Essays on Economic Theory

by

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Abstract

These four essays concern the theory of games and its application to economic theory. The first two, closely linked, chapters are an investigation into the foundational question of the sensitivity of the predictions of game theory to higher-order beliefs.

Impact of Higher-Order Uncertainty

with Muhamet Yildiz

In some games, the impact of higher-order uncertainty is very large, implying that present economic theories may be misleading as these theories assume common knowledge of the type structure after specifying the first or the second orders of beliefs. Focusing on normal-form games in which the players’ strategy spaces are compact metric spaces, we show that our key condition, called “global stability under uncertainty,” implies a variety of results to the effect that the impact of higher-order uncertainty is small. Our central result states that, under global stability, the maximum change in equilibrium strategies due to changes in players’ beliefs at orders higher than \( k \) is exponentially decreasing in \( k \). Therefore, given any need for precision, we can approximate equilibrium strategies by specifying only finitely many orders of beliefs.

Finite-Order Implications of Any Equilibrium

with Muhamet Yildiz

Present economic theories make a common-knowledge assumption that implies that the first or second-order beliefs determine all higher-order beliefs. We analyze the role of such a closing assumption at any finite order by instead allowing higher orders to vary arbitrarily. Assuming that the space of underlying uncertainty is sufficiently rich, we show that, under an arbitrary fixed equilibrium, the resulting set of possible outcomes must include all outcomes that survive iterated elimination of strategies that are never a strict best reply. For many games, this implies that, unless the game is dominance-solvable, every equilibrium will be highly sensitive to higher-order beliefs, and thus economic theories based on such equilibria may be misleading. Moreover, every equilibrium is discontinuous at each type for which two or more actions survive our elimination process. Conversely, the resulting set of possible outcomes must be contained in rationalizable strategy profiles. This yields a precise characterization in generic instances.

Price Dispersion and Loss Leaders

Dispersion in retail prices of identical goods is inconsistent with the standard model of price competition among identical firms, which predicts that all prices will be driven down to cost. One common explanation for such dispersion is the use of a loss-leader strategy, in which a firm prices one good below cost in order to attract a higher customer volume for profitable goods.
By assuming high transportation costs which indeed force each consumer to buy all desired goods at a single firm, we create the possibility of an effective loss-leader strategy. We find, however, that such a strategy cannot be effective in equilibrium, so that additional assumptions limiting price search or rationality must be introduced to explain price dispersion or loss leaders.

**Two Notes on the Blotto Game**

We exhibit a new equilibrium of the classic Blotto game in which players allocate one unit of resources among three coordinates and try to defeat their opponent in two out of three. It is well known that a mixed strategy will be an equilibrium strategy if the marginal distribution on each coordinate is $U[0, \frac{3}{2}]$. All known examples of such distributions have two-dimensional support. Here we exhibit a distribution which has one-dimensional support and is simpler to describe than previous examples. The construction generalizes to give one-dimensional distributions with the same property in higher-dimensional simplexes as well.

As our second note, we give some results on the equilibrium payoffs when the game is modified so that one player has greater available resources. Our results suggest a criterion for equilibrium selection in the original symmetric game, in terms of robustness with respect to a small asymmetry in resources.

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Chapter 1

Impact of Higher-Order Uncertainty

1.1 Introduction

Most economic theories are based on equilibrium analysis of models in which the players’ types (following Harsanyi (1967)) are simply taken as their beliefs about some underlying uncertainty, such as the marginal cost of a firm or the value of an object for a buyer, and rarely include a player’s beliefs about the other players’ beliefs about the underlying uncertainty. Using such a type structure implicitly assumes that, conditional on the first-order beliefs about some payoff-relevant uncertainty, all of a player’s higher-order beliefs are common knowledge.¹ Even the literature on global games (Carlsson and van Damme (1993)) and on forecasting others’ forecasts (Townsend (1983)) makes this assumption (in a finite-dimensional space of payoff uncertainty.)²

There is now an extensive literature, however, that emphasizes that in some games higher-order uncertainty has as large an impact on equilibrium behavior as lower-order uncertainty (see Rubinstein (1989), Kajii and Morris (1998) and Morris (2002)). As Rubinstein (1989) illustrates, the equilibria of a game in which a particular piece of information is common knowledge can be profoundly different from the equilibria of games in which this information is mutually

¹Here we use the standard terminology: a player’s first-order beliefs are his beliefs about the underlying uncertainty; his second-order beliefs are roughly his beliefs about the other players’ first-order beliefs, and so on.

²For an illustration of how a model with such an assumption can be deceptive regarding the impact of higher-order uncertainty, see Section 1.2.3.
known only up to some finite order — no matter how many orders we consider. Most importantly, when the higher-order beliefs have large impact, the present economic theories may be misleading. This large impact is also disturbing because it is hard to believe that we would ever know a player’s high-order beliefs with any precision. Without such knowledge, we cannot make accurate predictions when the impact of higher-order uncertainty is large. Moreover, assuming that higher-order beliefs correspond to higher-order reasoning, such a large impact implies that the bounds of rationality are at least as important as the basic incentives. This would necessitate a change of paradigm for analyzing these problems. Therefore, it is of fundamental importance to classify games in which high-order uncertainty has little impact.

In this paper, we provide a set of sufficient conditions under which high-order uncertainty has little impact. Our main sufficient condition is called “global stability under uncertainty.” It states that the variation in each player’s best response is always less than the variation in his beliefs about the others’ actions (according to the embedding metric defined later), multiplied by a constant $b$ that is less than 1. Under certain continuity assumptions, we show that global stability under uncertainty is closely related to the standard concept of global stability of best-response correspondence (under certainty). For games with one-dimensional strategy spaces, we further provide a simple second-order condition that guarantees global stability under uncertainty.

We consider finite-person games in which the strategy spaces are compact metric spaces and there is some payoff-relevant source of uncertainty that comes from a complete, separable metric space. We work in universal type space, where the players’ types are their entire hierarchy of beliefs about the underlying uncertainty, allowing players to entertain any coherent set of beliefs. We show that, when the best responses are always unique, global stability implies that our game is dominance-solvable. This may suggest that our condition is too strong. Nevertheless, we illustrate somewhere else (Weinstein and Yildiz (2004)) that, when the best responses are always unique, dominance solvability is a necessary condition for diminishing effect of higher-order beliefs. We will refer to equilibrium throughout the paper because our results also apply to games without unique best responses, which might not be dominance-solvable.

---

3For example, the Coase conjecture may fail when we introduce second-order uncertainty as shown by Feinberg and Skrzypacz (2002).
We fix a (Bayesian) Nash equilibrium of this game. Note that, since every type space can be embedded in universal type space, this corresponds to fixing an equilibrium for all type spaces simultaneously. Let us also fix a player’s beliefs up to a certain order $k$. Our main result states that, assuming global stability, the maximum variation in the player’s equilibrium strategy, as we vary all his higher-order beliefs, is at most $b^k$ times a constant. That means that, if we want to determine the equilibrium behavior within a certain margin of error (e.g., in order to check the validity of a certain theoretical prediction), we only need to specify finitely many orders of beliefs, where the required number of orders $k^*$ is a logarithmic function of the desired precision. In particular, the impact of an erroneous common knowledge assumption at orders higher than $k^*$ will be less than the specified bound. This is a contribution to the goal set out by Wilson (1987) of “successive reductions in the base of common knowledge required to conduct useful analyses of practical problems.”

We have so far focused on the maximum change in a player’s equilibrium strategy due to any change in his higher-order beliefs. We also investigate the relationship of the change in strategy to the size of the change in beliefs. Towards this goal, firstly, we define an “embedding metric” on beliefs at each order (as well as on beliefs about the other players’ actions). This metric has the crucial property of preserving the distances in lower-order beliefs when they are embedded in the space of higher-order beliefs as point masses, allowing us to sensibly compare variations at different orders. We ask how much a player’s strategy varies as we vary his belief at some order $k$ and keep all his other beliefs fixed. (To be able to do this without violating the coherency of his beliefs, we need an independence assumption about the different orders of beliefs, an assumption that is satisfied in traditional “independent private value” environments.) Now we can define the marginal impact of a change in $k$th-order beliefs as the variation in equilibrium strategies divided by the size of this change in beliefs as measured by our embedding metric. We show that, under global stability and the independence assumption, the marginal impact of changes in $k$th-order beliefs is at most $b^k$ times a constant. This formalizes our notion that, under global stability, the marginal impact of higher-order beliefs decreases exponentially. In that case, precision in lower-order beliefs will be much more important than the precision in higher-order beliefs in approximating a problem. It also follows that the players’ equilibrium behavior would not change much if they formed erroneous higher-order beliefs. These assertions
may all sound very natural; we should emphasize that they may easily fail when global stability does not hold. In particular, with linear best-responses, the marginal impact of \( k \)-th-order beliefs actually increases exponentially in \( k \) whenever global stability does not hold.

Assuming that best responses are always unique, one can show that global stability implies the contraction property of Nyarko (1997), who investigates the convergence to equilibrium in a general abstract model in the same vein as Townsend (1983). Under his contraction property, Nyarko (1997) shows that the unique equilibrium must be continuous with respect to the usual product topology on universal type space. Using this, one can further show that the maximum and the marginal impact of higher-order beliefs must eventually vanish as we consider higher and higher order beliefs. This does not, however, tell us how fast these impacts are diminishing or whether they diminish monotonically. Most importantly, Nyarko’s fixed-point argument would not tell us why higher-order beliefs must be less important under global stability and how this may be reversed if global stability fails. Our constructive proof with explicit bounds sheds light on these issues. Moreover, the framework we develop remains useful even when global stability fails.

The outline of the paper is as follows. In the next section, we illustrate the relation between stability and dampening impact of higher-order beliefs using games with linear best responses. In Section 1.7, we present our basic model with independence assumption and introduce the embedding metric; we introduce global stability in Section 1.4 and provide sufficient conditions and examples for it in Section 1.5. Our major results are presented in Section 2.3 with independence assumption, and our main result is extended beyond this assumption in Section 1.3. Section 2.9 reviews the relevant literature, and Section 2.10 concludes. Some proofs are relegated to the Appendix.

1.2 Examples with Linear Best Responses

We will now show how dampening impact of higher-order uncertainty is equivalent to stability in games with linear best-response functions, such as the linear Cournot duopoly. This illustrates the close relationship between these two concepts which we will establish in a broader context in the later sections.
1.2.1 Cournot Duopoly

Consider a Cournot duopoly where the inverse-demand function is given by

\[ P = \theta - Q \]

where \( P \) is the price of a good and \( Q = q_1 + q_2 \) where \( q_i \) is the supply of firm \( i \in N = \{1, 2\} \) and \( \theta \) is an unknown demand parameter. The costs are zero, so that the payoff function of firm \( i \) is

\[ u_i(q_1, q_2) = q_i(\theta - q_1 - q_2). \]

Everything other than \( \theta \) is common knowledge.

We do not make any informational assumption about \( \theta \); we want to allow variations in all levels of uncertainty. (See Subsection 1.2.3 for a traditional model of incomplete information—with strong informational assumptions.) Firm \( i \) has a probability distribution \( t_i^1 \) on \( \theta \), representing its beliefs about \( \theta \). Firm \( j \) has also a probability distribution \( t_j^1 \) on \( t_i^1 \), representing \( j \)'s beliefs about \( i \)'s beliefs about \( \theta \). In general, firm \( i \) has probability distribution \( t_i^k \) on \( t_j^{k-1} \), representing \( k \)th-order beliefs of firm \( i \). Firm \( i \)'s type is the entire list \( t_i = (t_i^1, t_i^2, \ldots) \).

A strategy profile \((q_1^*, q_2^*)\), where \( q_i^* : t_i \mapsto q_i^*(t_i) \) specifies firm \( i \)'s supply as a function of its type, is an equilibrium iff \( q_i^*(t_i) \) maximizes the expected payoff of type \( t_i \) given the strategy \( q_j^* \) of the other firm. That is, equilibrium strategy \( q_i^* \) will maximize the expected payoff

\[ E_i[q_i(\theta - q_i - q_j^*(t_j))] = q_i(\theta - q_i - E_i[q_j^*(t_j)]), \]

where expectation \( E_i \) will be determined by its beliefs \((t_i^1, t_i^2, \ldots)\) at all levels, as \( q_j^*(t_j) \) depends on the entire type \( t_j \). This implies that

\[ q_i^* = \frac{E_i[\theta]}{2} - \frac{1}{2} E_i[q_j^*(t_j)]. \quad (1.1) \]

Of course, we also have

\[ q_j^* = \frac{E_j[\theta]}{2} - \frac{1}{2} E_j[q_i^*(t_i)]. \quad (1.2) \]
Substituting (1.2) in (1.1), we can obtain

$$q_i^* = \frac{E_i[\theta]}{2} - \frac{E_iE_j[\theta]}{4} + \frac{1}{4} E_iE_j[q_i^*].$$

(1.3)

A further substitution of (1.1) in (1.3) would yield

$$q_i^* = \frac{E_i[\theta]}{2} - \frac{E_iE_j[\theta]}{4} + \frac{E_iE_jE_i[\theta]}{8} - \frac{1}{8} E_iE_jE_i[q_j^*].$$

Here $E_i[\theta]$ depends only on $t_i^1$, the beliefs of $i$ about the demand, $E_iE_j[\theta]$ depends only on $t_i^2$, the beliefs of $i$ about the beliefs of $j$ about the demand, and $E_iE_jE_i[q_j^*]$ depends on the third and all higher-order beliefs. In general,

$$q_i^* = \frac{E_i[\theta]}{2} - \frac{E_iE_j[\theta]}{4} + \frac{E_iE_jE_i[\theta]}{8} - \cdots + \frac{1}{2^k} \underbrace{E_iE_jE_i \cdots E_i[q_j^*]}_{k \text{ times}}$$

when $k$ is odd; the last term is $E_iE_jE_i \cdots E_i[q_j^*]/2^k$ when $k$ is even. In equilibrium, each firm’s supply will always be in $[0, 1]$; hence the absolute value of the last term is at most $1/2^k$. That is, if we fix the beliefs up to $k$th order, we know the equilibrium strategy $q^*$ up to an error of at most $1/2^k$.

This also implies that we can write the equilibrium strategy as a convergent series

$$q_i^* = \frac{E_i[\theta]}{2} - \frac{E_iE_j[\theta]}{4} + \frac{E_iE_jE_i[\theta]}{8} - \frac{E_iE_jE_iE_i[\theta]}{16} + \cdots$$

where the coefficient of the $k$th term is $1/2^k$. The significance of this formula is that the coefficients of expectations decrease exponentially as we go to higher-order expectations.

### 1.2.2 General Case with Linear Best Responses

The analysis above can be easily generalized to the case with linear best-response functions

$$BR_i = E_i[\theta + ba_j]$$
where $\theta$ is the underlying parameter and $a_j$ is the (unknown) action of player $j$. A strategy of a player is a function $s_i^*$ that determines which action $i$ takes as a function of his hierarchy of beliefs. Now, the equilibrium strategies satisfy

$$s_i^* = E_i[\theta] + bE_iE_j[\theta] + b^2E_iE_jE_i[\theta] + \cdots + b^kE_iE_jE_i\cdots E_i[s_j^*]$$  \hspace{1cm} (1.4)

when $k$ is odd. The absolute value of the coefficients will decrease exponentially, resulting in a convergent infinite series as above, if and only if $|b| < 1$.

- This corresponds precisely to the stability of the equilibrium of the complete information game under the best-response correspondence. (See Footnote 5.)

- When the equilibrium is unstable, the impact of higher-order beliefs in equilibrium is actually higher than that of lower-order beliefs, and one must know the higher-order beliefs to an increasingly high level of precision in order to predict behavior.

- Our derivation in this section relies only on the formation of higher-order expectations—not on the particular type space used. Hence it applies to any type structure.

- We are only able to use the substitution trick here to derive a simple formula because of the linearity of the best-response function. In the general case a player’s best response depends on the details of the entire distribution (as noted by Morris (2002)) and there is no direct relationship between a player’s best responses under certainty and uncertainty, rendering such elementary analysis impossible and requiring the more sophisticated tools of the following sections.

Note also that Morris and Shin (2003) and Morris (2002) obtain specific examples with linear best responses similar to ours in this section.

### 1.2.3 A Traditional Type Structure

We have ex ante $\theta \sim N(0,1)$, and each player $i$ gets a private signal $x_i = \theta + \varepsilon_i$ where $\varepsilon_i \sim N(0,(1-v)/v)$ for some $v \in (0,1)$ and $\theta$, $\varepsilon_1$, and $\varepsilon_2$ are all independent. For each $i$,
assume $BR_i = E[\theta + ba_j|x_i]$ for some $b \geq 0$, where $E[\theta|x_i] = vx_i$. The above is all common knowledge.

Check that, whenever $bv \neq 1$, we have a Bayesian Nash equilibrium $s^*$ with

$$s_i^* = \frac{vx_i}{1 - bv}. \quad (1.5)$$

When $bv < 1$, equilibrium seems intuitive. When $bv > 1$, however, counterintuitively the coefficient of $x_i$ is negative and hence $s_i^*$ is decreasing in $x_i$. Now, write $s_i^*$ as a series of higher order expectation as in (1.4). Since the $k$th-order expectation of $\theta$ is $E_iE_jE_i\ldots E_j[\theta] = v^kx_i$, we have

$$s_i^* = vx_i + bv^2x_i + b^2v^3x_i + \cdots + b^kE_iE_jE_i\ldots E_i[s_j^*].$$

Firstly, notice that when $bv > 1$, higher-order terms increase exponentially, yielding a divergent series. This explosively large impact of higher-order uncertainty, however, does not appear in the directly computed formula in (1.5). Second, when $bv < 1 < b$, we have a convergent series yielding seemingly intuitive formula in (1.5), despite the fact that marginal contributions of higher-order expectations increase exponentially. This is only because our single-dimensional type space forced the variations in higher-order expectations to decrease exponentially,\(^4\) compensating the increases in marginal contributions. But in the approximated real-life situation, the players will probably have higher-order doubts about this model. In that case, their higher-order expectations may vary significantly, leading to dramatically different behavior (under the equilibrium of more accurate model). In that case, the model's predictions about the behavior will be misleading, and considerations about higher-order beliefs within the model will yield a false sense of robustness.

1.3 Model

We consider a game among players $N = \{1, 2, \ldots, n\}$. The source of underlying uncertainty is a payoff-relevant parameter $\theta \in \Theta$ where $(\Theta, d)$ is a compact Polish space (i.e., a complete and separable metric space), where $d$ is a metric on set $\Theta$. Each player $i$ has action space $A_i$, which

\(^4\)This is a general phenomenon (see Samet (1998).)
is a compact metric space, and utility function \( u_i : \Theta \times A \to \mathbb{R} \) where \( A = \prod_i A_i \).

**Notation.** Given any list \( X_1, \ldots, X_n \) of sets, write \( X_{-i} = \prod_{j \neq i} X_j \), \( x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \) \( \in X_{-i} \), and \( (x_i, x_{-i}) = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \).

Likewise, for any family of functions \( f_j : X_j \to Y_j, j \in \mathbb{N} \), we define \( f_{-i} : X_{-i} \to Y_{-i} \) by \( f_{-i} (x_{-i}) = (f_j (x_j))_{j \neq i} \). Given any metric space \( (X, d) \), write \( \Delta (X) \) for the space of probability distributions on \( X \), suppressing the fixed \( \sigma \)-algebra on \( X \) which at least contains all open sets; when we use product spaces, we will always use the product \( \sigma \)-algebra.

We write \( d_i \) for the metric on \( A_i \) for each \( i \in \mathbb{N} \) and define the metric \( d_{-i} \) on \( A_{-i} \) by

\[
d_{-i} (a_{-i}, a'_{-i}) = \max_{j \neq i} d_j (a_j, a'_j).
\]

We now define the players’ hierarchy of beliefs about the underlying parameter \( \theta \), using the usual construction of the universal type space by Brandenburger and Dekel (1993), a variant of an earlier construction by Mertens and Zamir (1985). We will define our types using the auxiliary sequence \( \{ X_k \} \) of sets defined inductively by \( X_0 = \Theta \) and \( X_k = [\Delta (X_{k-1})]^n \times X_{k-1} \) for each \( k > 0 \). We endow each \( X_k \) with the weak topology and the \( \sigma \)-algebra generated by this topology, yielding a standard separable Borel space as \( \Theta \) is a Polish space. A player \( i \)'s first order beliefs are represented by a probability distribution \( \tau_i^1 \) on \( X_0 \), second order beliefs (about all players’ first order beliefs and the underlying uncertainty) are represented by a probability distribution \( \tau_i^2 \) on \( X_1 \), etc. Therefore, a type

\[
\tau_i = (\tau_i^1, \tau_i^2, \ldots)
\]

of a player \( i \) is a member of \( \prod_{k=1}^\infty \Delta (X_{k-1}) \). Since a player’s \( k \)-th order beliefs contain information about his lower order beliefs, we need the usual coherence requirements. We write \( T \) for the subset of \( (\prod_{k=1}^\infty \Delta (X_{k-1}))^n \) in which it is common knowledge that the players’ beliefs are coherent, i.e., the players know their own beliefs and their marginals from different orders agree. We will use the variables \( \tau, \tilde{\tau} \in T \) as generic type profiles. The beliefs of any type \( \tau_i \) about \( (\theta, \tau_{-i}) \) are represented by some probability distribution \( \kappa_{\tau_i} \in \Delta (\Theta \times T_{-i}) \).

A strategy of a player \( i \) is a measurable mapping \( s_i : T_i \to A_i \), that determines which action \( s_i (\tau_i) \) he would choose given his type \( \tau_i \). We fix a Bayesian Nash equilibrium \( s^* = \)
which must be such that \( s^*_i (\tau_i) \) maximizes the expected value \( E \left[ u_i (\theta, a_i, s^*_{-i} (\tau_{-i})) \mid \tau_i \right] \) of \( u_i (\theta, a_i, s^*_{-i} (\tau_{-i})) \) with respect to \( \kappa_{\tau_i} \) at each \( \tau_i \) and for each \( i \). Global stability implies existence and uniqueness of equilibrium for games with unique best responses.

### 1.4 Stability, Rationalizability, and Higher-order Uncertainty

We are now ready to present our sufficient condition for the dampening impact of higher order uncertainty: stability of equilibrium under the best-response function. The global stability of equilibrium is usually defined by the condition that the variation in the best response is less than the variation in the other players' strategies under certainty.\(^5\) We will first extend this notion to the best response function under uncertainty, which is not directly related to the best response function under certainty.

**Best Responses** Given any player \( i \) and any probability distribution \( \pi \) on \( \Theta \times A_{-i} \), we write \( BR_i (\pi) \) for the best response of player \( i \) when his beliefs about the underlying uncertainty \( \theta \) and the other players’ actions \( a_{-i} \) are represented by \( \pi \). Notice that we are taking the best response to be a function rather than a correspondence. Under certain conditions (e.g., when the strategy spaces are convex and utilities are strictly quasi-concave in own strategy), the best-response correspondence will indeed be singleton. In general, however, there may be multiple best responses. In those cases we will assume that the equilibrium uses a singleton selection from the best-response correspondence. In the former case, the global stability defined below will be a property of the game, while in the latter case, it will be a property of the equilibrium. Under the independence assumption, we will have \( \pi = t^1_i \times \mu \) for some \( t^1_i \in \Delta (\Theta) \) and \( \mu \in \Delta (A_{-i}) \). In that case, we will write \( BR_i (t^1_i, \mu) \) instead of \( BR_i (\pi) \). When it does not lead to any confusion, we will sometimes suppress some of the arguments (e.g., write \( BR_i (\mu) \) when \( t^1_i \) is fixed) or write it in the form of \( BR_i (\theta, a_{-i}; t_i) \), denoting the best response of player \( i \) when his type is \( t_i \), where \( \theta \) and \( a_{-i} \) are random variables.

\(^5\)The usual definition appears to be different. For instance, in two player games we only need that the product of maximum variations is less than 1. Of course, under this condition, we could rescale our metrics on each strategy space so that our definition is also satisfied.
Global stability under uncertainty

We say that global stability under uncertainty holds iff there exists some \( b \in [0, 1) \) such that, given any \( i \in N \) and any \( \pi, \pi' \in \Delta (\Theta \times A_{-i}) \) with \( \text{marg}_\Theta \pi = \text{marg}_\Theta \pi' \), we have

\[
d_i (BR_i (\pi), BR_i (\pi')) \leq b \tilde{d}_{-i} (\pi, \pi'),
\]

where

\[
\tilde{d}_{-i} (\pi, \pi') \equiv \inf_{\nu \in \chi_{\pi, \pi'}} \int d_{-i} (a_{-i}, a'_{-i}) \, d\nu (\theta, a_{-i}, a'_{-i})
\]

and

\[
\chi_{\pi, \pi'} = \{ \nu \in \Delta (\Theta \times A_{-i} \times A_{-i}) : \text{marg}_{12} \nu = \pi, \text{marg}_{13} \nu = \pi' \}.
\]

The required condition for global stability is the standard condition for Lipschitz continuity (of each \( BR_i \) with respect to the metric \( \tilde{d}_{-i} \) on \( \Delta (\Theta \times A_{-i}) \)) with the additional requirement that the constant \( b \), which can be thought of as an upper bound on the absolute value of the slope, be less than 1. Of course, this is the same as saying that for each \( i \) there is a \( b_i \in [0, 1) \) satisfying the above condition, since we can take \( b = \max \{ b_1, \ldots, b_n \} \).

Our first result states that global stability implies that our game is dominance-solvable. Notice that in our game, a strategy of a player \( i \) is a function from his entire type space to \( A_i \).

**Proposition 1** Assume that each player has single-valued best response correspondence, and assume global stability under uncertainty. Then, there exists unique rationalizable strategy profile \( s^* \), which is the unique equilibrium.

We prove this proposition in the appendix for the general model developed in Section 1.3. Our proof essentially shows that the diameter of the space of surviving strategies, measured as the maximum distance among available actions to any given type of any player, decreases by a factor of \( b \) at each round. Therefore, in the limit there can be at most one strategy profile. Since global stability implies the contraction property of Nyarko (1997), there exists an equilibrium \( s^* \), which will never be eliminated and hence will be the unique rationalizable strategy profile.

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6 Notice that the metric \( \tilde{d}_{-i} \) is identical to the embedding metric (defined in Section xxx) with the slight modification that the infimum is taken only for the joint distributions on \( (\Theta \times A_{-i})^2 \) under which the two copies of \( \theta \) are identical almost surely. Since we consider fewer distributions when we take the infimum, \( \tilde{d}_{-i} \) is at least as large as the usual embedding metric, and hence this modification weakens our stability condition.
The requirement that there is always a unique best response is not superfluous. For example, for the second-price auction with private values, there are multiple equilibria, and hence multiple rationalizable strategies, but the dominant-strategy equilibrium is globally stable and satisfies all of our other results.

The next result states that, under global stability, if a player \( i \) changes his beliefs about the other players’ beliefs, then the change in equilibrium strategy of player \( i \) can be at most \( b \) times the expected maximum change in the other players’ equilibrium strategies due to the change in their beliefs, under the original beliefs of \( i \).

**Proposition 2** Assume global stability under uncertainty for some \( b \in [0,1) \). Consider arbitrary \( i \) and arbitrary measurable functions \( \phi : \Theta \times \mathcal{T}_{-i} \to \Theta \times \mathcal{T}_{-i} \) and \( \phi_{-i} : \mathcal{T}_{-i} \to \mathcal{T}_{-i} \) with \( \phi(\theta, \tau_{-i}) = (\theta, \phi_{-i}(\tau_{-i})) \). Then, for any \( \tau_{i} \),

\[
d_i \left( s^*_i \left( \tau_{i} \circ \phi^{-1} \right), s^*_i \left( \tau_{i} \right) \right) \leq b \mathbb{E} \left[ d_{-i} \left( s^*_{-i} \left( \phi_{-i}(\tau_{-i}) \right), s^*_{-i} \left( \tau_{-i} \right) \right) \left| \tau_{i} \right| \right].
\]

**Proof.** Let \( \pi \) and \( \pi' \) be the distributions of \( (\theta, s^*_{-i}(\tau_{-i})) \) under \( \tau_{i} \) and \( \tau_{i} \circ \phi^{-1} \), respectively. Define the mapping \( \alpha : (\theta, \tau_{-i}) \mapsto (\theta, s^*_{-i}(\tau_{-i}), s^*_{-i} \left( \phi_{-i}(\tau_{-i}) \right)) \), which has distribution \( \kappa_{\tau_{i}} \circ \alpha^{-1} \) under \( \tau_{i} \). Clearly, \( \text{marg}_{12} (\kappa_{\tau_{i}} \circ \alpha^{-1}) = \pi \) and \( \text{marg}_{13} (\kappa_{\tau_{i}} \circ \alpha^{-1}) = \pi' \), i.e., \( \kappa_{\tau_{i}} \circ \alpha^{-1} \in \chi_{\pi, \pi'} \). Therefore, by global stability,

\[
d_i \left( s^*_i \left( \tau_{i} \circ \phi^{-1} \right), s^*_i \left( \tau_{i} \right) \right) = d_i \left( BR_{i} (\pi), BR_{i} (\pi') \right)
\leq \inf_{\nu \in \chi_{\pi, \pi'}} b \int d_{-i} (a_{-i}, a'_{-i}) d\nu (\theta, a_{-i}, a'_{-i})
\leq b \int d_{-i} (a_{-i}, a'_{-i}) d\kappa_{\tau_{i}} \circ \alpha^{-1} (\theta, a_{-i}, a'_{-i})
= b \mathbb{E} \left[ d_{-i} \left( s^*_{-i} \left( \tau_{-i} \right), s^*_{-i} \left( \phi_{-i}(\tau_{-i}) \right) \right) \left| \tau_{i} \right| \right].
\]

Proposition 2 expresses the idea that global stability is sufficient to guarantee that the impact of higher-order beliefs on equilibrium is diminishing. This is because when player \( i \) changes his beliefs from \( \tau_{i} \) to \( \tau_{i} \circ \phi^{-1} \), he is changing his “higher-order” beliefs about the other players’ “lower-order” beliefs. To see this, imagine that we change the other players’ \( k \)th order beliefs and make necessary changes in their beliefs at orders \( k+1 \) and higher according to some
fixed theory about information structure, making sure that the beliefs remain coherent.\textsuperscript{7} This corresponds to a transformation $\phi_{-i}$ that leaves the other players' $k-1$st order of beliefs intact, and the changes at orders $k+1$ and higher merely reflect the change in their $k$th-order beliefs according to the fixed theory. Now, player $i$'s first $k$ orders of beliefs are same under $T_i$ and $\tau_i \circ \phi^{-1}$. Under $\tau_i \circ \phi^{-1}$, his $k+1$st-order beliefs are changed to reflect his new beliefs about the other players' $k$th-order beliefs; the changes in his higher-order beliefs merely reflect the assumption that player $i$ still believes in the fixed theory about the information structure. Our proposition implies that, in that case, the change in equilibrium strategy of player $i$ due to the changes in his $k+1$st-order beliefs can be at most $b$ times the expected maximum change in the other players' equilibrium strategies due to the change in their $k$th-order beliefs, under the original beliefs of $i$.

1.5 Sufficient conditions for stability

In this section we present two sets of sufficient conditions for global stability under uncertainty. Both sets of conditions are closely related to global stability under certainty. We first present a general class of games where global stability under uncertainty is closely related to global stability under certainty. This class is characterized by Assumption 1a.

Assumption 1a  Best-response function of player $i$ takes the form of

$$BR_i (\pi) = f_i (E [g_i (\theta, a_{-i})])$$

where expectation is taken with respect to $\pi \in \Delta (\Theta \times A_{-i})$; $f_i : X \rightarrow A_i$ and $g_i : \Theta \times A_{-i} \rightarrow X$ are two Lipschitz continuous functions defined through some Banach space $(X, d_X)$; i.e., there exist $\alpha_i$ and $\beta_i$ such that $d_i (f_i (x), f_i (x')) \leq \alpha_i d_X (x, x')$ and $d_X (g_i (\theta, a_{-i}), g_i (\theta, a'_{-i})) \leq \beta_i d_{-i} (a_{-i}, a'_{-i})$.

Note that the functional form in (1.9) is satisfied whenever $u_i$ is analytical and the optimization problem has an interior solution. (The Taylor expansion for the first order condition would imply such a functional form, where $E [g_i]$ is the vector of all moments.) The more substantial

\textsuperscript{7}Such a theory can be a closing assumption at an order higher than $k$ or an independence assumption.
part of this assumption is that \( f_i \) and \( g_i \) are Lipschitz continuous. Under certainty, Assumption 1a yields a best response function

\[
BR_i (\theta, a_{-i}) = f_i (g_i (\theta, a_{-i})) = h_i (\theta, a_{-i}).
\]

Our equilibrium would be stable under the best response correspondence if

\[
d_i (h_i (\theta, a_{-i}), h_i (\theta, a'_{-i})) \leq b_i d_{-i} (a_{-i}, a'_{-i})
\]

at each \( \theta \) for some \( b_i < 1 \). The latter condition is slightly weaker than the following assumption.

**Assumption 1b** For each \( i \in N \), we have \( b_i \equiv \alpha_i \beta_i < 1 \).

**Proposition 3** Assumptions 1a and 1b imply global stability under uncertainty.

**Proof.** In the Appendix. □

That is, under Assumption 1a, global stability under uncertainty is implied by the existence of \( \alpha_i \)'s and \( \beta_i \)'s that satisfy Assumption 1b. Moreover, whenever \( f \) or \( g \) is the identity, global stability under certainty and uncertainty will be equivalent. Hence, there is a close link between these two concepts. Although Assumption 1 might not be easy to check in general, our next example presents a general class of games where these conditions can be easily checked.

**Example 1** For each \( i \in N \), take \( A_i = [\underline{x}, \bar{x}] \) for some \( \underline{x}, \bar{x} \in \mathbb{R} \) and

\[
u_i (\theta, a_i, a_{-i}) = \phi_i (a_i) g_i (\theta, a_{-i}) - c_i (a_i),
\]

where \( g_i : \Theta \times A_{-i} \to \mathbb{R} \) is a continuously differentiable function with \( |\partial g_i / \partial a_j| < \beta_i \) for each \( j \neq i \) and for some \( \beta_i \in \mathbb{R} \), and \( \phi_i \) and \( c_i \) are twice continuously differentiable functions with \( \phi_i' > 0, \phi_i'' < 0, c_i' > 0, \) and \( c_i'' \geq 0 \). Note that \( g_i \) is Lipschitz continuous with parameter \( \beta_i \) with respect to the changes in \( a_{-i} \). Check that

\[
BR_i (\pi) = f_i (E [g_i (\theta, a_{-i})])
\]

where \( f_i (z) \) is \( \underline{x} \) if \( z < c' (\underline{x}) / \phi' (\underline{x}) \), \( \bar{x} \) if \( z > c(\bar{x}) / \phi' (\bar{x}) \), and it is the unique solution \( x \) to the first order condition \( c' (x) / \phi' (x) = z \) otherwise. By the inverse-function theorem, \( f_i \) is
also Lipschitz continuous with parameter $\alpha_i = 1/\gamma_i$ where $\gamma_i = \min_{x \in [x_i, x]} \left( c'(x)/\phi'(x) \right)' > 0$. Therefore, global stability is satisfied whenever $b \equiv \max_{i \in N} \beta_i / \gamma_i < 1$.

Focusing on games where the agents’ strategy spaces are one-dimensional, our next result presents a simple sufficient condition for global stability, and hence for dampening impact of higher order uncertainty, in terms of second derivatives of the utility functions.

**Proposition 4** For each $i$, assume $A_i \subset \mathbb{R}$, $u_i (\theta, \cdot)$ is twice-continuously differentiable, $u_i (\theta, \cdot, a_{-i})$ is strictly concave, $\partial^2 u_i / \partial a_i^2$ is bounded away from zero, and

$$b_i \equiv \sum_{j \neq i} \frac{\max_{a, \theta} \left| \partial^2 u_i (\theta, a) / \partial a_i \partial a_j \right|}{\min_{a, \theta} \left| \partial^2 u_i (\theta, a) / \partial a_i^2 \right|} < 1. \quad (1.10)$$

Then, we have global stability under uncertainty whenever (i) $BR_i (\pi)$ is in the interior of $A_i$ for all $\pi$, or (ii) $A_i$ is convex.

**Proof.** In the Appendix. ■

1.6 **Maximum Impact of Higher-order Beliefs**

As argued in the Introduction, modelers would prefer not to have to specify the players’ higher-order beliefs, and the present economic theories rely on a general common knowledge assumption for all high-order beliefs. It is then very important to determine the accuracy with which we can predict a player’s equilibrium behavior if we only know his beliefs up to $k$th order and have no knowledge of his beliefs at higher orders. We are now ready to state and prove our main result, which states that global stability implies at least a certain level of accuracy.

**Proposition 5** Let $D_{s^*} = \max_{i \in N} \sup_{\tau_i^l, \tau_i^{l'} \in T} d_i (s_i^* (\tau_i^l), s_i^* (\tau_i^{l'})) \in \mathbb{R}$. Let also $\tau, \tilde{\tau} \in T$ be such that $\tau_i^l = \tilde{\tau}_i^l$ for all $l \leq k$ for some $k \geq 0$. Assume global stability under uncertainty for parameter $b$. Then, in the general model,

$$d_i (s_i^* (\tau_i^l), s_i^* (\tilde{\tau}_i^l)) \leq b^k D_{s^*}. \quad (1.11)$$

Notice that our result assumes only global stability and boundedness of the strategy space. Under these two assumptions we reach the conclusion that, if we know the beliefs up to a certain
order \( k \), we can know the equilibrium play within a bound of error that is an exponentially decreasing function of \( k \), bounding the maximum impact all the higher-order beliefs can have on equilibrium. Our result does not refer to any topology on the type space. Finally, \( D_s^* \) is chosen as a bound on the variations in equilibrium outcomes. If there are other known bounds on the equilibrium outcomes, then we can replace \( D_s^* \) with these bounds.

In certain cases, a modeler might want to predict the equilibrium behavior within a certain margin of error. For example, checking the validity of certain qualitative predictions of his theories may only require the knowledge of equilibrium strategies within a certain margin of error. Proposition 1 tells us how many orders of uncertainty he needs to specify. It implies that, given any \( \epsilon > 0 \) and any \( \tau \in T \), if we know \( t \) up to the order

\[
 k \geq \frac{\log(\epsilon) - \log(D_s^*)}{\log(b)},
\]

then we can compute the equilibrium strategies up to a maximum error of \( \epsilon \). If there is any known bound for equilibrium action, we can replace \( D_s^* \) with that bound. Notice that the expression on the right-hand side is increasing in \( b \) and decreasing in \( \epsilon \).

In the remainder of the section we prove our proposition. We start with the following technical lemma.

**Lemma 1** Let \( (X, \Sigma_X) \), \( (Y, \Sigma_Y) \), \( (Z, \Sigma_Z) \) be separable standard Borel spaces, and endow \( X \times Y \), \( Y \times Z \), \( X \times Z \), and \( X \times Y \times Z \) with the \( \sigma \)-algebras generated by the corresponding product topologies. Let probability measures \( P \) and \( P' \) on \( X \times Y \) and \( X \times Z \), respectively, be such that \( xP = xP' \). Then, there exists a probability measure \( \tilde{P} \) on \( X \times Y \times Z \) such that \( x_{XY} \tilde{P} = P \) and \( x_{XZ} \tilde{P} = P' \).

**Proof.** In the Appendix. \( \blacksquare \)

**Proof of Proposition 5.** Define \( \Omega = \Theta \times T \) to be the universal state space. This is the subset of the larger space \( \tilde{\Omega} = \Theta \times (\prod_{k=1}^{\infty} \Delta (X_{k-1}))^n \) in which coherency is common knowledge. By Brandenburger and Dekel (1993), \( \tilde{\Omega} \) is a Polish space, yielding a standard separable Borel space, and for every \( \tau = (\tau_1, \ldots, \tau_n) \in T \) and for every \( i \in N \), there exists a probability
distribution $\kappa_{\tau_i} \in \Delta (\Omega)$ such that
\[
\text{marg}_{X_{k-1}} \kappa_{\tau_i} = \tau_i^k \quad (\forall k),
\]
and $\kappa_{\tau_i} (\Omega) = 1$. Let
\[
\beta : (\theta, \tau) \mapsto (\theta, s_\tau^*(\tau)_{-i}),
\]
and write
\[
\pi_{\tau_i} = \kappa_{\tau_i} \circ \beta^{-1} \in \Delta (\Theta \times A_{-i})
\]
for the joint distribution of the underlying uncertainty and the other players' actions induced by $\tau_i$. Notice that $s_\tau^*(\tau)_{-i} = BR_i (\pi_{\tau_i})$.

We will use induction on $k$. For $k = 0$, this is true by definition. Fix any $k > 0$, and assume that the result is true for $k - 1$. Take any $\tau$ and $\tilde{\tau}$ as in the hypothesis. We have
\[
d_i (s_\tau^*(\tau), s_{\tilde{\tau}}^*(\tilde{\tau})) = d_i (BR_\tau (\pi_{\tau_i}), BR_{\tilde{\tau}} (\pi_{\tilde{\tau}_i}))
\leq b d_{-i} (\pi_{\tau_i}, \pi_{\tilde{\tau}_i})
\equiv b \inf_{\nu \in \chi_{\pi_{\tau_i}, \pi_{\tilde{\tau}_i}}} \int d_{-i} (a_{-i}, a'_{-i}) d\nu (\theta, a_{-i}, a'_{-i}),
\]
where the inequality is due to global stability and $\chi_{\pi_{\tau_i}, \pi_{\tilde{\tau}_i}}$ is defined by (1.8). The rest of the proof is devoted to constructing a $\nu \in \chi_{\pi_{\tau_i}, \pi_{\tilde{\tau}_i}}$ such that, under the induction hypothesis,
\[
\int d_{-i} (a_{-i}, a'_{-i}) d\nu (\theta, a_{-i}, a'_{-i}) \leq b^{k-1} D_{i*}.
\]
Combining (1.14) and (1.15), we obtain (1.11).

We will decompose $\bar{\Omega}$ as $\bar{\Omega} = \Theta \times L \times H$ where
\[
L = \prod_{l=1}^{k-1} \Delta (X_{l-1})^n \quad \text{and} \quad H = \prod_{l=k}^{\infty} \Delta (X_{l-1})^n
\]
are the spaces of lower and higher-order beliefs. For $k = 1$, we use the convention that $L$ is a singleton set, and $l \in L$ can simply be ignored in the following analysis for that case. Note that $X_{k-1} = \Theta \times L$.  

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By (2.1), we have probability distributions $\kappa_{\tau_i}$ and $\kappa_{\bar{\tau}_i}$ on $\Omega$ such that

$$
marg_{X_{k-1}} \kappa_{\tau_i} = \tau_i^k = \bar{\tau}_i^k = \marg_{X_{k-1}} \kappa_{\bar{\tau}_i},
$$

where the second equality is by our hypothesis. Since we have separable standard Borel spaces, by Lemma 1, there exists $\sigma \in \Delta (X_{k-1} \times H \times H)$ such that the marginals of $\sigma$ on the cross product of $X_{k-1}$ with the first and second copies of $H$ are

$$
\marg_{12} \sigma = \kappa_{\tau_i} \quad \text{and} \quad \marg_{13} \sigma = \kappa_{\bar{\tau}_i},
$$

respectively.

Now, consider $\nu = \sigma \circ \gamma^{-1} \in \Delta (\Theta \times A_{-i} \times A_{-i})$ where

$$
\gamma : (\theta, l, h_1, h_2) \mapsto (\theta, s_{-i}^*(l, h_1), s_{-i}^*(l, h_2)). \tag{1.17}
$$

Notice that the marginal of $\nu$ on the first copy of $\Theta \times A_{-i}$ is

$$
\marg_{12} \nu = \marg_{12} (\sigma \circ \gamma^{-1}) = (\marg_{12} \sigma) \circ \beta^{-1} = \kappa_{\tau_i} \circ \beta^{-1} = \pi_{\tau_i},
$$

and similarly $\marg_{13} \nu = \pi_{\bar{\tau}_i}$. Therefore, by definition, $\nu \in \chi_{\pi_{\tau_i}, \pi_{\bar{\tau}_i}}$.

We now prove (1.15). Take any $(\theta, a_{-i}, \bar{a}_{-i}) \in \gamma (X_{k-1} \times H \times H)$. By (1.17), $a_{-i} = s_{-i}^*(\hat{\tau}_{-i})$ and $\bar{a}_{-i} = s_{-i}^*(\bar{\tau}_{-i})$ for some type profiles $\hat{\tau} = (l, h_1)$ and $\bar{\tau} = (l, h_2)$, which agree up to the order $k - 1$ by (1.16). Then, by the induction hypothesis,

$$
d_{-i} (a_{-i}, \bar{a}_{-i}) \leq b^{k-1} D_{s^*}. \tag{1.18}
$$

Since $\text{supp} \nu \subset \gamma (X_{k-1} \times H \times H)$ (by construction), (1.18) implies (1.15).

1.7 Model with independence

We now define the players’ hierarchy of beliefs about the underlying parameter $\theta$. We confine ourself to the belief structures where a player’s beliefs are independent from his own beliefs at
other orders. We do this because we want to be able to (i) vary a player’s kth-order beliefs without worrying about the coherency of his beliefs and (ii) measure the impact of this change on equilibrium strategies without worrying about its impact through the changes in the player’s beliefs at other orders. (The independence assumption will be dropped in our main result.)

We define the beliefs (or type) of a player $i$ inductively. His first order beliefs (about $\theta$) are represented by a probability distribution $t_1^i \in \Delta_1 \equiv \Delta(\Theta)$ on $\Theta$. His kth-order beliefs (about $t_{k-1}^i$) are represented by a probability distribution $t_k^i \in \Delta_k \equiv \Delta(\Delta_{k-1}^{n-1})$ on $\Delta_{k-1}^{n-1}$. The type of a player $i$ is the list

$$t_i = (t_1^i, t_2^i, t_3^i, \ldots)$$

of all these probability distributions. We write $T_i$ for the set of all possible types $t_i$ of player $i$. We also write $T = \prod_i T_i$ for the set of all type profiles $t$. His beliefs are represented by the product measure $t_1^i \times t_2^i \times t_3^i \times \cdots$ of his beliefs $(t_1^i, t_2^i, t_3^i, \ldots)$ at each order; that is, given any $\prod_{k=0}^\infty X_k \subset \Theta \times T_{-i}$, the probability that he assigns to the event $\{(\theta, t_{-i}) \in \prod_{k=0}^\infty X_k\}$ is $\prod_{k=1}^\infty t_k^i(X_{k-1})$. (Here, of course, we have used the independence assumption.) We write $t \setminus t_k^i$ for the belief structure obtained by changing $t_k^i$ to $t_k^i$ in $t$; $t \setminus t_k^i$ and $t \setminus t_k^i$ are defined similarly.

Example 2 (Independent private value environment) Take any incomplete-information game with payoffs $u_i(a; \theta_i)$ for each $i$ where each $\theta_i \in \Theta_i$ is independently distributed with some probability distribution $P_i$ and privately known by player $i$, and this is common knowledge. This game can be embedded in our framework, by taking $\Theta = \cup_i \Theta_i$, $t_1^i = \delta_{\theta_i}$, $t_2^i = t_1^i \equiv P_{-i} \circ \xi^{-1}$ where $P_{-i} = \prod_{j \neq i} P_j$ and $\xi: \theta_{-i} \mapsto \prod_{j \neq i} \delta_{\theta_j}$, and taking $t_k^i = t_k^i \equiv \delta_{p_{k-1}^i}$ for each $k > 2$, where $\delta_x$ denotes the measure that puts probability 1 on $\{x\}$.

Embedding metric Throughout the paper, we will need a measure of the distances between probability distributions. We therefore introduce the following metric, which we will call embedding metric. Let $(X, d)$ be any metric space. Given any $\mu, \mu' \in \Delta(X)$, we first write

$$\Delta_{\mu, \mu'} = \{\nu \in \Delta(X \times X) | \text{marg}_1 \nu = \mu, \text{marg}_2 \nu = \mu'\}$$

(1.19)
for the set of all joint probability distributions with marginals \( \mu \) and \( \mu' \), where \( \operatorname{marg}_i \) is the marginal distribution on the \( i \)th copy of \( X \). Now we define our embedding metric \( d \) on \( \Delta(X) \) by setting

\[
d(\mu, \mu') = \inf_{\nu \in \Delta_{\mu, \mu'}} E_\nu [d(x_1, x_2)],
\]

where \( E_\nu \) is the expectation operator with respect to \( \nu \) and \((x_1, x_2)\) is a generic member of \( X \times X \). It is easy to verify that this is an extension in the following sense: if \( \mu \) and \( \mu' \) are point masses at \( x \) and \( x' \), respectively, then \( d(\mu, \mu') = d(x, x') \) — thus the notational convenience of using \( d \) for both metrics. An equivalent definition is given by

\[
d(\mu, \mu') = \inf_{Y \sim \mu, Y' \sim \mu'} E [d(Y, Y')]
\]

where \( \inf \) is taken over all pairs \( Y \) and \( Y' \) of \( X \)-valued random variables with distributions \( \mu \) and \( \mu' \), respectively, and coming from the same probability space, and \( E \) is the expectation operator on this space.

The embedding metric has the following property of preserving Lipschitz continuity; the proof is in the Appendix. Notice in the lemma that \( \mu \circ f^{-1} \) is the distribution of \( f(Y) \) for a random variable \( Y \sim \mu \).

**Lemma 2** Let \( (X, d_X) \) and \( (Z, d_Z) \) be two metric spaces, and \( f : X \to Z \) be such that

\[
d_Z(f(x), f(x')) \leq \lambda d_X(x, x') \quad (\forall x, x')
\]

for some \( \lambda \). Let also \( d_X \) and \( d_Z \) be the embedding metrics on \( \Delta(X) \) and \( \Delta(Z) \), respectively. Then,

\[
d_Z(\mu \circ f^{-1}, \mu' \circ f^{-1}) \leq \lambda d_X(\mu, \mu') \quad (\forall \mu, \mu').
\]

**1.8 Equilibrium Impact with Independence**

In this section, using the embedding metric defined above, we will put a natural metric on the type space, which will allow us to compare variations in different orders of the type space. We will show that, under the previously stated conditions, variations in higher-order beliefs have a
lower impact on equilibrium behavior than comparable variations in lower-order beliefs.

1.8.1 Embedding metric on beliefs

We now apply the embedding-metric construction inductively to define our embedding metric on beliefs of each player at each order. First, for \( k = 1 \), we extend \( d \) to \( \Delta_1 = \Delta (\Theta) \) by setting

\[
d(t^1_i, \tilde{t}^1_i) = \inf_{\theta \sim t^1_i, \theta' \sim \tilde{t}^1_i} E \left[ d(\theta, \theta') \right]
\]

at each \( t^1_i, \tilde{t}^1_i \in \Delta_1 \) and to \( \Delta^{n-1}_1 \) by setting

\[
d(t^1_{-i}, \tilde{t}^1_{-i}) = \max_{j \neq i} d(t^1_j, \tilde{t}^1_j)
\]

at each \( t^1_{-i}, \tilde{t}^1_{-i} \in \Delta^{n-1}_1 \). For any \( k > 1 \), we extend \( d \) to \( \Delta_k \) by setting

\[
d(t^k_i, \tilde{t}^k_i) = \inf_{Y \sim t^k_i, Y' \sim \tilde{t}^k_i} E \left[ d(Y, Y') \right],
\]

where \( Y \) and \( Y' \) take values in \( \Delta^{n-1}_{k-1} \) (whose generic member is \( t^{k-1}_{-i} \)), and to \( \Delta_k \) by setting

\[
d(t^k_{-i}, \tilde{t}^k_{-i}) = \max_{j \neq i} d(t^k_j, \tilde{t}^k_j)
\]

at each \( t^k_{-i}, \tilde{t}^k_{-i} \in \Delta^{n-1}_1 \).

1.8.2 Dampening impact of higher-order uncertainty

Assuming global stability, we will now find an upper bound for the change in equilibrium strategy caused by a change in any \( k \)th-order beliefs. When we consider comparable changes (according to \( d \)) at all orders \( k \), this bound will be decreasing exponentially in \( k \).

Proposition 6 Assume that, for each \( i \in N \), \( BR_i (\cdot, \mu) \) is Lipschitz continuous uniformly on \( \mu \), i.e.,

\[
d_i \left( BR_i \left( t^1_i; \mu \right), BR_i \left( \tilde{t}^1_i; \mu \right) \right) \leq \alpha d \left( t^1_i, \tilde{t}^1_i \right) \quad (\forall \mu, t^1_i, \tilde{t}^1_i)
\]

(1.22)

for some \( \alpha \in \mathbb{R} \). Assume also global stability under uncertainty for parameter \( b \). Then, in the
model with independence, for any $i$, $t_i$, $k$, and any $\tilde{t}_i^k$,

$$d_i \left( s_i^* \left( t_i \right), s_i^* \left( t_i \backslash \tilde{t}_i^k \right) \right) \leq \alpha b^{k-1} d \left( \tilde{t}_i^k, \tilde{t}_i^k \right).$$  \hspace{1cm} (1.23)

The conclusion can be spelled out as follows: Change the beliefs of a player at some order $k$ while all the other beliefs are fixed. The change in the equilibrium strategy due to this change in the beliefs is at most an exponentially decreasing function of $k$ times the change in the beliefs according to our embedding metric. In other words, the bound of the rate of change in equilibrium strategy as a function of $k$th-order belief is exponentially decreasing in $k$.

**Proof.** Firstly, for $k = 1$, (1.23) is just (1.22). Now assume that (1.23) holds at some $k - 1$, i.e., for all $j \in N$, $\hat{t}$, and $\hat{t}_j^{k-1}$,

$$d_j \left( s_j^* \left( \hat{t}_j \right), s_j^* \left( \hat{t}_j \backslash \hat{t}_j^{k-1} \right) \right) \leq \alpha b^{k-2} d \left( \hat{t}_j^{k-1}, \hat{t}_j^{k-1} \right).$$  \hspace{1cm} (1.24)

For any fixed $t$ and $i \in N$, let us define $f : \Delta_{k-1} \rightarrow A_i$ by setting

$$f \left( \hat{t}_i^{k-1} \right) = BR_i \left( t \backslash \hat{t}_i^{k-1} \right)$$

at each $\hat{t}_i^{k-1} \in \Delta_{k-1}$. Fix $\hat{t}_i = t_i$, so that our induction hypothesis (1.24) becomes

$$d \left( f \left( \hat{t}_i^{k-1} \right), f \left( \hat{t}_i^{k-1} \right) \right) \leq \alpha b^{k-2} d \left( \hat{t}_i^{k-1}, \hat{t}_i^{k-1} \right).$$

Then, by Lemma 2, for any $\tilde{t}_i^k$,

$$d \left( t_i^k \circ f^{-1}, \tilde{t}_i^k \circ f^{-1} \right) \leq \alpha b^{k-2} d \left( t_i^k, \tilde{t}_i^k \right).$$

Notice that $t_i^k \circ f^{-1}$ and $\tilde{t}_i^k \circ f^{-1}$ are the distributions of $s_i^*$ under $t_i$ and $t_i \backslash \tilde{t}_i^k$. Therefore, by global stability,

$$d_i \left( s_i^* \left( t_i \right), s_i^* \left( t_i \backslash \tilde{t}_i^k \right) \right) \leq \alpha b b^{k-2} d \left( t_i^k \circ f^{-1}, \tilde{t}_i^k \circ f^{-1} \right) \leq \alpha b b^{k-2} d \left( t_i^k, \tilde{t}_i^k \right) \leq \alpha b^{k-1} d \left( \tilde{t}_i^k, \tilde{t}_i^k \right).$$
Proposition 6 yields another bound for the maximum impact of higher-order beliefs as in Proposition 5. By adding the effects of changes at $k + 1$st and all higher orders, we obtain the following corollary. (The validity of this infinite summation follows from Proposition 5.)

**Corollary 1** Under the assumptions and the notation of Proposition 6, let $D_{\Theta} = \max_{\theta, \theta' \in \Theta} d(\theta, \theta')$. Let $t_i$, $\bar{t}_i$ be such that $t^l_i = \bar{t}^l_i$ for all $l \leq k$ for some $k > 1$. Then, in the model with independence,

$$d_i \left( s^*_i(t_i), s^*_i(\bar{t}_i) \right) \leq b^k \alpha D_{\Theta} / (1 - b).$$

### 1.8.3 Continuity in product topology

As we mentioned in the Introduction, global stability implies continuity of equilibrium with respect to the usual topology on the universal type space. In this section, we show a somewhat stronger continuity property.

Consider the topology of pointwise convergence under the embedding metric. Equilibrium strategy $s^*_i$ is continuous with respect to this topology if, for any sequence $\{t^m_i\}_{m \in \mathbb{N}}$ of types,

$$\left[ t^m_i \rightarrow \tilde{t}^k_i \quad \forall k \in \mathbb{N} \right] \Rightarrow \left[ s^*_i(t^m_i) \rightarrow s^*_i(\tilde{t}_i) \right],$$

where convergence of beliefs at each order is according to the embedding metric. Also, because the space of beliefs is compact under the embedding metric, this topology is metrized by the metric $d_b$ (called a Fréchet metric) defined by

$$d_b(t_i, \tilde{t}_i) = \sum_{k=1}^{\infty} b^{k-1} \cdot d_k(t^k_i, \tilde{t}^k_i),$$

where $b$ is any number in $(0, 1)$. Our next result states that, under global stability, the equilibrium strategy is Lipschitz continuous with respect to a Fréchet metric, and hence it is continuous in the product topology.

**Proposition 7** Under the assumptions and the notation of Proposition 6, in the model with independence, for each $i \in N$, the equilibrium strategy $s^*_i$ of player $i$ is Lipschitz continuous with respect to $d_b$. In that case, $s^*_i$ is continuous with respect to the product topology on type space generated by the embedding metric on beliefs at each order.
Proof. Fix any two types $t_i$ and $\tilde{t}_i$ of player $i$. For each $k \in \mathbb{N}$, define the type $t_{i,k}$ by setting

$$t_{i,k}^l = \begin{cases} t_i^l & \text{if } l \leq k, \\ \tilde{t}_i^l & \text{otherwise} \end{cases}$$

at each order $l$. We have

$$d_i \left( s^*_i \left( t_i \right), s^*_i \left( \tilde{t}_i \right) \right) \leq \sum_{k=1}^\infty d_i \left( s^*_i \left( t_{i,k}, \tilde{t}_i \right) \right) = \sum_{k=1}^\infty d_i \left( s^*_i \left( t_{i,k} \right), s^*_i \left( t_{i,k-1} \right) \right) \leq \alpha \sum_{k=1}^\infty d_k \left( t_{i,k}, \tilde{t}_i \right) = \alpha d_b \left( t_i, \tilde{t}_i \right),$$

where $\alpha$ is as defined in Proposition 6, proving the result. To see the first inequality, note that we can change $\tilde{t}_i$ to $t_i$ by changing $\tilde{t}_i^k$ to $t_i^k$ one at a time. Hence, by Proposition 1, for each $\epsilon > 0$, there exists an integer $l$ such that $d_i \left( s^*_i \left( t_i \right), s^*_i \left( \tilde{t}_i \right) \right) \leq \sum_{k=1}^l d_i \left( s^*_i \left( t_{i,k} \right), s^*_i \left( t_{i,k-1} \right) \right) + \epsilon \leq \sum_{k=1}^\infty d_i \left( s^*_i \left( t_{i,k} \right), s^*_i \left( t_{i,k-1} \right) \right) + \epsilon$. Since $\epsilon$ is arbitrary, this yields the inequality. The next equality is by definition; the next inequality Proposition 6, and the last equality is by definition. 

Corollary 2 Let $S^*$ be the set of Bayesian Nash Equilibria $s^*$ that use a singleton selection from best-response correspondence, and define $\Sigma^*$ by setting $\Sigma^*(t) = \{ s^*(t) | s^* \in S^* \}$ at each $t \in T$. Then, under the assumptions of Proposition 6, $\Sigma^*$ is lower semi-continuous with respect to the product topology on type space generated by the embedding metric on beliefs at each order.

Proof. Take any $t$, any $s^*(t) \in \Sigma^*(t)$, and sequence $t(n)$ that converges to $t$ in the topology above. By definition $s^*(t)$ is the value of a Bayesian Nash equilibrium $s^*$ at $t$. Then, by Proposition 7, $s^*(t(n)) \in \Sigma^*(t(n))$ converges to $s^*(t)$. 

1.9 Literature Review

Although there is a sizeable literature on the impact of higher-order uncertainty following Rubinstein (1989), the focus of most studies has been relaxation of common knowledge and lower semi-continuity of equilibrium in the worst-case scenarios, such as approximating common knowledge with common $p$-beliefs (Monderer and Samet (1989)), robustness of equilibrium against (possibly substantial) payoff uncertainty with small probability (Fudenberg, Kreps, and
Levine (1988) and Kajii and Morris (1997)), and strong topologies under which equilibrium is lower semi-continuous uniformly over all games (Monderer and Samet (1997) and Kajii and Morris (1998)). Most closely related to our work, Morris (2002) analyzes the impact of higher-order uncertainty within a model with linear best responses, reaching the conclusion that impact of higher-order beliefs can be arbitrarily large if we require a uniform bound over all games. Our focus differs in two ways. Firstly, we measure the impact of higher-order uncertainty within a single game (dropping the uniformity requirement). Second, while our sufficient condition implies continuity of best response, most of these papers analyze matrix games and naturally use the supremum metric on the mixed strategies, when the best response is generically discontinuous.

1.10 Conclusion

Present economic theories are mostly based on equilibrium analysis of models in which, conditional on only a few low orders of uncertainty, all higher-order beliefs are assumed to be common knowledge without any justification. We know, however, that in some games higher-order uncertainty has a profoundly large impact in equilibrium. In this paper we presented a sufficient condition, namely global stability under uncertainty, which guarantees that the impact of higher-order uncertainty is low. Using the universal type space, in which players can entertain any coherent set of beliefs, we have shown under this assumption that if we specify the players' beliefs up to some order \( k \), we will know their equilibrium behavior within a bound that decreases exponentially in \( k \) (cf. Proposition 5). That is, if a theoretical prediction requires knowledge of the strategies within a margin \( \epsilon \) of error, then the researcher can validate his theory by specifying first \( k(\epsilon) \) orders of beliefs, where \( k(\epsilon) \) is a logarithmic function of \( \epsilon \). Under a further independence assumption we also formalize our notion that, under stability, the marginal impact of higher-order uncertainty is (exponentially) decreasing in the order (cf. Propositions 2 and 6). That is, the problem must be approximated using lower-order uncertainty rather than higher-order uncertainty; this may be reversed when stability does not hold, as the impact of higher-order uncertainty may grow exponentially. In the latter case, we believe that accurate prediction using traditional analysis will be impossible.
When the best responses are always unique, we have a dominance-solvable game, and hence our analysis would not change if we considered refinements of equilibrium or non-equilibrium concepts, such as rationalizability. Nevertheless, in general, our use of normal-form representation and the solution concept of (unrestricted) Bayesian Nash equilibrium does impose an important limitation which requires further research. Many theories are based on extensive-form representations and use refinements, such as sequential rationality (Selten (1974), Kreps and Wilson (1982)). Their predictions are often driven by these refinements when equilibrium itself does not have any predictive power in their games. It is then crucial to extend our analysis to such a framework, using extensive-form constructions, such as Battigalli and Siniscalchi (1999).

1.11 Omitted Proofs

1.11.1 Proof of Lemma 2

Take any \( \mu, \mu' \in \Delta(X) \), and fix any \( \epsilon > 0 \). By definition of \( d_X (\mu, \mu') \), there exists \( \nu \in \Delta_{\mu,\mu'} \) such that
\[
E_{\nu} [d_X (x_1, x_2)] \leq d_X (\mu, \mu') + \epsilon. 
\] (1.25)

Define \( \bar{f} : X^2 \to Z^2 \) by \( \bar{f} (x_1, x_2) = (f (x_1), f (x_2)) \). Then, by definition, \( \nu \circ \bar{f}^{-1} \in \Delta_{\mu \circ f^{-1}, \mu' \circ f^{-1}} \).

Hence,
\[
d_Z (\mu \circ f^{-1}, \mu' \circ f^{-1}) \leq E_{\nu \circ \bar{f}^{-1}} [d_Z (z_1, z_2)] = E_{\nu} [d_Z (f (x_1), f (x_2))] \leq E_{\nu} [\lambda d_X (x_1, x_2)] = \lambda E_{\nu} [d_X (x_1, x_2)] \leq \lambda d_X (\mu, \mu') + \epsilon;
\]
since \( \epsilon \) is arbitrary, the result follows. [Here, the first inequality is by (1.20); the next equality is by change of variables, the next inequality is by the hypothesis, and the last inequality is by (1.25).] □
1.11.2 Proof of Proposition 1

We will now show that global stability implies that our game is dominance solvable. Beforehand, we formally define our elimination method and develop some notation needed in the proof.

**Rationalizability** Assume that, for each player $i$, his best response correspondence is single-valued, given by the best response function $BR_i$. (Recall that this is the case whenever $u_i$ is continuous and strictly quasi-concave, and $A_i$ is convex.) For each $i$, let $M_i \subset A_i^{Ti}$ be the set of all measurable functions $s_i : T_i \rightarrow A_i$, i.e., the set of all allowable strategies for player $i$. Define sets $S_i^k$, $k = 0, 1, \ldots$, iteratively as follows. Set $S_i^0 = A_i^{Ti}$. For each $k > 0$, let $\Sigma_{-i}^{k-1} = \Delta \left( S_{-i}^{k-1} \cap M_{-i} \right)$ be the set of all possible beliefs of player $i$ on other players’ allowable strategies that are not eliminated in the first $k-1$ rounds. Write $\pi_{\tau_i, \sigma_{-i}}$ for the induced beliefs of $i$ on $\Theta \times A_{-i}$ by his type $\tau_i$ and his belief $\sigma_{-i} \in \Delta (M_{-i})$ about the other players’ strategies. Write $S_i^k (\tau_i) = \left\{ BR_i (\pi_{\tau_i, \sigma_{-i}}) | \sigma_{-i} \in \Sigma_{-i}^{k-1} \right\}$ for the set of all best responses of $i$ with type $\tau_i$ against all his beliefs in $\Sigma_{-i}^{k-1}$, and set $S_i^k = \prod_{\tau_i \in T_i} S_i^k (\tau_i)$. The set of all rationalizable strategies for $i$ is $S_i^\infty = \bigcap_{k=0}^\infty S_i^k \cap M_i$.

**Proof of Proposition 1.** For each non-negative integer $k$, define

$$D_k = \sup_{i \in \mathbb{N}, \tau_i \in T_i, s_i, s'_i \in S_i^k} d_i (s_i (\tau_i), s'_i (\tau_i)).$$

We will show that $\lim_{k \to \infty} D_k = 0$; therefore, there cannot be any two distinct actions $s_i (\tau_i)$ and $s'_i (\tau_i)$ available to any type $\tau_i$ of any player $i$ in the limit of the process of elimination, showing the first part. The second part simply follows from the observation that $s_i^* \in S_i^k$ for each $k$, hence $s_i^* \in S_i^\infty$.

Towards showing that $\lim_{k \to \infty} D_k = 0$, assume global stability for some parameter $b$ and metric $d_{-i}$, and take any $k$, and any $i \in \mathbb{N}, \tau_i \in T_i, s_i, s'_i \in S_i^k$. By definition, $s_i (\tau_i) = BR_i (\pi_{\tau_i, \sigma_{-i}})$ and $s'_i (\tau_i) = BR_i (\pi_{\tau_i, \sigma'_{-i}})$ for some $\sigma_{-i}, \sigma'_{-i} \in \Sigma_{-i}^{k-1}$. Hence,

$$d_i (s_i (\tau_i), s'_i (\tau_i)) = d_i \left( BR_i \left( \pi_{\tau_i, \sigma_{-i}} \right), BR_i \left( \pi_{\tau_i, \sigma'_{-i}} \right) \right) \leq b d_{-i} \left( \pi_{\tau_i, \sigma_{-i}}, \pi_{\tau_i, \sigma'_{-i}} \right). \quad (1.26)$$

On the other hand, for each $\omega = (\theta, \tau)$, define $\mu_\omega = \delta_\omega \times (\sigma_{-i} \circ \rho_\omega^{-1})$ and $\mu'_\omega = \delta_\theta \times (\sigma'_{-i} \circ \rho_{\omega}^{-1})$, where $\rho_{s_{-i}} : s_{-i} \mapsto s_{-i} (\tau_{-i})$ and $\delta_\theta$ is the point mass at $\theta$; define also $\tilde{\mu}_\omega = \mu_\omega \times \mu'_\omega$. Notice that
\( \mu_\omega \) and \( \mu'_\omega \) are the probability distributions on \( \Theta \times A_{-i} \) conditional on \( \omega \), induced by beliefs \( \sigma_{-i} \) and \( \sigma'_{-i} \), respectively. Notice also that
\[
\pi_{\tau_i, \sigma_{-i}} = \int \mu_\omega (\cdot) d\kappa_{\tau_i}(\omega), \quad \pi_{\tau_i, \sigma'_{-i}} = \int \mu'_\omega (\cdot) d\kappa_{\tau_i}(\omega),
\]
and \( \nu \equiv \int \mu_\omega (\cdot) d\kappa_{\tau_i}(\omega) \in \Delta_{\pi_{\tau_i, \sigma_{-i}}, \pi_{\tau_i, \sigma'_{-i}}} \). Moreover, since \( \sigma_{-i}, \sigma'_{-i} \in \Sigma_{-i}^{k-1} \), given any \( \omega = (\theta, \tau) \), and any \(( (\theta, s_{-i}(\tau_{-i})), (\theta, s'_{-i}(\tau_{-i})) ) \) \( \in \tilde{\mu}_\omega \), we have
\[
\bar{d}_{-i}((\theta, s_{-i}(\tau_{-i})), (\theta, s'_{-i}(\tau_{-i}))) = d_{-i}(s_{-i}(\tau_{-i}), s'_{-i}(\tau_{-i})) \leq D_{k-1},
\]
yielding
\[
\bar{d}_{-i}(\pi_{\tau_i, \sigma_{-i}}, \pi_{\tau_i, \sigma'_{-i}}) \leq E_{\nu} [\bar{d}_{-i}(x_1, x_2)] = E_{\tilde{\mu}} [\bar{d}_{-i}(x_1, x_2)] \leq D_{k-1}. \tag{1.27}
\]
By combining, (1.26) and (1.27), we obtain
\[
d_{i}(s_{i}(\tau_{i}), s'_{i}(\tau_{i})) \leq bD_{k-1}.
\]
By taking the supremum on both sides, we obtain
\[
D_k \leq bD_{k-1}.
\]
Therefore, \( 0 \leq D_k \leq b^kD_0 \), showing that \( \lim_{k \to \infty} D_k = 0 \).

1.11.3 Proof of Lemma 1

Let \( \tilde{P} \equiv \times P = \times P' \). Since we have separable standard Borel spaces, there exists conditional probability \( P(\cdot|\cdot) : (\Sigma_X \times Y) \times (X \times Y) \to [0, 1] \) with respect to the \( \sigma \)-field \( \Sigma_X \times \{Y\} \), and we simply write \( P(B|x) \) for \( P(X \times B||x, y) \) where \( y \) can be chosen arbitrarily. We define \( P'(C|x) \) similarly for each \( C \in \Sigma_Z \). Notice that \( P(\cdot|x) \) and \( P'(\cdot|x) \) are probability distributions on \( (Y, \Sigma_Y) \) and \( (Z, \Sigma_Z) \), respectively.\(^8\) For each \( x \in X \), let
\[
\tilde{P}_x \equiv P(\cdot|x) \times P'(\cdot|x)
\]
be the product measure of \( P(\cdot|x) \) and \( P'(\cdot|x) \) on \( Y \times Z \), and define probability measure \( \tilde{P} \) by setting
\[
\tilde{P}(F) = \int \tilde{P}_x(F_Z) d\tilde{P}(x)
\]
\(^8\)See Parthasaraty (1967) for the results of probability theory in this proof.
at each measurable set $F \subseteq X \times Y \times Z$ where

$$F_x = \{(y,z) \in Y \times Z | (x,y,z) \in F \}.$$ 

Notice that, for any rectangle $\Theta \times B \times C \in \Sigma_X \times \Sigma_Y \times \Sigma_Z$, 

$$\tilde{P} (\Theta \times B \times C) = \int \chi_{\Theta} (x) P (B|x) P' (C|x) d\tilde{P} (x),$$

where $\chi_{\Theta}$ denotes the characteristic function of $\Theta$.

Now we show that $\tilde{P}$ satisfies the statement of the lemma. For each $\Theta \in \Sigma_X$ and $B \in \Sigma_Y$, we have

$$x_{XY} \tilde{P} (\Theta \times B) = \tilde{P} (\Theta \times B \times Z) = \int \chi_{\Theta} (x) P (B|x) P' (Z|x) d\tilde{P} (x) = \int \chi_{\Theta} (x) P (B|x) d\tilde{P} (x) \equiv P (\Theta \times B).$$

Since the probability measures $x_{XY} \tilde{P}$ and $P$ agree on the $\pi$-system of all rectangles $\Theta \times B$, which generates the entire $\sigma$-field on $X \times Y$, by Dynkin’s $\pi$-$\lambda$ Theorem they are equal. This is similarly true for $x_{XZ} \tilde{P}$ and $P'$.

### 1.11.4 Proof of Proposition 3

Under Assumptions 1a and 1b, take any $i \in N$. Firstly, if $\beta_i = 0$, then $g_i (\theta, a_{-i}) = g_i (\theta, a'_{-i}) = \tilde{g}_i (\theta)$ for each $(\theta, a_{-i}, a'_{-i})$, hence, for each $\pi, \pi'$ with $\Theta \pi = \Theta \pi'$, we have $BR_i (\pi) = f_i (E_\pi (\tilde{g}_i (\theta))) = f_i (E_{\pi'} (\tilde{g}_i (\theta))) = BR_i (\pi')$, yielding $d_i (BR_i (\pi), BR_i (\pi')) = 0 \leq b_i d_{-i} (\pi, \pi')$ for any $d_{-i}$.

Now assume that $\beta_i > 0$. Since $g_i$ is continuous and $\Theta \times A_{-i}$ is compact, there exists $M_i > 0$ such that

$$d_X (g_i (\theta, a_{-i}), g_i (\theta', a'_{-i})) \leq M_i \quad (\forall \theta, a_{-i}, \theta', a'_{-i}). \quad (1.28)$$

Define a metric $d_{\Theta,i}$ on $\Theta$ by setting $d_{\Theta,i} (\theta, \theta') = M_i / \beta_i$ at each distinct $\theta, \theta'$, and define $d_{-i}$
on $\Theta \times A_{-i}$ by
\[
d_{-i}((\theta, a_{-i}), (\theta', a'_{-i})) = d_{\Theta, i}(\theta, \theta') + d_{-i}(a_{-i}, a'_{-i}).
\]

Now, take any two random variables $(\theta, a_{-i}) \sim \pi$ and $(\theta', a'_{-i}) \sim \pi'$ that come from the same probability space and write $p$ for the probability that $\theta \neq \theta'$. Note that
\[
E \left[ d_{-i}((\theta, a_{-i}), (\theta', a'_{-i})) \right] = p M_i / \beta_i + E \left[ d_{-i}(a_{-i}, a'_{-i}) \right].
\] (1.29)

Moreover, we have
\[
d_i(BR_i(\pi), BR_i(\pi')) \leq \alpha_i E \left[ d_X \left( g_i(\theta, a_{-i}), g_i(\theta, a'_{-i}) \right) \right]
= \alpha_i E \left[ d_X \left( g_i(\theta, a_{-i}), g_i(\theta, a'_{-i}) \right) : \theta \neq \theta' \right]
+ \alpha_i E \left[ d_X \left( g_i(\theta, a_{-i}), g_i(\theta', a'_{-i}) \right) : \theta = \theta' \right]
\leq \alpha_i p M_i + \alpha_i \beta_i E \left[ d_{-i}(a_{-i}, a'_{-i}) : \theta = \theta' \right]
\leq \alpha_i p M_i + \alpha_i \beta_i E \left[ d_{-i}(a_{-i}, a'_{-i}) \right]
= b_i \left( p M_i / \beta_i + E \left[ d_{-i}(a_{-i}, a'_{-i}) \right] \right)
= b_i E \left[ \bar{d}_{-i}((\theta, a_{-i}), (\theta', a'_{-i})) \right],
\]
where the first inequality is derived as in the proof of Proposition 3, the next equality is by additivity, the next equality is by (1.28) and the Lipschitz continuity of $g_i$, the next inequality is by the non-negativity of $d_{-i}$, and the last two equalities are by definition of $b_i$ and (1.29).

Since $(\theta, a_{-i}) \sim \pi$ and $(\theta', a'_{-i}) \sim \pi'$ are arbitrary, this shows that $d_i(BR_i(\pi), BR_i(\pi')) \leq b_i E \left[ \bar{d}_{-i}(\pi, \pi') \right]$.

### 1.11.5 Proof of Proposition 4

Take any $\pi, \pi'$ in $\Delta(\Theta \times A_{-i})$ with $\operatorname{marg}_\Theta \pi = \operatorname{marg}_\Theta \pi' = t_i^1$. We will assume $BR_i(\pi)$ and $BR_i(\pi')$ are in the interior of $A_i$. (When $A_i$ is convex, we can take $BR_i(\pi)$ and $BR_i(\pi')$ as the unconstrained optima, as in that case the variations in the constrained optima are if anything less than the variations in unconstrained optima.) We write $u_i^1$, $u_{ii}^1$, and $u_{ij}^1$ for the first and second order partial derivatives of $u^i$ with respect to $a_i$, and the cross partial with respect $a_i$ and $a_j$, respectively. Firstly, since $BR_i(\pi)$ and $BR_i(\pi')$ are in the interior, the first order
conditions for optimization problems with $\pi$ and $\pi'$ yield

$$
\int u_i(\Theta, BR_i(\pi), a_{-i})d\pi(\Theta, a_{-i}) = 0
$$

(1.30)

and

$$
\int u_i(\Theta, BR_i(\pi'), a_{-i})d\pi'(\Theta, a_{-i}) = 0,
$$

(1.31)

respectively. Let

$$
J = \int u_i(\theta, BR_i(\pi), a_{-i})d\pi'(\theta, a_{-i})
$$

be the value of the derivative at $BR_i(\pi)$ for the optimization problem with $\pi'$. We will now find upper and lower bounds for $|J|$, and these bounds will yield (??). First we find an upper bound. Letting $\gamma$ be an arbitrary element of $X_{\pi,\pi'}$,

$$
|J| = \left| \int u_i(\theta, BR_i(\pi), a_{-i})d\pi'(\theta, a_{-i}) \right|
$$

(1.32)

$$
= \left| \int u_i(\theta, BR_i(\pi), a_{-i})d\pi'(\theta, a_{-i}) - \int u_i(\theta, BR_i(\pi), a_{-i})d\pi(\theta, a_{-i}) \right|
$$

$$
= \left| \int u_i(\theta, BR_i(\pi), a_{-i}) - u_i(\theta, BR_i(\pi), a'_{-i})d\gamma(\theta, a_{-i}, a'_{-i}) \right|
$$

$$
\leq \int \left[ \sum_{j \neq i} \max_a d_{ij} (a; \theta) \right] d\gamma(\theta, a_{-i}, a'_{-i})
$$

$$
\leq \sum_{j \neq i} \max_a d_{ij} (a; \theta) \int [d_{ij} (a_{-i}, a'_{-i})] d\gamma(\theta, a_{-i}, a'_{-i})
$$

(1.34)

Here the first equality is by definition, the second equality is by (1.30), and the following inequality is is by the triangle inequality. To derive the penultimate inequality, we write $u_i(BR_i(\pi), a'_{-i}) - u_i(BR_i(\pi), a_{-i})$ as the sum of the changes that we would get by changing each coordinate in turn, and apply the mean value theorem to each, obtaining

$$
\left| u_i(BR_i(\pi), a'_{-i}) - u_i(BR_i(\pi), a_{-i}) \right| \leq \sum_{j \neq i} \max_a \left| u_{ij} (a; \theta) \right| \left| a_j - a'_{j} \right|
$$

$$
\leq \sum_{j \neq i} \max_a \left| u_{ij} (a; \theta) \right| d_{-i} (a_{-i}, a'_{-i})
$$
where the last inequality is by our definition of the metric $d_{-i}$. To find our lower bound, we write

$$|J| = \left| \int u_i(\theta, BR_i(\pi), a_{-i}) d\pi'(\theta, a_{-i}) \right|$$

$$= \left| \int u_i(\theta, BR_i(\pi), a_{-i}) d\pi'(\theta, a_{-i}) - \int u_i(\theta, BR_i(\pi'), a_{-i}) d\pi'(\Theta, a_{-i}) \right|$$

$$= \left| \int u_i(\theta, BR_i(\pi), a_{-i}) - u_i(\theta, BR_i(\pi'), a_{-i}) d\pi'(\Theta, a_{-i}) \right|$$

$$\geq \int \left[ \min_a |u_{ii}^i(a; \theta)| \right] \left| BR_i(\pi) - BR_i(\pi') \right|$$

$$= \min_a |u_{ii}^i(a; \theta)| \left| BR_i(\pi) - BR_i(\pi') \right|. \quad (1.35)$$

Here the first and the second equalities are by definition and (1.31), respectively. The third equality is crucial; we have equality here because $U_i^i(a_{-i})$ is strictly decreasing, and hence $U_i^i(BR_i(\mu), a'_{-i}) - U_i^i(BR_i(\mu'), a'_{-i})$ never changes its sign. The inequality in the next line is again by the mean value theorem, and the last equality is because the term inside the expectation is a constant. Combining (1.34) and (1.35) and observing that $d_{-i}(\pi, \pi') = \inf_{\gamma_{\Delta(\Theta \times A_{-i})}} \int d_{-i}(a_{-i}, a'_{-i}) d\gamma(\theta, a_{-i}, a'_{-i})$ and that $\gamma$ was arbitrary, we obtain

$$|BR_i(\pi) - BR_i(\pi')| \leq d_{-i}(\pi, \pi') \sum_{j \neq i} \frac{\max_a |U_{ij}^i(a; \theta)|}{\min_a |u_{ii}^i(a; \theta)|}.$$

Check that $\max_a \int u_{ij}^i(a; \theta) dt_i^1(\theta) \leq \int \max_a |\partial^2 u_i(\theta, a) / \partial a_1 \partial a_j| dt_i^1(\theta)$ and $\min_a \int u_{ii}^i(a; \theta) dt_i^1(\theta) \geq \int \min_a |\partial^2 u_i(\theta, a) / \partial a_2^2| dt_i^1(\theta)$. Therefore,

$$\sum_{j \neq i} \frac{\max_a |U_{ij}^i(a; t_i^1)|}{\min_a |U_{ii}^i(a; t_i^1)|} \leq \sum_{j \neq i} \frac{\max_a |\partial^2 u_i(\theta, a) / \partial a_1 \partial a_j| dt_i^1(\theta)}{\min_a |\partial^2 u_i(\theta, a) / \partial a_2^2| dt_i^1(\theta)}$$

$$\leq \sum_{j \neq i} \frac{\max_a |\partial^2 u_i(\theta, a) / \partial a_1 \partial a_j|}{\min_a |\partial^2 u_i(\theta, a) / \partial a_2^2| dt_i^1(\theta)}$$

$$\leq \frac{\max_j \sum_{j \neq i} \max_a |\partial^2 u_i(\theta, a) / \partial a_1 \partial a_j|}{\min_a |\partial^2 u_i(\theta, a) / \partial a_2^2|} < 1,$$

completing the proof.
Bibliography


Chapter 2

Finite-order implications of any equilibrium

"Game theory ... is deficient to the extent it assumes other features to be common knowledge, such as one player’s probability assessment about another’s preferences or information. I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumption will the theory approximate reality." Wilson (1987)

2.1 Introduction

Nash equilibrium is the fundamental solution concept in modern economic analysis. It is well-understood that equilibrium outcomes may be highly dependent on informational assumptions. Nevertheless, most economic theories include a very specific informational assumption, usually without sufficient justification. They take a type structure, where a type is a fundamental payoff parameter or a belief (i.e., signal) about fundamentals, and assume that the specified type structure is common knowledge. Formally, they close the model after specifying the first and second-order beliefs.¹ That is to say, conditional on the first-order beliefs, all of the players’

¹Here the first-order beliefs are the beliefs about fundamentals (e.g., signals), and the second-order beliefs are beliefs about other players’ beliefs about fundamentals, specified through the joint distribution of the signals.
higher-order beliefs are assumed to be common knowledge. Since this and similar assumptions about higher-order beliefs may easily fail in the actual incomplete-information situation modeled, these theories may be misleading when the impact of higher-order beliefs on equilibrium behavior is large. There are examples that show that this impact might be large in some situations (Rubinstein (1989)), and we need to revise important theories, such as the Coase conjecture (Feinberg and Skrzypacz (2002)) and the surplus extraction property (Neeman (2004), Heifetz and Neeman (2003)), once we relax the common knowledge assumption. In this paper we characterize, for generic games, the set of predictions that are robust to assumptions about higher-order beliefs. Alternatively, we measure the sensitivity of behavior to these assumptions within a fixed equilibrium. Our results suggest that many more predictions are non-robust than previously thought.

These models typically have a large number of equilibria. Recognizing this, game theorists exerted tremendous effort in the last several decades to refine the concept of equilibrium, resulting in a multitude of refinement concepts. In application the researchers typically use these refinements (or some specific arguments) to focus on a particular equilibrium or small class of equilibria. Most of the statements in economic theory are predictions about the players' behavior according to these selected equilibria. Our methodology allows us to check the robustness of such predictions with respect to assumptions about higher-order beliefs.

Consider a situation where players have incomplete information about some payoff-relevant parameter. Imagine a researcher who has computed an equilibrium of this game in the universal type space, where a type of a player is given by the infinite hierarchy of his beliefs—his first-order beliefs, second-order beliefs, etc. The researcher would like to make a prediction about the action of a player \( i \) according to this equilibrium. Fix a type \( t_i \) of player \( i \) as his actual type, and write \( A_i^1 (t_i) \) for the set of all actions that are played by some alternative type of \( i \) whose first-order beliefs agree with \( t_i \). This set is the set of actions that the researcher cannot rule out if he only knows the first-order beliefs and assumes that the player plays according to the equilibrium. Similarly, write \( A_i^k (t_i) \) for the set of actions that the researcher cannot rule out if he only knows the first \( k \) orders of beliefs. Write \( A_i^\infty (t_i) \) for the limit of these (decreasing) sets as \( k \) approaches infinity, i.e., the set of all actions that cannot be ruled out by the researcher by looking at (arbitrarily many) finite orders of beliefs.
In a model that is closed at order $k$, all higher-order beliefs are determined by the first $k$ orders of beliefs and the assumption that is made to close the model. Fixing an equilibrium of this closed model, we wish to analyze how the model’s predictions according to this equilibrium are sensitive to the closing assumption. Dropping the closing assumption, one can construct a universal type space that contains the model as a subspace. Fix an equilibrium on the universal type space whose restriction to the model is the fixed equilibrium above. According to this equilibrium, the model predicts a unique action for each possible set of beliefs at orders 1 through $k$, namely the equilibrium action for the complete type implied by this set of beliefs and the closing assumption. But in the general model, every other action in $A^k_i(t_i)$ is played by a type whose first $k$ orders of beliefs will be exactly as this type (but will fail the closing assumption.) Therefore, we cannot rule out any action in $A^k_i(t_i)$ without resorting to the closing assumption. Therefore, a prediction based on the fixed equilibrium is robust to the closing assumption if and only if the prediction remains true for each selection from $A^k_i$.

Our main result gives a lower bound for $A^k_i(t_i)$. Assume that the space of underlying uncertainty is rich enough so that our fixed equilibrium has full range, i.e., every action is played by some type. (This assumption is without loss of generality. In Sections 2.6 and 2.8, we extend our results beyond this assumption.) For countable-action games, we show that $A^k_i(t_i)$ includes all actions which survive the first $k$ iterations of eliminating all actions which are never a strict best reply under $t_i$. That is, a prediction of a model that is closed at order $k$ is robust to the model’s assumptions about higher-order beliefs only if the prediction remains true for all actions that survive the first $k$ iterations of this elimination process. In particular, $A^\infty_i(t_i)$ includes all actions that survive iterated elimination of actions that cannot be a strict best reply.

Towards a full characterization of $A^k_i(t_i)$, we also provide an upper bound: $A^k_i(t_i)$ is a subset of the actions that survive the first $k$ iterations of eliminating strictly dominated actions, and hence $A^\infty_i(t_i)$ is a subset of the rationalizable actions. When there are no ties for best response, these elimination procedures lead to the same outcome, and therefore $A^\infty_i(t_i)$ is precisely equal to the set of rationalizable outcomes. That is, for generic games, a prediction of a model that is closed at order $k$ is robust to the model’s assumptions about higher-order beliefs if only if the prediction remains (approximately) true for all actions that survive $k$th-order iterated
dominance. We extend this characterization to *nice games*, where the action spaces are one-dimensional compact intervals and the utility functions are strictly concave in own action and continuous, as in many classical economic models.

To illustrate the main argument in the proof of the lower bound, we now explain why $A^1_i (t_i)$ includes all actions that survive the first round of elimination process. Let $t_i$ vary over the set of types that agree with $t_i$ at first order (i.e., concerning the underlying parameter) but may have any beliefs at higher orders (i.e., concerning the other players’ type profile.) Our full-range assumption implies that there are types $t_i$ with any beliefs whatsoever about other players’ equilibrium action profile. Given any action $a_i$ of $i$ that is a strict best reply to his fixed belief about the parameter and some belief about the other players’ actions, there is a type $t_i$ who has these beliefs in equilibrium, and therefore must play the strict best reply, $a_i$, in equilibrium. This argument will be formalized as part of an inductive proof of the main result.

Our result has serious implications on important research areas:

- **Wilson Doctrine:** Wilson (1987) proposes to approximate the unrestricted model by closing the model at higher orders, specifying more orders of beliefs, and hence weakening the common knowledge assumption. Assuming no ties and equilibria with full range, our result shows that this program will be successful for and only for the predictions that are true for all rationalizable outcomes. The proposed approximation is then possible under an equilibrium if and only if the game is dominance-solvable.

- **Continuity of Equilibrium:** It is well known that *some* Nash equilibria may be discontinuous in product topology and with respect to higher-order uncertainty, as in the electronic-mail game of Rubinstein (1989). There is an interest in understanding how severe this discontinuity is. Monderer and Samet (1989,1997) and Kajii and Morris (1998) have analyzed the weakest topologies that make the equilibrium continuous over *all* games (see also Milgrom and Weber (1985) for a continuity result.) These topologies are quite strong, but since they focus on the worst-case games, such as the electronic-mail game, it is not clear whether the equilibria used in applications will be highly sensitive to higher-order uncertainty. We show that *every equilibrium* with full range is discontinuous for *every game* at *every type* for which two or more actions survive our elimination process.
- **Equilibrium Refinements in Normal-form:** Our result shows that, for generic games with (potentially) rich spaces of fundamentals, an equilibrium refinement does not produce any new robust prediction.\(^2\) That is, any new prediction gained by the equilibrium refinement will also be dependent on the specific closing assumption and will be invalid under different assumptions on higher-order beliefs.

- **Global Games:** In this literature, one uses a limit argument to selects an equilibrium among multiple strict equilibria, but we show that every equilibrium with full range is discontinuous at the limit.

- **Ex-post Equilibrium:** Our result shows that predictions of ex-post equilibrium (computed within a traditional type space) need not be robust against higher-order beliefs—contrary to common perception.

There is a close link between higher-order reasoning and the impact of higher-order uncertainty within a fixed equilibrium with full range. When there are no ties, assuming \(k\)th-order mutual knowledge of payoffs and that the fixed equilibrium is played is equivalent to assuming \(k\)th-order mutual knowledge of rationality and common knowledge of payoffs.\(^3\) This implies that when the equilibrium is sensitive to high-order beliefs, the impact of high-order failures of rationality is also large.

Rationalizability characterizes the strategy profiles that are consistent with common knowledge of rationality (Tan and Werlang (1988)). The latter assumption is equivalent to equilibrium in a type space with epistemic types (describing players’ beliefs about the actions played). That is, there exists a type space—with epistemic types—in which each rationalizable strategy is played by a type. Brandenburger and Dekel (1987) further shows that rationalizability and a posteriori equilibrium, a refinement of subjective correlated equilibrium, yield the same payoff distributions. Therefore, once one allows for subjectivity in players’ priors and arbitrary type spaces (with possibly epistemic types), then any equilibrium prediction that universally holds for all equilibria must also hold for all rationalizable strategies. Allowing private information,

\(^2\)For dynamic games, which are usually non-generic, extensive-form refinements based on forward induction may lead to strong robust predictions (Battigalli and Siniscalchi (2003)).

\(^3\)The relationship between assumptions about rationality and payoff uncertainty is not straightforward; \(A_{ir}\) may differ from both rationalizability and iterative admissibility.
Battigalli and Siniscalchi (2003) introduces a notion of $\Delta$-rationalizability that incorporates a given restriction $\Delta$ on the first-order beliefs (about both underlying uncertainty and the strategies) into the elimination process. Once again, as they show, there exists a type space with epistemic types in which each $\Delta$-rationalizable strategy is played by some type in an equilibrium that satisfies the restriction $\Delta$. In contrast, we analyze the sensitivity of an arbitrary fixed equilibrium to the assumptions on higher-order beliefs and characterize the robust predictions by usual rationalizability in generic games, without using any epistemic types.

Our next section contains the basic definitions and preliminary results. We formulate our notion of robustness in Section 2.3. We present and prove our main results in Section 2.4. Our main theorem is extended to the nice games as a characterization in Section 2.5, and to mixed strategies and to the spaces of uncertainty that are not necessarily rich in Section 2.6. In Section 2.7, we present our discontinuity results and discuss their methodological implications for global games and robustness of equilibria. Section 2.8 contains a very negative result about Cournot oligopoly as an application. We review the literature in Section 2.9. Section 2.10 concludes. Some of the proofs are relegated to the appendix.

### 2.2 Basic Definitions and Preliminary Results

**Notation 1** Given any list $Y_1, \ldots, Y_n$ of sets, write $Y = \prod_i Y_i$, $Y_{-i} = \prod_{j \neq i} Y_j$, $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$, and $(y_i, y_{-i}) = (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)$. Likewise, for any family of functions $f_j : Y_j \to Z_j$, we define $f_{-i} : Y_{-i} \to Z_{-i}$ by $f_{-i}(y_{-i}) = (f_j(y_j))_{j \neq i}$. Given any metric space $(Y, d)$, we write $\Delta(Y)$ for the space of probability distributions on $Y$, suppressing the fixed $\sigma$-algebra on $Y$ which at least contains all open sets and singletons; we use the product $\sigma$-algebra in product spaces. The support of a probability distribution $\pi$ is denoted by $(\pi)$.

We consider a game with finite set of players $N = \{1, 2, \ldots, n\}$. The source of underlying uncertainty is a payoff-relevant parameter $\theta \in \Theta$ where $(\Theta, d)$ is a compact, complete and separable metric space, with $d$ a metric on set $\Theta$. Each player $i$ has action space $A_i$ and utility function $u_i : \Theta \times A \to \mathbb{R}$, where $A = \prod_i A_i$. We endow the game with the universal type space of Brandenburger and Dekel (1993), a variant of an earlier construction by Mertens and Zamir (1985), with an additional assumption that the players’ beliefs at each finite order have count-
able (or finite) support. Types are defined using the auxiliary sequence \(\{X_k\}\) of sets defined inductively by \(X_0 = \Theta\) and \(X_k = \left[\hat{\Delta} (X_{k-1})\right]^n \times X_{k-1}\) for each \(k > 0\), where \(\hat{\Delta} (X_{k-1})\) is the set of probability distributions on \(X_{k-1}\) that have countable (or finite) support. We endow each \(X_k\) with the weak topology and the \(\sigma\)-algebra generated by this topology. A player \(i\)'s first order beliefs (about the underlying uncertainty \(\theta\)) are represented by a probability distribution \(t_i^1\) on \(X_0\), second order beliefs (about all players' first order beliefs and the underlying uncertainty) are represented by a probability distribution \(t_i^2\) on \(X_1\), etc. Therefore, a type \(t_i\) of a player \(i\) is a member of \(\prod_{k=1}^\infty \hat{\Delta} (X_{k-1})\). Since a player's \(k\)th-order beliefs contain information about his lower-order beliefs, we need the usual coherence requirements. We write \(T = \prod_{i \in N} T_i\) for the subset of \(\left(\prod_{k=1}^\infty \hat{\Delta} (X_{k-1})\right)^n\) in which it is common knowledge that the players' beliefs are coherent, i.e., the players know their own beliefs and their marginals from different orders agree.

We will use the variables \(t_i, \hat{t}_i \in T_i\) as generic types of any player \(i\) and \(t, \hat{t} \in T\) as generic type profiles. For every \(t_i \in T_i\), there exists a probability distribution \(\kappa_{t_i}\) on \(\Theta \times T_{-i}\) such that

\[
t_i^k = \delta_{t_i^{k-1}} \times \text{marg}_{\Theta \times [\Delta (X_{k-2})]}^{N \setminus \{i\}} \kappa_{t_i}, \quad (\forall k)
\]

and \(t_i^1 = \text{marg}_\Theta \kappa_{t_i}\), where \(\delta_{t_i^{k-1}}\) is the probability measure that puts probability 1 on the set \(\{t_i^{k-1}\}\) and marg denotes the marginal distribution. Conversely, given any distribution \(\kappa_{t_i}\) on \(\Theta \times T_{-i}\), we can define \(t_i \in T_i\) via (2.1), as long as \(\text{marg}_{\Theta \times [\Delta (X_{k-2})]}^{N \setminus \{i\}} \kappa_{t_i}\) is always countable.

A strategy of a player \(i\) is any measurable function \(s_i : T_i \rightarrow A_i\). Given any type \(t_i\) and any profile \(s_{-i}\) of strategies, we write \(\pi (|t_i, s_{-i}) \in \Delta (\Theta \times A_{-i})\) for the joint distribution of the underlying uncertainty and the other players' actions induced by \(t_i\) and \(s_{-i}\); \(\pi (|t, s_{-i})\) is similarly defined for correlated mixed strategy profile \(s_{-i}\). For each \(i \in N\) and for each belief \(\pi \in \Delta (\Theta \times A_{-i})\), we write \(BR_i (\pi)\) for the set of actions \(a_i \in A_i\) that maximize the expected value of \(u_i (\theta, a_i, a_{-i})\) under the probability distribution \(\pi\). A strategy profile \(s^* = (s_1^*, s_2^*, \ldots)\) is a Bayesian Nash equilibrium iff at each \(t_i\),

\[
s_i^* (t_i) \in BR_i (\pi (|t_i, s^*_{-i}))
\]

Our type space is dense in universal type space, and any countable type space with no redundant type is embedded in our space.

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\(^4\text{This assumption is made to avoid technical issues related to measurability (see Remark 2.) Our type space is dense in universal type space, and any countable type space with no redundant type is embedded in our space.}\)
An equilibrium \( s^* \) is said to have full range iff

\[ s^* (T) = A. \]  

We extend our analysis beyond this full range assumption in Section 2.6. Moreover, this assumption is without loss of generality, as we can restrict the action space to \( s^* (T) \), by eliminating the actions that are never played in equilibrium \( s^* \). Finally, full range is implied by the assumption in the following lemma, which states that the state space \( \Theta \) is sufficiently rich. This assumption corresponds to allowing the broadest possible set of beliefs about other players’ payoffs, which should be allowed when we drop all common-knowledge assumptions.

**Lemma 3** Assume that, given any \( i \in N \), any \( \mu \in \Delta (A_{-i}) \), and any \( a_i \), there exists a probability distribution \( \nu \) on \( \Theta \) with countable support and such that

\[ BR_i (\nu \times \mu) = \{a_i\}. \]

Then, every equilibrium \( s^* \) has full range.

**Proof.** The proofs that are omitted in the text are in the appendix. ■

**Elimination Processes** We will use interim notions and allow correlations not only within players’ strategies but also between their strategies and the underlying uncertainty \( \theta \). Such correlated rationalizability is introduced by Battigalli (2003), Battigalli and Siniscalchi (2003) and Dekel, Fudenberg, and Morris (2003). Clearly, allowing such correlation only makes our sets larger. Since our main result is a lower bound in terms of these sets, this only strengthens our result. Moreover, our characterization provides yet another justification for this correlated rationalizability. Write \( M_i \) for the set of all measurable functions from \( \Theta \times T_i \) to \( A_i \). Towards defining rationalizability, define sets \( S^k_i [t_i], i \in N, t_i \in T_i, k = 0, 1, \ldots, \) iteratively as follows. Set \( S^0_i [t_i] = A_i \). For each \( k > 0 \), let \( \hat{S}^{k-1}_{-i} \subset M_{-i} \) be the set of all measurable functions \( f : \Theta \times T_{-i} \rightarrow A_{-i} \) such that \( f (\theta, t_{-i}) \in S^{k-1}_{-i} [t_{-i}] \) for each \( t_{-i} \). Let also \( \Sigma^{k-1}_{-i} \) be the set of all probability distributions on \( \hat{S}^{k-1}_{-i} \). Note that \( \Sigma^{k-1}_{-i} \) is the set of all possible beliefs of player \( i \) on
other players’ allowable actions that are not eliminated in the first \( k - 1 \) rounds. Write

\[ S^k_i [t_i] = \bigcup_{\sigma_{-i} \in \Sigma^{k-1}_{-i}} BR_i (\pi (\cdot | t_i, \sigma_{-i})) \]

for the set of all all actions \( a_i \) of \( i \) that are best reply against some of his beliefs in \( \Sigma^{k-1}_{-i} \). The set of all rationalizable actions for player \( i \) (with type \( t_i \)) is

\[ S^\infty_i [t_i] = \bigcap_{k=0}^\infty S^k_i [t_i] . \]

Next we define the set of strategies that survive iterative elimination of strategies that are never strict best reply, denoted by \( W^\infty_i [t_i] \), similarly. We set \( W^0_i [t_i] = A_i \) and

\[ W^k_i [t_i] = \left\{ a_i | BR_i (\pi (\cdot | t_i, \sigma_{-i})) = \{ a_i \} \text{ for some } \sigma_{-i} \in \Delta \left( \hat{W}^{k-1}_{i,-i} \right) \right\} , \]

where \( \hat{W}^{k-1}_{i,-i} \subset M_{-i} \) is the set of all functions \( f : \Theta \times T_{-i} \to A_{-i} \) such that \( f (\theta, t_{-i}) \in W^{k-1}_{-i} [t_{-i}] \) with probability 1 under \( t_i \). Finally, we set

\[ W^\infty_i [t_i] = \bigcap_{k=0}^\infty W^k_i [t_i] . \]

Notice that we eliminate a strategy if it is not a strict best-response to any belief on the remaining strategies of the other players. Clearly, this yields a smaller set than the result of iterative admissibility (i.e., iterative elimination of weakly dominated strategies).\(^5\) In some games, iterative admissibility may yield strong predictions. For example, in finite perfect information games it leads to backwards induction outcomes. Nevertheless, in generic normal-form games (as defined in Definition 1 below), all these concepts are equivalent and usually have weak predictive power.

**Remark 1** Any model can be embedded in the universal type space as long as there are no redundant types, i.e., multiple types with identical belief hierarchy. We will henceforth use the\(^5\)

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\(^5\)In particular, if we use non-reduced normal-form of an extensive-form game, many strategies will be outcome equivalent, in which case our procedure will eliminate all of these strategies. To avoid such over-elimination, we can use reduced-form, by representing all outcome-equivalent strategies by only one strategy.
terms model and subspace interchangeably. If there are redundant types in the model, one needs to consider a larger type space (Ely and Peski (2004)) in order to analyze the robustness of predictions. The results of such an analysis will, if anything, show more sensitivity to the assumptions about higher-order beliefs.

**Closing Assumption** We say that a type space (or model) \( \hat{T} \subset T \) is closed at order \( k \) if there are no two distinct \( t, \hat{t} \in \hat{T} \) such that \( t^l = \hat{t}^l \) for each \( l \leq k \). In that case, there is a well-defined, one-to-one function \( \gamma_\hat{T} : \text{Proj}_k(\hat{T}) \rightarrow T \), which we call the closing assumption for \( \hat{T} \), such that \( \gamma_\hat{T}(\text{Proj}_k(t)) = t \) for each \( t \in \hat{T} \), where \( \text{Proj}_k \) is the projection to the first \( k \) orders of beliefs.

2.3 Robustness to higher-order beliefs

We are interested in how robust equilibrium is against the failure of assumptions made at high orders, such as the failure of the common knowledge assumption at high orders. We now formalize our notion of robustness.

Consider a model \( \hat{T} \subset T \) that is closed at order \( k \). We want to check the robustness of the model's predictions according to a class of equilibria. Some of the equilibria of model \( \hat{T} \) will not be a part of any equilibrium defined globally on the universal type space. Since the existence of such equilibria depends on the closing assumption for \( \hat{T} \), we will not consider such equilibria. We therefore fix any family \( S^* \) of equilibria \( s^* \) on \( T \) and focus on equilibria \( s^*_{\hat{T}} \) on \( \hat{T} \), where \( s^*_{\hat{T}} \) is simply the restriction of \( s^* \) to \( \hat{T} \). Given any (propositional) formula \( Q \) with free variable in \( A^{\hat{T}} \), we say that \( Q \) is a prediction of \( \hat{T} \) with respect to \( S^* \) iff \( Q(s^*_{\hat{T}}) \) is true for each \( s^* \in S^* \). This formulation of prediction is very general, and a prediction need not depend on the actions of all types in \( \hat{T} \). For example, a prediction may state that the bidders with the lowest valuation bid 0 in a certain set of equilibria, which only depends on the actions of those types, or may state that the bidding function is increasing with the bidders’ valuations, which depends on all types. For any such prediction \( Q \), we say that \( Q \) is robust to (the assumptions about) higher-order beliefs iff \( Q(s^* \circ \gamma \circ \text{Proj}_k) \) is true for each \( s^* \in S^* \) and each alternative closing assumption \( \gamma \). To illustrate this, see the diagram in Figure 2-1: a prediction with respect to \( s^* \) is robust if it remains true for all functions defined by traversing any possible paths in the
Figure 2-1: Robustness to choice of closing assumption

diagram. That is, the prediction remains true independent of the choice of closing assumption.

We often need a prediction to be approximately true, e.g., the bidder with the lowest valuation does not bid more than \( \epsilon \). In our formulation, one can modify a prediction so that it is stated approximately and check the robustness of the modified prediction. Notice also that a model that is closed at order \( k \) can also be considered to be closed at order \( k' > k \). In checking robustness, one can choose \( k \) in order to check robustness to the assumptions about the beliefs at orders higher than \( k \). This is clearer in the following equivalent formulation.

Fix an equilibrium \( s^* \) and a type \( t_i \) of a player \( i \). According to equilibrium, he will play \( s_i^*(t_i) \). Now imagine a researcher who only knows the first \( k \) orders of beliefs of player \( i \) and knows that equilibrium \( s^* \) is played. All the researcher can conclude from this information is that \( i \) will play one of the actions in

\[
A_i^k [s^*, t_i] \equiv \{ s_i^*[\tilde{t}_i] | \tilde{t}_i^m = t_i^m \quad \forall m \leq k \}.
\]

Assuming, plausibly, that a researcher can verify only finitely many orders of a player’s beliefs, all a researcher can ever know is that player \( i \) will play one of the actions in

\[
A_i^\infty [s^*, t_i] = \bigcap_{k=0}^{\infty} A_i^k [s^*, t_i].
\]

**Notation 2** We write \( A^k [s^*, t] = \prod_i A_i^k [s^*, t_i], S^k [s^*, t] = \prod_i S_i^k [t_i], \text{ etc.} \)

The sets \( A_i^k [s^*, t_i] \) can be used to characterize the predictions of a model closed at order \( k \) that are robust to higher-order beliefs, as in the next lemma.
Lemma 4 A prediction $Q$ of a model $\hat{T}$ that is closed at order $k$, with respect to equilibria $S^*$, is robust to higher-order beliefs if and only if $Q(s)$ is true for each selection $s$ from the correspondence $A^k[s^*, \cdot]: \hat{T} \rightarrow 2^A$ and for each $s^* \in S^*$.

2.4 Main Results

We are now ready to prove our main result for countable-action games, i.e., games where each player $i$ has a countable or finite action space $A_i$.

Proposition 8 For any countable-action game, any equilibrium $s^*$ with full range, any $k \in \mathbb{N}$, $i \in \mathbb{N}$, and any $t_i$,

$$W^k_i [t_i] \subseteq A^k_i [s^*, t_i] \subseteq S^k_i [t_i];$$

in particular,

$$W^\infty_i [t_i] \subseteq A^\infty_i [s^*, t_i] \subseteq S^\infty_i [t_i].$$

Proof. We first show that $W^k_i [t_i] \subseteq A^k_i [s^*, t_i]$. For $k = 0$, the statement is given by the full-range assumption. For any given $k$ and any player $i$, write each $t_{-i}$ as $t_{-i} = (l, h)$ where $l = (t_{1_{-i}}, t_{2_{-i}}, \ldots, t_{k_{-i}})$ and $h = (t_{k_{-i}}, t_{k+1_{-i}}, \ldots)$ are the lower and higher-order beliefs, respectively. Let $L = \{l|\exists h: (l, h) \in T_{-i}\}$. The induction hypothesis is that

$$W^{k-1}_{-i} [l] \equiv \bigcup_{h'} W^{k-1}_{-i} [(l, h')] \subseteq A^{k-1}_{-i} [s^*, (l, h)] \quad (\forall (l, h) \in T_{-i}).$$

Fix any type $t_i$ and any $a_i \in W^k_i [t_i]$. We will construct a type $\tilde{t}_i$ such that $s^*_i (\tilde{t}_i) = a_i$ and the first $k$ orders of beliefs are same under $t_i$ and $\tilde{t}_i$, showing that $a_i \in A^k_i [s^*, t_i]$. Now, by definition, for some $\sigma_{-i} \in \Delta(W^{k-1}_{-i})$, $a_i$ is the unique best reply for type $t_i$ if $t_i$ assigns probability distribution $\sigma_{-i}$ on the other players’ strategies, i.e., $BR_i (\pi (\cdot | t_i, \sigma_{-i})) = \{a_i\}$. Let $P (\cdot | t_i, \sigma_{-i})$ be the probability distribution on $\Theta \times L \times A_{-i}$ induced by $\kappa_{t_i}$ and $\sigma_{-i}$. By the induction hypothesis, for each $(\theta, l, a_{-i}) \in suppP (\cdot | t_i, \sigma_{-i})$, $a_{-i} \in W^{k-1}_{-i} [l] \subseteq A^{k-1}_{-i} [s^*, (l, h)]$ for some $h$. Hence, there exists a mapping $\mu : suppP (\cdot | t_i, \sigma_{-i}) \rightarrow \Theta \times T_{-i},$

$$\mu : (\theta, l, a_{-i}) \mapsto \left(\theta, l, \tilde{h} (a_{-i}, \theta, l)\right), \quad (2.2)$$

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such that
\[ s^*_i \left( l, h(a_{-i}, \theta, l) \right) = a_{-i}. \] (2.3)

We define \( \tilde{t}_i \) by
\[ \kappa_{\tilde{t}_i} \equiv P(\cdot | t_i, \sigma_{-i}) \circ \mu^{-1}, \]
the probability distribution induced on \( \Theta \times T_{-i} \) by the mapping \( \mu \) and the probability distribution \( P(\cdot | t_i, \sigma_{-i}) \). Notice that, since \( t_i^k \) has countable support and the action spaces are countable, the set \( \text{supp}P(\cdot | t_i, \sigma_{-i}) \) is countable, in which case \( \mu \) is trivially measurable. Hence \( \kappa_{\tilde{t}_i} \) is well-defined. By construction of \( \mu \), the first \( k \) orders of beliefs (about \( (\theta, l) \)) are identical under \( t_i \) and \( \tilde{t}_i \):

\[
\text{marg}_{\Theta \times L} \kappa_{\tilde{t}_i} = \text{marg}_{\Theta \times L} P(\cdot | t_i, \sigma_{-i}) \circ \mu^{-1} = \text{marg}_{\Theta \times L} P(\cdot | t_i, \sigma_{-i}) = \text{marg}_{\Theta \times L} \kappa_{t_i},
\]
where the second inequality is by (2.2) and the last equality is by definition of \( P(\cdot | t_i, \sigma_{-i}) \). Moreover, using the mapping \( \gamma : (\theta, l, h) \mapsto (\theta, l, s^*_i (l, h)) \), we can check that the distribution induced by \( \kappa_{\tilde{t}_i} \) and \( s^*_i \) on \( \Theta \times L \times A_{-i} \) is

\[
P(\cdot | \tilde{t}_i, s^*_{-i}) \equiv \kappa_{\tilde{t}_i} \circ \gamma^{-1} = P(\cdot | t_i, \sigma_{-i}) \circ \mu^{-1} \circ \gamma^{-1} = P(\cdot | t_i, \sigma_{-i}),
\]
where the last equality is due to the fact that \( \mu \) is the inverse of the restriction of \( \gamma \) to \( \text{supp} \kappa_{\tilde{t}_i} \).

Therefore,
\[
\pi (\cdot | \tilde{t}_i, s^*_{-i}) = \text{marg}_{\Theta \times L} P(\cdot | \tilde{t}_i, s^*_{-i}) = \text{marg}_{\Theta \times L} P(\cdot | t_i, \sigma_{-i}) = \pi (\cdot | t_i, \sigma_{-i}).
\]

That is, the equilibrium beliefs of \( \tilde{t}_i \) about \( \Theta \times A_{-i} \) are identical to the beliefs of \( t_i \) about \( \Theta \times A_{-i} \) when \( t_i \) assigns probability distribution \( \sigma_{-i} \) on the other players’ strategies. Since \( a_i \) is the only best reply to these beliefs, \( \tilde{t}_i \) must play \( a_i \) in equilibrium:

\[
s^*_i (\tilde{t}_i) \in BR_i \left( \pi (\cdot | \tilde{t}_i, s^*_{-i}) \right) = BR_i (\pi (\cdot | t_i, \sigma_{-i})) = \{ a_i \}. \] (2.4)
To see the inclusion $A_k^* [s^*, t_i] \subseteq S_k^k [t_i]$, observe that for any $\tilde{t}_i$ with $\tilde{t}_i^m = t_i^m$ for each $m \leq k$, we have

$$s_i^* (\tilde{t}_i) \in S_i^\infty [\tilde{t}_i] \subseteq S_i^k [\tilde{t}_i] = S_i^k [t_i],$$

where the last equality is due to the fact that $S_i^k [t_i]$ depends only on the first $k$ orders of beliefs $(t_i^1, \ldots, t_i^k)$, completing the proof. [For a constructive but much longer proof of the last part, see our earlier working paper.]

Remark 2 Notice that the countability assumptions about the finite-order beliefs and the action spaces are used only to make sure that $\kappa_i$ is a well-defined probability distribution, or $\mu$ is measurable. In fact, whenever $\mu$ is measurable, our proof is valid. In the next section, we present another class of games in which $\mu$ is measurable; $\mu$ may not be measurable in general. These assumptions are not needed at all for the inclusion $A_k^* [s^*, t_i] \subseteq S_k^k [t_i]$.

The conclusion that $W_i^k [t_i] \subseteq A_i^k [s^*, t_i]$ can be spelled out as follows. Suppose that we know a player’s beliefs up to the $k$th order and do not have any further information. Suppose also that he has an action $a_i$ that survives $k$ rounds of iterated elimination of strategies that cannot be a strict best reply—for some type whose first $k$ orders of beliefs are as specified. Then, we cannot rule out that $a_i$ will be played in equilibrium $s^*$. Put it differently, if we have a model that is closed at order $k$ and if an action survives $k$ rounds of iterated elimination of strategies that cannot be a best reply for a type within the model, then we cannot rule out action $a_i$ as an equilibrium action for that type without invoking the closing assumption. Hence, any prediction that does not follow from the first $k$ steps of this elimination process is not robust to higher-order beliefs, no matter which set of equilibria is considered. Such a prediction depends crucially on the closing assumptions, and cannot be deduced from just the assumptions that lead to a specific equilibrium $s^*$. This suggests that, contrary to the current practice in economics, a researcher needs to justify his closing assumption at least as much as the other assumptions, such as the rationale for equilibrium selection. This is stated in the next proposition.

Proposition 9 Consider a countable-action game, a set $S^*$ of equilibria $s^*$, each with full range, and any model $\hat{T}$ that is closed at order $k$. If a prediction $Q$ of $\hat{T}$ with respect to $S^*$ is
robust to higher-order beliefs, then \( Q(s) \) is true for each selection \( s \) from the correspondence \( W^k_{|x|} \), defined on \( T \). Conversely, if a prediction \( Q(s) \) is true for each selection \( s \) from the correspondence \( S^k_{|x|} \), then \( Q \) is robust to higher-order beliefs.

**Proof.** The statement follows from Lemma 4 and Proposition 8. ■

Our next example shows that either of the inclusions in Proposition 8 may be strict. Hence, (i) some rationalizable strategies may not be in \( A_0^\infty \), showing the distinction between the results of Brandenburger and Dekel and ours, and (ii) \( A_0^\infty \) may include some weakly dominated strategies, distinguishing our result from the characterization of Brandenburger and Keisler (2000).

**Notation 3** For any \( \bar{\theta} \in \Theta \), write \( \bar{\theta}_i^{CPK}(\bar{\theta}) \) for the type of a player \( i \) who is certain that it is common knowledge that \( \theta = \bar{\theta} \).

**Example 3** Take \( N = \{1, 2\} \), \( \Theta = \{\theta_0, \theta_1\} \), and let the action spaces and the payoff functions for each \( \theta \) be given by

\[
\begin{array}{c|cc}
   & a^0 & a^1 \\
\hline
a^0 & 0,0 & 0,0 \\
a^1 & 0,0 & 1,1 \\
\end{array}
\]

(Note that \( \theta \) is not payoff relevant.) Define \( s^* \) by

\[
s^*_i(t_i) = \begin{cases} 
a^0 & \text{if } t_i = \theta^{CPK}_i(\theta_0); \\
a^1 & \text{otherwise.} \end{cases}
\]

Clearly, for each \( k \geq 1 \), we have \( W^k_{|x|}[t_i] = \{a^1\} \) and \( S^k_{|x|}[t_i] = \{a^0, a^1\} \) for each \( t_i \), while \( A^k_{|x|}[s^*; \theta^{CPK}_i(\theta_0)] = \{a^0, a^1\} \), and \( A^k_{|x|}[s^*; \theta^{CPK}_i(\theta_1)] = \{a^1\} \).

Proposition 8 yields a characterization whenever the payoffs are generic in the following (standard) sense.

**Definition 1** We say that the payoffs are generic at \( \theta \) iff there do not exist \( i, \) non-zero \( \alpha \in \mathbb{R}^{A_i} \), and distinct \( a_i, a'_i, a_{-i}, a'_{-i} \) such that (i) \( u_i(\theta, a_i, a_{-i}) = u_i(\theta, a'_i, a_{-i}) \) or (ii) \( \sum_{a_i} \alpha(a_i) u_i(\theta, a_i, a_{-i}) = \sum_{a_i} \alpha(a_i) u_i(\theta, a_i, a'_{-i}) = 0. \)

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When the payoffs are generic at \( \theta \) and it is common knowledge that \( \theta = \overline{\theta} \), then any action that is not strictly dominated will be a strict best reply against some belief (at each round), and hence the two elimination processes will be equivalent. In that case, Propositions 8 and 9 yield the following characterizations.

**Corollary 3** For any finite-action game and any equilibrium \( s^* \) with full range, if the payoffs are generic at some \( \theta \), then for each \( i \) and \( k \),

\[
A^k_i [s^*, t^i_{CK} (\theta)] = S^k_i [t^i_{CK} (\theta)] .
\]

**Corollary 4** Consider any \( \theta \) with generic payoffs. Any prediction \( Q \) of the complete-information model \( \hat{T} = \{ t^i_{CK} (\theta) \} \) with respect to any set of equilibria \( S^* \) is robust to higher-order beliefs if and only if \( Q \) is true for each action \( a \in S^k \left[ t^i_{CK} (\theta) \right] \).

That is, in generic finite-action games, a researcher’s predictions based on finite orders of players’ beliefs and an arbitrary set of equilibria with full range will be identical to the predictions that follow from rationalizability. This characterization will now be generalized to another widely-used class of games.

### 2.5 Nice games

We will now consider a class of “nice” games (Moulin (1984)), which are widely used in economic theory, such as imperfect competition, spatial competition, provision of public goods, theory of the firm, etc. We will show that \( A^k_i [s^*, t_i] = S^k_i [t_i] \) for each \( k \) whenever equilibrium \( s^* \) has full range.

**Definition 2** A game is said to be nice iff for each \( i \), \( A_i = [0, 1] \) and \( u_i (\theta, a_i, a_{-i}) \) is continuous in \( a = (a_i, a_{-i}) \) and strictly concave in \( a_i \).

We use the strict concavity assumption to make sure that a player’s utility function for any fixed strategy profile of the others is always single-peaked in his own action. (Single-peakedness is not preserved in presence of uncertainty.) We use the continuity assumption to make sure that a player’s strategy best response is continuous with respect to the other players’ strategies.
For the complete information types, our results in this section will be true under the weaker condition that $u_i(\theta, \cdot, a_{-i})$ is single-peaked with a maximand that is continuous in $a_{-i}$. Now, since our players always have unique best reply, our elimination processes will be equivalent, yielding the functional equation

$$W = S.$$  \hfill (2.5)

Moreover, our next lemma ensures that, despite our uncountable action spaces, we only need to consider countably many actions for types with countable supports, allowing us to circumvent the measurability issue discussed in Remark 2.

**Lemma 5** For any nice game, for any $i$, $t_i$, $k$, and any $a_i \in S_i^k[t_i]$, there exists $\hat{s}_{-i} \in \hat{S}_{-i}^{k-1}$ such that

$$BR_i(\pi (\cdot | t_i, \hat{s}_{-i})) = \{a_i\}.$$ 

Together with (2.5), Lemma 5 gives us our main result for this section.

**Proposition 10** For any nice game, let $S^*$ be any set of equilibria $s^*$, each with full range. Let also $\hat{\Theta} \times \hat{T}$ be a countable subset of $\Theta \times T$ such that for each $\hat{t}_i \in \hat{T}_i$, $\text{supp}\kappa_{\hat{t}_i} \subseteq \hat{\Theta} \times \hat{T}_{-i}$. Then, for any $k \in \mathbb{N}$, $i \in \mathbb{N}$, and $\hat{t}_i \in \hat{T}_i$,

$$S_i^k[\hat{t}_i] = A_i^k[s^*, \hat{t}_i];$$

in particular,

$$S_i^{\infty}[\hat{t}_i] = A_i^{\infty}[s^*, \hat{t}_i].$$

Moreover, if $\hat{T}$ is closed at order $k$, then a prediction $Q$ of $\hat{T}$ with respect to $S^*$ is robust to higher-order beliefs if and only if $Q(s)$ is true for each selection from $S_i^k$.

**Proof.** For any $a_i \in S_i^k[\hat{t}_i] = W_i^k[\hat{t}_i]$, by Lemma 5, there exists $\hat{s}_{-i} \in \hat{S}_{-i}^{k-1} = \hat{W}_{-i}^{k-1}$ such that $a_i$ is a strict best reply against $\pi (\cdot | t_i, \hat{s}_{-i})$. Since $\kappa_{\hat{t}_i}$ has countable support, $P (\cdot | \hat{t}_i, \hat{s}_{-i})$, the probability distribution induced by $\kappa_{\hat{t}_i}$ and $\hat{s}_{-i}$ on $\Theta \times L \times A_{-i}$, has a countable support:

$$\text{supp}P (\cdot | \hat{t}_i, \hat{s}_{-i}) = \{((\theta, l, \hat{s}_{-i} (\theta, l, h)) | (\theta, l, h) \in \text{supp}\kappa_{\hat{t}_i})\}.$$
Hence our proof of Proposition 8 applies. That is, there exists \( \tilde{t}_i \in T_i \) (not necessarily in \( \bar{T}_i \)) such that \( s^*_i(\tilde{t}_i) = a_i \) and \( \tilde{t}_i^m = \hat{t}_i^m \) for each \( m \leq k \), yielding the equality above. The last statement in the proposition follows from this inequality by Lemma 4.

### 2.6 Extensions

For ease of exposition, we have so far focused on pure strategy equilibria with full range. In this section, we will extend our results for mixed strategy equilibria and beyond the full-range assumption.

#### 2.6.1 Mixed Strategies

Since all equilibria in nice games are in pure strategies, we will focus on the countable-action games. Using interim formulation, we define a mixed strategy as any measurable function \( \sigma_i : T_i \to \Delta(A_i) \). A mixed strategy profile \( \sigma^* \) is Bayesian Nash equilibrium iff \( \text{supp} \sigma^*_i(t_i) \subseteq BR_i\left(\pi(\cdot|t_i, \sigma^*_i)\right) \) for each \( i \) and \( t_i \). Writing \( T^*_i = \{t_i|\text{supp} \sigma^*_i(t_i) = 1\} \) for the set of types who play pure strategies, we define a mapping \( s^*_i : T^*_i \to A_i \) by \( \text{supp} \sigma^*_i(t_i) = \{s^*_i(t_i)\} \).

We then use this "pure part of" \( \sigma^* \) to extend our previous definitions and results to mixed strategies. We say that \( \sigma^* \) has full range iff \( s^*_i(T^*_i) = A_i \) and set

\[
A^k_i[\sigma^*, t_i] = \left\{ s^*_i(\tilde{t}_i) | \tilde{t}_i \in T^*_i, \tilde{t}_i^m = t_i^m \forall m \leq k \right\},
\]

the set of all actions that are played with probability 1 under \( \sigma^* \) by some type \( \tilde{t}_i \) whose first \( k \) orders of beliefs are identical to those of \( t_i \). Clearly, every equilibrium has full range under the hypothesis of Lemma 3, i.e., when \( \Theta \) is sufficiently rich.

**Proposition 11** For any countable-action game, any (possibly mixed strategy) equilibrium \( \sigma^* \) with full range, any \( k \leq \infty \), \( i \in N \), and any \( t_i \),

\[
W^k_i[t_i] \subseteq A^k_i[\sigma^*, t_i] \subseteq S^k_i[t_i].
\]

**Proof.** In the proof of Proposition 8, insert \( s^\sigma \) everywhere \( s^* \) appears, and restrict the range of \( \mu \) and the domain of \( \gamma \) to \( \Theta \times T^*_{-i} \). Notice that, by (2.4), \( \tilde{t}_i \in T^*_i \). ■
That is, if $\sigma^*$ has full range (e.g., if $\Theta$ is sufficiently rich) and we know only the first $k$ orders of a player's beliefs, then for any $a_i \in W^k_i [t_i]$, we cannot rule out that $a_i$ is played with probability 1 according to $\sigma^*$. For $k \geq 1$, the full-range assumption can replaced by the weaker assumption that $A_i \subseteq \cup_t \text{supp}_i^* (t_i)$.

2.6.2 Without full range

Our full range assumption allowed us to consider large changes in higher-order beliefs. A researcher may be certain that it is common knowledge that the set of parameters are restricted to a small subset, or equivalently, the equilibrium considered may not vary much as the beliefs about the underlying uncertainty change. We will now present an extension of our main result to such cases.

Local Rationalizability For any $B_1 \times \cdots \times B_n \subset A$, define sets $S^k_i [B; t_i]$, $i \in N$, $k \in N$, $t_i \in T_i$, by setting

$$S^0_i [B; t_i] = B_i,$$

$$S^k_i [B; t_i] = \bigcup_{\sigma_{-i} \in \Delta (\hat{S}^k_{i-1} [B]; A_i)} BR_i (\pi \cdot | t_i, \sigma_{-i}) ,$$

where $\hat{S}^k_{i-1} [B] \subset M_{-i}$ is the set of all measurable functions $f : \Theta \times T_{-i} \rightarrow A_{-i}$ such that $f(\theta, t_{-i}) \in S^k_{i-1} [B; t_{-i}]$ for each $t_{-i}$. Notice that this is the same procedure as iterated strict dominance, except that the initial set is restricted to a subset. Unlike iterated strict dominance, these sets can become larger as $k$ increases. Hence we define the set of locally rationalizable strategies by

$$S^\infty_i [B; t_i] = \bigcap_{k=0}^{\infty} \bigcup_{m=k}^{\infty} S^m_i [B; t_i] .$$

Notice that the set $S^\infty_i [B; t_i]$ may be much larger than $B$. We define local version of $W^\infty$, similarly, by setting $W^0_i [B; t_i] = B_i$,

$$W^k_i [B; t_i] = \left\{ a_i \in A_i | BR_i (\theta^i, \sigma_{-i}) = \{ a_i \} \text{ for some } \sigma_{-i} \in \Delta (\hat{W}^k_{i-1} [B]) \right\} ,$$

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and \( W_i^\infty [B; t_i] = \bigcap_{k=0}^\infty \bigcup_{m=k}^\infty W_i^m [B; t_i] \). Notice that we consider all actions in our process, which is no longer an elimination process.

**Proposition 12** For any equilibrium \( s^* \), if the game has countable action spaces, then

\[
W_i^k [s^* (T); t_i] \subseteq A_i^k [s^*, t_i] \subseteq S_i^k [s^* (T); t_i] \quad (\forall i, k, t_i);
\]

if the game is nice, then with notation of Proposition 10, for any \( B \subseteq s^* (T) \),

\[
S_i^k [B; t_i] \subseteq A_i^k [s^*, t_i] = S_i^k [s^* (T); t_i] \quad (\forall i, k, t_i).
\]

The last statement implies that, for nice games, even the slight changes in very higher-order beliefs will have substantial impact on equilibrium behavior, unless the game is locally dominance-solvable. There are important games in which a slight failure of common knowledge assumption in very high orders leads to substantially different outcomes—as in Section 2.8.

### 2.7 Continuity of Equilibrium

It is well known that equilibrium may be discontinuous with respect to the product topology. In this section we will introduce our notion of continuity with respect to higher-order beliefs, which appears much weaker than continuity with respect to the product topology. We will show that even this weaker continuity property is violated in every equilibrium on a very large set.

#### 2.7.1 Equilibrium in pure strategies

We consider an arbitrary metric \( d \) on \( A \). A sequence \((a^m)_{m \in \mathbb{N}}\) is said to converge to some \( a \in A \) iff for each \( \epsilon > 0 \), there exists \( k \) such that \( d(a^m, a) < \epsilon \) for each \( m > k \). Given any subset \( B \subseteq A \), we write \( D(B) = \sup \{d(a, b) | a, b \in B\} \) for the diameter of \( B \).

**Definition 3** An equilibrium \( s^* \) is said to be continuous (with respect to product topology) at \( t \) iff for each sequence \((i[m])_{m \in \mathbb{N}}\) of type profiles

\[
\left[ i^k [m] \rightarrow t^k \quad \forall k \right] \Rightarrow \left[ s^* (i[m]) \rightarrow s^* (t) \right].
\]
An equilibrium \( s^* \) is said to be continuous with respect to higher-order beliefs at \( t \) iff for each \( \epsilon > 0 \), there exists \( k \) such that for each \( \tilde{t} \),

\[
[\tilde{t}^m = t^m \; \forall m \leq k] \Rightarrow d(s^*(\tilde{t}), s^*(t)) < \epsilon.
\]

The latter continuity concept is uniform continuity with respect to the product topology (on the type space) of discrete topologies on each order of beliefs. Of course, continuity with respect to discrete topology is much weaker than other topologies. The next lemma presents some basic facts.

**Lemma 6** For any equilibrium \( s^* \) and for any \( t \), the following are true.

1. If \( s^* \) is continuous at \( t \), then \( s^* \) is continuous with respect to higher-order beliefs at \( t \).

2. \( s^* \) is continuous with respect to higher-order beliefs at \( t \) iff \( D(A^k[s^*, t]) \to 0 \) as \( k \to \infty \).

3. If \( s^* \) is continuous with respect to higher-order beliefs at \( t \), then \( A^{\infty}[s^*, t] = \{s^*(t)\} \).

For nice games, Lemma 6.3 and Proposition 10 imply that if an equilibrium \( s^* \) with full range is continuous with respect to higher-order beliefs at \( t \), then \( S^{\infty}[t] = \{s^*(t)\} \), yielding the following discontinuity result. (One can obtain a similar result for countable-action games by replacing \( S^{\infty}[t] = \{s^*(t)\} \) with \( W^{\infty}[t] \leq 1 \).)

**Proposition 13** For any nice game, every equilibrium \( s^* \) with full range is discontinuous with respect to the higher-order beliefs (and the product topology) at each type profile \( t \) for which there are more than one rationalizable action profiles. In particular, if a nice game possesses an equilibrium \( s^* \) that is continuous with respect to higher-order beliefs or with respect to the product topology, then the game is dominance solvable.

### 2.7.2 Mixed-strategy equilibria in finite-action games

Endow the space of mixed action profiles, \( \Delta(A) \), with an arbitrary metric \( d \). Extend the definitions of continuity with respect to product topology and higher-order beliefs to mixed-strategy equilibria \( \sigma^* \) by replacing \( s^* \) with \( \sigma^* \) in Definition 3.
Definition 4 A mixed-strategy equilibrium \( \sigma^* \) is said to be weakly continuous with respect to higher-order beliefs at \( t \) iff there exists \( k \) such that for each \( \tilde{t} \),

\[
[\tilde{t}^m = t^m \quad \forall m \leq k] \Rightarrow \text{supp}(\sigma^* (\tilde{t})) \cap \text{supp}(\sigma^* (t)) \neq \emptyset.
\]

Now, continuity in product topology implies continuity with respect to higher-order beliefs. For finite-action games, the latter in turn implies weak continuity with respect to higher-order beliefs, as we show in the Appendix (see Lemma 9). There we also show that strong and weak continuity of \( \sigma^* \) at \( t \) with respect to higher-order beliefs imply that \( |\text{A}^\infty [\sigma^*, t]| \leq 1 \) and \( A^\infty [\sigma^*, t] \subseteq \text{supp}(\sigma^* (t)) \), respectively. This yields the following result.

Proposition 14 For any finite-action game and any equilibrium \( \sigma^* \) with full range, \( |W^\infty [t]| \leq 1 \) whenever (i) \( \sigma^* \) is continuous with respect to higher-order beliefs at \( t \), or (ii) \( \sigma^* \) is weakly continuous with respect to higher-order beliefs at \( t \) and \( \sigma^* (t) \) is pure.

Proof. Either of the conditions (i) and (ii) implies that \( |\text{A}^\infty [\sigma^*, t]| \leq 1 \) (see Lemma 9). Hence, by Proposition 11, \( |W^\infty [t]| \leq |\text{A}^\infty [\sigma^*, t]| \leq 1. \]

That is, for sufficiently rich \( \Theta \), every equilibrium is discontinuous with respect to higher-order beliefs (and the product topology) at each type profile for which two or more action profiles survive iterated elimination of strategies that cannot be a strict best reply. These type profiles include the generic instances of complete-information without dominance solvability. At each such type profile, even the weakest continuity property fails if the equilibrium actions are pure.

Example 4 Consider the coordinated attack game with payoff matrix

\[
\begin{array}{c|cc}
& \text{Attack} & \text{No Attack} \\
\hline
\text{Attack} & 1,1 & -2,0 \\
\text{No Attack} & 0,-2 & 0,0
\end{array}
\]

where there are two pure strategy equilibria: the efficient equilibrium (Attack, Attack) and the risk-dominant equilibrium (No Attack, No Attack). Since each action is a strict best reply, no action is eliminated in our elimination process. Therefore, Lemma 3 and Proposition 14 imply
that when we embed the coordinated-attack game in a rich type space as a type profile, every equilibrium must be discontinuous with respect to higher-order beliefs at that type profile.

Rubinstein's (1989) electronic-mail game presents a type space in which any equilibrium that selects the efficient equilibrium in the coordinated-attack game must be discontinuous with respect to higher-order uncertainty. In that example there is also a continuous equilibrium, which selects the risk-dominant action profile for each type profile. Our example shows that the latter continuous equilibrium is an artifact of the small type space utilized, and in fact in a rich type space, no equilibrium could have been robust against higher-order beliefs, and thus every equilibrium theory would have been sensitive to the assumptions about higher-order uncertainty, strengthening Rubinstein's position.

Following Carlsson and Van Damme (1993), the global games literature investigates the equilibria in nearby type profiles that are generated by a model that is closed at the first order. At these type profiles, the game is dominant solvable, and the resulting equilibrium action profile converges to the risk-dominant equilibrium as these type profiles approach the coordinated-attack game. In this way, they select the risk-dominant equilibrium. Our result uncovers a difficulty in this methodology: every equilibrium must be discontinuous at the limiting type profile, and an equilibrium selection argument based on continuity is problematic—as there is another path that we could have taken the limit in which we would have selected the other equilibrium. This is despite the fact that the equilibrium outcome is robust against higher-order beliefs in these nearby type profiles themselves.

On a positive note, Kajii and Morris (1997) show that the risk-dominant equilibrium is robust to incomplete information under common prior assumption. That is, if the common prior puts sufficiently high weight on the original game, then the incomplete information game has an equilibrium in which the risk-dominant equilibrium is played with high probability according to the common prior. Similar positive results are obtained by others, such as Ui (2001), Morris and Ui (2003). This suggests that, when there is a common prior, it may put low probability on the paths that converge to other equilibria. (The relationship between existence of nearby equilibria and existence of converging path for a fixed equilibrium is unknown.)
2.8 Application: Cournot Oligopoly

In a linear Cournot duopoly, the game is dominance-solvable, and hence Proposition 8 implies that higher-order beliefs have negligible impact on equilibrium. (This has also been shown by Weinstein and Yildiz (2003) and is also implied by a result of Nyarko (1996).) On the other hand, in a linear Cournot oligopoly with three or more firms, any production level that is less than or equal to the monopoly production is rationalizable, and hence Proposition 10 implies that a researcher cannot rule out any such output level for a firm no matter how many orders of beliefs he specifies. We will now show a more disturbing fact. Focusing complete-information types, $t^{CK}(\theta)$, for fairly general oligopoly models we will show that when there are sufficiently many firms, any such outcome will be in $S^1_\theta[B; t^{CK}(\theta)]$ for every neighborhood $B$ of $s^*(t^{CK}(\theta))$. Therefore, by Proposition 12, even a slight doubt about the model in very high orders will lead a researcher to fail to rule out any outcome that is less than monopoly outcome as a firm’s equilibrium output.

**General Cournot Model**  Consider $n$ firms with identical constant marginal cost $c > 0$. Simultaneously, each firm $i$ produces $q_i$ at cost $q_ic$ and sell its output at price $P(Q; \theta)$ where $Q = \sum_i q_i$ is the total supply. For some fixed $\theta$, we assume that $\Theta$ is a closed interval with $\theta \in \Theta \neq \{\theta\}$. We also assume that $P(0; \theta) > 0$, $P(\cdot; \theta)$ is strictly decreasing when it is positive, and $\lim_{Q \to \infty} P(Q; \theta) = 0$. Therefore, there exists a unique $Q$ such that

$$P(\hat{Q}; \theta) = c.$$  

(In order to have a nice game, we can impose an upper bound for $q$, larger than $\hat{Q}$, without affecting the equilibria.) We assume that, on $[0, \hat{Q}], P(\cdot; \theta)$ is continuously twice-differentiable and

$$P' + QP'' < 0.$$  

It is well known that, under the assumptions of the model, (i) the profit function, $u(q, Q; \theta) = q(P(q + Q) - c)$, is strictly concave in own output $q$; (ii) the unique best response $q^*(Q_{-i})$ to others' aggregate production $Q_{-i}$ is strictly decreasing on $[0, \hat{Q}]$ with slope bounded away from 0 (i.e., $\partial q^*/\partial Q_{-i} \leq \lambda$ for some $\lambda < 0$); (iii) equilibrium outcome at $t^{CK}(\theta)$, $s^*(t^{CK}(\theta))$, is
Lemma 7 In the general Cournot model, for any equilibrium \( s^* \), there exists \( \bar{n} < \infty \) such that for any \( n > \bar{n} \) and any \( B = [s^*_1(t^{CK}_1(\bar{\theta})) - \epsilon, s^*_1(t^{CK}_1(\bar{\theta})) + \epsilon]^n \subset A \) with \( \epsilon > 0 \), we have

\[
S_i^\infty [B; t^{CK}(\bar{\theta})] = [0, q^M] \quad (\forall i \in N),
\]

where \( q^M \) is the monopoly output under \( P(\cdot; \bar{\theta}) \).

Proof. Let \( \bar{n} \) be any integer greater than \( 1 + 1/|\lambda| \), where \( \lambda \) is as in (ii). Take any \( n > \bar{n} \).

By (iii), \( B = [q^0, \bar{q}^0]^n \) for some \( q^0, \bar{q}^0 \) with \( q^0 < \bar{q}^0 \). By (ii), for any \( k > 0 \), \( S^k [B; t^{CK}(\bar{\theta})] = [q^k, \bar{q}^k]^n \), where

\[
\bar{q}^k = q^* ( (n-1)q^{k-1} ) \quad \text{and} \quad q^k = q^* ( (n-1)q^{k-1} ).
\]

Define \( Q^k \equiv (n-1)q^k, \bar{Q}^k \equiv (n-1)\bar{q}^k, \) and \( Q^* = (n-1)q^* \), so that

\[
\bar{Q}^k = Q^* (Q^{k-1}) \quad \text{and} \quad Q^k = Q^* (Q^{k-1}).
\]

Since \( (n-1)\lambda < 1 \), the slope of \( Q^* \) is strictly less than \( -1 \). Hence \( Q^k \) decreases with \( k \) and becomes 0 at some finite \( \bar{k} \), and \( \bar{Q}^k \) increases with \( k \) and takes value \( Q^* (0) = (n-1)q^M \) at \( k+1 \).

That is, \( S^k [B; t^{CK}(\bar{\theta})] = [0, q^M]^n \) for each \( k > \bar{k} \). Therefore, \( S^\infty [B; t^{CK}(\bar{\theta})] = [0, q^M]^n \).}

Together with Proposition 12, this lemma yields the following.

Proposition 15 In the general Cournot model, assume that \( \Theta = [\bar{\theta} - \epsilon, \bar{\theta} + \epsilon] \) for arbitrarily small \( \epsilon > 0 \), and that the best-response function \( q^* (Q_{-i}; \theta) \) is a continuous and strictly increasing function of \( \theta \) at \( (Q_{-i}, \bar{\theta}) \) where \( Q_{-i} = (n-1)s^*_i(t^{CK}(\bar{\theta})) \) is the others’ aggregate output in equilibrium. Then,

\[
A_i^\infty [s^*, t^{CK}_i(\bar{\theta})] = [0, q^M] \quad (\forall i \in N),
\]

where \( q^M \) is the monopoly output under \( P(\cdot; \bar{\theta}) \).

Proof. By (i) above, we have a nice game. By the hypothesis, there exists \( B \subset s^* (T) \) as in Lemma 7. Hence, Lemma 7 and Proposition 12 imply

\[
[0, q^M] = S_i^\infty [B; t^{CK}_i(\bar{\theta})] \subseteq A_i^\infty [s^*, t^{CK}_i(\bar{\theta})] \subseteq [0, q^M],
\]
yielding the desired equality. ■

In Proposition 15, the assumption that $q^*(Q_{-i}; \theta)$ is responsive to $\theta$ guarantees that $\theta$ is a payoff-relevant parameter. Our proposition suggests that, with sufficiently many firms, any equilibrium prediction that is not implied by strict dominance will be invalid whenever we slightly deviate from the idealized complete information model. To see this, consider the confident researcher and his slightly skeptical friend in the Introduction. The former is confident that it is common knowledge that $\theta = \bar{\theta}$, while the latter is only willing to concede that it is common knowledge that $|\theta - \bar{\theta}| \leq \varepsilon$ and agrees with the $k$th-order mutual knowledge of $\theta = \bar{\theta}$. He is an arbitrarily generous skeptic; he is willing to concede the above for arbitrarily small $\varepsilon > 0$ and arbitrarily large finite $k$. Our proposition states that the skeptic nonetheless cannot rule out any output level that is not strictly dominated.

### 2.9 Literature Review

There is a sizeable literature that investigates the impact of higher-order beliefs in equilibrium. A branch of this literature investigates the robustness of some central predictions in economic theory. In mechanism design, all the surplus can be extracted from players in generic traditional type spaces (Cremer and Mclean (1988)). Neeman (2004) shows that this is not true when we introduce second-order uncertainty. In fact, this property generically fails within the set of common priors on the universal type space, as Heifetz and Neeman (2003) shows. In bargaining between a seller and a buyer with privately known valuation, the trade occurs immediately, as conjectured by Coase. Feinberg and Skrzypacz (2002) show that, when the buyer does not know whether the seller knows buyer’s valuation, there will be delay in all equilibria that satisfy a regularity condition.

In a more abstract level, under a stability condition, Nyarko (1996) shows that equilibrium is continuous on universal type space. In another paper (Weinstein and Yildiz (2003)), we have shown that a similar stability condition is sufficient for the maximum impact of higher-order beliefs to diminish exponentially. In a nice game, these stability conditions imply that the game is dominance solvable, which is shown in this paper to be a necessary condition for such a diminishing impact.
Morris (2002) analyzes the topology required for uniform continuity of equilibrium over a class of games in which the players try to predict the other players' hierarchy of beliefs. He shows that this topology is equivalent to the topology of uniform convergence and that traditional type spaces are non-generic with respect to the latter topology. By requiring uniform continuity, Morris (2002) focuses on worst-case scenarios, while we analyze the continuity of equilibrium within a fixed game. In fact, for that class of payoff functions the game is dominance-solvable, and therefore every equilibrium must be continuous.

Dekel, Fudenberg, and Morris (2004) characterize the topology that is required for upper- and lower-semicontinuity of rationalizability for general spaces, using again the notion of uniform continuity. While the topology for the upper-semicontinuity is the usual product topology, the topology for the lower-semicontinuity needs to be finer, yet the finite type-spaces are dense in the latter topology. The latter finding answers a question that has been open since Mertens and Zamir (1985) for rationalizability, showing that rationalizable strategies can be approximated using finite type spaces. This is consistent with our result because every equilibrium may be highly discontinuous while the rationalizability correspondence is smooth. One can easily construct simple examples in which the rationalizability correspondence is constant, while every equilibrium must be discontinuous at each type.6

Conceptually, our main result is related to the epistemic literature on rationalizability and equilibrium as follows: An important category of predictions are that of universal equilibrium predictions, which state that a proposition is true for all equilibria. We can use the results of Brandenburger and Dekel (1987) to analyze the robustness of these predictions: once one allows for subjectivity in players’ priors, then a universal equilibrium prediction will continue to hold only if it is also a universal prediction of iterated elimination of never-a-weak-best-reply strategies. Clearly, many existing results in economics fall into another category: local predictions. A researcher focuses on one equilibrium or a small class of equilibria and makes statements that are true for these equilibria. Our result concerns robustness of any such local prediction with respect to the assumptions on higher-order beliefs. It shows that once one

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6 For example, consider a complete information game in which every strategy is a unique best reply to a belief, but there is a unique Nash equilibrium. Scrambling the labels of strategies, construct new states, so that for each strategy profile there is a state at which the strategy profile is the unique Nash equilibrium. Consider the universal type space on these states. Now, $W^{\infty} = S^{\infty} = A$ is constant, but every equilibrium must be discontinuous at each type by Proposition 14.
allows for uncertainty about higher-order beliefs, then any local (or universal) equilibrium prediction must also be a universal prediction of iterated elimination of never-a-strict-best-reply strategies. We cover all local and universal predictions, generated by arbitrary equilibrium selection criteria. Hence, our lower bound becomes weak in certain cases. For example, when iterative admissibility has strong predictive power, our lower bound becomes weak, for it has to hold for refinements based on iterative admissibility, which is not sensitive to higher-order beliefs.

Battigalli and Siniscalchi (2003) explicitly link restrictions on the first-order beliefs (about both underlying uncertainty and the strategies) to the equilibrium outcomes via their notion of Δ-rationalizability. For sequential games, they obtain strong positive results for equilibria that incorporate forward induction reasoning (see also Battigalli and Siniscalchi (2002) and Feinberg (2002)). Their results suggest that, when Δ-rationalizability has strong predictive power, if an equilibrium satisfies the restriction Δ, then its predictions may be robust to the assumptions about higher-order beliefs.

Although our goals are different, we can, to some extent, use the methodology of Battigalli and Siniscalchi (2003) in our problem. Using this methodology, in cases with multiple Δ-rationalizable strategies, one may be able to construct an equilibrium that is sensitive to higher-order beliefs (on some type space). This stops short of our main results, which show that all equilibria are sensitive to higher-order beliefs whenever two or more actions survive the elimination process. This difference is especially important because, in response to the existence of sensitive equilibria, researchers typically invoke a refinement to assume them away. For

\footnote{In their model, a player $i$ has a “payoff type” $\theta_i$ and an epistemic type $e_i$, describing his beliefs about strategic uncertainty. For a fixed $k$, we can take $\theta_i = (t_i^1, \ldots, t_i^k)$, representing the first $k$ orders of his beliefs about fundamentals. They show that there exists a model with types of the form $(\theta_i, e_i)$, in which each Δ-rationalizable action $a_i(\theta_i)$ is played by some type $(\theta_i, e_i)$ in equilibrium $s^\ast$. For each type $(\theta_i, e_i)$, write $\tau(\theta_i, e_i)$ for the belief hierarchy of type $(\theta_i, e_i)$ about fundamentals. Suppose that this model happens to be such that

\[(\theta_i, e_i) \neq (\theta'_i, e'_i) \Rightarrow \tau(\theta_i, e_i) \neq \tau(\theta'_i, e'_i).\]  

Now represent the types with their belief hierarchies. By (2.6), there is a well-defined equilibrium $s^\ast$ on this type space, and each Δ-rationalizable action $a_i(\theta_i)$ is played by some type $\tau(\theta_i, e_i)$, whose first $k$ orders of beliefs is $\theta_i = (t_i^1, \ldots, t_i^k)$. The main difficulty here is to obtain (2.6), as we started with an epistemic type space, describing players' uncertainty about strategies. Battigalli and Siniscalchi have shown that, for generic cases, one can select a type space so that distinct types have distinct beliefs about $\theta_i$. This is, however, not enough to show (2.6). (Also, types of the form $\theta_i = (t_i^1, \ldots, t_i^k)$ seem to be non-generic as they contain payoff irrelevant information, but this is not crucial for their result.)}
example, the discontinuous equilibrium in electronic mail game is assumed away by focusing on
the risk-dominant equilibrium in the global games literature. We show that this is not possible.
This approach would also have other limitations. First, since the Δ-rationalizable sets depend
on the restriction Δ, we do not know the size of these sets, and we would not know when we
can construct such sensitive equilibria. (One may not be able to construct such an equilibrium,
even when the Δ-rationalizable set is large.) In contrast, for generic games, we characterize the
robust predictions using usual rationalizability. Second, one cannot obtain our discontinuity
results using this approach, for the equilibria constructed for different orders k may correspond
to different equilibria in universal type space.

Our result has counterparts in sophisticated and Bayesian learning models: the learning of
sophisticated agents leads to equilibrium if and only if the game is dominance solvable (Milgrom
and Roberts (1991)), and in a specific model, any sequence of rationalizable action profiles can be
a sample path in Bayesian learning (Nyarko (1996)). Universal type space allows heterogeneous
priors. In such environments, it is also difficult to interpret equilibrium as an outcome of a
non-sophisticated (myopic) learning process (Dekel, Fudenberg, and Levine (2004)).

2.10 Conclusion

It is a common practice in economics to close the model after specifying only the first or
second order beliefs, using a (mostly implicit) common knowledge assumption. In this paper,
we have investigated the role of this assumption in predictions according to an arbitrary fixed
equilibrium. Finding strong lower and upper bounds for the variations with respect to this
assumption, we have shown that it is this casually made common knowledge assumption that
drives any prediction that we could not have made already by iteratively eliminating strategies
that can never be strict best reply. In games like Cournot oligopoly, this implies that no
interesting conclusions could have been reached without making a precise common knowledge
assumption. Therefore, it is essential for assuring the reliability of theories to pay special care
to the closing assumption and justify it at least as much as the other assumptions.

When there are two or more actions that survive our elimination process, there is an inherent
unpredictability which cannot be avoided without making an assumption on infinitely many
orders of beliefs, as all of these actions are played with probability 1 by some types whose
finite-order beliefs agree for arbitrarily high orders. In that case, equilibrium is necessarily
discontinuous with respect to higher-order beliefs and in product topology. Moreover, when
there are no ties, there is a one-to-one relationship between this sensitivity to higher-order
beliefs and sensitivity to higher-order assumptions about players’ rationality. It then becomes
very difficult in analyzing these situations to justify the common knowledge of rationality as a
good approximating assumption.

2.11 Proofs and further results

Proof of Lemma 3. Take any \( i \) and any \( a_i \). Take any \( \gamma \in \Delta \left(T_{-i}\right) \) with countable support, and let \( \mu = \gamma \circ (s^\ast_{-i})^{-1} \in \Delta \left(A_{-i}\right) \). Let \( \nu \) be as in the hypothesis. Define \( t_i \) as the type such that
\[
\kappa_{t_i} = \nu \times \gamma.
\]
Notice that
\[
\pi \left(\cdot_{t_i}, s^\ast_{-i}\right) = \kappa_{t_i} \circ \beta^{-1} = (\nu \times \gamma) \circ \beta^{-1} = \nu \times \left(\gamma \circ (s^\ast_{-i})^{-1}\right) = \nu \times \mu.
\]
Hence,
\[
s^\ast(t_i) = BR_i \left(\pi \left(\cdot_{t_i}, s^\ast_{-i}\right)\right) = BR_i \left(\nu \times \mu\right) = a_i.
\]

Proof of Lemma 5. It follows from the following lemma.

Lemma 8 For any nice game and for any \( i \), \( t_i \), \( k \), the following are true.

1. \( S^k_i \left[t_i\right] = [a^k_i, \bar{a}^k_i] \) for some \( a^k_i, \bar{a}^k_i \in A_i \), which depend on \( t_i \).
2. For each \( a^k_i \in S^k_i \left[t_i\right] \), there exists \( \hat{s}_{-i} \in \hat{S}^{k-1}_{-i} \) such that
\[
BR_i \left(\pi \left(\cdot_{t_i}, \hat{s}_{-i}\right)\right) = \{a^k_i\}.
\]

Proof. We will use induction on \( k \). For \( k = 0 \), part 1 is true by definition. Assume that
part 1 is true for some \( k - 1 \), i.e., \( S^{k-1}_j \left[t_j\right] \) is a closed interval in \( A_j = [0, 1] \) for each \( j \). This
implies that \( \hat{S}^{k-1}_{-i} \) is a closed, convex metric space (with product topology).\(^8\) Moreover, by the
Maximum Theorem, \( BR_i \left(\pi \left(\cdot_{t_i}, s_{-i}\right)\right) \) is an upper-semi-continuous function of \( s_{-i} \). But by the
strict concavity assumption, \( BR_i \left(\pi \left(\cdot_{t_i}, s_{-i}\right)\right) \) is singleton, and hence the function \( \beta_i \left(\cdot_{t_i}\right) \)

\(^8\)Proof: Firstly, \( \prod_{j \neq t_i} S^k_j \left[t_j\right] \) is a compact space by Tychonoff’s theorem. But the space of all measurable
functions \( f : \Theta \times T_{-i} \rightarrow \mathbb{R}^{\Theta_{-i}} \) is closed. Hence, the intersection of these two spaces, namely \( \hat{S}^{k-1}_{-i} \), is compact.
Convexity of \( \hat{S}^{k-1}_{-i} \) follows from the facts that measurability is preserved under point-wise multiplication and addition and that the range is convex.
maps each \( s_{-i} \in \hat{S}^{k-1}_{-i} \) to the unique member of \( BR_i (\pi (|t_i, s_{-i})) \) is continuous. Since \( \hat{S}^{k-1}_{-i} \) is compact and convex, this implies that \( \beta_i \left( \hat{S}^{k-1}_{-i}; t_i \right) \) is compact and connected, and hence it is convex as it is unidimensional. That is, \( \beta_i \left( \hat{S}^{k-1}_{-i}; t_i \right) = [a^k, \bar{a}^k] \) for some \( a^k, \bar{a}^k \in A_i \). We claim that \( \beta_i \left( \hat{S}^{k-1}_{-i}; t_i \right) = S^k_i [t_i] \). This readily proves part 1. Part 2 follows from the definition of \( \beta_i \left( \hat{S}^{k-1}_{-i}; t_i \right) \).

Towards proving our claim, for each \((\theta, t_{-i}) \in \kappa_{t_i}\) and for each \( s_{-i} \in \hat{S}^{k-1}_{-i} \), define function \( U_i (\cdot; \theta, t_{-i}, s_{-i}) \) by setting \( U_i (a_i; \theta, t_{-i}, s_{-i}) = u_i (\theta, a_i, s_{-i} (\theta, t_{-i})) \) at each \( a_i \). Clearly, \( U_i \) is strictly concave, and for each \( \sigma_{-i} \in \Delta \left( \hat{S}^{k-1}_{-i} \right) \), the expected payoff of type \( t_i \) is

\[
\int U_i (a_i; \theta, t_{-i}, s_{-i}) \, d\kappa_{t_i} (\theta, t_{-i}) \, d\sigma_{-i} (s_{-i}).
\]

(2.7)

Now, take any \( a_i > \bar{a}^k_i \). Then, for each \((\theta, t_{-i}, s_{-i})\), by definition of \( \bar{a}^k_i \) and strict concavity of \( U_i (\cdot; \theta, t_{-i}, s_{-i}) \), we have \( U_i (a_i; \theta, t_{-i}, s_{-i}) < U_i (\bar{a}^k_i; \theta, t_{-i}, s_{-i}) \). It then follows from (2.7) that \( \bar{a}^k_i \) yields higher expected payoff than \( a_i \) for each \( \sigma_{-i} \in \Delta \left( \hat{S}^{k-1}_{-i} \right) \), and thus \( a_i \notin S_i [t_i] \).

Similarly, \( a_i \notin S_i [t_i] \) for each \( a_i < a^k_i \). ■

Proof of Lemma 6. Part 2 follows from the definitions, and Part 3 follows from Part 2 and the fact that \( D \left( A^k [s^*, t] \right) \leq D \left( A^k [s^*, t] \right) \) for each \( k \). To prove Part 1, take any \( \epsilon > 0 \) and any sequence \( \epsilon_k \) that converges to 0. For each \( k \), there exists \( \tilde{t} [k] \) such that \( \tilde{t} [k] = t^m \) for each \( m \leq k \) and \( d \left( s^* (\tilde{t} [k]), s^* (t) \right) \geq D \left( A^k [s^*, t] \right) / 2 - \epsilon_k \) so that

\[
0 \leq D \left( A^k [s^*, t] \right) \leq 2d \left( s^* (\tilde{t} [k]), s^* (t) \right) + 2\epsilon_k.
\]

(2.8)

But, by definition, for each \( m \) and each \( k > m \), \( \tilde{t}^m [k] = t^m \), and hence \( \tilde{t}^m [k] \to t^m \) as \( k \to \infty \). Hence, if \( s^* \) is continuous at \( t \), then as \( k \to \infty \), \( s^* (\tilde{t} [k]) \to s^* (t) \), and thus the right hand side of (2.8) converges to 0. That is, \( D \left( A^k [s^*, t] \right) \to 0 \), showing by part 2 that \( s^* \) is continuous with respect to higher-order beliefs at \( t \). ■

Lemma 9 For any finite-action game, the following propositions are ordered with logical implication in the following decreasing order.

1. \( \sigma^* \) is continuous with respect to product topology at \( t \).

2. \( \sigma^* \) is continuous with respect to higher-order beliefs at \( t \).
3. \( \sigma^* \) is weakly continuous with respect to higher-order beliefs at \( t \).

4. \( A^\infty [\sigma^*, t] \subseteq \text{supp}(\sigma^*(t)) \).

Moreover, (2) implies that \( |A^\infty [\sigma^*, t]| \leq 1 \).

**Proof.** Since mixed strategies can be considered as pure strategies with values in \( \Delta (A_i) \), by Lemma 6.1, (1) implies (2). To show that (2) implies (3), for each \( B \subseteq A \), write \( \Sigma_B = \{\alpha \in \Delta (A) | \text{supp} (\alpha) \subseteq B\} \), which is a compact set. Then, for each disjoint \( B \) and \( C \), \( d_{B,C} = \min \{d(\alpha_B, \alpha_C) | \alpha_B \in \Sigma_B, \alpha_C \in \Sigma_C\} > 0 \). Write \( d_{\min} \) for the minimum of \( d_{B,C} \) among all disjoint \( B \) and \( C \). Clearly, if \( d(\alpha, \alpha') < d_{\min} \), then the supports of \( \alpha \) and \( \alpha' \) have non-empty intersection. But (2) implies that there exists \( k \) such that whenever \( \tilde{t}^m = t^m \) for each \( m \leq k \), \( d(\sigma^*(\tilde{t}), \sigma^*(t)) < d_{\min}/2 \), whence \( \text{supp}(\sigma^*(\tilde{t})) \cap \text{supp}(\sigma^*(t)) \neq \emptyset \)—hence (3). (3) implies (4) because for each \( a \in A^\infty [\sigma^*, t] \) and each \( k \), there exists \( \tilde{t} \) such that \( \tilde{t}^m = t^m \) for each \( m \leq k \) and \( \text{supp}(\sigma^*(\tilde{t})) = \{a\} \). The last statement in the lemma also follows from the latter observation because \( \sigma^*(t) \) cannot be arbitrarily close to two distinct pure action profiles. \( \blacksquare \)


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Chapter 3

Price Dispersion and Loss Leaders

3.1 Introduction

There is plentiful empirical evidence of price dispersion, i.e. retail firms charging different prices for identical goods. One well-known retail strategy which results in price dispersion is the strategy of cutting prices on one good, known as a “loss leader,” in order to attract more store traffic and increase profits on other goods. Any price dispersion is, of course, contrary to the unique pure-strategy\(^1\) equilibrium prediction in the setting of Bertrand competition that all transactions will take place at marginal cost. This theoretical prediction requires the following assumptions:

1: All firms have identical costs.

2: Consumers have full information, at zero cost, about the prices charged by each firm.

Since the standard argument is given in the context of a market for a single good, when we are in a situation with multiple goods there is another, sometimes hidden, assumption.

3: Consumers are free to buy different goods at different firms, with zero transportation costs, so that the markets for different goods are “uncoupled”.

A considerable literature has focused on whether price dispersion can result from relaxing the second assumption, i.e. introducing search costs. In this paper I will keep assumption

\(^1\)There are mixed-strategy equilibria in which prices are above cost and firms make profits. However (in the single-good case) all such equilibria involve unbounded prices and are hence eliminated if consumers have a maximum willingness to pay. In this paper, we will only consider pure-strategy equilibria.
2, but will drop assumption 3. In particular, I will go to the other extreme and assume that customers are constrained to purchase their entire bundle of goods from a single firm. This abstracts the idea that it is time-consuming to do one’s shopping at multiple locations, essentially changing transportation costs from zero to infinity. Intuitively, one might expect that the resulting “coupled” markets could result in a firm being able to successfully employ a loss-leader strategy, cutting prices on good A below cost in order to attract customers who will buy the profitable good B.

In the single-good case, one proves that there is no equilibrium in which a firm makes positive profits by observing that in this case another firm could “undercut” the profitable firm, charging a slightly lower price and taking all the profits. This argument still has some validity in the multi-good case – now “undercutting” means choosing a price vector which is slightly lower for all goods. The situation is complicated, however, by the fact that when one firm undercuts another in an effort to steal its profits, it may attract a different clientele which buys goods in different proportion. If some goods are sold below cost, this may result in losses, confounding the undercutting argument. We will find, however, that under a relatively innocuous set of assumptions it is guaranteed that at least one firm can increase its profits by undercutting another. Our main assumption is that each consumer has inelastic demand (i.e. demands a fixed bundle). We also need a genericity assumption on the distribution of demand vectors that allows us to avoid the technical issue of ties in consumers’ firm selection. Our genericity assumption also lets us avoid a degenerate case in which two goods are always bought in a fixed proportion, which would allow one price to be raised and the other decreased with no impact on the decisions of any consumers. Under our assumptions, we will find that in equilibrium no one makes any profits, and at least two firms charge exactly marginal cost for all goods, just as in the case of one-good Bertrand competition.

Most of the literature on this topic assumes either bounded rationality, or limited information and search costs. This begs the question of why we want to consider a fully rational model. I feel that it is an important first step to see whether and to what extent a fully rational, complete-information model can explain this phenomenon before proceeding to behavioral and informational considerations. Here, we are able to prove a non-trivial result that loss leaders and price dispersion cannot result merely from linking markets through high transportation
costs. Therefore the observed phenomenon of price dispersion must be explained by some factor other than the mere coupling of markets for different goods, justifying exploration of the role of limited rationality and information.

3.2 Literature Review

Many models have been introduced in which consumers have limited information about prices, whether search costs are introduced explicitly or information is simply limited more directly. Lal and Matutes (1994) analyze a model with advertising, in which consumers must decide where to shop knowing only the prices of goods which the stores choose to advertise, and only observe the remaining prices once at the store. The consumers have rational expectations and therefore anticipate that unadvertised goods will be overpriced; nevertheless, in equilibrium both stores do employ a loss-leader strategy, advertising a particular good which is priced below cost. More recently, Spiegler (2005) analyzes a model without rational expectations. In this case, consumers randomly observe one price from each store and choose the store for which that one price provides the highest consumer surplus. This boundedly rational choice procedure, called $S(1)$, was introduced by Osborne and Rubinstein (1998). Spiegler finds that firms make positive profits in equilibrium. Also, the variance in prices increases as the number of firms increases, contrary to the usual intuition about competition. This is essentially because with a very large number of firms, the best way to get attention from $S(1)$ consumers is to have a small number of goods with an extremely low price. In fact, if the model did not include a hard lower bound on prices, the price of the loss leaders would become arbitrarily negative as the number of firms grew. An alternative model of limited search is analyzed by Chen, Iyer and Pazgal (2005) who assume that consumers have limited memories for prices. In particular, the consumers divide the set of possible prices into finitely many ranges and only remember which range a price is in. They find that this limited memory enables firms to extract a greater surplus.

On the empirical side, Walters and MacKenzie (1988) find that although the loss-leader strategy is certainly commonly used, it has no clear effect on store traffic or profits on other goods. This can potentially be explained by the Lal and Matutes model in which two firms
simultaneously use the same good as a loss leader. Sorensen (2001) looks at price dispersion in the retail market for prescription drugs. He fits the data to a model of costly search and indeed finds evidence that dispersion is an outcome of imperfect search. In particular, prices are less dispersed when there is more incentive to search, as with medications bought repeatedly for chronic conditions, and more dispersed when there is less incentive to search, as with consumers who are fully insured.

The role of this paper is to establish that price dispersion cannot occur in a world with full information, even with high transportation costs which limit the consumer to one firm. This baseline result makes it clear that models such as those above which have limited information or rationality as well as transportation costs are necessary to understand price dispersion.

### 3.3 Model

We have $K$ identical firms, each of which sells $N$ goods, with constant marginal cost $c_i$ for the $i$th good. There is a continuum of consumers of mass 1, with each consumer having inelastic demand, so that each individual’s demand is characterized by a vector in $\mathbb{R}^N$, specifying the quantity he purchases of each good. The distribution of demands is described by a probability distribution $P$ on $\mathbb{R}^N$. We assume that $P$ assigns zero mass to any hyperplane in $\mathbb{R}^N$. The consumers, who are only able to shop at one firm, select the firm which minimizes their cost (choosing randomly in case of ties.) A profile $(p_1, \ldots, p_K)$ of price vectors is an equilibrium if no firm could increase profits by changing prices (consumers then adjusting their firm choice.) As mentioned in the introduction, we will only consider pure-strategy equilibria.

We are interested in whether there exists a pure-strategy equilibrium in which firms earn positive profits. Note that if $N = 1$ we are in the case of standard Bertrand competition, and the only equilibrium outcome is for all firms to make zero profits, with at least two firms charging exactly marginal cost.

A few words are in order on the assumption that hyperplanes have zero mass, which I refer to as a genericity assumption. This terminology is warranted because any distribution on $\mathbb{R}^N$ is arbitrarily close to distributions which satisfy the assumption, in the following sense: if $X$ is any random variable on $\mathbb{R}^N$, and $Y$ is a random variable distributed independently $u[-\varepsilon, \varepsilon]$
on each coordinate, $X + Y$ satisfies the assumption, for arbitrarily small $\varepsilon$. Also note that this assumption is clearly necessary to exclude price dispersion. In particular, if two goods were always demanded in a fixed ratio, firms could always increase one price and decrease the other without any effect on equilibrium.

### 3.4 Main Result

In this section we will prove that under the assumptions given above, all firms will make zero profits in any equilibrium. We start with a lemma, which assures using our genericity assumption on the distribution of demands that ties in consumers’ firm selection will not be an issue for us.

**Lemma 10** Assuming all firms have distinct price vectors, the mass of consumers who prefer more than one firm equally is zero.

**Proof.** Take any pair of firms $i$ and $j$. The set of demands for which firms $i$ and $j$ will be preferred equally is \( \{ x \in \mathbb{R}^N : p_i \cdot x = p_j \cdot x \} \). This is a hyperplane and so by assumption its mass is zero. \(\blacksquare\)

We now proceed to the main result.

**Proposition 16** In any equilibrium, all firms make zero profits.

**Proof.** First note that no firm can make negative profits in equilibrium, because it can always assure itself of non-negative profits by charging at or above cost for all goods. Now suppose we have an equilibrium \( (p_1, \ldots, p_K) \). We can assume without loss of generality that all costs are zero, since we can replace each \( p_i \) by \( p_i - c \) and let costs be zero without affecting the profits or equilibrium conditions. Let \( S_i = \{ x \in \mathbb{R}^N : p_i \cdot x < p_j \cdot x, \forall j \neq i \} \) be the set of consumer demands for which firm $i$ is preferred to all other firms, and also let \( S_{i,j} = \{ x \in \mathbb{R}^N : p_i \cdot x < p_j \cdot x \wedge p_k \cdot x, \forall k \neq i, j \} \) be the set of demands for which $i$ is preferred and $j$ is second-best. Each firm’s profits are given by \( \pi_i = \int_{S_i} p_i \cdot x \, dP(x) \). Assume at least one \( \pi_i > 0 \). Define \( 1 \) as the vector \( (1, \ldots, 1) \in \mathbb{R}^N \) and define \( UC(i, j, \varepsilon) \) as the profit that firm $i$ would make if it undercut firm $j$ by switching to price vector \( p_j - \varepsilon \cdot 1 \). As discussed in the introduction, it is not automatically profitable to undercut a profitable firm. When firm
i undercuts firm \( j \) it does not only attract consumers from the set \( S_j \), which is known to be profitable, but also those from the set \( S_{i,j} \), because the price vector \( p_i \) is no longer available. We will be able to show, however, that we are not at equilibrium by showing that for some triple \((i, j, \epsilon)\), \( UC(i, j, \epsilon) > \pi_i \).

We will first consider the case in which some two firms \( i \) and \( j \) offer identical price vectors. If firms \( i \) and \( j \) make positive profit, either one could double its profits by decreasing prices slightly and attracting all consumers who go to these two firms (the argument also applies if there is a tie among more than two firms.) If firms \( i \) and \( j \) make zero profits and some other firm \( k \) makes positive profits, it will certainly be profitable for firm \( i \) to undercut firm \( k \) – since the price vector \( p_i \) is still available from firm \( j \), firm \( i \) will only attract consumers from the set \( S_k \) and no others, so the deviation will be profitable.

Henceforth we assume all price vectors are distinct, so that Lemma 1 applies. Let \( S(i, j, \epsilon) \) be the subset of demand space for which firm \( i \) is chosen after it switches to price vector \( p_j - \epsilon \cdot 1 \). Firm \( i \) will now be selected by all consumers who previously chose firm \( j \). It will also be chosen by those who previously chose firm \( i \) but liked firm \( j \) second best, because firm \( i \)'s old price vector is no longer available to them. Therefore, \( S_j \cup S_{i,j} \subset S(i, j, \epsilon) \).

Let \( A_{i,j,\epsilon} = S(i, j, \epsilon) - (S_j \cup S_{i,j}) \). We will now show that \( \mu(A_{i,j,\epsilon}) \to 0 \) as \( \epsilon \to 0 \). Any member of \( A_{i,j,\epsilon} \) prefers some firm \( k \neq i \) to firm \( j \). By Lemma 1, this preference is strict except on a set of measure zero. When the preference is strict, \( p_k \) will still be preferred to \( p_j - \epsilon \cdot 1 \) for small enough \( \epsilon \), proving that \( A_{i,j,\epsilon} \to \emptyset \), which implies the claim. This implies that \( UC(i, j, \epsilon) \to \int_{S_j \cup S_{i,j}} p_j \cdot x \ dP(x) \) as \( \epsilon \to 0 \).

Assume without loss of generality that firm \( 1 \) makes the smallest profit in our equilibrium. It will suffice to show that for some \( j \), \( \int_{S_j \cup S_{i,j}} p_j \cdot x \ dP(x) > \pi_1 \), for then there will exist an \( \epsilon \)
for which UC(1, j, ε) > π₁. Observe that

\[ \sum_{j=2}^{N} \int_{S_{j} \cup S_{1,j}} p_{j} \cdot xdP(x) = \sum_{j=2}^{N} \left[ \int_{S_{j}} p_{j} \cdot xdP(x) + \int_{S_{1,j}} p_{j} \cdot xdP(x) \right] \]

\[ \geq \sum_{j=2}^{N} \left[ \pi_{j} + \int_{S_{1,j}} p_{1} \cdot xdP(x) \right] \]

\[ = \sum_{j=2}^{N} \pi_{j} + \int_{\cup_{j} S_{1,j}} p_{1} \cdot xdP(x) \]

\[ = \sum_{j=2}^{N} \pi_{j} + \int_{S_{1}} p_{1} \cdot xdP(x) \]

\[ = \sum_{j=1}^{N} \pi_{j} \]

where the inequality comes from the revealed preference of customers in the set S₁,j; we know they pay less at firm 1 than at firm j.

Because there are N − 1 terms in the left-hand sum, the above inequality implies that for at least one j, \( \int_{S_{j} \cup S_{i,j}} p_{j} \cdot xdP(x) \geq \frac{\sum_{i=1}^{N} \pi_{i}}{N-1} > \frac{\sum_{i=1}^{N} \pi_{j}}{N} \geq \pi_{1} \), where the strict inequality is implied by at least one \( \pi_{j} > 0 \). Then firm 1 can improve its profit by undercutting firm j.

The inequality in (1) is precisely the stage of the proof at which we made use of the individually inelastic demands. In particular, this assumption implied that since firm 1’s customers, in aggregate, pay a non-negative surplus to firm 1, they would continue to pay a non-negative surplus if firm 1 vanished and they had to patronize their second choice. This is effectively what happens as we consider the outcome of firm 1 undercutting each of the other firms in turn; when he undercuts firm j he attracts not only firm j’s old customers, but also his old customers who liked j second best. Our assumptions on demands ensure that on average, across all firms he might undercut, this does not hurt him.

Proposition 1 leaves open the possibility of an equilibrium in which firms make profits on some customers but losses on others, but we will now see that this is not possible.

**Proposition 17** In any equilibrium, the set of consumers who pay exactly the marginal cost of their bundle has mass 1.
Proof. We will show that if this is not the case, then for some \( \epsilon \), any firm could make positive profits by switching to price vector \( c + \epsilon \cdot 1 \). Let \( S = \{ x \in \mathbb{R}^N : c \cdot x < p_i \cdot x, \forall i \} \) be the set of demands for which the best price available is above the cost of the bundle. Let \( S_\epsilon = \{ x \in \mathbb{R}^N : (c + \epsilon \cdot 1) \cdot x < p_i \cdot x, \forall i \} \) be the set of demands for which the prices \( c + \epsilon \cdot 1 \) are preferred to those currently available. Note that \( S = \bigcup_{n=1}^{\infty} S_n \).

Note that if \( P(S_\epsilon) > 0 \), then any firm can make profits by switching to price vector \( c + \epsilon \cdot 1 \), because it would attract consumers in \( S_\epsilon \) and all customers are paying above cost. Therefore, \( P(S_\epsilon) = 0 \). Then by countable additivity, \( P(S) = P(\bigcup_{n=1}^{\infty} S_n) \leq \sum_{n=1}^{\infty} P(S_n) = 0 \). Therefore, the set of customers who pay at most the cost of their bundle has mass 1. Suppose a non-zero mass of customers are paying below cost. Then, some firms would have to be suffering losses, but this cannot happen in equilibrium.

Proposition 2 still leaves open the possibility that although all consumers pay the cost of their bundle, some goods are priced above cost and others below. We will now see that our genericity assumption excludes this possibility.

Proposition 18 In any equilibrium, at least two firms have all prices equal to cost.

Proof. Note that if \( p_i \neq c \), then the set \( \{ x \in \mathbb{R}^N : c \cdot x = p_i \cdot x \} \) is a hyperplane and so has mass zero by assumption. Therefore, by the previous proposition, we must have \( p_i = c \) for some \( i \). If this were true only for firm \( i \), we could repeat the argument in the previous proposition to show that firm \( i \) could deviate and make profits by choosing an appropriate vector \( c + \epsilon \cdot 1 \), so it must hold for at least two firms.

Proposition 19 Firms with some prices not equal to cost have no (that is, mass zero) customers.

Proof. As argued above, customers who would pay the cost of their bundle at such a firm have mass zero, so this is implied by Proposition 2.

3.5 Conclusions and further directions

We have seen that in the above context, price dispersion cannot occur in a fully rational, complete-information model. Our model was also restricted to consumers who individually
have inelastic demand; the question remains open of whether we could get price dispersion without this restriction. Also, we have not characterized mixed-strategy equilibria of our model. This seems to be a difficult problem.
Bibliography


Chapter 4

Two Notes on the Blotto Game

4.1 Introduction

Consider a game in which the two players simultaneously select vectors from \([0, 1]^N\) whose coordinates sum to 1, and are considered to have won a coordinate if they select a higher number than their opponent in that coordinate. After Laslier and Picard (2002), we call the game in which a player’s payoff is the number of coordinates won minus the number lost the plurality game. Alternatively, the object could be simply to win a majority of coordinates. This is called the majority game. This game can be interpreted as a contest between politicians allocating advertising money among \(N\) states in a simplified electoral college system, where each state is won by the side with greater spending. The classic case \(N = 3\) was first described by Borel (1921), and equilibria were first given in Borel and Ville (1938). It is often called the Colonel Blotto game, as it could be interpreted as a model of resource allocation in warfare, assuming that even a small advantage in resources allocated to a given battle is enough to win that battle completely. In the special case that \(N = 3\) and budgets are equal, the majority game and plurality game coincide because a player can never win in 0 or 3 coordinates.

It is well-known that a mixed strategy given by a distribution on \([0, 1]^N\) whose marginal distribution on each coordinate is \(\text{Uniform} \ [0, \frac{2}{N}]\) will be an equilibrium strategy in the plurality game. Borel and Ville (1938) found two examples of such distributions for \(N = 3\), one with

\[\text{Uniform} \ [0, \frac{2}{3}]\]

\[\text{Uniform} \ [0, \frac{2}{3} - \frac{1}{3}]\]

\[\text{Uniform} \ [0, \frac{2}{3} - \frac{2}{3}]\]

\[\text{Uniform} \ [0, \frac{2}{3} - \frac{1}{3}]\]

\[\text{Uniform} \ [0, \frac{2}{3} - \frac{2}{3}]\]

\[\text{Uniform} \ [0, \frac{2}{3} - \frac{1}{3}]\]

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\[\text{Uniform} \ [0, \frac{2}{3} - \frac{2}{3}]\]

\[\text{Uniform} \ [0, \frac{2}{3} - \frac{1}{3}]\]
support on the inscribed disc in the triangular representation of the simplex and one, called the Hex equilibrium, with support on the full hexagon \( \{ x_1, x_2, x_3 \in [0, \frac{2}{3}] : x_1 + x_2 + x_3 = 1 \} \) which is the set of best responses in both equilibria. We will exhibit an equilibrium strategy here, with the same marginal distributions, that has one-dimensional support – in particular, its support consists of two line segments. The construction generalizes to give a solution for the \( N \)-dimensional plurality game. Specifically, we give a distribution on \( \{ x \in [0, 1]^N : \Sigma x_i = 1 \} \) with support on \( N - 1 \) parallel line segments and the desired marginals.

Also, there are no current results on the modification of the majority game in which players have different total wealth, i.e. one player picks a vector whose sum is 1 while the other picks a vector whose sum is \( r \). We will provide bounds on the equilibrium payoffs in such a game as a function of \( r \). We obtain particularly tight bounds when \( r \) is close to 1, thereby characterizing the marginal impact of a small advantage in available resources.

### 4.2 Recent Literature

Laslier and Picard (2002) apply equilibria of the Blotto game to analyze the redistribution of goods that results from two-party electoral competition. In particular, they give the Lorenz curve and determine other measures of the inequality that would result from the distributions prescribed by the disc equilibrium. Kvassov (2003) analyzes Blotto-style contests with the modification that players do not necessarily use all their available resources. This approach would be justified in the many applications in which resources have an alternative use or can be saved for the next period. Our framework, on the other hand, in which resources must be spent immediately or lost, would frequently be appropriate in the context of campaign spending or warfare. Kvassov also allows for asymmetric budgets. Unlike our results on asymmetric budgets in Section 4, which focus on the majority game, he focuses on a game in which the objective is to win as many coordinates as possible (plurality game). Clearly this would be appropriate in auctions or other contexts where each coordinate won has value, while the majority perspective would usually be appropriate in an electoral context.

The majority game is also relevant in the scenario studied by Szentes and Rosenthal (2003), a simultaneous auction for three objects (chopsticks) in which the marginal value of acquiring a
second object is high compared to the first. They are able to completely describe the equilibria of such auctions, which are closely related to Blotto games. The key difference is that the lower bidder for each object does not pay, whereas the usual Blotto game is similar to an all-pay auction. The all-pay condition is a sensible model when resources cannot be recovered, as in campaign spending or warfare.

### 4.3 One-Dimensional Equilibrium

In this section, we will exhibit a one-dimensional distribution on the \( N \)-dimensional simplex \( \Delta_N = \left\{ x \in \mathbb{R}^N : x_i \geq 0, \sum_{i=1}^{N} x_i = 1 \right\} \) whose marginal distribution on each coordinate is \( \text{Uniform} \left[ 0, \frac{2}{N} \right] \) and which is therefore an equilibrium of the plurality game. We can depict this distribution graphically in the case \( N = 3 \) as a uniform distribution on the two line segments pictured in Figures 1 and 2. Figure 1 depicts the simplex explicitly as a subset of \( \mathbb{R}^3 \), while in Figure 2 we have the usual two-dimensional representation which we will use henceforth. This is obtained by letting the plane of the page be the plane \( x_1 + x_2 + x_3 = 1 \). Notice that the distribution of \( x_3 \) is uniform on each line segment individually. Also, coordinate \( x_1 \) is distributed \( U \left[ 0, \frac{1}{3} \right] \) on the left-hand segment in Figure 2 and \( U \left[ \frac{1}{3}, \frac{2}{3} \right] \) on the right-hand segment, yielding the correct overall distribution. Similarly, coordinate \( x_2 \) is distributed \( U \left[ 0, \frac{1}{3} \right] \) on the right-hand segment and \( U \left[ \frac{1}{3}, \frac{2}{3} \right] \) on the left-hand segment. In general, our distribution will be uniform on \( N - 1 \) parallel line segments in the \( N \)-dimensional simplex, as described below.

**Proposition 20** Let \( T = 1 + 2 + \ldots + (N - 1) = \frac{N(N-1)}{2} \). The uniform distribution on the \( N - 1 \) parallel line segments whose endpoints are given by \( \frac{T}{k} \left( k, k + 1, \ldots, N - 2, 0, \ldots, k - 1, N - 1 \right) \) and \( \frac{T}{N-1} \left( k + 1, k + 2, \ldots, N - 1, 1, \ldots, k, 0 \right) \) for \( k = 0, 1, \ldots, N - 2 \) gives a distribution on \( \Delta_N \) whose marginal distribution on each coordinate is \( \text{Uniform} \left[ 0, \frac{2}{N} \right] \).

**Proof.** First notice that the coordinates of each endpoint sum to 1, so that each line segment is indeed contained in the simplex. Also, the distribution of the last coordinate \( x_N \), which plays a special role, is \( U \left[ 0, \frac{N-1}{T} \right] = U \left[ 0, \frac{2}{N} \right] \) on each segment. The distribution of the first coordinate is \( U \left[ \frac{1}{T}, \frac{k+1}{T} \right] \) on the \( k \)th segment, yielding the correct overall distribution as \( k \) runs from 0 to \( N - 2 \). A similar argument applies to coordinates \( x_2 \) through \( x_{N-1} \).
Figure 4-1:

Notice that we would get a different distribution if we relabelled the coordinates; if we take the average of the distributions formed by the possible labellings we get a distribution which, like the classic examples, is symmetric between the coordinates. In the 3-dimensional case this is a uniform distribution on the six-pointed star pictured in Figure 3.
Laslier and Picard (2002) compute the average Lorenz curve that would result if wealth were distributed as in the disc equilibrium. If we order a division of one unit of wealth among $N$ individuals so that $y_1 \leq y_2 \leq \ldots \leq y_n$, the Lorenz curve is defined by the partial sums $c_k(y) = \sum_{i=1}^{k} y_i$. Given the mixed strategy defined above, a straightforward computation shows that the expected values of these partial sums are $l_k(N) = \frac{k^2}{N^2}$. In the limit, this approximates an average Lorenz curve of $c(t) = t^2$; that is, the average proportion of total wealth held by the smallest fraction $t$ of the population is $t^2$. In contrast, for the disc equilibrium Laslier and Picard find that the corresponding curve is $c_d(t) = t - \frac{1}{4} \sin \pi t$, which lies above our curve, so that there is more inequality in our equilibrium. Indeed, they find that the limit of the Gini index of inequality (defined as twice the area between the Lorenz curve and the diagonal) is $\frac{1}{\pi}$, while for us it is $\frac{1}{3}$.
4.4 Asymmetric Budgets

In this section we will analyze the majority game in the case $N = 3$, with the modification that player 2 has a total budget of 1 unit but player 1 has a total budget of $r$. Note that in the plurality game payoffs are completely determined by the marginal distributions on each coordinate, because of the additively separable utility functions. This property does not hold in the majority game except in the special case of equal budgets. In general, this makes it more difficult to describe equilibria, but we will be able to show some results giving bounds on the equilibrium payoffs for different values of $r$. In particular, let $w(r)$ be the equilibrium probability of winning for player 1. This section will establish some properties of the function $w$.

It will be convenient to modify our tie-breaking rule and specify that the player with the larger budget wins all ties, as suggested by Kvassov (2003). This ensures that payoffs are weakly lower-semicontinuous, which along with the fact that we have a constant-sum game with compact action spaces, and discontinuities lie in a lower-dimensional space, allows us to apply a result of Dasgupta and Maskin (1986) to guarantee existence of a mixed-strategy
equilibrium.

We note that in two-player zero-sum games, it is appropriate to speak of equilibrium strategies rather than strategy profiles. Indeed, we can define an equilibrium strategy as one which guarantees that the player receives at least his equilibrium (or maximin) payoff, and any pair of equilibrium strategies will be an equilibrium in the usual sense. Similarly, we can speak of an \( \varepsilon \)-equilibrium strategy as one which guarantees that the player comes within \( \varepsilon \) of his maximin payoff.

We observe that changing \( r \) to \( \frac{1}{r} \) effectively interchanges the roles of the two players, so that we have the following:

**Fact 1:** \( w(\frac{1}{r}) = 1 - w(r) \).

Because of this symmetry, we will focus on the case \( r > 1 \). We will now specify exactly how much of an advantage is necessary for player 1 to guarantee victory.

**Proposition 21** \( w(r) = 1 \) if and only if \( r \geq \frac{3}{2} \).

**Proof.** If \( r \geq \frac{3}{2} \) player 1 can guarantee victory by choosing the vector \( (\frac{r}{3}, \frac{r}{3}, \frac{r}{3}) \); player 2 cannot win because beating player 1 in two coordinates would require more than \( \frac{2r}{3} \leq 1 \) unit of wealth.

Now suppose \( r < \frac{3}{2} \), and let player 2 use the strategy which is uniformly distributed on the simplex. Take any action of player 1, and assume without loss of generality \( x_1 \leq x_2 \leq x_3 \), so that \( x_1 + x_2 < \frac{2r}{3} < 1 \). Then the region of player 2’s action space in which he wins coordinates 1 and 2 is an equilateral triangle of side \( 1 - x_1 - x_2 \geq 1 - \frac{2}{3}r \), so for fixed \( r \), there is a positive lower bound on his winning probability proportional to \( (1 - \frac{2}{3}r)^2 \).

We also have the following:

**Proposition 22** If \( r \geq \frac{5}{4} \) then \( w(r) \geq \frac{2}{3} \).

**Proof.** Suppose player 1 uses an equal mixture among the three vectors given by \( (\frac{1}{3}, \frac{1}{3}, \frac{r-1}{3}) \) and its permutations. We claim that any vector chosen by player 2 defeats at most one of these three vectors, so player 1 is guaranteed to win at least two-thirds of the time. Suppose to the contrary, so that without loss of generality player 2 has a vector \( (x_1, x_2, x_3) \) which wins against \( (\frac{1}{2}, \frac{1}{2}, r-1) \) and \( (\frac{1}{2}, r-1, \frac{1}{2}) \). In order to win against the first of these vectors we must
have \( x_3 > r - 1 > \frac{1}{4} \) and in order to win against the second vector we must have \( x_2 > r - 1 > \frac{1}{4} \), implying that \( x_1 < \frac{1}{2} \). Then, we would have to win against both vectors in both the second and third coordinate, implying \( x_2 \) and \( x_3 \) are both greater than \( \frac{1}{2} \) which is impossible. 

As our main result in this section, we will determine the marginal impact of a player having a small advantage in available resources. The general idea is that the equilibrium strategies from the symmetric case will still approximate equilibrium strategies here. From this argument, we will get the following result, which gives tight bounds on \( w(r) \) when \( r \) is close to 1. It will turn out to be important that the stronger player uses an equilibrium strategy with bounded density. The Hex equilibrium has density proportional to \( \max_i |x_i - \frac{1}{3}| \), which is bounded above. Thus, the one-dimensional distribution given in the previous section, or the disc equilibrium, which has unbounded density near the boundary, would be inferior to the Hex strategy when budgets are slightly asymmetric, in the sense that they do not approximate equilibrium strategy as closely, giving the weaker player a higher maximum payoff.

**Proposition 23** There exists \( A > 0 \) such that for all \( r > 1 \),\( \frac{3}{2}r - 1 - A(r - 1)^2 \leq w(r) \leq \frac{3}{2}r - 1 \).

**Proof.** Assume that \( r > 1 \) and that player 2 employs a strategy which has marginal distribution \( U[0, \frac{2}{3}] \) on each coordinate. Then for any vector in player 1’s choice set \( \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = r\} \) his probability of winning in coordinate \( i \) is \( p_i = \min(\frac{3}{2}x_i, 1) \). Then \( p_1 + p_2 + p_3 \leq \frac{3}{2}r \), with equality if \( x_1, x_2, x_3 \leq \frac{2}{3} \). With \( r > 1 \), player 1 could win in one, two, or all three coordinates. Let the probability that he wins in exactly \( j \) coordinates be \( q_j \). We can now compute his expected number of coordinates won in two different ways, as \( q_1 + 2q_2 + 3q_3 = p_1 + p_2 + p_3 \). Thus, the probability that player 1 wins a majority is \( q_2 + q_3 = p_1 + p_2 + p_3 - (q_1 + q_2 + q_3) - q_3 = p_1 + p_2 + p_3 - 1 - q_3 \leq \frac{3}{2}r - 1 \). Therefore, \( w(r) \leq \frac{3}{2}r - 1 \).

Now, to get a lower bound on \( w(r) \), assume player 1 employs a scaled version of the Hex strategy which has marginal distribution \( U[0, \frac{2}{3}r] \) on each coordinate, and has maximum density \( \tilde{p} \). We will seek the action for player 2 that maximizes his winning probability. For each vector \( (y_1, y_2, y_3) \) in player 2’s choice set, his probability of winning each coordinate \( i \) is \( p_i = \min(\frac{3}{2}y_i, 1) \). Since player 2 will always win in 0, 1, or 2 coordinates, a derivation similar to that above gives that his winning probability is \( s_2 = p_1 + p_2 + p_3 - 1 + s_0 \) where \( s_j \) is the
probability that he wins exactly \( j \) coordinates. For a fixed action of player 2, the region in player 1’s action space for which player 2 wins in 0 coordinates will be an equilateral triangle of side \( r - 1 \), as depicted in Figure 4. He will want to choose his action so that this triangle is in a region of maximal density\(^2\), which is why it is important that the density of player 1’s strategy be bounded. We then have that player 2’s winning probability satisfies

\[
s_2 < p_1 + p_2 + p_3 - 1 + \frac{\sqrt{3}}{4}(r - 1)^2
\]

\[
\leq \frac{3}{2r} - 1 + \frac{\sqrt{3}}{4}(r - 1)^2
\]

Therefore

\[
w(r) \geq 1 - s_2^{\text{max}} \geq 2 - \frac{3}{2r} - \frac{\sqrt{3}}{4}(r - 1)^2
\]

Now we will use the expansion

\[
\frac{1}{r} = \frac{1}{1 - (1 - r)} = 1 + (1 - r) + (1 - r)^2 + (1 - r)^3 \ldots
\]

\[
< 1 + (1 - r) + (1 - r)^2
\]

when \( r > 1 \). Substituting this into the inequality above, we have

\[
w(r) \geq 2 - \frac{3}{2} \left[ 1 + (1 - r) + (1 - r)^2 \right] - \frac{\sqrt{3}}{4}(r - 1)^2
\]

\[
= \frac{3}{2}r - 1 - \left( \frac{\sqrt{3}}{4} + \frac{3}{2} \right)(r - 1)^2
\]

as desired. ■

In the course of the proof, we also showed

**Corollary 5** Fix \( r > 1 \) and let \( \varepsilon = A(r - 1)^2 \). Any equilibrium strategy of the symmetric game is an \( \varepsilon \)-equilibrium strategy for the weaker player. The Hex strategy is an \( \varepsilon \)-equilibrium strategy for the stronger player, while any other equilibrium strategy of the symmetric game with higher maximum density is not.

\(^2\)It may momentarily seem obscure that he wants to maximize his probability of losing all three coordinates. The key is that his expected number of coordinates won is constant, and winning exactly one coordinate is a useless waste of resources.
Figure 4-4: The simplex represents the action space of player 1, who is assumed to have fixed a mixed strategy. The point represents an action \((y_1, y_2, y_3)\) for player 2. The label in each region is the number of coordinates player 2 would win if 1’s action were in that region, so he is trying to maximize the total mass under player 1’s strategy of the regions labeled 2. This turns out to be equivalent to maximizing the mass of the triangle labeled 0.

Our bounds on \(w(r)\), which only differ by a second-order term for \(r\) close to 1, also yield the following result.

**Corollary 6** \(w'(1) = \frac{3}{2}\).

**Proof.** For the right derivative this is immediate. The left derivative follows from a short derivation using symmetry considerations (i.e. Fact 1). ■
4.5 Concluding Remarks

The proof of our final proposition suggests that an equilibrium strategy of the standard Blotto game will be more robust to small asymmetries in players' available resources when the density of the strategy is bounded above. It is relatively easy to show that the Hex strategy has the smallest maximum density of any distribution with the appropriate marginals, so that it is maximally robust from this point of view. The new distribution which we presented, of course, does not do well under this criterion, since it has one-dimensional support. It does have the aesthetic advantage of being extremely easy to describe and verify.
Bibliography


