Pricing and Efficiency in Wireless Cellular Data Networks

by

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Abstract:

In this thesis, we address the problem of resource allocation in wireless cellular networks carrying elastic data traffic. A recent approach to the study of large scale engineering systems, such as communication networks, has been to apply fundamental economic principles to understand how resources can be efficiently allocated in a system despite the competing interests and selfish behavior of the users. The most common approach has been to assume that each user behaves selfishly according to a payoff function, which is the difference between his utility derived from the resources he is allocated, and the price charged by the network's manager. The network manager can influence user behavior through the price, and thereby improve the system's efficiency. While extensive analysis along these lines has been carried out for wireline networks (see, for example, [10], [7], [23], [29], [21]), the wireless environment poses a host of unique challenges.

Another recent line of research for wireline networks seeks to better understand how the economic realities of data networks can impact the system's efficiency. In particular, authors have considered the case where the network manager sets prices in order to maximize profits rather than achieve efficient resource allocation; see [1] and references therein.

In this thesis, we make three contributions. Using a game theoretic framework, we show that rate-based pricing can lead to an efficient allocation of resources in wireless cellular networks carrying elastic traffic. Second, we use the game theoretic equilibrium notions as motivation for a cellular rate control algorithm, and examine its convergence and stability properties. Third, we study the impact of a profit-maximizing price setter on the system's efficiency. In particular, we show the surprising result that for a broad class of utility functions, including logarithmic and linear utilities, the profit maximizing price results in efficiency.

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Chapter 1

Introduction and Preliminaries

1.1 Motivation

In this thesis, we address the problem of resource allocation in wireless cellular networks carrying elastic data traffic. A recent approach to the study of large scale engineering systems, such as communication networks, has been to apply fundamental economic principles to understand how resources can be efficiently allocated in a system despite the competing interests and selfish behavior of the users. Game theory (see [40]) is a natural mathematical framework in this context. The most common approach has been to assume that each end-user behaves selfishly according to a payoff function, which depends on the utility derived from the amount of resources he is allocated, and the price charged by the network's manager. The network manager can influence user behavior through the price, and thereby improve the system's efficiency. This has proven to be a powerful methodology in the wireline networking community, and has given rise to promising new frameworks of study such as (to name a few) "selfish routing" (see [7]), the so-called "Kelly controllers" (see [23]), pricing based on duality (see [29]), and resource allocation based on market mechanisms (see [21]).

While extensive analysis along these lines has been carried out for wireline networks, wireless networks pose a host of unique challenges. As this thesis will show, several mathematical difficulties arise when attempting to apply even the most basic results from the game theory literature to a reasonable model for a wireless cellular network. This kind of challenge has hindered the research community from making the same level of progress seen for wireline networks.

Another recent line of research for wireline networks seeks to better understand how the economic realities of data networks can impact the system's efficiency. In particular, authors have
considered the case where the network manager sets prices in order to maximize profits rather than achieve efficient resource allocation, and have studied the associated efficiency loss; see [1] and references therein. Again, with few notable exceptions (such as [28]), profit-maximizing pricing for wireless networks has received little attention in the literature.

In this thesis, we make three contributions. First, we show that usage-based pricing can lead to an efficient allocation of resources in wireless cellular networks carrying elastic traffic. Second, we use the game theoretic equilibrium notions as motivation for a cellular rate control algorithm, and examine its convergence and stability properties. Finally, we study the impact of a profit-maximizing price setter on the system’s efficiency. In particular, we show the surprising result that for a large class of utility functions, including logarithmic and linear utilities, profit-maximizing pricing results in no efficiency loss.

1.2 Preliminaries for CDMA cellular systems

In this section, we briefly discuss some fundamental aspects of CDMA cellular systems to motivate our model; see [16] for an overview, and [51] for thorough coverage. We also discuss previous conventional approaches (those that do not employ pricing or game theoretic analysis) to the problem of resource allocation in such systems. The following brief discussion on CDMA, its technical merits, and important modelling issues can be found in [16].

Code division multiple access is one of the common multiple-access techniques in communication systems. It is known as a spread spectrum, or broadband, technique since, unlike other common multiple access schemes such as frequency division multiple access (FDMA) or time division multiple access (TDMA), the system’s entire bandwidth is shared at all times by all users. CDMA has numerous advantages which have led to its popularity. For example, implementation issues are simplified, since there is no need to schedule time or frequency slots as in TDMA or FDMA. Furthermore, CDMA has no hard constraint on the number of users admissible to the system, unlike TDMA and FDMA, where the number of users is limited to the number of time or frequency slots available.

The principle behind spread spectrum systems is that data is modulated over a large band-
width using a binary *pseudorandom sequence*, also known as a *spreading sequence* or a *signature*, before being transmitted. In order for the modulated signal to occupy the entire transmission bandwidth, this spreading sequence should be like white Gaussian noise. For downlink transmission (in which a central base station transmits data to each user), it seems reasonable to assume that the base station can choose unique and orthogonal signatures for all users (assuming that the number of users is less than the total number of orthogonal signature sequences available), and transmit the modulated data in a synchronous fashion. In this setting, each user can demodulate his received signal using his signature, and due to the orthogonality of signature sequences, there will be no interference caused by signals intended for other users. However, the number of possible signature sequences depends on the length of each signature (a system parameter), and it is possible that there will be more users than available orthogonal signatures. Furthermore, multipath distortion, a phenomenon inherent in wireless systems whereby the receiver receives several delayed versions of a signal superimposed on each other, can eliminate the orthogonality property of the signature sequences. Finally, for uplink transmission (in which each user has data to send to a central base station), it is not practical to coordinate the transmissions to be synchronous. Therefore, especially for uplink transmission, it is typically assumed that there is no correlation between the signature sequences of different users.

This motivates the following formula for the signal to interference ratio $\gamma_j$ for a user $j$; see [15]:

$$\gamma_j = \frac{W}{r_j} \frac{h_j p_j}{\sum_{i \neq j} h_i p_i + \sigma^2}.$$  \hspace{1cm} (1.2.1)

Here, user $j$ transmits data to the base station with a data rate $r_j$, power $p_j$, channel gain $h_j$, and with a spreading gain of $W/r_j$. (*Spreading gain* refers to the length of the binary signature sequence.) The quantity $\sum_{i \neq j} h_i p_i + \sigma^2$ is viewed as white Gaussian background noise by the receiver, where $\sigma^2$ is the system's background noise power.
1.2. Preliminaries for CDMA cellular systems

1.2.1 Measuring quality of service for systems carrying voice traffic

Early cellular systems only carried voice traffic, and transmitted data with a fixed rate of data transmission. In this case, quality of service is characterized by the Signal to Interference ratio (SIR) $\gamma_j$; see [15].

Because data for conversations must be transmitted in real time, the data must be sent with little delay, implying that retransmissions cannot be used to improve quality of service. Therefore, for voice traffic, acceptable communication is specified by some minimum SIR requirement. So long as a user's SIR is above a certain threshold $\gamma_{th}$, he will have an acceptable quality of service. Because the rate of data transmission is fixed, resource allocation is solely a transmission power control issue.

In light of this, many initial papers on CDMA power control deal with algorithms which allocate powers to meet an SIR threshold for all users, and deny service to some users if this objective is not feasible. One often considered class of algorithms that achieves this goal is SIR balancing algorithms, also known as “max min” algorithms; see [2], [12], [13], [35], [36], [55]. These sought to give all users the highest attainable common SIR.

1.2.2 Cellular power control infrastructure

Early power control papers considered a model in which a central coordinator controls all of the links from users to base stations in a system; see [55], [13]. However, such a controller would have very high information requirements and would need to perform complex computations, making this approach infeasible in practice. A more distributed approach, in which each base station acts autonomously, was favored. The fundamental system architecture for power control that was developed involved each base station tracking and updating the transmit powers of the mobile devices within its cell. A benefit of this architecture is that, since each cell can be viewed independently, when doing analysis one can restrict attention to single cell models without loss of generality, which is the approach taken in this thesis. Iterative power control algorithms and their convergence properties were studied; see, e.g., [12], [54], [56].
1.2.3 Third Generation systems

In contrast to earlier systems, Third Generation CDMA systems support variable data rates and support applications that are much more data-intensive than voice traffic. As a result, in addition to transmission powers, transmission data rates represent another controllable resource; see [42].

Furthermore, for non-real time applications such as file transfer, a user’s quality of service is not measured simply by his SIR, but instead by how fast data is transmitted to the base station without error. To quantify this, authors have introduced the notions of efficiency functions and effective data rates. An efficiency function $f$ maps a user’s SIR to a probability that a frame of data will be successfully received by the base station.

$$f(\gamma) \in [0,1] \quad (1.2.2)$$

The specific form of the efficiency function depends on the modulation scheme and error correction codes employed; see, e.g., [11],[43],[37]. A user transmitting with data rate $r_j$ and SIR $\gamma_j$ then has an effective data rate of

$$r_j f(\gamma_j). \quad (1.2.3)$$

Several approaches have been taken to consider resource allocation for third generation systems. For example, in [47], the users transmit with various data rates, which remain fixed over time, and an optimal power control scheme is derived to maximize the total effective data rate of the terminals. However, this approach treats data rates as fixed, and not as a controllable resource. In contrast, [22] treats both transmission powers and transmission data rates as variables for optimization. The objectives considered were minimizing total transmission power of the system, and minimizing total transmitted data rate of the system subject to constraints on effective data rates. Finally, [38] considered the objective of maximizing the total effective data rate of the system, optimizing over both rate and power.
1.3 Preliminaries for the application of game theory and pricing to wireless networks

There is an extensive body of literature that applies pricing and game theoretic principles to the analysis of wireline communication networks. However, as we will see in Chapter 2, wireless networks present a host of unique challenges that primarily arise due to non-convexities associated with the effective data rate function (1.2.3). As a first step, in this section, we discuss how pricing is used in different contexts, requirements that a practical pricing scheme must meet, and finally, discuss what constitutes a logical “resource” to price in the wireless setting. Throughout, we discuss models considered by other authors.

Uses of pricing in different contexts:

1. Economic: A service provider implements pricing in order to recoup the costs of running a network, and typically tries to maximize his revenue.

2. Differentiated Service: Users can pay different amounts to get varying qualities of service. This enables a network to satisfy heterogeneous users with various demands (i.e. various utility functions), who are transmitting data with various resource requirements.

3. Congestion Control: Prices are used as control parameters, and are employed to induce users into an efficient usage of a network’s limited resources.

While these issues have been studied extensively for the case of wireline networks (see [10] for a survey), only limited work has been done for wireless networks. In fact, research that involves pricing has almost exclusively focused on Pricing Motivation 3 above, neglecting Pricing Motivations 1 and 2.

A pricing scheme should also satisfy practical requirements regarding its implementation.

Pricing requirements:

1. Simple pricing structure: the pricing scheme should be easy for consumers to grasp, and for service providers to implement.
2. Consistent with current standards: the pricing scheme should fit in with current network infrastructure and standards.

Several pricing schemes have been considered in the literature. We can examine these schemes with the above considerations in mind.

### 1.3.1 Pricing power usage

Recall that a user’s SIR is adversely affected by the transmission power of other users (1). With this in mind, almost all previous models have considered usage-based pricing schemes that are linear functions of a user’s power usage; see, for example, [3], [18], [24], [25], [43], [53]. Letting \( q \) denote the unit price for power, we have

\[
\text{price} = qp_j.
\]

However, two issues arise with this pricing scheme. First, pricing the power implies that users will choose their own transmission power, which leads to a fully-distributed power control system. Recall that this model goes against the existing and well-studied cellular power control infrastructure, in which a base station coordinates the transmission powers of all the users in the cell; see Section 1.2. Therefore, this pricing mechanism is not compatible with existing standards and infrastructure (see Pricing Requirement 2).

Furthermore, power is an abstract quantity to price. A typical consumer may not understand this pricing method, and this violates Pricing Requirement 1.

For these reasons, many of the papers listed above use their pricing mechanisms as a starting point to develop new distributed algorithms that are a departure from the current cellular infrastructure. In contrast, this thesis seeks to use pricing as a means for congestion control within the cellular framework.
1.3. Preliminaries for the application of game theory and pricing to wireless networks

1.3.2 SIR based pricing

In [45], a different view is taken. Here, it is noted that the following is a fundamental system constraint for uplink data transmission:

\[ \sum_j r_j \gamma_j < W. \]

See [45], [46], [16]. This constraint is interpreted to mean that \( r_j \gamma_j \) represents the fundamental resource usage of user \( j \) in the CDMA uplink, and therefore a natural pricing mechanism will be a linear function of this quantity. Letting \( q \) represent the unit price, we have

\[ \text{price} = qr_j \gamma_j. \]

In [19], an auction based algorithm is proposed in which each user is charged a price that is linear in his SIR.

\[ \text{price} = q \gamma_j. \]

Although it is shown that such pricing schemes have desirable congestion control properties, they are abstract quantities to price (see Pricing Requirement 1).

1.3.3 Pricing rate

A pricing mechanism that has not, to our knowledge, been considered in previous models is the usage-based pricing of the data transmission rate. Letting \( q \) represent the unit price, we have

\[ \text{price} = qr_j. \]  \hspace{1cm} (1.3.1)

If one assumes that, given the data rates that users select, the base station manages power control issues, then this pricing mechanism fits in very well with the existing cellular infrastructure, and satisfies our Pricing Requirements. This is the pricing scheme we consider in this thesis.
1.4 Modeling user behaviour

In the game theoretic literature, user behavior is commonly modelled through the use of a utility function defined over the amount of resource assigned to the user. Ideally, a user’s utility function will reflect the fact that a user’s utility depends on some measurable performance criteria, such as the delay the user’s packets incur between transmission and reception. Defining utilities in this matter can lead to complexities, and therefore in the literature, utility functions are defined over the underlying resource that dictates the performance metrics. However, because third generation wireless systems carry various types of data whose quality of service depends on various parameters, it is not clear what this resource should be.

1.4.1 Utility functions for inelastic traffic

Examples of inelastic traffic, defined in [44] as traffic that is not tolerant to delay variations, include voice and real-time streams. User satisfaction for this type of traffic is characterized in a binary fashion: either the connection is acceptable, or it is not. Whether or not it is acceptable depends on whether the user’s SIR is above a threshold $\gamma_{th}$. Therefore, an appropriate utility function for voice users should depend only on the user’s SIR, and should be a step function; see [11], [44], and Figure 1.

Figure 1-1: The utility function for users sending voice traffic.
1.4.2 Utility functions for elastic data traffic

A more complex issue is how to characterize the utility of users sending elastic data traffic, which will be the focus of this thesis. Elastic traffic, defined in [44], refers to traffic that is tolerant to delay variations, such as file transfer or ordinary web browsing. A common assumption for utility functions of such traffic is that they are increasing, concave functions of their throughput. These assumptions are natural; the first says that more throughput yields more utility, and the second says that users get diminishing marginal returns for marginal increases in throughput.

\[ u_j(r_j f(\gamma_j)) \]

for user \( j \).

Figure 1-2: The utility function for users sending elastic traffic.

Recall that the achieved throughput of a data user is given by his effective data rate (cf. 1.2.3). Therefore, in [45] and [24], it is assumed that users derive utility from their effective data rate, giving a utility of \( u_j(r_j f(\gamma_j)) \) for user \( j \). This is the approach we follow in this thesis.

1.4.3 Risk Aversion

Risk aversion is a quantifiable characteristic of utility functions. Intuitively, it is a measure of how much a user prefers a particular expected payoff with certainty over the same expected payoff with uncertainty. In this section, we interpret the two measures of risk aversion, absolute risk aversion and relative risk aversion, that are used in this thesis. In the following, we give some main ideas from [4], which provides a thorough coverage of risk aversion.
1.4. Modeling user behaviour

Suppose an individual receives an amount $x \geq 0$ of some finite resource, and suppose his preferences over $x$ are described by a utility function $u : \mathbb{R}^+ \to \mathbb{R}^+$. Assume that $u$ is an increasing, concave function. In loose terms, the individual is risk averse at $x$ if he prefers to receive an amount $x$ of the resource with certainty rather than $x - \delta$ with probability $\frac{1}{2}$ and $x + \delta$ with probability $\frac{1}{2}$. It can be shown that strict concavity of the utility function implies that the individual will be risk averse for all $x$. There are two commonly used metrics used to quantify risk aversion.

The first is known as absolute risk aversion, and is defined by

$$A(x) \equiv -\frac{u''(x)}{u'(x)}. \quad (1.4.1)$$

For a simple interpretation, suppose the individual has an amount $x$ of the resource. He is offered a bet in which he can increase his amount of the resource to $x + \delta$ with probability $\pi$, or suffer a loss reducing his amount of the resource to $x - \delta$ with probability $(1 - \pi)$. For $\pi = 1$, clearly he will accept the bet, and for $\pi = 0$, he will reject it. It seems reasonable to quantify the risk aversion of the individual by the value of $\pi$ for which he is just indifferent to accepting or rejecting the bet. Let $\pi(x, \delta)$ denote this value of $\pi$ (it can be shown that such a value is guaranteed to exist, and is unique). It can be shown (see [4]) that for $\delta << 1$, $\pi(x, \delta)$ can be approximated as

$$\pi(x, \delta) = \frac{1}{2} + \frac{A(x)}{4}\delta + O(\delta^2),$$

where $O(\delta^2)$ represents terms of higher order in $\delta$. Thus, we see that absolute risk aversion can be interpreted as being related to the premium in expected return that the individual demands in order to accept uncertainty in his payoff.

The second common measure of risk aversion is relative risk aversion, and is defined by

$$R(x) \equiv -x\frac{u''(x)}{u'(x)}.$$ 

To interpret this, consider the same bet considered above, with the difference that the individual can win or lose an amount $\delta x$ that is proportional to his current wealth; i.e., he will either end
with $x + \delta x$ or $x - \delta x$. For $\delta << 1$, just as above, we can obtain the approximation

$$\pi(x, \delta x) = \frac{1}{2} + \frac{R(x)}{4}\delta + O(\delta^2).$$

Thus, relative risk aversion can be interpreted as being related to the premium in expected return that the individual demands in order to accept uncertainty in his payoff, where the premium is scaled by his current wealth.

Relative risk aversion plays a significant role in a central result of this thesis. In particular, we will show that if the relative risk aversion $R_j(x)$ is less than 1 for all users, and for all $x$, then the rate-based price which leads to an efficient allocation of resources is equivalent to the rate-based price that would be charged by a profit-maximizing service provider. Absolute risk aversion plays an important role in framing various results in this thesis in a physically interpretable form.

## 1.5 Quantifying Efficiency

In order to study the impact of pricing on efficiency, one needs to have well-defined and quantifiable measures of efficiency. While there are many notions of efficiency in the literature, the following are natural for elastic traffic:

Let $\mathcal{J}$ denote the set of users, indexed by $j$.

**Definition 1.** Given some fixed power allocation scheme, a throughput maximizing rate allocation is a rate vector $r^*$ that maximizes the total effective data rate of the system.

$$r^* \in \arg \max_r \left\{ \sum_j r_j f(\gamma_j) \right\}. \tag{1.5.1}$$

**Definition 2.** Given some fixed power allocation scheme, a rate vector $r^*$ is a Pareto efficient solution if there exists no other rate vector $r$ such that $u_j(r_j f(\gamma_j)) \geq u_j(r_j^* f(\gamma_j^*)), \forall j \in \mathcal{J}$ and $u_j(r_j f(\gamma_j)) > u_j(r_j^* f(\gamma_j^*))$ for some $j \in \mathcal{J}$.

These are the two notions of efficiency that we will study in this thesis. Maximizing the
total effective data rate of the system is a primary efficiency objective from an engineering perspective; when the objective in Definition 1 is satisfied, there is no outcome which can result in a higher utilization of the system. Pareto efficiency is a primary efficiency objective from a social perspective. When the objective in Definition 2 is satisfied, one cannot find a redistribution of resources which benefits some user without hurting some other user.

1.6 Contributions of this thesis

In this thesis, we make three contributions. First, we show that usage-based pricing can lead to an efficient allocation of resources in wireless cellular networks carrying elastic traffic. Second, we use the game theoretic equilibrium notions as motivation for a cellular rate control algorithm, and examine its convergence and stability properties. Third, we study the impact of a profit-maximizing price setter on the system's efficiency. We show the surprising result that for a large class of utility functions, a profit-maximizing price leads to efficiency.

In Chapter 2, we define a game theoretic framework which will form the basis of our study. We show the existence of a Nash equilibrium for any price \( q \), and in doing so deal with the non-convexities associated with utility functions in the wireless setting that have posed challenges in the research of wireless networks. We then employ limiting arguments to show that the Nash equilibrium can be approximated by a more tractable equilibrium notion in which an individual's unilateral change in action has a negligible impact on the overall system. This equilibrium notion is similar to the Wardrop equilibrium, first introduced in congestion analysis for transportation networks. We guarantee the existence of a price for which there exists an efficient equilibrium, and refer to this price as a Pigovian tax.

In Chapter 3, we examine stability properties of the Wardrop Equilibrium of Chapter 2. We use this as motivation for a congestion control algorithm based on the Wardrop Equilibrium, and study its convergence.

In Chapter 4, we consider the case of a profit-maximizing price setter. We characterize the resulting equilibrium. We then show the surprising result that for a large class of user utility functions, including logarithmic and linear utilities, the profit maximizing price is equal to the Pigovian tax.
1.7. Notation

Scalors and Vectors

We denote by $\mathbb{R}$ the set of real scalars. We denote by $\mathbb{R}^+$ the set of real and nonnegative scalars: $\mathbb{R}^+ = \{ x \in \mathbb{R} | x \geq 0 \}$.

We denote by $\mathbb{R}^n$ the set of $n$-dimensional real vectors. For any $x \in \mathbb{R}^n$, we use $x_i$ to indicate its $i$th component. Vectors in $\mathbb{R}^n$ will be viewed as column vectors. If $x \in \mathbb{R}^n = (x_1, ..., x_n)$, then we will use $x_{-i}$ to denote the components of $x$ other than $x_i$; i.e., $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a function of $n$ scalars $x_1, ..., x_n$, we let $f(x_i; x_{-i})$ denote the function $f$ as a function of $x_i$ while keeping the components $x_{-i}$ fixed.

Sequences

We denote a sequence of scalars, indexed by $n$, as $\{x(n)\}$. The sequence is said to converge if there exists a scalar $x$ such that for every $\epsilon > 0$, we have $|x(n) - x| < \epsilon$ for every $n$ greater than some integer $N$. The Bolzano-Weierstrass Theorem, used later in this thesis, states the following:

**Bolzano-Weierstrass Theorem:** A bounded sequence in $\mathbb{R}^n$ has at least one limit point.

If a sequence $\{x(n)\}$ converges to a limit point $x$, we write $x(n) \to x$.

The remainder of the notation used in this thesis is either explained within the document, or is self-evident.
Chapter 2

Wardrop Equilibrium for rate selection in wireless data networks

2.1 Model and Framework

We consider a single cell CDMA network, and focus on uplink data transmission. The cell consists of one base station, operated by a service provider, and $J$ users attempting to send data to the base station. Let $\mathcal{J}$ denote the set of users.

User $j$ transmits data to the base station with a data rate $r_j$, with power $p_j$, and with a spreading gain of $\frac{W}{r_j}$. The system bandwidth is therefore $W$. We assume that each user has a maximum transmission power of $p_{\text{max}}$. Let $r \in \mathbb{R}^J$ and $p \in \mathbb{R}^J$ denote the vectors of rates and powers, respectively. We also let $R = \sum_{j \in \mathcal{J}} r_j$ denote the sum rate of all users in the system. User $j$'s signal experiences a path loss of $h_j$. We assume there is a total background noise power of $\sigma^2$ at the base station. Based on these parameters, user $j$'s signal quality is quantified through his Signal to Interference Ratio (SIR)$^1 \gamma_j$, where

$$\gamma_j = \frac{W}{r_j} \frac{h_j p_j}{\sum_{i \neq j} h_i p_i + \sigma^2}. \quad (2.1.1)$$

(See [15], [16]). We assume $J$ and $W$ are large, so the contribution of an individual interferer is negligible. Then, (2.1.1) can be approximated as

$^1$A different convention is to refer to this quantity as the bit energy to noise density ratio $\frac{E}{N_0}$, in which case

$$\frac{h_j p_j}{\sum_{i \neq j} h_i p_i + \sigma^2}$$

is referred to as the SIR.
2.1. Model and Framework

\[ \gamma_j = \frac{W}{r_j \sum_{i \in J} h_i p_i} \]  \( \text{Eq} \ 2.1.2 \)

This type of approximation is used and justified in [16]. User \( j \) sends data to the base station in frames consisting of \( M \) bits, where \( M > 1 \). The probability that a frame transmitted by user \( j \) can be decoded by the base station without error is quantified by a function of \( \gamma_j \). We denote this function by \( f(\gamma_j) \), and refer to it as the efficiency function. The notion of the efficiency function has been well studied; see, e.g., [11],[37],[38],[43],[45]. The system’s efficiency function depends on the modulation scheme, which dictates the bit error rate \( BER \), and the error correction codes employed. A crude system with bit error rate \( P_e \) will have an efficiency of \( (1 - P_e)^M \); error correction codes will result in a more complicated expression. Below are examples of simple efficiency functions for various modulation schemes (see [43]), and the reader is referred to [11] for efficiency functions that incorporate error correction coding.

<table>
<thead>
<tr>
<th>Modulation Scheme</th>
<th>Bit Error Rate</th>
<th>Efficiency function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. DPSK</td>
<td>( \frac{1}{2} e^{-\gamma} )</td>
<td>((1 - \frac{1}{2} e^{-\gamma})^M)</td>
</tr>
<tr>
<td>2. Non-coherent FSK</td>
<td>( \frac{1}{2} e^{-\frac{\gamma}{2}} )</td>
<td>((1 - \frac{1}{2} e^{-\frac{\gamma}{2}})^M)</td>
</tr>
<tr>
<td>3. BPSK</td>
<td>( Q((2\gamma)^{\frac{1}{2}}) )</td>
<td>((1 - Q((2\gamma)^{\frac{1}{2}}))^M)</td>
</tr>
<tr>
<td>4. Coherent FSK</td>
<td>( Q((\gamma)^{\frac{1}{2}}) )</td>
<td>((1 - Q((\gamma)^{\frac{1}{2}}))^M)</td>
</tr>
</tbody>
</table>

**Table 1:** Efficiency functions for some simple system implementations.

We make only the following general assumptions on the system’s efficiency function, which hold for typical modulation schemes and error correction codes, including those discussed above.

**Assumption 1:** The efficiency function \( f : [0, \infty) \mapsto [0, 1) \) satisfies the following conditions (see Figure 2-1):

1. \( f \) is a strictly increasing, twice continuously differentiable function of \( \gamma \),
2. \( \lim_{\gamma \to 0} f(\gamma) = 0 \),
3. \( \lim_{\gamma \to 0} f'(\gamma) = 0 \),
4. There exists \( \bar{\gamma} \) such that \( \forall \ \gamma < \bar{\gamma}, f(\gamma) \) is strictly convex, and \( \forall \ \gamma > \bar{\gamma}, f(\gamma) \) is strictly
2.1. Model and Framework

concave,

5. Let $C \in \mathbb{R}^+$, and let $h(x) \equiv f\left(\frac{x}{1+cx}\right)$. There exists $\hat{x}$ such that $\forall \ x < \hat{x}$, $h(x)$ is strictly convex, and $\forall \ x > \hat{x}$, $h(x)$ is strictly concave,

6. All users have the same efficiency function $f(\cdot)$.

Assumptions 1.1 - 1.4 are natural for efficiency functions, and they can be seen by inspection of Figure 1. We note that Assumption 1.5 is a generalization of 1.4. Assumption 1.5 holds for all commonly considered efficiency functions, including those given in Table 1, with or without error correction codes; this can be proven for some classes of efficiency functions and shown by simulation for others where a proof is not tractable due to the complicated closed form expressions of their efficiency functions. Appendix B shows why Assumption 1.5 is necessary, and proves it for efficiency functions with bit error rates that are exponentially decaying functions of the SIR. Finally, Assumption 1.6 states that all users employ the same modulation scheme and error correction code. Since error correction decoding is done at the receiver, and all users are transmitting to the same base station, it is reasonable to assume that the same error correction coding will be used by all users. Furthermore, it is also reasonable to assume that all users

![Figure 2-1: The general form for the efficiency function $f_\gamma$, as a function of the signal quality $\gamma$. The point where $f(\gamma) = \gamma f'(\gamma)$ is considered later in the thesis.](image-url)
transmitting data to the same base station are using the same protocol, and thus will be using
the same modulation scheme.

User $j$'s effective data rate is given by $r_j f(\gamma_j)$. We assume that user $j$ derives a utility of
$u_j(r_j f(\gamma_j))$ based on his effective data rate.

**Assumption 2:** Assume that for each $j$, the utility function $u_j : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the
following conditions:

1. $u_j$ is a strictly increasing and twice continuously differentiable function.
2. $u_j$ is strictly concave.

Assumptions 2.1 and 2.2 are natural in the context of a data network. 2.1 says that users derive
more utility from a higher effective data rate. 2.2, which is appropriate for elastic data traffic,
says that higher effective data rates have diminishing marginal returns for users; see [44].

The service provider derives a profit of $qr_j$ from each user $j$ by collecting a payment $q$ per
each unit of data rate $r_j$. User $j$'s payoff is given by his utility less his total payment:

$$u_j(r_j f(\gamma_j)) - qr_j.$$  \hfill (2.1.3)

The following game will be the basis of our study.

**Definition 3.** The *Rate Selection Game* has $J$ players, consisting of the set of users $J$, who
act according to the following utilities and strategy spaces.

**User stage:** Given a price $q$, each user $j$ chooses $r_j$ to maximize his payoff.

$$\max_{r_j \geq 0} (u_j(r_j f(\gamma_j)) - qr_j).$$  \hfill (2.1.4)

**Power control stage:** Given the rate vector $r$ determined in the User Stage, the base station
assigns the transmit powers $p$ according to the following objective:

$$\max_{0 \leq p \leq p_{\text{max}}} \min_{0 \leq \gamma_j \leq \gamma_{\text{max}}} \gamma_j.$$  \hfill (2.1.5)
2.2. Power Control Solution

The equilibrium notion we will use is the Nash Equilibrium, which we now define (see [40]).

**Definition 4. (Nash Equilibrium)** Let \( r_{-j} \) denote the vector of rates \((r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_J)\). A Nash Equilibrium of the Rate Selection Game under price \( q \) is a vector \( r^* \geq 0 \) such that for all \( j \)

\[
 u_j \left( r_j^* f \left( \gamma \left( r_j^*; r_j^* \right) \right) \right) - qr_j^* \geq u_j \left( \overline{r}_j f \left( \gamma \left( \overline{r}_j; r_j^* \right) \right) \right) - q\overline{r}_j, \quad \text{for all } \overline{r}_j \geq 0.
\]

A Nash Equilibrium is a strategy profile \( r^* \) in which no player can increase his payoff with a unilateral change in strategy.

### 2.2 Power Control Solution

In this section, we characterize the power control objective (2.1.5). We will use this characterization to prove the existence of a Nash Equilibrium of the Rate Selection Game. It has been shown in [13], [35], and [55] that the "max-min" power control objective results in all users transmitting with the same SIR; hence, this objective is also known as SIR balancing. Furthermore, if user \( j \) has an SIR \( \gamma_j \) under a power allocation \( p \), he will have a higher SIR when all components of the power vector are scaled up uniformly; this is the scalability property of standard interference functions, defined in [28]. From this, it follows that some user must be allocated \( p_{\text{max}} \). The following proposition formalizes these results in the context of our model and assumptions, and also provides a complete characterization of the power control objective (2.1.5).

**Proposition 1.** For a rate vector \( r \), let \( p^* \) be an optimal solution of problem (2.1.5), and let \( \gamma_i^* \) be the corresponding SIR for user \( i \). Then, \( \gamma_i^* = \gamma_j^* \equiv \gamma^* \), \( \forall i, j \in \mathcal{J} \). Furthermore,

\[
\gamma^* = \frac{W}{r_k \eta_k + \sum r_j}, \tag{2.2.1}
\]

where \( \eta_k = \frac{a^2}{\lambda_k p_{\text{max}}} \) and \( k = \arg \min \left\{ \frac{h_k}{r_j} \right\} \).

\(^1\)The max-min power control described above, also known as SIR-balancing, is well studied and commonly considered as discussed in Section 1.2. Other power control methods could have been used in the power control stage. For example, a typical scheme is ensuring that each user's SIR \( \gamma_j \) is above some threshold; this is generally used for real-time traffic due to its strict BER requirements. However, in this paper we are focusing on the transmission of elastic data traffic. In this context, the max-min fair power control scheme is the most logical.
Proof: First, we show that \( y^*_i = y^*_j \), \( \forall i, j \in \mathcal{J} \). To find a contradiction, suppose that all users do not have the same SIR. Let \( \mathcal{M} = \arg \min_j \{ \gamma_j \} \). Then there exists \( i \) such that

\[
\frac{W}{r_i} \sum_{j \in \mathcal{J}} h_j p^*_j + \sigma^2 \geq \frac{W}{r_m} \sum_{j \in \mathcal{J}} h_j p^*_m + \sigma^2, \quad \forall m \in \mathcal{M},
\]

which implies that there exists some \( \delta > 0 \) such that

\[
\frac{W}{r_i} \sum_{j \in \mathcal{J}} h_j (p^*_j - \delta) + h_i \delta + \sigma^2 \geq \frac{W}{r_m} \sum_{j \in \mathcal{J}} h_j (p^*_m - \delta) + h_i \delta + \sigma^2, \quad \forall m \in \mathcal{M}. \tag{2.2.2}
\]

Therefore, the power allocation of \( p^* - \delta e_i \), where \( e_i \) is the \( i \)th unit vector, contradicts the optimality of \( p^* \).

We claim that \( p^*_j = p_{max} \) for some \( j \in \mathcal{J} \). To find a contradiction, suppose \( p_j < p_{max} \), for all \( j \in \mathcal{J} \). Then, there exists a scalar \( \alpha > 1 \) such that \( \alpha p^*_j \leq p_{max} \), for all \( j \). The power allocation \( \alpha p^* \) increases \( \gamma_j \), for all \( j \), contradicting the optimality of \( p^* \).

Next, we claim that \( p^*_k = p_{max} \), where \( k = \arg \min_j \{ \frac{h_j}{r_j} \} \). Denote the total interference experienced by each user as

\[
I = \sum_{i \in \mathcal{J}} h_i p_i + \sigma^2.
\]

To find a contradiction, let \( p^*_m = p_{max} \), where \( m \neq k \). Then

\[
\frac{h_m}{r_m} > \frac{h_k}{r_k},
\]

from which we obtain

\[
\frac{W}{r_m} \frac{h_m p_{max}}{I} > \frac{W}{r_k} \frac{h_k p_{max}}{I} \geq \frac{W}{r_k} \frac{h_k p^*_k}{I}.
\]

It follows that \( \gamma_k \neq \gamma_m \), which is a contradiction, so \( p^*_m \neq p_{max} \). Since some user must be allocated \( p_{max} \), we have \( p^*_k = p_{max} \). Since the SIR for all users must be equal, we have the following:

\[
\gamma^*_j = \frac{W}{r_k} \frac{h_k p_{max}}{I} = \frac{W}{r_j} \frac{h_j p^*_j}{I}, \quad \forall j. \tag{2.2.3}
\]

This implies

\[
p^*_j = \frac{h_k}{r_k} \frac{r_j}{h_j} \frac{p_{max}}{I}, \quad \forall j.
\]
and also

\[ h_j p_j^* = \gamma^* \frac{r_j}{W}, \quad \forall j. \]

Using the preceding relation, and the definition of \( I \), we have

\[ I = \sigma^2 + \sum h_j p_j^* = \frac{\sigma^2}{1 - \frac{\gamma^*}{W} \sum_{j \in \mathcal{J}} r_j}. \]

Substituting this into (2.2.3), we have

\[ \gamma^* = \frac{W h_k p_{max}}{r_k \sigma^2} \left( 1 - \frac{\gamma^*}{W} \sum_{j \in \mathcal{J}} r_j \right). \]

Solving for \( \gamma^* \), we have

\[ \gamma^* = \frac{W h_k p_{max}}{r_k \sigma^2 + h_k p_{max} \sum r_j}, \]

which, by the definition of \( \eta_k \), can be expressed as

\[ \gamma^* = \frac{W}{r_k \eta_k + \sum r_j}. \]

For the remainder of the thesis, \( \gamma \) will denote to the common SIR experienced by all users due to the power control objective (2.1.5). In some cases, to emphasize that the SIR under the max-min power control objective (2.1.5) is a function of only \( r \), we will use the notation \( \gamma(r) \).

## 2.3 Equilibrium characterization

In this section, we characterize the equilibrium of the Rate Selection Game. We establish the existence of a Nash equilibrium. We then show that for a large number of users \( J \) and bandwidth \( W \), the Nash Equilibrium can be approximated by another equilibrium notion that is both intuitively appealing and mathematically tractable. Motivated by the Wardrop Equilibrium used in the transportation network literature to model user behavior in a system with a large number of users (see [17] and [52]), we define this equilibrium as the Wardrop Equilibrium. We
show that it may not be unique, but give properties which will be useful to show uniqueness for the extensive game considered in Chapter 4. The definitions of a \textit{quasiconcave function} and a \textit{Nash equilibrium}, terms which are used in this section, are given in Appendix A.

2.3. Existence of a Nash Equilibrium

In order to show existence of a Nash Equilibrium, we will show that the utility function $u_j(r_jf(\gamma(r)))$ is a quasiconcave function of $r_j$ for all $j$. Existence of an equilibrium will then follow by Kakutani’s fixed point theorem (see Appendix A).

\textbf{Proposition 2.} User $j$’s utility function $u_j(r_jf(\gamma(r)))$ is a quasiconcave function of $r_j$.

\textbf{Proof:} The proof consists of four steps.

\textbf{Step 1:} $r_jf(\frac{W}{r_j})$ has a unique global maximum at $r_j^* = \arg\max\{r_jf(\frac{W}{r_j})\}$, is strictly increasing for $r_j < r_j^*$, and strictly decreasing for $r_j > r_j^*$.

\textbf{Proof:} Let $x = \frac{W}{r_j}$. All stationary points of $r_jf(x)$ must satisfy

$$\frac{\partial}{\partial r_j} \left\{ r_jf\left(\frac{W}{r_j}\right) \right\} = f(x) - xf'(x) = 0.$$  

The condition $f(x) = xf'(x)$ corresponds to the tangent of the line passing through the origin with the $f(x)$ curve; see Figure 2-1. This was observed in [45]. The strict concavity in Assumption 1.4 implies that the corresponding tangent point must exist. Assumptions 1.2, 1.3, and the strict concavity of 1.4 imply that the tangent point must be unique. (We note that $x = 0$ is not a possibility. This is because $r_j < \infty$, a fact which will be made clear in the analysis of Proposition 3).

Denote the value at which the tangent occurs as $x^*$. Since $x = \frac{W}{r_j}$ is strictly decreasing in $r_j$, there exists a unique $r_j^*$ satisfying $x^* = \frac{W}{r_j^*}$. We also have

$$\frac{\partial^2}{\partial r_j^2} \{r_jf(x)\} = xf''(x)\frac{x}{r_j^*}.$$
Because $x > 0$ and $r_j > 0$, this implies the following equivalence:

$$\frac{\partial^2}{\partial r_j^2} \{ r_j f(x) \} < 0 \iff f''(x) < 0.$$ 

Therefore, $r_j f(\frac{W}{r_j})$ is concave if and only if $f(x)$ is concave at $x = \frac{W}{r_j}$. Assumptions 1.2 and 1.3 imply that the tangent point must occur in the concave region of $f(x)$. Thus, the stationary point $r_j^*$ is a local maximum, and since it is a unique stationary point, it is the global maximum. This implies that $r_j f(\frac{W}{r_j})$ is increasing for $r_j < r_j^*$ and decreasing for $r_j > r_j^*$ (see Figure 2-2 for the general functional form).

![Figure 2-2: The functional form of $r_j f(\frac{W}{r_j})$ for typical $f(.)$.](image)

**Step 2:** $r_j f(\frac{W}{r_j+C_1})$, where $C_1 \in \mathbb{R}^+$ is an arbitrary constant, has a unique global maximum at $r_j^* = \arg \max \{ r_j f(\frac{W}{r_j+C_1}) \}$, is strictly increasing for $r_j < r_j^*$, and strictly decreasing for $r_j > r_j^*$.

**Proof:** Again, let $x = \frac{W}{r_j}$. Let $C = \frac{C_1}{W}$, and define the function $\tilde{f} : \mathbb{R}^+ \to \mathbb{R}^+$ to be:

$$\tilde{f}(x) = f \left( \frac{x}{1+Cx} \right).$$

Note that for $x = \frac{W}{r_j}$, we have

$$\tilde{f}(x) = f \left( \frac{W}{r_j+C_1} \right).$$  \hfill (2.3.1)
2.3. Equilibrium characterization

The following three properties are also satisfied:

\[ \frac{\partial}{\partial x} \left\{ f \left( \frac{x}{1 + Cx} \right) \right\} = f' \left( \frac{x}{1 + Cx} \right) \left( \frac{1}{1 + Cx} - \frac{Cx}{(1 + Cx)^2} \right) > 0, \quad \forall \ x, \]

\[ f \left( \frac{x}{1 + Cx} \right) = 0, \quad \text{for } x = 0, \]

\[ \lim_{x \to 0} \frac{\partial}{\partial x} \left\{ f \left( \frac{x}{1 + Cx} \right) \right\} = \lim_{x \to 0} f' \left( \frac{x}{1 + Cx} \right) \left( \frac{1}{1 + Cx} - \frac{Cx}{(1 + Cx)^2} \right) = 0. \]

Therefore, \( \bar{f}(x) \) satisfies Assumptions 1.1, 1.2, and 1.3. Furthermore, Assumption 1.5 implies that \( \bar{f}(x) \) satisfies Assumption 1.4. Therefore, all of the assumptions used in Step 1 hold, and due to (2.3.1), the result of Step 1 will apply to \( r_j f \left( \frac{\gamma_j}{r_j + C_1} \right) \).

**Step 3:** \( r_j f \left( \frac{\gamma_j}{C_2 r_j + C_1} \right) \), where \( C_1, C_2 \in \mathbb{R}^+ \) are arbitrary constants, has a unique global maximum at \( r_j^* = \arg \max \{ r_j f \left( \frac{\gamma_j}{C_2 r_j + C_1} \right) \} \), is strictly increasing for \( r_j < r_j^* \), and strictly decreasing for \( r_j > r_j^* \).

**Proof:** By Step 2, these conditions hold for the function \( r_j f \left( \frac{\gamma_j}{r_j + C_1} \right) \). Scaling the domain by \( C_2 \), we get

\[ C_2 r_j f \left( \frac{\gamma_j}{C_2 r_j + C_1} \right). \]

Scaling the range by \( C_2 \), we get

\[ r_j f \left( \frac{\gamma_j}{C_2 r_j + C_1} \right). \]

The results of Step 2 are preserved by these transformations.

**Step 4:** Let \( \gamma_{-j} \) denote the vector of rates \( (r_1, ..., r_{j-1}, r_{j+1}, ..., r_J) \). Using (2.2.1), denote user \( j \)'s effective data rate, as a function of \( r_j \) with \( \gamma_{-j} \) held fixed, as

\[ g(r_j; \gamma_{-j}) = r_j f \left( \frac{\gamma_j}{r_j + \sum_{i \in \mathcal{J}} r_i} \right). \]

Then \( g(r_j; \gamma_{-j}) \) is a continuous function of \( r_j \) which has a unique global maximum at \( r_j^* = \arg \max \{ g(r_j; \gamma_{-j}) \} \), is strictly increasing for \( r_j < r_j^* \), and strictly decreasing for \( r_j > r_j^* \).

**Proof:** Recall

\[ k = \arg \min_i \left\{ \frac{h_i}{r_i} \right\}, \quad \eta_k = \frac{\sigma^2}{h_k p_{\max}}. \]
We have
\[ \gamma(r_j; r_{-j}) = \frac{W}{r_k \eta_k + \sum_{i \in \mathcal{J}} r_i}. \] (2.3.2)

Note that there exists a value \( r'_j \) such that for \( r_j < r'_j, j \neq k \), and for \( r_j > r'_j, j = k \). For \( r_j < r'_j \) and \( r_j > r'_j \), \( g(r_j; r_{-j}) \) is continuous. For \( r_j = r'_j \), we have

\[ j = \arg \min_{i \in \mathcal{J}} \left\{ \frac{h_i}{r_i} \right\}. \]

By the definition of user \( k \),

\[ k = \arg \min_{i \in \mathcal{J}} \left\{ \frac{h_i}{r_i} \right\}. \]

This gives

\[ \frac{h_j}{r_j} = \frac{h_k}{r_k}, \] (2.3.3)

which implies

\[ r_j \eta_j = r_k \eta_k. \] (2.3.4)

(In the case where \( r_j = r'_j \), there are two users satisfying \( \arg \min_{i \in \mathcal{J}} \left\{ \frac{h_i}{r_i} \right\} \); to avoid ambiguity, in (2.3.3) and (2.3.4) we have let \( k \neq j \) denote the user who also satisfies this for \( r_j < r'_j \).)

It follows that

\[ \lim_{r_j \to r'_j^-} \gamma(r_j; r_{-j}) = \lim_{r_j \to r'_j^+} \gamma(r_j; r_{-j}), \]

and therefore

\[ \lim_{r_j \to r'_j^-} g(r_j; r_{-j}) = \lim_{r_j \to r'_j^+} g(r_j; r_{-j}), \]

so \( g(r_j; r_{-j}) \) is continuous everywhere.

At \( r'_j \), the derivative of \( g(r_j, r_{-j}) \) is discontinuous. Because \( \eta_k > 0 \), (2.3.2) implies

\[ \frac{\partial}{\partial r_j} \{ \gamma(r_j, r_{-j}) \} |_{r'_j^-} > \frac{\partial}{\partial r_j} \{ \gamma(r_j, r_{-j}) \} |_{r'_j^+}, \] (2.3.5)

from which it follows that

\[ \frac{\partial}{\partial r_j} \{ g(r_j, r_{-j}) \} |_{r'_j^-} > \frac{\partial}{\partial r_j} \{ g(r_j, r_{-j}) \} |_{r'_j^+}. \] (2.3.6)
2.3. Equilibrium characterization

For $r_j < r_j'$, let

$$C_1 = \sum_{i \neq j} r_i + r_k \eta_k, \quad C_2 = 1,$$

and for $r_j > r_j'$, let

$$C_1 = \sum_{i \neq j} r_i, \quad C_2 = (1 + \eta_j).$$

With these identifications, we see that $g(r_j; r_j)$ is a continuous function with a single discontinuity in its derivative at $r_j'$, and it takes the functional form given in Step 3 for both $r_j < r_j'$ and $r_j > r_j'$. There are 2 cases to consider:

The first case is if $g(r_j; r_j)$ is decreasing at $r_j^-$. By Step 3 there exists some $r_j^*$ such that $g(r_j, r_{-j})$ is increasing for $r_j < r_j^*$, and decreasing for $r_j \in (r_j^*, r_j')$. By Equation (2.3.6), $g(r_j, r_{-j})$ will be decreasing at $r_j'^+$, and by Step 3 it will be decreasing for all $r_j > r_j'$; see Figure 3-3a.

The second case is if $g(r_j; r_j)$ is increasing at $r_j^-$. By Step 2 it is increasing for $r_j < r_j'$. There are two subcases.

By Equation (2.3.6), $g(r_j, r_{-j})$ may be decreasing at $r_j^-$. In this subcase, we let $r_j^* = r_j'$, and by Step 3, $g(r_j, r_{-j})$ will be decreasing for all $r_j > r_j'$; see Figure 3-3b.

The second subcase is if $g(r_j, r_{-j})$ is increasing at $r_j'^+$. By Step 3 there exists some $r_j^*$ such that $g(r_j, r_{-j})$ will be increasing for $r_j < r_j^*$ and decreasing for $r_j > r_j^*$; see Figure 3-3c.

In either case, $g(r_j, r_{-j})$ has a unique global maximum $r_j^*$, is strictly increasing for $r_j < r_j^*$, and strictly decreasing for $r_j > r_j^*$. □

Step 5: $u_j(r_j f(\gamma(r)))$ is a quasiconcave function of $r_j$.

Proof: Because $u_j$ is a strictly increasing function, $u_j(r_j f(\gamma(r)))$ is strictly increasing for $r_j < r_j^*$, and strictly decreasing for $r_j > r_j^*$. Therefore, it is quasiconcave. □

Proposition 3. There exists a Nash Equilibrium of the Rate Selection Game.

Proof: The utility functions $u_j$ are continuous in $r$. By Proposition 2, they are quasiconcave in $r_j$. We next show that the action spaces of the users are restricted to a nonempty, convex, compact set, and then the existence of a Nash Equilibrium will follow from Kakutani's fixed
2.3. Equilibrium characterization

Point theorem. Consider some user \( j \). By Proposition 2, given any \( r_{-j} \), there exists a unique, finite optimal value for the objective

\[
\max_{r_j \geq 0} \{ u_j (r_j f(\gamma(r_j; r_{-j}))) \}.
\]

Take the maximum of this over all \( r_{-j} \), and denote the maximizing value by \( u_{j,\text{max}} \):

\[
 u_{j,\text{max}} = \max_{r_{-j}} \max_{r_j \geq 0} \{ u_j (r_j f(\gamma(r_j; r_{-j}))) \}.
\]

It follows that

\[
 u_j(r_j f(\gamma)) \leq u_{j,\text{max}}, \quad \forall \ r.
\]

This implies that for any price \( q \) and any \( r_{-j} \), there exists some rate \( r_{j,\text{max}} \) for which

\[
 u_j(r_j f(\gamma)) - qr_j < 0, \quad \forall \ r_j > r_{j,\text{max}}.
\]

Since user \( j \) can attain a payoff of 0 by transmitting at rate \( r_j = 0 \), it follows that user \( j \)'s rate \( r_j \) at a Nash Equilibrium must lie in the compact, convex interval \([0, r_{j,\text{max}}]\). A Nash Equilibrium exists by Kakutani’s fixed point theorem. \( \square \)
2.3.2 Large System Convergence of the Nash Equilibrium

In this section, we introduce the Wardrop Equilibrium by considering the Nash Equilibrium of Section 2.3.1 in the large system limit.

As a motivation for pursuing the Wardrop Equilibrium, consider the following necessary first order optimality conditions for a Nash Equilibrium, given for each user $i \neq k$,

\[
\begin{align*}
 u_i'(r_i f(\gamma)) \left[ f(\gamma) - r_i f'(\gamma) \frac{W}{(r_k \eta_k + \sum_j r_j)^2} \right] &= q, & r_i > 0, \\
&\leq q, & r_i = 0,
\end{align*}
\]

(2.3.7)

and for user $k$,

\[
\begin{align*}
 u_k'(r_k f(\gamma)) \left[ f(\gamma) - r_k f'(\gamma) \frac{W(1 + \eta_k)}{(r_k \eta_k + \sum_j r_j)^2} \right] &= q, & r_k > 0, \\
&\leq q, & r_k = 0.
\end{align*}
\]

(2.3.9)

The Nash Equilibrium is an appealing equilibrium concept in situations where a user's choice of actions impacts the system's parameters and thereby the behavior of other users, such as the wireless system we are considering here. As shown by (2.3.7) and (2.3.9), the Nash Equilibrium approach explicitly shows how user $j$'s decision on whether to change his strategy depends on the actions of the other users, as well as the impact that this change will have on the system's SIR. However, the Nash Equilibrium approach gives rise to several difficulties. First, as can be seen by (2.3.7) and (2.3.9), the Nash Equilibrium has a cumbersome mathematical characterization which leads to computational difficulties. Secondly, (2.3.7) and (2.3.9) depend on quantities such as the data rates of all users and the derivative of the efficiency function. Therefore, users have extensive information requirements in order to determine whether their first order optimality condition is satisfied.

However, if we restrict our attention to a many user, large bandwidth system, it seems reasonable to consider a more tractable equilibrium concept in which one user's unilateral
change in strategy has a negligible impact on the overall system. This motivates us to pursue an equilibrium notion along the lines of the Wardrop Equilibrium, first introduced in congestion analysis for transportation networks, in which the routing decisions of a single user are assumed to have a negligible impact on link congestion in a network; see [52] and [17]. Such an approach is intuitively justified in our case by observing the additive structure of the denominator in (2.2.1); if $J$ is large, then a change in $r_j$ for any user $j$ will have a negligible impact on $\gamma$.

Therefore, one expects users to view $\gamma$ as a constant when considering unilateral changes in their strategy. With such an equilibrium notion, one expects lower information requirements for each user to determine an optimal strategy. In fact, Chapter 3 presents a dynamic system based on the Wardrop Equilibrium in which users determine their strategy based only on the observed SIR $\gamma$, and local stability results are given for the system.

In order to formalize the Wardrop Equilibrium in our context, we use large system analysis, which is a common technique in characterizing CDMA systems. In such analysis, the number of users $J$ and the system bandwidth $W$ are both taken to infinity, but their ratio is held to a constant $\beta$; see, e.g., [16] and references therein.

In order to characterize the large system limit while maintaining mathematical structure, we use a replication argument, first employed by Debreu and Scarf [9] in the context of competitive equilibria in exchange economies and also used in [17]. In such an argument, one considers an “increasing sequence of systems” in which the $n^{th}$ system is constructed from the $(n - 1)^{th}$ system by scaling it in a symmetric fashion. In particular, the system’s bandwidth is scaled from $(n - 1)W$ to $nW$, and $J$ users are added to the system, where the $j^{th}$ new user has a utility function identical to the $j^{th}$ user of the original system. Replication arguments have previously been used in the analysis of other engineering systems, such as traffic systems [52] and wireline data networks [1]. Thus, we are using a replication argument commonly used in the economic modelling of large scale systems and economies as a tool in performing the large system analysis commonly used in the characterization of CDMA systems.

Formally, we consider a sequence of games $G(n)$. Define the game $G(n)$ to be the Rate Selection Game with $J$ classes of users, $\mathcal{N}_1, \ldots, \mathcal{N}_J$, with $n$ users in each class. Assume that all members of class $\mathcal{N}_j$ have the same utility function $u_j$. We denote the set of users in game $G(n)$ by $\mathcal{J}(n)$, and note that there are $nJ$ users in $\mathcal{J}(n)$. To maintain the constant of proportionality
2.3. Equilibrium characterization

\( \beta = \frac{W(n)}{nJ} \), we let \( W(n) = nW \). Finally, a user \( m \in J(n) \) transmits with rate \( r_m(n) \).

The SIR in game \( G(n) \), given by (2.2.1), is then

\[
\gamma(n) = \frac{nW}{r_k(n)\eta_k + \sum_j \sum_{m \in N_j} r_m(n)}.
\] (2.3.11)

The payoff in game \( G(n) \) to user \( i \) is

\[
u_i(r_i(n)f(\gamma(n))) - qr_i(n).\] (2.3.12)

**Proposition 4.** Let \( r(n) \) and \( \gamma(n) \) be a Nash Equilibrium rate vector and SIR for the game \( G(n) \) under price \( q \), and assume that \( r_m(n) > 0 \) for some \( m \). Then, we have

\[
\lim_{n \to \infty} r_j(n) = r_j, \] (2.3.13)

\[
\lim_{n \to \infty} \gamma(n) = \gamma, \] (2.3.14)

where

\[
\gamma = \frac{W}{\sum_{j \in J} r_j}
\]

and \( r \in \mathbb{R}^J \) and \( \gamma \in \mathbb{R} \) satisfy

\[
u'_j(r_jf(\gamma))f(\gamma) = q, \quad r_j > 0, \] (2.3.15)

\[
\leq q, \quad r_j = 0.
\]

**Proof:** First, we argue that the sequences \( \{r_i(n)\} \) and \( \{\gamma(n)\} \) have limits. By the analysis of Proposition 3, the Nash Equilibrium rate \( r_i(n) \) must lie in the compact interval \([0, r_{i,\text{max}}]\). By the Bolzano-Weierstrass theorem (see Section 1.7), \( \{r_j(n)\} \) has a convergent subsequence. Therefore, without loss of generality, we can assume \( r_i(n) \to r_i \) for all \( i \in J(n) \). Since \( r_m(n) > 0 \) for some \( m \) by assumption, it follows from (2.3.11) that there exists \( \gamma_{\text{max}} \) such that \( \gamma(n) \leq \gamma_{\text{max}} \). Furthermore, there exists \( \gamma_{\text{min}} \) such that \( \gamma(n) \geq \gamma_{\text{min}} \), where \( \gamma_{\text{min}} \) is (2.3.11) evaluated at \( r_i = r_{i,\text{max}} \) for all \( i \). Therefore, \( \gamma(n) \) lies in a compact interval for all \( n \), \( \{\gamma(n)\} \) has a convergent
2.3. Equilibrium characterization

subsequence by the Bolzano-Weierstrass theorem, and without loss of generality we can assume \( \gamma(n) \to \gamma \).

In order to prove (2.3.15), we will consider the following first order necessary user optimality condition for a Nash Equilibrium, which must be satisfied for all \( i \in J(n) \).

\[
u_i'(r_i(n)f(\gamma(n))) \left[ f(\gamma(n)) + r_i(n) \left( \frac{\partial}{\partial r_i(n)} \{ f(\gamma(n)) \} \right) \right] = q, \quad r_i(n) > 0, \\
\leq q, \quad r_i(n) = 0.
\]

In order to simplify this expression, we first show that

\[
r_i(n) = r_j(n) \quad \forall \ i, j \in N_m, \quad i \neq k, j \neq k, \quad \forall \ m,
\]

(i.e., with the possible exception of user \( k \), all users in the same class transmit with the same rate at the Nash equilibrium). To find a contradiction, suppose \( r_i(n) < r_j(n) \). We have

\[
u_i'(r_i(n)f(\gamma(n))) \left[ f(\gamma(n)) + r_i(n) \left( \frac{\partial}{\partial r_i(n)} \{ f(\gamma(n)) \} \right) \right] \\
\leq \\
u_j'(r_j(n)f(\gamma(n))) \left[ f(\gamma(n)) + r_j(n) \left( \frac{\partial}{\partial r_j(n)} \{ f(\gamma(n)) \} \right) \right].
\]

By Assumption 2.2, and since \( i \) and \( j \) have the same utility function, we have \( u_i'(r_i(n)f(\gamma(n))) \geq u_j'(r_j(n)f(\gamma(n))) \). This implies

\[
f(\gamma(n)) + r_i(n) \left( \frac{\partial}{\partial r_i(n)} \{ f(\gamma(n)) \} \right) \leq f(\gamma(n)) + r_j(n) \left( \frac{\partial}{\partial r_j(n)} \{ f(\gamma(n)) \} \right),
\]

and therefore

\[
r_i(n) \left( \frac{\partial}{\partial r_i(n)} \{ f(\gamma(n)) \} \right) \leq r_j(n) \left( \frac{\partial}{\partial r_j(n)} \{ f(\gamma(n)) \} \right).
\]

By inspection of (2.3.11), it is clear that \( \frac{\partial}{\partial r_i(n)} \{ f(\gamma(n)) \} = \frac{\partial}{\partial r_j(n)} \{ f(\gamma(n)) \} \) as long as \( i \neq k, j \neq k \). Since \( \frac{\partial}{\partial r_i(n)} \{ f(\gamma(n)) \} < 0 \), we have \( r_i \geq r_j \), a contradiction. A symmetric argument holds for \( r_i > r_j \).

Using (2.3.16), we can simplify (2.3.11). Denote \( r_i(n) \) for any \( i \in N_j \) such that \( i \neq k \) by
2.3. Equilibrium characterization

\( r_{N_j}(n) \), for all \( j \). Without loss of generality, let \( k \in N_k \). We have

\[
\gamma(n) = \frac{nW}{r_k(n)\eta_k + n \sum_{j \neq k} r_{N_j}(n) + (n-1)r_{N_k}(n) + r_k(n)}. \tag{2.3.17}
\]

Using (2.3.17), we can write the first order optimality condition for all users. If \( i \neq k \), by taking the derivative of (2.3.12), we have

\[
u_i'(r_i(n)f(\gamma(n))) \times 
\left[ f(\gamma(n)) - r_i(n)f'(\gamma(n)) \frac{nW}{(r_k(n)\eta_k + n \sum_{j \neq k} r_{N_j}(n) + (n-1)r_{N_k}(n) + r_k(n))^2} \right] = q, \quad r_j(n) > 0, \\
\leq q, \quad r_j(n) = 0.
\]

If \( i = k \), the first order optimality condition is

\[
u_i'(r_i(n)f(\gamma(n))) \times 
\left[ f(\gamma(n)) - r_i(n)f'(\gamma(n)) \frac{nW(1+\eta_k)}{(r_k(n)\eta_k + n \sum_{j \neq k} r_{N_j}(n) + (n-1)r_{N_k}(n) + r_k(n))^2} \right] = q, \quad r_j(n) > 0, \\
\leq q, \quad r_j(n) = 0.
\]

Consider the sequences given in the LHS of (2.3.18) and (2.3.19) for some user \( i \), which we denote \( \{x(n)\} \) and \( \{y(n)\} \) respectively. We have that \( \{x(n)\} \) and \( \{y(n)\} \) both approach the same limit as \( n \to \infty \):

\[
x(n), y(n) \to u_i'(r_i f(\gamma)) f(\gamma). \tag{2.3.20}
\]

We will now complete the proof of (2.3.15). Since \( r_i(n) \geq 0 \) for all \( n \), it follows from (2.3.18) and (2.3.19) that \( x(n) \leq q \) and \( y(n) \leq q \) for all \( n \). By (2.3.20), it then follows that

\[
u_i'(r_i f(\gamma)) f(\gamma) \leq q.
\]

If \( r_i > 0 \), then there exists \( \hat{n} \) such that \( r_i(n) > 0 \) for all \( n \geq \hat{n} \). From (2.3.18) and (2.3.19),
it follows that \( x(n) = q \) and \( y(n) = q \) for all \( n \geq \hat{n} \). (2.3.20) then implies

\[ u_i' (r_i f(\gamma)) f(\gamma) = q. \]

This proves (2.3.15).

Finally, we show (2.3.14). For this purpose, we first argue that \( r_k = r_{N_k} \), where \( r_k(n) \to r_k \) and \( r_{N_k}(n) \to r_{N_k} \); i.e., user \( k \), not considered in (2.3.16), also transmits with the same rate as the other users in his class at the Nash Equilibrium. For a contradiction, assume there exists \( i \in N_k \) such that \( r_i < r_k \). Then, using (2.3.18), (2.3.19), and (2.3.20), we have

\[ u_i'(r_i f(\gamma)) f(\gamma) \leq u_k'(r_k f(\gamma)) f(\gamma). \]

By Assumption 2, and since \( i \) and \( k \) have the same utility function, it follows that \( r_i \geq r_k \), a contradiction. A symmetric argument holds for \( r_i > r_k \), so \( r_k = r_{N_k} \).

Using this and (2.3.17), we have

\[ \gamma(n) \to \frac{W}{\sum_{j=1}^{J} r_{N_j}} = \frac{W}{\sum_{j \in J} r_j}. \]

This proves (2.3.14). \( \square \)

In the large system limit, users view \( \gamma \) as constant when optimizing. This follows from Equation (2.3.15), which is equivalent to the optimality condition of Equation (2.1.4) if \( \gamma \) is a constant. The interpretation is that users are "SIR-takers;" they do not anticipate the effect that their change in strategy will have on the SIR. This is analogous to [1] and [52], in which

\[ \sum_{j \in J} r_j >> r_k \eta_k \]

and (2.3.14) follows. This is typical of CDMA systems in general; background noise is often neglected in analysis due to CDMA’s "interference-limited" nature [50].
2.3. Equilibrium characterization

users are “congestion-takers.”

2.3.3 Defining the Wardrop Equilibrium

We now formally define the Wardrop Equilibrium, and prove its existence.

**Definition 5.** For a given price \( q \), a Wardrop Equilibrium of the User Stage is a rate vector \( r \) such that

\[
r_j \in \arg \max_{r_j \geq 0} \{ u_j(r_j f(\gamma)) - q r_j \}, \quad \forall j,
\]

where \( \gamma \) is assumed to be constant in the optimization, and satisfies

\[
\gamma = \frac{W}{\sum_{j \in J} r_j}.
\]

The results of the previous section show that the Wardrop Equilibrium is a good approximation of the Nash Equilibrium in a system with many users \( J \) and a large bandwidth \( W \).

**Proposition 5.** There exists a Wardrop Equilibrium of the Rate Selection Game.

**Proof:** Define the function \( B_j(\gamma) = \arg \max_{r_j \geq 0} \{ u_j(r_j f(\gamma)) - q r_j \} \); i.e., \( B_j(\gamma) \) is player \( j \)'s “best response” rate, given by (2.3.21), when the SIR is \( \gamma \). Note that by the strict concavity of \( u_j(\cdot) \), \( B_j(\gamma) \) is a continuous function that takes on the value of single real number for each \( \gamma \).

Showing the existence of a Wardrop Equilibrium is equivalent to showing the existence of an SIR \( \gamma \) which satisfies the following equation:

\[
\frac{W}{\sum_{j \in J} B_j(\gamma)} = \gamma.
\]

Since \( f(\cdot) \) is one-to-one, this is equivalent to showing the existence of an SIR \( \gamma \) which satisfies:

\[
f \left( \frac{W}{\sum_{j \in J} B_j(\gamma)} \right) = f(\gamma).
\]

Since \( f(\cdot) \) only takes on values in the compact, convex interval \([0, 1]\), and \( f(\cdot) \) is a continuous convex valued function, it follows from Kakutani’s fixed point theorem that there must exist an
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We now give the first order optimality conditions of (2.3.21).

\[ u'_j(r_j f(\gamma)) f(\gamma) \leq q, \quad r_j > 0, \]  
\[ \leq q, \quad r_j = 0, \]  
\[ (2.3.22) \]
\[ (2.3.23) \]

where

\[ \gamma = \frac{W}{\sum_{j \in J} r_j}. \]  
\[ (2.3.24) \]

By the strict concavity of \( u_j(\cdot) \), the preceding conditions are necessary and sufficient. It will often be more convenient to use the characterization (2.3.22 - 2.3.23) in future analysis.

2.3.4 Non-uniqueness of the Wardrop Equilibrium

In this section, we show by example that multiple Wardrop Equilibria may exist for a given price \( q \).

**Example 1:** Consider an instance of the Rate Selection Game with the following parameters:

\[ u_j(x) = 500 - 500e^{-\frac{x}{100}}, \quad \forall j, \]

\[ W = 9000, \]

\[ \frac{W}{J} = 9000, \]

\[ q = 0.1488. \]

Also, let the efficiency function be any function satisfying Assumption 1 and

\[ f(8) = 0.667, \quad f(9) = 0.9. \]  
\[ (2.3.25) \]

Since we have only placed restrictions on two points of the curve \( f(\cdot) \), one can imagine that such an efficiency function will exist. Constructing a closed form expression for such an efficiency function, however, is difficult, and so instead we present a piecewise approximation of such an
efficiency function.

Consider the following function, shown in Figure 2-4:

\[
f(\gamma) = \begin{cases} 
0, & 0 \leq \gamma \leq 6.07 \\
-\frac{16.776}{\gamma} + 2.764, & 6.07 \leq \gamma \leq 9 \\
-\frac{0.9}{\gamma} + 1, & 9 < \gamma \leq \infty.
\end{cases}
\]  

Figure 2-4: The function \( f(\gamma) \) considered in Example 1.

Although \( f \) violates Assumptions 1.1, 1.4, and 1.5, one can construct a continuously differentiable approximation of \( f \) such that (2.3.25) is satisfied, and this approximation will satisfy Assumptions 1.1 and 1.4. By simulation, namely by plotting \( f(x/(1+Cx)) \) for a range of values \( C \), we can convince ourselves that the approximation will also satisfy Assumption 1.5. Based on the analysis of Proposition 4, it is clear that users with the same utility function will transmit with the same rate at the Wardrop Equilibrium. Therefore, we let \( r \in \mathbb{R} \) denote the common transmission rate of all users, and (2.3.24) simplifies to

\[ \gamma = \frac{W}{Jr}. \]
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Let $r_1 = 1000$. We have the following:

\[
\frac{d}{dr_1} \left( r_1 f \left( \frac{W}{Jr_1} \right) \right) f \left( \frac{W}{Jr_1} \right) = e^{-\frac{r_1}{250f}} \frac{W}{Jr_1} f \left( \frac{W}{Jr_1} \right) = e^{-2f(9)} f(9) = q. \tag{2.3.27}
\]

Let $r_2 = 1125$. We have the following:

\[
\frac{d}{dr_2} \left( r_2 f \left( \frac{W}{Jr_2} \right) \right) f \left( \frac{W}{Jr_2} \right) = e^{-\frac{r_2}{250f}} \frac{W}{Jr_2} f \left( \frac{W}{Jr_2} \right) = e^{-2.25f(8)} f(8) = q. \tag{2.3.28}
\]

(2.3.27) and (2.3.28) imply that $r_1$ and $r_2$ are both Wardrop Equilibria.

2.3.5 Properties of Wardrop Equilibria

For a given price $q$, the Wardrop Equilibrium may not be unique. In this section we will study several properties of the Wardrop Equilibrium. In Chapter 4, we will use these properties to show that under a profit maximizing price, the Wardrop Equilibrium is unique.

**Proposition 6.** Let $\bar{r}$ and $\hat{r}$ be two Wardrop Equilibria for a given price $q$ with corresponding channel qualities $\bar{\gamma}$ and $\hat{\gamma}$. Then $\bar{\gamma} \neq \hat{\gamma}$.

**Proof:** Consider two Wardrop Equilibria $\bar{r}$ and $\hat{r}$, with corresponding SIR's $\bar{\gamma}$ and $\hat{\gamma}$. To arrive at a contradiction, assume $\bar{\gamma} = \hat{\gamma} \equiv \gamma$. Consider a user for whom $\bar{r}_j < \hat{r}_j$. From (2.3.22), (2.3.23), and (2.3.23), we have:

\[
u_j' (\bar{r}_j f(\gamma)) f(\gamma) \leq \nu_j' (\hat{r}_j f(\gamma)) f(\gamma),
\]

which implies

\[
u_j' (\bar{r}_j f(\gamma)) \leq \nu_j' (\hat{r}_j f(\gamma)).
\]

This, along with Assumption 2.2, gives

\[
\bar{r}_j f(\gamma) \geq \hat{r}_j f(\gamma),
\]

\[

\]
which implies
\[ r_j \geq r_j. \]

which yields a contradiction. A symmetric argument can be used to obtain a contradiction for users with \( r_j > r_j \). Therefore, \( r_j = r_j \) for all \( j \), implying that the Wardrop Equilibria are not distinct.

**Proposition 7.** Let \( \hat{\tau} \) and \( \hat{\tau} \) be two Wardrop Equilibria with corresponding SIR's \( \hat{\gamma} \) and \( \hat{\gamma} \). If there exists a user \( i \in J \) such that \( \hat{\tau}_i > 0 \) and \( \hat{\tau}_i > 0 \), then the following are equivalent.

1. \( \hat{\gamma} > \hat{\gamma} \).
2. \( \hat{r}_j f(\hat{\gamma}) \geq \hat{r}_j f(\hat{\gamma}) \), for all \( j \in J \).

**Proof:** To show that 1 implies 2, assume \( \hat{\gamma} > \hat{\gamma} \). By Assumption 1.1, \( f(\hat{\gamma}) > f(\hat{\gamma}) \). If \( \hat{r}_j \geq \hat{r}_j \), then \( \hat{r}_j f(\hat{\gamma}_j) \geq \hat{r}_j f(\hat{\gamma}) \). If \( \hat{r}_j < \hat{r}_j \), then

\[ u_j'(\hat{r}_j f(\hat{\gamma}_j)) f(\hat{\gamma}) \leq u_j'(\hat{r}_j f(\hat{\gamma})) f(\hat{\gamma}). \]

Since \( f(\hat{\gamma}) > f(\hat{\gamma}) \), this implies

\[ u_j'(\hat{r}_j f(\hat{\gamma}_j)) \leq u_j'(\hat{r}_j f(\hat{\gamma})). \]

which, by Assumption 2.2, implies

\[ \hat{r}_j f(\hat{\gamma}_j) \geq \hat{r}_j f(\hat{\gamma}). \]

To show 2 implies 1, assume \( \hat{r}_i f(\hat{\gamma}) \geq \hat{r}_i f(\hat{\gamma}) \). Since \( \hat{\tau} \) and \( \hat{\tau} \) are both Wardrop Equilibria,

\[ u_i' (\hat{r}_i f(\hat{\gamma})) f(\hat{\gamma}) = u_i' (\hat{r}_i f(\hat{\gamma})) f(\hat{\gamma}). \]

Since \( u(\cdot) \) is a strictly increasing function, this implies

\[ f(\hat{\gamma}) > f(\hat{\gamma}). \]
and since $f(\cdot)$ is a strictly increasing function, we have

$$\hat{\gamma} > \tilde{\gamma},$$

completing the proof.

\[ \square \]

\section*{2.4 Efficiency through pricing}

In this section, we solve for the optimal rate allocation with respect to two efficiency objectives: maximizing the sum of the effective data rates, or throughput, of all users, and Pareto efficiency. We show that the set of throughput maximizing rates is equivalent to the set of Pareto efficient rates by showing that both efficiency objectives have the same necessary and sufficient condition. We denote rate allocations which satisfy this condition as \textit{efficient}. We then show that an appropriately chosen price can result in a Wardrop Equilibrium with an efficient rate allocation.

\textbf{Definition 6.} We say that a rate vector $r^*$ is a \textit{throughput maximizing} rate allocation if it maximizes the total effective data rate of the system.

\begin{equation}
    r^* \in \arg \max_r \left\{ \sum_j r_j f \left( \frac{W}{R} \right) \right\}.
\end{equation}

\textbf{Definition 7.} We say that a rate vector $r^*$ is a \textit{Pareto efficient} rate allocation if there exists no other rate vector $r$ such that $u_j(r_j f(\gamma)) \geq u_j(r^*_j f(\gamma^*))$, $\forall$ $j \in J$ and $u_j(r_j f(\gamma)) > u_j(r^*_j f(\gamma^*))$ for some $j \in J$.

\textbf{Proposition 8.} Consider a rate vector $r$ with corresponding SIR $\gamma$. The following are equivalent.

1. $r$ is a throughput maximizing rate allocation.

2. $r$ is a Pareto efficient rate allocation.

3. $f(\gamma) = \gamma f'(\gamma)$
Proof: To show that 1 and 3 are equivalent, note that the objective in (2.4.1) is equivalent to

$$R^* \in \arg \max_{R \geq 0} \left\{ Rf \left( \frac{W}{R} \right) \right\},$$

where $R = \sum_{j \in \mathcal{J}} r_j$. From the analysis of Proposition 2, Step 1, we have that $Rf \left( \frac{W}{R} \right)$ has a unique global maximum $R^*$. Also from the analysis of Proposition 2, Step 1, $Rf \left( \frac{W}{R} \right)$ has one stationary point. This implies that the first order optimality condition

$$f(\gamma^*) = \gamma^* f'(\gamma^*),$$

where $\gamma^* = \frac{W}{R^*}$, is both a necessary and sufficient optimality condition.

![Functional form of the total effective data rate of the system as a function of $R$.](image)

We now show that 2 and 3 are equivalent. To show that 3 implies 2, let $r$ be a rate vector with corresponding SIR $\gamma$, and assume $f(\gamma) = \gamma f'(\gamma)$. To find a contradiction, assume that $r$ is not Pareto efficient. Then, there exists a rate vector $\tilde{r}$ such that $u_j(\tilde{r}_j f(\gamma)) \geq u_j(r_j f(\gamma))$, $\forall j \in \mathcal{J}$ and $u_j(\tilde{r}_j f(\gamma)) > u_j(r_j f(\gamma))$ for some $j \in \mathcal{J}$. Because $u_j$ is a strictly increasing function, it must be that

$$\sum_{j \in \mathcal{J}} \tilde{r}_j f(\gamma) > \sum_{j \in \mathcal{J}} r_j f(\gamma).$$

This implies that $\tilde{r}$ gives a higher total effective data rate. Since $f(\gamma) = \gamma f'(\gamma)$, this contradicts the equivalence of 1 and 3, which we have already proven.
To show that 2 implies 3, let \( r \) be a Pareto efficient rate vector, with sum rate \( \sum_{j \in \mathcal{J}} r_j = R \).

To find a contradiction, assume that \( f(\gamma) \neq \gamma f'(\gamma) \). Let \( \alpha \in \mathbb{R}^+ \) be some scalar, and consider the rate vector \( \alpha r \). The effective data rate of user \( j \) under \( \alpha r \) is

\[
\alpha r_j f \left( \frac{W}{\alpha R} \right).
\]

Taking the derivative of (2.4.2) with respect to \( \alpha \) gives

\[
\frac{\partial}{\partial \alpha} \left\{ \alpha r_j f \left( \frac{W}{\alpha R} \right) \right\} = r_j \left( f \left( \frac{W}{\alpha R} \right) - \frac{W}{\alpha R} f' \left( \frac{W}{\alpha R} \right) \right) \neq 0 \quad \text{for} \quad \alpha = 1, \quad \forall j \in \mathcal{J}.
\]

Therefore, there exists some \( \alpha \neq 1 \) such that \( \alpha r_j f \left( \frac{W}{\alpha R} \right) > r_j f \left( \frac{W}{R} \right) \), for all \( j \). Since \( u_j \) is a strictly increasing function and \( r \) was assumed to be Pareto efficient, this is a contradiction.

Maximizing the total effective data rate of the system is a primary efficiency objective from an engineering perspective, in that there is no outcome which can result in a higher utilization of the system. Pareto efficiency is a primary efficiency objective from a social perspective, in that one cannot find a redistribution of resources which benefits some user without hurting some other user. This section has shown that the set of rates which maximize the total effective data rate and the set of rates which are Pareto efficient are equivalent. This motivates the following definition.

**Definition 8.** Consider a rate vector \( r \) with corresponding SIR \( \gamma \). If \( f(\gamma) = \gamma f'(\gamma) \), then we say that \( r \) is **efficient**.

The condition \( f(\gamma) = \gamma f'(\gamma) \) has been considered in several other works. In particular (and not surprisingly), it was the optimality condition of [37] (see also [38]), which considered the objective of choosing the optimal rate vector \( r \) to maximize the total effective data rate of the system for a fixed power allocation \( p \). It was also a necessary user optimality condition in [45], in which users derived utility from their effective data rate and were charged a price \( \gamma_j r_j \). Finally, it was a Nash Equilibrium condition in [34] and [43], in which users derive utility from the energy efficiency of their mobile device.

The following Proposition shows that for an appropriately chosen price, the resulting Wardrop Equilibrium can be an efficient rate allocation.
Proposition 9. There exists a unique vector \((q^*, r^*)\) such that \(r^*\) is a Wardrop Equilibrium under the price \(q^*\) and \(r^*\) is an efficient rate allocation.

Proof: By the definition of the Wardrop Equilibrium (Definition 2.3.3) and Proposition 8, it is enough to show that there exists a unique solution \((q, r)\) to the following system of equations:

\[
\begin{align*}
\sum_{j \in J} r_j &= R^*, \quad (2.4.3) \\
\sum_{j \in J} u_j^'(r_j f\left(\frac{W}{R^*}\right)) f\left(\frac{W}{R^*}\right) &= q, \quad r_j > 0, \\
&\leq q, \quad r_j = 0,
\end{align*}
\]

where \(R^*\) is the unique solution to \(f\left(\frac{W}{R}\right) = \frac{W}{R} f\left(\frac{W}{R}\right)\).

First, we will show the existence of a solution. By using the change of variables \(g_j = r_j f\left(\frac{W}{R^*}\right)\) and \(\bar{q} = \frac{q}{f\left(\frac{W}{R^*}\right)}\), this is equivalent to finding a solution \((\bar{q}, g)\) to the following system of equations:

\[
\begin{align*}
\bar{u}_j'(g_j) &= \bar{q}, \quad r_j > 0, \\
&\leq \bar{q}, \quad r_j = 0,
\end{align*}
\]

\[
\sum_{j \in J} g_j = R^* f\left(\frac{W}{R^*}\right) \equiv G^*. \quad (2.4.6)
\]

Without loss of generality, rank the users according to decreasing \(u_j'(0)\), so we have

\[
u_1'(0) \geq u_2'(0) \geq ... \geq u_j'(0).
\]

Construct a solution in the following manner:

Set \(g_1(1) = G^*, \bar{q}(1) = u_1'(g_1(1))\) and \(g_i(1) = 0\) for \(i > 1\). If \(u_2'(0) \leq u_1'(g_1(1))\), then \((\bar{q}(1), g(1))\) is a solution to (2.4.5) and (2.4.6) and we are done. If \(u_2'(0) > u_1'(g_1(1))\), compute \((g_1(2), g_2(2))\) such that \(g_1(2) + g_2(2) = G^*\), and \(u_1'(g_1(2)) = u_2'(g_2(2))\). To see that such a pair \((g_1(2), g_2(2))\) is guaranteed to exist, note that

\[
u_1'(0) > u_2'(G^*),
\]
2.4. Efficiency through pricing

\[ u'_2(0) > u'_1(G^*) , \]

and \( g_2(2) = G^* - g_1(2) \).

Set \( \bar{q}(2) = u'_1(g_1(2)) = u'_2(g_2(2)) \), and \( g_i(2) = 0 \) for \( i > 2 \).

If \( u'_3(0) \leq u'_2(g_2(2)) \), then \((\bar{q}(2), g(2))\) is a solution to (2.4.5) and (2.4.6), and we are done.

If \( u'_3(0) > u'_2(g_2(2)) \), compute \((g_1(3), g_2(3), g_3(3))\) such that \( g_1(3) + g_2(3) + g_3(3) = G^* \), and \( u'_1(g_1(3)) = u'_2(g_2(3)) = u'_3(g_3(3)) \). To see that such a \((g_1(3), g_2(3), g_3(3))\) is guaranteed to exist, note that

\[ u'_2(0) > u'_2(G^*) \]

\[ u'_3(0) > u'_2(g_2(2)) \]

and \( g_3(3) = G^* - g_1(3) - g_2(3) \).

If we iterate in this fashion and no iteration \( j < J \) terminates with a solution to (2.4.5) and (2.4.6), then at the end of the \( J^{th} \) iteration we will have \( \sum_j g_j(J) = G^* \) and \( \bar{q}(J) = u'_1(g_1(J)) = \ldots = u'_J(g_J(J)) \). In this case, \((\bar{q}(J), g(J))\) is a solution to (2.4.5) and (2.4.6). Therefore, we are guaranteed to find a solution.

To show uniqueness, suppose, to arrive at a contradiction, that there is another price \( \hat{q} \) with

an efficient Wardrop Equilibrium \( \hat{r} \). If \( \hat{q} > q^* \), then from (2.4.3) it follows that \( \hat{r}_j < r^*_j \) for all \( j \) with \( r^*_j > 0 \). This implies \( \sum_j \hat{r}_j < \sum_j r^*_j = R^* \), contradicting (2.4.4). A symmetric argument

holds for \( \hat{q} < q^* \). Therefore, \( \hat{q} = q^* \), and in order to satisfy (2.4.3), it must be that \( \hat{r}_j = r^*_j \). \( \square \)

The price \( q^* \) can be interpreted as a Pigovian tax applied to offset the negative externality

that users impose on each other\(^1\) (see [30]).

Unfortunately, Example 1 (non-uniqueness) precludes the possibility that the Pigovian tax

is guaranteed to result in an efficient Wardrop Equilibrium; under the price \( q^* \), there may exist

another Wardrop Equilibrium \( r \neq r^* \) which is not efficient. In fact, an efficiency function

satisfying the conditions of Example 1 can be constructed such that \( r^*_j = 1000 \) for all \( j \) is an

efficient rate allocation, giving us an example. This issue will be addressed in Chapter 4, where

\(^1\)An externality occurs when a user's action indirectly impacts the utility of another user. In our model, when a user increases his data rate, he reduces the SIR of the system and thus reduces the effective data rate of all other users, creating a negative externality on them. The presence of negative externalities results in excessive data transmission, and thus undesirably high interference levels. Named after economist Arthur Pigou, Pigovian taxes are used to correct the resulting inefficiencies.
it is shown that the efficient Wardrop Equilibrium under the Pigovian tax is unique for a broad class of utility functions.

**Example 1, part 2:** Consider the instance of the Rate Selection Game described in Example 1. Figure 2-6 shows that \( r_1 = 1000 \) maximizes the system’s throughput. From Example 1, we know that \( r_1 = 1000 \) is a Wardrop Equilibrium under the price \( q = .1488 \), which along with the uniqueness of the Pigovian tax implies that \( q \) must be the Pigovian tax. But by Example 1, the Wardrop Equilibrium under \( q \) is not unique. •

![Figure 2-6](image)

Figure 2-6: The value of \( r \) that maximizes the throughput is 1000.
Chapter 3

Stability of the Wardrop Equilibrium and a rate control algorithm

3.1 Convergence of a dynamic system to the Wardrop Equilibrium

Chapter 2 introduced the Wardrop Equilibrium, and shows that it has the appealing property that each user only needs to know his own SIR $\gamma_j$ and the price $q$ to determine whether his current rate choice $r_j$ is optimal. It also shows that there is guaranteed to exist a Pigovian tax $q^*$ for which there is an efficient Wardrop Equilibrium. Therefore, it is reasonable to expect that the Wardrop Equilibrium notion may result in a price-based congestion control algorithm which has limited information requirements and communication overhead for the end-users, and results in an efficient rate allocation. In this chapter, we develop such an algorithm.

First, we describe a discrete time dynamic system in which each user chooses an optimal rate in each time index according to the Wardrop Equilibrium definition. This system has one or more fixed points, where each fixed point is a Wardrop Equilibrium. We characterize the local stability properties of the fixed points.

Next, we consider the special case of an efficient equilibrium and show that the necessary and sufficient local stability condition has a simple and easily interpretable form.

Finally, we motivate and describe a price-based rate control algorithm. In this algorithm, the service provider changes the price over time in order to drive the system to efficiency. We show that, even when the service provider has no knowledge of the users’ utility functions, the fixed point of the algorithm is an efficient Wardrop Equilibrium, and that this fixed point satisfies desirable local stability properties.

Appendix A defines relevant terms and gives background results on local stability analysis.
3.1.1 Local Stability of the dynamic system

We consider a discrete time system, where time is indexed by \( n \). At each time \( n \), each user \( j \in \mathcal{J} \) observes the system’s SIR \( \gamma(n) \), and chooses a rate \( r_j(n) \) to satisfy the Wardrop Equilibrium equations (2.3.22) and (2.3.23); i.e., they choose the rate that maximizes their payoff (2.1.3), viewing \( \gamma \) as a constant parameter. The rates \( r_j(n) \) therefore satisfy

\[
u_j'(r_j(n)f(\gamma(n)))f(\gamma(n)) = q, \quad r_j(n) > 0, \\
u_j'(r_j(n)f(\gamma(n)))f(\gamma(n)) \leq q, \quad r_j(n) = 0.
\]

The SIR is then updated by

\[
\gamma(n + 1) = \frac{W}{\sum_j r_j(n)}.
\]

An equilibrium \( r \) of this system, with corresponding SIR \( \gamma \), is a Wardrop Equilibrium of the Rate Selection Game under the price \( q \). We can equivalently express the SIR update of this system in a simpler form as follows. For notational clarity, let \( x(n) \) denote the value of the efficiency function \( f(.) \) at time \( n \), i.e. \( x(n) = f(\gamma(n)) \). We have the following SIR update equation, which can be obtained by substituting (3.1.1) into (3.1.2):

\[
x(n + 1) = f \left( \frac{Wx(n)}{\sum_{j \in \mathcal{J}_{\text{act}}} u_j^{-1} \left( \frac{q}{x(n)} \right)} \right).
\]

where \( \mathcal{J}_{\text{act}} = \{ j | r_j(n) > 0 \} \).

We note that \( u_j^{-1}(x) \) is not defined for \( x > u_j'(0) \). Consistent with the Wardrop Equilibrium notion (3.1.1) that user \( j \) chooses \( r_j = 0 \) if \( \frac{q}{f(\gamma)} \geq u_j'(0) \), we define

\[
u_j^{-1}(x) = 0, \quad x \geq u_j'(0)
\]

With this definition, we can write (3.1.3) as

\[
x(n + 1) = f \left( \frac{Wx(n)}{\sum_{j \in \mathcal{J}} u_j^{-1} \left( \frac{q}{x(n)} \right)} \right).
\]
3.1. Convergence of a dynamic system to the Wardrop Equilibrium

Let \( x \) denote the equilibrium of (3.1.5):

\[
x = f \left( \frac{Wx}{\sum_{j \in J} u_j^{-1} \left( \frac{q}{x} \right)} \right).
\]  

(3.1.6)

Let \( \gamma(q) \) denote the Wardrop Equilibrium SIR under the price \( q \); i.e., \( x = f(\gamma(q)) \). Let \( S(q) \) denote the partial derivative of the RHS of (3.1.6), for the price \( q \), with respect to \( x \) evaluated at the equilibrium \( x = f(\gamma(q)) \):

\[
S(q) = \frac{\partial}{\partial x} \left\{ f \left( \frac{Wx}{\sum_{j \in J} u_j^{-1} \left( \frac{q}{x} \right)} \right) \right\}_{x=f(\gamma(q))}.
\]  

(3.1.7)

Due to (3.1.4), the function \( u_j^{-1}(x) \) is not smooth at \( x = u_j'(0) \), and therefore the partial derivative in (3.1.7) may not exist for all \( x \); i.e., it may have differing left and right derivatives.

To see why this non-differentiability occurs, consider a user \( j \) and a time \( n \) and suppose that \( u_j'(0)x(n) = q \). For any SIR satisfying \( f(\gamma(n)) > x(n) \), the user will transmit with positive rate \( r_j(n) > 0 \) (cf. (3.1.1)). For any SIR satisfying \( f(\gamma(n)) < x(n) \), the user will transmit with rate \( r_j = 0 \) (cf. (3.1.1)). Therefore, a marginal increase in \( x(n) \) results in a marginal increase in \( r_j(n) \), whereas a marginal decrease in \( x(n) \) results in no marginal change in \( r_j(n) \).

For an example, see Figure (3-1) and note the differing left and right derivative at \( f(\gamma) \). Here, \( \max_j u_j'(0) = \frac{q}{f(\gamma)} \). For \( f(\gamma) < f(\gamma_c) \), all users choose a rate of \( r_j(n+1) = 0 \), so \( f(\gamma(n+1)) = 1 \).

For any \( f(\gamma) > f(\gamma_c) \), at least one user is active with strictly positive rate.

If such a situation arises at the equilibrium \( x = f(\gamma(q)) \), then the partial derivative in (3.1.7) may not exist. Therefore, we let \( S^+(q) \) and \( S^-(q) \) denote the right and left derivative, respectively, in (3.1.7). For the case \( S^-(q) \neq S^+(q) \), we define

\[
S(q) = \max_{S^+(q), S^-(q)} \{|S^-(q)|, |S^+(q)|\}.
\]

We present the following local stability result.

**Proposition 10.** Consider the system (3.2.6) under the price \( q \). This system is locally stable if and only if \( |S(q)| < 1 \).

**Proof:** Let \( x \) denote the equilibrium of this one-dimensional system. We linearize the system...
3.1. Convergence of a dynamic system to the Wardrop Equilibrium

with \( x(n) = x + e(n) \), where \( e(n) \) denotes the local perturbation from the equilibrium. If \( S^+(q) = S^-(q) \), the linearization of this one-dimensional system is given by

\[
e(n + 1) = S(q)e(n). \tag{3.1.8}
\]

Since \( S(q) \) is a scalar, the only eigenvalue of \( S(q) \) is simply \( S(q) \), so this system is locally stable if and only if \( |S(q^*)| < 1 \).

Now, suppose \( S^+(q) \neq S^-(q) \). Assume \( |S^+(q)| > |S^-(q)| \); the argument for \( |S^+(q)| < |S^-(q)| \) is symmetric. If \( e(n) > 0 \), then we have the linearization

\[
e(n + 1) = S^+(q)e(n).
\]

The system is locally stable if and only if \( |S^+(q)| = |S(q)| < 1 \).

If \( e(n) < 0 \), then we have the linearization

\[
e(n + 1) = S^-(q)e(n).
\]

If \( |S(q)| < 1 \), we have \( |S^-(q)| < |S^+(q)| < 1 \), so the system is locally stable. Therefore, the system is locally stable for arbitrary \( e(n) \) if and only if \( |S(q)| < 1 \).

We can visualize the Wardrop Equilibrium SIR \( \gamma(q) \) by plotting the right hand side of (3.2.6) as a function of \( x(n) \). The intersection of this curve with the line of slope 1 passing through the origin represents the equilibrium SIR, i.e., the SIR for which \( x(n + 1) = x(n) \). This is shown in Figure 3-1. \( S(q) \) can be interpreted as the slope of the curve shown in Figure 3-1 at its point of intersection with the line of slope 1.

3.1.2 Local Stability of efficient equilibria

In Section 2.4, we presented necessary and sufficient conditions for equilibria to be efficient. Furthermore, in Section 2.4, we proved the existence of a price, interpreted as a Pigovian tax, which induced an efficient equilibrium. Finally, in Section 3.1.1 we gave necessary and sufficient
conditions for a Wardrop Equilibrium to be locally stable. A natural question is: When are efficient equilibria locally stable? We answer this question next. We present the result in terms of the users’ absolute risk aversion $A_j(x)$, which is defined and interpreted in Section 1.4.3.

**Proposition 11.** Consider the system (3.2.6), and assume $q = q^*$ where $q^*$ denotes the Pigovian tax of Section 2.4. Let $r^*$ be an efficient Wardrop Equilibrium with corresponding SIR $\gamma^*$ under price $q^*$. Let $g^*_j = r^*_j f(\gamma^*)$ denote user $j$’s throughput, and let $G^* = \max_{R} R f(\frac{W}{R})$ denote the system’s maximum throughput. Let $J_{act} = \{j | q^* = u'_j(g^*)f(\gamma^*)\}$ denote the set of active users at the Wardrop Equilibrium. Then, this equilibrium is locally stable if and only if

$$
\sum_{j \in J_{act}} \frac{1}{A_j(g^*_j)} < 2G^*.
$$

**Proof:** We solve for $S(q^*)$, and note that at an efficient equilibrium, we have $f(\gamma^*) = \gamma^* f'(\gamma^*)$. 

![Figure 3-1: $x(n+1)$ as a function of $x(n)$ for a fixed price $q$. The equilibrium SIR $\gamma(q)$ is shown. Note that at $\gamma_c$, we have $\max_j u'_j(0) = \frac{q}{f(\gamma_c)}$. For $\gamma < \gamma_c$, all users choose a rate of $r_j(n+1) = 0$, so $f(\gamma(n+1)) = 1.$]
3.1. Convergence of a dynamic system to the Wardrop Equilibrium

Furthermore, note that \( \sum_j u_j^{-1}(\frac{q^*}{f(\gamma)}) = \sum_j v_j^* f(\gamma^*) = G^* \). Next, we use

\[
(u_j^{-1})'(\frac{q^*}{x^*}) = \begin{cases} 
\frac{1}{u_j''(u_j^{-1}(\frac{q^*}{x^*}))}, & j \in J_{\text{act}}, \\
0, & j \notin J_{\text{act}}
\end{cases}
\]

along with the necessary and sufficient condition of Proposition 10 to arrive at

\[
\left| \frac{q^*}{f(\gamma^*)} \sum_{j \in J_{\text{act}}} \frac{1}{u_j''(g_j^*)} \right| < 2G^*. \tag{3.1.10}
\]

Then, we use the first order condition \( q = u_j'(r_j f(\gamma)) f(\gamma) \) for all \( j \in J_{\text{act}} \), along with the definition of absolute risk aversion to complete the proof.

Example: Consider a system with the modulation scheme of non-coherent FSK, no error coding, 50 bits per frame, and \( W = 4.096 \times 10^6 \):

\[
f(\gamma) = (1 - e^{-\frac{\gamma}{T}})^{50}.
\]

We can calculate the maximum throughput as

\[
G^* = 347.17, \text{ kbps}.
\]

Assume all users have the same utility function, given by

\[
u_j(x) = 1 - e^{-bx} \quad \forall j.
\]

In this case, we can compute the absolute risk aversion for each user as

\[
A_j(x) = b, \quad \forall j.
\]
Therefore, the efficient equilibrium is locally stable if and only if

\[ \frac{J}{b} < 694340. \]

As more users join the system and \( J \) increases, it will eventually become necessary to increase \( G^* \), the system’s capacity, by, for example, increasing \( W \) in order to maintain local stability at the efficient equilibrium. •

\section{Dynamic pricing}

\subsection{Motivation}

In the stability analysis thus far, we have assumed that there exists some price \( q \) which is static over time. However, it seems reasonable that if the base station was able to dynamically change the price at each time \( n \), then it could improve the local stability of the system. This section will motivate this idea. We first have the following assumption which will hold for the remainder of this chapter.

\textbf{Assumption 4:} Assume that the Wardrop Equilibrium under any price \( q \) is unique.

We now present a sufficient condition under which Assumption 4 holds. The condition is expressed in terms of the users’ relative risk aversion, \( R_j(x) \), which is defined and interpreted in Section 1.4.3.

\textbf{Proposition 12.} If \( R_j(x) < 1 \) for all \( x \) and for all \( j \), then Assumption 4 holds.

\textbf{Proof:} Since \( R_j(x) < 1 \), (1.4.1) implies

\[ u_j''(x)x + u_j'(x) > 0, \quad \forall j, \forall x. \]

We therefore have

\[ \frac{\partial}{\partial \{f(\gamma)\}} \{u_j'(r_jf(\gamma))f(\gamma)\} = u_j''(r_jf(\gamma))r_jf(\gamma) + u_j'(r_jf(\gamma)) > 0, \quad \forall j, \forall r_j. \]

Therefore, \( u_j'(r_jf(\gamma))f(\gamma) \) is strictly increasing in \( f(\gamma) \). Let \( r_j(n) \) denote user \( j \)'s rate choice
after observing SIR $\gamma(n)$, and $r(n)$ denote user $j$'s rate choice after observing SIR $\gamma(n)$, where $\hat{\gamma}(n) < \gamma(n)$. We claim that $\hat{r}_j(n) < r_j(n)$. Too see this, note that since $f(\gamma)$ is a strictly increasing function of $\gamma$, we have for all $j \in J_{\text{act}}$

$$u_j'(\hat{r}_j(n)f(\hat{\gamma}(n)))f(\hat{\gamma}(n)) = q$$

$$= u_j'(r_j(n)f(\gamma(n)))f(\gamma(n))$$

$$> u_j'(r_j(n)f(\hat{\gamma}(n)))f(\hat{\gamma}(n)).$$

By the strict concavity of $u_j$ (cf. Assumption 2.2), $\hat{r}_j(n) < r_j(n)$ for all $j$. Therefore,

$$f(\hat{\gamma}(n + 1)) > f(\gamma(n) + 1)).$$

We will now complete the proof. Suppose, for a contradiction, that there are two fixed points $x$ and $\hat{x}$ of (3.1.5), where each fixed point satisfies (3.1.6). Assume, without loss of generality, that $\hat{x}(n) < x(n)$. Recall that we are using the notation $f(\gamma(n)) = x(n)$, and so we have $f(\hat{\gamma}(n)) < f(\gamma(n))$. Since $x(n) = x(n + 1)$ and $\hat{x}(n) = \hat{x}(n + 1)$, we have

$$x(n) - \hat{x}(n) = x(n + 1) - \hat{x}(n + 1).$$

But $x(n) - \hat{x}(n) > 0$, and by (3.2.4) we have $x(n + 1) - \hat{x}(n + 1) < 0$, a contradiction. This implies that there can only be one fixed point $x$ satisfying $x(n + 1) = x(n)$. Since the set of fixed points of the dynamic system are equivalent to the set of Wardrop Equilibria, the Wardrop Equilibrium must be unique.

The condition $R_j(x) < 1$ for all $j$ and for all $x$ will appear in a central result in Chapter 4 as well.

Consider the following system: At each time $n$, each user $j \in J$ observes the system’s SIR $\gamma(n)$ and price $q(n)$, and chooses a rate $r_j(n)$ to satisfy the Wardrop Equilibrium equations (2.3.22) and (2.3.23) (i.e., they choose the rate that maximizes their payoff (2.1.3), viewing $\gamma$
as a constant parameter):

\[ u'_j(r_j(n)f(\gamma(n)))f(\gamma(n)) = q(n), \quad r_j(n) > 0, \]  
\[ \leq q(n), \quad r_j(n) = 0. \]  

(3.2.5)

The SIR is then updated by

\[ \gamma(n + 1) = \frac{W}{\sum_j r_j(n)}. \]  

(3.2.6)

Finally, the price is updated as a function of \( r(n) \)

\[ q(n + 1) = s(r(n)), \]  

(3.2.7)

where \( s : \mathbb{R}^J \rightarrow \mathbb{R}^+ \) is a price update rule chosen by the service provider. We now state the service provider’s objective.

**Service provider objective:** Let \( q^* \) denote the Pigovian tax, under which the rate allocation \( r^* \) is an efficient Wardrop Equilibrium with corresponding SIR \( \gamma^* \). The service provider’s objective is to choose the mapping \( s \) such that \((q^*, r^*)\) is a locally stable equilibrium of the system (3.2.5 - 3.2.7).

We can equivalently express the SIR update of this system in a simpler form as follows. For notational clarity, let \( x(n) \) denote the value of the efficiency function \( f(\cdot) \) at time \( n \), i.e. \( x(n) = f(\gamma(n)) \). We have the following SIR update equation, which can be obtained by substituting (3.2.5) into (3.2.6):

\[ x(n + 1) = f \left( \frac{Wx(n)}{\sum_j u'_j^{-1} \left( \frac{g(n)}{x(n)} \right)} \right). \]  

(3.2.8)

If the service provider knows the utility function of all users, and the value of the efficient SIR \( \gamma^* \), then the service provider can satisfy its objective by choosing \( q(n) \) to satisfy the following equation:

\[ \frac{Wx(n)}{\sum_j u'_j^{-1} \left( \frac{g(n)}{x(n)} \right)} = \gamma^*. \]  

(3.2.9)

If the service provider is able to solve the nonlinear equation (3.2.9) for the single unknown \( q(n) \),
3.2. Dynamic pricing

the system is guaranteed to be locally stable at \( r^* \) under this price choice. For any perturbation from this equilibrium, the system will immediately return to the equilibrium.

However, the price update (3.2.9) requires knowledge of the utility functions \( u_j \); a more realistic assumption is that the service provider must update the price with no knowledge of the utility functions. Furthermore, even with knowledge of the utility functions, (3.2.9) may be difficult to solve. In the next section, we propose a pricing scheme with simple computation which requires no knowledge of the utility functions.

3.2.2 A dynamic pricing scheme to achieve efficiency

We make the following assumption on the service provider’s information structure.

**Assumption 6:** Assume that the service provider does not know the utility functions \( u_j \). Furthermore, assume that the service provider knows the efficiency function \( f(\cdot) \), and can compute the efficient SIR \( \gamma^* \) satisfying \( f(\gamma^*) = \gamma^* f'(\gamma^*) \).

The system under consideration is

\[
\begin{align*}
x(n + 1) &= f \left( \frac{W x(n)}{\sum_j u_j^{-1} \left( \frac{q(n)}{x(n)} \right)} \right), \quad (3.2.10) \\
q(n + 1) &= s(r(n)). \quad (3.2.11)
\end{align*}
\]

where \( s(\cdot) \) is a price update rule chosen by the service provider. As before, let \( \gamma^* \) denote the efficient SIR satisfying \( f(\gamma^*) = \gamma^* f'(\gamma^*) \), and let \( q^* \) be such that \( (q^*, r^*) \) is a Wardrop Equilibrium with SIR \( \gamma^* \). Finally, let \( \gamma(q) \) denote the Wardrop Equilibrium SIR under a general price \( q \). We have the following.

**Proposition 13.** Let \( \gamma(q) \) be the equilibrium SIR under the price \( q \). \( \gamma(q) \) is a nondecreasing function of \( q \).

**Proof:** For a contradiction, assume that there exists a price \( \tilde{q} > q \) with Wardrop Equilibrium SIR \( \gamma(\tilde{q}) \) such that \( \gamma(\tilde{q}) < \gamma(q) \). Since \( f(0) = 1 \), the RHS of (3.2.10) is 1 for \( x(n) = 0 \). Furthermore, since we have assumed that there is a unique Wardrop Equilibrium under \( q \), it
3.2. Dynamic pricing

follows that

\[ f \left( \frac{Wx(n)}{\sum_j u_j^{-1} \left( \frac{q}{x(n)} \right)} \right) > x(n) \quad \text{for} \quad x(n) < f(\gamma(q)). \]

Since \( f(\cdot) \) is a strictly increasing function and \( u_j^{-1} \) is a strictly decreasing function, we have

\[ f \left( \frac{Wx(n)}{\sum_j u_j^{-1} \left( \frac{q}{x(n)} \right)} \right) > x(n) \quad \text{for} \quad x(n) < f(\gamma(q)). \]

Therefore,

\[ f \left( \frac{Wf(\gamma(q))}{\sum_j u_j^{-1} \left( \frac{\gamma(q)}{f(\gamma(q))} \right)} \right) > f(\gamma(q)), \]

which contradicts \( \gamma(q) \) being an equilibrium SIR.

We have the following Corollary which follows from the analysis of Proposition 13.

**Corollary 1.** Let \( q^* \) denote the Pigovian tax. Let \( S(q) \) be defined as in (3.1.7). We have \( S(q^*) < 0. \)

**Proof:** Let \( x^* \) denote the equilibrium of (3.2.12), and let \( \gamma^* \) denote the corresponding SIR. As shown in Proposition 13,

\[ f \left( \frac{Wx(n)}{\sum_j u_j^{-1} \left( \frac{q^*}{x(n)} \right)} \right) > x(n) \quad \text{for} \quad x(n) < x^*. \]

Since \( x^* \) is an equilibrium,

\[ f \left( \frac{Wx^*}{\sum_j u_j^{-1} \left( \frac{q^*}{x^*} \right)} \right) = x^*. \]

Since, by assumption, \( x^* \) is the unique equilibrium, it follows that the slope of

\[ f \left( \frac{Wx(n)}{\sum_j u_j^{-1} \left( \frac{q}{x(n)} \right)} \right) \]

at \( x(n) = x^* \), which is equivalent to \( S(q^*) \), must be negative.

We consider the following price update for (3.2.11). The SIR update is shown for convenience:

\[ x(n + 1) = f \left( \frac{Wx(n)}{\sum_j u_j^{-1} \left( \frac{q(n)}{x(n)} \right)} \right), \quad (3.2.12) \]

\[ q(n + 1) = q(n) + \alpha(f(\gamma^*) - f(\gamma(n))). \quad (3.2.13) \]
3.2. Dynamic pricing

where \( \alpha \) is some constant positive stepsize. This price update scheme is motivated by Proposition 13. Based on the nondecreasing property of \( \gamma(q) \), we consider an algorithm in which, at time \( n \), the service provider observes the SIR \( \gamma(n) \). If \( \gamma(n) < \gamma^* \), the service provider will increase the price. If \( \gamma(n) > \gamma^* \), the service provider will decrease the price.

Before studying the local stability properties of (3.2.12 - 3.2.13), we observe that such a price update has a natural economic interpretation known as \textit{price tâtonnement} (see [33]). If, at some time \( n \), \( \gamma(n) < \gamma^* \), then the sum rate of all users \( r \) satisfies \( \sum_j r_j(n) > \sum_j r_j^* \). We can interpret this as an "excess demand" for rate; the users are transmitting with such high data rates that it is hurting the system's overall throughput. When this excess demand is present, the service provider increases the price according to (3.2.13). Similarly, we can view the case where \( \sum_j r_j(n) < \sum_j r_j^* \) as "excess supply;" the system can support higher data rates and improve total throughput. In this case, the service provider reduces the price. This kind of dynamic price adjustment as a response to excess supply or demand is commonly referred to as \textit{price tâtonnement} in the economics literature. In the context of (3.2.13), the quantity \( f(\gamma^*) - f(\gamma(n)) \) is used as a simple and easily computable measure of the excess supply or demand.

We now study the local stability properties of this algorithm, and present conditions on the stepsize values that will establish local stability. Recall that the system with static pricing is locally stable if and only if \( |S(q^*)| < 1 \). We will show that the system (3.2.12), (3.2.13) will be locally stable for a larger range of values of \( S(q^*) \). We linearize the system (3.2.12), (3.2.13) around its equilibrium point \((x^*, q^*)\), where \( x^* = f(\gamma^*) \). The Jacobian is given by

\[
J = \begin{bmatrix}
S(q^*) & -L \\
-\alpha & 1 \\
\end{bmatrix}
\]

where

\[
L = \frac{\sum_j (u_j^{x-1}) \left( \frac{q^*_j}{x^*} \right)}{\sum_j u_j^{x-1} \left( \frac{q^*_j}{x^*} \right)}. \tag{3.2.14}
\]

This value of \( L \) is found by solving for the appropriate partial derivative, and then simplifying.
by substituting the efficiency condition \( f(\gamma^*) = \gamma^* f'(\gamma^*) \). Consistent with (3.1.4), we define
\[
(u_j^{-1})'(x) = 0, \quad x \geq u_j'(0).
\]
The eigenvalues of \( J \) are given by:
\[
\lambda_1 = \frac{S(q^*)}{2} + \frac{1}{2} + \frac{\sqrt{(S(q^*) + 1)^2 - 4(S(q^*) - L\alpha)}}{2}, \quad (3.2.15)
\]
\[
\lambda_2 = \frac{S(q^*)}{2} + \frac{1}{2} - \frac{\sqrt{(S(q^*) + 1)^2 + 4(S(q^*) - L\alpha)}}{2}. \quad (3.2.16)
\]
Define \( \lambda \) as
\[
\lambda = \arg\max_{\lambda_1, \lambda_2} \{|\lambda_1|, |\lambda_2|\}.
\]
The system is locally stable if and only if \(|\lambda| < 1\). Note that for \(-1 < S(q^*) < 0\), we can choose the stepsize
\[
\alpha = \frac{S(q^*)}{L},
\]
and we will have \(|\lambda| < 1\). This is not surprising; if the system with static pricing is locally stable, we can find a stepsize such that the system with dynamic pricing is locally stable. Because of this, and since we have shown \( S(q^*) < 0 \), we will focus on the case in which \( S(q^*) \leq -1 \). In this case, we have \( \lambda = \lambda_1 \):
\[
\lambda = \frac{S(q^*)}{2} + \frac{1}{2} - \frac{\sqrt{(S(q^*) + 1)^2 - 4(S(q^*) - L\alpha)}}{2}. \quad (3.2.17)
\]
We have the following two Propositions, which show the improvement in local stability as a result of dynamic pricing.

**Proposition 14.** Consider the system the system (3.2.12). For \( \alpha = \frac{1}{L} \), (3.2.13) is locally stable if and only if \(|S(q^*)| < 1.5\).

**Proof:** Figure 3-2 shows a plot of \( \lambda \) as a function of \( S(q^*) \) for \(|L\alpha| = 1\). As shown, \(|\lambda| < 1\) is equivalent to \(|S(q^*)| < 1.5\). \( \square \)

Proposition 14 shows the improved local stability that results from dynamic pricing; recall that the system under the static price \( q^* \) is locally stable if and only if \(|S(q^*)| < 1\). Therefore,
for $-1.5 < S(q^*) < -1$, dynamic pricing with the stepsize specified in Proposition 14 stabilizes an otherwise unstable system.

**Proposition 15.** There exists $\alpha$ such that the system (3.2.12), (3.2.13) is locally stable if and only if $|S(q^*)| < 3$.

**Proof:** For $|S(q^*)| \geq 3$, we have $\frac{S(q^*)}{2} + \frac{1}{2} \leq -1$. This implies $\Re\{\lambda\} \leq -1$, and therefore $|\lambda| \geq 1$.

For $|S(q^*)| < 3$, choose $\alpha = \frac{4S(q^*) - (S(q^*) + 1)^2}{4L}$ and $|\lambda| < 1$.

Proposition 15 shows that dynamic pricing can stabilize an otherwise unstable system for $-3 < S(q^*) < 1$. However, this proposition is weaker than Proposition 14 in the sense that the stepsize of Proposition 14 is guaranteed to stabilize the system for any $|S(q^*)| < 1.5$, whereas a stepsize that stabilizes the system for $|S(q^*)| = 2$ may not stabilize it for arbitrary $S(q^*) \in (-3, 0)$.

The stepsizes, and the extent to which the local stability properties improve, are given in terms of the quantities $L$ and $S(q^*)$. We now present a result that relates $L$ and $S(q^*)$ to system parameters.
Proposition 16. Let \((q^*, r^*)\) be the efficient Wardrop Equilibrium of the Rate Selection Game, and let \(\gamma^*\) be the associated Wardrop Equilibrium SIR. Let \(g_j^* \equiv r_j^* f(\gamma^*)\) denote the equilibrium throughput of user \(j\), and let \(G^*\) denote the total system throughput. Let \(J_{\text{act}} = \{j | r_j^* > 0\}\) denote the set of active users at the Wardrop Equilibrium. We have

\[
L = \frac{\sum_{j \in J_{\text{act}}} u_j^*(g_j^*)}{G^*} \quad (3.2.18)
\]

\[
S = 1 - \frac{\sum_{j \in J_{\text{act}}} \frac{1}{\Lambda_j(g_j^*)}}{G^*} \quad (3.2.19)
\]

**Proof:** These can be derived by taking appropriate partial derivatives, and making use of the following:

Because the equilibrium is efficient, we have

\[
f(\gamma^*) = \gamma^* f'(\gamma^*).
\]

By the definition of the Wardrop Equilibrium, we have

\[
q^* = u_j'(g_j^*) f(\gamma^*).
\]

This can be rewritten as

\[
u_j^{-1} \left( \frac{q^*}{x^*} \right) = g_j^*.
\]

We also have

\[
(u_j^{-1})' \left( \frac{q^*}{x^*} \right) = \frac{1}{u_j' \left( u_j^{-1} \left( \frac{q_j^*}{x^*} \right) \right)}.
\]

Substituting these into 3.2.14 gives the expression for \(L\). Substituting these into 3.1.7 after taking the partial derivative gives the expression for \(S\). 

Propositions 14 and 15 show that the appropriate stepsize is inversely proportional to \(|L|\). By Proposition 16, we see that, in general, the required stepsize is increasing in the second derivative of the users' utility functions. Furthermore, \(|S|\) is decreasing in the absolute risk aversion of the users, so we expect improved local stability properties of the system with higher.
absolute risk aversions of the users.

The analysis presented in this section focuses on local, rather than global, stability analysis. A system is globally stable if it converges to the equilibrium from any initialization. Understanding the global stability properties of the static and dynamic pricing schemes presented in this chapter is left as a future direction. However, local stability analysis is the first step to validating the convergence properties of an algorithm. The numerical simulation in the next section shows that the dynamic pricing scheme proposed in this chapter has promise in terms of its global stability properties.

### 3.2.3 Numerical simulation

In this subsection, we illustrate how dynamic pricing can stabilize an otherwise unstable system. We assume all users have the same utility function, given by

\[ u_j(x) = 1 - e^{-x} \quad \forall j. \]

We use an efficiency function

\[ f(\gamma) = (1 - e^{-\gamma})^5, \]

and \( \frac{W_f}{J} = 2.3 \). We want the system to be stable at a Wardrop Equilibrium \( r^* \) under the price \( q^* = 0.3 \). We initialize the system with \( f(\gamma(0)) = 0.1 \). With static pricing, where \( q(n) \equiv q^* = 0.3 \) for all \( n \), the system is unstable and oscillates. With dynamic pricing, we initialize \( q(0) = 0.4 \), choose a stepsize of \( \alpha = 0.05 \), and the system converges to the equilibrium \( (q^*, r^*) \).
Figure 3-3: An instance of the dynamic system presented in this chapter under both static (top) and dynamic (bottom) pricing.
Chapter 4

Profit-maximizing pricing

In Part 1, we showed that an appropriately chosen price, interpreted as a Pigovian tax, can lead to an efficient rate allocation. However, the use of such a Pigovian tax to promote efficiency assumes that the agent setting prices is interested in maximizing the system's efficiency. Another realistic assumption is that the prices are set by a service provider, who is a selfish agent that acts strategically by setting the price in order to maximize profit. This motivates us to ask the following fundamental question: If prices are set by a profit maximizing agent, whom we refer to as the Service Provider, what is the resulting efficiency loss?

The purpose of this chapter is to answer this question. We show that despite the selfish behavior of all agents involved, for a broad class of user utility functions there is no efficiency loss; i.e. the profit maximizing price is equal to the Pigovian tax of Chapter 2.

Furthermore, we show that the Wardrop Equilibrium is always unique under a profit maximizing price, implying that this price will not result in an inefficient Wardrop Equilibrium as opposed to the efficient Wardrop Equilibrium.

We model this situation by adding a Service Provider stage to the Rate Selection Game considered in Chapter 2.

Definition 9. The Price-Rate Control Game has J+1 players, consisting of the set of users $J$ and the Service Provider, who act sequentially according to the following utilities and strategy spaces.

1. Service provider stage: The service provider chooses a price $q$ to maximize its total profit, given that each user will choose a rate to maximize his payoff.
4.1. Efficiency through profit maximizing prices

\[
\begin{align*}
\max_{q \geq 0} q \sum_{j} r_j \\
\text{s.t. } r_j \in \arg\max_{r_j \geq 0} \{u_j(r_j f(y(r))) - qr_j\}, \quad \forall j.
\end{align*}
\] (4.0.1)

2. User stage

3. Power Control stage

where the User stage and Power Control stage are given in Definition 3

Characterizing the optimal price from the service provider’s viewpoint corresponds to finding the Subgame Perfect Equilibrium (SPE) of this sequential game (see [40]). Every price \(q\) defines a different subgame, and this subgame corresponds to the game that was studied in Chapter 2. Thus, for every price \(q\), there is an associated Wardrop Equilibrium \(r(q)\). The set of \(r(q)\) for all possible prices \(q\) corresponds to an equilibrium path, and the service provider’s objective is to choose a price \(q\) on this equilibrium path such that \(r(q)\) satisfies the objective 4.0.1.

4.1 Efficiency through profit maximizing prices

In this section, we study the Service Provider stage of the Price-Rate Control Game. We show that despite the non-uniqueness of the Wardrop Equilibrium that was shown in Chapter 2, the Wardrop Equilibrium under profit maximizing pricing is unique. We then isolate a broad class of utility functions that guarantee the subgame perfect equilibrium will be efficient. We conclude the chapter with examples.

To find the subgame perfect equilibrium, we substitute the Wardrop Equilibrium conditions (2.3.22) and (2.3.23) into the Service Provider Objective (4.0.1):
4.1. Efficiency through profit maximizing prices

\[ \begin{align*}
\max_{r,q} & \quad q \sum_j r_j & (4.1.1) \\
\text{s.t.} & \quad u'_j(r_j f(\gamma)) f(\gamma) = q, & r_j > 0, \\
& \quad \leq q, & r_j = 0, & \forall j.
\end{align*} \]

Equivalently, (4.1.1) can be written as:

\[ \begin{align*}
\max_{r,q} & \quad \sum_j u'_j(r_j f(\gamma)) f(\gamma) r_j & (4.1.2) \\
\text{s.t.} & \quad u'_j(r_j f(\gamma)) f(\gamma) = q, & r_j > 0, \\
& \quad \leq q, & r_j = 0, & \forall j.
\end{align*} \]

**Definition 10.** The Subgame Perfect Equilibria (SPE) of the Price-Rate Control Game are the optimal solutions to (4.1.2).

We have shown in Example 1 that for a price \( q \), the Wardrop Equilibrium may not be unique. However, the following Proposition shows that the service provider can always find a profit maximizing price for which the Wardrop Equilibrium is unique.

**Proposition 17.** (Uniqueness) There exists a profit maximizing price \( \hat{q} \) for which the Wardrop Equilibrium is unique.

**Proof:** Figure 4-1 plots \( R f(\frac{W}{R}) \) (the system throughput) as a function of \( R \). \( R^* \) denotes the value of \( R \) which maximizes the system throughput. We will use this figure in the following arguments. We first prove the following lemmas.

**Lemma 17.1:** There is guaranteed to exist an SPE \((\hat{q}, \hat{r})\) with \( 0 < \hat{R} \leq R^* \), where \( \hat{R} = \sum \hat{r}_j \).

**Proof:** Consider any Wardrop Equilibrium given a price \( \overline{q} \). Let \( \overline{r} \) be the equilibrium rate vector, let \( \overline{R} = \sum \overline{r}_j \), and let \( \overline{R} = \frac{W}{\overline{R}} \). Assume that

\[ R^* < \overline{R} < \infty. \]

We will show that there is guaranteed to exist another Wardrop Equilibrium under a different
4.1. Efficiency through profit maximizing prices

Figure 4-1: The total effective data rate of the system as a function of $R$. Lemma 17.1 of the proof shows that for any Wardrop Equilibrium with $R > R^*$, there is another Wardrop Equilibrium under a different price with equal profit for the service provider such that $R < R^*$.

price $q$ with the same profit, whose rates $r$ satisfy $0 < R < R^*$.

Because $Rf\left(\frac{W}{R}\right)$ is strictly increasing from 0 to its maximum value for $0 < R < R^*$, and strictly decreasing for $R^* < R < \infty$ (cf. proof of Proposition 2, Step 1) (see Figure 4-1), there exists a unique $R < \bar{R}$ such that

$$Rf\left(\frac{W}{R}\right) = \bar{R}f\left(\frac{W}{R}\right).$$

Furthermore, it must be that $0 < R < R^*$. Let $\gamma = \frac{W}{R}$. Define $K$ as

$$K = \frac{f(\gamma)}{f(\bar{\gamma})}. \tag{4.1.3}$$

Define $r_j$, for all $j$, as

$$r_j = \frac{\bar{r}_j}{K}, \quad \forall j.$$

Note that $W/\sum r_j = \gamma$. We have

$$r_j f(\gamma) = \bar{r}_j f(\bar{\gamma}), \quad \forall j. \tag{4.1.4}$$
Because \( \bar{r} \) is a Wardrop Equilibrium, we have

\[
    u_j'(\bar{r}_jf(\gamma))f(\gamma) = \bar{q}, \quad r_j > 0, \\
    \leq \bar{q}, \quad r_j = 0.
\]

(4.1.3), (4.1.4), and (4.1.5) imply

\[
    u_j'(r_jf(\gamma))f(\gamma) = K\bar{q}, \quad r_j > 0, \\
    \leq K\bar{q}, \quad r_j = 0,
\]

(4.1.6)

Therefore, for a price \( q = K\bar{q}, r \) is a Wardrop Equilibrium. Furthermore, (4.1.4) implies

\[
    \sum_j \{u_j'(\bar{r}_jf(\gamma))f(\gamma)\bar{r}_j\} = \sum_j \{u_j'(r_jf(\gamma))f(\gamma)r_j\}.
\]

Therefore, the service provider profits under \((q, r)\) and \((\bar{q}, \bar{r})\) are equal. Because we can find some Wardrop Equilibrium with \( R \in (0, R^*) \) to match the profit of any Wardrop Equilibrium outside this range, there must exist a profit maximizing Wardrop Equilibrium \( \bar{R} \) such that \( 0 < \bar{R} \leq R^* \). This implies there exists an SPE \((\hat{q}, \hat{r})\) such that \( 0 < \bar{R} \leq R^* \).

The next lemma shows that any two SPE’s under the same price will both have the same sum data rate.

**Lemma 17.2:** Let \((\hat{q}, \hat{r})\) be an SPE such that \( 0 < \hat{R} \leq R^* \). Then for any other Wardrop Equilibrium \( \bar{r} \) under \( \hat{q} \), we have \( \bar{R} = \hat{R} \).

**Proof:** If \( \bar{R} > \hat{R} \), then

\[
    \bar{q} \sum r_j > \hat{q} \sum \bar{r}_j,
\]

which is a contradiction, since \( \hat{r} \) is a profit maximizing equilibrium. If \( \bar{R} < \hat{R} \), then

\[
    \bar{\gamma} = \frac{W}{\bar{R}} > \frac{W}{\hat{R}} = \hat{\gamma}.
\]
For $0 < R \leq R^*$, $R \frac{W}{R}$ is increasing. Therefore,

$$\tilde{R} \frac{W}{R} < \hat{R} \frac{W}{R},$$

which implies

$$r_{j}^{\top} f(\tilde{\gamma}) < r_{j}^{\top} f(\hat{\gamma}), \text{ for some } j.$$

(4.1.7) and (4.1.8) contradict Proposition 7. Therefore, $\tilde{R} = \hat{R}$. $\square$

We now return to the proof of the Proposition. Let $\tilde{q}$ be the price such that $(\tilde{q}, \tilde{r})$ is an SPE, and $0 \leq \tilde{R} \leq R^*$; such a price is guaranteed to exist by Lemma 17.1. Assume there is another Wardrop Equilibrium $\check{r}$. From Lemma 17.2, $\tilde{R} = \check{R}$, so $\tilde{\gamma} = \check{\gamma}$. This contradicts Proposition 6. $\square$

There may be an SPE $(\tilde{q}, \tilde{r})$ such that $\tilde{R} > R^*$. However, due to the non-uniqueness of the Wardrop Equilibria, the service provider has no guarantee that the Wardrop Equilibrium $\tilde{r}$ will be the result when the price is set to $\tilde{q}$. There may be another non profit maximizing Wardrop Equilibrium $\hat{r} \neq \check{r}$ for the price $\hat{q}$.

Proposition 17 shows that it is in the service provider's best interest to restrict the feasible region of the Service Provider objective (4.1.1) to the region $0 \leq R \leq R^*$; in doing so, the service provider ensures the uniqueness of the Wardrop Equilibrium under the profit maximizing price. Therefore, for the remainder of the paper, we restrict attention to the following region:

$$\{r \mid 0 \leq R \leq R^*\},$$

where

$$R^* = \arg \max_{R} \left\{ R \frac{W}{R} \right\}.$$

We next present conditions which ensure that the SPE of the Price-Rate Control Game results in an efficient rate allocation. For this purpose, we first prove the following two lemmas. The first lemma shows that the Service Provider Objective (4.1.1) is equivalent to a simpler problem, in which we only consider users who transmit with strictly positive rate at the SPE.
Lemma 18.1: Let $(\hat{q}, \hat{\mathbf{r}})$ be an SPE. Define the set

$$J_{SPE} = \{ j | \hat{r}_j > 0 \}.$$  

Let $[\hat{r}_j]_{j \in J_{SPE}} \in \mathbb{R}^{|J_{SPE}|}$ denote the vector consisting of components of the rate vector $\hat{r} \in \mathbb{R}^J$ corresponding to users in the set $J_{SPE}$. Then, $(\hat{q}, [\hat{r}_j]_{j \in J_{SPE}})$ is an optimal solution to

$$\begin{aligned}
& \max_{q, [\mathbf{r}]_{J_{SPE}}} \quad q \sum_{j \in J_{SPE}} r_j \\
& \text{s.t.} \quad u_j'(r_j f(\gamma)) f(\gamma) = q, \quad \forall j \in J_{SPE} \\
& \quad q \geq 0, \quad r_j \geq 0, \quad \forall j \in J_{SPE},
\end{aligned}$$

where

$$\gamma = \frac{W}{\sum_{j \in J_{SPE}} \hat{r}_j}.$$  

Proof: To arrive at a contradiction, suppose there exists $(\bar{q}, [\bar{r}_j]_{j \in J_{SPE}})$ such that

$$\bar{q} \sum_{j \in J_{SPE}} \bar{r}_j > \hat{q} \sum_{j \in J_{SPE}} \hat{r}_j. \quad (4.1.10)$$

Let

$$\bar{\gamma} = \frac{W}{\sum_{j \in J_{SPE}} \bar{r}_j},$$

$$\bar{R} = \sum_{j \in J_{SPE}} \bar{r}_j.$$

Also, let $\gamma(\bar{q}), [r_j(\bar{q})]_{j \in J}$, and $R(\bar{q})$ denote the Wardrop Equilibrium SIR, rate vector, and sum rate under the price $\bar{q}$. We claim that

$$R(\bar{q}) \geq \bar{R}. \quad (4.1.11)$$
4.1. Efficiency through profit maximizing prices

To arrive at a contradiction, suppose $R(q) < \tilde{R}$. This implies that $\gamma(q) > \tilde{\gamma}$. We then have

$$u'_j(r_j(q)f(\gamma(q)))f(\gamma) < u'_j(r_j(q)f(\gamma(q)))f(\gamma(q)) \leq \hat{q}$$

$$= u'_j(\bar{r}_j f(\bar{\gamma}))f(\bar{\gamma}), \quad \forall j \in \mathcal{J}_{SPE}.$$

By the strict concavity of $u_j$, this implies

$$r_j(q)f(\gamma(q)) > \bar{r}_j f(\bar{\gamma}), \quad \forall j \in \mathcal{J}_{SPE}.$$ 

Because $\bar{r}_j = 0$ for $j \notin \mathcal{J}_{SPE}$, we have

$$r_j(q)f(\gamma(q)) \geq \bar{r}_j f(\bar{\gamma}), \quad \forall j \in \mathcal{J}. \quad (4.1.12)$$

Summing (4.1.12) over all $j \in \mathcal{J}$, we obtain

$$R(q)f\left(\frac{W}{R(q)}\right) \geq \hat{R}f\left(\frac{W}{\hat{R}}\right). \quad (4.1.13)$$

We are restricted to the region $0 \leq R \leq R^*$ by (4.1.9), which implies $Rf(W/R)$ is a strictly increasing function of $R$ (cf. proof of Proposition 8). We then have $R(q) \geq \hat{R}$, which is a contradiction. From (4.1.10) and (4.1.11), we have

$$\hat{q} \sum_{j \in \mathcal{J}} r_j(q) \geq \hat{q} \sum_{j \in \mathcal{J}_{SPE}} \hat{r}_j > \hat{\bar{q}} \sum_{j \in \mathcal{J}_{SPE}} \bar{r}_j = \hat{\bar{q}} \sum_{j \in \mathcal{J}} \bar{r}_j,$$

which is a contradiction, since $(\hat{q}, \hat{r})$ is an SPE, which must be profit maximizing \hfill \Box

Hence, when considering the Service Provider objectives (4.1.1) and (4.1.2), we can restrict attention to the set of users transmitting with strictly positive data rate. The next lemma expresses the Service Provider objective in a form which will allow us to more easily see when the efficiency condition of Section 2.4 will be met.

**Lemma 18.2** Let $(\hat{q}, \hat{r})$ be an SPE, with sum rate $\hat{R} = \sum_{j \in \mathcal{J}} \hat{r}_j$, and SIR $\hat{\gamma} = \frac{W}{\hat{R}}$. Then, $(\hat{k}, \hat{g})$, 

4.1. Efficiency through profit maximizing prices

where $\hat{k} = \frac{\hat{q}}{\hat{f}(\hat{\gamma})}$ and $\hat{g}_j = \hat{r}_j f(\hat{\gamma})$, is an optimal solution to

\[
\max_{k, \{g_j\}_j \in \mathcal{JSPE}} \sum_{j \in \mathcal{JSPE}} u'_j(g_j)g_j \tag{4.1.15}
\]

s.t. \[
u'_j(g_j) = k, \quad \forall j \in \mathcal{JSPE}, \tag{4.1.16}
\]
\[
\sum_{j \in \mathcal{JSPE}} g_j \leq \max_R \left\{ Rf\left(\frac{W}{R}\right) \right\}, \tag{4.1.17}
\]
\[
g_j \geq 0, \quad \forall j \in \mathcal{JSPE}. \tag{4.1.18}
\]

**Proof:** To arrive at a contradiction, suppose there exists $\bar{g}$ and $\bar{k}$ such that

\[
\sum_{j \in \mathcal{JSPE}} u_j(\bar{g}_j)\bar{g}_j > \sum_{j \in \mathcal{JSPE}} u_j(\hat{g}_j)\hat{g}_j,
\]

\[
u'_j(\bar{g}_j) = \bar{k}, \quad \forall j \in \mathcal{JSPE}, \tag{4.1.19}
\]
\[
\sum_{j \in \mathcal{JSPE}} \bar{g}_j \leq \max \left\{ Rf\left(\frac{W}{R}\right) \right\}. \tag{4.1.20}
\]

We construct a vector $\bar{\tau}$ such that

\[
\bar{g}_j = \bar{r}_j f(\frac{W}{R}), \quad \forall j \in \mathcal{JSPE}, \tag{4.1.21}
\]

where $\bar{R} = \sum_j \bar{r}_j$. Such an $\bar{\tau}$ is guaranteed to exist due to (4.1.20). We have

\[
\bar{k} f\left(\frac{W}{R}\right) \sum_{j \in \mathcal{JSPE}} \bar{r}_j = \sum_{j \in \mathcal{JSPE}} u'_j\left(\bar{r}_j f\left(\frac{W}{R}\right)\right)\bar{r}_j f\left(\frac{W}{R}\right) \tag{4.1.22}
\]

\[
= \sum_{j \in \mathcal{JSPE}} u'_j(\bar{g}_j)\bar{g}_j \tag{4.1.23}
\]

\[
> \sum_{j \in \mathcal{JSPE}} u'_j(\hat{g}_j)\hat{g}_j
\]

\[
= \sum_{j \in \mathcal{JSPE}} u'_j(\hat{r}_j f(\hat{\gamma}))\hat{r}_j f(\hat{\gamma})
\]

\[
= \hat{q} \sum_{j \in \mathcal{JSPE}} \hat{r}_j. \tag{4.1.24}
\]

(4.1.22) and (4.1.23) follow from (4.1.19) and (4.1.21). (4.1.24) follows from the SPE feasibility.
4.1. Efficiency through profit maximizing prices

Writing the Service Provider objective as (4.1.15)-(4.1.18) has two benefits. First, it eliminates the mathematical complexity of dealing with the efficiency function by expressing user j’s utility in terms of a single variable gj. The effect of the efficiency function is captured by the single real number \( \max_R \{ Rf(\frac{W}{R}) \} \) in (4.1.17); this number represents the maximum total effective data rate the system can support. Second, isolating conditions for which the SPE is throughput maximizing and Pareto optimal becomes equivalent to isolating conditions for which the constraint (4.1.17) is tight. This will simplify our future analysis.

The interpretation of the service provider problem, when written in the form of (4.1.15)-(4.1.18), is as follows: the vector g represents the vector of effective data rates. The service provider chooses any price (and implicitly a corresponding allocation of effective data rates), subject to 2 constraints: (4.1.16) says that the user optimality conditions must be satisfied, and (4.1.17) says that the service provider cannot allocate more throughput than the system can support.

We next present a class of utility functions which guarantee that the subgame perfect equilibrium is efficient, as defined in Section 2.4. These utility functions are characterized by their relative risk aversions \( R_j(x) \), which was defined and interpreted in Section 1.4.3.

**Proposition 18.** Consider the Price-Rate Control Game. If \( R_j(x) < 1 \), \( \forall j, \forall x \), then the subgame perfect equilibrium is efficient.

**Proof:** From Lemmas 18.1 and 18.2, the Service Provider Objective can be written as:

\[
\max_{k, g_j \in \mathcal{SPE}} \sum_{j \in \mathcal{SPE}} u_j'(g_j) g_j \\
\text{s.t.} \quad u_j'(g_j) - k = 0, \quad \forall j \in \mathcal{SPE},
\]

\[
\sum_{j \in \mathcal{SPE}} g_j \leq \max_R \left\{ Rf \left( \frac{W}{R} \right) \right\},
\]

\[
g_j \geq 0, \quad \forall j \in \mathcal{SPE}.
\]

The constraint gradients are linearly independent (see Appendix C), so the problem admits Lagrange multipliers. Therefore, we can assign Lagrange multipliers \( \lambda_1, \ldots, \lambda_J \) to the first group.
4.1. Efficiency through profit maximizing prices

of constraints (4.1.26) and a Lagrange multiplier \( \mu \leq 0 \) to the remaining constraint (4.1.27) to arrive at the following Lagrangian function:

\[
L(k, g, \lambda, \mu) = \sum_{j \in \mathcal{J}_{SPE}} u'_j(g_j)g_j + \sum_{j \in \mathcal{J}_{SPE}} \lambda_j(u'_j(g_j) - k) + \mu \left( \sum_{j \in \mathcal{J}_{SPE}} g_j - \max_R \left\{ Rf \left( \frac{W}{R} \right) \right\} \right).
\]

By taking derivatives with respect to \( g_j \) and \( k \) respectively, and using the first order optimality conditions, we obtain the following:

\[
\begin{align*}
\sum_{j \in \mathcal{J}_{SPE}} u''_j(g_j)g_j + u'_j(g_j) + \lambda_ju'_j(g_j) + \mu &= 0, \quad \forall j \in \mathcal{J}_{SPE}, \\
- \sum_{j \in \mathcal{J}_{SPE}} \lambda_j &= 0.
\end{align*}
\]

We claim that \( \mu \neq 0 \) in (4.1.29). To arrive at a contradiction, assume that \( \mu = 0 \). Summing (4.1.29) over \( j \) and using (4.1.30), we have

\[
\sum_j \left\{ \frac{u'_j(g_j) + u''_j(g_j)g_j}{u''_j(g_j)} \right\} = 0.
\]

If \( R_j(x) < 1 \), for all \( j \) and for all \( x \), then

\[
u'_j(x) + xu''_j(x) > 0, \quad \forall j, \quad \forall x,
\]

which implies

\[
\sum_j \left\{ \frac{u'_j(g_j) + u''_j(g_j)g_j}{u''_j(g_j)} \right\} < 0,
\]

yielding a contradiction. This implies that \( \mu \neq 0 \), and by the complementary slackness condition, constraint (4.1.27) holds with equality; see [44]. Therefore, the SPE is throughput maximizing, and by Proposition 8 it is efficient. \( \square \)

One can view \( \mu \) as the marginal cost of the constraint (4.1.27) from the service provider’s viewpoint (i.e., if the service provider were able to increase the system’s capacity by a marginal amount \( \delta \), the resulting profits would increase by \( \delta \mu \)). When the relative risk aversion condition of Proposition 18 holds, this marginal cost is guaranteed to be positive, so the limiting factor
4.1. Efficiency through profit maximizing prices

for the service provider's profits will always be the maximum throughput of the system; see Figure 4.1.

An important implication of this fact is that the service provider always has an incentive to upgrade the system's capacity by, for example, increasing the bandwidth \( W \) or implementing a better error correction code to improve the characteristics of the efficiency function \( f(\cdot) \).

\[ Rf\left(\frac{W}{R}\right) \]

Figure 4-2: System throughput \( Rf\left(\frac{W}{R}\right) \) as a function of \( R \). For \( \mu \neq 0 \), the operating point is throughput maximizing and Pareto optimal, shown in the Figure. For any other operating point, \( \mu = 0 \).

**Example 2:** Consider the class of utility functions \( u_j(x) = a_j \ln(1 + b_j x) \), where \( a_j \) and \( b_j \) are user dependent parameters. In this case \( R_j(x) = \frac{b_j x}{1+b_j x} < 1 \), implying that the SPE will be Pareto optimal and throughput maximizing regardless of the values of \( a_j \) and \( b_j \).

**Example 3:** Consider the class of utility functions \( u_j(x) = a_j x^{b_j} \), where \( a_j \) and \( b_j \) are user dependent parameters, and \( 0 \leq b_j \leq 1 \) for concavity. We have \( R_j(x) = b_j - 1 < 1 \), so regardless of the \( a_j \) and \( b_j \), the SPE will be Pareto optimal and throughput maximizing.

**Example 4:** Consider linear utility functions \( u_j(x) = a_j x + b_j \) where \( a_j > 0 \) and \( b_j \geq 0 \). \( R_j(x) \) is not well defined for linear utility functions, but users with linear utilities have no risk aversion, and are typically referred to as "risk-neutral." Therefore, we expect the SPE to be efficient. For this case, Condition 1 of Proposition 18 simplifies to

\[ a_j + \mu = 0. \]
Therefore, $\mu \neq 0$, proving that the outcome is throughput maximizing and Pareto optimal.

For classes of utility functions where $R_j(x) < 1$ is not guaranteed to hold for all $j$ or for all $x$, it is still possible that system throughput will be maximized. However, whether or not this occurs will depend on the specific parameters of the utility functions.

**Example 5:** Consider the class of utility functions $u_j(x) = a_j(1 - e^{b_jx})$. We have $R_j(x) = b_jx$. Thus, the SPE may not be system efficient. In the next section we will show that whether or not this occurs depends on the specific values of the $b_j$.

## 4.2 Characterization of the SPE

In this section, we explicitly characterize the SPE price. We also present examples which demonstrate how we can characterize the SPE operating point.

### 4.2.1 SPE Price characterization

We first give the following preliminary result, which shows that if the SPE is efficient, then the SPE price and rate vector must be uniquely defined.

**Lemma 19.1** Consider an efficient SPE $(q^*, r^*)$ of the Price-Rate Control Game with corresponding SIR $\gamma^*$. There does not exist another efficient SPE $(q, r)$.

**Proof:** By Proposition 8, $f(\gamma^*) = \gamma^* f'(\gamma^*)$ if and only if

$$\sum_{j \in J} r_j^* f(\gamma^*) = \max_{R} \left\{ Rf \left( \frac{W}{R} \right) \right\}.$$  

To find a contradiction, suppose there exists another SPE $(q, r)$ with corresponding SIR $\gamma$ such that $\sum r_j f(\gamma) = \max_{R} \{Rf(\frac{W}{R})\}$. By Proposition 8, $\gamma^* = \gamma$, and $R^* = R$. If $q \neq q^*$, we have

$$q \sum r_i = qR \neq q^* R^* = q^* \sum r_i^*,$$

which is a contradiction since an SPE must be profit maximizing. Therefore, $q = q^*$. If $r_i \neq r_i^*$,
4.2. Characterization of the SPE

for some \( i \in J_{SPE} \), then we have

\[
q = \frac{u'_i(r_i f(\gamma)) f(\gamma)}{u''_i(r_i f(\gamma)) f(\gamma)} \neq \frac{u'_i(r'_i f(\gamma^*)) f(\gamma^*)}{u''_i(r'_i f(\gamma^*)) f(\gamma^*)} = q^*,
\]

which is a contradiction. \( \square \)

**Proposition 19.** Let \((q, r)\) be an SPE to the Rate Selection Game, with corresponding SIR \( \gamma \) and sum rate \( R \). Let \( g_j = r_j f(\gamma) \) denote user \( j \)'s effective data rate. Let \( J_{SPE} = \{ j | r_j > 0 \} \).

Then,

\[
q = \begin{cases} 
-Rf(\gamma)/\sum_{j \in J_{SPE}} \frac{1}{u''_j(r_j f(\gamma))}, & \text{if } f(\gamma) \neq \gamma f'(\gamma), \\
\frac{u'_j(r'_j f(\gamma^*)) f(\gamma^*)}{u''_j(r'_j f(\gamma^*) f(\gamma^*)}, & \text{for any } j, \text{ if } f(\gamma) = \gamma f'(\gamma),
\end{cases}
\]

where \( r^* \) and \( \gamma^* \) are defined in Lemma 19.1.

**Proof:** Using user \( j \)'s first order optimality condition \( q = \frac{u'_j(r_j f(\gamma)) f(\gamma)}{u''_j(r_j f(\gamma)) f(\gamma)} = u'_j(g_j) f(\gamma) \) and (4.1.29) of Proposition 18, we have

\[
u''_j(g_j) g_j f(\gamma) + q + \lambda_j u''_j(g_j) f(\gamma) + \mu f(\gamma) = 0, \quad j \in J_{SPE}.
\]

This implies that

\[
\lambda_j = \frac{u''_j(g_j) g_j f(\gamma) + q + \mu f(\gamma)}{-u''_j(g_j) f(\gamma)}.
\]

Summing over \( j \) and (4.1.30), we have

\[
\sum_{j \in J_{SPE}} \left\{ \frac{q + u''_j(g_j) g_j f(\gamma) + \mu f(\gamma)}{u''_j(g_j) f(\gamma)} \right\} = 0,
\]

which implies

\[
\sum_{j \in J_{SPE}} \left\{ \frac{q}{u''_j(g_j) f(\gamma)} + g_j + \frac{\mu}{u''_j(g_j)} \right\} = 0.
\]

Letting \( g_j = r_j f(\gamma) \), we have

\[
R f(\gamma) + \frac{q}{f(\gamma)} \sum_{j \in J_{SPE}} \frac{1}{u''_j(r_j f(\gamma))} + \mu \sum_{j \in J_{SPE}} \frac{1}{u''_j(r_j f(\gamma))} = 0,
\]
from which we obtain
\[ q = \frac{Rf^2(\gamma)}{-\sum_{j \in J_{SPE}} u_j'(r_jf(\gamma))} - \mu f(\gamma). \]  

(4.2.2)

If \( f(\gamma) \neq \gamma f'(\gamma) \), then \( \mu = 0 \) by the complementary slackness condition of (4.1.27); see [44]. If \( f(\gamma) = \gamma f'(\gamma) \), then Lemma 19.1 applies. \( \square \)

Proposition 19 shows that if \( f(\gamma) \neq \gamma f'(\gamma) \), the SPE price depends on the second derivatives of the user utility functions. But surprisingly, if \( f(\gamma) = \gamma f'(\gamma) \), this dependency no longer holds; no matter what the second derivatives of the user utility functions are, the SPE price and rates are uniquely defined.

### 4.2.2 Characterizing the SPE for efficient equilibria

In situations where the SPE is guaranteed to be efficient, such as when the relative risk aversion condition of Proposition 18 holds, computing various quantities at the SPE is straightforward. One can solve the equation \( f(\gamma) = \gamma f'(\gamma) \) to uniquely determine the SPE signal quality \( \hat{\gamma} \). Furthermore, we have

\[
\text{SPE sum rate of data transmission: } \hat{R} = \frac{W}{\hat{\gamma}},
\]

\[
\text{SPE system throughput: } \hat{R}f(\hat{\gamma}),
\]

\[
\text{SPE profit: } \hat{q}\hat{R}.
\]

where \( \hat{q} \) is given in Proposition 19.

**Example 6:** Assume the condition of Proposition 18 holds. We assume that \( W = 4.096 \times 10^6 \) chips per second, and that the modulation scheme is non-coherent FSK with no error coding and 50 bits per frame. This gives an efficiency function of

\[ f(\gamma) = \left(1 - \frac{1}{2}e^{-\frac{\gamma}{2}}\right)^{50}. \]

Then, we can calculate several system parameters with no other knowledge of the utility...
functions. Using the method described above, we have:

Signal quality: $\gamma = 9.579$

Frame success rate: $f(\gamma) = 0.8119$

Total uplink data transmission rate: $\sum r_j = 427.602$ kbps.

Effective uplink data rate: $\sum r_j f(\gamma) = 347.17$ kbps.

These results hold for any utility functions, as long as they satisfy the condition of Proposition 18. For the special case of homogenous users, all the users will choose the same data rate at equilibrium, and so they will each get an equal share of the net throughput. For 50 users, this results in each user achieving an effective data rate of 6.943 kbps.

4.2.3 Characterizing the SPE for inefficient equilibria

Example 7: To illustrate the point that efficiency may or may not result if the utility functions do not satisfy the condition of Proposition 18, we consider the family of utility functions from Example 5. Specifically, we consider the case of homogeneous users and let

$$u_j(x) = u(x) - e^{-Kx}, \quad \forall j.$$ 

For the systems parameters given in Example 6 and for 50 users, it can be shown that if $K < 1/6943$, we will get a system and Pareto efficient solution, and the same numerical results given in Section 4.2.2 apply. However, if $K > 1/6943$, each user's equilibrium effective data rate will be $1/K$. For example, if $K = 1/500$, then each user's equilibrium effective data rate will be 500 bps, a significant loss of efficiency when compared to 6.943 kbps. In fact, there is no theoretical limit on the efficiency loss, since choosing an arbitrarily large value of $K$ under this scenario can yield an arbitrarily small system throughput.
Chapter 5

Extension to general networks

In this section, we consider the potential of applying the results of this thesis to the general network setting, including wireline networks. Previous game theoretic models for general networks have modelled congestion by including an additive latency cost function in the user's payoff (see, for example, [1] or [7]). For the case of a single link shared by all users, the payoff function is:

\[ u_j(r_j) - l(R)r_j - qr_j, \]  

(5.0.1)

where \( l(\cdot) \) is a flow dependent latency function and \( l(R)r_j \) represents the monetary cost incurred by the user due to the congestion delays. This latency function depends on the sum data rate of all users.

In this section, we show how the Wardrop Equilibrium characterization of Chapter 2 for wireless cellular networks directly lends itself to the study of more general networks in which multiple users access shared links. The resulting payoff function may be more natural than the additive latency term given in Equation (5.0.1). Instead of viewing delays caused by congestion as a separate cost in the payoff, we instead assume that congestion causes delays which reduce the net throughput of the transmitter. Therefore, we consider a payoff of the form:

\[ u_j(r_j C(R)) - qr_j. \]  

(5.0.2)

where we refer to \( C(R) \in (0,1) \) as the congestion function, which scales the transmitted data rate \( r_j \) down to the realized throughput \( r_j C(R) \).

We will show that for reasonable network models of protocols often used in general networks, including wireline, the function \( C(R) \) satisfies exactly the same assumptions as the efficiency function \( f(R) \), where for clarity we view \( f(\cdot) \) as a function of \( R \) instead of \( \gamma = \frac{W}{R} \).
We illustrate the idea with an example.

**Example 2:** Consider the slotted Aloha multiple access protocol; see [6]. Let \( R \) represent the rate of attempted transmissions and \( RC(R) \) represents the packet delivery rate. In a simple ALOHA multiple access model, \( C(R) = e^{-R} \). This corresponds to an efficiency function of \( C(R) \equiv f(\gamma) = e^{-\frac{1}{\gamma}} \), which can be shown to satisfy Assumption 1. 

The same procedure can be applied to more complex models of Aloha, such as unslotted Aloha:

\[
C(R) = e^{-2R},
\]

One can follow the method of Example 2 to find the equivalent efficiency function \( f(\gamma) \), and verify that Assumption 1 is satisfied.

Intuitively, the key similarity between wireless efficiency functions and congestion functions like those mentioned is shown in Figure 5-1. This shows the system’s total effective data rate as a function of total data rate transmitted by the users. When the total input data rate \( R \) is low, there is very little interference, and \( f(\gamma) \cong 1 \). Therefore, the total effective data rate is approximately equal to the input data rate. This corresponds to the linear regime of Figure 5-1, labelled as Region I. As the input data rate increases, the system suffers from more interference, and the effective data rate is less than the input data rate; Region II. As the input data rate continues to grow, interference levels result in the effective data rate decreasing, until it eventually decays towards 0 in the limit; Region III.

This is the same phenomenon that occurs in typical general multiple access networks; a general discussion of this is given in [48].

We note that viewing users are “congestion-takers” in wireline networks is a widely-used and well-justified assumption (see, for example, [1]). Therefore, the Wardrop Equilibrium notion of Chapter 2 extends to the case of these general networks.

Based on the preceding discussion, we see that the interference model developed for wireless CDMA networks employing an SIR-balancing power control objective directly applies to other network settings with the identification \( C(R) \sim f(R) \).
Figure 5-1: The total effective data rate as a function of the total input rate.
Chapter 6

Conclusion and future directions

6.1 Main contributions

This thesis considers the problem of resource allocation in wireless cellular networks carrying elastic data traffic. It follows in the footsteps of other authors who have used fundamental economic principles, most notably pricing, the modeling of end-user behavior with utility functions, and equilibrium analysis, in the study of other large scale engineering systems. Albeit with slight difficulty due to the non-convex functional forms associated with the wireless setting, the thesis showed that fundamental results seen in wireline networks also hold for the wireless case considered here. Notably, there exists a price under which the network’s limited resources are used with full efficiency.

However, due to the unique characteristics of the wireless setting, this thesis shows surprising results that would not be expected in the wireline network setting. In particular, profit maximizing pricing by a monopolist does not, in general, lead to a loss of efficiency. Expressed in terms consistent with previous literature [1], there is no monopolist’s markup from the Pigovian tax for a broad class of utility functions.

To arrive at the results, we focus on the Wardrop Equilibrium notion. We first prove the existence of a Nash Equilibrium, but show that this particular equilibrium notion suffered from two flaws: lack of mathematical tractability, and extensive end-user information requirements that make it impractical to implement. To address these issues, we shift focus to the Wardrop Equilibrium notion by employing two well-known limiting arguments in conjunction: large system analysis, taken from the spread spectrum wireless network literature, and replication, taken from the economics literature.

The Wardrop Equilibrium, and the fact that a Pigovian tax exists, inspire a rate-control
algorithm. Its stability properties are studied, and then improved upon with dynamic pricing.

6.2 A future direction

This thesis represents a first step in the application of pricing and equilibrium analysis to wireless networks. While this thesis provided a comprehensive study for rate control in cellular networks, the multi-hop wireless network case should prove to have a rich structure.

This thesis considers uplink in a cellular network, which gives rise to many simplifying symmetries: All users attempt to use the same frequency band to send data to the same destination. The max-min power control results in all users experiencing the same signal quality. The limiting arguments used to arrive at the Wardrop Equilibrium notion result in all users having the same marginal impact on the system’s SIR; the notion of a “bottleneck user” is swept away in the limit.

As a result, one can think of this as a single resource problem. In full analogy with wireline networks, it is as if there is a single wired link that goes to the base station, and all users transmit on, and congest, this link. The extension presented in Chapter 5 adds rigor to this viewpoint.

Literature in wireline networks has shown that if one is allowed to choose a price for each link in the network, efficiency gains can be realized. Due to the single link interpretation of the cellular system studied in this thesis, it is not surprising that only one price was required to achieve efficiency.

The most interesting future direction involves breaking the symmetry by allowing there to be multiple destinations and routing choices for the transmitters. For multiple destinations, it is reasonable to expect that each destination will need to have its own price in order to achieve efficiency. If nodes are also given flexibility to make routing decisions, we can expect another layer of heterogeneous pricing to emerge in order to achieve efficiency.

Finally, this thesis was restricted to elastic data users. There are fundamental differences in the behavior of transmitters of elastic data versus inelastic data; see Chapter 1.4. There will undoubtedly be numerous mathematical challenges in extending this kind of analysis to the step utility functions associated with transmitters of inelastic data.
6.2. A future direction

This thesis provides another example of how the study of a network based on pricing, utility functions, and equilibrium analysis can lead to algorithms with very low communication overhead, but which differentiate the service offered to users with heterogeneous preferences. A unifying framework in this context must extend these features to general network topologies and include both elastic and inelastic traffic streams. If successful, such a framework will provide the backdrop for promising resource allocation algorithms and protocols.
Appendix

7.1 Appendix A

In this appendix, we recall the background for basic results in game theory and stability analysis.

7.1.1 Nash Equilibrium and its existence

See [40] for a thorough coverage of game theory concepts.

Definition 11. (Quasiconcavity) Let $f$ be a function defined over a convex set $S$. Let $U_c = \{x \in S | f(x) \geq c\}$ denote the upper level set of $f$ for $c$. The function $f$ is quasiconcave if $U_c$ is convex for all $c$.

A pure strategy Nash Equilibrium is an action profile in which no player can improve his payoff through unilateral deviation. Formally:

Definition 12. (Nash Equilibrium) A Nash Equilibrium of the Rate Selection Game under price $q$ is a vector $r^* \in [0, r_{max}]$ such that for all $j$

$$u_j (r_j^* f (\gamma (r_j^*; r_{-j}^*))) - qr_j^* \geq u_j (r_j f (\gamma (r_j; r_{-j}^*))) - qr_j^*, \text{ for all } r_j^* \in [0, r_{max}].$$

Kakutani’s Fixed Point Theorem is commonly used to prove the existence of a Nash Equilibrium.

Proposition 20. (Kakutani) Consider a mapping $f : A \to A$, where $A$ is a nonempty, convex, compact subset of a finite dimensional Euclidean space. If $f(x)$ is nonempty for all $x \in A$, if
$f(x)$ is convex for all $x \in A$, and if $f(x)$ has a closed graph, then there exists a fixed point $x \in f(x)$.

The following existence Proposition follows from Kakutani's Fixed Point Theorem.

**Proposition 21.** Consider a game with $J$ players, where each player $j$ chooses an action $a_j$ in some action space $A_j$, and receives and a payoff function $u_j(a)$. If $A_j$ is a nonempty, convex, and compact subset of a Euclidean space, and if $u_j(a_j; a_{-j})$ is continuous in $a$ and quasiconcave in $a_i$, then there exists a Nash Equilibrium.

### 7.1.2 Stability of dynamic systems

This section defines terms and gives preliminary results relevant to the stability analysis presented in Chapter 3. Unless otherwise stated, the material is paraphrased from [8].

Suppose that the discrete time system

$$x(k + 1) = f(x(k)), \quad (7.1.1)$$

has an equilibrium point $x^*$. Without loss of generality, assume $x^* = 0$ (if the equilibrium is at any other value, the following will hold after a change of variables that moves the equilibrium to the origin). We have the following definition.

**Definition 13.** A system is locally stable around its equilibrium point (at the origin) if it satisfies the following two conditions:

1. Given any $\epsilon > 0$, there exists $\delta_1 > 0$ such that if $\|x(t_0)\| < \delta_1$, then $\|x(t)\| < \epsilon$, for all $t > t_o$.
2. There exists $\delta_2 > 0$ such that if $\|x(t_0)\| < \delta_2$, then $x(t) \to 0$ as $t \to \infty$.

Write

$$f(x) = hx + e(x),$$
where $h$ is the Jacobian associated with the linearization of (7.1.1) and $e(x)$ denotes higher order terms; i.e.,

$$\lim_{\|x\| \to 0} \frac{\|e(x)\|}{\|x\|} = 0.$$  

We refer to the system

$$x(k + 1) = hx(k),$$  

(7.1.2)

as the linearized system. We have the following.

**Proposition 22.** If the linearized system (7.1.2) is locally stable, then the equilibrium point of the system (7.1.1) is locally stable.

**Proposition 23.** Consider a linear system of the form (7.1.2), and let $\lambda_1, ..., \lambda_n$ denote the eigenvalues of $h$. If $|\lambda_i| < 1$ for all $i$, then the system is locally stable.

### 7.2 Appendix B

In this section, we justify Assumption 1.5, stated here for convenience.

**Assumption 1.5** Let $C \in \mathbb{R}^+$ and let $h(x) \equiv f(\frac{x}{1+C})$. There exists $\hat{x}$ such that $\forall x < \hat{x}$, $h(x)$ is strictly convex, and $\forall x > \hat{x}$, $h(x)$ is strictly concave.

First, we show that this assumption is needed for the proof of Proposition 2, Lemma 2.3. Consider the following function $f$:

$$f(x) = \begin{cases} 
  x^2, & x \leq .5 \\
  x - .25, & .5 \leq x \leq .9 \\
  10x - 8.35, & .9 \leq x \leq .935 \\
  1, & x \geq .935 
\end{cases}$$  

(7.2.1)

Although $f$ violates Assumptions 1.1 and 1.4, one can construct a function $f_\epsilon$ which is a continuously differentiable approximation of $f$; i.e. an $\epsilon$-perturbed version of $f$ that satisfies Assumptions 1.1-1.5 by "smoothing" the corners of $f$ and slightly convexifying the linear portions.
Let $h(x) = f(x)$, and let $h_C(x) = f\left(\frac{x}{1+C}\right)$. $h(x)$ is plotted for $C = 1$. For our purposes, it is enough to note that $h_C(x)$ will look like $h(x)$ with rounded corners. The proof of Lemma 2.3 relied on the fact that there is only one point on the $h_C(x)$ curve which is a tangent point to a line passing through the origin. $h_C(x)$ will have 3 such points; the corresponding points on $h(x)$ are marked by arrows. Since each of these points corresponds to a stationary point of $r_j f\left(\frac{W}{r_j+C}\right)$, Lemma 2.3 does not follow. In fact, by plotting $r_j f\left(\frac{W}{r_j+C}\right)$ for $W = 1, C_1 = 1$, one can see that one of the stationary points is a local minimum and the other two are local maxima, implying that the function is not quasiconcave.

However, Assumption 1.5 is natural for efficiency functions. It can be proven for all efficiency functions with no error correction codes in which the bit error rate decays exponentially as a function of $\gamma$, and can be shown by simulation for other efficiency functions.

**Proposition 24.** Consider any modulation scheme with BER $Ae^{-B\gamma}$, where $A$ and $B$ are positive constants. Then, Assumption 1.5 holds.

**Proof:** In this case, we have

$$f\left(\frac{x}{1+Cz}\right) = (1 - Ae^{-B\gamma})^M.$$  \hspace{1cm} (7.2.2)

After taking the second derivative of (7.2.2), setting it equal to 0, and simplifying, we have the
following condition:

\[ e^{-\frac{x}{1+\varepsilon x}}(A(M - 1) + 1) - 1 = \left(1 - e^{-\frac{x}{1+\varepsilon x}}\right)2C(1 + Cx). \]  

(7.2.3)

Proving that Assumption 1.5 holds is equivalent to showing that there is one unique point satisfying (7.2.3). The left hand side of (7.2.3) is strictly positive for \( x = 0 \) (recall that \( M > 1 \) by assumption), and is strictly decreasing for all \( x > 0 \). The right hand side is 0 for \( x = 0 \), and is strictly increasing and unbounded for \( x > 0 \). Therefore, there is a unique value \( \hat{x} \) for which (7.2.3) holds.

\[ \square \]

7.3 Appendix C

In order for the Karush-Kuhn-Tucker conditions to be necessary at an optimal solution to an optimization problem of the form (4.1.25) - (4.1.27), all equality constraint gradients and active inequality constraint gradients must be linearly independent. In this section, we prove this linear independence.

The constraint gradients of (4.1.26) and (4.1.27) are given in the following matrix; the the \( ij^{th} \) entry is the partial derivative of the \( i^{th} \) constraint with respect to the \( j^{th} \) variable. Rows 1
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to \( J \) correspond to constraint (4.1.26), row \( J + 1 \) corresponds to constraint (4.1.27), columns 1 to \( J \) correspond to the variables \( g_1 \) to \( g_J \), and column \( J + 1 \) corresponds to the variable \( k \).

\[
A = \begin{bmatrix}
  u''(g_1) & 0 & 0 & \ldots & 0 & -1 \\
  0 & u''(g_2) & 0 & \ldots & 0 & -1 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & u''(g_J) & -1 \\
  1 & 1 & 1 & \ldots & 1 & 0
\end{bmatrix}
\]

Let \( a_i \) denote the \( i \)th column vector. To find a contradiction, assume that there exists \( \lambda_1, \ldots, \lambda_J \) such that \( \sum_{j=1}^{J} \lambda_j a_j = a_{J+1} \). Since \( a_i(J+1) = 1 \) for all \( i = 1, \ldots, J \), and \( a_{J+1}(J+1) = 0 \), it must be that \( \lambda_i < 0 \) for some \( i = 1, \ldots, J \). Since \( \lambda_i u''(g_i) = -1 \), it follows that \( u''(g_i) > 0 \).

This contradicts Assumption 2.2. Therefore, the matrix \( A \) is full rank, and the rows are linearly independent. \( \Box \)


BIBLIOGRAPHY


