also have $\tilde{W} = \tilde{X}^\vee \rtimes W^\mathbf{o}$ and $W^\mathbf{o} \subset W_C$, it induces an isomorphism $\overline{\tilde{X}^\vee} \to \tilde{W}^C$.

This identification, and its analogue for $\overline{X^\vee}$, give a natural bijection

$$R(k, \mathbf{G}) = Z^1(K/k, \tilde{W}^C)/B^1(K/k, W^C). \tag{3.5}$$

Write $\overline{N}$ for $N_{\overline{G}}(\overline{T})$, and $\overline{N}^C$ for the preimage of $\tilde{W}^C$ in $\overline{N}$. By Lemma 2.2.3 of [3], the map

$$Z^1(K/k, \overline{N}^C) \to Z^1(K/k, \tilde{W}^C) \tag{3.6}$$

is surjective. For each $b \in Z^1(K/k, \tilde{W}^C)$, choose a lift $\overline{b} \in Z^1(K/k, \overline{N}^C)$; this cocycle specifies an inner form of $\mathbf{G}$. Write F_b for the automorphism $\mathrm{Ad}(\overline{b}) \circ F$ of G, $\overline{G}$, or G_{ad}, and also for the automorphism bF of $X^\vee$, $\tilde{X}^\vee$, $X^\vee_{ad}$ or of the associated apartments.

More generally, for $x \in \tilde{W}$, we write F_x for the composite xF inside the automorphism group of $X^\vee$, $\tilde{X}^\vee$, $X^\vee_{ad}$, or the associated apartments, and for the map $\mathrm{Ad}(x) \circ F$ on the associated tori. We will only use this notation when F_x is a finite–order automorphism of all of these objects. We will write $Z^1(F_x, -)$ in place of $Z^1(\langle F_x \rangle, -)$.

3.3.3 Assigning representations to cocycles

We now construct a Deligne–Lusztig representation for each $\lambda \in Z^1(F_w, \tilde{X}^\vee)$.

Since $\lambda \in Z^1(F_w, \tilde{X}^\vee)$, the element $t_\lambda w F \in \tilde{X}^\vee \rtimes \langle \mathrm{Fr} \rangle$ has finite order, and hence fixes a point $x_0 \in \mathcal{A}(G, T)$. Because C is a fundamental domain for the action of W_C, there exists $r \in W_C$ such that $rx_0 \in \overline{C}$, the closure of the alcove C. Factor $r = r_C r^\mathbf{o}$, where $r_C \in X^\vee \cap W_C$ and $r^\mathbf{o} \in W^\mathbf{o}$. The point $x = rx_0$ is in the closure of C; let J be the facet of $\mathcal{A}(G,T)$ in which it lies. The transformation $r(t_\lambda wF)r^{-1}$ fixes the point x and stabilizes the facet J.

Under the factorization $\tilde{W} = W_C \rtimes \tilde{W}^C$, suppose

$$rt_\lambda wF(r^{-1}) = zb \tag{3.7}$$

with $z \in W_C$ and $b \in \tilde{W}^C$. Then

$$x = r(t_\lambda wF)r^{-1}x = (rt_\lambda wF(r^{-1})F)x = (zbF)x = z(bFx).$$

We have both $bFx \in \overline{C}$ and $z(bFx) \in \overline{C}$. Since C is a fundamental domain for the action of W_C, this implies that $bFx = z(bFx) = x$. Thus, the element x is F_b–fixed and the facet J is F_b–stable. We conclude that J^{F_b} is a nonempty facet in the apartment $\mathcal{A}(G,T)^{F_b}$ of $\mathbf{G}_b$ over k.

The map $S_\phi \to R(k,\mathbf{G})$ is induced from the map $\tilde{W} \to \tilde{W}^C$ sending t_λ to b. Since W_C is normal in $\tilde{W}$, the map does not depend on the choice of r; in particular it does not depend on the representative λ of ρ.

We have $z \in W_J$, the subgroup of W_C generated by reflections through hyperplanes containing J. The group W_J may be naturally identified with the Weyl group $W(\mathsf{G}_J, \mathsf{T})$ of the finite group G_J. Because J is F_b–stable, the Frobenius automorphism F_b gives an $\mathfrak{f}$–structure on the finite reductive group $\mathsf{G}_J = G_J/G_J^+$. By the Lang–Steinberg theorem, there exists $p \in G_J$ such that $p^{-1}F_b(p) = z$. Then $S = \mathrm{Ad}(p)T$ is an F_b–stable maximal k–torus in G_J. Write S_{F_b} for S with the k–structure given by F_b.

The torus S_{F_b} is minisotropic because the image of zb in $\overline{W}$ is elliptic. Indeed, suppose $\omega \in X^{\vee F}$ such that $\mathrm{Ad}(p)\circ\omega \in \mathrm{Hom}_k(\overline{k}^\times, S_{F_b})$. Then $F_b(\mathrm{Ad}(p)\circ\omega(F^{-1}a)) = \mathrm{Ad}(p)\circ\omega(a)$ for all $a \in \overline{k}^\times$. Conjugating by p^{-1}, this amounts to $\mathrm{Ad}(z)F_b(\omega(F^{-1}a)) =$

$\omega(a)$. But

$$\begin{aligned}
\mathrm{Ad}(z)F_b(\omega(F^{-1}a)) &= \mathrm{Ad}(zb)\omega(a) \\
&= \mathrm{Ad}(rt_\lambda wF(r^{-1}))\omega(a) \\
&= \mathrm{Ad}(rt_\lambda F(r^{-1})) \circ \mathrm{Ad}(F(r)wF(r^{-1}))(\omega(a)) \\
&= \mathrm{Ad}({}^{F(r)}w)(\omega(a)) \qquad (3.8)
\end{aligned}$$

where the last line uses the fact that $rt_\lambda F(r^{-1}) \in \overline{T}$ since F and $N_G(T)$ preserve $\overline{T}$. We have deduced that ${}^{F(r)w}\omega = \omega$; since w (and hence ${}^{F(r)}w$) is elliptic, this forces ${}^{F(r^{-1})}\omega \in X_*((Z(G))^0)$, so $\omega \in X_*((Z(G))^0)$, as desired.

Factor $zb = t_{\lambda_0}w_0$ with $t_{\lambda_0} \in \tilde{X}^\vee$ and $w_0 \in W^{\mathbf{o}}$. Write $T_{F_{w_0}}$ for T with the k–structure given by F_{w_0}. Then $\mathrm{Ad}(p) : T_{F_{w_0}} \to S_{F_b}$ is an isomorphism of k–groups: for $s \in S$, $s = \mathrm{Ad}(p)t$, we have

$$F_b(\mathrm{Ad}(p)t) = F_b(ptp^{-1}) = pp^{-1}F_b(ptp^{-1})pp^{-1} = \mathrm{Ad}(p)(\mathrm{Ad}(z)F_b(t)) = \mathrm{Ad}(p)(F_{w_0}(t)).$$

In particular, there are natural isomorphisms

$$^LS_{F_b} \cong {}^LT_{F_{w_0}} \cong \check{T} \rtimes_{F_{\check{w}_0}} \Gamma \qquad (3.9)$$

where $\check{w}_0 \in \check{W}$ is dual to $w_0 \in W^{\mathbf{o}}$.

The local Langlands correspondence for abelian groups [8] gives a bijection between tame Langlands parameters of a torus and its depth zero characters; the details we need are in [3], section 4.3 in the semisimple case, and in [13] reductive groups. First, we need to extract from ϕ a Langlands parameter ϕ^T for $\mathbf{T}_{F_{w_0}}$. Let $\check{G}_{ab} = \check{G}/[\check{G}, \check{G}]$ be the maximal abelian quotient of $\check{G}$.

Lemma 3.2 ([13]) *The canonical map*

$$\check{T}/(1 - \check{w}_0F)\check{T} \to \check{G}_{ab}/(1 - F)\check{G}_{ab} \qquad (3.10)$$

is a bijection. The left hand side is in natural bijection with the equivalence classes of Langlands parameters ϕ^T for $\mathbf{T}_{F_{w_0}}$ such that $\alpha_{\phi^T}\big|_{\mathcal{I}} = \alpha_{\phi}\big|_{\mathcal{I}}$ inside $\check{T}$.

Take $\check{r} \in N_{\check{G}}(\check{T})$ such that the image of $\check{r}$ in $\check{W}$ is dual to the image of r in $\overline{W}$. Choose ϕ^T to correspond to the image of $\mathrm{Ad}(\check{r})(\phi(\mathrm{Fr}))$ in the right hand side of equation (3.10).

Let $\mathsf{S} = S/S \cap G_J^+$. Our choice of uniformizer ϖ for k gives a decomposition $T^{F_{w_0}} = (T^{F_{w_0}})^0 \times (X^\vee)^{F_{w_0}}$ into compact and hyperbolic parts; thereby we get a decomposition

$$\mathrm{Hom}(T^{F_{w_0}}, \mathbb{C}^\times) = \mathrm{Hom}((T^{F_{w_0}})^0, \mathbb{C}^\times) \otimes \mathrm{Hom}(T^{F_{w_0}}/(T^{F_{w_0}})^0, \mathbb{C}^\times) \tag{3.11}$$

of depth zero characters into characters of S^{F_b} and unramified characters of the k–torus $T^{F_{w_0}}$. The tame Langlands correspondence for abelian groups produces a depth zero character of $T^{F_{w_0}}$ from ϕ^T; suppose its decomposition above is $\theta^T_{\lambda,r} \otimes \chi^T_{\lambda,r}$. Then $\mathrm{Ad}(p)^*(\theta^T_{\lambda,r})$ factors to give a character $\theta_{\lambda,r}$ of the torus S^{F_b} of the finite group $\mathsf{G}_J^{F_b}$. This character is in general position because ϕ was regular semisimple. Take $\chi_{\lambda,r}$ to be the unramified character $\mathrm{Ad}(p)^*(\chi^T_{\lambda,r})$ of $T^{F_{w_0}}$. Because w_0 is elliptic, T_{w_0} is an minisotropic torus in G, so this unramified character must factor to a character of the center $Z_b(k)$ of $G_b(k)$.

The representation $\pi_{\lambda,r}$ we associate to λ and our choice of r is

$$\pi_{\lambda,r} := \mathrm{Ind}_{Z_b(k)(G_J)_b(k)}^{G_b(k)} \chi_{\lambda,r} \otimes \varepsilon((G_J)_{F_b}, S_{F_b}) R^{(G_J)_b}_{(S,\theta_{\lambda,r})} \tag{3.12}$$

where $\varepsilon(G_{J F_b}, S_{F_b})$ is the sign

$$\varepsilon(G_J, S) = (-1)^{\mathrm{rank}_k((G_J)_{F_b}) - \mathrm{rank}_k(S_{F_b})} \tag{3.13}$$

and $R^{(G_J)_b}_{(S,\theta_{\lambda,r})}$ is the Deligne–Lusztig generalized character associated to the geometric conjugacy class containing $(\mathsf{S}, \theta_{\lambda,r})$ for the finite reductive group G_J/G_J^+; the product is the character of an irreducible representation of $\mathsf{G}_J^{F_b}$, which we inflate to $G_J^{F_b}$. The proof of Theorem 4.5.1 of [3] goes through to show that $\pi_{\lambda,r}$ is irreducible.

3.3.4 Independence of choices

Now we show that the representation $\pi_{\lambda,r}$ is independent of our choice of r in the preceding subsection. Suppose that $r' \in W_C$ and $r'x_0 \in \overline{C}$. Then $r' \in W_J \cdot r$. Via $\tilde{W} = W_C \rtimes \tilde{W}^C$, factor $r'(t_\lambda w)F(r'^{-1}) = z'b'$. Because W_C is normal in $\tilde{W}$ and $r' \in W_J r$, $b' = b$. Take $p' \in G_J$ such that $p'^{-1}F_b(p') = z'$. Let $S' = \mathrm{Ad}(p')T$, again a minisotropic, F_b–stable maximal k–torus. It is clear from our construction that $\mathrm{Ad}(p'p^{-1})^*$ sends the geometric conjugacy class (S,θ) constructed from r and ϕ to (S',θ'), and that $\pi_{\lambda,r} = \pi_{\lambda,r'}$.

Furthermore, the representation π_λ depends only on the class $\rho = [\lambda]$ of λ in S_ϕ. Suppose that $\lambda = \lambda' + (1 - wF)\nu$ for some $\nu \in X^\vee$. Then $t_\lambda = t_\nu t_{\lambda'} wF(t_\nu^{-1})w^{-1}$ and

$$t_\lambda wF = t_\nu(t_{\lambda'}wF)t_\nu^{-1} \tag{3.14}$$

Let x_0' be the point fixed by $t_{\lambda'}wF$. Then $x_0 = t_\nu x_0'$ is fixed by $t_\lambda wF$. Given $r' \in W_C$ with $r'x_0' \in \overline{C}$, and $r \in W_C$ such that $rx_0 \in \overline{C}$, let $r^C = r'(rt_\nu)^{-1} \in W^C$. We have $r^C bF(r^{C^{-1}}) = b'$ by equation (3.14), which establishes that these two elements have the same image in W/W_C, and the fact that C is F–stable, which shows that the left hand side belongs to W^C.

The facet J' associated to ϕ and λ' is $r^C J$. Let $\bar{b} \in Z^1(F, \overline{N}^C)$ be a lift of b. For any lift $\overline{r^C}$ of r^C, $\mathrm{Ad}(\overline{r^C})$ induces an isomorphism of k–groups between $\mathbf{G}_{\bar{b}}$ and $\mathbf{G}_{\overline{b'}}$, where $\overline{b'} = \overline{r^C}\bar{b}F(\overline{r^C}^{-1})$: for $x \in \mathbf{G}_{F_{\bar{b}}}$,

$$\mathrm{Ad}(\overline{r^C})(F_{\bar{b}}(x)) = \mathrm{Ad}(\overline{r^C}\bar{b})F(x) = \mathrm{Ad}(\overline{b'}F(\overline{r^C}))F(x) = F_{\overline{b'}}(\mathrm{Ad}(\overline{r^C})x).$$

Because $r^C J = J'$, $\mathrm{Ad}(r^C)$ sends the subgroup G_J to $G_{J'}$. Suppose $p \in G_J$ and $S = \mathrm{Ad}(p)T$ such that $p^{-1}F_b(p) = z$, with z as above. Then $p' = \mathrm{Ad}(\overline{r^C})p \in G_{J'}$

satisfies

$$\begin{aligned} p'^{-1}F_{\overline{b'}}(p') &= \overline{r^C}p^{-1}\overline{r^C}^{-1}\mathrm{Ad}(\overline{r^C}bF(\overline{r^C}^{-1}))F(\overline{r^C}p\overline{r^C}^{-1}) \\ &= \overline{r^C}p^{-1}\overline{r^C}^{-1}(\overline{r^C}\overline{b}F(p)\overline{b}^{-1}\overline{r^C}^{-1}) \\ &= {}^{\overline{r^C}}(p^{-1}F_{\overline{b}}(p)) \\ &= z'. \end{aligned}$$

We conclude that $\mathrm{Ad}(p')T$ is a torus in $G_{J'}$, of type $z' \in W_{J'}$ with respect to the $F_{\overline{b'}}$ $\mathfrak{f}$–structure on $\mathsf{G}_{J'}$.

The characters χ^T_ϕ and θ^T_ϕ constructed above depend only on ϕ and not on ρ. Thus, the pairs $(\mathsf{S}, \theta) = (\mathrm{Ad}(p)T, \mathrm{Ad}(p)^*\theta^T_\phi)$ and $(\mathsf{S}', \theta') = (\mathrm{Ad}(p')T, \mathrm{Ad}(p')^*\theta^T_\phi)$ are identified under $\mathrm{Ad}(r^C) : G_J \to G_{J'}$, and the representation $\pi_{\lambda',r'} = \pi_{\lambda,r}$. We have already shown that $\pi_{\lambda,r}$ is independent of r; we conclude that we get the same representation $\pi_{(\phi,\rho)} = \pi_{\lambda,r}$ for all choices of λ above ϕ, and all choices of r.

Finally, we consider $\check{G}$–conjugate extended Langlands parameters. Suppose that $(\phi', \rho') = ({}^g\phi, g \cdot \rho)$ for some $g \in \check{G}$, and that ϕ and ϕ' both are in good position. This forces $g \in N_{\check{G}}(\check{T})$.

Let $a \in W^{\mathbf{o}}$ be the element dual to the class of g in $\check{W}$. Say λ represents $\rho \in S_\phi$. If $\check{w} \in \check{W}$ is the element associated to ϕ, we have $\check{w}' = g * \check{w}$ associated to ϕ', and $\lambda' = {}^g\lambda$ represents $\rho' \in S_{\phi'}$. On the p–adic side, $t_{\lambda'} = {}^a t_\lambda$ and $w' = a * w$. Because $a \in W_C$ and $a(t_\lambda wF)a^{-1} = t_{\lambda'}w'F$, the facet $J \subset \overline{C}$ determined by ϕ and λ matches J' corresponding to ϕ' and λ'. From the fact that $t_\lambda wF$ and $t_{\lambda'}w'F$ are W–conjugate and that $x = x'$, the elements $z \in W_J$ and $b \in \tilde{W}^C$ determined by (ϕ, λ) and (ϕ', λ') also match. In particular, both extended Langlands parameters determine the same $w_0 \in W^{\mathbf{o}}$. We deduce that the characters χ_ϕ and $\chi_{\phi'}$ are equal, and that the geometric conjugacy classes (S, θ) and (S', θ') match, because they are determined by the downward arrows in the commuting diagram

$$\begin{tikzcd} T_w \arrow[rr, "\mathrm{Ad}(a)"] \arrow[dr, "\mathrm{Ad}(r)"'] & & T_{w'} \arrow[dl, "\mathrm{Ad}(r')"] \\ & T_{w_0} & \end{tikzcd}$$

in which the horizontal map is dual to the corresponding isomorphism $\mathrm{Ad}(g) : \check{T} \rtimes_{F_w} \Gamma \to \check{T} \rtimes_{F_{w'}} \Gamma$ of L–groups. Thus $\pi_{(\phi,\rho)} = \pi_{(\phi',\rho')}$.

3.3.5 Exhaustion of Deligne–Lusztig representations

Consider a stable rational form on $\mathbf{G}$, represented by some element $b \in Z^1(K/k, \tilde{W}^C)$. Let π be a representation of $\mathbf{G}_b$ of Deligne–Lusztig type. Write

$$\pi = \mathrm{Ind}_{Z(k)(G_J)_b(k)}^{G_b(k)}(\chi \otimes \varepsilon(G_J, S) R^{G_J}_{(\mathsf{S},\theta)})$$

for some $J \subset \mathcal{A}(G,T)$, some unramified character χ, and some geometric conjugacy class (S,θ) of G_J, with S minisotropic for the F_b–structure on G_J. Because C is a fundamental domain for the action of W_C, we may and do assume that $J \subset \overline{C}$. We wish to find ϕ and ρ such that $\pi = \pi_{(\phi,\rho)}$.

Take $p \in G_J$ such that $\mathrm{Ad}(p)T = S$. Let $z = p^{-1}F_b(p) \in W_J$. Because S is minisotropic, the image of zb in $\overline{W}$ is elliptic (reversing the logic of equation (3.8)). Factor $zb = t_\lambda w$ with $\lambda \in \tilde{X}^\vee$ and $w \in W^{\mathbf{o}}$. We claim that $\lambda \in Z^1(F_w, \tilde{X}^\vee)$, in other words, that $t_\lambda w F$ has finite order. The facet J is stabilized by $\mathrm{Ad}(b) \circ F$ and fixed by z. The element bF belongs to the finite group $\tilde{W}^C \rtimes \langle F \rangle$ (where we quotient by F^m if the quasi–split form $\mathbf{G}$ splits over a degree m unramified extension), and z has finite order because it belongs to the finite group G_J. Because the Weyl group W_J is F_b–stable, $\mathrm{Ad}(zb) \circ F$ must have finite order, as desired.

Consider the character $\theta^T = \mathrm{Ad}(p^{-1})^*\theta$ of T^{F_w} and the unramified character $\chi^T = \mathrm{Ad}(p^{-1})^*\chi$ of T^{F_w}. Let ϕ^T be the Langlands parameter of T_{F_w} corresponding to

$\theta^T \otimes \chi^T$ under equation (3.11). Let $n' \in N_{\check{G}}(\check{T})$ be a lift of $\check{w}$. Under the bijection (3.10), the class of n' might not correspond to the class determined by ϕ^T; however, by multiplying by a suitable element of $\check{T}$, we may replace n' by $n \in N_{\check{G}}(\check{T})$ that does correspond to ϕ^T and still is a lift of $\check{w}$. We define the tame regular semisimple elliptic Langlands parameter ϕ for $\mathbf{G}$ by the conditions $\alpha_\phi|_{\mathcal{I}} = \alpha_{\phi^T}|_{\mathcal{I}}$ and $\alpha_\phi(\mathrm{Fr}) = n$. Let ρ be the class of the cocycle λ in S_ϕ. Then $\pi_{(\phi,\rho)} = \pi$.

3.3.6 Non–duplication of representations

Suppose (ϕ, ρ) and (ϕ', ρ') are extended Langlands parameters over a stable rational form $[y] \in R(k, \mathbf{G})$, represented by $y \in Z^1(\langle F \rangle, \tilde{W}^C)$ under equation (3.5). Let $\mathbf{G}_y$ be the group inner to $\mathbf{G}$ with a k–structure in which $\Gamma_{K/k}$ acts by $F_{\overline{y}}$, where $\overline{y} \in Z^1(F, \overline{N}^C)$ descends to y. We show that if $\pi_{(\phi',\rho')} = {}^g\pi_{(\phi,\rho)}$ for some $g \in \mathbf{G}_y(k)$, then there exists $\check{g} \in {}^L\mathbf{G}$ such that $(\phi', \rho') = {}^{\check{g}}(\phi, \rho)$.

Without loss of generality, we may assume that ϕ and ϕ' are in good position, determining $\check{w}, \check{w}' \in \check{W}$, and that $\check{w}$ and $\check{w}'$ and the representatives λ and λ' of ρ and ρ' are chosen such that $t_\lambda wF$ and $t_{\lambda'} w'F$ fix points x and x' in $\overline{C}$. Take z, b, J, S, θ, z', b', J', S', and θ' associated to (ϕ, λ) and (ϕ', λ') as in previous subsections.

First, we replace (λ, w) and (λ', w') with more felicitous choices, so that our representations may all be constructed inside $\mathbf{G}_y$. Because (ϕ, λ) and (ϕ', λ') both map to $[y]$, there exist $c, c' \in W^C$ such that $c * b = y = c' * b'$. Let $\check{c}, \check{c}' \in N_{\check{G}}(\check{T})$ be dual to the images of c and c' in $\overline{W}$; we then have $\check{c}(\check{b}F)\check{c}^{-1} = \check{y}F$. Replacing (ϕ, λ) by $({}^c\phi, {}^c\lambda)$ and (ϕ', λ') by $({}^{c'}\phi', {}^{c'}\lambda')$, we may assume that $b = y = b'$, on top of our previous assumptions.

Write

$$\begin{aligned} \pi = \pi_{(\phi,\rho)} &= \mathrm{Ind}_{Z(k)(G_J)_b(k)}^{G_b(k)} \chi \otimes \varepsilon(G_J, S) R_{(\mathsf{S},\theta)}^{(G_J)_b} \\ \pi' = \pi_{(\phi',\rho')} &= \mathrm{Ind}_{Z(k)(G_{J'})_{b'}(k)}^{G_{b'}(k)} \chi' \otimes \varepsilon(G_{J'}, S') R_{(\mathsf{S}',\theta')}^{(G_{J'})_{b'}} \end{aligned}$$

By Theorem 5.2 of [11], the types $(G_J, \varepsilon(G_J, S)R_{(S,\theta)})$ and $(G_{J'}, \varepsilon(G_{J'}, S')R_{(S',\theta')})$ must be associate. Using the fact G_J is a maximal parahoric in $\mathbf{G}_y$, this means there exists $g \in G_y(k)$ such that $gJ = J'$ and ${}^g\left(\varepsilon(G_J, S)R^{(G_J)_y}_{(S,\theta)}\right) \cong \left(\varepsilon(G_{J'}, S')R^{(G_{J'})_y}_{(S',\theta')}\right)$ as representations of $(G_{J'})_y$; we may take g to be in $(N_G(T))^{F_y}$. Consequently, $({}^g\mathsf{S}, {}^g\theta)$ and (S', θ') belong to the same geometric conjugacy class, so there exists $h \in N_{G_{J'}}(T)$ such that $({}^{hg}S, {}^{hg}\theta) = (S', \theta')$ and $h^{-1}F_y(h) \in T$. Let a be the image of hg in W^{F_y}.

If $\check{a} \in N_{\check{G}}(\check{T})$ is dual to the image of a in $\overline{W}$, then $({}^{\check{a}}\phi, {}^{\check{a}}\rho)$ is an extended Langlands parameter that yields the facet J', the k–structure given by F_y, and the pair (S', θ'). We know that $a(t_\lambda wF)a^{-1} = z''yF$ for some $z'' \in W_{J'}$. On the other hand, with y fixed, z'' is uniquely determined by the pair (S', θ'), so $z'' = z'$. Since

$$a(t_\lambda wF)a^{-1} = z'yF = t_{\lambda'}w'F, \tag{3.15}$$

both $\alpha_{{}^{\check{a}}\phi}(\mathrm{Fr})$ and $\alpha_{\phi'}(\mathrm{Fr})$ have image $\check{w}'$ in $\check{W}$. Because $\check{w}'$ is elliptic, there exists $\check{t} \in \check{T}$ such that $\alpha_{{}^{\check{t}\check{a}}\phi}(\mathrm{Fr}) = \alpha_{\phi'}(\mathrm{Fr})$. Both Langlands parameters yield the character θ' of S'; since this character is regular, we must have ${}^{\check{t}\check{a}}\phi = \phi'$. If the element $a \in W_C$ is decomposed as $a^t a^{\circ}$ with $a^t \in X^\vee \cap W_C$ and $a^{\circ} \in W^{\circ}$, then ${}^{a^{\circ}}w = w$, so by equation (3.15),

$${}^{\check{t}\check{a}}\lambda = {}^{\check{a}}\lambda = \lambda' \bmod (1 - wF)X^\vee. \tag{3.16}$$

Thus $({}^{\check{t}\check{a}}\phi, \check{t}\check{a} \cdot \rho) = (\phi', \rho')$ as desired. □

3.4 Behavior in SL_2

As an example of how Langlands parameter extensions and their corresponding representations behave with respect to coverings, we consider the group $\mathbf{G} = \mathbf{SL_2}$. Here, $H^1(k, \mathbf{G}) = 1$ by Kneser's Theorem [5], but $R(k, \mathbf{G}_{ad}) = \mathbb{Z}/2\mathbb{Z}$. Take $\tau = -1 \in R(k, \mathbf{G})$. Then $\mathbf{G}_\tau(k) = SL_1(D)$, the norm–one elements of a central simple algebra D of dimension 4 over k. Let k_2 be the unramified quadratic extension of k. We may write $D = k_2 \oplus k_2 a$, where $a \in D$ satisfies $axa^{-1} = {}^{\mathrm{Fr}}x$ for $x \in k_2$ and $a^2 = -1$.

Recall the covering $\eta : \mathbf{G} \to \overline{\mathbf{G}}$, where $\overline{\mathbf{G}} = \mathbf{PGL_2}$. Suppose ϕ_{ad} is a tame regular semisimple elliptic Langlands parameter for $\mathbf{PGL_2}$. We have an L–homomorphism ${}^L\eta : {}^L\mathbf{PGL_2} \to {}^L\mathbf{SL_2}$, providing a Langlands parameter $\phi = {}^L\eta \circ \phi_{ad}$ for $\mathbf{SL_2}$. We have ${}^L\mathbf{PGL_2} = \tilde{G} \times \Gamma$ and ${}^L\mathbf{SL_2} = \check{G} \times \Gamma$, where $\tilde{G} = SL_2(\mathbb{C})$ and $\check{G} = PGL_2(\mathbb{C})$. Fix Cartan and Borel subgroups $\check{T} \subset \check{B} \subset \check{G}$, and let $\tilde{T} \subset \tilde{B}$ be their covers in $\check{G}$. Suppose ϕ_{ad} to be in good position with respect to $\tilde{T}$, so that $\alpha_{\phi_{ad}}(\mathrm{Fr})$ represents the nontrivial element of $W(\tilde{G}, \tilde{T}) = W(\check{G}, \check{T})$. Then

$$S_{\phi_{ad}} = \tilde{G}^{\phi_{ad}} = \tilde{T}^{\tilde{w}} = \{\pm 1\} = Z(SL_2(\mathbb{C})) = R(k, \mathbf{G}_{ad}). \tag{3.17}$$

Each fiber of $r : S_{\phi_{ad}} \to H^1(k, \mathbf{G}_{ad})$ has a unique element, and DeBacker and Reeder have constructed corresponding representations $\pi_{(\phi_{ad},1)}$ of $\mathbf{PGL_2}(k)$ and $\pi_{(\phi_{ad},-1)}$ of $PGL_1(D)$.

Consider an apartment $\mathcal{A} = \mathcal{A}(G,T)$ of SL_2; it may naturally be identified with a particular apartment of PGL_2. Choose a vertex $\mathbf{o}$ as the origin, and let α be an affine root that vanishes there. Let C be the chamber between the hyperplanes where α and $\alpha+1$ respectively vanish. There is a generator λ for $\tilde{X}^\vee = X^\vee_{PGL_2}$ sending α to $\alpha+1$; the element 2λ is a generator for $X^\vee = X^\vee_{SL_2}$.

The Weyl group $\tilde{W} = W(PGL_2, T_{ad})$ acts transitively on the vertices of $\mathcal{A}$, but $W = W(SL_2, T)$ has two orbits, consisting of the vertices where the roots $\alpha + k$ vanish for k odd and even, respectively. Let J be the vertex where α vanishes and J' be the vertex where $\alpha+1$ vanishes. Let $w \in W^{\mathbf{o}}$ be the nontrivial reflection. Through the above constructions, ϕ_{ad} determines a character $\theta_{ad}^{T_{ad}}$ of $T_{ad}^{F_w}$, and ϕ determines its restriction θ^T to T^{F_w}.

If $\overline{S}$ is an anisotropic torus of $(PGL_2)_J$, then $S = \eta^{-1}(\overline{S})$ is an anisotropic torus of $(SL_2)_J$. Because the restriction of $\pi_{(\phi_{ad},1)}$ to $(PGL_2)_J$ contains $-1 \cdot R_{(\overline{S},\theta_{ad})}$, the restriction of $\eta^*(\pi_{(\phi_{ad},1)})$ must contain $-1 \cdot R_{(S,\theta)}$.

The types $(J, R_{(S,\theta)})$ and $(J', R_{(S',\theta')})$ are not associate, so we have found two inequivalent irreducible representations in the restriction of $\pi_{(\phi_{ad},1)}$ to G. These representations appear as representations $\pi_{(\phi,\rho)}$ for $\rho \in S_\phi$ as follows. By equation (3.4), we calculate

$$S_\phi = Z^1(\langle s_\alpha \rangle, \tilde{X}^\vee)/B^1(\langle s_\alpha \rangle, \tilde{X}^\vee) = \tilde{X}^\vee/2X^\vee = \tilde{X}^\vee/4\tilde{X}^\vee \tag{3.18}$$

and by equation (2.21), we have

$$R(k, \mathbf{G}) = \overline{\tilde{X}^\vee}/(1-F)\overline{X^\vee} = \overline{\tilde{X}^\vee} = \tilde{X}^\vee/X^\vee = \tilde{X}^\vee/2\tilde{X}^\vee. \tag{3.19}$$

The map $S_\phi \to R(k, \mathbf{G})$ corresponds to $\tilde{X}^\vee/4\tilde{X}^\vee \to \tilde{X}^\vee/2\tilde{X}^\vee$, and each fiber has two elements.

Over the split form in $R(k, \mathbf{G})$, we have the classes in S_ϕ corresponding to 1 and t_λ^2. Both 1 and t_λ^2 are in W_C, so the element b of equation (3.7) is 1. The corresponding facets are those fixed by s_α and $t_\lambda^2 s_\alpha$, namely $J = \mathbf{o}$ and $J' = t_\lambda \mathbf{o}$. Any irreducible representation π of SL_2 in the restriction of $\pi_{(\phi_{ad},1)}$ has one of the two K–types $(J, -R_{(S,\theta)})$ or $(J', -R_{(S',\theta')})$, but $\pi_{(\phi,1)}$ and $\pi_{(\phi,t_\lambda^2)}$ are the unique irreducible representations with these properties. We have precisely

$$\mathrm{Res}_{\mathbf{G}(k)} \pi_{(\phi_{ad},1)} = \pi_{(\phi,1)} \oplus \pi_{(\phi,t_\lambda^2)}. \tag{3.20}$$

Over the fiber of $\tau = -1 \in R(k, \mathbf{G})$, we have the classes corresponding to t_λ and t_λ^3. Write ρ for the class of t_λ in S_ϕ. The elements $t_\lambda s_\alpha$ and $t_\lambda^3 s_\alpha$ preserve x and $t_\lambda x$, where x is the barycenter of C. However, $s_\alpha \in W^{\mathbf{o}}$ satisfies

$${}^{s_\alpha}(t_\lambda s_\alpha) = t_\lambda^{-1} s_\alpha = t_\lambda^3 s_\alpha \bmod (1 - wF)X^\vee_{SL_2} \tag{3.21}$$

Let

$$\tilde{n} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in SL_2(\mathbb{C})$$

and let $\check{n}$ be its image in $N_{\check{G}}(\check{T})$. Let $\phi' = {}^{\check{n}}\phi$. Observe that $R_\phi = R_{\phi'}$ in $\tilde{G}$. We have $({}^{\check{n}}\phi, \check{n} \cdot \rho) = (\phi', \rho^{-1})$. Both (ϕ, ρ) and (ϕ', ρ^{-1}) determine the same character θ^T of T^{F_w} (see the discussion under Lemma 3.10), and the same representation of $\mathbf{G}_\tau(k)$. The representation arising from the character ${}^{s_\alpha}(\theta^T)$, on the other hand, arises from either $({}^{\check{n}}\phi, \rho)$ or the equivalent extended parameter (ϕ, ρ^{-1}).

The identification of these representations works differently than over the split fiber. There, the representations associated to the extended parameters $(\phi, 1)$ and $({}^{\check{n}}\phi, 1)$ match because the nontrivial Weyl group $(W(G_J, T))^{F_1}$ transposes the corresponding parameters for the L–group of the split torus T.

Chapter 4

Induction

4.1 Compatibility of stable rational forms

Suppose $\mathbf{G}$ is a quasi–split connected reductive group defined over a p–adic field k, with k–stable Cartan and Borel subgroups $\mathbf{T} \subset \mathbf{B}$. Consider a standard parabolic subgroup $\mathbf{P} = \mathbf{LU}$ of $\mathbf{G}$ such that $\mathbf{P}$, $\mathbf{L}$, and $\mathbf{U}$ are each defined over k. Let (ϕ^L, ρ^L) be an extended Langlands parameter for $\mathbf{L}$.

Following [17], we define the *infinitesimal character* of a Langlands parameter ϕ for $\mathbf{G}$ to be $\phi|_{\mathcal{W}}$. The inclusion of $\mathbf{L}$ in $\mathbf{G}$ induces an L–homomorphism $\iota : {}^L\mathbf{L} \to {}^L\mathbf{G}$ on the dual group side. By a slight abuse of terminology, we say that a Langlands parameter ϕ^L for $\mathbf{L}$ is the infinitesimal character of ϕ if the composite $\iota \circ \phi^L|_{\mathcal{W}} = \phi|_{\mathcal{W}}$. Whenever we view an infinitesimal character as a Langlands parameter, we consider the Langlands parameter to be trivial on the nilpotent part of the Weil–Deligne group.

Let $\mathcal{B}(G)$ be the (extended) building of G. The inclusion of L in G determines a map $\mathcal{B}(L) \to \mathcal{B}(G)$. Consider a K–split, maximally k–split torus T of L. There is an embedding $\mathcal{A}(L,T) \to \mathcal{A}(G,T)$. Both apartments are affine spaces of the same dimension, but $\mathcal{A}(G,T)$ has more hyperplanes.

The based root datum for $\mathbf{L}$ is $(X, \Delta_L, X^\vee, \Delta_L^\vee)$ for some subsets $\Delta_L \subset \Delta$ and $\Delta_L^\vee \subset \Delta^\vee$. Let $\tilde{X}_L^\vee$ and $\tilde{X}_G^\vee$ be the cocharacter lattices of the quotients $\overline{\mathbf{L}}$ and $\overline{\mathbf{G}}$

constructed in Chapter 2. These differ, but both lattices embed into $X^\vee \otimes \mathbb{R}$. Write $\tilde{X}^\vee$ for the intersection $\tilde{X}_L^\vee \cap \tilde{X}_G^\vee$ inside this affine space, and $\overline{\tilde{X}^\vee}$ for $\tilde{X}^\vee/\mathbb{Z}\Delta_L^\vee$. There is a natural map (but not necessarily a surjection) $\overline{\tilde{X}^\vee} \to \tilde{X}_L^\vee/\mathbb{Z}\Delta_L^\vee$, so that we get a map $Z^1(k, \overline{\tilde{X}^\vee}) \to R(k, \mathbf{L})$.

Suppose $\tau \in R(k, \mathbf{G})$ is a stable rational form for which $\mathbf{L}$ is defined over k. The criterion for a translation $t_\lambda \in \tilde{X}_G^\vee$ to preserve $\mathcal{A}(L, T)$ is that it live in the subgroup $\tilde{X}^\vee$. Therefore, representatives of τ must come from $Z^1(k, \overline{\tilde{X}^\vee})$.

Let τ^L be the stable rational form of $\mathbf{L}$ defined by the extended Langlands parameter (ϕ^L, ρ^L) corresponding to π^L above. Suppose ϕ is a Langlands parameter of $\mathbf{G}$ with infinitesimal character ϕ^L. If the rational form in $H^1(k, \mathbf{L}_{ad})$ associated to τ^L arises from a rational form on G, then τ^L is represented by a cocycle in $Z^1(k, \overline{\tilde{X}^\vee})$. Let $\tau \in R(k, \mathbf{G})$ be the stable rational form of $\mathbf{G}$ obtained from (any) representative of τ^L via the map $Z^1(k, \overline{\tilde{X}^\vee}) \to Z^1(k, \overline{\tilde{X}^\vee}_G)$.

Although the map $\check{L} \to \check{G}$ might not lift to a map $\tilde{L} \to \tilde{G}$, there is a relationship between the parameter extensions S_{ϕ^L} and S_ϕ. From the lattices

$$\begin{aligned} \tilde{X}^\vee &= (X^\vee_{\mathbf{G}_{ad}} \cap X^\vee_{\mathbf{L}_{ad}}) + X^\vee && (4.1) \\ \tilde{X} &= \{\chi \in X : \langle \chi, \omega \rangle \in \mathbb{Z} \text{ for all } \omega \in \tilde{X}^\vee\} && (4.2) \end{aligned}$$

we obtain the based root datum $(\tilde{X}, \tilde{X}^\vee, \Delta_L, \Delta_L^\vee)$ of a quotient of $\mathbf{L}$ that covers $\overline{\mathbf{L}}$. Calling the corresponding reductive algebraic group $\mathbf{L}_1$, we may factor the map $\eta = \eta_L : \mathbf{L} \to \overline{\mathbf{L}}$ of Chapter 2 as

$$\mathbf{L} \xrightarrow{\eta_1} \mathbf{L}_1 \xrightarrow{\eta_2} \overline{\mathbf{L}} \tag{4.3}$$

and the dual map $\pi = \pi_L$ as

$${}^L\overline{L} \xrightarrow{{}^L\eta_2} {}^LL_1 \xrightarrow{{}^L\eta_1} {}^LL. \tag{4.4}$$

Through Proposition 2.1, the image of $Z^1(k, \overline{\tilde{X}^\vee})/B^1(k, \overline{X^\vee}_L)$ in $R(k, \mathbf{L})$ may be identified with ${}^L\eta_2{}^*(\mathrm{Hom}(\pi_0({}^L\eta_1{}^{-1}(Z({}^L\mathbf{G}))), \mathbb{C}^\times))$. Recall that $R_{\phi^L} = \pi_L^{-1}(\check{L}^{\phi^L})$, and $S_{\phi^L} = \mathrm{Hom}(\pi_0(R_{\phi^L}), \mathbb{C}^\times)$. The fiber

$$S_{\phi^L}(\tau^L) = {}^L\eta_2{}^*(\{\sigma \in \mathrm{Hom}(\pi_0({}^L\eta_1{}^{-1}(\check{L}^{\phi^L})), \mathbb{C}^\times) : \sigma|_{{}^L\eta_1{}^{-1}(Z({}^L\mathbf{L}))} = \tau^L\}). \tag{4.5}$$

The based root data show that there is a natural map $\tilde{\iota} : {}^L\mathbf{L_1} \to {}^L\overline{\mathbf{G}}$, and we have:

$$\begin{tikzcd}
{}^L\mathbf{L_1} \arrow[rr, "\tilde{\iota}"] \arrow[d, "{}^L\eta_1"'] & & {}^L\overline{G} \arrow[d, "\pi"] \\
{}^L\mathbf{L} \arrow[rr, "\iota"] & & {}^L\mathbf{G} \\
& \mathcal{W} \arrow[ul, "\phi^L"] \arrow[ur, "\phi|_{\mathcal{W}}"'] &
\end{tikzcd} \tag{4.6}$$

A comparison of S_ϕ and S_{ϕ^L} will require an analysis of the various ϕ with infinitesimal character ϕ, which we will perform later in this chapter. The simplest example comes from the Langlands parameter $\phi_0 = \iota \circ \phi^L$. Then we have maps ${}^L\eta_1{}^{-1}(\check{L}^{\phi^L}) \xrightarrow{\tilde{\iota}} {}^L\overline{G}$, inducing a restriction map $S_{\phi_0}(\tau) \to S_{\phi^L}(\tau^L)$. For other ϕ with infinitesimal character ϕ^L, we do not always expect elements of $S_\phi(\tau)$ to determine elements of $S_{\phi^L}(\tau^L)$, particularly if there exist $\rho_1, \rho_2 \in S_{\phi^L}(\tau^L)$ such that $\mathrm{Ind}_{\mathbf{LU}}^{\mathbf{G}}(\pi_{(\phi^L,\rho_1)} \otimes 1_U) = \mathrm{Ind}_{\mathbf{LU}}^{\mathbf{G}}(\pi_{(\phi^L,\rho_2)} \otimes 1_U)$.

4.2 Parabolic induction and infinitesimal characters

The case of principal series gives a prototype for the relationship between parabolic induction of representations and extensions of Langlands parameters. Consider the situation where $\mathbf{L} = \mathbf{T}$ and $\mathbf{P} = \mathbf{B}$. Identify ${}^L\mathbf{T}$ with the subgroup $\check{T} \rtimes \Gamma$ of ${}^L\mathbf{G}$. Given an unramified character κ of $\mathbf{T}(k)$ corresponding to a Langlands parameter

$\phi^T : \mathcal{W} \to {}^L\mathbf{T}$, the work of Kazhdan and Lusztig [4] placed the subquotients of $\mathrm{Ind}_{\mathbf{B}(k)}^{\mathbf{G}(k)}(\kappa \otimes 1_U)$ in correspondence with some of the pairs (N, ρ), with $N \in \check{\mathfrak{g}}$ such that $\mathrm{Ad}(\phi^T(\mathrm{Fr}))N = qN$, and ρ a representation of the component group of the centralizer $Z_{\check{G}}(\phi^T(\mathrm{Fr})) \cap Z_{\check{G}}(N)$, trivial on $Z({}^L\mathbf{G})$. (Here, q is the order of $\mathfrak{f}$.) These pairs may be regarded as certain pairs (ϕ, ρ) with ϕ a Langlands parameter having infinitesimal character ϕ^T, and $\rho \in S_\phi(1)$.

When one tries to remove the word "certain" in the above correspondence, one obtains not only principal series but the class of all *unipotent* representations [9], [10]. We wish to describe Lusztig's result in a particularly suggestive way.

We have fixed a K–split maximally k–split torus $\mathbf{T}$ and a Borel subgroup $\mathbf{TU}$ of $\mathbf{G}$. Following, *e.g.*, Moy and Prasad [12] Section 6.3, we assign a Levi subgroup $\mathbf{M}^J$ of a standard parabolic subgroup of $\mathbf{G}$ to every Γ_k–stable facet $J \subset \mathcal{A}(G, T)$ as follows. Let C be the maximal $\mathfrak{f}$–split torus contained in the center of the reductive quotient G_J of the parahoric subgroup G_J. Lift C to T to obtain a subtorus T of S. The group $M = Z_G(C)$ is a Levi subgroup of G with the desired properties. Because M contains T, there is a natural embedding $\mathcal{A}(M, T) \to \mathcal{A}(G, T)$, and J is a minimal F–stable facet in $\mathcal{A}(M, T)$. We take $\mathbf{M}^J = \mathbf{M}$. Let $\mathbf{N}^J$ be the unipotent radical of the parabolic subgroup generated by $\mathbf{M}^J$ and $\mathbf{U}$.

Assume now that $\mathbf{G} = \mathbf{G}_{ad}$ and $\mathbf{G}$ is split. Let $\tau \in Z^1(k, \mathbf{G})$, and suppose that $\mathbf{P} = \mathbf{MN}$ is a F_τ–stable parabolic subgroup of $\mathbf{G}$. Given an unramified Langlands parameter ϕ with infinitesimal character ϕ^T and $\rho \in S_\phi(\tau) = S_\phi^{pure}(\tau)$, Lusztig assigns a representation π of $\mathbf{G}_\tau(k)$ to (ϕ, ρ) in [9], depending only on the $\check{G}$–conjugacy class of this pair.

Note that in ${}^L\mathbf{T}$, the component group S_{ϕ^T} is trivial. Lusztig's correspondence has the following property:

> Suppose $\tau \in R(k, \mathbf{G})$, and $J \subset \mathcal{A}(G, T)$ is an F_τ–stable facet. Let σ^M be a unipotent cuspidal representation of G_J, $\mathbf{M} = \mathbf{M}^J$, and κ be an

unramified character of the center of $\mathbf{M}_\tau(k)$. Put

$$\pi^M = \kappa \otimes c - \mathrm{Ind}_{(M_J)_\tau(k)}^{M_\tau(k)} \sigma^M.$$

If π is an irreducible subquotient of $\mathrm{Ind}_{\mathbf{M}_\tau^J(k)\mathbf{N}_\tau^J(k)}^{\mathbf{G}_\tau(k)}(\pi^M \otimes 1_{\mathbf{N}_\tau(k)})$, then π corresponds to an extended Langlands parameter (ϕ, ρ) such that $\rho \in S_\phi(\tau)$ and $\phi|_{\mathcal{W}} = \phi^T$, where ϕ^T is the Langlands parameter of ${}^L\mathbf{T}$ corresponding to κ.

When we think of the word "unramified" as meaning "geometric conjugacy class $(T, 1)$" (the unipotent cuspidal representations), it suggests an immediate generalization of this property. Let $\mathbf{M}$ be a k–stable parabolic subgroup of $\mathbf{G}$, and π^M be a Deligne–Lusztig representation of $\mathbf{M}$. From our result in the previous chapter, it determines a pair (ϕ^M, ρ^M), with $\phi^M : \mathcal{W} \to {}^L\mathbf{M}$ a Langlands parameter for $\mathbf{M}$ and $\rho^L \in S_{\phi^M}$. Let $b \in Z^1(K/k, \tilde{W}^C)$, $J \subset \mathcal{A}(M, T)$ be as in the previous chapter. We have $M^J = M$. Let $\mathbf{N} = \mathbf{N}^J$. The type of π^M may be represented by (M_J, σ^M) for some representation $\sigma^M = c - \mathrm{Ind}_{M_J}^{M(k)} \epsilon R_{(\mathsf{S},\theta)}$.

The stable rational form τ^L associated to b determines a stable rational form $\tau \in R(k, \mathbf{G})$, as above. We conjecture that the following property will appear in a Langlands correspondence between depth zero representations and tame extended Langlands parameters:

> Suppose π is an irreducible subquotient of $\mathrm{Ind}_{\mathbf{M}_\tau(k)\mathbf{N}_\tau(k)}^{\mathbf{G}_\tau(k)}(\pi^M \otimes 1_{\mathbf{N}_\tau(k)})$. Then π corresponds to an extended Langlands parameter (ϕ, ρ) such that $\phi|_{\mathcal{W}} = \phi^M$.

We would like to add a compatibility condition between ρ and ρ^M to this statement.

4.3 Langlands parameters with a fixed infinitesimal character

Now we return to a standard k–stable parabolic $\mathbf{P} = \mathbf{LU}$ and a tame regular semisimple elliptic Langlands parameter ϕ^L for $\mathbf{L}$ in good position. Let π^L be the corresponding supercuspidal representation from Chapter 3. We consider the Langlands parameters for $\mathbf{G}$ with infinitesimal character ϕ^L.

Put $\phi_0 = \iota \circ \phi^L$. Fix a generator s for $\alpha_{\phi_0}(\mathcal{I})$, and let $\check{w}$ be the image of $\alpha_{\phi_0}(\mathrm{Fr})$ in $\check{W}$. Recall that $\tilde{T}$ is the preimage of $\check{T}$ in $\tilde{G}$. From our description of the based root data for $\tilde{L}$ and $\tilde{G}$, it is clear that $\tilde{L} = \pi^{-1}(\check{L})$, which is a parabolic subgroup of $\tilde{G}$. The element s is regular semisimple in $\check{L}$, and we have $\check{L}^{\phi^L} = \check{T}^{\check{w}}$ and $R_{\phi^L} = \tilde{T}^{\check{w}}$.

The group $R_{\phi_0} = \pi^{-1}(\check{G}^{\phi_0})$ is also the centralizer of $\pi^{-1}(\phi_0)$ in $\tilde{G}$. Recall that the derived groups of $\tilde{G}$ and $\tilde{L}$ are simply connected. By a theorem of Steinberg, the centralizer of a semisimple element in $\tilde{G}$ is connected. Suppose $\tilde{s} \in \tilde{G}$ with $\pi(\tilde{s}) = s$. Then $\tilde{G}^{\tilde{s}}$ is a connected, psuedo–Levi subgroup of $\tilde{G}$. Given $\alpha \in \Phi$, write $\tilde{U}_\alpha$ for the corresponding root subgroup of $\tilde{G}$. There exists a subset $\Phi_\phi \subset \Phi$ such that $\tilde{G}^{\tilde{s}} = \left\langle \tilde{T}, \tilde{U}_\alpha : \alpha \in \Phi_\phi \right\rangle$ (*cf.* Theorem 3.5.6 in [2]). Because s is regular semisimple for $\tilde{L}$, $\Phi_\phi \cap \Phi_L = \emptyset$.

Given ϕ^L tame regular semisimple elliptic, a Langlands parameter ϕ of G with the property that $\phi|_{\mathcal{W}} = \phi^L$ is determined by an element $N \in \check{\mathfrak{g}}$ satisfying certain conditions. To define ϕ from ϕ^L and such an N, we use the decomposition of the Weil–Deligne group $\mathcal{W}' = \mathcal{W} \ltimes \mathbb{C}$, requiring that for $z \in \mathbb{C}$, we have $\phi(z) = exp(zN)$. Put

$$\mathcal{N}(\phi^L) = \{N \in \check{\mathfrak{g}} : \phi \text{ as defined above defines a Langlands parameter}\}. \tag{4.7}$$

Write ϕ_N for the Langlands parameter corresponding to $N \in \mathcal{N}(\phi^L)$; this notation extends our definition of ϕ_0.

We can characterize the set $\mathcal{N}(\phi^L)$ as follows. The element $\breve{w}$ determines an automorphism of Φ; we call the set of α' such that $\breve{w}^i\alpha = \alpha'$ for some i the *$\breve{w}$-orbit of* α, and denote it by $[\alpha]$. Let n be the minimal positive integer such that $\breve{w}^n \in \check{T}$. Let $\Phi(\phi^L)$ denote the set of $\breve{w}$-orbits $[\beta]$ in $\check{\Phi}$ such that for some (equivalently all) $\alpha \in [\beta]$:

1. $\alpha(s) = 1$
2. $Ad(\breve{w}^n)N_\alpha = q^n N_\alpha$

Fix a Chevalley basis $\{H_\omega, X_\alpha\}$ for $\check{\mathfrak{g}}$. Given $[\alpha] \in \Phi(\phi^L)$, fix $\alpha \in [\alpha]$. Let $k = |[\alpha]|$. Certainly k divides n. Put

$$X_{[\alpha]} = \sum_{i=0}^{k-1} q^{-i} Ad(w^i)(X_\alpha). \tag{4.8}$$

We have $Ad(w)X_{[\alpha]} = qX_{[\alpha]}$. Write $\check{\mathfrak{g}}_{[\alpha]} = \mathbb{C}X_{[\alpha]}$. This vector space is independent of the choices of α and X_α.

Proposition 4.1 *We have*

$$\mathcal{N}(\phi^L) = \oplus_{[\alpha]\in\Phi(\phi^L)} \check{\mathfrak{g}}_{[\alpha]}. \tag{4.9}$$

Proof. From the structure of the Weil–Deligne group, $\mathcal{N}(\phi^L)$ consists of the $N \in \check{\mathfrak{g}}$ such that $\phi(\mathcal{I})$ acts trivially on N and such that $Ad(\phi(Fr))N = qN$. It is clear that the right hand side of equation 4.9 is contained in $\mathcal{N}(\phi^L)$. Conversely, suppose $N \in \mathcal{N}(\phi^L)$. Necessarily N is nilpotent; the projections $\check{\mathfrak{g}} \to \check{\mathfrak{g}}_\alpha$ determine $N_\alpha \in \check{\mathfrak{g}}_\alpha$ such that $N = \sum_{\alpha\in\check{\Phi}} N_\alpha$. Define complex constants c_α by $N_\alpha = c_\alpha X_\alpha$. Because $\mathrm{Ad}(\breve{w})N = qN$, we must have $c_{\breve{w}^{-1}\alpha} = q^{-1}c_\alpha$. We deduce that $N = \sum_{\alpha\in\Phi(\phi^L)} c_\alpha X_{[\alpha]}$. $\square$

Bibliography

[1] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr. *The Langlands classification and irreducible characters for real reductive groups*, volume 104 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[2] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[3] Stephen DeBacker and Mark Reeder. Depth–zero supercuspidal l–packets and their stability. *Preprint*, 2004.

[4] David Kazhdan and George Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.*, 87(1):153–215, 1987.

[5] Martin Kneser. Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern. II. *Math. Z.*, 89:250–272, 1965.

[6] Robert E. Kottwitz. Stable trace formula: cuspidal tempered terms. *Duke Math. J.*, 51(3):611–650, 1984.

[7] Robert E. Kottwitz. Stable trace formula: elliptic singular terms. *Math. Ann.*, 275(3):365–399, 1986.

[8] Robert P. Langlands. Representations of abelian algebraic groups. *Pacific J. Math.*, (Special Issue):231–250, 1997. Olga Taussky-Todd: in memoriam.

[9] George Lusztig. Classification of unipotent representations of simple p-adic groups. *Internat. Math. Res. Notices*, (11):517–589, 1995.

[10] George Lusztig. Classification of unipotent representations of simple p-adic groups. II. *Represent. Theory*, 6:243–289 (electronic), 2002.

[11] Allen Moy and Gopal Prasad. Unrefined minimal K-types for p-adic groups. *Invent. Math.*, 116(1-3):393–408, 1994.

[12] Allen Moy and Gopal Prasad. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helv.*, 71(1):98–121, 1996.

[13] Mark Reeder. Some supercuspidal l–packets of positive depth. *Preprint*, 2005.

[14] Robert Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

[15] John Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

[16] Jacques Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[17] David A. Vogan, Jr. The local Langlands conjecture. In *Representation theory of groups and algebras*, volume 145 of *Contemp. Math.*, pages 305–379. Amer. Math. Soc., Providence, RI, 1993.