AN APPLICATION OF THE MALLIAVIN CALCULUS TO INFINITE DIMENSIONAL DIFFUSIONS

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ABSTRACT

Diffusions on the infinite product of a compact manifold are defined, and their finite dimensional marginals studied. It is shown that under reasonable hypotheses, the marginals possess smooth densities. An estimate on the densities is obtained which is independent of the number of dimensions with respect to which the marginals are taken. An application to statistical mechanics is discussed.

I Introduction

The study of the regularity of infinite dimensional diffusions presents certain difficulties. One cannot expect the transition probability functions to admit densities, even in the independent context. It is clear, however, in this context, that the finite dimensional marginals will, under reasonable hypotheses, admit densities. Equally clear is the fact that the uniform norms of the densities become unbounded as the number of dimensions with respect to which the marginal is taken goes to infinity. For certain applications it is necessary to have an estimate on the marginals which remains bounded as the number of dimensions increases. To see what sort of estimate may work, we will look at the independent case in more detail.

Let M be a compact Riemannian manifold, and σ its Riemannian measure, which we may assume is a probabilitty measure. Suppose that $x_k(t)$, $k \quad Z$ are independent diffusions on M whose transition probability functions $P_k(t,*)$ admit densities $p_k(t,*)$ with respect to σ. Then the transition probability function for the diffusion $\{x_k\}_{k \quad Z}$ is $P(t,*) = \prod_{k \in Z} P_k(t,*)$, and if $p_{(N)}(t,*)$ denotes the density for the marginal of $P(t,*)$ with respect to $\sigma^{\{k:\ |k| \leq N\}}$, then $p_{(N)}(t,*) = \prod_{-N \leq k \leq N} p_k(t,*)$. Although $\sup_{\eta}(p_{(N)}(t,\eta)) \longrightarrow \infty$ as $N \longrightarrow \infty$,

the ratio

$\mathrm{grad}_{\eta_k} p_{(N)}(t,\eta)/p_{(N)}(t,\eta) = \mathrm{grad}_{\eta} p_k(t,\eta_k)/p_k(t,\eta_k)$

remains bounded as $N \quad \infty$. The dimension independent estimate which we obtain in a more general setting involves this ratio.

The method of proof for all the results is the Malliavin calculus. Techniques of partial differential equations do not lend themselves to the infinite dimensional setting, because, as we have seen, we have to take marginal distributions, and these marginals do not, in general, satisfy any autonomous equation. On the other hand, Malliavin's calculus is well suited to the study of marginal distributions. Furthermore, ratios like the one in the preceding paragraph occur naturally when integrating by parts, on R, if $\mu(dx) = p(x)dx$ then $\int f'(x)\mu(dx) = -\int f(x)p'(x)/p(x)\mu(dx)$ for $f \in C_0^\infty(R)$. The Malliavin calculus allows us to integrate by parts on Wiener space. The main ideas for the proofs are derived from [A], where the same results are proved on the infinite dimensional torus.

Section II is devoted to a discussion of diffusions on M, section III to the Malliavin calculus on M, and section IV to the regularity of diffusions on M. (Sections II - IV result from private communication with D. Stroock.) The regularity of finite dimensional marginals of diffusions on M^Z is covered in section V, and the dimension - independent estimate on these marginals is covered in the following section. Finally, an application to statistical mechanics is discussed in the last section.

II. Diffusions on Manifolds

Let M be a compact manifold and V_j, $0 \leq j \leq d$ be vector fields on M. Set

$$(1) \qquad L = \sum_{j=1}^{d} (V_j)^2 + V_0$$

In this section, we will describe what is meant by 'the diffusion on M generated by L'.

Denote by m the dimension of M, and for some $D \geq m$ let $i: M \longrightarrow R^D$ be an imbedding. Then there are smooth functions $W_j: R^D \longrightarrow R^D$, $j = 0,\ldots,d$ satisfying

$$(2) \qquad \begin{array}{ll} \text{i)} & \text{For } x \in i(M),\ W_j(x) = i_*(V_j(i^{-1}(x))) \\ \text{ii)} & W_j^{\,i} \text{ and each of its derivatives is bounded on } R^D \text{ for} \\ & 0 \leq j \leq d \text{ and } 1 \leq i \leq D. \end{array}$$

Set $\Theta = \{\theta \in C([0,\infty),R^d): \theta(0) = 0\}$ and let $\theta(t) = \{\theta^j(t): 1 \leq j \leq d\}$ be the position of θ at time $t \geq 0$. Set $\mathcal{B}_t = \sigma(\theta(s): 0 \leq s \leq t)$ and $\mathcal{B} = \sigma(\theta(s): 0 \leq s)$. Let W be Wiener measure on $(\Theta,\mathcal{B})$. For $x \in R^D$, let $x(*,x)$ be the unique solution to the equation

$$(3) \qquad x(T,x) = x + \sum_{j=1}^{d} \int_0^T W_j(x(t,x)) \circ d\theta^j(t) + \int_0^T W_0(x(t,x))dt.$$

<u>Lemma(4)</u> If $x \in i(M)$ then $x(t,x) \in i(M)$ for all $t \geq 0$.

Proof. The following set up will be useful in this proof and elsewhere. For $i \geq 1$, let U_i, U_i', and $\tilde{U}_i$ $1 \leq i \leq r$ be precompact open subsets of R^D so that

i) $\overline{U_i} \subset\subset U'_i$. $\overline{U'_i} \subset\subset \tilde{U}_i$

(5) ii) $\bigcup_{i=1}^{\infty} U_i \supseteq R^D$ and $\exists r < \infty$ $\bigcup_{i=1}^{r} \tilde{U}_i \supseteq i(M)$

iii) For each i $\pi_i \in C_b^{\infty}(\overline{\tilde{U}_i})$ so that $(\tilde{U}_i, \pi_i)$ is a coordinate chart in R^D and

$$M \cap \overline{\tilde{U}_i} = \{\pi_i^{m+1} = \dots = \pi_i^{D} = 0\} \cap \overline{\tilde{U}_i}$$

For $x \in R^D$ let $n(x) = \min\{n \geq 1: x \in \tilde{U}_n\}$. Define $\tau_0 = 0$ and for $i \geq 1$, $\tau_i = \inf\{t \geq \tau_{i-1}: x(t,x) \notin \tilde{U}_{n(x(\tau_{i-1},x))}\}$.

By the strong Markov property, it will suffice to show that if $x \in M$ then $x(t \wedge \tau_1) \in i(M)$ for $t \geq 0$. Set $z = \pi_{n(x)}x$. Then, by Ito's formula,

$$z(T,z) = z + \sum_{j=1}^{d} \int_0^T \tilde{W}_j(z(t,z)) \circ d\theta^j(t) + \int_0^T \tilde{W}_0(z(t,z))dt,$$

$T \leq \tau_1$, where $\tilde{W}_j = (\partial\pi/\partial x)W_j$. Since $\tilde{W}_j^{\,n}(z(t \wedge \tau_1, z)) = 0$ for $z \in M$, $0 \leq j \leq d$ and $m+1 \leq n \leq D$, the proof is complete.

Now, set $\Omega = C([0,\infty),M)$ and for $\omega \in \Omega$ let $\eta(t,\omega) \in M$ be the position of ω at time $t \geq 0$. Set $\mathcal{M}_t = \sigma(\eta(s): 0 \leq s \leq t)$ and $\mathcal{M} = \sigma(\eta(s): 0 \leq s)$. For $\eta \in M$ we can define the measure P_η on $(\Omega, \mathcal{M})$ as the distribution of $i^{-1} \circ y(*, i(\eta))$ under W.

Theorem(6) For all $\eta \in M$, P_η described above is the unique probability measure on $(\Omega, \mathcal{M})$ such that $P_\eta(\eta(0)=\eta)=1$ and $(f(\eta(t))-\int_0^t Lf(\eta(s))ds, \mathcal{M}_t, P_\eta)$ is a martingale for every f in $C^\infty(M)$. Finally, the family $\{P_\eta : \eta \in M\}$ is Feller continuous and strong Markov.

Proof. That P_η satisfies the desired conditions is clear. The rest of the proof can be taken from chapter 6 of [3].

Remark(7) We can define diffusions on non - compact M in the same way if we assume that there are W_j, $0 \leq j \leq d$ satisfying (2).

Remark(8) From now on we will assume, for notational convenience, that $M \subseteq R^D$ and i is the identity.

III. The Malliavin Calculus on Manifolds

In sections IV - VI we will prove certain regularity results about the transition probability functions for diffusions on manifolds. We first need some results about the Malliavin calculus.

Let $W_j : R^D \to R^D$, $j=0,\dots,d$ satisfy ii) of (2) and let $x(t,x)$ be the solution to the system (3). Let $A(t,x) = ((\langle x^k(t,x), x^n(t,x)\rangle))_{1 \le k \le D, 1 \le n \le D}$ be the Malliavin covariance matrix. (For $\Phi, \Psi \in \mathcal{E}$ (see [1]), $\langle \Phi, \Psi \rangle = \mathcal{L}(\Phi\Psi) - \Psi\mathcal{L}\Phi - \Phi\mathcal{L}\Psi$, where $\mathcal{L}$ is the Ornstein - Uhlenbeck operator.) Then, for $\pi \in C_b^\infty(R^D, R^D)$

$$((\langle \pi^k(x(t,x)), \pi^n(x(t,x))\rangle))_{1 \le k \le D, 1 \le n \le D} = \partial\pi/\partial x(x(t,x))A(t,x)\partial\pi^*/\partial x(x(t,x)).$$

Hence, $A(t,x) \in T_{x(t,x)}(R^D)^{\otimes 2}$. (If N is a manifold and $p \in N$, $T_p(N)$ means the tangent space to N at p, and $T^*_p(N)$ means the cotangent space.) We will show, when $x \in M$ and W_j is an extension to R^D of a vector field V_j on M, $0 \le j \le d$, that $A(t,x) \in T_{x(t,x)}(M)^{\otimes 2}$.

We can make a selection of the map $x \mapsto x(*,x)$ so that for $T > 0$,

$(t,x) \in [0,T] \times R^D \mapsto x(t,x) \in C^{0,\infty}([0,\infty]\ R^D)$. Then, setting $X(t,x) = ((\,(\partial x^i/\partial x^j)(t,x)\,))_{1 \le i,j \le D}$, we have

$$(9) \quad X(T,x) = I + \sum_{j=1}^{d} \int_0^T (\partial W_j/\partial x)(x(t,x))X(t,x)\circ d\theta^j(t) + \int_0^T (\partial W_0/\partial x)(x(t,x))X(t,x)dt$$

Thus, X is invertible. Furthermore, (see [1])

$$(10) \qquad A(T,x) = \sum_{j=1}^{d} \int_0^T [X(t,T,x)W_j(x(t,x))]^{\otimes 2} dt$$

where $X(t,T,x) = X(T,x)X^{-1}(t,x)$. Since $X(T,x)\colon T_x(M) \to T_{x(t,x)}(M)$ and $W_j(x(t,x)) \in T_{x(t,x)}(M)$, $A(T,x) \in (T_{x(T,x)}(M))^{\otimes 2}$.

IV. Regularity

Suppose that M is a compact Riemannian manifold and that its Riemannian metric, g_M is the same as the metric it inherits from R^D. Let σ be the positive measure on M associated with the Riemannian structure. We may as well assume that σ is a probability measure.

Let $\{V_{k,j}\}_{0 \leq j \leq d}$ be a collection of vector fields on M so that

(11) for all x in M, $\{V_j(x)\}_{1 \leq j \leq d}$ spans the tangent space to M at x.

Let $P(t,\eta_0,\Gamma) = P_{\eta_0}(\eta(t) \in \Gamma)$ for $\eta_0 \in M$ and $\Gamma \subseteq M$. The goal of this section is to show that, for $t>0$, $P(t,\eta_0,d\eta)$ admits a density $p(t,\eta_0,\eta)$ with respect to σ which is smooth in η.

Fix $1 \leq i \leq r$ and set $U = U_i$, $U' = U_i'$ and $\pi = \pi_i$, where U_i, U_i', and π_i are defined in (5). We will show that $P(t,\eta_0,*)$ admits a density on $M \cap U$.

Choose a smooth function ρ on R^D so that $0 \leq \rho \leq 1$ and $\bar{U} \subset\subset \{\rho = 1\} \subset\subset \mathrm{supp}(\rho) \subset\subset U'$. Define the measure μ on R^m by $\mu(t,\eta_0,\Gamma) = E^W[\rho(x(t,\eta_0)\ z_{(m)} \in \Gamma\ ]$, where $z = \pi x$ and $z_{(m)} = \{z^1,\ldots,z^m\}$. Clearly we will be finished once we show that μ admits a smooth density. In order to do so, it suffices to show that for all $f \in C_0(R^m)$ and all multi-indices $\alpha \in N^m$,

$|\int D^\alpha f(\xi)\mu(t,\eta_0,d\xi)| \leq C_\alpha(t)||f||_\infty$, where

$D^\alpha = \partial^{|\alpha|}/(\ \partial z^{\alpha_1} \ \ldots \ \partial z^{\alpha_m})$.

Fix $1 \leq k \leq m$, and set $F(z) = f(z_{(m)})$. Then, for $1 \leq p \leq D$ and $x(t,\eta_0) \in U'$, $\langle F(z(t,\eta_0)),z^p(t,\eta_0)\rangle =$ $(\partial F/\partial z^q)(z(t,\eta_0))\tilde{A}^{q,p}(t,\eta_0)$ where $\tilde{A}(t,\eta_0) =$

$$\sum_{q=1}^{D} \partial\pi/\partial x(x(t,\eta_0))A(t,\eta_0)\partial\pi/\partial x^*(x(t,\eta_0)).$$ (So, by

section III, $\tilde{A}^{p,q}(t,\eta_0) = \langle z^p(t,\eta_0), z^q(t,\eta_0)\rangle$ and $\tilde{A}^{p,q}(t,\eta_0) = 0$ if $m+1 \leq p \vee q \leq D$ and $x(t,\eta_0) \in U'$.) Thus, if $x(t,\eta_0) \in U'$, $(\partial F/\partial z^q)(z(t,\eta_0)) =$

$$\sum_{p=1}^{m} (\tilde{A}_{(m)})_{p,q} \langle F(z(t,\eta_0)), z^p(t,\eta_0)\rangle$$

where $((\ (\tilde{A}_{(m)})_{p,q}\ ))_{1 \leq p,q \leq m} = (\tilde{A}_{(m)})^{-1}$, and $\tilde{A}_{(m)}$ is the upper left hand m by m submatrix of $\tilde{A}$.

Assume, for the moment, that

$$(12)\ \chi_{U'}(x(t,\eta_0))/\det(\tilde{A}_{(m)}(t,\eta_0)) \in \bigcap_{p=1}^{\infty} L^p(W)$$

Then, (see Lemma 3.4 in [2]), $\rho'(x(t,\eta_0))(\tilde{A}_{(m)})_{q,p}$ for $1 \leq q,p \leq m$ and any $\rho' \in C_0^{\infty}(U')$. If $\Phi \in \mathcal{E}$, setting $\Psi = \rho'(x(t,\eta_0))\Phi$, we may define $H_k(\Psi) =$

$$-\sum_{q=1}^{m} \Big[\langle z^q(t,\eta_0), (\tilde{A}_{(m)})_{q,k}(t,\eta_0)\ \Psi\ \rangle + 2(\tilde{A}_{(m)})_{q,k}(t,\eta_0)\Psi\mathcal{L}z^q(t,\eta_0)\Big]$$

Then, choosing $\rho' \in C_0^{\infty}(U')$ with $0 \leq \rho' \leq 1$ and $\operatorname{supp}(\rho) \subset\subset \{\rho' = 1\}$,

$$E^W[\{\partial F/\partial z^k\}(z(t,\eta_0))\rho(t,\eta_0)] =$$

$$E^W[\{\partial(F(\rho' \circ \pi^{-1}))/\partial z^k\}(z(t,\eta_0))\rho(t,\eta_0)] =$$

$E^W[F(z(t,\eta_0))\rho'(x(t,\eta_0))H_k(\rho(t,\eta_0))]$

We thus have the desired bound on $|\int D^\alpha f(\xi)\mu(t,\eta,\xi)|$ for $|\alpha| \leq 1$. For general α, the bound can be obtained by induction.

It remains to show (12). By (11), there is a positive ε with

$$\sum_{j=1}^{d} (X_M(t,T,\eta_0)V_j(x(t,\eta_0)))^2 \geq$$

$$\varepsilon X_M(t,T,\eta_0)g_M^{-1}(x(t,\eta_0))X_M(t,T,\eta_0)^*$$

Hence, if $x(T,x)\in U'$, $(A_{(m)})^{-1}(T,\eta_0) \leq$

$$(1/(\varepsilon T^2))\int_0^T\Big[(\partial\pi^{-1}/\partial x)(x(t,\eta_0))^*X^{-1}(t,T,\eta_0)^*$$

$$X^{-1}(t,T,\eta_0)(\partial\pi^{-1}/\partial x)(x(t,\eta_0))\Big]_{(m)}dt$$

Setting $Y(t,T,\eta_0) = X^{-1}(t,T,\eta_0)^*X^{-1}(t,T,\eta_0)$,

$1/\det(A_{(m)}(T,x)) \leq$

$$\left[(1/(\varepsilon T^2 m))\int_0^T \mathrm{Tr}\Big[\{(\partial\pi^{-1}/\partial x)(x(t,\eta_0))^*Y(t,T,\eta_0)(\partial\pi^{-1}/\partial x)(x(t,\eta_0))\}_{(m)}\Big]dt\right]^m$$

But, $\mathrm{Tr}([(\partial\pi^{-1}/\partial x)(x(t,\eta_0))^*Y(t,T,\eta_0)(\partial\pi^{-1}/\partial x)(x(t,\eta_0))]_{(m)})$ $\leq C\mathrm{Tr}(Y(t,T,\eta_0))$ if $x(t,\eta_0)\in U'$, and $\mathrm{Tr}(Y(t,T,\eta_0))$ can easily be estimated in $L^p(W)$.

We have now completed the proof of the following theorem.

<u>Theorem (13)</u> Let $\lambda>1$ and $t>0$. $P(t,\eta_0,d\eta)$ admits a density

$p(t,\eta_0,\eta)$ which is smooth in η. The uniform norms on p and its derivatives can be bounded independent of $1/\lambda \leq t \leq \lambda$ and $\eta_0 \in M$.

V. Infinite Dimensional Diffusions

We want to extend the results of section IV to diffusions on M^Z. However, to avoid certain technicalities, we will prove results about $M^{[-K,K]}$, where K is a large integer, and $[-K,K] = \{k : |k| \leq K\}$. This will suffice if we show that the results so obtained do not depend on K.

In this context, let $\Omega = C([0,\infty),M^{[-K,K]})$ and $\Theta = C([0,\infty),(R^d)^{[-K,K]})$. Define $\eta = \{\eta : |k| \leq K\}$, $\mathcal{M}_t$, $\mathcal{M}$, $\theta = \{\theta_k^j : |k| \leq K,\ 1 \leq j \leq d\}$, $\mathcal{B}$, $\mathcal{B}_t$, and W as before.

Let $R \in Z^+$. Suppose that for $|k| \leq K$ and $0 \leq j \leq d$, $V_{k,j} : M^{[-K,K]} \rightarrow T_M$ satisfies

(14)

i) $V_{k,j}(\eta) \in T_M(\eta_k)$ for $\eta \in M^{[-K,K]}$

ii) $V_{k,j}$ is smooth and $V_{k,j}$ and each of its derivatives is bounded independent of k and j

iii) $V_{k,j}$ depends only on $\{\eta_n\}_{k-R \leq n \leq k+R}$

Then there are functions $W_{k,j} : (R^D)^{[-K,K]} \rightarrow R^D$ satisfying

(15)

i) $W_{k,j}$ is smooth and $W_{k,j}$ and each derivative is bounded independent of k and j

ii) $W_{k,j}$ depends only on $\{y_n\}_{k-R \leq n \leq k+R}$ for $y \in (R^D)^{[-K,K]}$

iii) For $\eta \in M^{[-K,K]}$, $W_{k,j}(I(\eta)) = i_* V_{k,j}(\eta)$ where $I = i^{\otimes[-K,K]}$

For y $(R^D)^{[-K,K]}$, let $y(t,y) = \{y_k(t,y) : |k| \leq K\}$ denote the unique solution to the system of equations

$$(16)\quad y_k(T) = y_k + \sum_{j=1}^{d} \int_0^T W_{k,j}(y(t)) \circ d\theta_k^j(t) + \int_0^T W_{k,0}(y(t))dt$$

Theorem (6) yields the following.

Corollary(17) Let $V_{k,j}$, $|k| \le K$, $0 \le j \le d$ satisfy (14). For f $C^\infty(M^{[-K,K]})$, define

$$Lf(\eta)= \sum_{|k| \le K} \{1/2 \sum_{j=1}^{d}(V_{k,j})^2 f(\eta)+V_{k,0}f(\eta)\}.$$

($V_{k,j}f$ is formed by fixing η_n, $n \neq k$ and acting $V_{k,j}$ on f as a function of η_k.) For $\eta \in M^{[-K,K]}$ and $y = I(\eta)$ let $y(*,y)$ be the solution to (16) with $\{W_{k,j}\}_{|k| \le K, 1 \le j \le d}$ satisfying (15). Let P_η on $(\Omega, \mathcal{M})$ be the distribution of $I^{-1} \circ y(*,I(y))$ under W. Then P_η is the unique probability measure on $(\Omega,\mathcal{M})$ such that $P_\eta(\eta(0)=\eta)=1$ and $(f(\eta(t))-\int_0^t Lf(\eta(s))ds, \mathcal{M}_t, P_\eta)$ is a martingale for every f in $C^\infty(M^{[-K,K]})$.
Finally, the family $\{P_\eta : \eta \ M^Z\}$ is Feller continuous and strong Markov.

Assume that we are given $V_{k,j}$ for which (14) holds and so that $(\forall \varepsilon > 0)$ $(\forall |k| \le K)$ $(\forall \eta \in M^{[-K,K]})$ $(\forall \gamma \in T_M{}^*(\eta_k))$,

$$\sum_{j=1}^{d} \langle \gamma, V_{k,j}(\eta_k) \rangle^2 \ge \varepsilon |\gamma|^2. \tag{18}$$

Let $P_{(N)}(t,\eta_0,\Gamma)$
$= P_{\eta_0}(\eta_{(N)}(t) \in \Gamma)$ for $\eta_0 \in M^{[-K,K]}$ and $\Gamma \subseteq M^{[-N,N]}$.

Theorem (19) Let $\lambda > 1$ and $t > 0$. $P_{(N)}(t,\eta_0,\eta)$ admits a density $p_{(N)}(t,\eta_0,\eta)$ which is smooth in η. The uniform norms of $p_{(N)}$ and its derivatives can be bounded independent of $1/\lambda \le t \le \lambda$ and $\eta_0 \in M^{[-K,K]}$.

Proof. For $y \in (R^D)^{[-K,K]}$ let $x \in R^{[-KD+1,(K+1)D]}$ be defined by $x^{kD+n} = (y_k)^n$, $|k| \leq K$, $1 \leq n \leq D$. Let $x_{(m)} \in R^{[-KD+1,(K+1)D]}$ be defined by $x^{km+n} = x^{kD+n}$, $|k| \leq K$ and $1 \leq n \leq m$. Set $y_{(N)} = (y_{-N},\dots,y_N)$ and $x_{(m,N)} = ((x_{(m)})_{-Nm+1},\dots,(x_{(m)})_{(N+1)m})$. Similarly, for a matrix $B=((B^{i,j}))_{-KD+1 \leq i,j \leq (K+1)D}$ define $B_{(m)} = ((\ (B_{(m)})^{i,j}\ ))_{-Km+1 \leq i,j \leq K(m+1)}$ by $(B_{(m)})^{pm+p',qm+q'} = B^{pD+p',qD+q'}$ for $-K \leq p,q \leq K$ and $1 \leq p',q' \leq m$. Define $B_{(m,N)} = ((\ (B_{(m)})^{i,j}\ ))_{-mN+1 \leq i,j \leq (N+1)m}$.

Let U_i, U_i', and π_i, $1 \leq i \leq r$ be as in (5) and let $\{i_k\}_{-N \leq k \leq N}$ be a sequence of integers between 1 and r. Set $U = \prod_{|k| \leq N} U_{i_k}$ and $U' = \prod_{|k| \leq N} U_{i_k}'$. Define

$$\pi(y)_k = \begin{cases} \pi_{i_k}(y_k) & -N \leq k \leq N \\ y_k & \text{otherwise} \end{cases}$$

for $y \in R^{D[-K,K]}$ with $y_k \in \tilde{U}_{i_k}$. Choose $\rho \in C_0^\infty(U')$ so that $U \subset\subset \{\rho = 1\}$ and $0 \leq \rho \leq 1$. Define $\mu(t,\eta_0,\Gamma) = E^W[\rho(y_{(N)}(t,\eta_0))\chi_\Gamma((\pi x)_{(m,N)}(t,\eta_0))]$ for $\Gamma \subseteq R^{[-Nm+1,(N+1)m]}$. Since, by section 6 of [2], $x^i(t,\eta_0)$ is bounded (in the sense of Malliavin's calculus) independent of K and $-KD+1 \leq i \leq (K+1)D$, the proof in section IV will work here to show that μ admits a smooth density if we can show that

$$(20)\quad \chi_{U'}(y_{(N)}(t,\eta_0))/\det(\tilde{A}_{(m,N)}(t,\eta_0)) \in \bigcap_{p=1}^{\infty} L^P(W),$$

where $\tilde{A}(t,\eta_0) = (\partial\pi/\partial x)(x(t,\eta_0))A(t,\eta_0)(\partial\pi/\partial x^*)(x(t,\eta_0))$

and $A(t,\eta_0) = ((\ \langle x^p(t,\eta_0), x^q(t,\eta_0)\rangle\ ))_{-KD+1 \le p,q \le (K+1)D}$.

Define $\tilde{W}_{k,j} \in R^Z$ for $|k| \le K$, $0 \le j \le d$ by

$$(\tilde{W}_{k,j})^{nD+n'} = \begin{cases} (W_{k,j})^{n'} & n=k \text{ and } 1 \le n' \le D \\ 0 & \text{otherwise} \end{cases}$$

Set $R_{k,j}=\partial\tilde{W}_{k,j}/\partial x$ and let X be the unique solution to

$$X(t,T,y) = I + \sum_{|k| \le K} \sum_{j=1}^{d} \int_t^T R_{k,j}(y(s,y))X(t,s,y)\circ d\theta_k^{\,j}(s) +$$

$$\sum_{|k| \le K} \int_t^T R_{k,0}(y(s,y))X(t,s,y)ds$$

Then $A(T,y) = \sum_{|k| \le K} \sum_{j=1}^{d} \int_0^T (X(t,T,y)W_{k,j}(y(t,y)))^2 dt.$

Suppose that $\{i_k\}_{|k| \le K}$ is an extension of the given sequence $\{i_k\}_{|k| \le N}$ with $1 \le i_k \le r$ for $|k| \le K$. Then, as in Theorem (13), using Lemma (2.18) from [A], if

$x \in \prod_{k=-K}^{K} U_{i_k}$, then $(1/\det(\tilde{A}_{(m,N)}(t,\eta_0)))^{1/(2N+1)} \le$

$1/(\varepsilon(2N+1)mT^2)\int_0^T \mathrm{Tr}\Big[((\partial\pi/\partial x)(x(t,\eta_0))Y(t,T,\eta_0)(\partial\pi/\partial x^*)(x(t,\eta_0)))_{(m,N)}\Big]dt$

where $Y(t,T,\eta_0) = X^{-1}(t,T,\eta_0)^*X^{-1}(t,T,\eta_0)$. The integrand is bounded by a constant times the trace of $(Y(t,T,\eta_0))_{(N)}$, where the constant depends only on $\{i_k\}_{|k| \le N}$ and

$\mathrm{Tr}(Y(t,T,\eta_0)_{(N)})$ is estimated as in section 6 of [2].

VI A Dimension - Independent Estimate

The estimates on the marginals obtained in the preceding section are dependent on the number of dimensions for which we are taking the marginal. In this section we obtain, under an additional hypothesis, an estimate which does not have this dependence. Specifically, for $|k| \leq K$, define

$$G_t(\eta,k) = \int |\mathrm{grad}_k p(t,\eta,\xi)/p(t,\eta,\xi)|^2 p(t,\eta,\xi)\sigma^{[-K,K]}(d\xi)$$

Theorem (21) Suppose that $V_{k,j}$ $|k| \leq K$, $0 \leq j \leq d$ satisfy (14) and (18). Assume in addition that for $j = 1,\ldots d$ and $n \neq k$, $V_{k,j}$ is independent of η_n. Then $\lambda > 1$

$$\sup_{|k| \leq K} \quad \sup_{1/\lambda \leq t \leq \lambda} \quad \sup_{\eta \in M^{[-K,K]}} G_t(\eta,k) \leq C < \infty$$

where C does not depend on K.

Proof. First we need two lemmas.

Lemma (22) Fix $|k| \leq K$. Suppose that for every $0 \leq j \leq d$ and $n \neq k$, $V_{k,j}$ is independent of η_n and that for all $n \neq k$, and $0 \leq j \leq d$ $V_{n,k}$ does not depend on η_k. Let y be the solution to (16) for $\{W_{k,j}\}_{|k| \leq K, 0 \leq j \leq d}$ satisfying (15). Then

$\langle y_k^p(t,\eta), y_n^q(t,\eta)\rangle = 0$ for $n \neq k$, $t \geq 0$, $\eta \in (R^D)^{[-K,K]}$ and $1 \leq p,q \leq D$.

Proof. See the proof of Lemma (3.8) in [1].

Lemma (23) Let F be a finite subset of the integers and set $\tilde{W}_{k,0} = \chi_{F^c}(k) W_{k,0}$ for $|k| \leq K$, where $\{W_{k,j}\}_{|k| \leq K, 0 \leq j \leq d}$ satisfies (15). Let z be the solution to the system (16) with $\tilde{W}_{k,0}$

replacing $W_{k,0}$. Set $a_k = \sum_{j=1}^{d} W_{k,j}{}^2$. Then for $k \in F$, there is $c_k \in C_b{}^{\infty}(R^D)^{[-K,K]} \longrightarrow R^D$ so that $a_k c_k$

$$= b = W_0 + (1/2)\sum_{j=1}^{d} W_{k,j}(W_{k,j}).$$

$(W_{k,j}(W_{k,j})$: $(R^D)^{[-K,K]}$ R^D is defined by

$$[W_{k,j}(W_{k,j})]^n = \sum_{p=1}^{D} W_{k,j}{}^p(\partial/\partial\eta_k{}^p)(W_{k,j}{}^n).)$$

If $S(t) = \exp\Big[\ \sum\sum\int_0^t \langle c,W_{k,j}\rangle(z(s,\eta))d\theta_k{}^j(s)$

$+\int_0^t \langle c,b\rangle(z(s,\eta))ds\Big]$, then $(S(t),\mathcal{B}_t,W)$ is a martingale and so there is a probability measure P on with P(A) = $E^W[R(t),A]$. Finally, if $y(*,\eta)$ is the solution to the system (16) then $E^P[f(z(t,\eta))] = E^W[f(y(t,\eta))]$ for all bounded measurable f on $(R^D)^{[-K,K]}$.

Proof. Let U_i, $1 \leq i \leq r$ be as described in (5) and let ρ_i be a partition of unity subordinate to U_i. Choose c_{k_i} so that for $y_k \in U_i$, $a_k(\eta)c_{k_i}(\eta) = b_k(\eta)$. Set $c_k(\eta) = \sum \rho_i(\eta_k)c_{k_i}(\eta)$. Then c_k is smooth and $a_k c_k = b_k$. The rest of the lemma follows from Cameron - Martin - Girsanov theory. (See [2].)

Fix $|k| \leq K$ and $1 \leq i \leq r$, and define U_i, U_i', and π_i as in (5). Set $U = \{\xi \in M^{[-K,K]}: \xi_k \in U_i\}$, and $U' = \{\xi \in M^{[-K,K]}: \xi_k \in U_i'\}$. Define π on U' by

$$(\pi(\xi))_n = \begin{cases} \pi_i(\xi_k) & n = k \\ \xi_n & \text{otherwise} \end{cases}$$

We will show

$$\int \chi_U(\xi)\,|\mathrm{grad}_k p(t,\eta,\xi)/p(t,\eta,\xi)|^2 p(t,\eta,\xi)\sigma^{[-K,K]}(d\xi)$$

is bounded in the required manner.

Choose $\{W_{r,j}\}_{|r| \leq K, 0 \leq j \leq d}$ so that (15) holds and set

$$\tilde{W}_{k,j} = \begin{cases} W_{r,j} & j \neq 0 \text{ or } |r-k| > R \\ 0 & \text{otherwise} \end{cases}$$

Denote by $y(t,y)$ the solution th the system (16) and by $w(t,y)$ the solution with $\tilde{W}_{r,j}$ replacing $W_{r,j}$. Let S and P be as in Lemma (23) witth $F = \{r: |r-k| \leq R\}$.

Define $\alpha_k(t,\eta) = ((\langle w_k^p(t,\eta), w_k^q(t,\eta)\rangle))_{1 \leq p,q \leq D}$. Then

$$\alpha_k(t,\eta) = \sum_{j=1}^{d} \int_0^t (e_k(s,t,\eta)V_{k,j}(w(t,\eta)))^2 dt$$

where $e_k(t,T,\eta) = I +$

$$\sum_{j=1}^{d} \int_0^T (\partial W_{k,j}/\partial w_k)(w(s,\eta))e_k(s,T,\eta)\circ d\theta_k^j.$$

For $w(t,x) \in U'$, set $\tilde{\alpha}_k(t,\eta) = (\partial\pi/\partial w_k)(w_k(t,\eta))\alpha_k(t,\eta)(\partial\pi^*/\partial w_k)(w_k(t,\eta))$ and $((((\tilde{\alpha}_k)_{(m)})_{p,q}))_{1 \leq p,q \leq m} = ((\tilde{\alpha}_k)_{(m)})^{-1}$.

If $F \in C_0^\infty(U')$ and $\rho \in C_0^\infty(U')$ then $\rho(w_k(t,\eta))((\tilde{\alpha}_k)_{(m)})_{p,q} \in \mathcal{E}$ for $1 \leq p,q \leq m$, and for $\Psi \in \mathcal{E}$, $E^W[(\partial/\partial z_k^n)(F\circ\pi^{-1})(z(t,\eta))\rho(w_k(t,\eta))\Psi]$

$= -E^W[F^\circ\pi^{-1}(z(t,\eta))H_k{}^n(\rho(w_k(t,\eta))\Psi)]$

where $H_k{}^n(\Phi) =$

$$\sum_{q=1}^{m}\Big[\ \langle z^q(t,\eta),((\tilde{\alpha}_k)_{(m)})_{q,n}(t,\eta)\Phi\rangle + 2((\tilde{\alpha}_k)_{(m)})_{q,n}(t,\eta)\Phi\mathscr{f}\, z^q(t,\eta)\Big],$$

and $z = \pi w$.

Choose $\rho \in C_0{}^\infty(U_i{}')$ with $0 \leq \rho \leq 1$ and $U_i \subset\subset \{\rho = 1\}$. Define the measure ν on $M^{[-K,K]}$ by

$\nu(dw) = \rho(w)(\pi^{-1})_*(\partial/z_k{}^n)p(t,\eta,w)\sigma^{[-N,N]}(dw)$.

Then by Lemma (3.6) in [1] the theorem will be proved once we find a function $\widehat{\Psi} \in L^2(P)$ with the $L^2(P)$ norm of Ψ bounded independent of the desired quantities and with $E^\nu[f] = E^P[f(w(t,\eta))\Psi]$ for every $f \in C^\infty(M^{[-K,K]})$.

Integrating by parts, $E^\nu[f] =$ $-E^P[\partial/\partial z_k((f\rho)^\circ\pi^{-1}h)(z(t,\eta))(1/h(z(t,\eta)))]$, where $h(z)$ $= (|g(z_k)|)^{1/2}$. (Here $g(*)$ is the Riemannian metric expressed in the coordinates π_i.) Thus, $E^\nu[f] =$

$-E^W[\partial/\partial z_k((f\rho)^\circ\pi^{-1}h)(z(t,\eta))(S(t)/h(z(t,\eta)))] =$

$E^W[(f\rho^\circ\pi^{-1}h)(z(t,\eta))H_k{}^n((S(t)/h(z(t,\eta)))\rho_1(w_k(t,\eta)))]$

where $\rho_1 \in C_0{}^\infty(U_i{}')$ is chosen so that $0 \leq \rho_1 \leq 1$ and $\{\rho_1 = 1\} \subset\subset \mathrm{supp}(\rho)$.

So, $E^\nu[f] = E^P[f(w(t,\eta))\Psi]$ with $\Psi =$

$(\rho(w(t,\eta))h(z(t,\eta))/S(t))H_k{}^n((S(t)/h(z(t,\eta)))\rho_1(w_k(t,\eta)))$

and $\bar{\Psi}$ can be estimated in $L^2(P)$.

Remark (24) For $|k| \leq N \leq K$, set $G_t(\eta,k,N) =$

$$\int |\mathrm{grad}_k p_{(N)}(t,\eta,\xi)/p_{(N)}(t,\eta,\xi)|^2 p_{(N)}(t,\eta,\xi)\sigma^{[-N,N]}(d\xi)$$

Then $G_t(\eta,k,N) \leq G_t(\eta,k)$.

Remark (25) The obvious analogues to the preceding results hold on $M^{[-K,K]^\beta}$ for β any positive integer, and with the same proofs.

VII Application

In section 4 of [A], results are proved about the ergodic properties of a certain class of diffusions on the infinite dimensional torus. There, the specific energy function is introduced and shown to be a Liapunov function. That is, denoting by $h(\mu)$ the specific energy of a measure μ on $T^{Z^{\beta}}$, $\beta \in Z^{+}$, and by P_t the Markov semigroup associated with the diffusion, $h(P_t{*}\mu)$ is nondecreasing for $t > 0$, where $P_t{*}$ is the adjoint of P_t.

The analogous results hold for a class of diffusions on $M^{Z^{\beta}}$, and with the same proofs, once we show that the specific energy function is finite.

A collection of smooth functions $\mathcal{J} = \{J_F: M^{Z^{\beta}} \rightarrow R: F \subseteq Z^{\beta}$, $|F| = \text{cardinality}(F) < \infty\}$ is called a potential if $J_F(\eta)$ depends only on η_k, $k \in F$, and J_F is invariant under permutations of the indices of F. $\mathcal{J}$ is called finite range if there is an $R \in Z^{\beta+}$ so that if $k,n \in F$ and $|k-n| > R$ then $J_F = 0$. $\mathcal{J}$ is shift invariant if for any $F \subseteq Z^{\beta}$ and $k \in Z^{\beta}$, $J_{F+k}(\eta) = J_F(S^{-k}\eta)$ where $(S^{-k}\eta)_n = \eta_{n-k}$.

Assume that $\mathcal{J}$ is a shift invariant and finite range potential. For $k \in Z^{\beta}$, define the energy at site k, $H_k(\eta)$ by

$$H_k(\eta) = \sum_{F\ \ k} J_F(\eta)$$

For μ a probability measure on $M^{Z^{\beta}}$, define the specific free energy $h(\mu)$ as follows. Set $\Delta_k = \{n: |n| \leq Rk\}$. If the marginal density of μ on M^{Δ_n} has a density with respect to σ^{Δ_n}, denote it by $\mu^{(n)}(\eta_{\Delta_n})$ and set $h_n(\mu) =$

$$\int_{M^{\Delta_n}} \sum_{F\subseteq\Delta_n} J_F(\eta_{\Delta_n})\mu^{(n)}(\eta_{\Delta_n})\sigma^{\Delta_n}(d\eta_{\Delta_n}) +$$

$$\int_{M^{\Delta_n}} \mu^{(n)}(\eta_{\Delta_n})\log(\mu^{(n)}(\eta_{\Delta_n}))\sigma^{\Delta_n}(d\eta_{\Delta_n})$$

If not, set $h_n(\mu) = \infty$. Let $h(\mu) = \overline{\lim}_{n\ \infty}(2nR+1)^{-\beta}h_n(m)$.

Let $c:M^{Z^\beta}\longrightarrow(0,\infty)$ be a smooth function depending only on coordinates η_k for $|k| \leq R$ and define $c_k(\eta) = c(S^{-k}\eta)$. Suppose that $e^{H_0(\eta)}c_k(\eta)$ depends only on η_0. Set

$$Lf = \sum_{k\in Z^\beta} e^{H_k(\eta)}\,\mathrm{div}_k(c_k(\eta)\mathrm{grad}_k f(\eta))$$

for $f\in C^\infty(M^{Z^\beta})$ which depend only on finitely many coordinates. Denote by $P(t,\eta,*)$ the transition probability function for the diffusion associated with L. For μ a probability measure on M^{Z^β}, set $\mu_t(*) = \int p(t,\eta,*)\mu(d\eta)$.

<u>Theorem(26)</u> $h(\mu_t) < c(t) < \infty$.

Proof. For $f\in C^1(M^n)$, $\int f^2(\eta)\log(f(\eta))\sigma^{\Delta_n}(d\eta)$

$$\leq C\int|\mathrm{grad}(f(\eta))|^2\sigma^{\Delta_n}(d\eta) +$$

$$\int f^2(\eta)\sigma^{\Delta_n}(d\eta)\log(\int f^2(\eta)\sigma^{\Delta_n}(d\eta))$$

where C does not depend on n. (This is the logarithmic Sobolev inequality on M^{Δ_n}.) Thus, setting $f(\eta_{\Delta_n}) = (\mu_t^{(n)}(\eta_{\Delta_n}))^{1/2}$,

$$\int_{M^{\Delta_n}} \mu_t^{(n)}(\eta_{\Delta_n})\log(\mu_t^{(n)}(\eta_{\Delta_n}))\sigma^{\Delta_n}(d\eta_{\Delta_n})$$

$$\leq C \int_{M^{\Delta_n}} \sum_{F\leq\Delta_n} |\mathrm{grad}_k\mu_t^{(n)}(\eta_{\Delta_n})|^2/\mu_t^{(n)}(\eta_{\Delta_n})\sigma^{\Delta_n}(d\eta_{\Delta_n})$$

since $\int f^2(\eta_{\Delta_n})\sigma^{\Delta_n}(\eta_{\Delta_n}) = 1$. Furthermore,

by Lemma (3.3) in [1], and Theorem (21),

$$\int_{M^{\Delta_n}} \sum_{F\leq\Delta_n} |\mathrm{grad}_k\mu_t^{(n)}(\eta_{\Delta_n})|^2/\mu^{(n)}(\eta_{\Delta_n})\sigma^{\Delta_n}(d\eta_{\Delta_n})$$

can be bounded independent of n. Since xlogx is convex, $0 \leq$

$$\int_{M^{\Delta_n}} \mu_t^{(n)}(\eta_{\Delta_n})\log(\mu_t^{(n)}(\eta_{\Delta_n}))\sigma^{\Delta_n}(d\eta_{\Delta_n})$$

$\leq C(2nR+1)^{\beta}$. Also,

$$\left|\int_{M^{\Delta_n}} \sum_{F\leq\Delta_n} J_F(\eta_{\Delta_n})\mu^{(n)}(\eta_{\Delta_n})\sigma^{\Delta_n}(d\eta_{\Delta_n})\right|$$

$$\leq \left[\sup_{\eta\ M^{Z^\beta}} \sum_{F\ 0} J_F(\eta)\right](2nR+1)^d.$$

Combining the preceding two statements, we obtain the desired bound on $h(\mu_t)$. The following theorems are proved as in section (4) of [1].

Theorem (27) For $0<t_1 \leq t_2$, $h(\mu_{t_1}) \leq h(\mu_{t_1})$.

Theorem(28) If μ is shift invariant (in terms of the shift on Z^β) and stationary for the diffusion generated by L then μ is reversible for this diffusion.

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