Nonlinear Dynamics of a Rotating, Extending Spacecraft Appendage

by

Roger Mark Hain

B.S.E. Princeton University (1986)

SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
May, 1988

Signature of Author ______________________________
Department of Aeronautics and Astronautics
May 27, 1988

Certified by ______________________________
Dr. Achille Messac
Thesis Supervisor, C. S. Draper Laboratory, Inc.

Certified by ______________________________
Prof. John Dugundji
Thesis Supervisor, Prof. of Aeronautics and Astronautics

Accepted by ______________________________
Prof. Harold Y. Wachman
Chairman, Departmental Graduate Committee

Contents

List of Figures

Abstract

The second order nonlinear integral-partial differential equations of motion are derived using Newton's method for a rotating spacecraft, modelled as a 2-beam-and-hub system whose appendages have time-varying lengths. These equations are transformed to ordinary differential equation form using separation of variables and a Galerkin's method approach. The resulting o.d.e.'s are numerically integrated using a 4^{th} order Runge-Kutta routine. The results of several important subcases of the equations are shown to duplicate those of other researchers. For the completely nonlinear, rotating and time-varying beam length case, results of an analysis of the WISP space experiment are shown. It is found that the inclusion of nonlinear terms is critically important in certain cases.

Acknowledgements

I wish to thank the many people who helped make this thesis possible through their guidance and support.

Dr. Achille Messac provided the opportunity for me to do this research at Draper Laboratory, suggested this thesis topic and has always made himself available to answer my questions. His insight and encouragement have truly been invaluable. Professor John Dugundji has contributed a great deal of his time and effort and drawn on his great experience to provide excellent assistance to this work.

Many others at Draper have helped along the way. Stephen Gates has provided valuable recommendations and suggestions. Joel Storch, Scott Hunziker and Alex Gruzen have all made contributions. There are many more, whose friendship and camaraderie has made these two years much more enjoyable.

I owe much to my parents, whose love and support is always present and whose faith in me never falters.

Finally, Regina Rosales has been immensly supportive, caring and above all, patient.

My heartfelt thanks go to all.

This report was prepared at The Charles Stark Draper Laboratory, Inc. Publication of this report does not constitute approval by the Draper Laboratory of the findings or conclusions contained herein. It is published for the exchange and stimulation of ideas.

Roger M. Hain

Roger M. Hain

Permission is hereby granted by The Charles Stark Draper Laboratory, Inc. to the Massachusetts Institute of Technology to reproduce any or all of this thesis.

Nomenclature

a_1	:	component of appendage acceleration
a_2	:	component of appendage acceleration
EI	:	appendage stiffness (Fig. 2.2)
$\vec{F}(x,t)$	:	appendage internal force (Fig. 2.3)
$F_1(x,t)$	:	component of force-vector $\vec{F}$
$F_2(x,t)$	:	component of force-vector $\vec{F}$
$\vec{H}(x,t)$	:	system angular momentum
I_h	:	hub's instantaneous mass moment-of-inertia about the the $\hat{k}$-axis
$l(t)$	:	appendage instantaneous length. (Fig. 2.2)
l_r	:	appendage reference length
m	:	appendage linear mass density in undeformed state (Fig. 2.2)
$\vec{M}(x,t)$	:	appendage internal moment vector (Fig. 2.3)
M_3	:	component of $\vec{M}$ in the $\hat{k}$ direction
q_i	:	generalized coordinate
Q_i	:	generalized force
$\vec{R}(x,t)$	:	inertial vector locating an appendage differential element (Fig. 2.1)

$S(x,t)$	:	appendage shear (Fig. 2.4)
$T(x,t)$	:	appendage tension (Fig. 2.4)
T_o	:	torque applied at hub center
T_r	:	reference time
T	:	kinetic energy
$u(x,t)$	:	longitudinal elastic deformation of appendage centerline particle corresponding to coordinate x (Fig. 2.3)
$\vec{V}(x,t)$	:	inertial time derivative of $\vec{R}$
$v(x,t)$	:	lateral elastic deformation of appendage centerline particle corresponding to coordinate x (Fig. 2.3)
V	:	potential energy
$x(t)$	:	particle coordinate along appendage length in undeformed state (Fig. 2.3)
$\beta(x,t)$	:	rotational deformation of appendage centerline (Fig. 2.3)
κ	:	beam curvature
$\gamma_i(\xi)$	:	i^{th} modeshape of a cantilevered beam
$\eta_i(\tau)$	:	coefficient of i^{th} modeshape
ϕ	:	angle between x-axis and tangent to deformed beam (Fig. B.1)
τ	:	nondimensional time variable
$\theta(t)$	:	hub angular rotation angle (Fig. 2.1)

ξ	:	nondimensional spatial variable
$(\hat{i}, \hat{j}, \hat{k})$	:	axes which define hub-attached body-frame (x, y) (Fig. 2.2)
(X, Y)	:	inertial frame centered at hub center of mass (Fig. 2.1)
$\frac{d}{dt}$	:	inertial time derivative of any vector; or, total time derivative of any scalar function
$\dot{g} = \frac{\partial g}{\partial t}$	:	partial time derivative of any scalar function g
$g_x = \frac{\partial g}{\partial x}$	:	partial derivative of any scalar function g with respect to x
g'	:	partial derivative of any scalar function g with respect to ξ
$\mathring{g}$	:	partial derivative of any scalar function g with respect to τ

Chapter 1

Introduction

1.1 Historical Perspective

There are many spacecraft missions in which it is important to understand the he dynamics of deploying appendages. This issue has been studied most notably in regard to two separate situations; deployment dynamics of spinning satellites with flexible appendages, and shuttle flight experiments involving deployment of long flexible booms.

Messac [1] modelled dynamics of deployment of a spinning satellite with flexible appendages using linear equations. Lips and Modi [2] developed nonlinear equations of motion for a detailed model of a deploying spacecraft appendage. The simplified linear two dimensional equations were solved and individual structural and dynamics effects were examined. Lips and Modi [3] also examine the three dimensional deployment problem, and identify instabilities not apparent in the case of planar rotation. Additionally, it was found that an offset or shifting center of mass had negligible effect on the dynamic response. Hughes [4] discusses attitude dynamics of spinning satellites during extension of long appendages. Maximum bending moments are identified and give rise to restrictions on extension rate and initial nutation angle. Weeks [5] develops a linear analysis of a nonrotating space structure composed of a beam and membrane to model the NASA Solar Array Flight Experiment. Hughes [6] applies a general deployment dynamics analysis to the Communications Technology Satellite by making several simplifying assumptions and outlines a suggested solution methodology. Cloutier [7] examines synchronous deployment of masses about a rotating spacecraft. Booms connect the masses to the spacecraft and are considered flexible in the derivation of the equations and rigid during solution. Honma [8] derives equations of motion for constant speed extension of tip masses connected by massless wires to a spinning satellite. The linearized equations are solved.

Tabarok, et. al., [9] provide a thorough treatment of the dynamics of an extending cantilevered beam, and solve linear equations for two sample cases of constant extension rate. Gates [10], [11] derives and numerically integrates linear equations of motion for a spinning central hub with flexible extending appendages. Lips, et. al., [12] examine dynamics issues of the WISP space experiment, specifically, response to vernier thruster torque and constant spin rate. Dow et. al., [13] use a process of lumped mass discretization to investigate dynamics behavior of satellite antenna deployment, including effects of thermal bending, solar pressure, and a magnetic damper boom.

1.2 Problem Definition

This work is meant to provide the capability to determine the dynamics response to a prescribed forcing function for a rotating spacecraft with extending flexible appendages undergoing moderate displacements. Additionally, the capability to alter the analysis to model special subcases is also desired.

1.3 Thesis Overview

Chapter 2 presents the definition of the mathematical model of the spacecraft structure and shows the development of the equilibrium equations of the entire structure and also of a single typical beam element. These equations, along with other constitutive relations, are used to derive the integral partial differential equations which govern the rotational and vibrational motion of the structure.

Chapter 3 outlines the transformation of the partial differential equations to ordinary differential equations. A change of variables is made to nondimensionalize space and time coordinates and make computation simpler. The lateral beam displacement is assumed to be equal to a summation of time-varying cantilevered beam modes and the p.d.e.'s are transformed to ordinary differential equations. These o.d.e.'s are rewritten in first order state vector form to facilitate numerical integration.

Chapter 4 transforms the nonlinear, time-varying rotational and vibrational o.d.e.'s to some special subcases. Results which duplicate those found in several existing papers are presented along with new data calculated from the complete nonlinear extending equation.

Chapter 2

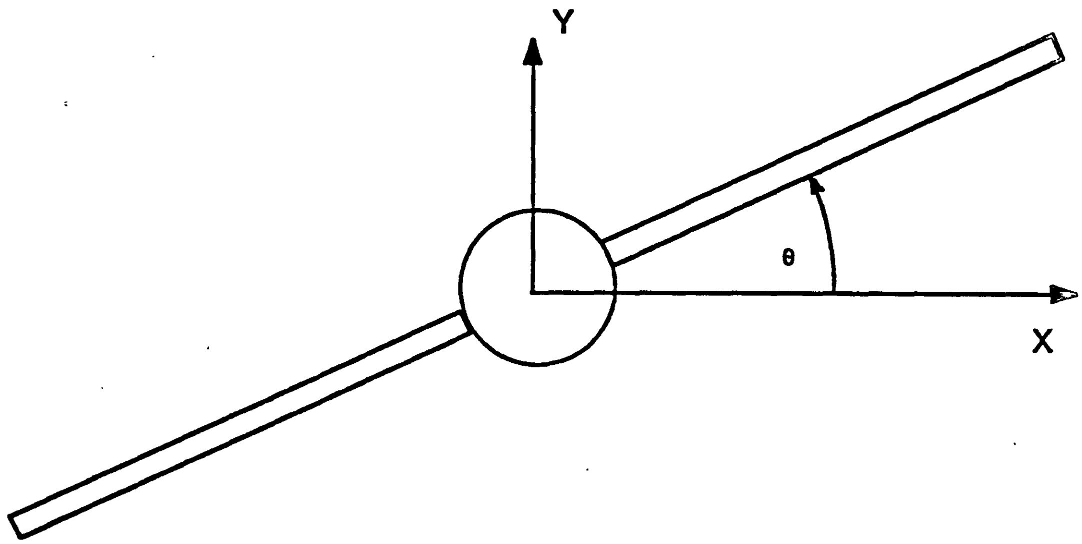
Equations of Motion

In this section the system model is introduced, the simplifying assumptions are listed and the system kinematics are defined. The translational and rotational forms of Newton's second law are applied to an element of mass of the beam to determine the vibrational partial differential equation of motion for the structure. The rotational partial differential equation of motion is obtained by applying the law of conservation of angular momentum to the structure. As in [1], a Newton's method approach was preferred over Hamiton's principle because it offers a rigorous and concise method for (1) analyzing the time-varying structural configuration of the model, and (2) extending the analysis to the nonlinear case. Additionally, this method does not require an explicit formulation of the system strain energy.

2.1 System Model

The spacecraft structure is shown in Fig. 2.1. Although this analysis models a hub with two appendates, a minor change allows extension of the analysis to any even number of appendages. Each of the two appendages is modelled as a cantilevered Bernoulli-Euler beam extending from the central body, or hub, of the spacecraft. Although the hub contributes rotational inertia to the system, its radius is considered small and is set equal to zero to simplify the equations.

Figure 2.1: Spacecraft Structure



Assumptions

The following simplifications are made:

1. The beams undergo "large-but-moderate" elastic deflection. While the deflections encountered are larger than those usually associated with small deflections, they are not "large". Eq. 2.2 defines the order of the nonlinear terms retained in the analysis.

2. The length of the beam is an arbitrary, prescribed function of time, given by the variable $l(t)$.

3. Both beams have identical structural properties and extension rate .

4. The mass density, m, and the stiffness, EI, are both constant along the length of the beam.

5. The structure undergoes only antisymmetric elastic deformation.

6. Rigid and flexible body motion occur only within the plane of rotation.

7. The beams are axially rigid. As the beam deflects it does not "stretch" axially; it's length remains that which is prescribed by $l(t)$. Appendix A examines the consequences of this assumption.

2.2 System Kinematics

The motion of the vehicle in its undeformed state is described by the hub orientation angle, θ, and the longitudinal coordinate of a particle along the appendages's centerline x (See Fig. 2.2). It is assumed that during deformation a centerline particle undergoes a longitudinal displacement, u, and a lateral displacement, v, with respect to its original undeformed location. Only antisymmetric deformation modes are considered, so that the extension rate, the material and geometric properties, and the forcing functions do not differ from one appendage to the next. Therefore the variables $x(t)$, $v(t)$, and $u(t)$ are equal for all appendages.

The hub-attached frame, (x, y), is fixed to the undeformed beam and rotates with respect to the inertial frame, (X, Y). Using the above kinematic variables, the inertial vector $\vec{R}$ locates a centerline particle on the beam, shown in Fig. 2.2. $\vec{R}$ is expressed in terms of the above kinematic variables in the hub-attached set of dextral axes $(\hat{i}, \hat{j}, \hat{k})$ as

$$\vec{R} = \{x(t) - u(x(t), t),\ v(x(t), t),\ 0\} \tag{2.1}$$

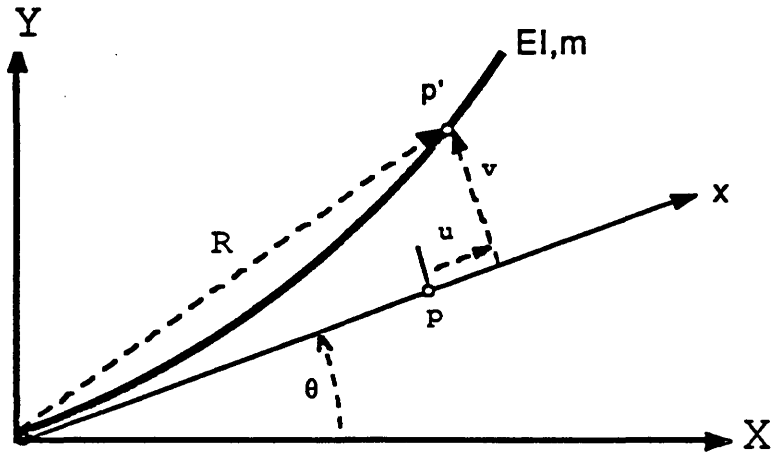
Since it is assumed that the appendage elastic deformation is large but moderate, terms of order v^2 are retained in this analysis, but terms of order v^3 or higher are eliminated. It is also assumed that the axial deformation gradient, u_x, is of the same order as v_x^2, and the square of either of these values is negligible when compared to unity:

$$u_x^2 \sim v_x^4 \ll 1 \tag{2.2}$$

Under the above assumptions the rotational deformation of the centerline, β, is defined such that (see Fig. 2.3)

$$\cos\beta = \frac{1 + u_x}{\sqrt{(1 - u_x)^2 + (v_x)^2}} \approx 1 + u_x \tag{2.3}$$

Figure 2.2: Location of Mass Particle on Deformed Beam



and

$$\sin\beta = \frac{v_x}{\sqrt{(1+u_x)^2+(v_x)^2}} \approx v_x \tag{2.4}$$

in which only $\cos\beta$ differs from the often invoked linear approximation. Note that the denominator in the above fractions is equal to 1 + a fourth order term when the axial rigidity relation is used (Eq. A.4).

2.3 Application of Newton's Laws

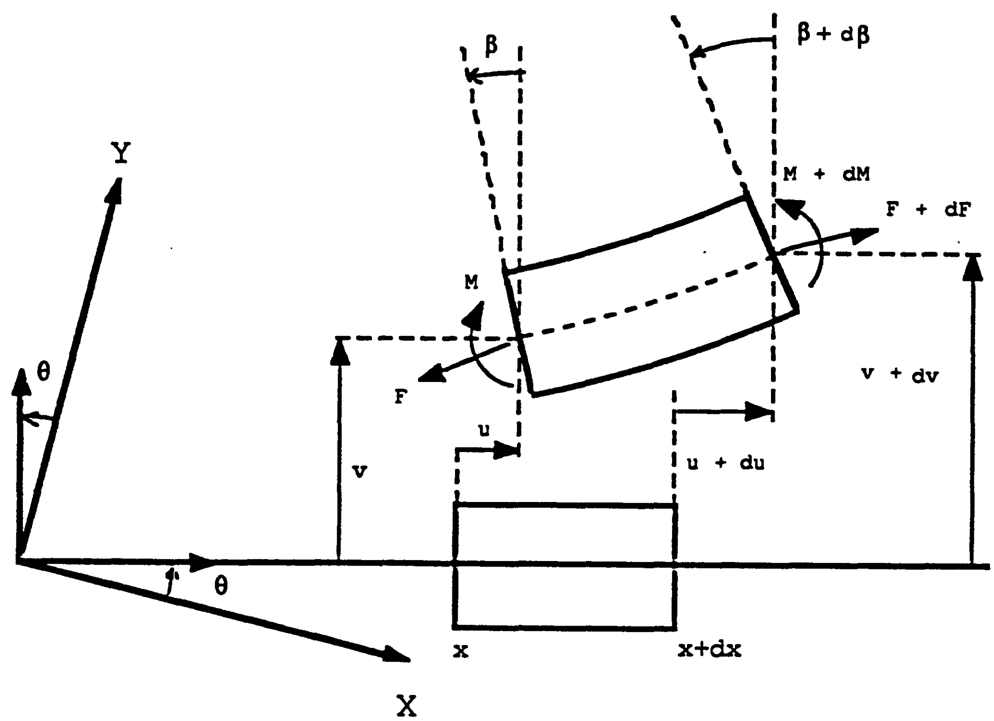
Applying Newton's second law to a particle of mass on the beam, dm, shown in Fig. 2.3, produces this equation for translational motion:

$$-\vec{F} + (\vec{F} + \frac{\partial \vec{F}}{\partial x}dx) = m dx \frac{d\vec{V}}{dt} \tag{2.5}$$

which is simplified to

$$\frac{\partial \vec{F}}{\partial x} = m\frac{d\vec{V}}{dt} \tag{2.6}$$

Figure 2.3: Beam Element Forces and Displacements



$$dF = \frac{\partial F}{\partial x}dx \qquad d\beta = \frac{\partial \beta}{\partial x}dx \qquad du = \frac{\partial u}{\partial x}dx \qquad dv = \frac{\partial v}{\partial x}dx$$

Newton's second law also yields the following equation which governs the particle's rotational motion.

$$-\vec{M} + (\vec{M} + \frac{\partial \vec{M}}{\partial x} dx) + \vec{R} \times (-\vec{F}) + \vec{R}(x + dx) \times (\vec{F} + \frac{\partial \vec{F}}{\partial x} dx)$$

$$= \frac{d}{dt}(\vec{R} \times (m dx)\vec{V}) \tag{2.7}$$

Eq. 2.7 can be simplified by expanding it as

$$\frac{\partial \vec{M}}{\partial x} + \frac{\partial \vec{R}}{\partial x} \times \vec{F} + \vec{R} \times \frac{\partial \vec{F}}{\partial x} = m\frac{d\vec{R}}{dt} \times \vec{V} + m\vec{R} \times \frac{d\vec{V}}{dt} \tag{2.8}$$

Noting that $\vec{V} = d\vec{R}/dt$ and substituting for $\partial \vec{F}/\partial x$ using Eq. 2.6 leads to

$$\frac{\partial \vec{M}}{\partial x} = -\frac{\partial \vec{R}}{\partial x} \times \vec{F} \tag{2.9}$$

Referring to Fig. 2.4, the appendage internal force, $\vec{F}$, is expressed in the body-frame as

$$\vec{F} = \{F_1,\ F_2,\ 0\} \tag{2.10}$$

where

$$F_1 = T(1 + u_x) - Sv_x \tag{2.11}$$

$$F_2 = Tv_x + S(1 + u_x) \tag{2.12}$$

The inertial velocity vector, $\vec{V}$, also expressed in the body-frame, is given by

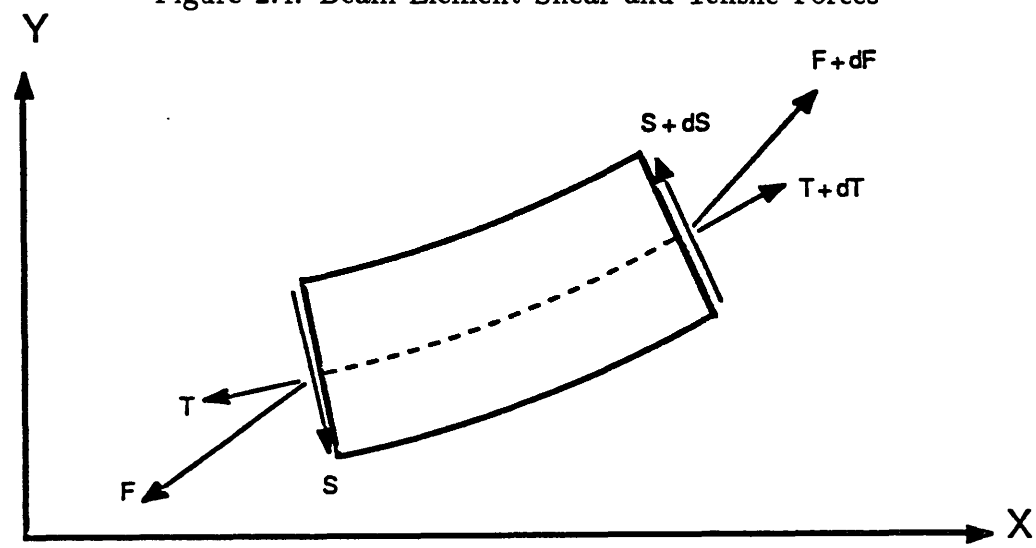
$$\vec{V} = \frac{d\vec{R}}{dt} = \{V_1,\ V_2,\ 0\} \tag{2.13}$$

where

$$\begin{aligned} V_1 &= \dot{x} + \dot{x}u_x + \dot{u} - (\vec{\omega} \times \vec{R}) \cdot \hat{i} \\ &= \dot{x}(1 + u_x) + \dot{u} - \dot{\theta}v \end{aligned} \tag{2.14}$$

$$\begin{aligned} V_2 &= \dot{v} + \dot{x}v_x + (\vec{\omega} \times \vec{R}) \cdot \hat{j} \\ &= \dot{v} + \dot{x}v_x + \dot{\theta}(x + u) \end{aligned} \tag{2.15}$$

Figure 2.4: Beam Element Shear and Tensile Forces



$$dF = \frac{\partial F}{\partial x}dx \qquad dS = \frac{\partial S}{\partial x}dx \qquad dT = \frac{\partial T}{\partial x}dx$$

and the internal moment vector, $\vec{M}$, is given by

$$\vec{M} = \{0,\ 0,\ M_3\} \tag{2.16}$$

Taking the partial derivative of Eq. 2.16 with respect to x yields

$$\frac{\partial \vec{M}}{\partial x} = \frac{\partial}{\partial x} M_3 \hat{k} \tag{2.17}$$

Calculating $\partial \vec{R}/\partial x$ from Eq. 2.1

$$\frac{\partial \vec{R}}{\partial x} = \{1 + u_x,\ v_x,\ 0\} \tag{2.18}$$

Substituting the above two equations along with Eq. 2.10 into Eq. 2.9 yields

$$\frac{\partial M_3}{\partial x} = -\{F_2(1 + u_x) - F_1 v_x\} \tag{2.19}$$

Using the expressions for F_1 and F_2 from Eqs. 2.11 and 2.12, respectively, results in

$$\frac{\partial M_3}{\partial x} = -\{(Tv_x + S(1 - u_x))(1 + u_x) - (T(1 + u_x) - Sv_x)v_x\} \qquad (2.20)$$

or,

$$\frac{\partial M_3}{\partial x} = -\{Tv_x + S + Su_x + Tv_x u_x - Su_x + Su_x^2 - Tv_x - Tv_x u_x + Sv_x^2\} \quad (2.21)$$

Since $u_x \sim v_x^2$ and only terms of order 2 or less are retained in this analysis, the equation is reduced to:

$$\frac{\partial M_3}{\partial x} = -\{S(1 - 2u_x - v_x^2)\} \qquad (2.22)$$

which, after invoking the assumption of axial rigidity (see Appendix A), becomes

$$\frac{\partial \vec{M}_3}{\partial x} = -S \qquad (2.23)$$

or, using the moment displacement relation shown in appendix B,

$$EIv_{xxx} = -S \qquad (2.24)$$

2.4 Vibrational Equation of Motion

The equations which govern the vibrational equilibrium of the beam can be obtained by expanding Eq. 2.6. The acceleration vector is computed from Eq. 2.13 as

$$\begin{aligned} \frac{d\vec{V}}{dt} = \frac{d^2\vec{R}}{dt^2} &= \left\{\frac{dV_1}{dt}, \frac{dV_2}{dt}, 0\right\} + \vec{\omega} \times \vec{V} \\ &= \left\{\frac{dV_1}{dt} - V_2\dot{\theta}, \frac{dV_2}{dt} + V_1\dot{\theta}, 0\right\} \end{aligned} \qquad (2.25)$$

Differentiating Eqs. 2.14 and 2.15 yields

$$\frac{dV_1}{dt} = \ddot{x} - \dot{u}_x\dot{x} - u_{xx}\dot{x}^2 - \ddot{u} + \dot{u}_x\dot{x} - \ddot{\theta}v - \dot{\theta}(\dot{v} + v_x\dot{x}) + u_x\ddot{x} \qquad (2.26)$$

$$\frac{dV_2}{dt} = \ddot{v} + 2\dot{v}_x\dot{x} + v_{xx}\dot{x}^2 + v_x\ddot{x} + \ddot{\theta}(x + u) + \dot{\theta}(\dot{x} + u_x\dot{x} + \dot{u}) \tag{2.27}$$

and from direct substitution of Eqs. 2.14 and 2.15,

$$-V_2\dot{\theta} = -\dot{\theta}(\dot{v} + v_x\dot{x} + \dot{\theta}x + \dot{\theta}u) \tag{2.28}$$

$$V_1\dot{\theta} = \dot{\theta}(\dot{x} + u_x\dot{x} + \dot{u} - \dot{\theta}v) \tag{2.29}$$

Finally, combining Eqs. 2.25 through 2.29,

$$\frac{d\vec{V}}{dt} = \{a_1,\ a_2,\ 0\} \tag{2.30}$$

where

$$a_1 = \ddot{x}(1 + u_x) + 2\dot{u}_x\dot{x} + \ddot{u} + u_{xx}\dot{x}^2 - \ddot{\theta}v - 2\dot{\theta}(\dot{v} + v_x\dot{x}) - \dot{\theta}^2(x + u) \tag{2.31}$$

$$a_2 = \ddot{v} + 2\dot{v}_x\dot{x} + v_{xx}\dot{x}^2 + v_x\ddot{x} + \ddot{\theta}(x + u) + 2\dot{\theta}(\dot{x} + u_x\dot{x} + \dot{u}) - \dot{\theta}^2v \tag{2.32}$$

Eq. 2.6, is now written in the $\hat{i}$ direction as

$$\frac{\partial}{\partial x}(T + Tu_x - Sv_x) = ma_1 \tag{2.33}$$

and in the $\hat{j}$ direction as

$$\frac{\partial}{\partial x}(S + Su_x + Tv_x) = ma_2 \tag{2.34}$$

The vibrational equation of motion, Eq. 2.33, is integrated with respect to x to obtain

$$(T + Tu_x - Sv_x)|_x^l = m\int_x^l a_1dx \tag{2.35}$$

Noting that at $x = l, T = S = 0$,

$$T = (-m\int_x^l a_1dx + Sv_x)(1 + u_x)^{-1} \tag{2.36}$$

This result is substituted into Eq. 2.34, leading to

$$\frac{\partial}{\partial x}\left[(-m\int_x^l a_1dx + Sv_x)(1 + u_x)^{-1}v_x + S + Su_x\right] = ma_2 \tag{2.37}$$

Using the expresion for the shear force from Eq. 2.24,

$$\frac{\partial}{\partial x}\left[(-m\int_x^l a_1 dx - EIv_{xxx}v_x)(1+u_x)^{-1}v_x - EIv_{xxx} - EIv_{xxx}u_x\right] = ma_2 \tag{2.38}$$

Since $u_x^2 \ll 1$, $|u_x| < 1$, so $(1+u_x)^{-1}$ can be represented as the infinite series:

$$1 + u_x + higher\ order\ terms \tag{2.39}$$

Using this form of the expression and neglecting terms of third or higher order yields

$$\frac{\partial}{\partial x}\left[(-m\int_x^l a_1 dx)v_x - EIv_{xxx}\right] = ma_2 \tag{2.40}$$

or

$$-EIv_{xxxx} - \frac{\partial}{\partial x}\left[m\int_x^l a_1 dx v_x\right] = ma_2 \tag{2.41}$$

Substituting the expressions for a_1 and a_2 from Eqs. 2.31 and 2.32,

$$\begin{aligned}
-EIv_{xxxx} - \frac{\partial}{\partial x}(mv_x\int_x^l & \ddot{x}(1-u_x) + 2\dot{u}_x\dot{x} - \ddot{u} - u_{xx}\dot{x}^2 \\
& -\ddot{\theta}v - 2\dot{\theta}(\dot{v} - v_x\dot{x}) - \dot{\theta}^2(x+u)\, dx) \\
= m(\ddot{v} - 2\dot{v}_x\dot{x} - v_{xx}\dot{x}^2 - v_x\ddot{x} - \ddot{\theta}(x-u) & + 2\dot{\theta}(\dot{x} + u_x\dot{x} - \dot{u}) - \dot{\theta}^2 v)
\end{aligned} \tag{2.42}$$

or

$$\begin{aligned}
& -\frac{EI}{m}v_{xxxx} - \ddot{v} - \ddot{x}v_x - 2\dot{x}\dot{v}_{xx} - \dot{x}^2 v_{xx} \\
& -\frac{\partial}{\partial x}\left\{v_x\int_x^l(\ddot{x}(1+u_x) + 2\dot{x}\dot{u}_x + \ddot{u} + \dot{x}^2 u_{xx})dx\right\} \\
& -\left[x - u - \frac{\partial}{\partial x}\left\{v_x\int_x^l v dx\right\}\right]\ddot{\theta} \\
& -\left[v - \frac{\partial}{\partial x}\left\{v_x\int_x^l(x+u)dx\right\}\right]\dot{\theta}^2 \\
& -2\left[\dot{x} + \dot{x}u_x - \dot{u} - \frac{\partial}{\partial x}\left\{v_x\int_x^l(\dot{v} + \dot{x}v_x)dx\right\}\right]\dot{\theta} = 0
\end{aligned} \tag{2.43}$$

Eliminating all terms third order and higher leads to the final form of the vibrational equation as:

$$
\begin{aligned}
&-\frac{EI}{m}v_{xxxx} - \ddot{v} - \ddot{x}v_x - 2\dot{x}\dot{v}_x - \dot{x}^2 v_{xx} - \frac{\partial}{\partial x}\left\{v_x \int_x^l \ddot{x}\,dx\right\} \\
&-\left[x + u + \frac{\partial}{\partial x}\left\{v_x \int_x^l v\,dx\right\}\right]\ddot{\theta} \\
&+\left[v + \frac{\partial}{\partial x}\left\{v_x \int_x^l x\,dx\right\}\right]\dot{\theta}^2 \\
&-2\left[\dot{x} + \dot{x}u_x + \dot{u} - \frac{\partial}{\partial x}\left\{v_x \int_x^l (\dot{v} + \dot{x}v_x)dx\right\}\right]\dot{\theta} = 0
\end{aligned}
\tag{2.44}
$$

2.5 Rotational Equation of Motion

The governing rotational dynamics equation for the structure is

$$\frac{d\vec{H}}{dt} = \{0,\ 0,\ T_0\}^T \tag{2.45}$$

where T_o is the torque applied at the hub's center of mass. $\vec{H}$ is the system angular momentum, given by

$$\vec{H} = I_h\dot{\theta}\hat{k} + 2\int_0^{l(t)} m\vec{R} \times \vec{V}dx \tag{2.46}$$

Substituting for the vectors $\vec{R}$ and $\vec{V}$ leads to

$$T_o = I_h\ddot{\theta} + 2m\frac{d}{dt}\int_0^{l(t)}\{(\dot{v} + v_x\dot{x} + \dot{\theta}(x+u))(x+u) - (\dot{x}(1+u_x) + \dot{u} - \dot{\theta}v)v\}dx \tag{2.47}$$

Retaining only second and lower order terms in v and gradients of v yields

$$T_0 = I_h\ddot{\theta} - 2m\frac{d}{dt}\int_0^{l(t)}\{\dot{\theta}(x^2 + v^2 + 2xu) + \dot{x}(xv_x - v) + x\dot{v}\}dx \tag{2.48}$$

Evaluating some integrals and partially applying the differentiation operator yields the final form of the rotational equation as

$$\begin{aligned}
T_o &= \ddot{\theta}\left[I_h + \frac{2}{3}ml^3 + 2m\int_0^{l(t)}\{v^2 + 2ux\}dx\right] \\
&+ 2m\dot{\theta}\left[\dot{l}l^3 + \frac{d}{dt}\int_0^{l(t)}\{v^2 + 2ux\}dx\right] \\
&+ 2m\dot{l}\left[2l\dot{v}(l,t) + \dot{l}lv_x(l,t) - \dot{l}v(l,t)\right] + 2m\ddot{l}lv(l,t) \\
&+ 2m\int_0^{l(t)}\left\{x\ddot{v} - 2\dot{l}\dot{v} - 2\ddot{l}v\right\}dx
\end{aligned} \tag{2.49}$$

Eq. 2.44, the vibrational equation of motion, and Eq. 2.49, the rotational equation of motion, are the nonlinear integral partial differential equations which completely govern the rigid-body and flexible-body motion of the system as it undergoes rotation and extension. They are shown together at the beginning of the next section.

Chapter 3

Solution of Equations

3.1 Governing Equations

Eq. 2.4, the vibrational equation of motion, and Eq. 2.49, the rotational equation of motion, are rewritten here as

Vibrational Equation

$$
\begin{aligned}
&-\frac{EI}{m}v_{xxxx} - \ddot{v} - \ddot{x}v_x - 2\dot{x}\dot{v}_x - \dot{x}^2 v_{xx} - \frac{\partial}{\partial x}\left\{v_x \int_x^l \ddot{x}\,dx\right\} \\
&-\left[x + u - \frac{\partial}{\partial x}\left\{v_x \int_x^l v\,dx\right\}\right]\ddot{\theta} \\
&-\left[v - \frac{\partial}{\partial x}\left\{v_x \int_x^l x\,dx\right\}\right]\dot{\theta}^2 \\
&-2\left[\dot{x} + \dot{x}u_x + \dot{u} - \frac{\partial}{\partial x}\left\{v_x \int_x^l (\dot{v} + \dot{x}v_x)dx\right\}\right]\dot{\theta} = 0 \qquad (3.1)
\end{aligned}
$$

Rotational Equation

$$
\begin{aligned}
T_o &= \ddot{\theta}[I_h + \frac{2}{3}ml^3 + 2m\int_0^{l(t)}\{v^2 + 2ux\}dx] \\
&+ 2m\dot{\theta}[\dot{l}l^3 + \frac{d}{dt}\int_0^{l(t)}\{v^2 + 2ux\}dx] \\
&+ 2m\dot{l}[2l\dot{v}(l,t) + \dot{l}lv_x(l,t) - \dot{l}v(l,t)] + 2m\ddot{l}lv(l,t) \\
&- 2m\int_0^{l(t)}\{x\ddot{v} - 2\dot{l}\dot{v} - 2\ddot{l}v\}dx
\end{aligned}
\tag{3.2}
$$

3.2 Solution Methodology

To solve the two partial differential equations presented above and determine the rotational and vibrational response of the beam to a particular forcing function, the two equations of motion are transformed into a system of ordinary differential equations in $\eta(t)$. This is done by first transforming the equations of motion to functions of nondimensional spatial and temporal variables. Second, a separation of variables technique is applied and Galerkin's method is used to transform the equations to ordinary differential equation form. These ordinary differential equations are then written in state vector representation, making numerical integration feasible.

3.3 Change of Variables

The functional dependence of all functions is changed so that any functions of $x(t), t$ become functions of the nondimensional variables ξ, τ, i.e.,

$$\{x(t), t\} \rightarrow \{\xi, \tau\} \tag{3.3}$$

where ξ and τ are given by

$$\xi = \frac{x}{l} \tag{3.4}$$

and

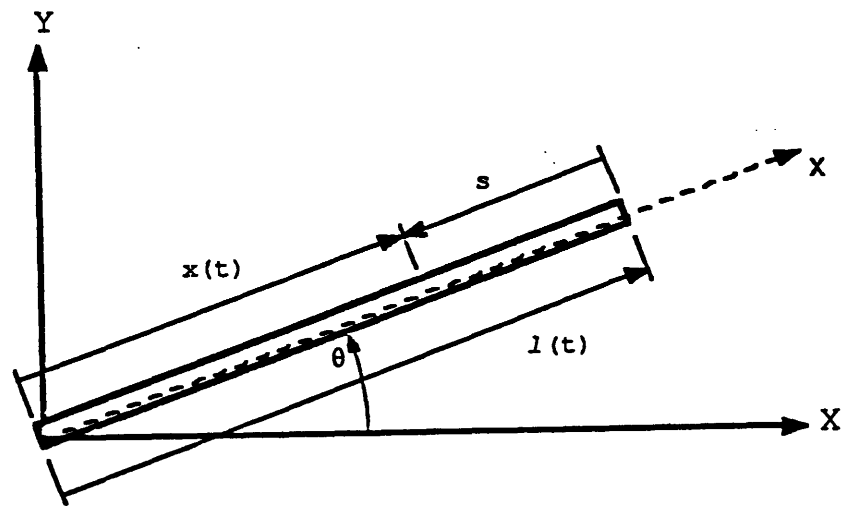
$$\tau = \frac{t}{t_r} \tag{3.5}$$

The Jacobean matrix is given for this transformation by the following formula:

$$\left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial t} \end{array} \right\} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \tau}{\partial x} \\ \\ \frac{\partial \xi}{\partial t} & \frac{\partial \tau}{\partial t} \end{bmatrix} \left\{ \begin{array}{c} \frac{\partial}{\partial \xi} \\ \\ \frac{\partial}{\partial \tau} \end{array} \right\} \tag{3.6}$$

Three of the four derivatives to be calculated in the Jacobean matrix are straightforward. The calculation of the fourth, $\partial\xi/\partial t$, is slightly more involved and is presented in the following paragraph.

Figure 3.1: Coordinate of Material Point on Beam



It is first noted that a material point on the underformed beam can be identified not only by the usual time varying coordinate, x, but also by the time independent coordinate, s. $s = 0$ at the tip of the beam, as shown in Fig. 3.1. The relationship between x and s is given by:

$$x(t) = l(t) - s \tag{3.7}$$

The scalar time derivative of this expression yields the relationship

$$\dot{x} = \dot{l} \tag{3.8}$$

With this equation in mind, ξ is written as a function of x and t as

$$\xi(x,t) = \xi(x(t),t) = \frac{x(t)}{l(t)} \tag{3.9}$$

Then

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial x}\frac{dx}{dt} + \frac{\partial \xi}{\partial t} \tag{3.10}$$

Using Eqs. 3.4 and 3.8,

$$\frac{d\xi}{dt} = \frac{1}{l}\dot{x} - \frac{x\dot{l}}{l^2} = \frac{\dot{l}}{l} - \xi\frac{\dot{l}}{l} \tag{3.11}$$

The Jacobean requires the partial time derivative of ξ, and it is seen from the second term in Eq. 3.11 that

$$\frac{\partial \xi}{\partial t} = -\xi\frac{\dot{l}}{l} \tag{3.12}$$

Eq. 3.6 becomes:

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{1}{l} & 0 \\ -\xi\frac{\dot{l}}{l} & \frac{1}{t_r} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \tau} \end{Bmatrix} \tag{3.13}$$

Using Eq. 3.13 the derivatives of v present in the partial differential equations of motion are evaluated as

$$\frac{\partial v}{\partial x} = \frac{1}{l}\frac{\partial v}{\partial \xi} \tag{3.14}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{l^2}\frac{\partial^2 v}{\partial \xi^2} \tag{3.15}$$

$$\frac{\partial^4 v}{\partial x^4} = \frac{1}{l^4}\frac{\partial^4 v}{\partial \xi^4} \tag{3.16}$$

$$\frac{\partial v}{\partial t} = -\xi\frac{\dot{l}}{l}\frac{\partial v}{\partial \xi} + \frac{1}{t_r}\frac{\partial v}{\partial \tau} \tag{3.17}$$

$$\frac{\partial^2 v}{\partial t^2} = \xi^2\frac{\dot{l}^2}{l^2}\frac{\partial^2 v}{\partial \xi^2} + \left[\frac{2\xi\dot{l}^2}{l^2} - \frac{\xi\ddot{l}}{l}\right]\frac{\partial v}{\partial \xi} - \frac{2\xi}{t_r}\frac{\dot{l}}{l}\frac{\partial^2 v}{\partial \xi \partial \tau} + \frac{1}{t_r^2}\frac{\partial^2 v}{\partial \tau^2} \tag{3.18}$$

$$\frac{\partial^2 v}{\partial x \partial t} = \frac{1}{l}\left[-\frac{\dot{l}}{l}\frac{\partial v}{\partial \xi} - \xi\frac{\dot{l}}{l}\frac{\partial^2 v}{\partial \xi^2} + \frac{1}{t_r}\frac{\partial^2 v}{\partial \xi \partial \tau}\right] \tag{3.19}$$

From this point on the following notation is used:

$$\frac{\partial(\ )}{\partial\xi} = (\ )'$$

$$\frac{\partial(\ )}{\partial\tau} = \overset{\circ}{(\ )}$$

where ( ) denotes any function.

Next, all functions and operators in the equations of motion are altered so that functions of x and t and differentiations and integrations with respect to x and t are written as functions of ξ and/or τ and differentiation and integration with respect to ξ or τ, respectively.

Substituting $v(\xi,\tau)$ for $v(x(t),t)$ in the vibrational equation and making use of the above relations, simplifying, evaluating one of the integrals and rearranging terms yields the desired form of the vibrational equation, comparable to Eq. 3.1, with all expressions functions of ξ and τ.

$$\begin{aligned}
&\frac{EIt_\tau^2}{ml^4}v'''' - (\xi^2 - 2\xi + 1)\left(\frac{\overset{\circ}{l}}{l}\right)^2 v'' + (1-\xi)\left[\frac{\overset{\circ\circ}{l}}{l} - 2\frac{\overset{\circ}{l}^2}{l^2}\right]v' \\
&+ (1-\xi)\frac{2\overset{\circ}{l}}{l}\overset{\circ}{v}' \overset{\circ\circ}{v} - \frac{\overset{\circ\circ}{l}}{l}\frac{\partial}{\partial\xi}[v'(1-\xi)] \\
&- \left[l\xi - u - \frac{1}{l}\frac{\partial}{\partial\xi}\left\{v'\int_\xi^1 v\,d\xi\right\}\right]\overset{\circ\circ}{\theta} \\
&- \left[v - \frac{1}{2}\frac{\partial}{\partial\xi}\left\{v'(1-\xi^2)\right\}\right]\overset{\circ}{\theta}^2 \\
&- 2\left[\overset{\circ}{l}(1 - \frac{1}{l}u_\xi) - \overset{\circ}{u} - \frac{1}{l}\frac{\partial}{\partial\xi}\left\{\int_\xi^1 v'\left(\frac{-\xi\overset{\circ}{l}}{l}v' + \overset{\circ}{v} - \frac{\overset{\circ}{l}}{l}d\xi\right)v'\right\}\right]\overset{\circ}{\theta} = 0
\end{aligned} \tag{3.20}$$

Similarly, substituting $v(\xi,\tau)$ for $v(x(t),t)$, making use of relations 3.14 - 3.19 and simplifying the rotational equation yields its desired form, comparable with Eq. 3.2, with all expressions functions of ξ and τ, rather than x and t, as

$$\frac{T_0 t_\tau^2}{2m} = \left\{\frac{I_h}{2m} + \frac{l^3}{3} + l\int_0^1 (v^2 + 2u\xi l)d\xi\right\}\overset{\circ\circ}{\theta}$$

$$+ \left\{ l^2 \mathring{l} + \frac{l}{t_r} \frac{\partial}{\partial \tau} \int_0^1 (v^2 + 2u\xi l) d\xi \right\} \mathring{\theta}$$
$$+ (\overset{\circ\circ}{l} l - \overset{\circ 2}{l}) v(1) - \overset{\circ 2}{l} v'(1) + 2 \mathring{l} l \mathring{v}(1)$$
$$+ \int_0^1 \left\{ \xi^3 \overset{\circ 2}{l} v'' - 2\xi^2 \mathring{l} l \mathring{v}' + l^2 \xi \overset{\circ\circ}{v} - \overset{\circ\circ}{l} lv \right.$$
$$\left. -2 \mathring{l} l \mathring{v} + (2\xi^2 \overset{\circ 2}{l} + 2 \overset{\circ 2}{l} \xi - \xi^2 \overset{\circ\circ}{l} l) v' \right\} d\xi \qquad (3.21)$$

3.4 Separation of Variables

It is assumed that the lateral displacement of the beam, v, is equal to a summation of admissible functions multiplied by time dependent coefficients. This is written in vector notation as

$$v(\xi, \tau) = l_r \vec{\gamma}^T(\xi) \vec{\eta}(\tau) \qquad (3.22)$$

where l_r is a reference length and γ and η are nondimensional and given by

$$\vec{\gamma}^T(\xi) = (\gamma_1, \gamma_2, \ldots, \gamma_n)$$
$$\vec{\eta}^T(\tau) = (\eta_1, \eta_2, \ldots, \eta_n)$$

n is arbitrary and is equal to the number of admissible functions included in the analysis. The admissible function γ_i is assumed to be equal to the ith modeshape of a uniform slender cantilevered beam of length l. Note that this means γ_i is orthogonal to γ_j for $i \neq j$. Explicitly, this modeshape is given by

$$\gamma_i(\xi) = [\cos \varepsilon_i \xi - \cosh \varepsilon_i \xi - \beta_i (\sin \varepsilon_i \xi - \sinh \varepsilon_i \xi)] \qquad (3.23)$$

where

$$\beta_i = \frac{\cos \varepsilon_i + \cosh \varepsilon_i}{\sin \varepsilon_i - \sinh \varepsilon_i} \qquad (3.24)$$

This modeshape satisfies the boundry conditions

$$\vec{\gamma}(0) = \frac{d\vec{\gamma}}{d\xi}(0) = \frac{d^2\vec{\gamma}}{d\xi^2}(1) = \frac{d^3\vec{\gamma}}{d\xi^3}(1) = 0 \qquad (3.25)$$

Note that the fourth derivative of γ with respect to x is equal to a constant multiple of γ itself. Specifically,

$$\frac{d^4\vec{\gamma}}{d\xi^4} = \mathbf{\Lambda}\vec{\gamma} \tag{3.26}$$

where $\mathbf{\Lambda}$ is the diagonal matrix given by

$$\mathbf{\Lambda} = \begin{bmatrix} \varepsilon_1^4 & & & \mathbf{0} \\ & \varepsilon_2^4 & & \\ & & \ddots & \\ \mathbf{0} & & & \varepsilon_n^4 \end{bmatrix} \tag{3.27}$$

Additionally, this modeshape satisfies the following orthonormality relation

$$\int_0^1 \vec{\gamma}\vec{\gamma}^T d\xi = \mathbf{I} \tag{3.28}$$

where $\mathbf{I}$ is the identity matrix.

3.5 Transformation to O.D.E. form

3.5.1 Transformation of Vibrational Equation to O.D.E. form

To transform the partial differential equation governing vibrational motion, Eq. 3.20, to an ordinary differential equation the substitution of Eq. 3.22 is made. The equation is then premultiplied by $\vec{\gamma}$ and integrated over ξ from 0 to 1, removing the dependence on ξ from the equation.

Substituting $v = l_r\vec{\gamma}^T(\xi)\vec{\eta}(\tau)$ into the vibrational equation, premultiplying by $\vec{\gamma}(\xi)$ and integrating from 0 to 1 yields

$$\frac{EIl_r t_r^2}{ml^4}\int_0^1 \vec{\gamma}\vec{\gamma}''''^T d\xi\vec{\eta} - \left(\frac{\overset{\circ}{l}}{l}\right)^2 l_r \int_0^1 (\xi^2 - 2\xi - 1)\vec{\gamma}\vec{\gamma}''^T d\xi\vec{\eta}$$

$$+\frac{\overset{\circ\circ}{l}\, l_r}{l}\int_0^1 (1-\xi)\vec{\gamma}\vec{\gamma}'^T d\xi\vec{\eta} - \frac{2\,\overset{\circ}{l}^2\, l_r}{l^2}\int_0^1 (1-\xi)\vec{\gamma}\vec{\gamma}'^T d\xi\vec{\eta} + \frac{2\,\overset{\circ}{l}\, l_r}{l}\int_0^1 (1-\xi)\vec{\gamma}\vec{\gamma}'^T d\xi\ \overset{\circ}{\vec{\eta}}$$

$$-\,l_r\int_0^1 \vec{\gamma}\vec{\gamma}^T d\xi\ \overset{\circ\circ}{\vec{\eta}} - \frac{\overset{\circ\circ}{l}\, l_r}{l}\int_0^1 \vec{\gamma}\frac{\partial}{\partial\xi}\left[(1-\xi)\vec{\gamma}^T\right] d\xi\vec{\eta}$$

$$-\overset{\circ\circ}{\theta}\int_0^1 \vec\gamma l\xi d\xi + \overset{\circ\circ}{\theta}\int_0^1 \vec\gamma u d\xi - \frac{\overset{\circ\circ}{\theta}\, l_r^2}{l}\int_0^1 \vec\gamma \frac{\partial}{\partial\xi}\left[\vec\gamma'^T\vec\eta\int_\xi^1 \vec\gamma'^T\vec\eta d\xi\right]d\xi$$

$$-\overset{\circ}{\theta}^2 l_r \int_0^1 \vec\gamma\vec\gamma^T d\xi\vec\eta - \frac{\overset{\circ}{\theta}^2 l_r}{2}\int_0^1(1-\xi^2)\vec\gamma\vec\gamma'^T d\xi\vec\eta + \overset{\circ}{\theta}^2 l_r\int_0^1 \xi\vec\gamma\vec\gamma'^T d\xi\vec\eta$$

$$-2\overset{\circ}{\theta}\overset{\circ}{l}\int_0^1\vec\gamma\left(1-\frac{1}{l}u_\xi\right)d\xi - 2\overset{\circ}{\theta}\int_0^1\vec\gamma\,\overset{\circ}{u}\,d\xi - \frac{2\overset{\circ}{\theta}\, l_r^2}{l}\int_0^1\vec\gamma\frac{\partial}{\partial\xi}\left[\vec\gamma'^T\vec\eta\int_\xi^1\frac{-\xi\overset{\circ}{l}}{l}\vec\gamma'^T\vec\eta d\xi\right]d\xi$$

$$-\frac{2\overset{\circ}{\theta}\, l_r^2}{l}\int_0^1\vec\gamma\frac{\partial}{\partial\xi}\left[\vec\gamma'^T\vec\eta\int_\xi^1\vec\gamma^T d\xi\,\overset{\circ}{\vec\eta}\right]d\xi - \frac{2\overset{\circ}{\theta}\, l_r^2}{l}\int_0^1\vec\gamma\frac{\partial}{\partial\xi}\left[\vec\gamma'^T\vec\eta\frac{\overset{\circ}{l}}{l}\int_\xi^1\vec\gamma'^T d\xi\vec\eta\right]d\xi = 0 \tag{3.29}$$

Evaluating integrals and rearranging terms yields

$$l_r\overset{\circ\circ}{\vec\eta} - \frac{2\overset{\circ}{l}\, l_r}{l}\mathbf{N}\overset{\circ}{\vec\eta} + l\overset{\circ\circ}{\theta}\left\{\vec W\right\} - 2\overset{\circ}{\theta}\overset{\circ}{l}\left\{\vec Z\right\}$$

$$-\left\{\frac{l_r\overset{\circ\circ}{l}}{l}(\mathbf{N}-\mathbf{P}) - \frac{l_r\overset{\circ}{l}^2}{l^2}\mathbf{Q} - \frac{EIl_rt_r^2}{ml^4}\mathbf{\Lambda}\right\}\vec\eta - \overset{\circ}{\theta}^2 l_r\vec\eta + \frac{\overset{\circ}{\theta}^2 l_r}{2}\mathbf{B}\vec\eta$$

$$-\overset{\circ\circ}{\theta}\frac{l_r^2}{2l}\mathbf{G}^T\vec\eta - \frac{\overset{\circ\circ}{\theta}\, l_r^2}{l}\mathbf{G}\vec\eta$$

$$-\frac{2\overset{\circ}{\theta}\, l_r^2\overset{\circ}{l}}{l^2}(\mathbf{S}-\mathbf{G})\vec\eta - \frac{2\overset{\circ}{\theta}\, l_r^2}{l}\mathbf{G}\overset{\circ}{\vec\eta}$$

$$-\frac{\overset{\circ}{\theta}\, l_r^2}{l}\left(2\mathbf{G}^T\overset{\circ}{\vec\eta} - \frac{\overset{\circ}{l}}{l}\mathbf{G}^T\vec\eta\right) - \frac{\overset{\circ}{\theta}\overset{\circ}{l}\, l_r^2}{l^2}\mathbf{H}^T\vec\eta = 0 \tag{3.30}$$

So that each individual term is nondimensional, the equation is multiplied by $1/l_r$. The complete nonlinear, extending vibrational equation is:

$$\overset{\circ\circ}{\vec\eta} - \left\{\frac{2\overset{\circ}{l}}{l}\mathbf{N} - \frac{2\overset{\circ}{\theta}\, l_r}{l}\left[\mathbf{G}^T-\mathbf{G}\right]\right\}\overset{\circ}{\vec\eta} + \frac{\overset{\circ\circ}{\theta}\, l}{l_r}\left\{\vec W\right\} - \frac{2\overset{\circ}{\theta}\overset{\circ}{l}}{l_r}\left\{\vec Z\right\}$$

$$-\left\{\frac{\overset{\circ\circ}{l}}{l}(\mathbf{N}-\mathbf{P}) - \frac{\overset{\circ}{l}^2}{l^2}\mathbf{Q} - \frac{EIt_r^2}{ml^4}\mathbf{\Lambda}\right\}\vec\eta + \overset{\circ}{\theta}^2\mathbf{D}\vec\eta \tag{3.31}$$

$$-\frac{\overset{\circ\circ}{\theta}\, l_r}{l}\left\{\frac{1}{2}\mathbf{G}^T-\mathbf{G}\right\}\vec\eta - \frac{2\overset{\circ}{\theta}\, l_r\overset{\circ}{l}}{l^2}\left\{\frac{1}{2}\mathbf{G}^T+\mathbf{G}-\mathbf{S}+\frac{1}{2}\mathbf{H}^T\right\}\vec\eta = 0$$

where

$$\mathbf{G} = \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \cdots & \vec{g}_n \end{bmatrix} \qquad \vec{g}_i = \frac{1}{\varepsilon_i^4}\mathbf{A}_i\vec{\eta} \tag{3.32}$$

$$\mathbf{S} = \begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \cdots & \vec{s}_n \end{bmatrix} \qquad \vec{s}_i = \mathbf{R}_i\vec{\eta} \tag{3.33}$$

$$\mathbf{H} = \begin{bmatrix} \vec{h}_1 & \vec{h}_2 & \cdots & \vec{h}_n \end{bmatrix} \qquad \vec{h}_i = \mathbf{F}_i\vec{\eta} \tag{3.34}$$

$$\mathbf{F}_i = \int_0^1 \vec{\gamma}'\gamma_i\vec{\gamma}'^T\,d\xi \tag{3.35}$$

$$\mathbf{R}_i = \int_0^1 \vec{\gamma}'\gamma_i(\xi - 1)\vec{\gamma}'^T\,d\xi \tag{3.36}$$

$$\mathbf{A}_i = \int_0^1 \vec{\gamma}'\gamma_i'''\vec{\gamma}'^T\,d\xi \tag{3.37}$$

$$\mathbf{N} = \int_0^1 (1 - \xi)\vec{\gamma}\vec{\gamma}'^T\,d\xi \tag{3.38}$$

$$\mathbf{P} = \int_0^1 (1 - \xi)\vec{\gamma}'\vec{\gamma}'^T\,d\xi \tag{3.39}$$

$$\mathbf{Q} = \int_0^1 (1 - \xi)^2\vec{\gamma}'\vec{\gamma}'^T\,d\xi \tag{3.40}$$

$$\mathbf{B} = \int_0^1 (1 - \xi^2)\vec{\gamma}'\vec{\gamma}'^T\,d\xi \tag{3.41}$$

$$\mathbf{D} = \frac{1}{2}\mathbf{B} - \mathbf{I} \tag{3.42}$$

$$\mathbf{\Lambda} = \text{Diagonal matrix, } \Lambda_i = \varepsilon_i^4 \tag{3.43}$$

$$\left\{\vec{W}\right\} = \mathbf{\Lambda}^{-1}\vec{\gamma}''(0) = \{-2\varepsilon_i^{-2}\} \qquad i = 1, 2, \ldots, n \tag{3.44}$$

$$\left\{\vec{Z}\right\} = \mathbf{\Lambda}^{-1}\vec{\gamma}'''(0) = \{2\frac{\beta_i}{\varepsilon_i}\} \qquad i = 1, 2, \ldots, n \tag{3.45}$$

Numerical values of all the above integrals and matricies are given in appendix E.

3.5.2 Transformation of Rotational Equation to O.D.E. form

The substitution of Eq. 3.22 is made to Eq. 3.21, the rotational equation of motion. The integrals and derivatives in the equation are carried out, resulting in

$$
\begin{aligned}
\frac{T_0 t_r^2}{2m} = & \left(\frac{I_h}{2m} - \frac{l^3}{3}\right) \overset{\circ\circ}{\theta} + \mathring{l}\, l^2\, \mathring{\theta} + l_r l^2 \left\{\vec{W}\right\}^T \overset{\circ\circ}{\vec{\eta}} \\
& + 2\, \mathring{l}\, l l_r \left(2\left\{\vec{W}\right\}^T + \left\{\vec{Z}\right\}^T\right) \mathring{\vec{\eta}} + 2(\overset{\circ\circ}{l}\, l l_r + \mathring{l}^2\, l_r)\left[\left\{\vec{W}\right\}^T + \left\{\vec{Z}\right\}^T\right]\vec{\eta} \\
& - \overset{\circ\circ}{\theta}\, l_r^2 l \vec{\eta}^T \mathbf{D}\vec{\eta} - 2\, \mathring{\theta}\, l_r^2 l \vec{\eta}^T \mathbf{D}\, \mathring{\vec{\eta}} - \mathring{\theta}\mathring{l}\, l_r^2 \vec{\eta}^T \mathbf{D}\vec{\eta}
\end{aligned}
\tag{3.46}
$$

To nondimensionalize each individual term, the rotational equation is multiplied by $1/l^3$. The complete nonlinear extending rotational ordinary differential equation is

$$
\begin{aligned}
\frac{T_0 t_r^2}{2ml^3} = & \left(\frac{I_h}{2ml^3} + \frac{1}{3} - \left(\frac{l_r}{l}\right)^2 \vec{\eta}^T \mathbf{D}\vec{\eta}\right) \overset{\circ\circ}{\theta} \\
& - \left(\frac{\mathring{l}}{l} - \frac{l_r^2}{l^2}\left\{\vec{\eta}^T \frac{\mathring{l}}{l}\mathbf{D} + 2\, \mathring{\vec{\eta}}^T\, \mathbf{D}\right\}\vec{\eta}\right) \mathring{\theta} \\
& - \frac{l_r}{l}\left\{\vec{W}\right\}^T \overset{\circ\circ}{\vec{\eta}} - \frac{2\, \mathring{l}\, l_r}{l^2}\left(2\left\{\vec{W}\right\}^T - \left\{\vec{Z}\right\}^T\right) \mathring{\vec{\eta}} \\
& - 2\left(\frac{\overset{\circ\circ}{l}\, l_r}{l^2} - \frac{\mathring{l}^2\, l_r}{l^2}\right)\left[\left\{\vec{W}\right\}^T - \left\{\vec{Z}\right\}^T\right]\vec{\eta}
\end{aligned}
\tag{3.47}
$$

recall from definitions given of the arrays in the vibrational equation that

$$\mathbf{D} = \frac{1}{2}\mathbf{B} - \mathbf{I}$$

$$\mathbf{B} = \int_0^1 (1 - \xi^2)\vec{\gamma}\vec{\gamma}'^T d\xi$$

$$\left\{\vec{W}\right\} = \{-2\varepsilon_i^{-2}\} \qquad i = 1, 2, \ldots, n$$

$$\left\{\vec{Z}\right\} = \{2\frac{\beta_i}{\varepsilon_i}\} \qquad i = 1, 2, \ldots, n$$

3.6 State Vector Representation

For computational convenience the equations are rewritten in the form

$$\mathbf{M}\overset{\circ\circ}{\vec{Y}} + \mathbf{C}\overset{\circ}{\vec{Y}}_1 + \mathbf{K}Y = \vec{F} \tag{3.48}$$

where

$$\vec{Y} = \left\{ \begin{array}{c} \theta \\ \vec{\eta} \end{array} \right\} = \left\{ \begin{array}{c} \theta \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{array} \right\} \tag{3.49}$$

and

$$\vec{F} = \left\{ \begin{array}{c} T_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right\} \tag{3.50}$$

Since the nonlinear equations of motion account for the arbitrary variable length of the beam, matricies **M**, **C** and **K** form a nonlinear, time-varying system. They are expressed as

<u>Mass Matrix</u>

$$\mathbf{M} = \left[\begin{array}{ccccc} M_{RR} & \vdots & & M_{RV} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \vdots & & & \\ M_{VR} & \vdots & & M_{VV} & \\ & \vdots & & & \end{array} \right] \tag{3.51}$$

where

$$\begin{aligned} M_{RR} &= \frac{1}{3} - \left(\frac{l_r}{l}\right)^3 \frac{I_h}{2ml_r^3} - \left(\frac{l_r}{l}\right)^2 \vec{\eta}^T \mathbf{D} \vec{\eta} \\ M_{RV} &= \frac{l_r}{l} \left\{ \vec{W} \right\}^T \\ M_{VR} &= \frac{l}{l_r} \left\{ \vec{W} \right\} + \frac{l_r}{l} \left(\frac{1}{2}\mathbf{G}^T - \mathbf{G} \right) \vec{\eta} \\ M_{VV} &= \mathbf{I} \end{aligned} \tag{3.52}$$

Damping Matrix

$$\mathbf{C} = \begin{bmatrix} C_{RR} & \vdots & C_{RV} \\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\ & \vdots & \\ C_{VR} & \vdots & C_{VV} \\ & \vdots & \end{bmatrix} \tag{3.53}$$

where

$$\begin{aligned}
C_{RR} &= \frac{\mathring{l}}{l_r}\left(\frac{l_r}{l}\right) - 2\left(\frac{l_r}{l}\right)^2 \mathring{\vec{\eta}}\,\mathbf{D}\vec{\eta} - \frac{\mathring{l}}{l_r}\left(\frac{l_r}{l}\right)^3 \vec{\eta}\mathbf{D}\vec{\eta} \\
C_{RV} &= 2\frac{\mathring{l}}{l_r}\left(\frac{l_r}{l}\right)^2\left(2\left\{\vec{W}\right\}^T - \left\{\vec{Z}\right\}^T\right) \\
C_{VR} &= -2\frac{\mathring{l}}{l_r}\left\{\vec{Z}\right\}^T - 2\frac{l_r}{l}\left[\mathbf{G}^T - \mathbf{G}\right]\mathring{\vec{\eta}} \\
C_{VV} &= 2\frac{\mathring{l}}{l_r}\left(\frac{l_r}{l}\right)\mathbf{N}
\end{aligned} \tag{3.54}$$

Stiffness matrix

$$\mathbf{K} = \begin{bmatrix} K_{RR} & \vdots & K_{RV} \\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\ & \vdots & \\ K_{VR} & \vdots & K_{VV} \\ & \vdots & \end{bmatrix} \tag{3.55}$$

where

$$\begin{aligned}
K_{RR} &= 0 \\
K_{RV} &= 2\left\{\frac{\mathring{\mathring{l}}}{l_r}\left(\frac{l_r}{l}\right)^2 - \left(\frac{\mathring{l}}{l_r}\right)^2\left(\frac{l_r}{l}\right)^3\right\}\left[\left\{\vec{W}\right\}^T + \left\{\vec{Z}\right\}^T\right] \\
K_{VR} &= 0
\end{aligned}$$

$$K_{VV} = \left\{ \frac{\overset{\circ\circ}{l}}{l_r} \left(\frac{l_r}{l} \right) (\mathbf{N} - \mathbf{P}) - \left(\frac{\overset{\circ}{l}}{l_r} \right)^2 \left(\frac{l_r}{l} \right)^2 \mathbf{Q} + \frac{EIt_r^2}{ml_r^4} \left(\frac{l_r}{l} \right)^4 \mathbf{\Lambda} \right\} + \overset{\circ}{\theta}^2 \mathbf{D}$$

$$-2\,\overset{\circ}{\theta}\,\frac{\overset{\circ}{l}}{l_r} \left(\frac{l_r}{l} \right)^2 \left\{ \frac{1}{2}\mathbf{G}^T + \mathbf{G} - \mathbf{S} + \frac{1}{2}\mathbf{H}^T \right\} \tag{3.56}$$

(3.57)

The first order state equation form of Eq. 3.48 is obtained by making the substitution

$$\vec{X} = \left\{ \begin{array}{c} y \\ \overset{\circ}{y} \end{array} \right\} \tag{3.58}$$

This leads to

$$\overset{\circ}{\vec{X}} = \mathbf{L}\vec{X} + \vec{U} \tag{3.59}$$

where

$$\mathbf{L} = \left[\begin{array}{ccc} 0 & \vdots & \mathbf{I} \\ \cdots\cdots & \cdots & \cdots\cdots \\ -\mathbf{M}^{-1}\mathbf{K} & \vdots & -\mathbf{M}^{-1}\mathbf{C} \end{array} \right] \; ; \; \vec{U} = \left[\begin{array}{c} 0 \\ \cdots\cdots \\ \mathbf{M}^{-1}\vec{F} \end{array} \right] \tag{3.60}$$

This equation is the first order form of the dynamics equations of motion for the spacecraft structure. In this form the equations are easily integrated numerically using a 4^{th} order Runge - Kutta scheme. The system response to either an applied torque or a prescribed angular acceleration is determined this way using the FORTRAN computer program in Appendix F.

Chapter 4

Numerical Results

Certain simplifications of the spacecraft equations of motion provide interesting special cases for comparison with existing data. Section 4.1 presents the nonrotating extending case, the rotating nonextending case, and the linearized, rotating extending case. Section 4.2 examines new findings of improved WISP spacecraft response due to an analysis including nonlinear terms.

4.1 Special Cases

4.1.1 Nonrotating, Extending Appendage Case

Reduced Equations

The linear nonrotating, extending equation is:
<u>Vibrational Equation</u>

$$\overset{\circ\circ}{\vec{\eta}} + \frac{2\overset{\circ}{l}}{l}\mathbf{N}\overset{\circ}{\vec{\eta}} + \left\{ \frac{\overset{\circ\circ}{l}}{l}(\mathbf{N}-\mathbf{P}) - \frac{\overset{\circ}{l}^{2}}{l^{2}}\mathbf{Q} + \frac{EIt_r^2}{ml^4}\mathbf{\Lambda} \right\} \vec{\eta} = 0 \qquad (4.1)$$

Note that there is no difference between the linear and nonlinear equations for this case.

Test Results

Tabarok, B., et. al., [9] published numerical results of a clamped - free beam modelled with a single cantilevered modeshape extending at a constant rate of 108 m/s. The beam stiffness divided by the mass per unit length is

$$\frac{EI}{m} = 1.56 \times 10^8 \; m^4/sec^2$$

The graphs shown in Fig. 4.1 were created by simulating these same conditions and concur precisely with those published in [9]. The graphs show the tip displacement as a function of time for two different sets of "initial" conditions. The top two graphs, case A, correspond to $\eta(t = .5) = 0$, $\dot{\eta}(t = .5) = .1360827635 \; sec^{-1}$. The bottom graphs, case B, correspond to $\eta(t = .5) = 2.520051176 \times 10^{-3}$, $\dot{\eta}(t = .5) = 0 \; sec^{-1}$. The curves were obtained by starting at t=.5 seconds and solving the equations both backward and forward in time.

The unusual values of the "initial" conditions are necessary since η is proportional to some function of instantaneous length multiplied by the modeshape coefficients used by Tabarrok.

4.1.2 Rotating, Nonextending Appendage Case

Reduced Equations

The linear rotating, nonextending equations are:
Vibrational Equation

$$\overset{\circ\circ}{\vec{\eta}} - \overset{\circ\circ}{\theta}\left\{\vec{W}\right\} - \frac{EIt_r^2}{ml^4}\Lambda\vec{\eta} = 0 \tag{4.2}$$

Rotational Equation

$$\frac{T_0 t_r^2}{ml^3} = \left(\frac{I_h}{ml^3} + \frac{1}{3}\right)\overset{\circ\circ}{\theta} + \left\{\vec{W}\right\}^T \overset{\circ\circ}{\vec{\eta}} \tag{4.3}$$

The nonlinear rotating, nonextending equations are:
Vibrational Equation

$$\overset{\circ\circ}{\vec{\eta}} - 2\,\overset{\circ}{\theta}\left[\mathbf{G}^T - \mathbf{G}\right]\overset{\circ}{\vec{\eta}} - \overset{\circ\circ}{\theta}\left\{\vec{W}\right\}$$

$$-\frac{EIt_r^2}{ml^4}\mathbf{\Lambda}\vec{\eta}-\overset{\circ}{\theta}^2\,\mathbf{D}\vec{\eta}+\overset{\circ\circ}{\theta}\left\{\frac{1}{2}\mathbf{G}^T-\mathbf{G}\right\}\vec{\eta}=0$$

<u>Rotational Equation</u>

$$\begin{aligned}\frac{T_0t_r^2}{ml^3} &= \left(\frac{I_h}{ml^3}+\frac{1}{3}+\vec{\eta}^T\mathbf{D}\vec{\eta}\right)\overset{\circ\circ}{\theta} \\ &\quad +2l\,\overset{\circ}{\vec{\eta}}^T\,\mathbf{D}\vec{\eta}\,\overset{\circ}{\theta}+\left\{\vec{W}\right\}^T\overset{\circ\circ}{\vec{\eta}}\end{aligned}\tag{4.4}$$

Test Results

Ryan [14] presents numerical results of a spin-up maneuver of a constant length cantilevered beam with the following properties:

$$EI = 1.4\times10^4 Nm^2 \qquad m = 1.2\ kg/m \qquad l = 10\ m$$

The beam rotation is prescribed by

$$\dot{\theta}(t)=\begin{cases}\frac{2}{5}\left[t-\left(\frac{7.5}{\pi}\right)\sin\frac{\pi t}{7.5}\right] \text{ rad/sec} & 0<t<15\text{sec}\\ 6 \text{ rad/sec} & t>15\text{sec}\end{cases}$$

and is modelled with the first 3 natural modes of a cantilevered beam. The results shown in Fig. 4.2, which duplicate Ryan's "New Theory" results, use the same beam properties but only include the first cantilevered modeshape. The graphs show various parameters for the nonlinear case on the left and the linear case on the right.

4.1.3 Rotating, Extending Appendage Case (Linear)

Reduced Equations

The linear rotating, extending equations are:
<u>Vibrational Equation</u>

$$\begin{aligned}&\overset{\circ\circ}{\vec{\eta}}-\frac{2\,\overset{\circ}{l}}{l}\mathbf{N}\,\overset{\circ}{\vec{\eta}}-\frac{\overset{\circ\circ}{\theta}\,l}{l_r}\left\{\vec{W}\right\}-\frac{2\,\overset{\circ\circ}{\theta l}}{l_r}\left\{\vec{Z}\right\}\\ &-\left\{\frac{\overset{\circ\circ}{l}}{l}(\mathbf{N}-\mathbf{P})-\frac{\overset{\circ}{l}^2}{l^2}\mathbf{Q}-\frac{EIt_r^2}{ml^4}\mathbf{\Lambda}\right\}\vec{\eta}=0\end{aligned}\tag{4.5}$$

Rotational Equation

$$\frac{T_0 t_r^2}{ml^3} = \left(\frac{I_h}{ml^3} + \frac{1}{3}\right)\overset{\circ\circ}{\theta} + \frac{\mathring{l}}{l}\,\mathring{\theta}$$
$$-\frac{l_r}{l}\left\{\vec{W}\right\}^T \overset{\circ\circ}{\vec{\eta}} + \frac{2\,\mathring{l}\,l_r}{l^2}\left(2\left\{\vec{W}\right\}^T + \left\{\vec{Z}\right\}^T\right)\mathring{\vec{\eta}}$$
$$+2\left(\frac{\overset{\circ\circ}{l}\,l_r}{l^2} + \frac{\mathring{l}^2\,l_r}{l^2}\right)\left[\left\{\vec{W}\right\}^T + \left\{\vec{Z}\right\}^T\right]\vec{\eta} \tag{4.6}$$

Test Results

A test of a linear, rotating and retracting system was made by Stephen Gates [10], [11]. The model used was identical to the one used to develop the equations of motion in Chapter 2. The system charactereistics were:

$$I_h = 746770.8333 \ kg \cdot m^2 \qquad EI = 1676 \ Nm^2 \qquad m = .335 \ kg/m$$

The length of each beam was originally 150 meters. At t = 10 seconds the beams are retracted at the rate of 1 meter per second. After 135 seconds the retraction stops and the structure continues to rotate. The hub inertia was picked to be 746770.8333 so that the final rigid body inertia of the system would be half of the initial inertia. The results shown in Fig. 4.3 match those computed by Gates. The final angular velocity oscillates about a value equal to approximately twice the inital angular velocity, as expected from the smaller rotational inertia. Angular momentum is conserved.

4.2 Nonlinear Rotating Extending Appendage Analysis

The nonlinear, rotating extending beam equations are the complete nonlinear equations of the system given in Chapter 3. For convenience, they are repeated here.

Vibrational Equation

$$\overset{\circ\circ}{\vec{\eta}} + \left\{\frac{2\,\mathring{l}}{l}\mathbf{N} - \frac{2\,\mathring{\theta}\,l_r}{l}\left[\mathbf{G}^T - \mathbf{G}\right]\right\}\mathring{\vec{\eta}} + \frac{\overset{\circ\circ}{\theta}\,l}{l_r}\left\{\vec{W}\right\} - \frac{2\,\mathring{\theta}\mathring{l}}{l_r}\left\{\vec{Z}\right\}$$

$$-\left\{\frac{\overset{\circ\circ}{l}}{l}(\mathbf{N}-\mathbf{P}) - \frac{\overset{\circ}{l}^{2}}{l^2}\mathbf{Q} + \frac{EIt_r^2}{ml^4}\mathbf{\Lambda}\right\}\vec{\eta} + \overset{\circ}{\theta}^{2}\,\mathbf{D}\vec{\eta} \tag{4.7}$$

$$-\frac{\overset{\circ\circ}{\theta}\, l_r}{l}\left\{\frac{1}{2}\mathbf{G}^T - \mathbf{G}\right\}\vec{\eta} - \frac{2\,\overset{\circ}{\theta}\, l_r\, \overset{\circ}{l}}{l^2}\left\{\frac{1}{2}\mathbf{G}^T + \mathbf{G} - \mathbf{S} + \frac{1}{2}\mathbf{H}^T\right\}\vec{\eta} = 0$$

<u>Rotational Equation</u>

$$\begin{aligned}
\frac{T_0 t_r^2}{2ml^3} = & \left(\frac{I_h}{2ml^3} + \frac{1}{3} - \left(\frac{l_r}{l}\right)^2 \vec{\eta}^T\mathbf{D}\vec{\eta}\right)\overset{\circ\circ}{\theta} \\
& -\left(\frac{\overset{\circ}{l}}{l} - \frac{l_r^2}{l^2}\left\{\vec{\eta}^T\frac{\overset{\circ}{l}}{l}\mathbf{D} - 2\,\overset{\circ}{\vec{\eta}}^{T}\,\mathbf{D}\right\}\vec{\eta}\right)\overset{\circ}{\theta} \\
& +\frac{l_r}{l}\left\{\vec{W}\right\}^T \overset{\circ\circ}{\vec{\eta}} - \frac{2\,\overset{\circ}{l}\, l_r}{l^2}\left(2\left\{\vec{W}\right\}^T - \left\{\vec{Z}\right\}^T\right)\overset{\circ}{\vec{\eta}} \\
& -2\left(\frac{\overset{\circ\circ}{l}\, l_r}{l^2} - \frac{\overset{\circ}{l}^{2}\, l_r}{l^2}\right)\left[\left\{\vec{W}\right\}^T - \left\{\vec{Z}\right\}^T\right]\vec{\eta}
\end{aligned} \tag{4.8}$$

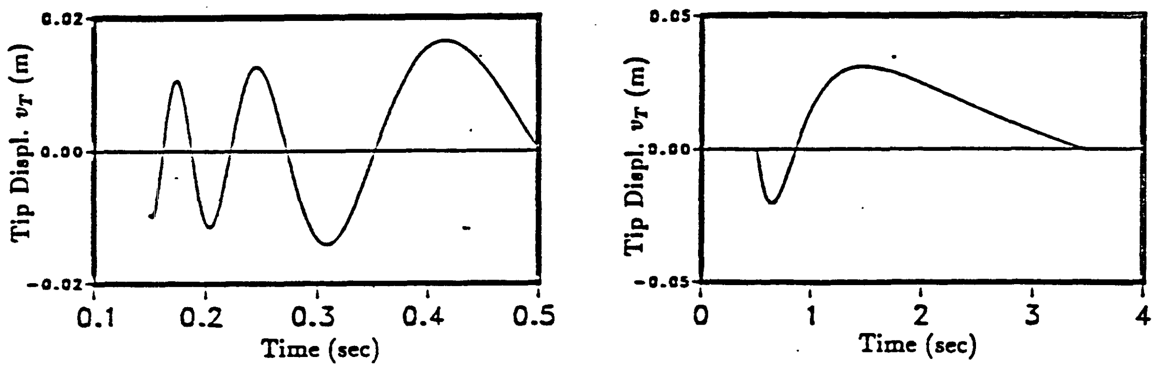
Fig. 4.4 shows two comparison tests of a linear extending beam analysis on the left and a nonlinear extending beam on the right. The spacecraft modelled was the WISP structure, whose properties are given by [12].

$$I_h = 1.2\times 10^6\ kg\cdot m^2 \qquad EI = 1676\ \ Nm^2 \qquad m = .335\ \ kg/m$$

The structure is rotating at an initial angular velocity of .1 rad/sec and the 2 beams are each initially 150 meters long. During a 500 second interval the beams are retracted to a length of 100 meters, using a smooth retraction path. In the linear case 2 modes were included in the analysis. The nonlinear case required 5 modes and the solution may attain 5-10% better accuracy by including yet higher modes. The differences between the linear and nonlinear simulation for this apparently "slow and gentle" retraction are dramatic. The maximum tip displacement of 208 meters for the linear case is well beyond the bounds of small displacement theory. However, the maximum displacement in the of nonlinear analysis of 22 meters is approximately 20% of the beam length, well within the moderate displacement assumption.

The stiffening of the structure due to the inclusion of nonlinear $\dot{\theta}^2$ terms is the primary cause of smaller displacements shown in the nonlinear analysis. Omitting nonlinear terms with η^2 or $\dot{\eta}^2$ elements was found to show only a 1-5% greater deflection for this case.

case A



case B

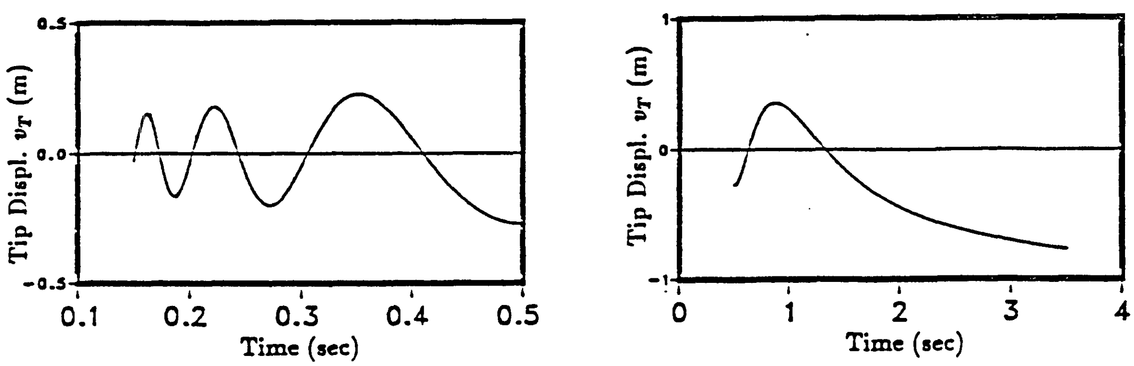


Figure 4.1: Nonrotating Extending beam

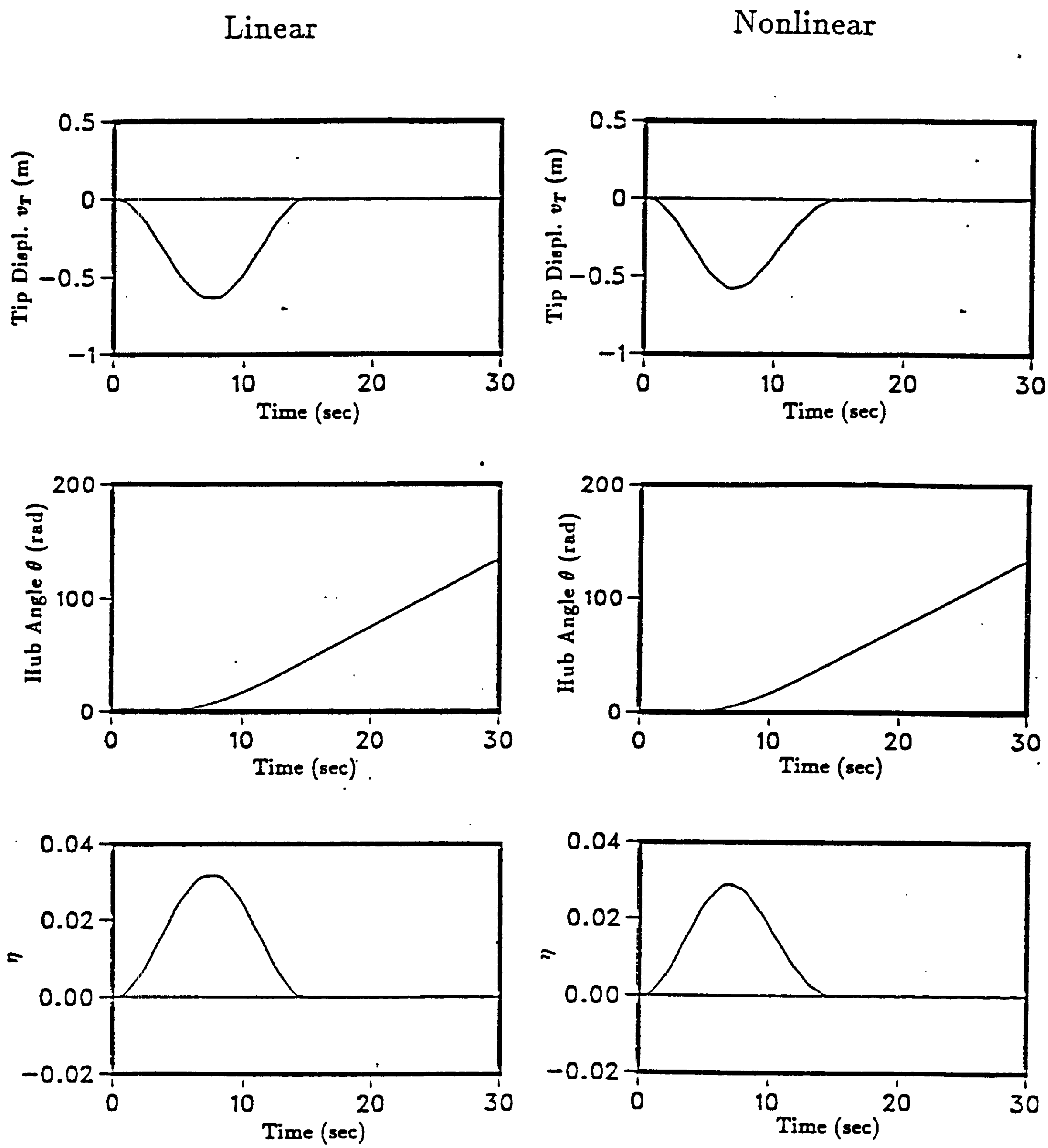


Figure 4.2: Rotating, Nonextending beam

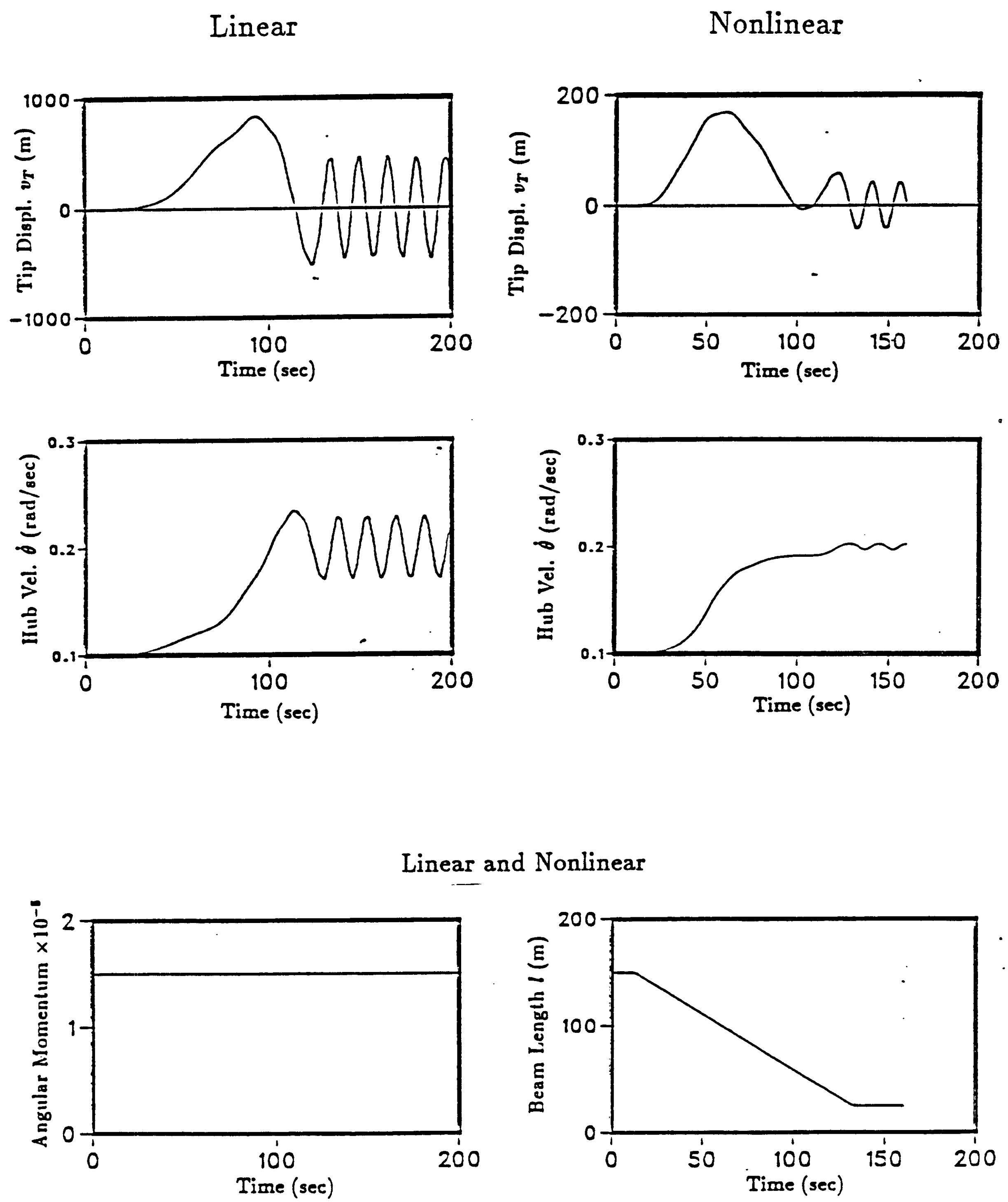


Figure 4.3: Linear, Rotating and Extending beam

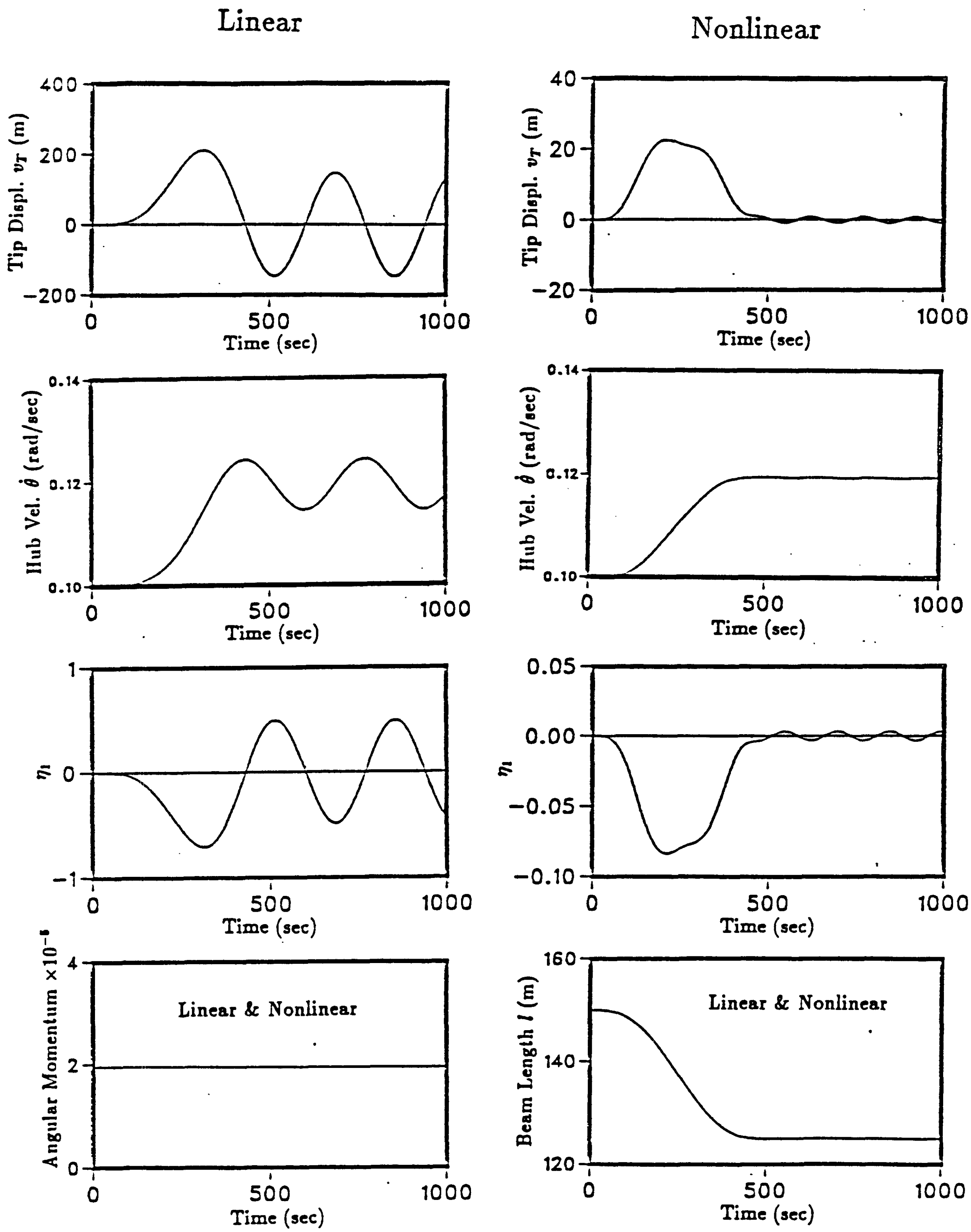


Figure 4.4: WISP analysis

Chapter 5

Conclusions

The equations of motion for a 2-beam-and-hub spacecraft model including large-but-moderate beam deflection and time-varying beam length are identified and written in integral partial differential equation form as Eqs. 3.1 and 3.2. The solution of these equations through transformation to ordinary differential equation form and numerical integration with a 4^{th} order Runge-Kutta method has been shown to corroborate results of previous research efforts for various simplified subcases.

The computer simulation has been demonstrated to provide complete nonlinear dynamics analysis of a spacecraft model with time-varying beam lengths. Specifically, an analysis of the WISP space mission has shown the values of beam tip displacement for a gentle retraction maneuver. Additionally, it has been demonstrated through this simulation that it is sometimes necessary to include second order nonlinear effects in apparently gentle maneuvers for acceptable accuracy.

Recommendations

A useful extension to this work would be a graph of maximum tip displacement for different cases of beam stiffness, rotation and extension rate. Another useful addition would be the identification of the limits of the analysis goverened by the assumption of the inclusion of only second order of lower terms in the lateral displacement variable. In particular, if angular acceleration and velocity are equal in order of magnitude to the mode-shape coefficient, η, additional terms should be included in the derivation to maintain consistency.

Major efforts in the future might include the addition of out-of-plane deflection of the beam, the investigation of using a more physically accurate structural model for the beam, such as a Timoshenko beam rather than a Bernoulli-Euler beam, or the extension of this analysis to a rotation, extending plate.

Bibliography

[1] Messac, A., et. al., "Dynamics of a Spinning Spacecraft During Extension of Flexible Appendages", AIAA Dynamics Specialist Conference, Paper No. 84-1021-CP, presented at the AIAA Dynamics Specialists Conference, Palm Springs, CA May 17-18, 1984.

[2] Lips, K.W., and Modi, V.J., "Transient Attitude Dynamics of Satellites with Flexible Appendages ",*Acta Astronautica*, Vol. 8, Oct. 1978, pp. 797-815.

[3] Lips, K.W., and Modi, V.J., "Three-Dimensional Response Characteristics for Spacecraft with Deploying Flexible Appendages", *Journal of Guidance, Control, and Dynamics*, Vol. 4, No. 6, November-December 1981, pp. 650-656.

[4] Hughes, P.C., "Dynamics of a Spin-Stabilized Satellite During Extension of Rigid Booms", C.A.S.I. Transactions, Vol. 5, No.1, March 1972, pp. 11-14.

[5] Weeks, G.E., "Dynamics Analysis of a Deployable Space Structure", *Journal of Spacecraft*, Vol. 23, No. 1, January-February 1986, pp. 102-107

[6] Hughes, P.C., "Deployment Dynamics of the Communication Technology Satellite - A Progress Report", Paper N76-28328, *Symposium on the Dynamics and Control of Nonrigid Space Vehicles*, Fracati, Italy, May 1976.

[7] Cloutier, J.G., "Dynamics of Deployment of Extendible Booms from Spinning Space Vehicles", *Journal of Spacecraft*, Vol. 5, No. 5, May 1968, pp. 547-552.

[8] Honma, M., "Dynamics of Spinning Satellite with Flexible Appendages Extending at a Constant Speed", *Proceedings of the 12th I.S.T.S.*, Tokoyo, 1977, pp. 313-318.

[9] Tabarrok, B., C.M. Leech, Y.I. Kim, "On the Dynamics of an Axially Moving Beam", *Journal of the Franklin Institute*, Vol. 297, No. 3, March 1974, pp. 201-220.

[10] Gates, Stephen, "Planar dymanics of the Deployment of Flexible Beams from a Spinning Rigid Body", *Draper Laboratory Internal Memorandum* December 9, 1986

[11] Gates, Stephen, "An Approximation Method for the Numerical Solution of the Deployment Dynamics Equations", *Draper Laboratory Internal Memorandum* January 6, 1987

[12] Lips, K.W., et. al., "Dynamics and Control Characteristics for the WISP 300m Dipole Antenna/Suttle Configuration" *AAS/AIAA Astrodynamics Specialist Conference*, paper AAS 85-365

[13] Dow, P. C., et. al., "Dynamic Stability of a Gravity Gradient Stabalized Satellite having Long Flexible Antennas", *AIAA/JACC Guidance and Control Conference*, Seattle, Washington, 1966, pp. 285-303

[14] Ryan, R. R., "Flexible Body Dynamics Benchmark Problem # 1: Uniform Cantilever Beam Spin-up", Multibody Dynamics Conference, 1987, Pasadena California

[15] Tsuchiya, K., "Dynamics of a Spacecraft During Extension of Flexible Appendages", *Journal of Guidance, Control and Dynamics*, Vol 6, No. 2, March-April 1983 pp. 100-103.

[16] Jankovic, M.S., Comments of "Dynamics of a Spacecraft During Extension of Flexible Appendages", *Journal of Guidance, Control, and Dynamics*, Vol 7, No. 1, Januarry-February 1984, p. 128.

[17] Dugundji, J., "Simple Expressions for Higher Vibration Modes of Uniform Euler Beams" *AIAA Journal*, Technical Note, to be published.

[18] Hildebrand, Francis B., *Advanced Calculus for Applications*, Prentice-Hall, 1976

[19] Washizu, K., *Variational Methods in Elasticity and Plasticity - 3rd Edition*, Pergamon Press, 1982, p. 209.

Appendix A

Consequences of Axial Rigidity

This appendix addresses the consequences of the assumed axial rigidity of the neutral axis.

Relationship Between Longitudinal and Lateral Coordinate

It is assumed in this analysis that (See Eq. 2)

$$u_x^2 \sim v_x^4 \ll 1$$

and that the position vector expressed in the body frame is given by (See Eq. 1)

$$\vec{R} = \{x(t) + u(x,t), v(x,t), 0\}$$

It then follows that an element of length along the deformed beam, ds, is given by

$$ds = \sqrt{(dx + du)^2 + dv^2} \tag{A.1}$$

$$= dx\sqrt{(1 + u_x)^2 + v_x^2} \tag{A.2}$$

Expanding the term under the radical and using a Taylor series approximation yields

$$ds \approx dx(1 + u_x + \frac{1}{2}v_x^2) \tag{A.3}$$

Since the assumed axial rigidity of the neutral-axis requires that $ds = dx$, Eq. A.3 implies that

$$u_x = -\frac{1}{2}v_x^2 \tag{A.4}$$

which is consistent with the assumption that u_x is of the same order as v_x^2 (Eq. 2.2).

Appendix B

Moment Displacement Relation for "Large-but-Moderate" Deflection

This appendix shows two methods which explain that the well known moment-displacement relation for linear small-displacement analysis remains valid for the present case where

$$u_x^2 \sim v_x^4 \ll 1$$

Method 1: Elasticity

The elastic displacement components may be written as

$$u_1 = u - z\frac{\partial v}{\partial x} \tag{B.1}$$

$$u_2 = v \tag{B.2}$$

$$u_3 = 0 \tag{B.3}$$

where z is the coordinate perpendicular to the neutral axis.

The general strain-displacement relation reads

$$e_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\{(\frac{\partial u_1}{\partial x})^2 + (\frac{\partial u_2}{\partial x})^2 + (\frac{\partial u_3}{\partial x})^2\}$$

$$= \frac{\partial u_1}{\partial x}(1 + \frac{1}{2}\frac{\partial u_1}{\partial x}) + \frac{1}{2}\{(\frac{\partial u_2}{\partial x})^2 + (\frac{\partial u_3}{\partial x})^2\} \tag{B.4}$$

Since u_{1_x} is the derivative of the axial displacement coordinate u, and u_x is of the order of v_x^2, u_{1_x} is a fourth order term and can be eliminated. Eq. B.4 may be approximated as

$$e_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\{(\frac{\partial u_2}{\partial x})^2 + (\frac{\partial u_3}{\partial x})^2\} \tag{B.5}$$

which, upon substitution for the displacement coordinates, becomes

$$e_{11} = u_x + \frac{1}{2}v_x^2 - zv_{xx} \tag{B.6}$$

The moment-displacement is then readily obtained by evaluating

$$M = -\int_{-\frac{h}{2}}^{\frac{h}{2}} bz\sigma_{11}dz \tag{B.7}$$

where $\sigma_{11} = Ee_{11}$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} bEz\{u_x + \frac{1}{2}v_x^2 - zv_{xx}\}dz \tag{B.8}$$

$$= EIv_{xx} \tag{B.9}$$

Method 2: Curvature

The moment, $M(s)$, is defined as

$$M(s) = EI\kappa \tag{B.10}$$

and the beam curvature, κ, is defined by

$$\kappa \equiv \frac{d\phi}{ds} \tag{B.11}$$

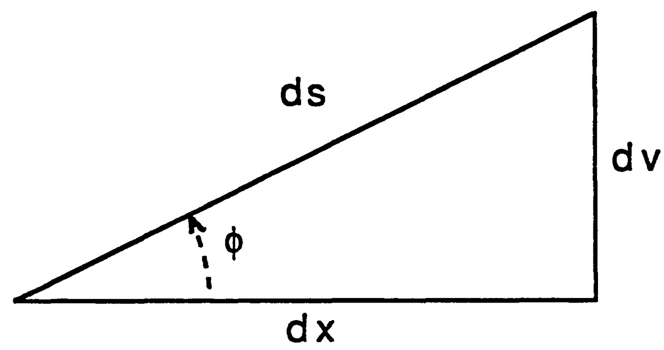
From Fig. B.1 it can be seen that

$$\sin(\phi) = \frac{dv}{ds} \tag{B.12}$$

Differentiating both sides of this equation with respect to s yields

$$\frac{d^2v}{ds^2} = \cos\phi\frac{d\phi}{ds} \tag{B.13}$$

Figure B.1: Beam Arc Length vs. Coordinate Length



and using this relation with Eq. B.11, κ can be expressed as

$$\kappa = \frac{d\phi}{ds} = \frac{\frac{d^2v}{ds^2}}{\cos\phi} = \frac{\frac{d^2v}{ds^2}}{\sqrt{1-\sin^2\phi}} \tag{B.14}$$

Using the expression for $\sin\phi$ from Eq. B.12 yields

$$\kappa = \frac{\frac{d^2v}{ds^2}}{\sqrt{1-(\frac{dv}{ds})^2}} = \frac{d^2v}{ds^2}(1-(\frac{dv}{ds})^2)^{-\frac{1}{2}} \tag{B.15}$$

This result can be expanded in the binomial series

$$\kappa = \frac{d^2v}{ds^2}(1+\frac{1}{2}(\frac{d^2v}{ds^2})^2 - \frac{3}{8}(\frac{d^2v}{ds^2})^4 + \ldots) \tag{B.16}$$

Since this analysis includes no nonlinear terms higher than second order, every term but the first can be eliminated to yield the approximation

$$\kappa \approx \frac{d^2v}{ds^2} \tag{B.17}$$

Note that this value is the same as the linear approximation to κ.

Substituting Eq. B.17 into Eq. B.10 and noting that axial rigidity demands that $ds = dx$ yields the identical result as Eq. B.9, namely,

$$M = EIv_{xx} \tag{B.18}$$

Appendix C

Alternate Cantilvered Beam Modeshape Form

The traditional form of the equation for the modeshapes of a clamped-free (cantilevered) beam was shown in Eq. 3.23 and is repeated here for clarity

$$\gamma_i(\xi) = \cos(\epsilon_i \xi) - \cosh(\epsilon_i \xi) - \beta_i(\sin(\epsilon_i \xi) - \sinh(\epsilon_i \xi)) \tag{C.1}$$

where

$$\beta_i = \frac{\cos(\epsilon_i) + \cosh(\epsilon_i)}{\sin(\epsilon_i) - \sinh(\epsilon_i)} \tag{C.2}$$

It is observed [17] that the numerical calculation of the above form of $\gamma_i(\xi)$ is prone to certain computer related inaccuracies. In particular, the hyperbolic functions sinh and cosh generally have much larger magnitudes than the trigonometric functions sin and cos. For large values of the argument $\epsilon_i \xi$, a problem known as "catastrophic cancellation" results in the loss of significant digits in $\gamma_i(\xi)$.

This problem can be avoided by eliminating additions of very large numbers of opposite algebraic sign. A new form of the modeshapes can be obtained through algebraic manipulation by adding and subracting $\sinh(\epsilon_i \xi)$ to Eq. C.1. The resulting form of the equation is

$$\gamma_i(\xi) = \cos(\epsilon_i \xi) - e^{-\epsilon_i \xi} - \beta_i \sin(\epsilon_i \xi) + (\beta_i - 1)\sinh(\epsilon_i \xi) \tag{C.3}$$

where

$$\beta_i - 1 = \frac{\cos(\epsilon_i) + e^{-\epsilon_i} - \sin(\epsilon_i)}{\sin(\epsilon_i) + \sinh(\epsilon_i)} \tag{C.4}$$

All four terms in Eq. C.3 are of order 1 or less, resulting in a more numerically stable equation. The ε_i and β_i coefficients for the first 5 modes are given in appendix E.

Appendix D

Energy Method Derivation of the Equations of Motion

In order to verify that the derivation using Newton's method for the nonlinear equations of the nonextending beam was performed correctly, the same analysis was repeated using Lagrange's equations. This appendix outlines the process of that energy method derivation.

It is desired to calculate the transverse deflection of a rotating, extending beam. The assumptions are the same as those used in the Newton's method derivation (see section 2.1). Fig. 2.1 shows the beam conventions. The variables used in this analysis are the same as those used for the Newton's method analysis.

Lagrange's Equations

Lagrange's equations of motion are shown below

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad i = 1, 2, ..., n + 1 \tag{D.1}$$

where
- T is the kinetic energy of the system
- V is the potential energy of the system
- q_i is the i^{th} generalized coordinate, $\theta, \eta_1, \ldots, \eta_n$
- Q_i is the generalized force applied at coordinate i
- n is the number of flexible beam modes

<u>Kinetic Energy</u>

The vector $\vec{R}$ locates any point p' on the deformed beam, as shown in Fig. 2.2. The kinetic energy of the beam, T, is given by

$$T = \frac{1}{2}\int_0^l m\dot{\vec{R}}^2 dx \tag{D.2}$$

where m is the mass/length of the beam
l is the length of the beam

The vector $\vec{R}$ can be represented by

$$\vec{R} = X\hat{i} + Y\hat{j} \tag{D.3}$$

and Eq. D.2 can then be rewritten as

$$T = \frac{1}{2}\int_0^l m(\dot{X}^2 + \dot{Y}^2)dx \tag{D.4}$$

Fig. 2.2 also shows the transverse displacement v, and the axial displacement u, of any point p along the undeformed neutral axis of the beam to its deformed position. p'. X and Y locate p' and are given by

$$X = (x + u)\cos\theta - v\sin\theta \tag{D.5}$$

$$Y = (x + u)\sin\theta + v\cos\theta \tag{D.6}$$

Using these expressions, the kinetic energy can be expressed in terms of θ and v as

$$T = \frac{1}{2}\int_0^l m[\dot{u}^2 - 2v\dot{\theta}\dot{u} + (v^2 + x^2 + u^2 + 2xu)\dot{\theta}^2 + 2\dot{v}\dot{\theta}(x + u) + \dot{v}^2]dx \tag{D.7}$$

Potential Energy

The potential energy of a bending beam is given by

$$V = \frac{1}{2}\int_0^l EI\kappa^2 dx \tag{D.8}$$

where the curvature, κ, is shown in appendix B to be

$$\kappa = \frac{\partial^2 v}{\partial x^2} \tag{D.9}$$

Combining this result with Eq. D.8, the potential energy equation of a beam bending with moderate angle displacements is given by

$$V = \frac{1}{2}\int_0^l EI(\frac{\partial^2 v}{\partial x^2})^2 dx \tag{D.10}$$

Assumed Modes Solution

It is assumed that v can be represented as a summation of orthogonal modes:

$$v = \sum_{i=1}^{n} l_r \gamma_i(\xi)\eta_i(\tau) \tag{D.11}$$

with derivatives

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{n} \frac{l_r}{tr}\gamma_i(\xi)\frac{d\eta_i}{d\tau}(\tau) \tag{D.12}$$

$$\frac{\partial v}{\partial x} = \sum_{i=1}^{n} \frac{\partial \gamma_i}{\partial \xi}(\xi)\eta_i(\tau) \tag{D.13}$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{i=1}^{n} \frac{1}{l_r}\frac{\partial^2 \gamma_i}{\partial \xi^2}(\xi)\eta_i(\tau) \tag{D.14}$$

where the expressions have been non-dimensionalized such that

$$\tau = \frac{t}{t_r} \tag{D.15}$$

$$\xi = \frac{x}{l_r} \tag{D.16}$$

and t_r is a constant reference time
l_r is a constant reference length

It can be shown that these modes satisfy the orthonormality relation

$$\int_0^1 \gamma_i(\xi)\gamma_j(\xi)d\xi = \delta_{ij} \tag{D.17}$$

In the remaining analysis the following conventions are introduced:

$$\frac{d(\ )}{d\xi} \equiv (\ )'$$

$$\frac{d(\ )}{d\tau} \equiv (\overset{\circ}{\ })$$

$$\sum_{i=1}^{n} \equiv \sum_{i}$$

The expression for the axial displacement u was given as Eq. A.4 and is repeated here.

$$u(x) = -\frac{1}{2}\int_0^x (\frac{\partial v}{\partial \alpha})^2 d\alpha \tag{D.18}$$

Note that α is simply a dummy variable for x. With the substitutions given in Eq. D.11 through Eq. D.14, u and its time derivatives become

$$u = -\frac{l_r}{2}\int_0^{\xi} \sum_{i=1}^{n} \gamma_i' \eta_i \sum_{j=1}^{n} \gamma_j' \eta_j d\xi \tag{D.19}$$

$$\frac{\partial u}{\partial t} = -\frac{l_r}{t_r}\int_0^{\xi} \sum_{i=1}^{n} \gamma_i' \overset{\circ}{\eta}_i \sum_{j=1}^{n} \gamma_j' \eta_j d\alpha \tag{D.20}$$

$$(\frac{\partial u}{\partial t})^2 = \frac{l_r^2}{t_r^2}\int_0^{\alpha} \sum_{i=1}^{n} \gamma_i' \overset{\circ}{\eta}_i \sum_{j=1}^{n} \gamma_j' \overset{\circ}{\eta}_j \sum_{k=1}^{n} \gamma_k' \overset{\circ}{\eta}_k \sum_{l=1}^{n} \gamma_l' \overset{\circ}{\eta}_l \, d\xi \tag{D.21}$$

Once again using the expressions given by Eq. D.11 through Eq. D.14, the quantity $\dot{X}^2 - \dot{Y}^2$ is written as

$$\frac{l^2}{t_r^2}\sum_i\sum_j\sum_k\sum_l \overset{\circ}{\eta}_i\overset{\circ}{\eta}_j \eta_k\eta_l \int_0^{\xi} \gamma_i'\gamma_j'\gamma_k'\gamma_l' d\alpha - \frac{2l^2\overset{\circ}{\theta}}{t_r^2}\sum_i\sum_j\sum_k \eta_i\eta_j \overset{\circ}{\eta}_k \gamma_i \int_0^{\xi} \gamma_j'\gamma_k' d\alpha -$$

$$\frac{\overset{\circ}{\theta}^2}{t_r^2}\left\{\sum_i l^2\eta_i^2 - l^2\xi^2 + \frac{l^2}{4}\sum_i\sum_j\sum_k\sum_l \eta_i\eta_j\eta_k\eta_l \int_0^{\xi} \gamma_i'\gamma_j'\gamma_k'\gamma_l' d\alpha - l^2\xi\sum_i\sum_j \eta_i\eta_j \int_0^{\xi} \gamma_i'\gamma_j' d\alpha\right\}$$

$$+\sum_i \frac{l^2}{t_r^2}\overset{\circ}{\eta}_i^2 - \frac{l^2\overset{\circ}{\theta}}{t_r^2}\sum_i \overset{\circ}{\eta}_i\gamma_i(\sum_j\sum_k \eta_j\eta_k \int_0^{\xi}\gamma_j'\gamma_k' d\alpha - 2\xi) \tag{D.22}$$

Solving Lagrange's Equations

In this section the individual terms of Lagrange's equations are evaluated. The first generalized coordinate, q_1, is the rotational degree of freemdom, θ. The remaining q_i's, $i = 2, \ldots, n+1$ correspond to η_1 through η_n. The first step in evaluating the first term of Lagrange's equations is to evaluate $\partial T/\partial q_i$ for $i = 1, q_1 = \theta$. This expression is

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}\int_0^l m(\dot{X}^2 + \dot{Y}^2)dx\right]$$

$$= \frac{ml^3}{t_r}\int_0^1 \sum_i \sum_j \sum_k \eta_i \eta_j \overset{\circ}{\eta}_k \gamma_i \int_0^\alpha \gamma_j' \gamma_k' d\xi d\xi$$

$$-\frac{ml\overset{\circ}{\theta}}{t_r}\int_0^1 \left\{ \sum_i l^2\eta_i^2 + l^2\xi^2 + \frac{l^2}{4}\sum_i\sum_j\sum_k\sum_l \eta_i\eta_j\eta_k\eta_l \int_0^\xi \gamma_i'\gamma_j'\gamma_k'\gamma_l' d\alpha \right.$$

$$\left. -l^2\xi \sum_i\sum_j \eta_i\eta_j \int_0^\xi \gamma_i'\gamma_j' d\alpha \right\} d\xi$$

$$-\frac{ml^3}{2t_r}\int_0^1 \sum_i \overset{\circ}{\eta}_i \gamma_i \left(\sum_j\sum_k \eta_j\eta_k \int_0^\xi \gamma_j'\gamma_k' d\alpha - 2\xi \right) d\xi \tag{D.23}$$

The same expression for any vibrational degree of freedom, $\eta_m, 2 \leq m \leq n$, is

$$\frac{\partial T}{\partial \dot{\eta}_m} = \frac{\partial}{\partial \dot{\eta}_m}\left[\frac{1}{2}\int_0^l m(\dot{X}^2 + \dot{Y}^2)dx\right]$$

$$= \frac{ml^3}{t_r}\sum_i\sum_j\sum_k \overset{\circ}{\eta}_j \eta_k \eta_l \int_0^1\int_0^\xi \gamma_m'\gamma_j'\gamma_k'\gamma_l' d\alpha d\xi$$

$$-\frac{ml^3\overset{\circ}{\theta}}{t_r}\sum_i\sum_j \eta_i\eta_j \int_0^1 \gamma_i \int_0^\xi \gamma_j'\gamma_m' d\alpha d\xi + \frac{ml^3}{t_r}\overset{\circ}{\eta}_m$$

$$-\frac{ml^3\overset{\circ}{\theta}}{2t_r}\int_0^1 \gamma_m \left(\sum_i\sum_j \eta_i\eta_j \int_0^\xi \gamma_i'\gamma_j' d\alpha - 2\xi \right) d\xi \tag{D.24}$$

The second step in evaluating the first term of Lagrange's equations is to take the time derivative of the previous two equations. The time derivative of Eq. D.23 is

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{ml^3}{t_r^2}\int_0^1 \sum_i\sum_j\sum_k (\overset{\circ}{\eta}_i \eta_j \overset{\circ}{\eta}_k - \eta_i \overset{\circ}{\eta}_j \overset{\circ}{\eta}_k - \eta_i\eta_j \overset{\circ\circ}{\eta}_k)\gamma_i \int_0^\xi \gamma_j'\gamma_k' d\alpha d\xi$$

$$-\frac{ml\overset{\circ\circ}{\theta}}{t_r^2}\int_0^1 \left\{ \sum_i l^2\eta_i^2 - l^2\xi^2 + \frac{l^2}{4}\sum_i\sum_j\sum_k\sum_l \eta_i\eta_j\eta_k\eta_l \int_0^\xi \gamma_i'\gamma_j'\gamma_k'\gamma_l' d\alpha \right.$$

$$\left. -l^2\xi \sum_i\sum_j \eta_i\eta_j \int_0^\xi \gamma_i'\gamma_j' d\alpha \right\} d\xi$$

$$-\frac{ml\overset{\circ}{\theta}}{t_r^2}\int_0^1 \left\{ \sum_i 2l^2\eta_i \overset{\circ}{\eta}_i + l^2\sum_i\sum_j\sum_k\sum_l \overset{\circ}{\eta}_i \eta_j\eta_k\eta_l \int_0^\xi \gamma_i'\gamma_j'\gamma_k'\gamma_l' d\alpha \right.$$

$$-2l^2\xi \sum_i \sum_j \mathring{\eta}_i \, \eta_j \int_0^{\xi} \gamma_i' \gamma_j' d\alpha \Bigg\} d\xi$$

$$-\frac{ml^3}{2t_r^2} \int_0^1 \Bigg\{ \sum_i \overset{\circ\circ}{\eta}_i \, \gamma_i \left(\sum_j \sum_k \eta_j \eta_k \int_0^{\xi} \gamma_j' \gamma_k' d\alpha - 2\xi \right)$$

$$+2 \sum_i \mathring{\eta}_i \, \gamma_i \sum_j \sum_k \mathring{\eta}_j \, \eta_k \int_0^{\xi} \gamma_j' \gamma_k' d\alpha \Bigg\} d\xi \qquad \text{(D.25)}$$

The time derivative of Eq. D.24 is

$$\frac{ml^3}{t_r^2} \sum_j \sum_k \sum_l (\overset{\circ\circ}{\eta}_j \, \eta_k \eta_l + 2 \, \mathring{\eta}_j \mathring{\eta}_k \, \eta_l) \int_0^1 \int_0^{\xi} \gamma_m' \gamma_j' \gamma_k' \gamma_l' d\alpha d\xi$$

$$-\frac{ml^3 \, \overset{\circ\circ}{\theta}}{t_r^2} \sum_i \sum_j \eta_i \eta_j \int_0^1 \gamma_i \int_0^{\xi} \gamma_j' \gamma_m' d\alpha d\xi$$

$$-\frac{ml^3 \, \mathring{\theta}}{t_r^2} \sum_i \sum_j (\mathring{\eta}_i \, \eta_j + \eta_i \, \mathring{\eta}_j) \int_0^1 \gamma_i \int_0^{\xi} \gamma_j' \gamma_m' d\alpha d\xi$$

$$-\frac{ml^3 \, \overset{\circ\circ}{\theta}}{2t_r^2} \int_0^1 \gamma_m \left(\sum_j \sum_k \eta_j \eta_k \int_0^{\xi} \gamma_j' \gamma_k' d\alpha - 2\xi \right) d\xi + \frac{ml^3}{t_r^2} \overset{\circ\circ}{\eta}_m$$

$$-\frac{ml^3 \, \mathring{\theta}}{2t_r^2} \int_0^1 \gamma_m \sum_j \sum_k (\mathring{\eta}_j \, \eta_k - \eta_j \, \mathring{\eta}_k) \int_0^{\xi} \gamma_j' \gamma_k' d\alpha d\xi \qquad \text{(D.26)}$$

The second term of Lagrange's equations evaluated at $q_1 = \theta$ is

$$\frac{\partial T}{\partial \theta} = \frac{\partial}{\partial \theta} \left[\frac{1}{2} \int_0^l m(\dot{X}^2 - \dot{Y}^2) dx \right] = 0 \qquad \text{(D.27)}$$

The second term evaluated at $q_i = \eta_m$ is

$$\frac{\partial T}{\partial \eta_m} = \frac{\partial}{\partial \eta_m} \left[\frac{1}{2} \int_0^l m(\dot{X}^2 + \dot{Y}^2) dx \right]$$

$$= \frac{ml^3}{t_r^2} \sum_i \sum_j \sum_k \mathring{\eta}_i \mathring{\eta}_j \, \eta_k \int_0^1 \int_0^{\xi} \gamma_i' \gamma_j' \gamma_k' \gamma_m' d\alpha d\xi$$

$$-\frac{ml^3 \overset{\circ}{\theta}}{t_r^2} \sum_j \sum_k \eta_j \overset{\circ}{\eta}_k \int_0^1 \gamma_m \int_0^\xi \gamma_j' \gamma_k' d\alpha d\xi + \sum_i \sum_k \eta_i \overset{\circ}{\eta}_k \int_0^1 \gamma_i \int_0^\xi \gamma_k' \gamma_m' d\alpha d\xi$$

$$+\frac{m \overset{\circ}{\theta}}{2t_r^2} \{ 2\eta_m l^3 + l^3 \sum_i \sum_j \sum_k \eta_i \eta_j \eta_k \int_0^1 \int_0^\xi \gamma_i' \gamma_j' \gamma_k' \gamma_m' d\alpha d\xi$$

$$-2l^3 \sum_i \eta_i \int_0^1 \xi \int_0^\xi \gamma_i' \gamma_m' d\alpha d\xi \} - \frac{ml^3 \overset{\circ}{\theta}}{t_r^2} \sum_i \sum_j \overset{\circ}{\eta}_i \eta_j \int_0^1 \gamma_i \int_0^\xi \gamma_j' \gamma_m' d\alpha d\xi \quad \text{(D.28)}$$

The third term of Lagrange's equations is $\partial V / \partial q_i$. For $i = 1$ this term becomes

$$\frac{\partial V}{\partial \theta} = 0 \quad \text{(D.29)}$$

For the vibrational degrees of freedom, $\eta_m, 2 \leq m \leq n + 1$, the third term is

$$\frac{\partial V}{\partial \eta_m} = \frac{\partial}{\partial \eta_m} \left[\frac{1}{2} \int_0^l EI \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx \right] \quad \text{(D.30)}$$

$$= \frac{EI}{l} \sum_i \eta_i \int_0^1 \gamma_i'' \gamma_m'' d\xi \quad \text{(D.31)}$$

Final Equations

The rotational equation of motion for the system is obtained by combining the first, second and third terms for $q_i = q_1 = \theta$ and multiplying the equation by t_r^2/ml^3. This equation is

$$\frac{T_0 t_r^2}{ml^3} = \sum_i \sum_j \sum_k (\overset{\circ}{\eta}_i \eta_j \overset{\circ}{\eta}_k - \eta_i \overset{\circ}{\eta}_j \overset{\circ}{\eta}_k - \eta_i \eta_j \overset{\circ\circ}{\eta}_k) \int_0^1 \gamma_i \int_0^\xi \gamma_j' \gamma_k' d\alpha d\xi$$

$$+ \overset{\circ\circ}{\theta} \int_0^1 \Bigg[\sum_i \eta_i^2 + \xi^2 + \frac{1}{4} \sum_i \sum_j \sum_k \sum_l \eta_i \eta_j \eta_k \eta_l \int_0^\xi \gamma_i' \gamma_j' \gamma_k' \gamma_l' d\alpha$$

$$-2\xi \sum_i \sum_j \overset{\circ}{\eta}_i \eta_j \int_0^\xi \gamma_i' \gamma_j' d\alpha \Bigg] d\xi$$

$$\overset{\circ}{\theta} \int_0^1 \Bigg[\sum_i 2\eta_i \overset{\circ}{\eta}_i - \sum_i \sum_j \sum_k \sum_l \overset{\circ}{\eta}_i \eta_j \eta_k \eta_l \int_o^\xi \gamma_i' \gamma_j' \gamma_k' \gamma_l' d\alpha$$

$$-2\xi \sum_i \sum_j \overset{\circ}{\eta}_i \, \eta_j \int_0^\xi \gamma_i' \gamma_j' d\alpha \Bigg] d\xi$$

$$-\frac{1}{2}\int_0^1 \left[\sum_i \overset{\circ\circ}{\eta}_i \, \gamma_i \left(\sum_j \sum_k \eta_j \eta_k \int_0^\xi \gamma_j' \gamma_k' d\alpha - 2\xi \right) + 2 \sum_i \sum_j \sum_k \overset{\circ}{\eta}_i \overset{\circ}{\eta}_j \, \eta_k \gamma_i \int_0^\xi \gamma_j' \gamma_k' d\alpha \right] d\xi \tag{D.32}$$

Combining the first, second and third terms whose degree of freedom is η_m where $2 \leq m \leq n-1$, dividing by ml and rearranging terms results in the n equations which govern the motion of the of the beam's n vibrational modes. This vibrational equation is

$$\frac{l^2}{t_r^2} \sum_j \sum_k \sum_l (\overset{\circ\circ}{\eta}_j \, \eta_k \eta_l + 2 \overset{\circ}{\eta}_j \overset{\circ}{\eta}_k \, \eta_l) \int_0^1 \int_0^\xi \gamma_m' \gamma_j' \gamma_k' \gamma_l' d\alpha d\xi$$

$$+\frac{l^2 \overset{\circ\circ}{\theta}}{t_r^2} \sum_i \sum_j \eta_i \eta_j \int_0^1 \gamma_i \int_0^\xi \gamma_j' \gamma_m' d\alpha d\xi + \frac{l^2 \overset{\circ}{\theta}}{t_r^2} \sum_i \sum_j \overset{\circ}{\eta}_i \, \eta_j \int_0^1 \gamma_i \int_0^\xi \gamma_j' \gamma_m' d\alpha d\xi$$

$$-\frac{l^2 \overset{\circ\circ}{\theta}}{2t_r^2} \left\{ \sum_j \sum_k \eta_j \eta_k \int_0^1 \gamma_m \int_0^\xi \gamma_j' \gamma_k' d\alpha d\xi - 2 \int_0^1 \gamma_m \xi d\xi \right\} + \frac{l^2}{t_r^2} \overset{\circ\circ}{\eta}_m$$

$$-\frac{l^2 \overset{\circ}{\theta}}{2t_r^2} \sum_j \sum_k (\overset{\circ}{\eta}_j \, \eta_k + 3\eta_k \overset{\circ}{\eta}_k) \int_0^1 \gamma_m \int_0^\xi \gamma_j' \gamma_k' d\alpha d\xi$$

$$-\frac{l^2}{t_r^2} \sum_i \sum_j \sum_k \overset{\circ}{\eta}_i \overset{\circ}{\eta}_j \, \eta_k \int_0^1 \int_0^\xi \gamma_i' \gamma_j' \gamma_k' \gamma_m' d\alpha d\xi + \frac{l^2 \overset{\circ}{\theta}}{t_r^2} \sum_i \sum_j \overset{\circ}{\eta}_i \, \eta_j \int_0^1 \gamma_i \int_0^1 \gamma_j' \gamma_m' d\alpha d\xi$$

$$-\frac{l^2 \overset{\circ}{\theta}^2}{t_r^2} \{ \eta_m - \sum_i \eta_i \int_0^1 \xi \int_0^\xi \gamma_i' \gamma_m' d\alpha d\xi \} + \frac{EI}{ml^2} \sum_i \eta_i \int_0^1 \gamma_i'' \gamma_m'' d\xi \tag{D.33}$$

When all third order nonlinear terms are eliminated it can be seen that these eqations match the second order nonlinear equations for a nonextending beam that are derived in section 3.2 using Newton's method. Namely,

<u>Rotational Equation</u>

$$\frac{T_0 t_r^2}{ml^3} = \overset{\circ\circ}{\theta} \int_0^1 \left[\sum_i \eta_i^2 + \xi^2 - 2\xi \sum_i \sum_j \overset{\circ}{\eta}_i \, \eta_j \int_0^\xi \gamma_i' \gamma_j' d\alpha \right] d\xi$$

$$\mathring{\theta}\int_0^1\left[\sum_i 2\eta_i\,\mathring{\eta}_i - 2\xi\sum_i\sum_j \mathring{\eta}_i\,\eta_j\int_0^\xi \gamma_i'\gamma_j'\,d\alpha\right]d\xi + \int_0^1\left[\sum_i \overset{\circ\circ}{\eta}_i\,\gamma_i\xi\right]d\xi \qquad \text{(D.34)}$$

<u>Vibrational Equation</u>

$$-\frac{l^2\,\overset{\circ\circ}{\theta}}{t_r^2}\sum_i\sum_j \eta_i\eta_j\int_0^1\gamma_i\int_0^\xi \gamma_j'\gamma_m'\,d\alpha d\xi + \frac{l^2\,\mathring{\theta}}{t_r^2}\sum_i\sum_j \mathring{\eta}_i\,\eta_j\int_0^1\gamma_i\int_0^\xi \gamma_j'\gamma_m'\,d\alpha d\xi$$

$$-\frac{l^2\,\overset{\circ\circ}{\theta}}{2t_r^2}\left\{\sum_j\sum_k \eta_j\eta_k\int_0^1\gamma_m\int_0^\xi \gamma_j'\gamma_k'\,d\alpha d\xi - 2\int_0^1\gamma_m\xi\,d\xi\right\} + \frac{l^2}{t_r^2}\,\overset{\circ\circ}{\eta}_m$$

$$-\frac{2l^2\,\mathring{\theta}}{t_r^2}\sum_j\sum_k \mathring{\eta}_j\,\eta_k\int_0^1\gamma_m\int_0^\xi \gamma_j'\gamma_k'\,d\alpha d\xi + \frac{l^2\,\mathring{\theta}}{t_r^2}\sum_i\sum_j \mathring{\eta}_i\,\eta_j\int_0^1\gamma_i\int_0^1 \gamma_j'\gamma_m'\,d\alpha d\xi$$

$$-\frac{l^2\,\mathring{\theta}^2}{t_r^2}\{\eta_m - \sum_i \eta_i\int_0^1\xi\int_0^\xi \gamma_i'\gamma_m'\,d\alpha d\xi\} + \frac{EI}{ml^2}\sum_i \eta_i\int_0^1 \gamma_i''\gamma_m''\,d\xi \qquad \text{(D.35)}$$

Appendix E

Calculation of Inertial Integrals

The inertial integrals are calculated by implementing the same 4^{th} order Runge-Kutta numerical integration scheme used for the solution of the ordinary differential equations. This is done by first restating the integral as

$$I = \int_a^b f(\alpha)\, d\alpha$$

Let

$$\frac{\partial y}{\partial x} = f(x)$$

Then

$$y(x) = \int_a^x f(\alpha)\, d\alpha$$

Therefore

$$I = y(b)$$

Thus, the integrals to be evaluated are restated in the form

$$y(a) = 0$$

$$y(b) = I$$

$$\frac{\partial y}{\partial x} = f(x)$$

The matricies, vectors, and 3-dimensional arrays evaluated for the first 5 modes are presented below.

$$\varepsilon_i = \begin{Bmatrix} 1.8751040 \\ 4.6940911 \\ 7.8547574 \\ 10.995540 \\ 14.137168 \end{Bmatrix} \qquad \beta_i = \begin{Bmatrix} 0.73409551 \\ 1.01846731 \\ 0.99922449 \\ 1.00003355 \\ .999998550 \end{Bmatrix}$$

$$W = \{-2\varepsilon_i^{-2}\} \qquad i = 1, 2, \ldots, n \qquad Z \doteq \{2\frac{\beta_i}{\varepsilon_i}\} \qquad i = 1, 2, \ldots, n$$

$$W = \begin{Bmatrix} -0.56882574 \\ -0.09076679 \\ -0.03241637 \\ -0.01654234 \\ -0.01000703 \end{Bmatrix} \qquad Z = \begin{Bmatrix} 0.78299175 \\ 0.43393589 \\ 0.254425297 \\ 0.181898022 \\ 0.14147084 \end{Bmatrix}$$

$$\mathbf{N} = \int_0^1 (1-\xi)\vec{\gamma}\vec{\gamma}'^T \, d\xi$$

$$\mathbf{N} = \begin{bmatrix} 0.500000000 & -0.654951274 & -0.228695631 & -0.116424680 & -0.070391535 \\ & 0.500000000 & -1.63740540 & -0.754049957 & -0.446425915 \\ & & 0.500000000 & -2.75996685 & -1.36488152 \\ \mathbf{N}_{ij} = \mathbf{N}_{ji}\,,\ i \neq j & & & 0.500000000 & -3.81629467 \\ & & & & 0.500000000 \end{bmatrix}$$

$$\mathbf{P} = \int_0^1 (1-\xi)\vec{\gamma}'\vec{\gamma}'^T\, d\xi$$

$$\mathbf{P} = \begin{bmatrix} 1.57087803 & -0.422320366 & -1.07208443 & -0.873137712 & -0.762325704 \\ & 8.64714241 & 1.89005470 & -3.64338493 & -3.06280518 \\ & & 24.9521179 & 8.33829021 & -7.14108658 \\ & & & 51.4591064 & 19.0191345 \\ SYM. & & & & 87.7923279 \end{bmatrix}$$

$$\mathbf{Q} = \int_0^1 (1-\xi)^2\vec{\gamma}'\vec{\gamma}'^T\, d\xi$$

$$\mathbf{Q} = \begin{bmatrix} 0.755083621 & 0.527069807 & -0.559411228 & -0.653448880 & -0.616500556 \\ & 4.33783531 & 3.44129372 & -1.46306705 & -2.34727764 \\ & & 14.1851864 & 10.1280355 & -1.97334003 \\ & & & 30.8074341 & 20.8979492 \\ SYM. & & & & 53.9824982 \end{bmatrix}$$

$$\mathbf{B} = \int_0^1 \vec{\gamma}'(1-\xi^2)\vec{\gamma}'^T\, d\xi$$

$$\mathbf{B} = \begin{bmatrix} 2.38667297 & -1.37171078 & -1.58475876 & -1.09282684 & -0.908150852 \\ & 12.9564495 & 0.338815749 & -5.82370186 & -3.77833366 \\ & & 35.7190399 & 6.54854393 & -12.3088331 \\ & & & 72.1107788 & 17.1403198 \\ SYM. & & & & 121.602158 \end{bmatrix}$$

The individual matricies in the 3-dimensional array **A** are:

$$\mathbf{A}_i = \int_0^1 \vec{\gamma}' \gamma_i''' \vec{\gamma}'^T \, d\xi$$

$$\mathbf{A}_1 = \begin{bmatrix} 26.3730469 & -18.4942169 & -16.4187469 & -10.6408157 & -8.85824776 \\ & 143.575958 & -6.98174000 & -62.6827240 & -36.2099762 \\ & & 387.446777 & 47.0517883 & -136.093369 \\ & & & 774.354004 & 141.532944 \\ SYM. & & & & 1299.25513 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -103.011948 & 943.927979 & -500.062988 & -315.640625 & -200.515244 \\ & -1035.77344 & 2791.21387 & -935.910645 & -1313.64062 \\ & & -198.720245 & 6105.23437 & -1010.10547 \\ & & & 1749.54321 & 10988.7031 \\ SYM. & & & & 4578.32031 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -95.1386108 & -1169.06592 & 7525.31641 & -2989.66797 & -1591.58350 \\ & 8629.83203 & -7941.68359 & 18619.4961 & -5608.66406 \\ & & 8120.63281 & -8052.46094 & 36451.8984 \\ & & & 12760.1094 & -4508.98828 \\ SYM. & & & & 19744.2578 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} -63.8195801 & -1020.20850 & -5466.99609 & 29050.1289 & -10226.5352 \\ & -4973.36328 & 38029.4922 & -30049.8594 & 66596.8125 \\ & & -22767.8203 & 28633.0039 & -33103.4414 \\ & & & -31183.5156 & 39418.5000 \\ SYM. & & & & -25577.2422 \end{bmatrix}$$

$$\mathbf{A}_5 = \begin{bmatrix} -52.9176788 & -690.144531 & -4080.48560 & -16356.9219 & 79572.0625 \\ & -7092.67187 & -16095.1992 & 112466.875 & -81406.8125 \\ & & 126513.500 & -58208.1680 & 77607.4375 \\ & & & 71793.1250 & -84504.9375 \\ SYM. & & & & 85213.5000 \end{bmatrix}$$

The individual matricies in the 3-dimensional array **R** are:

$$\mathbf{R}_i = \int_0^1 \vec{\gamma}'\gamma_i(\xi - 1)\vec{\gamma}'^T \, d\xi$$

$$\mathbf{R}_1 = \begin{bmatrix} 1.21197987 & -1.94127560 & -0.388771832 & -0.019708782 & -0.024174653 \\ & 6.74166107 & -3.82725906 & -2.31213284 & 0.149746001 \\ & & 15.4504280 & -6.10586548 & -6.23327637 \\ & & & 28.2660675 & -9.12326527 \\ SYM. & & & & 45.2233734 \end{bmatrix}$$

$$\mathbf{R}_2 = \begin{bmatrix} 1.33617401 & 0.558530092 & -4.28103256 & -0.869393528 & -0.184457898 \\ & 3.18760109 & 4.25081539 & -11.7612181 & -4.06399441 \\ & & 16.8501129 & 10.5886698 & -23.2678528 \\ & & & 37.8424072 & 18.8473969 \\ SYM. & & & & 65.3193970 \end{bmatrix}$$

$$\mathbf{R}_3 = \begin{bmatrix} 0.142476380 & 2.91268063 & 0.208138227 & -6.43809891 & -1.34964752 \\ & 1.54717636 & 1.93412209 & 0.650911570 & -17.6205444 \\ & & 8.28947353 & 14.5578251 & 1.97157288 \\ & & & 24.4875031 & 32.1750183 \\ SYM. & & & & 43.7328033 \end{bmatrix}$$

$$\mathbf{R}_4 = \begin{bmatrix} 0.023667935 & 0.494373679 & 4.81118393 & 0.110539675 & -8.42709446 \\ & 6.65889835 & 2.05972958 & 1.17983818 & -0.964184463 \\ & & -2.02299213 & 5.05331612 & 14.1483288 \\ & & & 9.66746330 & 19.4558868 \\ SYM. & & & & 24.4349213 \end{bmatrix}$$

$$\mathbf{R}_5 = \begin{bmatrix} 0.007216453 & 0.112661123 & 0.858858705 & 6.69171238 & 0.068360447 \\ & 1.91680527 & 11.0725946 & 2.40212440 & 0.770909429 \\ & & 4.14834881 & -5.08257866 & 3.24874496 \\ & & & 1.08731937 & 7.00313854 \\ SYM. & & & & 15.0084724 \end{bmatrix}$$

The individual matricies in the 3-dimensional array **F** are:

$$\mathbf{F}_i = \int_0^1 \vec{\gamma}' \gamma_i \vec{\gamma}'^T \, d\xi$$

$$\mathbf{F}_1 = \begin{bmatrix} -5.19606400 & 13.1679697 & -10.9491253 & 11.0040989 & -10.9939117 \\ & -42.7950287 & 47.7574310 & -37.8047485 & 37.9628143 \\ & & -91.7760010 & 88.2697449 & -61.9667358 \\ & & & -157.153137 & 137.578903 \\ SYM. & & & & -237.838669 \end{bmatrix}$$

$$\mathbf{F}_2 = \begin{bmatrix} -1.34139824 & -4.48821831 & 16.2434387 & -11.1599646 & 11.2348652 \\ & 10.0089064 & -33.8573914 & 58.2068939 & -38.0131378 \\ & & 9.70805740 & -67.7286987 & 109.659058 \\ & & & 2.28693485 & -107.944382 \\ SYM. & & & & -13.3862858 \end{bmatrix}$$

$$\mathbf{F}_3 = \begin{bmatrix} 0.027705602 & -3.92985725 & -4.24450207 & 20.1404114 & -11.7530155 \\ & -4.73762131 & 12.7214613 & -31.2336121 & 68.0973053 \\ & & -33.4746552 & 13.0978031 & -57.6179504 \\ & & & -71.4166718 & 9.82317257 \\ SYM. & & & & -112.574493 \end{bmatrix}$$

$$\mathbf{F}_4 = \begin{bmatrix} -0.024980824 & 0.229944646 & -7.71646309 & -4.18601418 & 24.1781006 \\ & -10.6361389 & -3.66717815 & 13.4617872 & -31.9645386 \\ & & 21.5275269 & -29.4557343 & 11.2699614 \\ & & & 23.4571533 & -63.5717468 \\ SYM. & & & & 25.5609894 \end{bmatrix}$$

$$\mathbf{F}_5 = \begin{bmatrix} 0.004678264 & -0.191102862 & 0.877098024 & -11.7086391 & -4.16287994 \\ & -0.054310787 & -19.3674316 & -1.53450775 & 13.7879753 \\ & & -5.42145252 & 27.1802368 & -27.1806793 \\ & & & -27.0269470 & 27.0218506 \\ SYM. & & & & -60.3184509 \end{bmatrix}$$

Appendix F

FORTRAN code

=== OUTPUT FROM XPRTUTIL FOR RMH3834 ===

AT 01:39:41 ON 05/18/88 - RMH3834.THESIS.FORT

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C     SOLVE.FORT - CALCULATES SPATIAL VARIABLE RESPONSE OF A NONLINEAR C
C                  AND INEXTENSIBLE ROTATING 2 BEAM DYNAMIC SYSTEM     C
C                                                                      C
C     NS    : NUMBER OF STATES                                         C
C                                                                      C
C                                                                      C
C                                                                      C
C                                                                      C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XM(11,11),SS(22),IFLAG(4),EPS(10),A(10,10,10),D(10,10),
     &          SSP(22),GAMMAL(10),GAMAPL(10),PMAT(15),BETA(10),PSI(5,5)

      COMMON /PRSCRB/ PREVAR
      COMMON /FLAG/ IFLAG
      COMMON /MATRIX/ A,D,EPS,PMAT,BETA
      COMMON /PARAM/ V1,V2,V3
      COMMON /TORQ/ XINRTA,THETAR
      COMMON /CNSTNT/ EI,XMASSL,TR,XLEN,XLENTO,XIHUB,XLENP,XLENPP
      COMMON /COUNT/ KSPEC
      COMMON /GAMA/ GAMMAL,GAMAPL
      COMMON /TEST/ VIBTST,PSI


C_____READ VALUES FOR USER SUPPLIED CONSTANTS

      READ (7,*) T
      READ (7,*) N
      READ (7,*) TR
      READ (7,*) IMAX
      READ (7,*) DT
      READ (7,*) IFLAG(1)
      READ (7,*) IFLAG(2)
      READ (7,*) IFLAG(3)
      READ (7,*) IFLAG(4)
      READ (7,*) EI
      READ (7,*) XMASSL
      READ (7,*) THETAR
      READ (7,*) XLENTO
      READ (7,*) XIHUB

      NS = 2+2*N

C______INITIAL CONDITIONS
            DO 101 I=1,NS
            SS(I) = 0.0D0
 101        CONTINUE

C
C______2 MODE WISP TEST INITIAL CONDITION
C
      SS(N+2)=.1D0
```

```
C
C______READ VALUES FOR MATRICES A, D; VECTORS EPS, PMAT
C
      CALL SINPUT(N)

C
C______ASSEMBLE TIME INVARIANT PORTIONS OF MATRICES A AND D, AND
C       CALCULATE GAMMA(X=L) FOR ALL MODES
C
      CALL SETUP(N)
      VTIPMX = 0.D0
      XLEN = XLENTO

C______INTEGRATE EQUATIONS AND WRITE STATE

      DO 102 I=1,IMAX+1

C       IF (MOD((I-1),10).EQ.0) THEN

          IF (KSPEC.EQ.1) THEN
            WRITE(6,*) 'IN SOLVE, SS =',(SS(J),J=1,NS)
          ENDIF

          SUM = 0.D0
          DO 103 J = 1,N
            SUM = SUM+GAMMAL(J)*SS(J+1)
 103      CONTINUE

          VTIP = XLENTO*SUM
          IF (DABS(VTIP).GT.DABS(VTIPMX)) THEN
            VTIPMX = VTIP
          ENDIF

          WRITE (17,1001) T*TR,(SS(L),L=1,NS),PREVAR,VTIP,XLEN,ANGMOM
          WRITE (37,1001) T*TR,VIBTST
 1001     FORMAT(1X,1PD12.5,3(1PD19.11),
     &                       /,13X,3(1PD19.11),
     &                       /,13X,3(1PD19.11),
     &                       /,13X,3(1PD19.11),
     &                       /,13X,3(1PD19.11))

C       ENDIF

C       IF (MOD((I-1),50).EQ.0) THEN
C         KSPEC = 1
C       ELSE
          KSPEC = 0
C       ENDIF

        IF (ABS(SS(N+3)).GT.1.D5) STOP

        CALL RUNGE(T,DT,SS,SSP,NS,N)

        CALL MOMNTM(SS,N,ANGMOM)

 102  CONTINUE

      WRITE (6,*) 'MAX TIP DISPL. = ',VTIPMX
```

```
      STOP
      END

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C                 <<<<<<<<<  RUNGE  >>>>>>>>>                          C
C                                                                      C
C     THIS SUBROUTINE INTEGRATES THE EQUATIONS. IT ALSO WRITES THE     C
C     STATES.                                                          C
C                                                                      C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      SUBROUTINE RUNGE(T,DT,SS,SSP,NS,N)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION SSI(100),SSPI(100),D1(100),D2(100),D3(100),D4(100),
     &          SS(100),SSP(100),EPS(10)

           TI = T

          IF (KSPEC.EQ.1) THEN
            WRITE(6,*) 'IN RUNGE, SS ='
            WRITE(6,*)  (SS(J),J=1,NS)
          ENDIF

           DO 10 I=1,NS
           SSI(I) = SS(I)
10         CONTINUE
           IC=1
           CALL DERIV (TI,SSI,SSPI,IC,NS,N)

          IF (KSPEC.EQ.1) THEN
            WRITE(6,*) 'IN RUNGE, SSI ='
            WRITE(6,*)  (SSI(J),J=1,NS)
          ENDIF

           DO 20 I=1,NS
           D1(I) = DT*SSPI(I)
20         CONTINUE

C__       __COPY STATES AND...
C
29         TI = T +DT/2.D0
           DO 30 I=1,NS
           SSI(I) = SS(I)  + D1(I)/2.D0
30         CONTINUE
           IC=2
           CALL DERIV (TI,SSI,SSPI,IC,NS,N)
           DO 40 I=1,NS
           D2(I) = DT*SSPI(I)
40         CONTINUE
C
           DO 50 I=1,NS
           SSI(I) = SS(I)  + D2(I)/2.D0
50         CONTINUE
           IC=3
           CALL DERIV (TI,SSI,SSPI,IC,NS,N)
           DO 60 I=1,NS
           D3(I) = DT*SSPI(I)
```

```
60          CONTINUE
C
            TI = T + DT
            DO 70 I=1,NS
            SSI(I) = SS(I) + D3(I)
70          CONTINUE
            IC=4
            CALL DERIV (TI,SSI,SSPI,IC,NS,N)
            DO 80 I=1,NS
            D4(I) = DT*SSPI(I)
80          CONTINUE
C
            T=TI
            DO 90 I=1,NS
            SS(I) = SS(I) + ( D1(I)+2.D0*D2(I)+2.D0*D3(I)+D4(I) )/6.D0
90          CONTINUE

999   RETURN
      END
C
C
C-----------------------------------------------------------------------C
C                                                                       C
      SUBROUTINE DERIV (TI,SS,SSP,IC,NS,N)
C                                                                       C
C                                                                       C
C                                                                       C
C-----------------------------------------------------------------------C

C SET UP VARIABLES
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(10,10,10),FARRAY(10,10,10),RARRAY(10,10,10),
     &          D(10,10),EPS(10),XM(11,11),ETATD(10),
     &          S(10,10),XH(10,10),XGMAT(10,10),EXGMAT(10),
     &          SS(22),G(10,10),C(11,11),XK(11,11),GAMMAL(10),
     &          XMINV(11,11),XNMK(11,11),XNMC(11,11),GAMAPL(10),
     &          PMF(11,1),SSP(22),F(11,1),XTRANS(11,11),BETA(10),
     &          XMRV(10),XKVV(10),IFLAG(4),XMVR(10),PSI(5,5),
     &          XN(10,10),XP(10,10),XQ(10,10),PMAT(15)

      COMMON /PRSCRB/ PREVAR
      COMMON /CNSTNT/ EI,XMASSL,TR,XLEN,XLENTO,XIHUB,XLENP,XLENPP
      COMMON /EXTMAT/ XN,XP,XQ
      COMMON /FLAG/ IFLAG
      COMMON /MATRIX/ A,D,EPS,PMAT,BETA
      COMMON /MTCNST/ XMRR,XMRV,XKVV,XMVR
      COMMON /PARAM/ V1,V2,V3
      COMMON /TORQ/ XINRTA,THETAR
      COMMON /XNLMAT/ XGMAT,FARRAY,RARRAY
      COMMON /COUNT/ KSPEC
      COMMON /GAMA/ GAMMAL,GAMAPL
      COMMON /TEST/ VIBTST,PSI

      PI = 4.D0*DATAN(1.D0)



C
C______BEFORE ACTUALLY ASSEMBLING THE MASS AND STIFFNESS
```

```
C       MATRICES, PERFORM SOME PRELIMINARY NONLINEAR CALCULATIONS
C       IF NONLINEARITIES OR EXTENSION ARE TO BE INCLUDED IN THE
C       ANALYSIS.
C

      IF (IFLAG(2).EQ.1) THEN
C
C______CALCULATE ETA TRANSPOSE * D
C
        DO 201 I = 1,N
          X = 0.0D0
          DO 202 J=1,N

            X = X+SS(1+J)*D(J,I)

202       CONTINUE
          ETATD(I) = X
201     CONTINUE

C
C______CALCULATE THE TWO DIMENSIONAL G MATRIX FORMED BY MULTIPLYING
C       THE THREE DIMENSIONAL ARRAY A BY ETA
C
      DO 203 I = 1,N
        DO 204 J = 1,N
          X = 0.0D0
          DO 205 K = 1,N

            X = X + A(I,J,K)*SS(1+K)

205       CONTINUE
          G(J,I) = 1.0D0/EPS(I)**4 * X
204     CONTINUE
203   CONTINUE

C
C______IF, IN ADDITION TO GEO. NONLINEAR TERMS EXTENSION IS INCLUDED,
C       CALCULATE THE TWO DIMENSIONAL MATRICIES S AND XH FORMED BY
C       MULTIPLYING THE THREE DIMENSIONAL ARRAYS R AND F BY ETA
C

      IF (IFLAG(4).EQ.1) THEN

      DO 371 I = 1,N
        DO 372 J = 1,N
          X1 = 0.0D0
          X2 = 0.0D0
          DO 373 K = 1,N

            X1 = X1 + RARRAY(I,J,K)*SS(1+K)
            X2 = X2 + FARRAY(I,J,K)*SS(1+K)

373       CONTINUE
          S(J,I) = X1
          XH(J,I) = X2
372     CONTINUE
371   CONTINUE

C
C______ALSO CALCULATE ETA TRANSPOSE TIMES XGMAT
```

```
C

      DO 351 I = 1,N
        X=0.D0
        DO 352 J = 1,N

          X = X + SS(1+J)*XGMAT(J,I)

 352    CONTINUE
        EXGMAT(I) = X
 351  CONTINUE

      ENDIF
      ENDIF

C
C______CALL SUBROUTINE SETUPX TO PERFORM PRELIMINARY CALCULATIONS
C       IF EXTENSION IS INCLUDED
C

      IF (IFLAG(4).EQ.1) THEN

        CALL SETUPX(N,TI)

      ENDIF
C
C______BEGIN ASSEMBLING MATRICES - START BY ZEROING ALL MATRICIES
C

      DO 101 I = 1,N+1
        DO 102 J = 1,N+1

          XM(I,J) = 0.0D0
          C(I,J) = 0.D0
          XK(I,J) = 0.D0

 102    CONTINUE
 101  CONTINUE

C
C______INCLUDE THE LINEAR NONEXTENTING TERMS OF THE XM MATRIX
C

      XM(1,1) = XMRR
      DO 103 I = 1,N
        XM(I+1,I+1) = 1.0D0
        XM(1,I+1) = XMRV(I)
        XM(I+1,1) = XMVR(I)
 103  CONTINUE


C
C______INCLUDE THE LINEAR NONEXTENTING TERMS OF THE XK MATRIX
C

      DO 104 I = 1,N
        XK(I+1,I+1) = XKVV(I)
 104  CONTINUE
```

```
C
C______INCLUDE GEOMETRIC NONLINEAR ROTATIONAL TERMS IF SUCH TERMS
C      ARE IN THE ANALYSIS.
C

      IF (IFLAG(2).EQ.1) THEN

C
C______CALCULATE ETATD * ETA, STORE IN VARIABLE X, ADD
C      X TO XMRR TO CALCULATE COMPLETE XM(1,1) TERM
C

         X = 0.0D0
         DO 107 I = 1,N
           X = X + ETATD(I)*SS(1+I)
107      CONTINUE

         XM(1,1) = XMRR + (XLENTO/XLEN)**2*X

C
C______INCLUDE NONLINEAR ELEMENTS IN MVR SECTION OF XM MATRIX
C


         DO 108 I=1,N
           X = 0.0D0
           DO 109 J = 1,N
             X = X+.5D0*G(J,I)*SS(1+J)-G(I,J)*SS(1+J)
109        CONTINUE
           XM(I+1,1) = XM(I+1,1) + (XLENTO/XLEN)*X
108      CONTINUE

C
C______CALCULATE MATRIX C, STARTING WITH CRR
C

C
C______CRR TERM
C
         DO 120 I = 1,N
           C(1,I+1) = 2.D0*SS(N+2)*(XLENTO/XLEN)**2*ETATD(I)
 120     CONTINUE

C
C______IF EXTENSION IS INCLUDED, ADD ADDITIONAL GEOMETRIC
C      NONLINEAR TERMS TO C(1,1)
C

         IF (IFLAG(4).EQ.1) THEN

           X = 0.D0
           DO 353 I = 1,N
C            X = X + EXGMAT(I)*SS(1+I)
             X = X + ETATD(I)*SS(1+I)
 353       CONTINUE
           C(1,1) = C(1,1) + XLENTO**2*XLENP/XLEN**3*X

         ENDIF

C
```

```
C______CALCULATE GEOMETRIC NONLINEAR CVR TERMS
C
        DO 121 I = 1,N
          X = 0.0D0
          DO 122 J = 1,N
            X = X + G(J,I)*SS(2+N+J)-G(I,J)*SS(2+N+J)
122       CONTINUE
          C(I+1,1) = 2.D0*(XLENTO/XLEN)*X
121     CONTINUE

C
C______IF EXTENSION IS INCLUDED, ADD EXTENDING GEO NL TERMS
C

        IF (IFLAG(4).EQ.1) THEN

          DO 355 I = 1,N
            DO 356 J = 1,N

              XK(1+I,1+J) = XK(I+1,J+1) - 2.D0*SS(2+N)*XLENTO
     &                      *XLENP/XLEN**2*(1.D0/2.D0*(G(J,I)
     &                      +XH(J,I))+G(I,J)-S(I,J))

 356        CONTINUE
 355      CONTINUE

      ENDIF
      ENDIF

C
C_______INCLUDE GYROSCOPIC NONLINEAR TERMS TO THE K MATRIX, IF SUCH
C       TERMS ARE IN THE ANALYSIS
C

      IF (IFLAG(3).EQ.1) THEN

        DO 126 I = 1,N
          DO 127 J = 1,N
            XK(1+I,1+J) = XK(1+I,1+J)-D(I,J)*SS(2+N)**2
127       CONTINUE
126     CONTINUE

      ENDIF

C
C______IF EXTENSION IS INCLUDED, ADD THE APPROPRIATE LINEAR TERMS TO
C      THE DAMPING AND STIFFNESS MATRICIES
C

      IF (IFLAG(4).EQ.1) THEN

        C(1,1) = C(1,1) + XLENP/XLEN

        DO 802 I = 1,N
          C(I+1,1) = C(I+1,1)-4.D0*XLENP/XLENTO*BETA(I)/EPS(I)
          C(1,I+1) =C(1,I+1)+4.D0*XLENP*XLENTO/XLEN**2*(-2.D0/EPS(I)**2
     &               +BETA(I)/EPS(I))
 802    CONTINUE

        DO 803 I = 1,N
```

```
          DO 804 J = 1,N

            C(I+1,J+1) = 2.DO*XLENP/XLEN*XN(I,J)

 804      CONTINUE
 803     CONTINUE

         DO 805 I = 1,N
           DC 806 J = 1,N

             XK(I+1,J+1) = XK(I+1,J+1)+XLENPP/XLEN*(XN(I,J)-XP(I,J))
     &                     -(XLENP/XLEN)**2*XQ(I,J)

 806       CONTINUE
 805     CONTINUE

         DO 807 I=1,N
           XK(1,I+1) = 4.DO*(XLENPP*XLENTO/XLEN**2+XLENP**2*XLENTO
     &                /XLEN**3)*(-1.DO/EPS(I)**2+BETA(I)/EPS(I))
 807     CONTINUE

       ENDIF

C
C______IF ANGULAR VELOCITY IS PRESCRIBED, BRANCH TO A DIFFERENT
C      SECTION OF THE PROGRAM TO CALCULATE THE STATE EQUATIONS
C

      IF (IFLAG(1).EQ.0) GO TO 1000

C
C______ASSEMBLE FORCING FUNCTION ARRAY, F
C
      CALL TORQUE(TI,X)
      PREVAR = X
      F(1,1) = PREVAR

C
C______SET UP REMAINDER OF VECTOR F.
C

      DO 130 I = 2,N+1
        F(I,1) = 0.DO
 130  CONTINUE

C
C______PUT EQUATIONS IN COMPLETE STATE VECTOR FORM
C      - FIRST FIND M INVERSE
C
      DO 131 I = 1,N+1
        DO 132 J = 1,N+1

          XTRANS(I,J) = XM(I,J)

132     CONTINUE
131   CONTINUE

      CALL DLINRG(N+1,XTRANS,11,XMINV,11)
```

```
C      WRITE (6,*) 'XMINV '
C      DO 707 I=1,N+1
C        WRITE (6,*) XMINV(I,1),XMINV(I,2),XMINV(I,3)
C 707  CONTINUE

C
C______CALCULATE ELEMENTS OF N MATRIX
C

      CALL MATMLT (11,11,11,N+1,N+1,N+1,XMINV,XK,XNMK)
      CALL MATMLT (11,11,11,N+1,N+1,N+1,XMINV,C,XNMC)
      CALL MATMLT (11,1,11,N+1,1,N+1,XMINV,F,PMF)

C
C______STATE VECTOR EQUATION
C

      DO 140 I = 1,N+1
        SSP(I) = SS(N+1+I)
140   CONTINUE

      DO 141 I = 1,N+1
        X = 0.0D0
        DO 142 J = 1,N+1
          X = X - XNMK(I,J)*SS(J)-XNMC(I,J)*SS(N+1+J)
142     CONTINUE
        SSP(N+1+I) = X + PMF(I,1)
141   CONTINUE

C
C______CALCULATE VIBRATIONAL TEST
C

      IF (IC.EQ.1) THEN

      D0 = 0.D0
      D1 = 0.D0
      D2 = 0.D0
      NTPSIN = 0.D0
      NDPSIN = 0.D0

      DO 1301 I = 1,N
        D0 = D0 + GAMMAL(I)*SS(1+I)
        D1 = D1 + GAMAPL(I)*SS(1+I)
        D2 = D2 + GAMMAL(I)*SSP(2+N+I)
 1301 CONTINUE

      DO 1302 I = 1,N
        DO 1303 J = 1,N
          NTPSIN = NTPSIN + SS(1+I)*PSI(I,J)*SS(1+J)
          NDPSIN = NDPSIN + SS(2+N+I)*PSI(I,J)*SS(1+J)
 1303   CONTINUE
 1302 CONTINUE

      RHS = XLENPP/XLEN*D1-SSP(2+N)/XLEN*D0
     &      -2.D0*SS(2+N)*XLENT0/XLEN*D0*D1-SS(2+N)**2*D1

      LHS = D2+SSP(2+N)/XLENT0*(XLEN-XLENT0**2/(2.D0*XLEN)*NTPSIN)
     &      +2.D0*SS(2+N)/XLENT0*(XLENP+XLENP*XLENT0**2/(2.D0*XLEN**2)
```

```
     &      *NTPSIN-XLENTO**2/XLEN*NDPSIN)-SS(2+N)**2*DO

      VIBTST = RHS-LHS

      ENDIF


      GO TO 999

C
C______SINCE ANGULAR ACCEL. HAS BEEN PRESCRIBED, CALCULATE THE STATE
C       EQUATIONS IN THIS SIMPLER MANNER
C       - FIRST CALCULATE THETPP (PRESCRIBED ANGULAR VELOCITIES)
C         NOTE: THETPP NUST BE IN NONDIMENSIONAL FORM
C

C
C______THIS SECTION PRESCRIBES A BANG-BANG MANEUVER
C

C1000 IF (TI.GT.1.DO) THEN
C       THETPP=0.DO
C     ELSE
C       IF (TI.GT..5DO) THEN
C         THETPP = -4.DO*.175DO/TR**2
C       ELSE
C         THETPP = 4.DO*.175DO/TR**2
C       ENDIF
C     ENDIF

C
C______THIS SECTION PRESCRIBES THE R.R.RYAN SPINUP MANEUVER
C

 1000 IF (TI.GT.1.DO) THEN
        THETPP=0.DO
      ELSE
        THETPP = TR**2*2.DO/5.DO*(1.DO-DCOS(2.DO*PI*TI))
      ENDIF

C
C______THIS SECTION ELIMINATES THE ROTATIONAL DEGREE OF FREEDOM
C       BY PRESCRIBING ZERO ANGULAR VELOCITY AND ACCELERATION
C
C
C1000 CONTINUE
C
C       THETPP=0.DO


C
C______CALCULATE STATE VECTOR
C


      DO 301 I = 1,N+1
        SSP(I) = SS(N+1+I)
 301  CONTINUE

      SSP(N+2) = THETPP
```

```
      DO 302 I = 1,N
        SUM = 0.D0
        DO 303 J = 1,N+1

          SUM = SUM + XK(I+1,J)*SS(J) + C(I+1,J)*SS(N+1+J)

 303    CONTINUE
        SSP(N+2+I) = -SUM-XM(I+1,1)*THETPP
 302  CONTINUE


      PREVAR = THETPP
999   CONTINUE

      IF (KSPEC.EQ.1.AND.IC.EQ.1) THEN
C
C______WRITE ALL INFORMATION FOR DEBUGGING PURPOSES
C

      WRITE (6,*) 'A'
      DO 1101 I = 1,N
        DO 1102 K = 1,N
          WRITE(6,*) (A(I,K,J),J=1,N)
1102    CONTINUE
        WRITE (6,*) 'THAT WAS A',I
1101  CONTINUE

      WRITE (6,*) 'R'
      DO 1103 I = 1,N
        DO 1104 K = 1,N
          WRITE(6,*) (RARRAY(I,K,J),J=1,N)
1104    CONTINUE
        WRITE (6,*) 'THAT WAS R',I
1103  CONTINUE

      WRITE (6,*) 'F'
      DO 1105 I = 1,N
        DO 1106 K = 1,N
          WRITE(6,*) (FARRAY(I,K,J),J=1,N)
1106    CONTINUE
        WRITE (6,*) 'THAT WAS F',I
1105  CONTINUE

      WRITE (6,*) 'M'
      DO 704 I=1,N+1
        WRITE (6,*) (XM(I,J),J=1,N+1)
  704 CONTINUE

      WRITE (6,*) 'C'
      DO 705 I=1,N+1
        WRITE (6,*) C(I,1),C(I,2),C(I,3)
  705 CONTINUE

      WRITE (6,*) 'K'
      DO 706 I=1,N+1
        WRITE (6,*) XK(I,1),XK(I,2),XK(I,3)
  706 CONTINUE

      WRITE (6,*) 'XN'
      DO 774 I=1,N
```

```
        WRITE (6,*) (XN(I,J),J=1,N)
  774 CONTINUE

      WRITE (6,*) 'XP'
      DO 775 I=1,N
        WRITE (6,*) (XP(I,J),J=1,N)
  775 CONTINUE

      WRITE (6,*) 'XQ'
      DO 776 I=1,N
        WRITE (6,*) (XQ(I,J),J=1,N)
  776 CONTINUE

      WRITE (6,*) 'G'
      DO 707 I=1,N
        WRITE (6,*) G(I,1),G(I,2)
  707 CONTINUE

      WRITE (6,*) 'S'
      DO 756 I=1,N
        WRITE (6,*) S(I,1),S(I,2)
  756 CONTINUE

      WRITE (6,*) 'XH'
      DO 757 I=1,N
        WRITE (6,*) XH(I,1),XH(I,2)
  757 CONTINUE

      WRITE (6,*) 'EPS(1),EPS(2)'
      WRITE (6,*) EPS(1),EPS(2),EPS(3),EPS(4),EPS(5)
      WRITE (6,*) 'BETA(I),XKVV(I),EXGMAT(I),I'

      DO 708 I = 1,N
        WRITE(6,*) BETA(I),XKVV(I),EXGMAT(I),I
  708 CONTINUE
      WRITE (6,*) ' '
      WRITE (6,*) 'XLENPP,XMRR =',XLENPP,XMRR
      WRITE (6,*) ' '
      WRITE (6,*) 'V1,V2,V3 =',V1,V2,V3
      WRITE (6,*) 'EI,XMASSL,XLEN =',EI,XMASSL,XLEN
      WRITE (6,*) 'XLENTO,XIHUB,XLENP =',XLENTO,XIHUB,XLENP

      WRITE(6,*) 'PI,TR =',PI,TR
      WRITE(6,*) ' '
      WRITE(6,*) 'STATE DERIVATIVE VECTOR - TIME =',TI*TR
      WRITE(6,*) 'SSP(1) =',SSP(1)/TR,'SS(1) =',SS(1)
      WRITE(6,*) 'SSP(2) =',SSP(2)/TR,'SS(2) =',SS(2)
      WRITE(6,*) 'SSP(3) =',SSP(3)/TR,'SS(3) =',SS(3)
      WRITE(6,*) 'SSP(4) =',SSP(4)/TR**2,'SS(4) =',SS(4)/TR
      WRITE(6,*) 'SSP(5) =',SSP(5)/TR**2,'SS(5) =',SS(5)/TR
      WRITE(6,*) 'SSP(6) =',SSP(6)/TR**2,'SS(6) =',SS(6)/TR
      WRITE(6,*) '  '

      ENDIF


      RETURN
      END
```

```
C
C-----------------------------------------------------------------------C
C                                                                       C
      SUBROUTINE SINPUT(N)
C                                                                       C
C                                                                       C
C                                                                       C
C-----------------------------------------------------------------------C
C


      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(10,10,10),RARRAY(10,10,10),FARRAY(10,10,10),
     &          XJ(10,10),T(10,10),E(10,10),XGMAT(10,10),
     &          D(10,10),EPS(10),PMAT(15),XN(10,10),B(10,10),
     &          XP(10,10),XQ(10,10),BETA(10),PSI(5,5)
      COMMON /EXTMAT/ XN,XP,XQ
      COMMON /MATRIX/ A,D,EPS,PMAT,BETA
      COMMON /XNLMAT/ XGMAT,FARRAY,RARRAY
      COMMON /TEST/ VIBTST,PSI

C
C______READ VALUES FOR B AND A MATRICES
C

      DO 101 I = 1,5
        DO 102 J = 1,5

          READ (9,*) B(I,J)

 102    CONTINUE
 101  CONTINUE


      DO 103 I = 1,5
        DO 104 J = 1,5
          DO 105 K = 1,5

            READ (9,*) A(I,J,K)
            READ (22,*) FARRAY(I,J,K)
            READ (23,*) RARRAY(I,J,K)

 105      CONTINUE
 104    CONTINUE
 103  CONTINUE

C
C______READ IN VALUES FOR EPS(I)
C

      DO 106 I=1,N
        READ (8,*) EPS(I)
 106  CONTINUE

      DO 201 I = 1,5
        DO 202 J = 1,5

          READ (18,*) XN(I,J)

 202    CONTINUE
```

```
  201   CONTINUE

        DO 203 I = 1,5
          DO 204 J = 1,5

            READ (19,*) XP(I,J)

  204     CONTINUE
  203   CONTINUE

        DO 205 I = 1,5
          DO 206 J = 1,5

            READ (20,*) XQ(I,J)

  206     CONTINUE
  205   CONTINUE

C
C______READ IN MATRIX PSI
C

        DO 876 I = 1,5
          DO 875 J = 1,5

            READ (34,*) PSI(I,J)

  875     CONTINUE
  876   CONTINUE


C
C_______READ IN VALUES FOR POLYNOMIAL MATRIX PMAT NEEDED FOR
C       TORQUE COMPUTATION
C

        DO 107 I = 1,10
          READ (10,2001) PMAT(I)
  107   CONTINUE

  2001 FORMAT (2X,1PD23.15)
C
C______SET UP MATRIX D
C

        DO 108 I = 1,N
          DO 109 J = 1,N

            D(I,J) = -.5D0*B(I,J)

  109     CONTINUE
          D(I,I) = 1.0D0+D(I,I)
  108   CONTINUE


        WRITE(6,*) 'IN SUBROUTINE SINPUT MATRIX XGMAT EQUALS:'
        DO 711 I = 1,N
          DO 712 J = 1,N
            WRITE(6,*) XGMAT(I,J)
  712     CONTINUE
```

```
 711  CONTINUE

      RETURN
      END



C
C-----------------------------------------------------------------------C
C                                                                       C
      SUBROUTINE SETUP(N)
C                                                                       C
C                                                                       C
C                                                                       C
C-----------------------------------------------------------------------C
C

      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XMRV(10),XKVV(10),A(10,10,10),D(10,10),EPS(10),
     &          PMAT(15),GAMMAL(10),GAMAPL(10),BETA(10),XMVR(10),
     &          IFLAG(10)
      COMMON /CNSTNT/ EI,XMASSL,TR,XLEN,XLENTO,XIHUB,XLENP,XLENPP
      COMMON /FLAG/ IFLAG
      COMMON /MATRIX/ A,D,EPS,PMAT,BETA
      COMMON /MTCNST/ XMRR,XMRV,XKVV,XMVR
      COMMON /PARAM/ V1,V2,V3
      COMMON /TORQ/ XINRTA,THETAR
      COMMON /GAMA/ GAMMAL,GAMAPL

C
C______IF EXTENSION OCCURS, DO NOT EVALUATE THE MATRIX CONSTANTS,
C      WHICH ARE NOW FUNCTIONS OF TIME.
C

      IF (IFLAG(4).EQ.1) GO TO 123

C
C______CALCULATE THE RIGID BODY INERTIA OF THE SYSTEM
C

      XINRTA = 2.D0*XMASSL*XLENTO**3

C
C______SET TIME DEPENDENT LENGTH EQUAL TO INITIAL LENGTH
C

      XLEN = XLENTO

C
C______EVALUATE V1, V2, V3
C

      V1 = EI*TR**2/(XMASSL*XLENTO**4)
      V2 = XIHUB/XINRTA
      V3 = TR**2/XINRTA

C
C______EVALUATE MATRIX M
C      FIRST CALCULATE MRR
C
```

```
      XMRR = V2+1.0D0/3.0D0

C
C_______NEXT CALCULATE MRV
C

      DO 112 I = 1,N
        XMRV(I) = -2.0D0/EPS(I)**2
        XMVR(I) = XMRV(I)
 112  CONTINUE

C
C_______CALCULATE THAT PART OF KVV WHICH IS NOT TIME DEPENDENT
C       AND ASSIGN IT TO THE MATRIX XKVV
C

      DO 123 I = 1,N
        XKVV(I) = V1*EPS(I)**4
 123  CONTINUE

C
C_______CALCULATE THE VALUE OF GAMMA(X) AT X=L
C

      DO 901 I = 1,N

        SI = DSIN(EPS(I))
        CO = DCOS(EPS(I))
        SH = DSINH(EPS(I))
        CH = DCOSH(EPS(I))

        BETA(I) = (CO+CH)/(SI+SH)
        BETAM1 = (CO+DEXP(-EPS(I))-SI)/(SI+SH)

        GAMMAL(I) = CO-DEXP(-EPS(I))-BETA(I)*SI+BETAM1*SH
        GAMAPL(I) = EPS(I)*(-SI+DEXP(-EPS(I))-BETA(I)*CO+BETAM1*CH)

 901  CONTINUE





      RETURN
      END




C
C----------------------------------------------------------------------C
C                                                                      C
      SUBROUTINE SETUPX(N,TAU)
C                                                                      C
C                                                                      C
C                                                                      C
C----------------------------------------------------------------------C
C

      IMPLICIT REAL*8 (A-H,O-Z)
```

```
      DIMENSION XMRV(10),XKVV(10),A(10,10,10),D(10,10),EPS(10),
     &          PMAT(15),GAMMAL(10),BETA(10),XMVR(10)
      COMMON /CNSTNT/ EI,XMASSL,TR,XLEN,XLENTO,XIHUB,XLENP,XLENPP
      COMMON /MATRIX/ A,D,EPS,PMAT,BETA
      COMMON /MTCNST/ XMRR,XMRV,XKVV,XMVR
      COMMON /PARAM/ V1,V2,V3
C
C______COMPUTE PI
C

      PI = 4.D0*DATAN(1.D0)

C
C______STEVE'S WISP CASE
C      UPDATE LENGTH(TAU) & FIRST AND SECOND DERIVATIVE (W.R.T. TAU)
C
C
C       IF (TAU.LE.10) THEN
C           XLENPP = 0.D0
C           XLENP = 0.D0
C           XLEN = 150.D0
C
C         ELSE IF (TAU.LT.15) THEN
C           XLENPP = -5.D0*PI/48.D0*DSIN(PI/5.D0*(TAU-10.D0))
C           XLENP = 25.D0/48.D0*(DCOS(PI/5.D0*(TAU-10.D0))-1.D0)
C           XLEN = 25.D0/48.D0*(5.D0/PI*DSIN(PI/5.D0*(TAU-10.D0))
C    &          -TAU+10.D0)+150.D0
C
C         ELSE IF (TAU.LT.130) THEN
C           XLENPP = 0.D0
C           XLENP = -25.D0/24.D0
C           XLEN = 150.D0-125.D0/48.D0-25.D0/24.D0*(TAU-15.D0)
C
C         ELSE IF (TAU.LT.135) THEN
C           XLENPP = 5.D0*PI/48.D0*DSIN(PI/5.D0*(TAU-130.D0))
C           XLENP = -25.D0/48.D0*(DCOS(PI/5.D0*(TAU-130.D0))+1.D0)
C           XLEN = -25.D0/48.D0*(5.D0/PI*DSIN(PI/5.D0*(TAU-130.D0))
C    &              +TAU-130.D0)+150.D0-125.D0/48.D0-25.D0*115.D0/24.D0
C
C         ELSE
C           XLENPP = 0.D0
C           XLENP = 0.D0
C           XLEN = 25.D0
C
C       ENDIF

C
C______THESIS  WISP CASE
C      UPDATE LENGTH(TAU) & FIRST AND SECOND DERIVATIVE (W.R.T. TAU)
C

C      SET VALUES FOR INITIAL LENGTH, FINAL LENGTH, MANEUVER TIME

         XLI = XLENTO
         XLF = 125.D0
         TM = 500.D0

         VEL = (XLF-XLI)*2.D0/TM

       IF (TAU.LT.TM) THEN
```

```
          XLEN = XLI + 1.D0/2.D0*VEL
     &           *(TAU-TM/(2.D0*PI)*DSIN(TAU/TM*2.D0*PI))
          XLENP = 1.D0/2.D0*VEL*(1.D0-DCOS(TAU/TM*2.D0*PI))
          XLENPP = PI/TM*VEL*DSIN(TAU/TM*2.D0*PI)

        ELSE

          XLEN = XLF
          XLENP = 0.D0
          XLENPP = 0.D0

        ENDIF


C
C_______FRANKLIN INSTITUTE TEST (TEST 8)
C

C        XLENPP = 0.D0
C        XLENP = 108.D0
C        XLEN = XLENP*TAU

C
C_______CALCULATE NONDIMENSIONAL (TIME VARYING) CONSTANTS V1, V2, V3
C

      V1 = EI*TR**2/(XMASSL*XLEN**4)
      V2 = XIHUB/(2.D0*XMASSL*XLEN**3)
      V3 = TR**2/(2.D0*XMASSL*XLEN**3)

C
C_______CALCULATE MASS MATRIX COMPONENTS
C

        XMRR = V2+1.D0/3.D0

        DO 101 I = 1,N
          XMRV(I) = -2.D0*XLENTO/(XLEN*EPS(I)**2)
          XMVR(I) = -2.D0*XLEN/(XLENTO*EPS(I)**2)
          XKVV(I) = V1*EPS(I)**4
 101    CONTINUE

      RETURN
      END
C-----------------------------------------------------------------------C
C                                                                       C
      SUBROUTINE TORQUE(TI,X)
C                                                                       C
C                                                                       C
C                                                                       C
C-----------------------------------------------------------------------C
C

      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION IFLAG(4)
      COMMON /TORQ/ XINRTA,THETAR
      COMMON /CNSTNT/ EI,XMASSL,TR,XLEN,XLENTO,XIHUB,XLENP,XLENPP
```

```
      PI = 4.D0*DATAN(1.D0)

C
C______IF TI.GT.1 SET TORQUE TO ZERO AND RETURN
C
      IF (TI.GT.1.D0) THEN
        X = 0.D0
        RETURN
      ENDIF

C
C_______FIRST DETERMINE ANGULAR ACCELERATION FOR A 1 RADIAN MANEUVER
C

C BANG-BANG TORQUES
C
C       IF (TI.LT..5D0) THEN
C         ONERAD = 4.D0/TR**2
C       ELSE
C         ONERAD = -4.D0/TR**2
C       ENDIF

C SINE WAVE TORQUES
C
C       ONERAD = 2.0D0*PI/TR**2*DSIN(2.D0*PI*TI/TR)


C SUB-OPTIMAL TORQUES
C
C       CALL TORQX(TI,ONERAD,PMAT)

C NO TORQUE
C
        ONERAD = 0.D0

C
C_______NEXT, RESCALE THE MANEUVER FOR THE ACTUAL THETAR
C

        X = THETAR*ONERAD

C
C_______MULTIPLY ANGULAR ACCELERATION BY
C       RIGID BODY INERTIA TO DETERMINE THE TORQUE
C

        X = XINRTA*X

        RETURN
        END


C----------------------------------------------------------------------C
C                                                                      C
      SUBROUTINE MOMNTM (SS,N,ANGMOM)
C                                                                      C
C                                                                      C
C                                                                      C
C----------------------------------------------------------------------C
```

```
C

      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION SS(22),D(10,10),A(10,10,10),EPS(10),PMAT(15),
     &          BETA(10),ETATD(10),IFLAG(4),GAMMAL(10),GAMAPL(10),
     &          PSI(5,5)
      COMMON /CNSTNT/ EI,XMASSL,TR,XLEN,XLENTO,XIHUB,XLENP,XLENPP
      COMMON /FLAG/ IFLAG
      COMMON /MATRIX/ A,D,EPS,PMAT,BETA


C
C______CALCULATE ETA TRANSPOSE * D (ETATD FROM DERIV USES PREVIOUS
C         QUARTER TIME STEP VALUE OF SS(I), SO A NEW ETATD NEEDS TO
C         BE CALCULATED HERE)
C
      DO 201 I = 1,N
        X = 0.0D0
        DO 202 J=1,N

          X = X+SS(1+J)*D(J,I)

 202    CONTINUE
        ETATD(I) = X
 201  CONTINUE

      SUM1 = XLEN**3*SS(N+2)/3.D0

      X1 = 0.D0
      X2 = 0.D0
      X3 = 0.D0
      X4 = 0.D0

      DO 101 I = 1,N

        X1 = X1 + ETATD(I)*SS(1+I)
        X2 = X2 + SS(1+I)*2.D0*BETA(I)/EPS(I)
        X3 = X3 - SS(1+I)*2.D0/EPS(I)**2
        X4 = X4 - SS(2+N+I)*2.D0/EPS(I)**2

 101  CONTINUE

      IF (IFLAG(2).EQ.0) X1 = 0.D0

      SUM1 = SUM1 + XLEN*XLENTO**2*SS(2+N) * X1
     &            + 2.D0*XLEN*XLENP*XLENTO * X2
     &            + 2.D0*XLEN*XLENP*XLENTO * X3
     &            + XLEN**2*XLENTO * X4

      ANGMOM = 2.D0*XMASSL/TR*SUM1 + XIHUB*SS(2+N)/TR

       RETURN
       END
```