A SYSTEMATIC APPROACH TO MODELING FOR STABILITY-ROBUSTNESS IN CONTROL SYSTEM DESIGNS

by

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A Systematic Approach to Modeling for Stability-Robustness in Control System Designs

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ABSTRACT

The following is a presentation of an approach to modelling which facilitates the design of robust control systems. Specifically, a complex system is described by a set of linear, time-invariant models, each one representing the system at some particular operating point. Next, a controller is designed at each operating point with the intention of having the control system simply switch to the appropriate controller based upon the current operating condition. In order to ensure the robustness of this control scheme, constraints on the design of each controller are found which will ensure that no right-half plane poles appear when the actual operating condition is in between the modelled operating points. These constraints take the form of boundaries in the frequency domain which the controller's Bode magnitude plot must not violate. If these constraints prevent the controller from meeting its performance requirements, they can be relaxed, but only at the expense of adding more modelled operating points each with their own associated controllers.

This system modeling methodology is demonstrated through its application to the design of autopilots for two different tail controlled missile models. The first model varies with altitude and is used to show the ability of this approach to guarantee stability in time-invariant control systems. The second model is not time-invariant and is used to demonstrate that this method can also be useful in the design of time-varying systems.

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BIOGRAPHICAL NOTE

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INTRODUCTION

1.1 Linear Systems Concepts

In the real world, all physical systems are nonlinear. However, (1) with only a very few exceptions, we can use linear models to describe these systems provided that we sufficiently restrict the region of operating conditions over which we expect them to be valid. Hence, we can represent a physical system by a set of linear models if each model covers a small enough region of the operating conditions. Whenever possible, we use linear models to describe physical systems because of the great mathematical simplification which accompanies linearization. Unlike nonlinear systems, there are many powerful methods for dealing with linear systems. Ever since the early 1930s, (2) extensive research has been done in the area of linear dynamical systems. This research has led to many important frequency domain techniques. However, during the 1960s, the interest in time-varying problems and aerospace problems with time domain characteristics led to a renewal of interest in state-space techniques. This research produced many important design methods including pole-shifting controllers, quadratic regulators, state estimators and state observers.

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Linear systems are ones which have linearity with respect to their initial state and input and have an output which is the sum of their zero-input and zero-state response. Furthermore, linear systems can be either time-invariant or time-varying in nature. Linear, time-invariant (LTI) systems are by far the easiest to deal with mathematically. Moreover, the stability of LTI systems can be determined from the eigenvalues of the system. If all of the eigenvalues of the system are in the left-half of the complex plane, then we know that the system is stable. Linear, time-varying systems, on the other hand, are not so easily analyzed. Unlike the LTI case, there is no general closed-form solution for the state transition matrix, making exact solutions difficult. Furthermore, questions regarding stability cannot be answered by simply checking the system's characteristic root locations.
1.2 Problem Statement

A typical approach to design a control system for a complex physical system is to first develop a family of linear, time-invariant models, each one associated with some particular operating point in the physical system's envelope. Next, compensation is designed around each of these models, and finally some scheme of "gain scheduling" is devised to switch the control system to the appropriate compensator given the current operating conditions. If the initial modelling is accurate, the final system should be able to meet a reasonable set of specifications. However, if the model done during the initial phases of the design process is inadequate, then the designer may have to go back and rethink his modelling decisions and redesign the whole control system. Clearly, such an iterative design methodology is not very efficient. What we would like is some method of determining the reasonableness of the family of linear, time-invariant models which we initially develop. If the modelling does not appear to be adequate, more or different operating points can be added to correct the problem before an attempt is made to design the control system. In this way, the designer can approach the task of designing the control system in a more systematic and efficient fashion.
We can define a reasonable family of models as one in which each model can be utilized to design a feedback control system which meets the necessary design specifications at the operating point and at the same time is at least stable over some specified region around the operating point. Assuming that the control gain scheduling scheme is designed using these same specified regions to choose which compensator will be used, this definition of reasonableness implies that a control system can be designed which will always be stable, even though the system is designed using a finite number of linear, time-invariant models. Moreover, it also implies that this system will not sacrifice too much performance, as measured at the operating points, to achieve this guarantee of stability.

What we require is a method of determining the compensator design constraints for each model which will ensure stability within the specified region around the operating point. If these constraints are so severe that they prevent the control system from meeting its necessary design specifications, then the modelling is inadequate. However, if the constraints are realistic, then the modelling is reasonable as far as this particular operating point is concerned. We also know what constraint we should use in our design of this individual compensator in order to ensure stability of the final system.
2.1 Lehtomaki's Result

In an effort to determine what constraints are necessary to ensure stability within some specified region of operating conditions, we begin by considering the following result obtained by Lehtomaki (4). If a system with an open-loop transfer function $T(s)$ and a closed-loop transfer function $C(s)$ is perturbed, the new system can be represented by the open-loop transfer function $T'(s)$ and the closed-loop transfer function $C'(s)$. Now, $C'(s)$ will have no right-half plane poles if the following conditions hold:

1.) a) $T(s)$ and $T'(s)$ have the same number of right-half plane poles.
   
   b) If $T'(s)$ has any poles on the imaginary axis, $T(s)$ has these same poles too.
   
   c) $C(s)$ has no right-half plane poles.

2.) $|C(s)| < |E_m(s)|^{-1}$ where $E_m(s) = T'(s)T^{-1}(s) - 1$ is the multiplicative model error.
This result is based on the Nyquist stability criterion which states that (5), for a closed-loop system to be stable, the Nyquist plot of its open-loop transfer function must encircle the -1 point in the clockwise direction as many times as the number of right-half plane poles in the open-loop transfer function. From condition 1c, we know that \( C(s) \) is stable, and therefore the Nyquist plot of \( T(s) \) encircles the -1 as many times as the number of right-half plane poles of \( T(s) \).

From condition 1a, we see that \( T(s) \) and \( T'(s) \) have the same number of right-half plane poles, so for \( C'(s) \) to be stable, the Nyquist plot of \( T'(s) \) must have the same number of encirclements of the -1 point as \( T(s) \) does. Therefore, if we can continuously deform the Nyquist diagram of \( T(s) \) into the Nyquist diagram of \( T'(s) \) without passing the Nyquist locus through the critical point -1, and changing the number of encirclements, then \( C'(s) \) will be stable.

Fig. 1 shows that for any frequency the corresponding point of the Nyquist locus of the single-input single-output open-loop transfer function \( T(j\omega) \) can be represented by a vector sum (6).
\[ T(jw) = D(w) - 1 \]  
(eq. 1)

equivalently,

\[ D(w) = T(jw) + 1 \]  
(eq. 2)

Now, we define \( T'(jw) \) as follows, where \( L(jw) = T'(jw)T^{-1}(jw) \) is the model error.

\[ T'(jw) = L(jw)T(jw) \]  
(eq. 3)

Subtracting \( T'(jw) \) from both sides

\[ T'(jw) - T(jw) = L(jw)T(jw) - T(jw) \]  
(eq. 4)

or,

\[ T'(jw) - T(jw) = (L(jw) - 1)T(jw) \]  
(eq. 5)
Fig. 2 shows that for any frequency the distance between the corresponding point on the Nyquist locus of \( T(j\omega) \) and the corresponding point on the locus of \( T'(j\omega) \) is given by \( |T'(j\omega) - T(j\omega)| \). In order for the Nyquist locus of \( T'(s) \) to continuously deform into the Nyquist locus of \( T(s) \) without passing through -1, the distance \( |T'(j\omega) - T(j\omega)| \) must be less than the distance from \( T(j\omega) \) to the -1 point for every frequency. This can be expressed as:

\[
|T'(j\omega) - T(j\omega)| < |D(\omega)| \quad \text{(eq. 6)}
\]

substituting eq. 2,

\[
|T'(j\omega) - T(j\omega)| < |T(j\omega) + 1| \quad \text{(eq. 7)}
\]

substituting eq. 5,

\[
|(L(j\omega) - 1)T(j\omega)| < |T(j\omega) + 1| \quad \text{(eq. 8)}
\]

or,

\[
|L(j\omega) - 1| < |(T(j\omega) + 1)T^{-1}(j\omega)| \quad \text{(eq. 9)}
\]

using \( C(j\omega) = T(j\omega)(T(j\omega) + 1)^{-1} \),
\[
\left| L(j\omega) - 1 \right| < \left| C^{-1}(j\omega) \right| \quad \text{(eq. 10)}
\]

or,
\[
\left| C(j\omega) \right| < \left| L(j\omega) - 1 \right|^{-1} = \left| E_m(j\omega) \right|^{-1} \quad \text{(eq. 11)}
\]

Equation 11 is the same as condition 2. Therefore, condition 2 ensures that we can continuously deform the Nyquist diagram of \( T(s) \) into the Nyquist diagram of \( T'(s) \) without changing the number of encirclements of the critical point -1.

2.2 Numerical Example

In order to illustrate how these conditions relate to an actual system, we consider the following simple numerical example. The open-loop transfer function of the system is given by:
\[
T(s) = \frac{1}{s + 1} \quad \text{(eq. 12)}
\]

The open-loop transfer function of the perturbed system is given by:
\[
T'(s) = \frac{2}{s + 2} \quad \text{(eq. 13)}
\]
As we can see, $T(s)$ and $T'(s)$ each have no right-half plane poles and no poles on the imaginary axis. Therefore, conditions 1a and 1b are satisfied. The closed-loop transfer function of the system is given by:

$$C(s) = \frac{T(s)(T(s) + 1)}{s + 1} = \frac{1}{s + 2}$$  \hspace{1cm} \text{(eq. 14)}

This closed-loop transfer function has no right-half plane poles so condition 1c is satisfied. The model error is:

$$L(s) = T'(s)T^{-1}(s) = \frac{2(s + 1)}{(s + 2)}$$  \hspace{1cm} \text{(eq. 15)}

Hence, the multiplicative model error is:

$$E_m(s) = L(s) - 1 = \frac{s}{s + 2}$$  \hspace{1cm} \text{(eq. 16)}

In order that condition (2) be satisfied, the Bode magnitude plot curve of the closed-loop transfer function must lie below the reciprocal of the magnitude of the multiplicative model error. From Fig. 3, we can see that this is indeed the case. Hence, we know that the closed-loop transfer function of the perturbed system must have no poles in the right-half plane. The closed-loop transfer function of the perturbed system is:
This transfer function has one pole at -4 which is in the left-half plane as it must be. It should be remembered that condition two is a sufficient but not a necessary condition, since it was derived assuming the worst possible phase at each and every frequency. Hence, the multiplicative model error boundary in the frequency domain is a worst case constraint, and the poles of the closed-loop system may still reside in the left-half plane even if it is violated.

2.3 Theory of the Modelling Methodology

The result obtained by Lehtomaki gives a set of constraints which will ensure the closed-loop stability of a perturbed system. This result is now extended to guarantee the stability of a closed-loop system whose plant model lies within a specified range of operating conditions. First, let \( G_n(s) \) be the transfer function of the nominal plant model at the specific operating point and let \( G_a(s) \) be the transfer function of any other plant model within the specified region of operating conditions. Also, let \( K(s) \) represent the compensator. If we associate \( G_n(s) \) with the original system and \( G_a(s) \) with the perturbed system, then \( T(s) = K(s)G_n(s) \) and \( T'(s) = K(s)G_a(s) \).
Since both open-loop transfer functions have the same compensator, 
$T(s)$ and $T'(s)$ will have the same number of right-half plane 
poles if $G_n(s)$ and $G_a(s)$ have the same number of right-half 
plane poles. Similarly, $T(s)$ and $T'(s)$ will have their poles 
on the imaginary axis, if any, in the same locations if $G_n(s)$ 
and $G_a(s)$ have their poles on the imaginary axis, if any, in 
the same locations.

Solving eq. 3 for the model error,

$$L(j\omega) = T'(j\omega)T^{-1}(j\omega)$$  \hspace{1cm} (eq. 18)

using $T(j\omega) = K(j\omega)G_n(j\omega)$ and $T'(j\omega) = K(j\omega)G_a(j\omega)$,

$$L(j\omega) = G_a(j\omega)G_n^{-1}(j\omega)$$  \hspace{1cm} (eq. 19)

combining eq. 19 and eq. 11,

$$\left| \frac{C(j\omega)}{L(j\omega)} \right| < \left| G_a(j\omega)G_n^{-1}(j\omega) - 1 \right|^{-1} = \left| E_m(j\omega) \right|^{-1}$$  \hspace{1cm} (eq. 20)

Equation 20 states that the gain of the closed-loop system designed 
using the plant model at the given operating point must at all 
frequencies be less than the reciprocal of the multiplicative model 
error between this plant and any other plant model in the specified 
range of operating conditions. Hence, within some specified region 
of operating conditions, the closed-loop system will have no poles 
in the right-half plane if the following are true:
1.) All of the plant models within the specified region have the same number of right-half plane poles.

2.) All of the plant models have their poles on the imaginary axis, if any, in the same locations.

3.) The closed-loop system is stable at the specified operating point.

4.) The closed-loop system at the specified operating point has a Bode magnitude plot which lies below the reciprocal of the multiplicative model error between the plant at the specified operating point and any other plant model in the specified region of operating conditions.

In practice, conditions (1) and (2) above can be checked by observing the plant pole locations as a function of the operating conditions within the specified region. If the number of unstable poles or the locations of the imaginary axis poles changes then the specified region of operating conditions covered by the plant model at the operating point is too large. If this is the case, adjustments will have to be made to the region of the operating envelope covered by the plant model at this particular operating point. Further, more plant models at different operating points may have to be used in order to adequately model the system.
Condition (3) above is a basic consideration of any control system design, and therefore we can assume that regardless of the type of compensator we choose, this condition will be met. Condition (4) can be checked graphically by plotting the reciprocal of the magnitude of the multiplicative model error on a Bode plot for a representative set of operating conditions within the specified range of operating conditions. The region of the Bode plot which lies below all of these curves is the allowable region for the Bode plot of the closed-loop system at the specified operating point. If this constraint is unacceptable (it may limit the bandwidth of the closed-loop system to such an extent that the system performance will not meet necessary specifications), then again, the specified range of operating conditions covered by this plant model is too large. Adjustments will have to be made to the operating envelope of the plant model, and more plant models at other operating points may have to be added. If the constraints on the closed-loop frequency response are acceptable, when we have achieved adequate modelling of the real plant within the specified region by a linear, time-invariant model at the chosen operating point.
APPLICATIONS

3.1 Missile Model with Variable Altitude

In order to illustrate how this system modelling methodology can be applied to aid in the design of a control system for a plant model, we will consider the design of an autopilot for a simple flight vehicle. Specifically, we will examine the problem of designing an autopilot for a tail-controlled missile (appendix) whose actual aerodynamic model changes as a function of the flight conditions. We would like to be able to design the missile's control system using a nominal plant model which is valid at some particular set of flight conditions. However, we know that even if the autopilot is designed to perform well at this particular operating point, there is no guarantee that the system will even be stable at some other operating point. In order to overcome this problem, we will use the modelling methodology to develop a set of constraints on the autopilot design which will ensure that the closed-loop control system designed using the plant model at a particular operating point will be at least stable over a specific range of operating conditions. This guarantee of stability follows from the fact that the modelling methodology will ensure that all of the poles of this LTI system are strictly in the left-half plane. Moreover, by adjusting the range of operating conditions over which we expect the closed-loop autopilot system to be stable, we can control the severity of the design constraints and hence still meet the necessary design specifications of the control system. Of course,
if we limit the range of operating conditions covered by the control system designed at our chosen operating point, we will have to utilize a model of the system at another operating point to design a control system which covers the neglected range of operating conditions.

The nominal missile plant model is derived assuming that its velocity is Mach two at 2,000 feet altitude. In this application, we assume that the Mach number remains constant but that the altitude at which the missile operates can vary up to 40,000 feet. As the altitude at which the missile flies increases, the velocity of the missile (assuming constant Mach number) decreases. Also, the atmospheric density decreases. The decrease in velocity and atmospheric density lead to a corresponding decrease in dynamic pressure. This in turn causes the aerodynamic derivatives $M_\alpha$ and $M_\delta$ to increase and $Z_\alpha$ and $Z_\delta$ to decrease. Since the missile model is based on the values of missile velocity, $M_\alpha$, $M_\delta$, $Z_\alpha$ and $Z_\delta$, it is clear that any change in altitude will have a direct effect on the dynamics of the plant model. The actual plant models at 10,000, 20,000, 30,000, and 40,000 feet are as follows:

at 10,000 feet

$$\dot{X}(t) = \begin{bmatrix} 0 & -3591.48 \\ 1 & -3.236 \end{bmatrix} X(t) + \begin{bmatrix} -5306.09 \\ -1.814 \end{bmatrix} \hat{o}(t)$$

$$Y(t) = \begin{bmatrix} 0 & 6973.4 \end{bmatrix} X(t) + \begin{bmatrix} 3909.6 \end{bmatrix} \hat{o}(t)$$
at 20,000 feet,

\[
\dot{X}(t) = \begin{bmatrix} 0 & -2401.3 \\ 1 & -2.2483 \end{bmatrix} X(t) + \begin{bmatrix} -3547.71 \\ -1.2605 \end{bmatrix} \xi(t)
\]

\[
Y(t) = \begin{bmatrix} 0 & 4662.5 \end{bmatrix} X(t) + \begin{bmatrix} 2614.0 \end{bmatrix} \xi(t)
\]

at 30,000 feet,

\[
\dot{X}(t) = \begin{bmatrix} 0 & -1553.8 \\ 1 & -1.516 \end{bmatrix} X(t) + \begin{bmatrix} -2295.6 \\ -0.85 \end{bmatrix} \xi(t)
\]

\[
Y(t) = \begin{bmatrix} 0 & 3016.9 \end{bmatrix} X(t) + \begin{bmatrix} 1691.4 \end{bmatrix} \xi(t)
\]

at 40,000 feet,

\[
\dot{X}(t) = \begin{bmatrix} 0 & -970.0 \\ 1 & -0.972 \end{bmatrix} X(t) + \begin{bmatrix} -1433.1 \\ -0.5454 \end{bmatrix} \xi(t)
\]

\[
Y(t) = \begin{bmatrix} 0 & 1883.5 \end{bmatrix} X(t) + \begin{bmatrix} 1055.96 \end{bmatrix} \xi(t)
\]
Our goal is to design an autopilot for the missile, using the nominal plant model, which is guaranteed to be stable at any altitude in the specified range. Assuming that the missile plant's poles remain strictly in the left-half plane regardless of the altitude, this can be achieved by designing a compensator which gives a closed-loop system whose Bode magnitude plot lies below the reciprocal of the multiplicative model error between the nominal plant model and the actual plant model over the specified range of altitudes. If this constraint limits the closed-loop bandwidth too severely, preventing the system from meeting its necessary performance specifications, the range of altitudes covered by this particular operating point and its associated control system can be reduced. The remaining range of altitudes would then have to be covered by another closed-loop control system designed using a second plant model at another operating point. Finally, an altimeter would be required to switch to the appropriate compensator depending upon the altitude of the missile.

Fig. 4A shows the locations of the missile plant's poles as a function of the altitude. From this plot, we can see that as the missile operates at higher and higher altitudes, the complex conjugate pair
of plant model poles become more oscillatory and more heavily damped. Moreover, the plant's poles always remain strictly in the left-half plane. In other words, all of the missile models have the same number of right-half plane poles (none) and no poles on the imaginary axis. Hence, we can guarantee the stability of the closed-loop autopilot system by simply keeping the nominal (2,000 foot altitude plant model) Bode magnitude plot curve of the closed-loop system below the curves of the reciprocal of the multiplicative model error over the specified range of operating altitudes. Fig. 4B shows the curves of the reciprocal of the multiplicative model error between the nominal 2,000 foot altitude plant model and several other plant models at different altitudes. From this figure, we can see that the curves of the reciprocal of the multiplicative model error corresponding to higher and higher altitudes crossover the zero decibel line at increasingly lower frequencies. Furthermore, since the allowable region for the closed-loop frequency response curve lies below these curves, we can see that as we try to guarantee closed-loop stability at higher altitudes, the constraints on the bandwidth of our closed-loop system become more severe. Hence, there is a trade-off between guaranteeing the closed-loop stability of the system at higher and higher altitudes and performance as measured at the chosen 2,000 foot altitude operating point.
Since the constraints on the design of our closed-loop control system are only sufficient and not necessary conditions for closed-loop stability at other altitudes, we will first examine the consequences of designing a closed-loop system which violates the constraints. Compensator $K_1$ (Fig. 5A) was chosen to give a closed-loop system with a frequency response of unity (for good command-following performance) up to about 25 radians/second before rolling off. With this much closed-loop system bandwidth, the performance of the closed-loop system is quite good as demonstrated by its step response (Fig. 5B). This step response has a rise time of only about one tenth of a second with no overshoot or oscillation. However, this result is only valid at an altitude of 2,000 feet. As we can see from Fig. 5C, the closed-loop frequency response curve of this system cuts across all of the curves of the reciprocal of the multiplicative model error. This means that there is no guarantee that at higher altitudes, this compensator will yield a closed-loop system which is even stable. From Fig. 5D, we see that before even reaching 10,000 feet, the changes in the missile's plant model, due to the difference in altitude, causes poles to appear in the right-half plane of the closed-loop system. Clearly, it will be necessary to reduce the bandwidth of the closed-loop system and heed the multiplicative model error constraints in order to construct a system which will not go unstable at higher altitude.
The second compensator that we will consider (Fig. 6A), is designed to give a closed-loop system with a frequency response of unity up to about 15 radians/second before it rolls off. This system is much like the previous one which we considered except that it has slightly less closed-loop bandwidth. Note that with this reduction in bandwidth, the step response of the closed-loop system at the 2,000 foot operating point (Fig. 6B) is correspondingly a little slower. From Fig. 6C, we can see that the reduction in the closed-loop system's bandwidth has caused the closed-loop frequency response curve to roll off below the reciprocal of the multiplicative model error at 10,000 feet, but the closed-loop frequency response curve still cuts across the other model error curves. Therefore, we know that up to 10,000 feet in altitude, the closed-loop system is guaranteed to be stable. Fig. 6D shows that up to 10,000 feet, the closed-loop system's poles do remain in the left-half plane, but above 10,000 feet, they move to the right-half plane causing the system to become unstable. With this autopilot, we have successfully extended the maximum altitude at which our control system will still be stable, but only at the expense of system performance at the nominal 2,000 foot altitude. Furthermore, we still have not designed a compensator that can control the missile up to an altitude of 40,000 feet.

In order to design a compensator which will yield a stable closed-loop system up to an altitude of 40,000 feet, we try a third compensator (Fig. 7A). This compensator was chosen to yield a closed-loop
system with even less bandwidth than the previous one, having a frequency response of unity up to only about one radian/second before rolling off. As we would expect, the step response of the closed-loop system (Fig. 7B) at 2,000 feet is now much slower than in the last two cases, with a rise time of about two and one half seconds. However, from Fig. 7C, we can see that with this compensator, the closed-loop system at the nominal 2,000 foot operating point has a Bode magnitude plot which rolls off below all of the reciprocal of the multiplicative model error curves. Even though we have lost a significant amount of performance at the nominal operating point of 2,000 feet, the closed-loop autopilot system is now guaranteed to be stable up to the 40,000 foot altitude goal. The plot of the closed-loop pole locations as a function of altitude (Fig. 7D) confirms that all of the poles remain in the left-half plane over the specified range of altitudes.

Assuming that this constraint on the bandwidth of the closed-loop system is acceptable, we have completed the task of modelling the missile sufficiently to meet our design goals. We know that as long as we observe the constraints on the closed-loop system's frequency response, we can use the nominal 2,000 foot missile plant model to design an autopilot which will be at least stable all the way up to 40,000 feet in altitude. However, if the degradation in performance at the nominal 2,000 foot operating point is not acceptable, the range of altitudes
covered by the control system designed at this operating point will have to be decreased. Assuming this to be the case, we can decide to use two different plant models in our overall autopilot system. The first model will be that for the nominal 2,000 foot altitude and we will select the second one to be the plant model at the 40,000 foot operating point. The nominal plant model for 2,000 foot altitude will be used to design the low altitude control system, and the 40,000 foot model will be used to design a separate high altitude control system. Using an altimeter, the missile will select the appropriate control system by switching to the correct compensator based on the current altitude.

Of course, we have already designed a good low altitude control system. Compensator $K_2$ yields a closed-loop system with good performance characteristics at the 2,000 foot altitude while at the same time guaranteeing stability up to 10,000 feet. All that remains to be considered is the nature of the high altitude control system which must have guaranteed stability over the range of altitudes from 10,000 to 40,000 feet.

The constraints on the design of the high altitude control system are determined in a fashion completely analogous to the way in which we found the constraints on our low altitude control system. Thus, the closed-loop system at 40,000 feet must have a Bode magnitude curve which rolls off below all of the curves of the reciprocal of the
multiplicative model error between the plant model at 40,000 feet and
the plant models at altitudes down to 10,000 feet. Fig. 8 depicts
the curves of the reciprocal of the multiplicative model error for
10,000, 20,000 and 30,000 feet, and Fig. 9A gives the block diagram
of a closed-loop system designed to have a closed-loop Bode magnitude
curve which rolls off below all of them (Fig. 9B), guaranteeing
stability over the specified altitude range. From Fig. 9C, we can
see that the step response of this system has a rise time which is about
twice as fast as the unacceptably slow step response at 2,000 feet with
compensator $k_3$. Also the plot of the closed-loop pole locations as a function
of altitude (Fig. 9D) shows that all of the closed-loop poles do indeed
remain in the left-half plane from 10,000 to 40,000 feet.

By using two different models of the missile (each with its own
closed-loop frequency domain design constraints to ensure stability
over a specified range of flight conditions around the model's
operating point) instead of just one, we have improved the performance
of our autopilot system. Moreover, by incorporating more plant models
at other altitude operating points in our autopilot system, we could
further enhance its performance. And, assuming we continue to apply
this system modelling methodology, we can also continue to guarantee
the stability of the closed-loop autopilot system for flight conditions
between those chosen operating points.
3.2 Time-Varying Missile Model

In order to illustrate the usefulness of this system modelling methodology in dealing with time-varying systems, we consider the following problem. The nominal missile model which we have derived describes the dynamics of the system at rocket motor burnout. We know that in the actual missile the amount of propellant in the rocket motor affects not only the total mass of the missile but also the position of its center of gravity. Moreover, as the rocket motor burns, the amount of propellant which it carries is constantly decreasing. Hence, until the point of rocket motor burnout, the missile's total mass and the position of its center of gravity are also constantly changing.

As the mass of the missile decreases due to the burning of the motor, the missile's pitch moment of inertia correspondingly decreases. This in turn causes the aeroderivatives $M_\alpha$ and $M_\delta$ to decrease. Also, the decreasing mass causes $Z_\alpha$ and $Z_\delta$ to increase. The burning of the rocket motor also causes the center of gravity of the missile to move further and further forward. This in turn causes the partial derivatives of the moment coefficient, $C_{\alpha \alpha}$ and $C_{\alpha \delta}$, to decrease. This has the effect of also decreasing the aeroderivatives $M_\alpha$ and $M_\delta$. Since our missile model is based on the values of $M_\alpha$, $M_\delta$, $Z_\alpha$, and $Z_\delta$, it is in actuality a time-varying system during the early phase of flight while the rocket motor is still burning.
We can approach this problem by treating the actual time-varying model of the missile as a general plant model and the nominal, time-invariant model at motor burnout as a plant model at a particular operating point in time. We can then define the times between the rocket motor's ignition and its burnout as the specified range of operating conditions in which we want our closed-loop autopilot system to have all of its poles in the left-half plane. Assuming that the plant's poles remain only in the left-half plane while the rocket motor is burning, this can be achieved by simply designing a closed-loop autopilot (using the plant model at the motor burnout operating point) whose Bode magnitude plot lies below the reciprocal of the multiplicative model error curve at each point in time in the range of operating conditions.

In order to study the problem further, it is assumed that the rocket motor burns its fuel at a constant rate over a period of five seconds. Also, the center of gravity of the missile moves at a constant rate from a point located two thirds of the way down the missile's body as measured from the missile's nose tip to a point located one half of the way down the missile. If the rocket motor begins firing at t=0, then at t=5 burnout will occur, and from that point onward, the missile model will be time-invariant and remain the same as it is at t=5 for the remainder of its flight. The position of the missile's center of gravity is given by:
\[ X_{cg} = 32.55 - 1.6268t \] (eq. 1)

the missile's mass is given by,

\[ m = 0.828125 - 0.09937t \] (eq. 2)

the missile's pitch moment of inertia is given by,

\[ I = 1.10925 - 0.13311t \] (eq. 3)

the partial derivatives of the moment coefficients are,

\[ C_{m\alpha} = -0.279 - 3.2174t \] (eq. 4)
\[ C_{m\delta} = -15.1866 - 1.8033t \] (eq. 5)

the aeroderivatives are,

\[ M\alpha = \frac{-36.688 - 423.09t}{1.10925 - 0.13311t} \] (eq. 6)
\[ M\delta = \frac{-1997.04 - 237.1336t}{1.10925 - 0.13311t} \] (eq. 7)
\[ Z\alpha = \frac{3122.33}{0.828125 - 0.09937t} \] (eq. 8)
\[ Z\delta = \frac{1750.53}{0.828125 - 0.09937t} \] (eq. 9)
the time-varying state space plant model is,

\[
\dot{X}(t) = \begin{bmatrix}
0 & -36.688 - 423.09t \\
1.10925 - 0.13311t & -1.4081 \\
0 & 0.828125 - 0.09937t
\end{bmatrix} X(t) + \begin{bmatrix}
-1997.04 - 237.1336t \\
1.10925 - 0.13311t & -0.78945 \\
0 & 0.828125 - 0.09937t
\end{bmatrix} \delta(t)
\]

\[
Y(t) = \begin{bmatrix}
0 & 3122.33 \\
0.828125 - 0.09937t & 1750.53
\end{bmatrix} X(t) + \begin{bmatrix}
-1997.04 - 237.1336t \\
1.10925 - 0.13311t & -0.78945 \\
0 & 0.828125 - 0.09937t
\end{bmatrix} \delta(t)
\]

Fig. 10A shows how the time-varying plant's poles change with time while the rocket motor is burning. As the rocket motor burns, this complex conjugate pair of poles moves to higher and higher real and imaginary frequencies. Also, the poles are always in the left-half of the complex plane. Since the number of unstable plant poles (none) remains the same over the entire specified range of operating conditions and there are no poles on the imaginary axis, we can conclude that the closed-loop autopilot for this missile will always have its poles in the left-half plane if the closed-loop system at the specified operating point (t=5 plant) has a Bode magnitude plot which lies below the reciprocal of the multiplicative model error at each point in time over the specified range. Fig. 10B shows plots of the reciprocal of the multiplicative model error between the nominal t=5 plant at the chosen operating point and several other plant models at other operating...
points in time. From the plots, we can see that the curves corresponding to earlier and earlier times become increasingly more restrictive in their constraints on the closed-loop system bandwidth. Hence, these increasingly restrictive constraints degrade the achievable performance of the closed-loop system. This means that we are faced with a trade-off between closed-loop system performance and keeping all of the closed-loop poles in the left-half plane at earlier and earlier times in the missile's flight using the autopilot designed using the t=5 plant. The reason for this is that when the system is at an operating point in time before t=5, we have a situation which is equivalent to having the t=5 plant with some unmodelled high frequency dynamics which can cause the closed-loop system to have right-half plane poles if its bandwidth is too high. As time decreases from t=5, the poles move to lower and lower frequencies. This means that model error between the t=5 plant model and earlier plant models occurs at successively lower frequencies. As these "unmodelled high frequency dynamics" move to lower and lower frequencies, the closed-loop system bandwidth must correspondingly be lowered to keep all of the poles in the left-half plane.

In order to ensure that all of the closed-loop poles are in the left-half plane, we must design a compensator that will give us a system whose closed-loop Bode plot lies below all of the curves in Fig. 10B, but for the sake of illustration, we will begin by exploring
the consequences of violating this constraint. Fig. 11A shows the block diagram of a possible autopilot for the missile. This autopilot was designed to give a closed-loop frequency response to unity, for good command-following, up to about 11 radians/second before it rolls off. From Fig. 11B, we can see that this autopilot design does have good command-following performance as demonstrated by its step response. However, since this system was designed using the t=5 plant model, these results are only valid during the phase of flight after rocket motor burnout. Fig. 11C shows the closed-loop frequency response superimposed on a pilot of the reciprocal of the multiplicative model error curves. From this graph, we can see that the constraint necessary to ensure that the closed-loop poles are in the left-half plane is violated at all of the time points being considered. Fig. 11D shows the locations of the closed-loop poles, and as we might expect, from t=1 to t=4, there are closed-loop poles in the right-half plane. Furthermore, Fig. 11E, which shows the time-varying, closed-loop step response simulation of the system, shows the catastrophic results of this autopilot design. From the beginning of the flight, until about t=4, the missile experiences oscillations which very quickly increase to extremely large amplitudes. After t=4, these oscillations begin to decay, but they reach such high amplitudes that the missile probably would not survive. It is important to note that since the closed-loop system is known to be stable after
rocket motor burnout, the system output is bounded, and therefore by
definition the closed-loop system is stable. The problem is not that
the system is unstable, but that its relative stability characteristics
are so poor as to be unacceptable. Even though the closed-loop autopilot
system has good performance characteristics at the chosen operating point
(the t=5, rocket motor burnout plant model), we will have to redesign
it because of its inadequate performance over the specified range of
time in which it must operate.

With the aim of developing a closed-loop autopilot system with
a good performance after rocket motor burnout and adequate relative
stability characteristics during motor burn, we consider a new
compensator. This new compensator, Fig. 12A, was designed to give a
system with a little less closed-loop bandwidth, rolling off at about
8 radians/second. From Fig. 12B, we can see that this reduction in
bandwidth causes a corresponding reduction in performance as evidenced
by the slower step response of the closed-loop system with the t=5
plant model. Fig. 12C shows the closed-loop frequency response curve
superimposed on a plot of the reciprocal of the multiplicative model
error curves. Since the bandwidth of the closed-loop system has been
reduced, its Bode plot curve no longer violates the constraint
necessary to ensure that there are no closed-loop, right-half plane
poles, at least at t=2, 3 and 4. At t=1, the constraint is still
violated because the closed-loop Bode plot curve lies above the
reciprocal of the multiplicative model error curve at some frequencies. From the plot of the time-varying, closed-loop poles, Fig. 12D, we can see that at t=1 there are closed-loop, right-half plane poles, but from t=2 to burnout, all of the closed-loop poles are in the left-half plane as they must be. The time-varying step response simulation, Fig. 12E, shows that up until about two seconds into the flight, there are oscillations of increasing amplitude, but after this, the oscillations steadily decrease in amplitude, and the output settles on the correct value. Also, unlike the previous case, the amplitude of the oscillations never reach extreme levels. With this compensator, we have significantly improved the relative stability of the closed-loop system prior to rocket motor burnout, but at the expense of performance after motor burnout.

Finally, we consider a third compensator, Fig. 13A, which gives a closed-loop system with even less bandwidth, rolling off at about 5 radians/second. The closed-loop system's step response after motor burnout, Fig. 13B, is correspondingly a little slower, but now we have reduced the closed-loop system bandwidth enough to meet the constraint necessary to ensure that all of the closed-loop poles are in the left-half plane. Fig. 13C shows that the closed-loop Bode plot curve lies below all of the reciprocal of the multiplicative model error curves, and Fig. 13D shows that the poles do indeed remain in the left-half plane.
Fig. 13E shows that with all of the poles in the left-half plane over the specified range of time, the relative stability of the system while the rocket motor is burning is very good. This time-varying step response simulation shows a slight initial overshoot with only small amplitude oscillations which quickly decay away. Although keeping the closed-loop poles in the left-half plane is key to predicting the relative stability of the closed-loop system in this example, it must be remembered that this is not always true when dealing with time-varying systems. However, because of the difficulties associated with dealing with time-varying systems, a relatively straight-forward technique, such as this, can prove to be a useful tool for a system designer who is dealing with problems arising from the time-varying nature of the system's model.

Assuming that the autopilot's performance after rocket motor burnout is acceptable, we have solved our problem of taking into account the time-varying nature of the missile model. If our design specifications call for better performance, we could time-schedule the compensators. For example, we could simply begin the flight with this compensator and switch to the second compensator which we considered at two seconds into the flight in order to improve the missile's performance after rocket motor burnout. In this way, we can meet design specifications related to the missile's performance after rocket motor burnout by limiting the specified operating range covered
by the compensator associated with the t=5 operating point.
SUMMARY AND CONCLUSIONS

The first major example which we considered involved an autopilot design for a tail-controlled missile whose aerodynamic model varied as a function of altitude. The approach chosen consisted of selecting a specific altitude operating point and using the model of the missile at that operating point to design the autopilot. Since the operating point was chosen to be at 2,000 feet and the autopilot was required to be stable all the way up to 40,000 feet, this autopilot design was a natural application for the system modelling methodology.

Because all of the poles of the altitude varying missile model were strictly in the left-half plane, closed-loop autopilot stability could be guaranteed over a range of altitudes by keeping the Bode magnitude curve of the closed-loop system at the specified operating point below the curves of the reciprocal of the multiplicative model error between the nominal plant model at the specified operating point and the plant models at the other altitudes within the specified range. The first compensator produced a closed-loop system which violated the constraint at 10,000 feet and above and had poles in the right-half plane at 10,000 feet and above, demonstrating the need for this method of guaranteeing stability at high altitudes. The second compensator met the constraint up through 10,000 feet and produced a closed-loop system which was stable up through
10,000 feet, showing that the constraints on the closed-loop system at the chosen operating point did indeed ensure stability. The third compensator produced a closed-loop system which met the constraints all the way up to 40,000 feet, giving an autopilot system which was stable up to 40,000 feet but which lacked adequate performance characteristics because the stability constraints imposed overly severe restrictions on its closed-loop bandwidth.

From these results, we conclude that it is necessary to reduce the range of altitudes covered by the 2,000 foot missile model and to use more than one aerodynamic model of the missile in order to design an acceptable autopilot system. Therefore, we decide to retain the 2,000 foot altitude model to describe the missile up to 10,000 feet and chose a second aerodynamic model, at the 40,000 foot operating point, to cover the range from 10,000 to 40,000 feet in altitude. The autopilot system would now consist of two compensators, one for low altitude flight and another for high altitude flight, with an altimeter being used to select the appropriate compensator. Finally, we determine the constraints on the closed-loop system designed at the 40,000 foot operating point which would ensure stability all the way down to 10,000 feet. Hence, by using the system modelling methodology, we are able to determine not only how many aerodynamic missile models we should use in our overall autopilot system design but also the range of altitudes associated with
each model. Further, we can establish what constraints on the closed-loop system designed using each model should be observed in order to guarantee stability over the entire range of operating altitudes.

The second major example concerns the application of the system modelling methodology to a time-varying model of the tail controlled missile. Since the modelling methodology only guarantees that the closed-loop system has all of its poles in the left-half plane, there is no assurance from this method that the closed-loop time-varying system would be stable. Moreover, there is no guarantee that keeping all of the closed-loop poles in the left-half plane would even improve the relative stability of the closed-loop, time-varying system. However, in some time-varying systems, including the one presented in this example, keeping all of the poles in the left-half plane does favorably influence the relative stability of the system. Hence, the system modelling methodology proves to be a potentially useful tool in dealing with this time-varying system.

The time-varying nature of this missile model stems from the effects of the burning of the rocket motor's fuel. As the fuel burns, the missile's mass and center of gravity are constantly changing, which in turn cause the aerodynamic model of the tail controlled missile to continuously vary with time. Of course, after all of the fuel is burned, the missile model becomes time-invariant and remains the same as it was at the point of rocket motor burnout. Analogous to the nominal 2,000 foot altitude operating point model
in the first example, the missile aerodynamic model at rocket motor burnout is chosen as the nominal model around which the autopilot system would be designed. The problem was to ensure that the closed-loop autopilot designed at this operating point would have all of its poles in the left-half plane during the period of time when the rocket motor was burning. Since the poles of the time-varying missile model remains strictly in the left-half plane, this could be achieved by simply keeping the Bode magnitude plot of the closed-loop system at the nominal operating point below the reciprocal of the multiplicative model error between the nominal missile model and the actual missile model at each point in time.

The first compensator presented produces a closed-loop system which violates this constraint during most of the time the rocket motor is burning. The result is closed-loop poles in the right-half plane and an unacceptable time-varying step response with large amplitude oscillations. The second compensator meets the constraints from two seconds into the flight and onward and produces a system which had right-half plane poles only up to two seconds into the flight. The time-varying step response is much improved with the missile experiencing increasing amplitude oscillations only during the period of flight when the poles were in the right-half plane. Finally, the third compensator meets the constraints at all of the time points and produced a system with no poles in the right-half plane. The time-varying step response is very good with no oscillations.
of increasing amplitude, demonstrating that the system modelling methodology could be used to facilitate the design of time-varying systems as well as time-invariant ones.

In both the altitude varying and time-varying model examples, the system modelling methodology is used to ensure that the closed-loop control system designed at a particular operating point would still have all of its poles in the left-half plane over a specified range of operating conditions. In the altitude-varying model example, the problem consists of designing an autopilot for a missile which could fly at any one of a range of different altitudes. Hence, the problem is time-invariant, and the assurance of having all of the closed-loop poles in the left-half plane is a guarantee of stability. In the time-varying model example, there was no such guarantee of stability, however, keeping all of the closed-loop poles in the left-half plane is shown numerically to be necessary to prevent periods of increasing amplitude oscillations.

In both examples, we also saw the trade-off which can exist between maximizing system performance at the nominal operating point and guaranteeing that the closed-loop poles will remain in the left-half plane over an increasingly greater range of operating conditions. In the altitude-varying model example, we find that it is unreasonable to expect a single model of the missile at one operating point to describe the system well enough to allow the autopilot to meet both the stability
requirements and the necessary performance specifications at the nominal operating point. The system modelling methodology is able to handle this problem by reducing the range of operating conditions over which the control system designed at the nominal operating point is expected to be stable, and by introducing a second operating point with its own control system to cover the neglected range of operating conditions. Hence, it is demonstrated that the system modelling methodology, by generating a sufficient number of system models each with its own frequency domain design constraints, could ensure that the closed-loop control system would have no right-half plane poles over some specified range of operating conditions and that the performance specifications at the operating points would be met.
REFERENCES


APPENDIX

Derivation of the Nominal Tail Controlled Missile Model

Using the nomenclature of Fig. 14, the missile's force and moment equations in body axes are:

\[ m\ddot{X}_B = T - qS_{\text{REF}}C_A \]  
(eq. 1)

\[ m\ddot{Y}_B = qS_{\text{REF}}C_N \]  
(eq. 2)

\[ I\ddot{\phi} = qS_{\text{REF}}dC_M \]  
(eq. 3)

where,

- \( m \) = missile mass
- \( T \) = motor thrust
- \( q \) = dynamic pressure
- \( S_{\text{REF}} \) = missile reference area (cross-section)
- \( I \) = pitch moment of inertia
- \( d \) = reference diameter
- \( C_A \) = axial force coefficient
- \( C_N \) = normal force coefficient
- \( C_M \) = moment coefficient
Assuming constant velocity along the X axis we have:

\[ m\ddot{Y}_B = q_{S_{REF}}^C N \]  
\[ I\ddot{\phi} = q_{S_{REF}}^{dC_M} \]  

(eq. 4)  
(eq. 5)

The acceleration along the Y axis may be expressed as:

\[ \ddot{Y}_B = V_M \cos \phi \]  
\[ (eq. 6) \]

Assuming small \( \alpha \) gives:

\[ \ddot{Y}_B = V_M \dot{\gamma} \]  
\[ (eq. 7) \]

Combining eq. 4 and eq. 5 with eq. 7

\[ mV_M \dot{\gamma} = q_{S_{REF}}^C N \]  
\[ I\ddot{\phi} = q_{S_{REF}}^{dC_M} \]  

(eq. 8)  
(eq. 9)

Expanding these equations in a Maclaurin series:

\[ mV_M \dot{\gamma} = q_{S_{REF}}^C ( C_N + C_{N\alpha} \alpha + C_{N\delta} \delta ) \]  
\[ (eq. 10) \]

\[ I\ddot{\phi} = q_{S_{REF}}^{dC_M} ( C_M + C_M\alpha \alpha + C_M\delta \delta ) \]  
\[ (eq. 11) \]
We now define the following:

\[
Z_x = \frac{C_{N_x} qS_{REF}}{m} \quad \text{(eq. 12)}
\]

\[
Z_\delta = \frac{C_{N_\delta} qS_{REF}}{m} \quad \text{(eq. 13)}
\]

\[
M_x = \frac{C_{M_x} qS_{REF}^d}{I} \quad \text{(eq. 14)}
\]

\[
M_\delta = \frac{C_{M_\delta} qS_{REF}^d}{I} \quad \text{(eq. 15)}
\]

Using equations 10 through 15 and assuming \(C_N\) and \(C_M\) are zero at \(x = 0\) and \(\delta = 0\), we can describe the missile model by the block diagram of Fig. 15. And, from the block diagram, we can see that the state-space description of the missile model is given by:

\[
\begin{bmatrix}
\dot{\theta}(t) \\
\dot{\alpha}(t) \\
Y(t)
\end{bmatrix} =
\begin{bmatrix}
0 & M_x \\
1 & -Z_x/V_M \\
0 & Z_\alpha
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}(t) \\
\dot{\alpha}(t) \\
\alpha(t)
\end{bmatrix} +
\begin{bmatrix}
M_\delta \\
-Z_\delta/V_M \\
Z_\delta
\end{bmatrix} \delta(t)
\]
Using the nomenclature of Fig. 16, the normal force coefficient and moment coefficient for a supersonic tail controlled missile are given by (7):

\[
C_N = 2\alpha + 1.5S_p \frac{\alpha^2}{S} + \frac{8S_{\text{tail}}(\alpha + \delta)}{S\sqrt{M^2 - 1}} + \frac{8S_{\text{wing}}\alpha}{S\sqrt{M^2 - 1}} \tag{eq. 16}
\]

\[
C_M = 2\alpha(X_{cg} - X_{cpn}) + 1.5S_p \frac{\alpha^2(X_{cg} - X_{cpb})}{d} + \frac{8S_{\text{wing}}(X_{cg} - X_{cpw})}{Sd\sqrt{M^2 - 1}}
\]

\[
+ 8(\alpha + \delta)S_{\text{tail}}(X_{cg} - X_{hl}) \frac{1}{Sd\sqrt{M^2 - 1}} \tag{eq. 17}
\]

where,

\[
S_p = \text{missile planform area} = (L-L_1)d + 2Ld/3
\]

\[
S = \text{missile reference area} = \frac{\pi d^2}{4}
\]

\[
S_{\text{tail}} = \text{tail area} = 0.5h_T(C_{RT} + C_{TT})
\]

\[
S_{\text{wing}} = \text{wing area} = 0.5h_w(C_{RW} + C_{TW})
\]

\[
X_{cg} = \text{C.G. distance from nose tip} = 0.5L
\]

\[
X_{cpn} = \text{C.P. of nose from tip} = 2L_1/3
\]

\[
X_{cpb} = \text{C.P. body from nose tip} = (L_1+L)/2
\]

\[
X_{cpw} = \text{C.P. wing from nose tip} = L_1+L_2+0.7C_{RW}+0.2C_{TW}
\]

\[
X_{hl} = \text{C.P. tail from nose tip} = L-0.3C_{RT}-0.2C_{TT}
\]

\[
M = \text{Mach number}
\]
Using the dimensions of Fig. 17 we have (8):

\[
\begin{align*}
S_p &= 172 \text{ in}^2
S &= 11.0 \text{ in}^2 \\
S_{\text{tail}} &= 9.9 \text{ in}^2 \\
S_{\text{wing}} &= 3 \text{ in}^2 \\
d &= 3.75 \text{ in.}
\end{align*}
\]

\[
\begin{align*}
X_{\text{cg}} &= 24.416 \text{ in.} \\
X_{\text{cpn}} &= 5.7 \text{ in.} \\
X_{\text{cpb}} &= 28.7 \text{ in.} \\
X_{\text{cpw}} &= 30.8 \text{ in.} \\
X_{\text{hl}} &= 46.25 \text{ in.}
\end{align*}
\]

Assuming that the missile flies at a constant speed of Mach two, the partial derivatives of the normal force and moment coefficients are:

\[
\begin{align*}
C_{Nx} &= 7.42 \\
C_{N\delta} &= 4.16 \\
C_{Mx} &= -16.38 \\
C_{M\delta} &= -24.2
\end{align*}
\]

Also, (7) assuming an average surface to air missile density of 0.05 lbs./in\(^3\) and a typical mass ratio of 0.6, we can estimate the missile's mass as follows:

\[
\text{volume of missile} = 530 \text{ in}^3.
\]
missile launch weight = 530 in.³ x 0.05 lbs/in³ = 26.5 lbs.

missile weight at motor burnout = 26.5 lbs. x 0.4 = 10.6 lbs.
missile mass = 0.33125 slugs

Hence, we can estimate the pitch moment of inertia by:

\[
I = m(0.25(d/2)^2 + (L)^2/12) = 0.4437 \text{ slug-ft}^2. \quad (\text{eq. 18})
\]

Furthermore, if the missile flies at an altitude of 2,000 feet, its velocity at Mach two will be 2217.4 ft/sec. and the dynamic pressure will be 5509 lbs./ft². Now, using the values we have obtained for the partial derivatives of the normal force and moment coefficients, the missile's mass, pitch moment of inertia, diameter, reference area and dynamic pressure we have:

\[
Z_\alpha = 9425.9 \text{ lbs./slug}
\]
\[
Z_\delta = 5284.6 \text{ lbs./slug}
\]
\[
M_\alpha = -4854.56 \text{ lbs./slug-ft.}
\]
\[
M_\delta = -7172.18 \text{ lbs./slug-ft.}
\]

Hence, the nominal state-space missile model at rocket motor burnout is:

\[
\begin{bmatrix}
\dot{\theta}(t) \\
\dot{x}(t)
\end{bmatrix}
= \begin{bmatrix}
0 & -4854.56 \\
1 & -4.251
\end{bmatrix}
\begin{bmatrix}
\theta(t) \\
x(t)
\end{bmatrix}
+ \begin{bmatrix}
-7172.18 \\
-2.3832
\end{bmatrix} \delta(t)
\]

\[
Y(t) = \begin{bmatrix}
0 & 9425.9 \\
0 & 5284.6
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}(t) \\
\dot{x}(t)
\end{bmatrix}
+ \begin{bmatrix}
5284.6 \\
0
\end{bmatrix} \delta(t)
\]
FIGURE 1

VECTOR SUM ILLUSTRATION

ARBITRARY POINT ON THE NYQUIST LOCUS OF $T(jw)$
FIGURE 2

VECTOR DIFFERENCE ILLUSTRATION

POINT ON THE NYQUIST LOCUS OF $T(jw)$ CORRESPONDING TO THE FREQUENCY $w$

POINT ON THE NYQUIST LOCUS OF $T'(jw)$ CORRESPONDING TO THE FREQUENCY $w$
figure 3

bode plot of the reciprocal of the multiplicative model error and the closed-loop transfer function

key
solid line: $|E_m(s)|^{-1}$
dashed line: $|C(s)|$
FIGURE 4A

PLANT POLES AS A FUNCTION OF ALTITUDE

LEGEND
- □ 2,000 ft.
- ∆ 10,000 ft.
- × 20,000 ft.
- ○ 30,000 ft.
- ▽ 40,000 ft.
FIGURE 4B

RECIPROCAL OF THE MULTIPLICATIVE MODEL
ERROR AS A FUNCTION OF ALTITUDE FOR THE
2,000 FOOT ALTITUDE OPERATING POINT

FROM LEFT TO RIGHT THE CURVES CROSSING THE
0 dB LINE ARE $|E_m(j\omega)|^{-1}$ AT 40,000, 30,000,
20,000 and 10,000 FEET WITH MODEL ERROR
NORMALIZED TO UNITY AT DC.
FIGURE 5A
CLOSED-LOOP SYSTEM BLOCK DIAGRAM WITH
COMPENSATOR K1

\[ K_1(s) = \frac{-0.004 (s+2.1+69.6j)(s+2.1-69.6j)}{s(s+85.5+45.16j)(s+85.5-45.16j)} \]
FIGURE 5B

STEP RESPONSE WITH COMPENSATOR K1 AT
THE 2,000 FOOT OPERATING POINT
FIGURE 5C

CLOSED-LOOP FREQUENCY RESPONSE (AT 2,000 FOOT ALTITUDE OPERATING POINT WITH COMPENSATOR K1) AND MODEL ERROR CURVES

SOLID LINES: $|E_m(j\omega)|^{-1}$ CURVES
DOTTED LINES: CLOSED-LOOP FREQUENCY RESPONSE
FIGURE 5D

CLOSED-LOOP POLE LOCATIONS AS A FUNCTION OF ALTITUDE WITH COMPENSATOR K1

LEGEND
- □ 2,000 ft.
- △ 10,000 ft.
- × 20,000 ft.
- ○ 30,000 ft.
- ▽ 40,000 ft.
FIGURE 6A

CLOSED-LOOP SYSTEM BLOCK DIAGRAM WITH
COMPENSATOR K2

\[ K_2(s) = \frac{-0.00167(s+2.1+69.6j)(s+2.1-69.6j)}{s(s+70.5+28.98j)(s+70.5-28.98j)} \]
FIGURE 6B

STEP RESPONSE WITH COMPENSATOR K2 AT THE
2,000 FOOT OPERATING POINT
CLOSED-LOOP FREQUENCY RESPONSE (AT 2,000 FOOT ALTITUDE OPERATING POINT WITH COMPENSATOR K2) AND MODEL ERROR CURVES

SOLID LINES: $|E_m(j\omega)|^{-1}$ CURVES

DOTTED LINES: CLOSED-LOOP FREQUENCY RESPONSE
FIGURE 6D

CLOSED-LOOP POLE LOCATIONS AS A FUNCTION OF ALTITUDE WITH COMPENSATOR K2

LEGEND

□ 2,000 ft.
△ 10,000 ft.
× 20,000 ft.
○ 30,000 ft.
▽ 40,000 ft.
FIGURE 7A

CLOSED-LOOP SYSTEM BLOCK DIAGRAM WITH COMPENSATOR $K_3$

$$K_3(s) = \frac{-8.66 \times 10^{-5} (s+2.1+69.6j)(s+2.1-69.6j)}{s(s+65.45+5.18j)(s+65.45-5.18j)}$$
FIGURE 7B

STEP RESPONSE WITH COMPENSATOR K3 AT
THE 2,000 FOOT ALTITUDE OPERATING POINT

(output vs. time graph)

TIME (SEC)

OUTPUT
FIGURE 7C

CLOSED-LOOP FREQUENCY RESPONSE (AT 2,000 FOOT ALTITUDE OPERATING POINT WITH COMPENSATOR K3) AND MODEL ERROR CURVES

SOLID LINES: $|E_m(j\omega)|^{-1}$ CURVES
DOTTED LINE: CLOSED-LOOP FREQUENCY RESPONSE
FIGURE 7D

CLOSED-LOOP POLE LOCATIONS AS A FUNCTION
OF ALTITUDE WITH THE COMPENSATOR K3

LEGEND
□ 2,000 ft.
△ 10,000 ft.
× 20,000 ft.
○ 30,000 ft.
▼ 40,000 ft.
FROM LEFT TO RIGHT THE $\frac{1}{E_{m}(j\omega)}$ CURVE REACHING ITS MINIMUM POINT IS $\frac{1}{E_{m}(j\omega)}$ AT 30,000, 20,000 AND 10,000 FEET WITH MODEL ERROR NORMALIZED TO UNITY AT DC.
FIGURE 9A

CLOSED-LOOP SYSTEM BLOCK DIAGRAM WITH
COMPENSATOR K₄

\[ K₄(s) = \frac{-0.003(s+.486+31.1j)(s+.486-31.1j)}{s(s+53.5+8.35j)(s+53.5-8.35j)} \]
FIGURE 9B

CLOSED-LOOP FREQUENCY RESPONSE (AT THE 40,000 FOOT ALTITUDE OPERATING POINT WITH COMPENSATOR K4) AND MODEL ERROR CURVES

SOLID LINES: $|E_m(j\omega)|^{-1}$ CURVES

DOTTED LINE: CLOSED-LOOP FREQUENCY RESPONSE
FIGURE 9C

STEP RESPONSE WITH COMPENSATOR K4 AT THE

40,000 FOOT ALTITUDE OPERATING POINT
FIGURE 9D

CLOSED-LOOP POLE LOCATIONS AS A FUNCTION OF
ALTITUDE WITH COMPENSATOR K4

LEGEND
\( \Delta \) 10,000 ft.
\( \times \) 20,000 ft.
\( \circ \) 30,000 ft.
\( \triangledown \) 40,000 ft.
FIGURE 10A

TIME-VARYING PLANT POLES
FIGURE 10B

RECIProCAL OF THE MULTIPLICATIVE
MODEL ERROR FOR $t=1, 2, 3$ AND 4

From left to right the curves crossing the 0 dB
line are $|E_m(j\omega)|^{-1}$ at $t=1, 2, 3$ and 4 with model
error normalized to unity at DC.
CLOSED-LOOP SYSTEM BLOCK DIAGRAM WITH COMPENSATOR $K_5$

$$K_5(s) = \frac{-0.0018(s+2.1+69.6j)(s+2.1-69.6j)}{s(s+70.5+28.98j)(s+70.5-28.98j)}$$
FIGURE 11B

STEP RESPONSE WITH COMPENSATOR K5
FIGURE 11C

CLOSED-LOOP FREQUENCY RESPONSE WITH COMPENSATOR K5 AND MODEL ERROR CURVES

DOTTED LINE: CLOSED-LOOP FREQUENCY RESPONSE WITH t=5 PLANT

SOLID LINE: FROM LEFT TO RIGHT THE CURVES CROSSING THE 0 dB LINE ARE $|E_m(j\omega)|^{-1}$ AT $t=1, 2, 3$ AND 4
FIGURE 11D

TIME-VARYING POLE PLOT WITH COMPENSATOR K5
FIGURE 11E

TIME-VARYING STEP RESPONSE WITH COMPENSATOR K5
FIGURE 12A

CLOSED-LOOP SYSTEM BLOCK DIAGRAM
WITH COMPENSATOR K6

\[ K_6(s) = \frac{-1.819 \times 10^{-4} (s+2.1+69.6j)(s+2.1-69.6j)}{s(s+40.79)(s+33.71)} \]
FIGURE 12B

STEP RESPONSE WITH COMPENSATOR K6
FIGURE 12C

CLOSED-LOOP FREQUENCY RESPONSE WITH
COMPENSATOR K6 AND MODEL ERROR CURVES

DOTTED LINE: CLOSED-LOOP FREQUENCY
RESPONSE WITH t=5 PLANT

SOLID LINES: FROM LEFT TO RIGHT THE CURVES
CROSSING THE 0 dB LINE ARE
$E_m(j\omega)^{-1}$ AT t=1, 2, 3 AND 4
FIGURE 12D

TIME-VARYING POLE PLOT WITH COMPENSATOR K6
FIGURE 12E

TIME-VARYING STEP RESPONSE WITH COMPENSATOR K6
FIGURE 13A

CLOSED-LOOP SYSTEM BLOCK DIAGRAM WITH

COMPENSATOR K7

\[
K_7(s) = \frac{-3.2277 \times 10^{-5} (s+2.1+69.6j)(s+2.1-69.6j)}{s(s+16.5+9.53j)(s+16.5-9.53j)}
\]
FIGURE 13B

STEP RESPONSE WITH COMPENSATOR K7
FIGURE 13C

CLOSED-LOOP FREQUENCY RESPONSE WITH COMPENSATOR K7 AND MODEL ERROR CURVES

DOTTED LINE: CLOSED-LOOP FREQUENCY RESPONSE WITH t=5 PLANT

SOLID LINES: FROM LEFT TO RIGHT THE CURVES CROSSING THE 0 dB LINE ARE $|E_m(j\omega)|^{-1}$ AT $t=1,2,3$ AND 4
FIGURE 13D

TIME-VARYING POLE PLOT WITH COMPENSATOR K7
FIGURE 13E

TIME-VARYING STEP RESPONSE WITH
COMPENSATOR K7
FIGURE 14

BODY AXES DIAGRAM OF THE TAIL CONTROLLED MISSILE
FIGURE 15

BLOCK DIAGRAM OF THE MISSILE MODEL
FIGURE 16

MISSILE NOMENCLATURE
FIGURE 17

MISSILE DIMENSIONS

DIMENSIONS IN INCHES