AN ANALOGUE

FOR MOTOR-VEHICLE VIBRATION

by

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B.S. in M.E. Finland's Institute of Technology

1948

Submitted to the Department of Mechanical Engineering

on May 16, 1952 in partial fulfillment

of the requirements for the degree of

MASTER OF SCIENCE

from the

Massachusetts Institute of Technology

1952

Signature of Author ....

Department of Mechanical Engineering, May 16, 1952

Professor in Charge of Research

Chairman of Departmental

Committee on Graduate Students ...

ME
Thesis
1952

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A motor-vehicle with its springs and pneumatic tires forms a very complicated vibrational system. Although the direct mathematical solution may be obtained for systems of one or two degrees of freedom, the problem of the complete vehicle is best solved experimentally.

The purpose of this thesis was to design and build a mechanical analogue for motor-vehicle vibration. It consists of a body and two unsprung masses. The body is suspended by springs from the two masses which in turn are suspended by stiff springs representing the tires.

In designing a model it is necessary to retain certain dimensionless quantities which are determined through the methods of dimensional analysis. The wheel-base of the model was chosen to be approximately one fifth of that of the average American car. The spring and tire stiffness and the weight distribution may be varied in large limits. For shock absorbers electro-

magnetic dampers were introduced. This type of damper makes use of the fact that the force opposing the motion of a conductor in a magnetic field is proportional to the velocity of the conductor. By changing the exciting current in the electromagnet which produces the magnetic field, the amount of damping may be controlled. With a recording device the apparatus shows the motion of the body directly and can be conveniently used for study on problems of motor-vehicle vibration as well as demonstration purposes.

Thesis supervisor: C. Fayette Taylor

Title: Professor of Automotive Engineering

ACKNOWLEDGEMENT

The writer wishes to take this opportunity to express his sincere appreciation to Professor C. F. Taylor for his constant interest, kind encouragement, and helpful advice throughout his supervision of this thesis. Other aid I am happy to acknowledge came from Professor J. E. Forbes. His suggestions were an invaluable aid in designing the electromagnetic damper. The kind interest taken by all members of the staff of Sloan Automotive Laboratory has done much to make this thesis both interesting and pleasant for the writer.

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I. INTRODUCTION

The purpose of a motor-vehicle suspension is to protect the vehicle and the passangers or load from the road shocks. The suspension problem is very important and it has been the subject of extensive research both in this country and Europe (References 1...10). Dr. Haley's thesis on vibrational characteristics of automotive suspensions which includes a large bibliography, may be mentioned here as a study made previously at Massachusetts Institute of Technology [11].

Unfortunately, the motor-vehicle with its springs and pneumatic tires forms a very complicated system. Even when only vertical displacements are considered and the effect of seat cushioning is neglected, the system still has seven degrees of freedom.

Mr. P.E. Mercier proposed a solution for the vibration problem of the complete vehicle in his article on vehicle suspensions [12]. He considered a system consisting a body and four wheel masses. He assumed also some kind of interaction between different wheels, i.e. the load of the suspension member corresponding to one wheel caused by the motion of another.

In order to simplify the mathematical procedure Mercier assumed complete symmetry of the vehicle. Further for the sake of mathematical simplicity he did not take damping forces into account. He justified this by claiming that when elastic characteristics are rationally determined the required degree of damping is low. It is true that damping has very little effect on the frequency of the vibration. However, it significantly affects the amplitude, especially with steady-state disturbance at resonance frequency when damping is essential.

In this thesis the problem of vehicle vibration is considered using a mechanical analogue. The problem has been simplified by neglecting the rotations of the body and the two axles about the longitudinal axis of the vehicle. These motions may be studied separately. In other words, motions are considered only in the vertical plane parallel to the direction of travel of the vehicle. The number of degrees of freedom is thus reduced to four, namely the "bouncing" and "pitching" of the body and the up-and-down motions of the two axles.

II: MATHEMATICAL CONSIDERATION

The Single Mass System

To start with the simplest kind of vibration problem let us consider first the single mass system as shown in Fig. 1. The motion is determined by a second order linear differential equation. By simple calculations the complete solution for free vibrations can always be found [13] and is of the form:

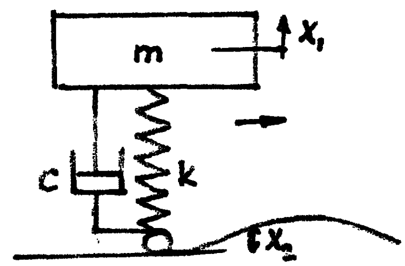


Fig. 1.

$$x_1 = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{1}$$

where $$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

and A_1 and A_2 are arbitrary constants which depend on the initial conditions.

The values for s can be real or complex. In the former case the motion is not vibration but an exponential curve to the equilibrium position. The smallest amount of damping for which this occurs is

$$c_c = 2\sqrt{mk} \tag{2}$$

This is called the critical damping. The damping coefficient is usually expressed in fractions of the critical value.

Using the fact that the natural undamped frequency is

$$\omega_n = \sqrt{\frac{k}{m}} \tag{3}$$

the solution of the motion can be expressed in another, more convenient form:

$$x = A\ e^{-\frac{c}{c_c}\omega_n t} \sin \left(\sqrt{1-\left(\frac{c}{c_c}\right)^2}\ \omega_n t + \phi\right. \tag{4}$$

The coefficient A and the phase angle ϕ depend again on the initial conditions. The value

$$\sqrt{1-\left(\frac{c}{c_c}\right)^2}\ \omega_n$$

is called the damped natural frequency. We see that a reasonable amount of damping has very small effect on the natural frequency. A damping coefficient which is half of the critical value reduces the natural frequency only by 13.4 per cent.

The solution for a forced vibration is also found if the forcing function (the form of the road surface) and the initial conditions are known. A road surface of sinusoidal form may be defined as a function of time thus:

$$x_2 = a_0 \, (1 - \cos \omega t) \mathbb{1}(t) \tag{5}$$

The initial vertical displacement and velocity of the mass are assumed to be zero. The following expression is then found for the motion of the mass as a function of time:

$$\frac{x_1(t)}{a_0} = 1 - A_1 \sin(\omega t + \phi_1) + A_2 e^{-\frac{c}{c_c}\omega_n t} \sin\left(\sqrt{1-\left(\tfrac{c}{c_c}\right)}\,\omega_n t + \phi_2\right) \tag{6}$$

where the coefficients A_1 and A_2 and the phase angles ϕ_1 and ϕ_2 are functions of damping ratio and frequency ratio:

$$\begin{aligned}
A_1 &= \frac{\sqrt{1 + (2\, c/c_c \;\omega/\omega_n)^2}}{\sqrt{(2\, c/c_c \;\omega/\omega_n)^2 + (\omega/\omega_n - 1)^2}} \\
A_2 &= \frac{\omega/\omega_n}{\sqrt{\left[1-(c/c_c)^2\right]\left[(2\, c/c_c \;\omega/\omega_n)^2 + (\omega/\omega_n - 1)^2\right]}} \\
\phi_1 &= \tan^{-1} \frac{(2\, c/c_c \;\omega/\omega_n)^2 + (\omega/\omega_n - 1)^2}{2\, c/c_c \,(\omega/\omega_n)^3} \\
\phi_2 &= \tan^{-1} \frac{\sqrt{1 - (c/c_c)^2}\left[1 - (\omega/\omega_n)^2\right]}{c/c_c \left[1 + (\omega/\omega_n)^2\right]}
\end{aligned} \tag{7}$$

The motion is described by three terms, namely a constant, a sinusoidal term varying at the forcing frequency and a damped sinusoid at the natural damped frequency of the system. The third term decreases very rapidly and some time after the disturbance has begun the system is said to be in steady-state vibration, because the motion is purely periodic. Coefficient A_1 is the ratio of the amplitude of the motion to the amplitude of the road surface. If it is plotted as function of frequency ratio for different damping ratios the well-known chart of Fig. 2 is obtained.

The Two Mass System

For the two mass system with the notations of Fig. 3 the differential equations are:

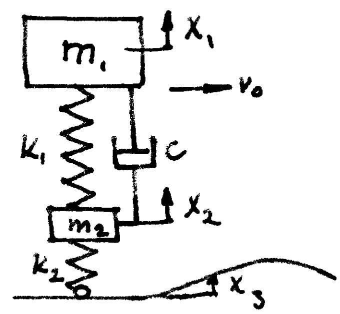


Fig. 3.

$$m_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) + k_2(x_2 - x_3) = 0 \qquad (8)$$

The natural undamped frequencies of the system are found to be the roots of the equation [14]:

$$\omega^4 - \left[\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right]\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \qquad (9)$$

which are:

$$\omega_{1,2}^2 = \frac{1}{2}\left[\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right] \pm \frac{1}{2}\sqrt{\left[\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right]^2 - 4\frac{k_1 k_2}{m_1 m_2}}$$

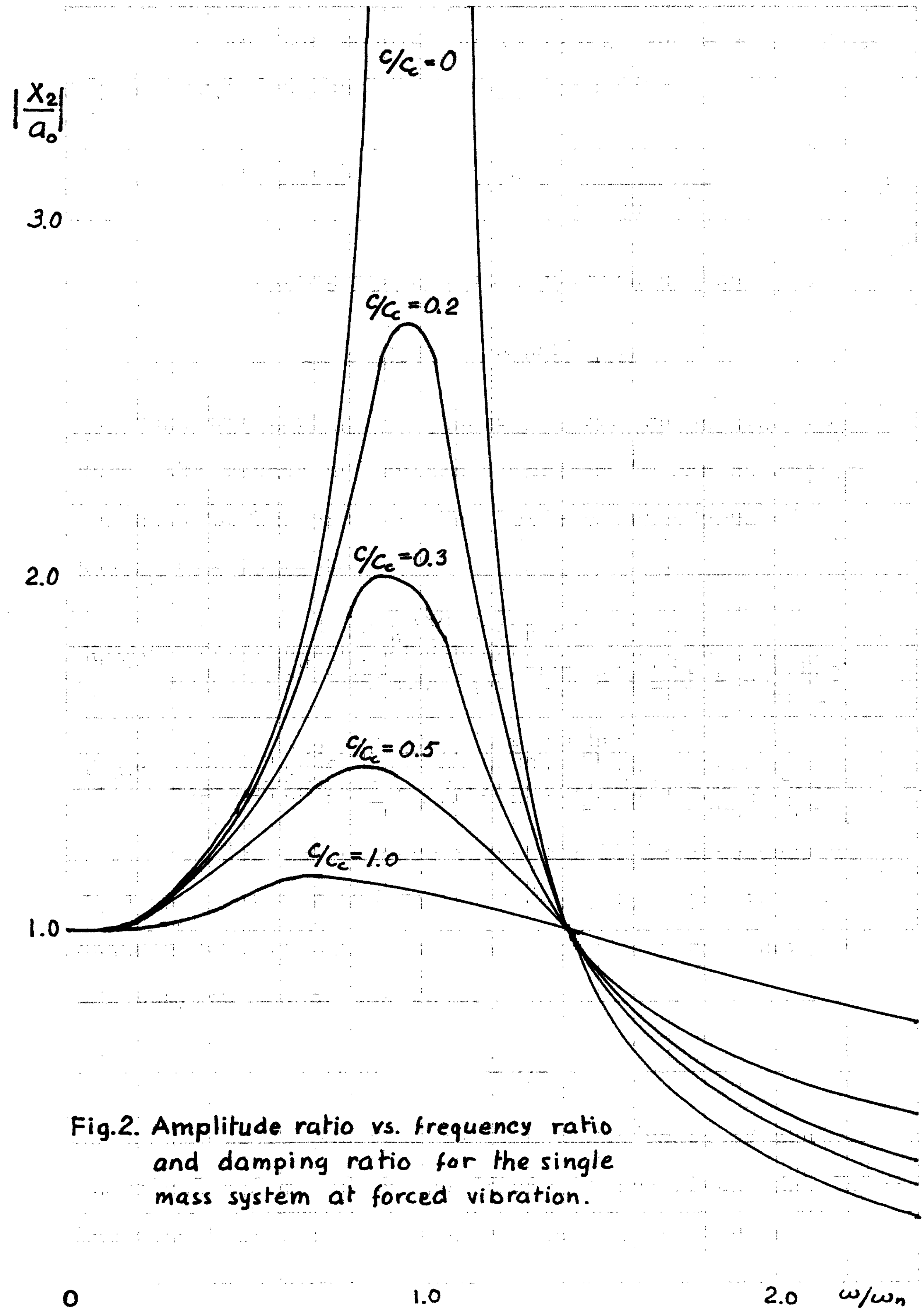


Fig.2. Amplitude ratio vs. frequency ratio and damping ratio for the single mass system at forced vibration.

For the motor-vehicle $k_2 \gg k_1$ and $m_1 \gg m_2$. Then the natural frequencies are approximately:

$$\omega_1^2 = \frac{k_1 + k_2}{m_2} \qquad \omega_2^2 = \frac{k_1 k_2}{k_1 + k_2} \frac{1}{m_1} \tag{10}$$

Assuming again a road surface of the form:

$$x_3 = a_0 (1 - \cos \omega t)\, 1(t)$$

and that the initial displacements and velocities are zero, the method of Laplace transform [15] can be applied for solving the problem. The Laplace transforms of x_1 and x_2 are found to be:

$$\bar{X}_1(p) = \frac{a_0 \omega^2 \frac{k_2}{m_2} \frac{k_1}{m_2} \left(\frac{c}{k_1} p + 1 \right)}{p (p^2 + \omega^2) \left[p^4 + c \left(\frac{1}{m_1} + \frac{1}{m_2} \right) p^3 + \left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} \right) p^2 + \frac{c k_2}{m_1 m_2} p + \frac{k_1 k_2}{m_1 m_2} \right]}$$
$$\bar{X}_2(p) = \frac{a_0 \omega^2 \frac{k_2}{m_2} \left(p^2 + \frac{c}{m_1} p + \frac{k_1}{m_1} \right)}{p (p^2 + \omega^2) \left[p^4 + c \left(\frac{1}{m_1} + \frac{1}{m_2} \right) p^3 + \left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} \right) p^2 + \frac{c k_2}{m_1 m_2} + \frac{k_1 k_2}{m_1 m_2} \right]} \tag{11}$$

For the inverse transform it is necessary to find the roots of the fourth order polynomial in the denominator. If the system is oscillatory the roots are all complex and of the form:

$$p = -\alpha \pm i\beta$$

It is possible to find these roots if the numerical values of the coefficients are given. Dr. Haley has done this for some single cases in his thesis [11].

The general solution as algebraic function of the coefficients is not obtainable as in the case of one degree of freedom.

The steady-state solution may be obtained quite easily in terms of the parameters of the system without evaluation of the roots of the fourth order polynomial. This is accomplished by assuming that the solutions are of the form:

$$\begin{aligned} x_1 &= A_1 e^{i\omega t} \\ x_2 &= A_2 e^{i\omega t} \end{aligned} \tag{12}$$

If these are substituted into (8) and the equations are solved for x_1 and x_2, the amplitude ratios are found to be:

$$\begin{aligned} \left|\frac{X_1}{a_0}\right| &= \sqrt{\frac{\left(\frac{k_1 k_2}{m_1 m_2}\right)^2 + \left(\frac{c\omega k_2}{m_1 m_2}\right)^2}{\left[\omega^4 - \left(\frac{k_1}{m_1} + \frac{k_1+k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2}\right]^2 + \left(\frac{c\omega k_2}{m_1 m_2}\right)^2 \left(1 - \omega^2 \frac{m_1+m_2}{k_2}\right)^2}} \\ \left|\frac{X_2}{a_0}\right| &= \sqrt{\frac{\left(\frac{k_2}{m_2}\right)^2\left(\omega^2 - \frac{k_1}{m_1}\right)^2 + \left(\frac{c\omega k_2}{m_1 m_2}\right)^2}{\left[\omega^4 - \left(\frac{k_1}{m_1} + \frac{k_1+k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2}\right]^2 + \left(\frac{c\omega k_2}{m_1 m_2}\right)^2 \left(1 - \omega^2 \frac{m_1+m_2}{k_2}\right)^2}} \end{aligned} \tag{13}$$

The first part of the denominator is the same as the equation for determining the natural frequencies. This is clear because without damping the amplitude goes to infinity if the forcing frequency is the same as the natural frequency of the system.

The Body with Distributed Mass

As next step let us consider a body with distributed mass and two parallel springs. This is also a system of two degrees of freedom.

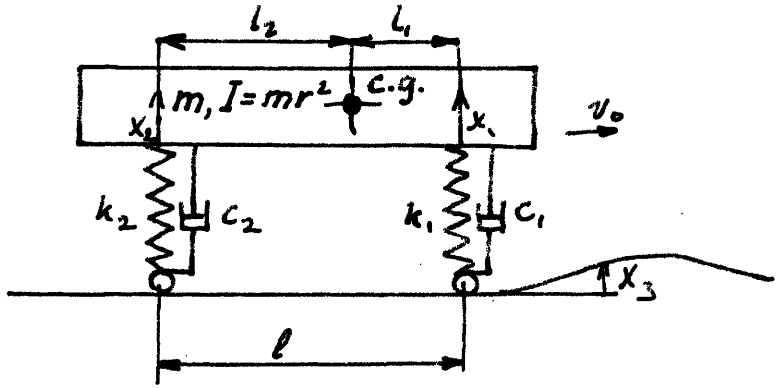


Fig. 4.

With the notations of Fig. 4 the differential equations of the system are:

$$\begin{aligned} m\frac{l_2^2+r^2}{l^2}\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 \quad + m\frac{l_1 l_2 - r^2}{l^2}\ddot{x}_2 &= 0 \\ m\frac{l_1 l_2 - r^2}{l^2}\ddot{x}_1 + \frac{m(l_1^2+r^2)}{l^2}\ddot{x}_2 + c_2\dot{x}_2 + k_2 x_2 &= 0 \end{aligned} \tag{14}$$

The natural undamped frequencies are the roots of the equation:

$$\omega^4 - \left(\frac{k_1}{m}\frac{l_1^2+r^2}{r^2} + \frac{k_2}{m}\frac{l_2^2+r^2}{r^2}\right)\omega^2 + \frac{l^2}{r^2}\frac{k_1 k_2}{m^2} = 0 \tag{15}$$

A particularly interesting case occurs when the ratio

$$\frac{l_1 l_2 - r^2}{l^2}$$

which is called the mass coupling, is zero. Then the set of differential equations (14) reduces to two separate equations and the motions of x_1 and x_2 are independent of each other. The natural frequencies are then simply:

$$\omega_1 = \sqrt{\frac{k_1 l}{m l_2}} \qquad \omega_2 = \sqrt{\frac{k_2 l}{m l_1}} \tag{16}$$

Another exception is the case when

$$k_1 l_1 = k_2 l_2$$

or spring constants are proportional to the wheel loadings. This means that the static deflections in front and rear are equal. Then the natural motions are pure up and down motions parallel to itself and rocking about the center of gravity. The natural frequencies of these motions are:

$$\omega_1 = \sqrt{\frac{k_1 + k_2}{m}} \qquad \omega_2 = \sqrt{\frac{k_1 l_1^2 + k_2 l_2^2}{I}} \tag{17}$$

If mass coupling is also zero these frequencies are equal.

Let us assume that the initial displacements and velocities are zero and the exciting motion under the wheel 1 has the same form as previously:

$$x_3 = a_0 (1 - \cos \omega t)\, 1(t)$$

The Laplace transforms of the motions are:

$$\bar{X}_1(p) = \frac{a_0\omega^2[a_1p^3 + (a_3+a_4)p^2 + a_6p + a_7]}{p(p^2+\omega^2)[p^4 + (a_1+a_2)p^3 + (a_3+a_4+a_5)p^2 + a_6p + a_7]}$$

$$\bar{X}_2(p) = \frac{-a_0\omega^2 p\left(\frac{l_1l_2}{r^2} - 1\right)\left(\frac{c_1}{m}p + \frac{k_1}{m}\right)}{p(p^2+\omega^2)[p^4 + (a_1+a_2)p^3 + (a_3+a_4+a_5)p^2 + a_6p + a_7]} \qquad (18)$$

where the constants $a_1 \ldots a_7$ are:

$$a_1 = \frac{c_1}{m}\frac{l_1^2+r^2}{l^2} \qquad a_5 = \frac{k_2}{m}\frac{l_2^2+r^2}{l^2}$$

$$a_2 = \frac{c_2}{m}\frac{l_2^2+r^2}{l^2} \qquad a_6 = \frac{c_1k_2+c_2k_1}{m^2}$$

$$a_3 = \frac{c_1c_2}{m^2} \qquad a_7 = \frac{k_1k_2}{m^2}$$

$$a_4 = \frac{k_1}{m}\frac{l_1^2+r^2}{l^2}$$

For the inverse transform the roots of the fourth order polynomial in the denominator are required. The amplitudes for steady-state vibration are found without evaluation of the roots through a method used previously for the two mass system and they are:

$$\left|\frac{X_1}{a_0}\right| = \sqrt{\frac{[(a_3+a_4)\omega^2 - a_7]^2 - \omega^2(a_1\omega^2 - a_6)^2}{[\omega^4 - (a_3+a_4+a_5)\omega^2 + a_7]^2 + \omega^2[(a_1+a_2)\omega^2 - a_6]^2}}$$

$$\left|\frac{X_2}{a_0}\right| = \sqrt{\frac{\omega^4\left[\left(\frac{c_1}{m}\right)^2\omega^2 + \left(\frac{k_1}{m}\right)^2\right]\left(\frac{l_1l_2}{r^2} - 1\right)^2}{[\omega^4 - (a_3+a_4+a_5)\omega^2 + a_7]^2 + \omega^2[(a_1+a_2)\omega^2 - a_6]^2}} \qquad (19)$$

The Complete Vehicle

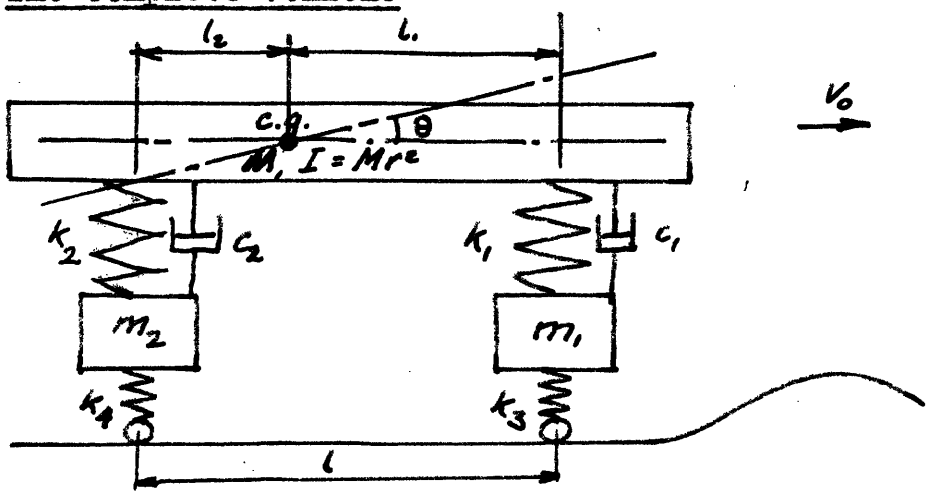


Fig. 5.

The vibrational system of the complete vehicle as chosen for this study is shown in Fig. 5. It is a combination of the two previous cases and has four degrees of freedom. The following coordinates are used:

x = rise of the body at center of gravity

θ = rotation of the body about center of gravity (positive counterclockwise)

x_1 = rise of the body at wheel 1

x_2 = " " " 2

x_3 = " of the mass m_1

x_4 = " " m_2

x_5 = " of the ground under wheel 1

x_6 = " " " 2

The coordinates x and θ can be expressed in terms of x_1 and x_2:

$$x = \frac{l_2}{l} x_1 + \frac{l_1}{l} x_2 \tag{20}$$

$$\theta = \frac{1}{l}(x_1 - x_2)$$

Let F_1 and F_2 be the algebraic sum of the spring and damping forces at front and rear respectively. From the balance of forces and moments acting on the body we get:

$$\begin{aligned} -M\ddot{x} + F_1 + F_2 &= 0 \\ -Mr^2\ddot{\theta} + F_1 l_1 - F_2 l_2 &= 0 \end{aligned} \tag{21}$$

Substituting the values for x and θ from (20) and the values for F_1 and F_2:

$$\begin{aligned} F_1 &= -c_1(\dot{x}_1 - \dot{x}_3) - k_1(x_1 - x_3) \\ F_2 &= -c_2(\dot{x}_2 - \dot{x}_4) - k_2(x_2 - x_4) \end{aligned} \tag{22}$$

we get the differential equations for the motion of the body:

$$\begin{aligned} M\frac{l_2^2+r^2}{l^2}\ddot{x}_1 + M\frac{l_1 l_2 - r^2}{l^2}\ddot{x}_2 + c_1(\dot{x}_1 - \dot{x}_3) + k_1(x_1 - x_3) &= 0 \\ M\frac{l_1^2+r^2}{l^2}\ddot{x}_2 + M\frac{l_1 l_2 - r^2}{l^2}\ddot{x}_1 + c_2(\dot{x}_2 - \dot{x}_4) + k_2(\dot{x}_2 - \dot{x}_4) &= 0 \end{aligned} \tag{23}$$

The differential equations for the unsprung masses are found to be:

$$\begin{aligned} m_1\ddot{x}_3 - c_1(\dot{x}_1 - \dot{x}_3) - k_1(x_1 - x_3) + k_3(x_3 - x_5) &= 0 \\ m_2\ddot{x}_4 - c_2(\dot{x}_2 - \dot{x}_4) - k_2(x_2 - x_4) + k_4(x_4 - x_6) &= 0 \end{aligned} \tag{24}$$

Let us consider a sinusoidal forcing function which hits the rear wheel l/v_0 later than the front wheel:

$$x_5 = a_0 (1-\cos\omega t)\, 1(t)$$
$$x_6 = a_0 \left[1-\cos\omega\left(t-\frac{l}{v_0}\right)\right] 1\left(t-\frac{l}{v_0}\right) \tag{25}$$

If all the velocities and displacements are zero for t=0, the Laplace transforms of the equations (23) and (24) are:

$$M\frac{l_2^2+r^2}{l^2}p^2\bar{x}_1 + M\frac{l_1l_2-r^2}{l^2}p^2\bar{x}_2 + c_1p(\bar{x}_1-\bar{x}_3)+k_1(\bar{x}_1-\bar{x}_3)=0$$

$$M\frac{l_1^2+r^2}{l^2}p^2\bar{x}_2 + M\frac{l_1l_2-r^2}{l^2}p^2x + c_2p(\bar{x}_2-\bar{x}_4)+k_2(\bar{x}_2-\bar{x}_4)=0$$

$$m_1p^2\bar{x}_3 - c_1p(\bar{x}_1-\bar{x}_3) - k_1(\bar{x}_1-\bar{x}_3) + k_3\bar{x}_3 = k_3a_0\left(\frac{1}{p}+\frac{-p}{p^2+\omega^2}\right) \tag{26}$$

$$m_2p^2\bar{x}_4 - c_2p(\bar{x}_2-\bar{x}_4) - k_2(\bar{x}_2-\bar{x}_4) + k_4\bar{x}_4 = k_3a_0e^{-\frac{l}{v_0}p}\left(\frac{1}{p}+\frac{-p}{p^2+\omega^2}\right)$$

After elimination of $\bar{x}_3$ and $\bar{x}_4$:

$$\left[\frac{l_2^2+r^2}{l^2}p^4 + \left(\frac{c_1}{M}+\frac{c_1}{m_1}\frac{l_2^2+r^2}{l^2}\right)p^3 + \left(\frac{k_1}{M}+\frac{k_1+k_3}{m_1}\frac{l_2^2+r^2}{l^2}\right)p^2 + \frac{k_3c_1}{Mm_1}p + \frac{k_1k_3}{Mm_1}\right]\bar{x}_1$$
$$+\frac{l_1l_2-r^2}{l^2}p^2\left(\frac{c_1}{m_1}p+\frac{k_1}{m_1}\right)\bar{x}_2 = \left(\frac{c_1}{m_1}p+\frac{k_1}{m_1}\right)\frac{k_3}{M}\frac{a_0\omega^2}{p(p^2+\omega^2)}$$

$$\left[\frac{l_1^2+r^2}{l^2}p^4 + \left(\frac{c_2}{M}+\frac{c_2}{m_2}\frac{l_1^2+r^2}{l^2}\right)p^3 + \left(\frac{k_2}{M}+\frac{k_2+k_4}{m_2}\frac{l_1^2+r^2}{l^2}\right)p^2 + \frac{k_4c_2}{Mm_2}p + \frac{k_2k_4}{Mm_2}\right]\bar{x}_2 \tag{27}$$
$$+\frac{l_1l_2-r^2}{l^2}p^2\left(\frac{c_2}{m_2}p+\frac{k_2}{m_2}\right)\bar{x}_1 = \left(\frac{c_2}{m_2}p+\frac{k_2}{m_2}\right)\frac{k_4}{M}\frac{a_0\omega^2e^{-l/v_0\,p}}{p(p^2+\omega^2)}$$

These equations can be solved for $\bar{x}_1$ and $\bar{x}_2$ by the use of determinants. The denominator will be:

$$D = p\,(p^2+\omega^2)(a_8 p^8 + a_7 p^7 + \cdots + a_1 p + a_0) \qquad (28)$$

where the coefficients are:

$$a_8 = \lambda_1\lambda_2 - \lambda^2 = \frac{l_1^2+r^2}{l^2}\,\frac{l_2^2+r^2}{l^2} - \left(\frac{l_1 l_2 - r^2}{l^2}\right)^2$$

$$a_7 = \lambda_2\left(\lambda_1\frac{c_2}{m_2}+\frac{c_2}{M}\right)+\lambda_1\left(\lambda_2\frac{c_1}{m_1}+\frac{c_1}{M}\right)-\lambda^2\left(\frac{c_1}{m_1}+\frac{c_2}{m_2}\right)$$

$$a_6 = \lambda_2\left(\lambda_1\frac{k_2+k_4}{m_2}+\frac{k_2}{M}\right)+\lambda_1\left(\lambda_2\frac{k_1+k_3}{m_1}+\frac{k_1}{M}\right)+\left(\lambda_2\frac{c_1}{m_1}+\frac{c_1}{M}\right)\left(\lambda_1\frac{c_2}{m_2}+\frac{c_2}{M}\right)$$
$$-\lambda^2\left(\frac{k_1+k_3}{m_1}+\frac{k_2+k_4}{m_2}+\frac{c_1c_2}{m_1m_2}\right)$$

$$a_5 = \lambda_2\frac{c_2}{M}\frac{k_4}{m_2}+\lambda_1\frac{c_1}{M}\frac{k_3}{m_1}+\left(\lambda_2\frac{c_1}{m_1}+\frac{c_1}{M}\right)\left(\lambda_1\frac{k_2+k_4}{m_2}+\frac{k_2}{M}\right)$$
$$+\left(\lambda_1\frac{c_2}{m_2}+\frac{c_2}{M}\right)\left(\lambda_2\frac{k_1+k_3}{m_1}+\frac{k_1}{M}\right)-\lambda^2\left(\frac{c_1}{m_1}\frac{k_2+k_4}{m_2}+\frac{c_2}{m_2}\frac{k_1+k_3}{m_1}\right)$$

$$a_4 = \lambda_2\frac{k_2}{M}\frac{k_4}{m_2}+\lambda_1\frac{k_1}{M}\frac{k_3}{m_1}+\left(\lambda_2\frac{c_1}{m_1}+\frac{c_1}{M}\right)\frac{c_2}{M}\frac{k_4}{m_2}$$
$$+\left(\lambda_1\frac{c_2}{m_2}+\frac{c_2}{M}\right)\frac{c_1}{M}\frac{k_3}{m_1}+\left(\lambda_2\frac{k_1+k_3}{m_1}+\frac{k_1}{M}\right)\left(\lambda_1\frac{k_2+k_4}{m_2}+\frac{k_2}{M}\right)$$
$$-\lambda^2\frac{k_1+k_3}{m_1}\,\frac{k_2+k_4}{m_2}$$

$$a_3 = \left(\lambda_2\frac{c_1}{m_1}+\frac{c_1}{M}\right)\frac{k_2}{M}\frac{k_4}{m_2}+\left(\lambda_1\frac{c_2}{m_2}+\frac{c_2}{M}\right)\frac{k_1}{M}\frac{k_3}{m_1}+\left(\lambda_2\frac{k_1+k_3}{m_1}+\frac{k_1}{M}\right)\frac{c_2}{M}\frac{k_4}{m_2}$$
$$+\left(\lambda_1\frac{k_2+k_4}{m_2}+\frac{k_2}{M}\right)\frac{c_1 k_3}{M\,m_1}$$

$$a_2 = \left(\lambda_2\frac{k_1+k_3}{m_1}+\frac{k_1}{M}\right)\frac{k_2k_4}{M m_2}+\left(\lambda_1\frac{k_2+k_4}{m_2}+\frac{k_2}{M}\right)\frac{k_1k_3}{M m_1}+\frac{c_1c_2k_3k_4}{M^2 m_1 m_2}$$

$$a_1 = \frac{c_1k_3k_2k_4}{M^2 m_1 m_2}+\frac{c_2k_4k_1k_3}{M^2 m_1 m_2}$$

$$a_0 = \frac{k_1k_2k_3k_4}{M^2 m_1 m_2}$$

$$\lambda = \frac{l_1l_2-r^2}{l^2}, \quad \lambda_1 = \frac{l_1^2+r^2}{l^2}, \quad \lambda_2 = \frac{l_2^2+r^2}{l^2}$$

For inverse transform it is necessary to find the roots of the eighth order polynomial in p. If the numerical values of the coefficients are known, the roots can be found using Graeffe's method [16], but it is quite too laborous for any practical purposes. Therefore the mathematical solution of the vibration problem of the vehicle is not obtainable. The only way to find the motions is the use of analogues.

III. SIMILITUDE OF VIBRATION SYSTEMS

Dimensional Analysis

Dimensional analysis treats the general forms of equations that describe natural phenomena. Applications of dimensional analysis abound in nearly all fields of engineering, particularly in fluid mechanics and heat-transfer theory [17]. If experimental methods are to be applied the scope of the results can frequently be increased through the use of dimensional analysis. In designing a model of a vibrating system it is necessary to retain certain dimensionless quantities just as in any other model making.

The procedure in the application of dimensional analysis is first to list the various fundamental physical factors or dimensions which enter into the problem. These quantities can be determined by examining the differential equations of the simpler problems. These and their fundamental units are:

1. Mass - $m = W/g$ — FT^2/L
2. Spring constant - k — F/L
3. Damping constant - c — FT/L
4. Natural frequency - ω_n — $1/T$
5. Forced frequency - ω — $1/T$
6. Displacement - x — L

7. Wheelbase - l L

8. Moment of inertia - I FT^2L

The time might be considered but it does not seem necessary because it is the reciprocal of the frequency. The force does not enter in because except inertia, spring, and damping forces no external forces are acting in the system.

The terms involve three fundamental units, F, T, and L. The number of π functions that may be determined is the number of factors (8) minus the number of fundamental units (3), or five π functions. Among tha variables listed three are selected which contain all three of the fundamental dimensions among them. These are the mass, the wheelbase, and the natural frequency. These are set down with one of the other variables written after them. If α , β, and γ are arbitrary powers to be assigned to the three quantities selected, then the dimensional equations can be written as:

$$
\begin{aligned}
\pi_1 &= m^\alpha l^\beta \omega_n^\gamma k = \left(\frac{FT^2}{L}\right)^\alpha (L)^\beta \left(\frac{1}{T}\right)^\gamma \frac{F}{L} \\
\pi_2 &= m^\alpha l^\beta \omega_n^\gamma c = \quad " \quad " \quad " \quad \frac{FT}{L} \\
\pi_3 &= m^\alpha l^\beta \omega_n^\gamma \omega = \quad " \quad " \quad " \quad \frac{1}{T} \\
\pi_4 &= m^\alpha l^\beta \omega_n^\gamma l = \quad " \quad " \quad " \quad L \\
\pi_5 &= m^\alpha l^\beta \omega_n^\gamma I = \quad " \quad " \quad " \quad FT^2L
\end{aligned}
\tag{29}
$$

The sum of the exponents for each of the fundamental units must be zero. Therefore for π_1:

$$\alpha + 1 = 0 \qquad \text{(Powers of force, F)}$$

$$2\alpha - \gamma = 0 \qquad \text{(Powers of time, T)}$$

$$-\alpha + \beta - 1 = 0 \qquad \text{(Powers of Length, L)}$$

When these equations are solved simultaneously the values of the exponents are:

$$\alpha = -1$$

$$\beta = 0$$

$$\gamma = -2$$

so that the first dimensionless function π_1 is

$$\pi_1 = \frac{k}{m\,\omega_n^2}$$

The remaining functions may be evaluated in a like manner:

$$\pi_2 = \frac{c}{m\,\omega_n} \qquad \pi_4 = \frac{x}{l}$$

$$\pi_3 = \frac{\omega}{\omega_n} \qquad \pi_5 = \frac{I}{m\,l^2}$$

If these five dimensionless quantities for one system are equal to the respective quantities of another system, then the two systems are similar.

Use of Differential Equations

For the complete vehicle which has three masses, four different spring constants, and two damping constants, the dimensional analysis approach is not sufficient. More definite method is changing variables in the differential equations to dimensionless form.

The differential equations for the complete vehicle are, combining (23), (24), and (25):

$$
\begin{aligned}
&M\frac{l_2^2+r^2}{l^2}\ddot{x}_1 + M\frac{l_1 l_2 - r^2}{l^2}\ddot{x}_2 + c_1(\dot{x}_1-\dot{x}_3) + k_1(x_1-x_3) = 0 \\
&M\frac{l_1^2+r^2}{l^2}\ddot{x}_2 + M\frac{l_1 l_2 - r^2}{l^2}\ddot{x}_1 + c_2(\dot{x}_2-\dot{x}_4) + k_2(x_2-x_4) = 0 \\
&m_1\ddot{x}_3 - c_1(\dot{x}_1-\dot{x}_3) - k_1 x_1 + (k_1+k_3)x_3 = k_3 a_0(1-\cos\omega t) \\
&m_2\ddot{x}_4 - c_2(\dot{x}_2-\dot{x}_4) - k_2 x_2 + (k_2+k_4)x_4 = k_4 a_0\left[1-\cos\omega\left(t-\frac{l}{V_0}\right)\right]
\end{aligned}
\qquad (30)
$$

If a change of variables is made using

$$
\bar{X} = \frac{x}{l}, \qquad T = \omega_n t \qquad (31)
$$

When the dimensionless terms are differentiated, the results are:

$$
dx = l\,d\bar{X}, \qquad dt = \frac{dT}{\omega_n}
$$

$$
\dot{x} = \frac{dx}{dt} = l\omega_n\frac{d\bar{X}}{dT} = l\omega_n\dot{\bar{X}}
$$

$$
\ddot{x} = \frac{d^2x}{dt^2} = l\omega_n^2\frac{d^2\bar{X}}{dT^2} = l\omega_n^2\ddot{\bar{X}}
$$

These derivatives are substituted in the original equations:

$$M\frac{l_2^2+r^2}{l^2}l\omega_n^2\ddot{X}_1+M\frac{l_1l_2-r^2}{l^2}l\omega_n^2\ddot{X}_2+c_1l\omega_n(\dot{X}_1-\dot{X}_3)+k_1l(X_1-X_3)=0$$

$$M\frac{l_1^2+r^2}{l^2}l\omega_n^2\ddot{X}_2+M\frac{l_1l_2-r^2}{l^2}l\omega_n^2\ddot{X}_1+c_2l\omega_n(\dot{X}_2-\dot{X}_4)+k_2l(X_2-X_4)=0$$

$$m_1l\omega_n^2\ddot{X}_3-c_1l\omega_n(\dot{X}_1-\dot{X}_3)-k_1lX_1+(k_1+k_3)lX_3=k_3a_0(1-\cos\omega t) \qquad (32)$$

$$m_2l\omega_n^2\ddot{X}_4-c_2l\omega_n(\dot{X}_2-\dot{X}_4)-k_2lX_2+(k_2+k_4)lX_4=k_3a_0\left[1-\cos\left(\omega t-\frac{l}{v_0}\right)\right]$$

Since X and T are dimensionless, their derivatives are dimensionless. If each equation is divided by the coefficient of the second derivative, it makes all terms dimensionless:

$$\frac{l_2^2+r^2}{l^2}\ddot{X}_1+\frac{l_1l_2-r^2}{l^2}\ddot{X}_2+\frac{c_1}{M\omega_n}(\dot{X}_1-\dot{X}_3)+\frac{k_1}{M\omega_n^2}(X_1-X_3)=0$$

$$\frac{l_1^2+r^2}{l^2}\ddot{X}_2+\frac{l_1l_2-r^2}{l^2}\ddot{X}_1+\frac{c_2}{M\omega_n}(\dot{X}_2-\dot{X}_4)+\frac{k_2}{M\omega_n^2}(X_2-X_4)=0$$

$$\ddot{X}_3-\frac{c_1}{m_1\omega_n}(\dot{X}_1-\dot{X}_3)-\frac{k_1}{m_1\omega_n^2}X_1+\frac{k_1+k_3}{m_1\omega_n^2}X_3=\frac{k_3}{m_1\omega_n^2}\frac{a_0}{l}\left(1-\cos\frac{\omega}{\omega_n}T\right) \qquad (33)$$

$$\ddot{X}_4-\frac{c_2}{m_2\omega_n}(\dot{X}_2-\dot{X}_4)-\frac{k_2}{m_2\omega_n^2}X_2+\frac{k_2+k_4}{m_2\omega_n^2}X_4=\frac{k_4}{m_2\omega_n^2}\frac{a_0}{l}\left[1-\cos\frac{\omega}{\omega_n}\left(T-\frac{v_0}{\omega_n l}\right)\right]$$

The dimensionless quantities are then:

$$\frac{l_1}{l},\quad \frac{l_2}{l},\quad \frac{r^2}{l_1l_2},\quad \frac{c_1}{M\omega_n},\quad \frac{c_2}{M\omega_n},$$

$$\frac{k_1}{M\omega_n^2},\quad \frac{k_2}{M\omega_n^2},\quad \frac{k_1}{m_1\omega_n^2},\quad \frac{k_2}{m_2\omega_n^2}$$

$$\frac{k_1+k_3}{m_1\omega_n^2},\quad \frac{k_2+k_4}{m_2\omega_n^2},\quad \frac{a_0}{l},\quad \frac{\omega}{\omega_n},\quad \frac{v_0}{\omega_n l}$$

If these quantities are equal in two systems also their ratios are equal. This way following quantities may be determined:

$$\frac{k_1 l_1}{k_2 l_2}, \quad \frac{c_1 l_1}{c_2 l_2}, \quad \frac{m_1}{M}, \quad \frac{m_2}{M}$$

The final conditions that two vibration systems are similar, are that following dimensionless constants in one system are equal to corresponding constants in another system:

1. l_1/l - Weight distribution
2. $r^2/l_1 l_2$ - Mass coupling
3. $\frac{c_1}{M\omega_n}$ - Relative damping
4. $\frac{k_1}{M\omega_n^2}$ - Static deflection
5. $\frac{k_1 l_1}{k_2 l_2}$ - Spring coupling
6. $\frac{c_1 l_1}{c_2 l_2}$ - Damping coupling
7. $\frac{m_1}{M}, \frac{m_2}{M}$ - Relative weight of unsprung mass
8. $\frac{k_3}{k_1}, \frac{k_4}{k_2}$ - Relative tire stiffness
9. $\frac{a_0}{l}$ - Amplitude/wheelbase ratio
10. $\frac{\omega}{\omega_n}$ - Frequency ratio
11. $\frac{v_0}{\omega_n l}$ - The ratio of driving speed to the product of natural frequency and the wheelbase

IV. DESCRIPTION OF THE ANALOGUE

General

The main purpose of the analogue was the use for demonstration of vibrational characteristics of the motor-vehicle. It was therefore quite natural to think it first as a small-scale model of the actual vehicle as shown in Fig. 7, using compression springs in the place of springs and tires. However, the guidance needed in this type of analogue would involve too much friction and the application of road disturbance would not be convenient.

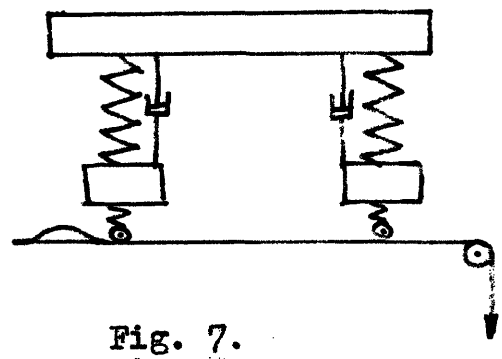


Fig. 7.

For smaller friction and easier guidance the whole system was turned upside down as shown in Fig. 8. Although a little more imagination is needed in order to understand that this represent an automobile, it was found very succesful.

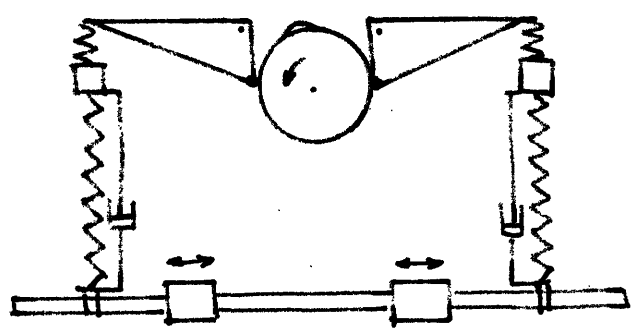


Fig. 8.

No guides are needed to prevent other than vertical motions because the system is always stabile in the field of gravity. The friction is thus reduced to a minimum. Further the road disturbance is easily represented by a single rotating cam which affects both front and rear wheels through rocker arms.

Weight Distribution

The wheelbase of the analogue was chosen to be 24 inches which is approximatelly one fifth of the wheelbase of the average American car. The weight of the analogue does not need to be proportional to the third power of the linear ratio as for some other models, because the laws of similitude do not include any relation between the mass and the legth in this case.

It was desired to have a large range of weight distribution. Therefore the "chassis" was made of 1"x$\frac{1}{4}$" duralumium bar, 40" long, which weighed only 1.01 lb. Four movable steel weights were made weighing 1.10 lb each. The heaviest part of the analogue are the electromagnetic dampers which will be described later. They weigh 3.38 lb each. The weight of the whole body is

Chassis	1 x 1.01 lb	= 1.01 lb
Weights	4 x 1.10 "	= 4.40 "
Dampers	2 x 3.38 "	= 6.76 "
		12.17 lb

The moment of inertia of the chassis and the dampers is:

$$I_c = \frac{ml^2}{12} = \frac{1.01 \times 40^2}{12} = 135 \text{ lbm-in}^2$$

$$I_D = m\left(\frac{l}{2}\right)^2 = 6.76 \times 12^2 = 975 \text{ lbm-in}^2$$

$$1110 \text{ lbm-in}^2$$

The variation in the moment of the inertia is due to the change of position of the weights. The minimum is obtained when all four weights are in the center of the bar, and the maximum when they are at both ends of the bar:

$$I_{min} = 4.4 \times 1.5^2 = 10 \text{ lbm-in}^2$$

$$I_{max} = 4.4 \times 18.5^2 = 1505 \text{ lbm in}^2$$

The minimum and the maximum radii of gyration of the body are:

$$r_{min} = \sqrt{\frac{1110 + 10}{12.17}} = 9.6 \text{ in.}$$

$$r_{max} = \sqrt{\frac{1110 + 1505}{12.17}} = 14.35 \text{ in.}$$

The minimum and the maximum mass coupling at symmetrical weight distribution are:

$$\frac{r_{min}^2}{l_1 l_2} = \frac{9.6^2}{12^2} = .64$$

$$\frac{r_{max}^2}{l_1 l_2} = \frac{14.35^2}{12^2} = 1.43$$

The mass coupling can be varied in larger range than in actual vehicle. In order to get zero mass coupling ($r^2/l_1 l_2 = 1$) at symmetrical weight distribution the moment of inertia must be equal to the mass multiplied by square of the half of the wheelbase:

$$I = m\left(\frac{l}{2}\right)^2 = 12.17 \times 12^2 = 1751 \text{ lbm in.}$$

This may be obtained so that two weights are left in the center and two are put 17.075" from the center. This distance is marked on the bar as well as the center for easier weight installation.

Springs

Because of the large size of the damper it was found more convenient to use a set of two parallel springs between the body and the unsprung masses instead of a single spring.

For different static deflections and weight distributions a large number of springs of different rates was needed. The maximum and minimum spring loads which were half of corresponding axle loads, were estimated 4 lb and 2 lb. The static deflections between 4 in. and 8 in. were desired corresponding the frequencies 94...66 cycles per minute. The maximum and minimum spring rates were then found:

$$k_{max} = \frac{F_{max}}{\delta_{min}} = \frac{4}{4} = 1 \text{ lb/in.}$$

$$k_{min} = \frac{F_{min}}{\delta_{max}} = \frac{2}{8} = .25 \text{ lb/in.}$$

Extension type springs were ordered from Hardware Products Company, Boston. These are made of high quality spring steel and have initial tension which makes the actual static deflection smaller than what is the effective deflection. A table of these springs and their properties is given below.

Outside diameter in.	3/8			1/2		
Wire diameter in.	031	047	062	047	062	094
Max. load lb.	4.2	15.	35.	11.	25.	85.
Max. extension in.	3.4	1.2	.7	2.4	1.2	.4
Initial tension lb.	.9	3.0	7.0	2.2	5.0	17.
Spring rate lb/in.	1.0	9.6	43.	3.7	16.	153

The figures for maximum extension and spring rate are for springs one inch long. The maximum load and initial tension remain constant for any length.

In order to keep the weight and the free length of the springs as small as possible the suspension springs were made 3/8" outside diameter .031" wire diameter. The maximum load for this spring is just above the maximum required.

For "tire" spring the load varies between 4 and 8 lb. Static deflections from .8" to .4" were wanted because the natural frequency of tires is approximately ten times that of the suspension springs. Spring rates are then:

$$k_{max} = \frac{F_{max}}{\delta_{min}} = \frac{8}{.4} = 20 \text{ lb/in.}$$

$$k_{min} = \frac{F_{min}}{\delta_{max}} = \frac{4}{.8} = 5 \text{ lb/in.}$$

A set of springs of seven different lengths were ordered for both purposes. All springs were carefully tested and the actual spring rates are listed below:

Suspension springs 3/8 x .031		Tire springs 3/8 x .047	
Length	Rate	Length	Rate
1"	1.26	½"	20.0
1½"	.833	¾"	13.8
2"	.605	1"	10.0
2½"	.495	1¼"	8.48
3"	.410	1½"	6.90
3½"	.345	1¾"	6.13
4"	.298	2"	5.25

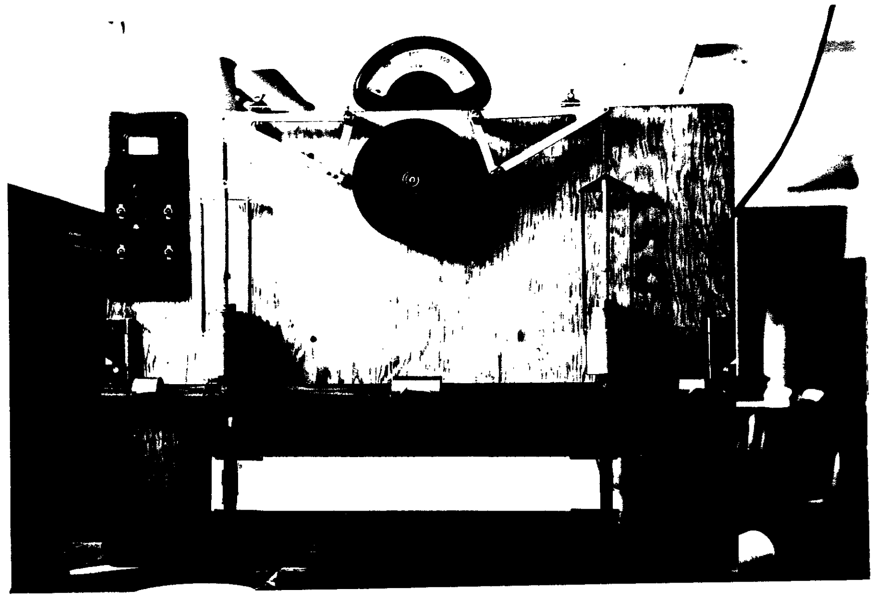


Fig. 9. The analogue.

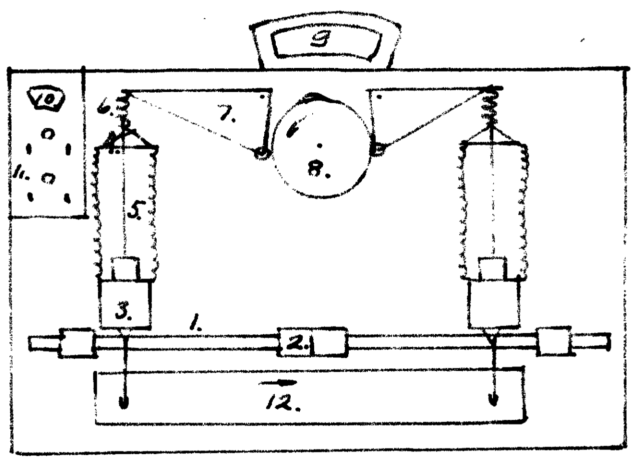


Fig. 10. Schematical representation.

1. Chassis bar
2. Movable weight
3. Damper
4. Unsprung mass
5. Suspension springs
6. "Tire" springs
7. Rocker arm
8. Cam
9. Speedometer for the cam
10. Ammeter for the dampers
11. Switchboard
12. Recording paper

The Driving Mechanism

For driving the cam which produces the road shocks a variable speed electric motor is used. Through a reduction gear the speed scale is 0...400 rpm, but 30 rpm is practically the lowest speed possible.

The relation between the cam rpm and the vehicle speed in mph whish has to be represented, may be found through following reasoning. During one half of cam revolution the analogue "travels" a distance equal to the wheelbase. At n rpm the distance traveled in one hour is

$$60 \times n \times 2l$$

If the wheelbase is measured in feet, rpm scale must be multiplied by

$$\frac{120 \cdot l}{5280} = .0227 \times l$$

in order to convert the speed into mph. For an average car of 10 feet wheelbase this coefficient is .227.

The Recording Mechanism

For measuring the amplitude of the motion a recording mechanism is provided which uses regular 3½" wide adding machine tape. The tape is driven by an electric motor through a three-speed gearbox. The speeds for the tape are .3", 1.0", and 5.0" per second.

V. ELECTROMAGNETIC DAMPER

General Theory

If a conductor moves in a magnetic field an electromotive force is induced in the conductor. The electromotive force causes a current which in turn causes a force opposing the motion of the conductor. The force is proportional to the magnetic flux density and the relative velocity of the conductor with respect to the field.

This phenominon occurs in so called eddy-current brakes used in laboratories to measure the mechanical output of a motor. It may be succesfully applied to represent an automotive shock absorber. It is ideal for this purpose because the damping is linear, i.e. proportional to the first power of the velocity. In a fluid dashpot there is always in addition to the linear viscous damping more or less hydrodynamic damping which is proportional to the square of the velocity. Linear damping is ideal not only for the simplicity of mathematical treatment but also from comfort standpoint.

The control of an electromagnetic damper is also easier than in a fluid damper. This is accomplished by changing the exciting current in the electromagnet. The only drawback is considerably heavier weight.

It was figured, however, that even when one half of the weight of the body would be concentrated at the places of spring attachment in form of dampers, the desired range of weight distribtuion would still be obtained by using light aluminium bar for chassis and heavier movable weights. This gave for each damper an approximate weight of three pounds.

In order to use the material most efficiently a round magnet as shown in Fig. 11, was suggested. A cylinder of non-magnetic material of high conductivity moves up and down in the circular air gap. The magnet has to be attached to the body and the moving cylinder to the axle mass. Otherwise the unsprung mass would become too large.

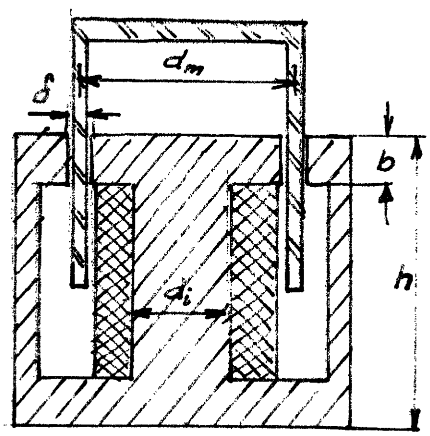


Fig. 11.

If the cylinder moves with a velocity v in a magnetic field which has flux density B, the electromotive force induced in the cylinder is in volts:

$$E = B \times \pi d_m \times v \times 10^{-8} \tag{34}$$

The resistance of the piece of the cylinder in the air gap is in ohms:

$$R = \frac{\varrho \pi d_m}{b \delta} \tag{35}$$

where ϱ is the resistivity of the material (for copper $\varrho = 6.79 \times 10^{-7}$ ohm in.) and b and δ are the dimensions of the cross section of the air gap. Electric current in the cylinder is the electromotive force divided by the resistance:

$$I = \frac{E}{R} = \frac{B b \delta v}{\varrho} \times 10^{-8} \tag{36}$$

The force which opposes the motion of the cylinder is proportional to the current, flux density, and the legth of the conductor:

$$F = f I \pi d_m B \tag{37}$$

f is proportionality factor which das the value one when force is measured in newtons, flux density in webers/m^2, current in amperes, and the length in meters. If the English system is used f has to be calculated:

1 newton = .224 lb

1 weber/m^2 = 64500 lines/sq.in.

1 meter = 39.37 in.

$$f = \frac{.224}{64500 \times 39.37} = 8.83 \times 10^{-8}$$

By substituting the expression for current (36) into (37) and using the English units we get:

$$F = 8.83 \times 10^{-16} \frac{1}{\rho} B^2 b \delta \pi d_m v \tag{38}$$

The damping coefficient is equal to the force divided by the velocity:

$$C = \frac{F}{v} = 8.83 \times 10^{-16} \frac{1}{\rho} B^2 b \delta \pi d_m \tag{39}$$

Where: c = damping coefficient (lb.sec/in.)

ρ = resistivity of the material (ohm in.)

B = flux density (lines/sq.in.)

b = height of the air gap (in.)

δ = length of the air gap (in.)

d_m = mean diameter of the air gap (in.)

Design of the Magnet

If certain damping coefficient is desired, the flux density required in the air gap is found from (39):

$$B_a = \sqrt{\frac{\rho c}{8.83 \times 10^{-16} \delta b \pi d_m}} \tag{40}$$

If leakage is neglected the total flux in the air gap is equal to the flux in the core

$$\Phi = A_a B_a = A_c B_c \tag{41}$$

from which:

$$B_a = \frac{d_1^2}{4 d_m b} B_c \tag{42}$$

In order to reduce the number of variables in the expression for B_a (40) the following step is made:

$$B_a = \frac{B_a^2}{B_a} = \frac{4 \varrho c}{8.83 \cdot 10^{-16} \pi d \, d_i^2 \, B_c} \tag{43}$$

Magnetic field intensity in the air gap is

$$K_a = \mu_o B_a \tag{44}$$

where μ_o is the permeability of free space. If B_a is measured in liner/sq.in. and K_a in amp.turns/in. then it has the value

$$\mu_o = 3.192$$

The magnetomotive force required is

$$F_m = K_a \delta + K_c \cdot l_c \tag{45}$$

where l_c is the path of the lines in the core. In the first approximation the path from the inner core to the outer core may be neglected. If the height of the core is h and the value for K_a is substituted:

$$F_m = \frac{4 \varrho c}{8.83 \times 10^{-16} \pi \, d_i^2 \, B_c} + 2 K_c h \tag{46}$$

The cross sectional area available for the coil:

$$A = \tfrac{1}{2} h \, (d_m - d_i - \delta) \tag{47}$$

The number of ampere turns which can be safely used is:

$$N I = C_s A = \tfrac{1}{2} C_s h \, (d_m - d_i - \delta) \tag{48}$$

C_s is the safety carrying capasity of the wire. For small size wire in confined spaces it is approximately:

$$C_s = 2000 \text{ amp/sq.in.}$$

The magnetomotive force is equal to the number of amperetums:

$$\frac{4\varrho c}{8.83\times10^{-16}\pi\, d_i^2\, B_c} + 2K_c h = \tfrac{1}{2}C_s h(d_m - d_i - \delta) \quad (49)$$

from which h may solyed:

$$h = \frac{4\varrho c}{8.83\times10^{-16}\pi d_i^2 B_c\left[\frac{1}{2}C_s(d_m-d_i-\delta)-2K_c\right]} \quad (50)$$

For certain value of core diameter this expression has a minimum. This is found by differentiating (50) with respect to d_i and setting that equal to zero. The optimum core diameter will be:

$$d_i = \frac{d_m - \delta - 4\frac{K_c}{C_s}}{3} \quad (51)$$

The magnetizing curve for cold rolled steel is shown in Fig. 12.[18] In order to use the material efficiently rather high flux density is desired. Corresponding to the assumed maximum flux density 120 kilolines/sq.in. the maximum field intensity is found by interpolation:

$$K_c = 300 \text{ amp.turns/in.}$$

The length of the air gap was assumed 3/32" and the mean diameter 2" because this size of copper tube for the damper cylinder was available. The optimum core diameter can then be calculated:

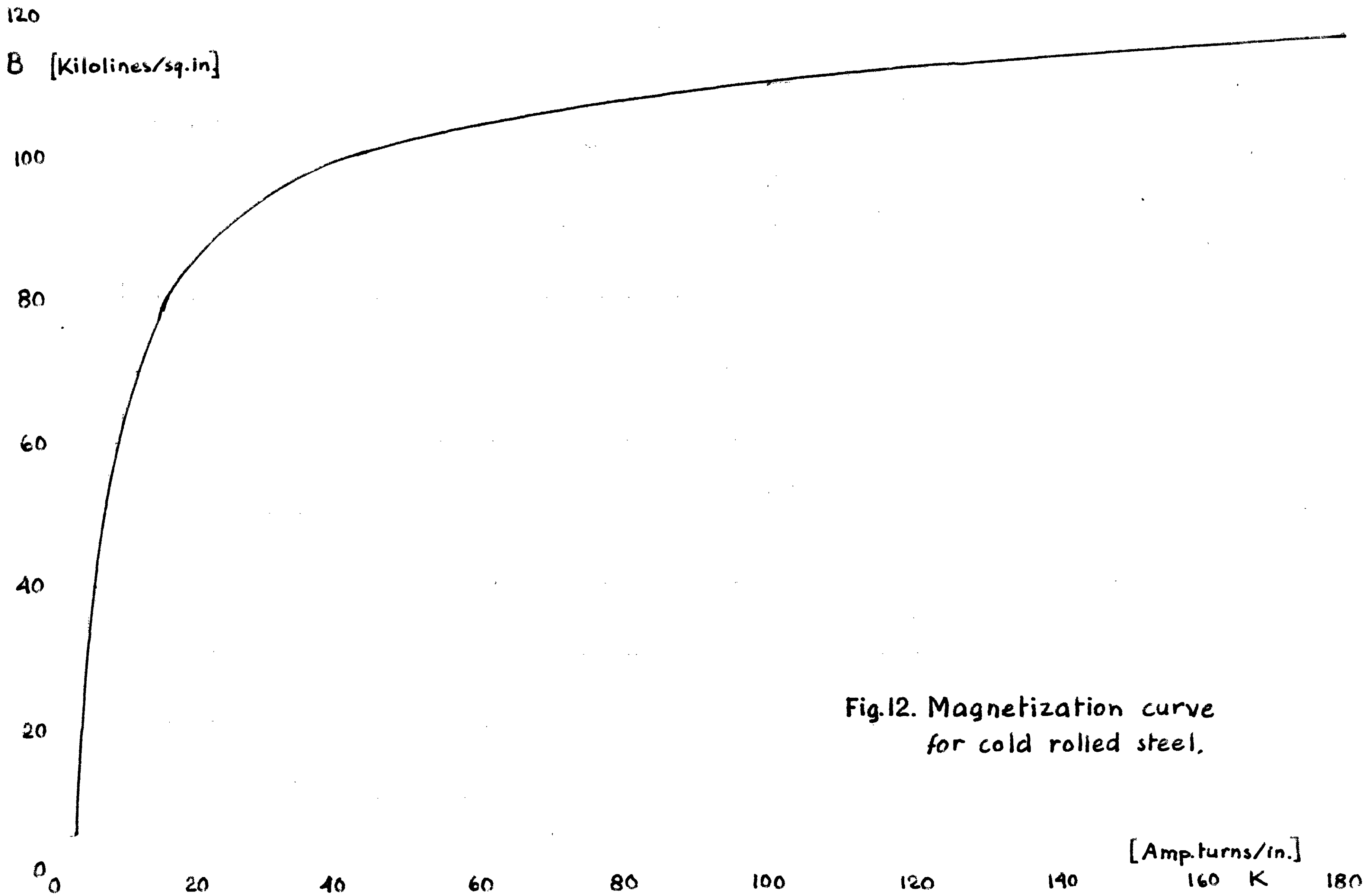


Fig.12. Magnetization curve
for cold rolled steel.

$$d_i = \frac{2 - \frac{3}{32} - 4\frac{300}{2000}}{3} = 0.875 \text{ in.}$$

Corresponding value for h is found from eq.(51)

$$h = 2.86 \text{ in.}$$

Because in calculation of the length of the path in the core the horizontal part was neglected the total height of the magnet was designed 2¾". The height of the air gap was assumed ¼".

The magnetizing curve for the magnet can be plotted be means of Fig. 12 and equation (45) which is after necessary calculations:

$$F_m = 11.25 \; B_c + 5.72 \; K_c \qquad (52)$$

The damping coefficient c may be determined as function of magnetomotive force by using equation (39) which is after calculations:(B_c in kilolines/sq.in.)

$$c = .187 \; (B_c/100)^2 \qquad (53)$$

The calculated values for F_m and c are in the table below and the damping coefficient vs. magnetomotive force is plotted in Fig. 13.

B_c kilolines/in^2	K_c amp.turns/in	F_m amp.turns	c lb.sec/in.
20	4	248	.0075
40	6	484	.0300
60	9	727	.0673
70	12	856	.0916
80	16	992	.1196
90	23	1145	.1514
100	40	1354	.1870
110	100	1810	.2265
120	300	3066	.2690

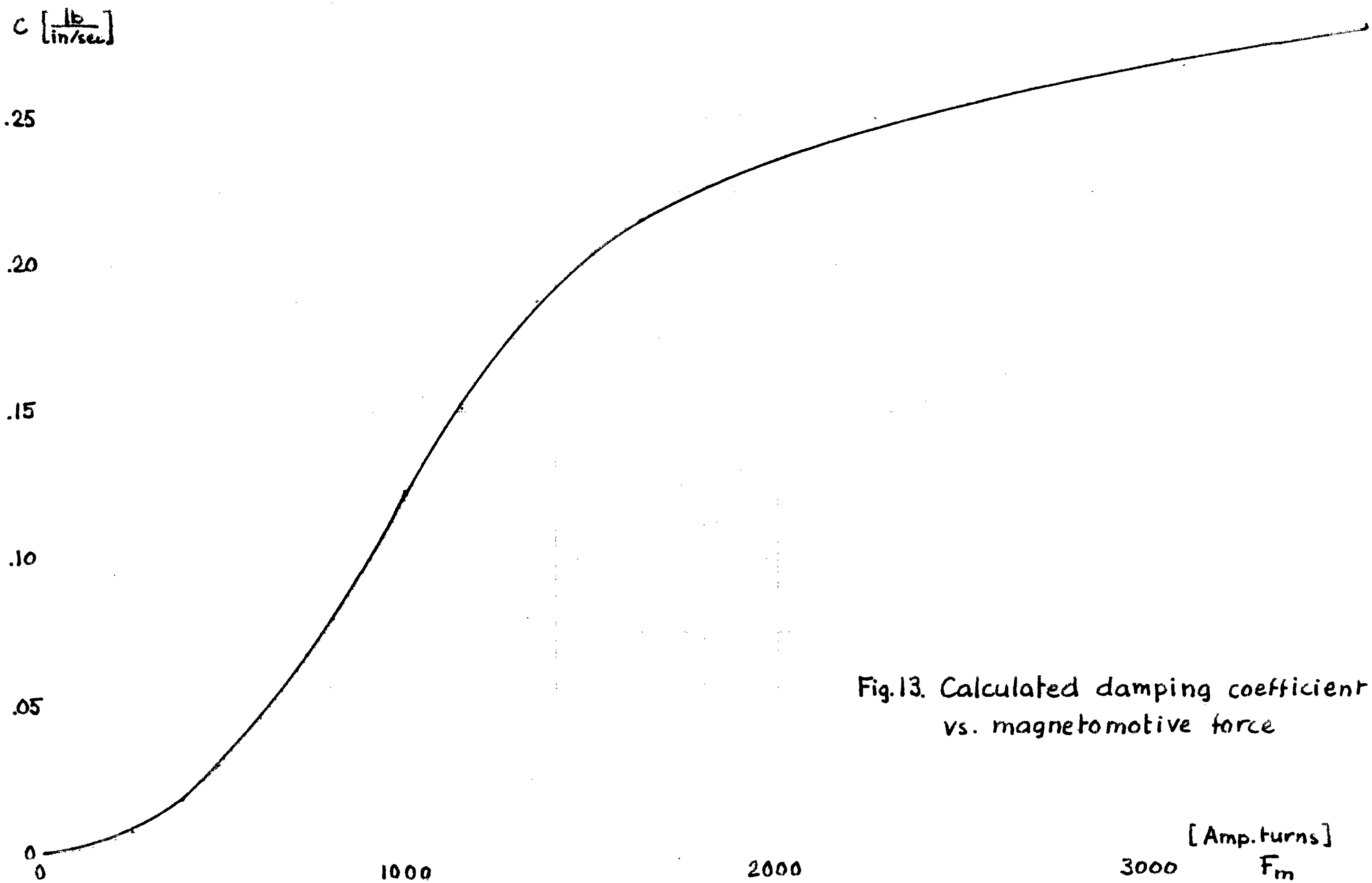


Fig. 13. Calculated damping coefficient vs. magnetomotive force

The critical damping coefficient for average axle load (6 lb.) and spring rate (1 lb/in.) of the analogue is

$$C_c = 2\sqrt{km} = 2\sqrt{\frac{6\times 1}{32.2\times 12}} = .25 \ \frac{\text{lb}\cdot\text{sec}}{\text{in}}$$

The number of ampere turns which gives this amount of damping is 2600 according to Fig. 13.

For coil it was suggested to use wire No 20, which specifications are [19]:

Diameter	31.96 mils
Area	1096 circ.mils
Resistance	0.672 ohm/in^3

The volume of the coil is (Fig. 14):

$$V = \frac{\pi}{4}\left[\left(1\tfrac{7}{8}\right)^2 - \left(\tfrac{7}{8}\right)^2\right]\times 2\tfrac{1}{4} = 5.1 \ \text{in}^3$$

Resistance:

$$R = 5.1 \times 0.672 = 3.425 \ \Omega$$

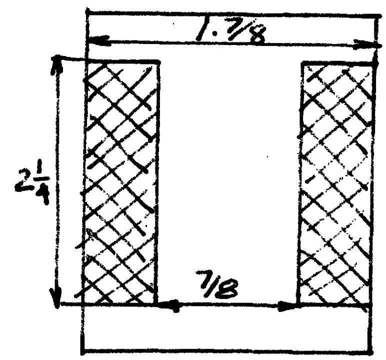


Fig. 14.

When the coil was made there were 14 layers, 55 turns in each, which means 770 turns total. In order to obtain critical damping coefficient the current must be:

$$I = \frac{F_m}{N} = \frac{2600}{770} = 3.38 \ \text{A}$$

The voltage required is:

$$E = R\,I = 3.425 \times 3.38 = 11.6 \ \text{V}$$

The maximum current is a little high for safety carrying capasity of the wire. The cross section of the wire is only 292 circular mils per ampere when at least 500 is recommended in Radio Engineers Handbook. Because the maximum current is used only for short periods, this considered satisfactory. It was found that it did not cause any damage even when the maximum current was left on for several minutes.

In order to keep friction as small as possible only one wire was connected to each damper. The other terminal is connected to "ground" where the current is lead through the suspension springs. Two reostats are used for controlling the current, one for the total current and another for the ratio between the dampers. The wiring diagram of the damping system is shown in Fig.15.

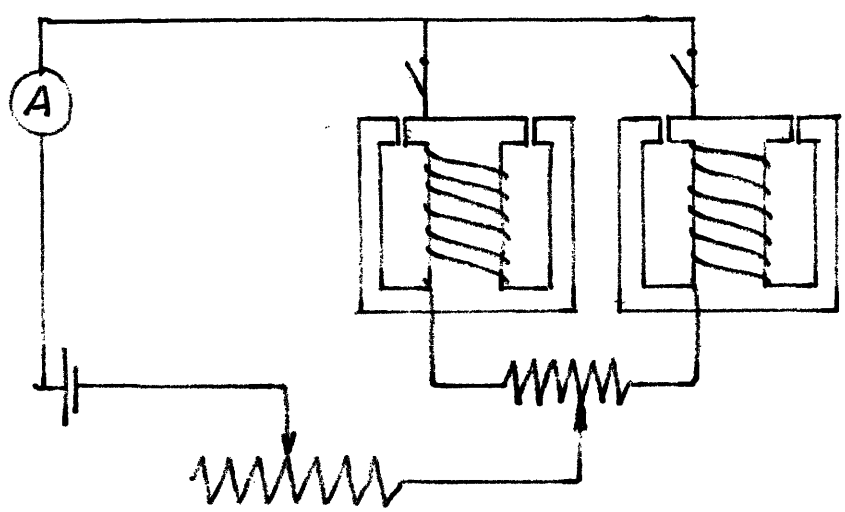


Fig. 15.

VI. TEST RESULTS

For using the analogue it is necessary to find the relation between exciting current and damping coefficient. The direct measuring of the damping force is difficult but it can be conveniently determined by examining free or steady-state vibrations.

Free Vibrations

The rate of diminishing of free vibrations depends on the damping. The relation is shown in Fig. 16 where the ratio of successive half-cycle amplitudes is plotted against relative damping coefficient [20].

It was found that after turning off the exciting current the remaining magnetism in the magnet caused a small amount of damping. To eliminate this the first run was made without copper cylinder in the air gap. There was still some dry friction left due to the recorder pens but it is so small it can be neglected. The first two runs were made at low recorder speed (.3 in./sec.) but later only the intermediate speed (1.0 in./sec.) was used, because this made dry friction even for up and down stroke. The vibration curves, test data are given in Fig. 17...25.

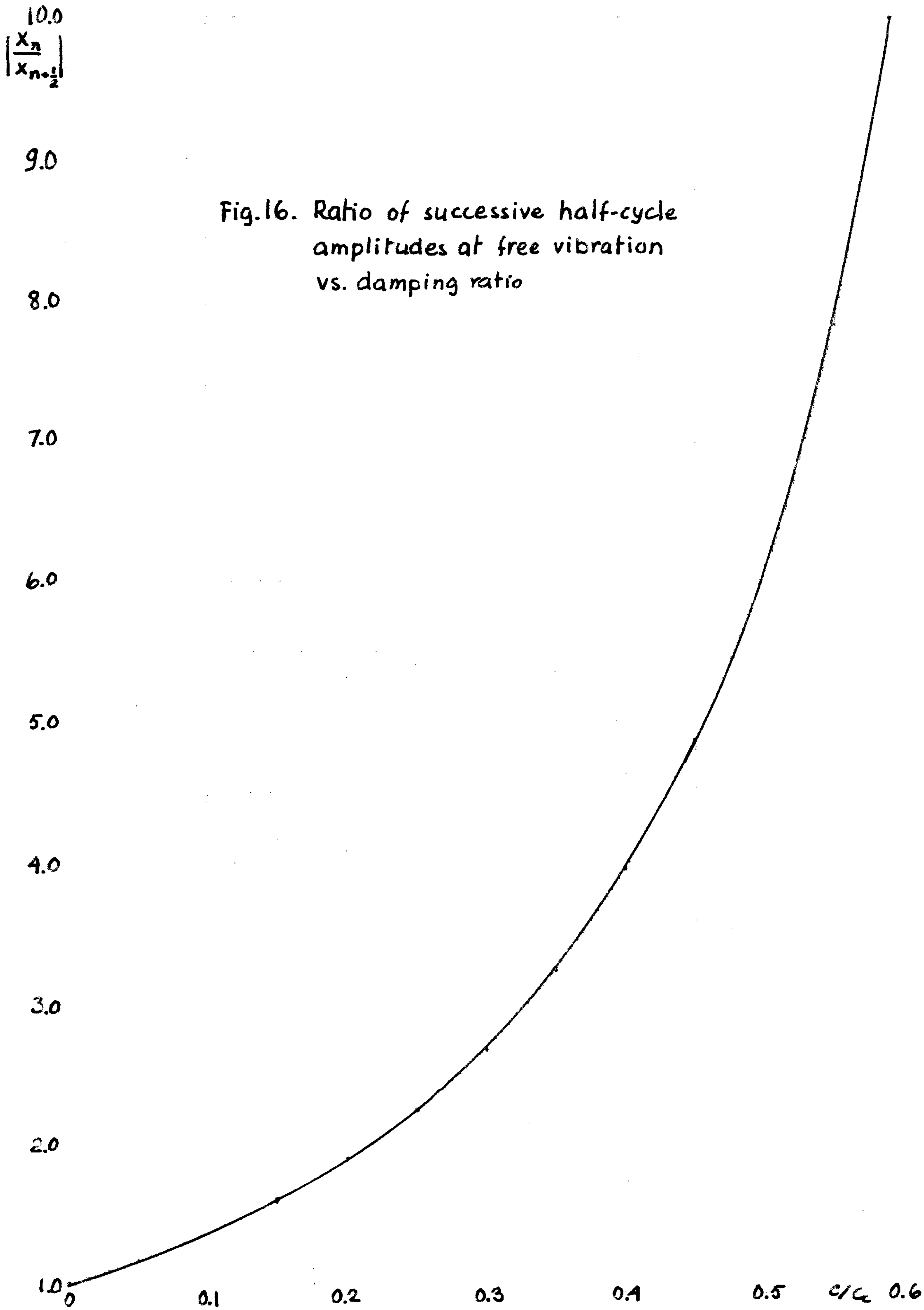
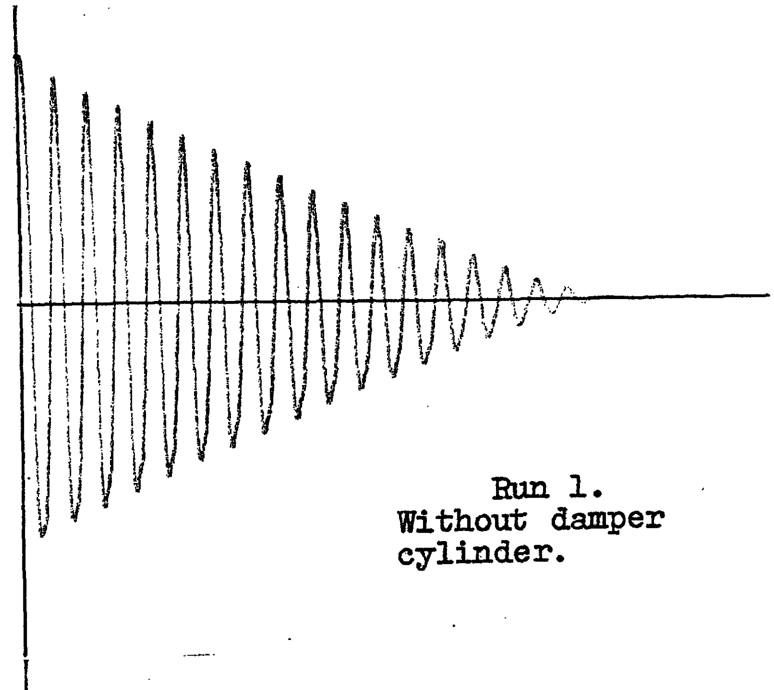
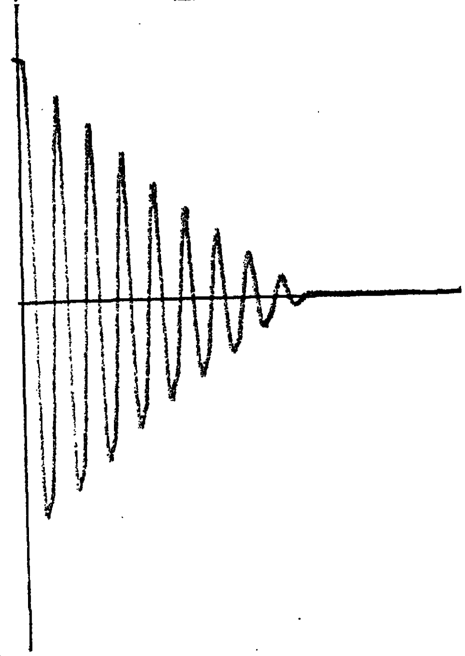


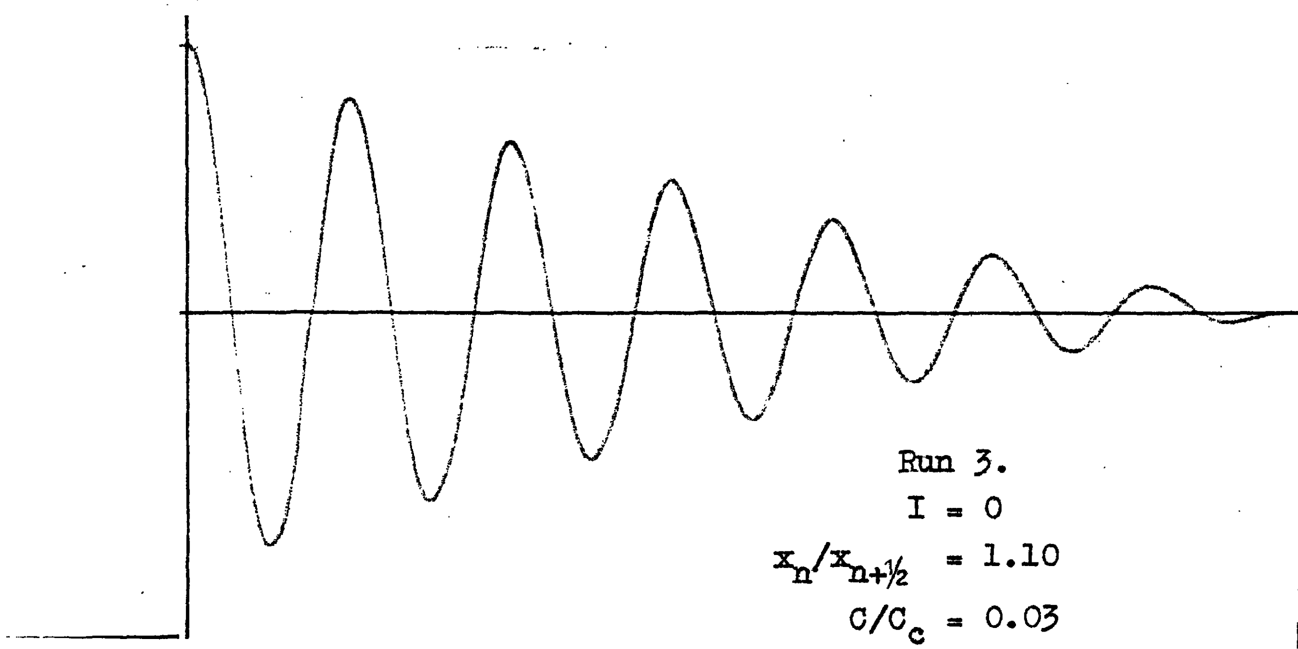
Fig.16. Ratio of successive half-cycle amplitudes at free vibration vs. damping ratio



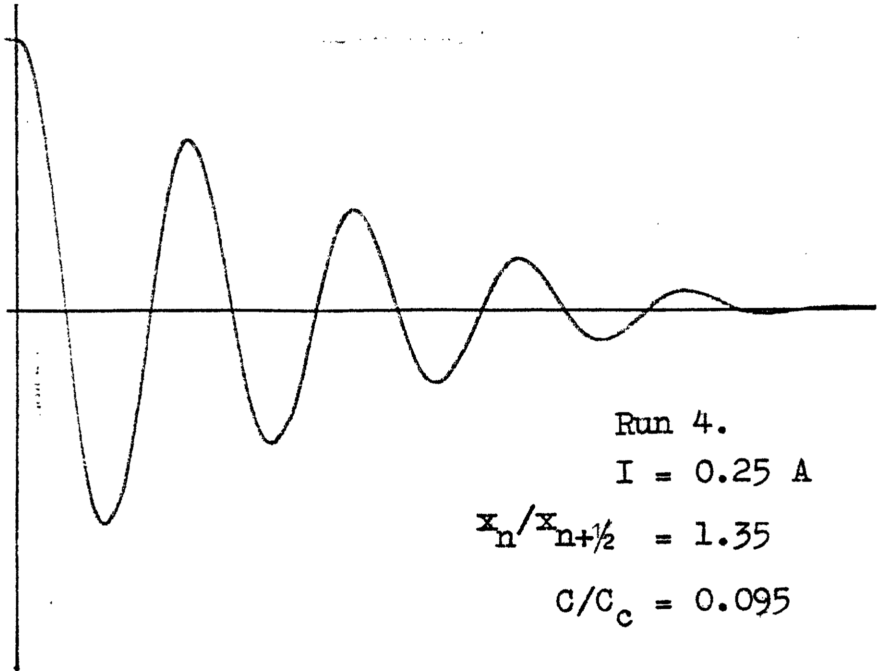
Run 1.
Without damper
cylinder.

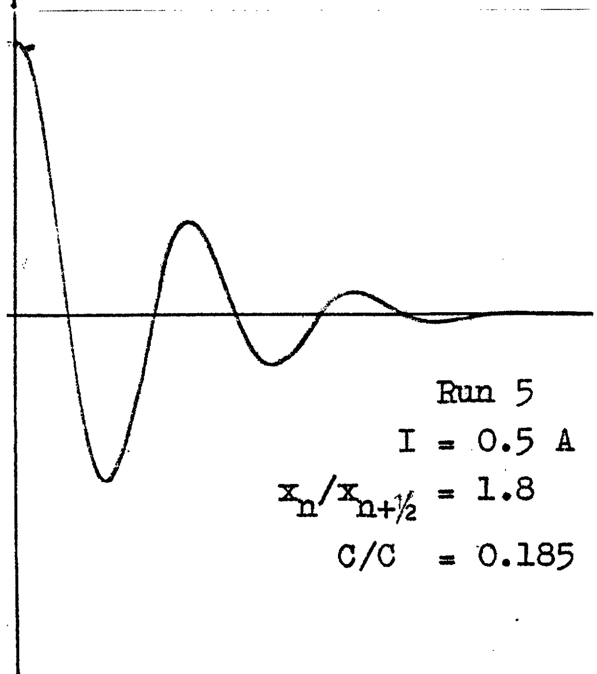


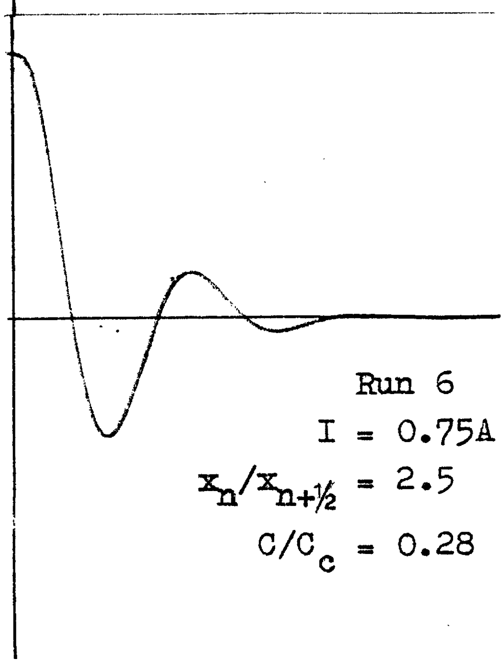
Run 2.
With damper
cylinder
I = 0

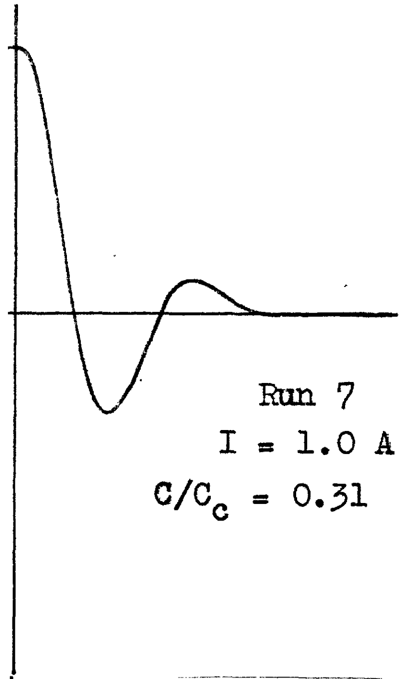


Run 3.
I = 0
$x_n/x_{n+\frac{1}{2}}$ = 1.10
C/C_c = 0.03

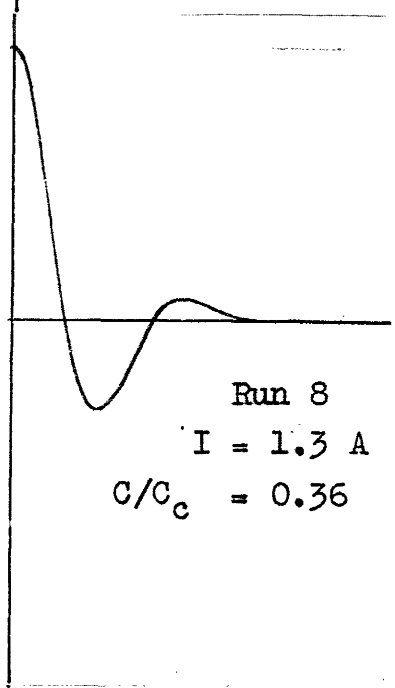

Run 4.
I = 0.25 A
$x_n/x_{n+½}$ = 1.35
C/C_c = 0.095


Run 5
I = 0.5 A
$x_n/x_{n+½}$ = 1.8
C/C = 0.185

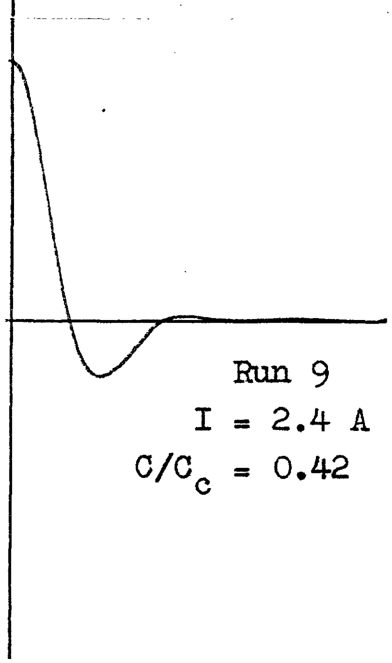

Run 6
I = 0.75A
$x_n/x_{n+½}$ = 2.5
C/C_c = 0.28



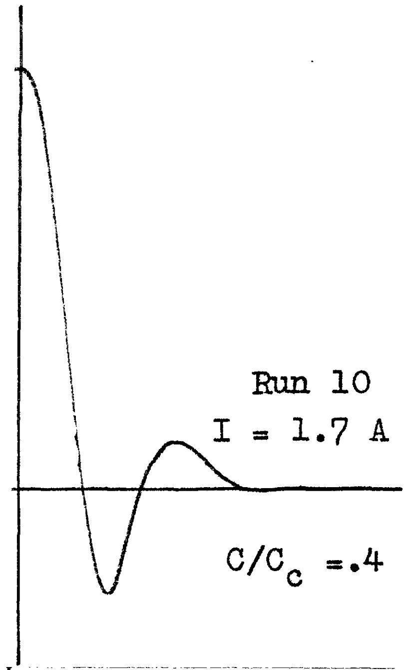
Run 7
I = 1.0 A
C/C_c = 0.31



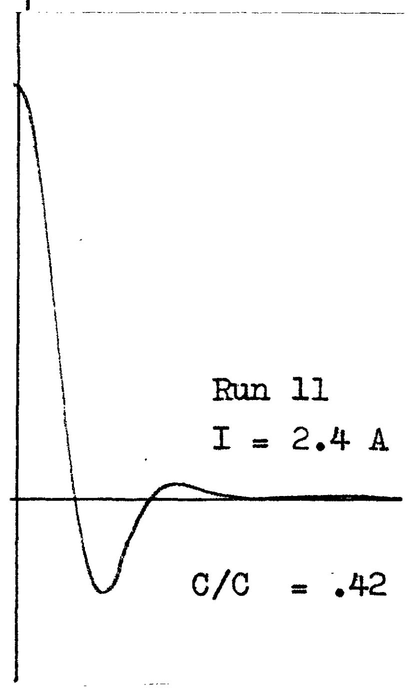
Run 8
I = 1.3 A
C/C_c = 0.36



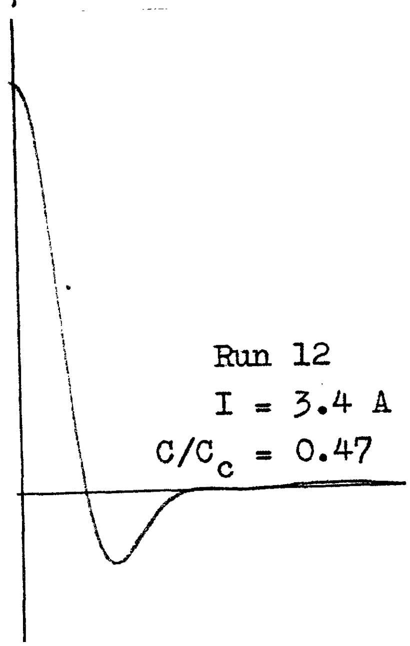
Run 9
I = 2.4 A
C/C_c = 0.42



Run 10
I = 1.7 A
C/C_c =.4



Run 11
I = 2.4 A
C/C = .42



Run 12
I = 3.4 A
C/C_c = 0.47

Forced Vibrations

The damping coefficient can be determined also by using forced vibrations. Some time after the forced vibration has started the transient term dies out and only the steady-state part is left. The test is best done at resonance frequency because the amplitude variation for different damping coefficients is then largest. The ratio of steady-state amplitude to the forcing amplitude is plotted against the relative damping coefficient for $\omega = \omega_n$ in Fig. 26.

The forcing function was provided by a single exentric cam which produced a bump 1/6" high. At the end of the rocker arms the displacement will be twice as much, i.e. 1/3". A symmetric weight distribution with zero mass coupling was used for these tests as well as for free vibrations. The spring coefficients for tires and suspension springs were 6.9 lb/in and .82 lb/in respectively. The critical damping ratio is then

$$C_c = 2\sqrt{km} = 2\sqrt{\frac{.82 \times 6.9 \times 12.17}{(.82 + 6.9) \times 2 \times 12 \times 32.2}} = .214$$

In Fig. 33 the relative damping ratio is plotted against the exciting current. The test results coinside reasonably well. The absolute damping coefficient is shown in Fig. 34 as function of the exciting current.

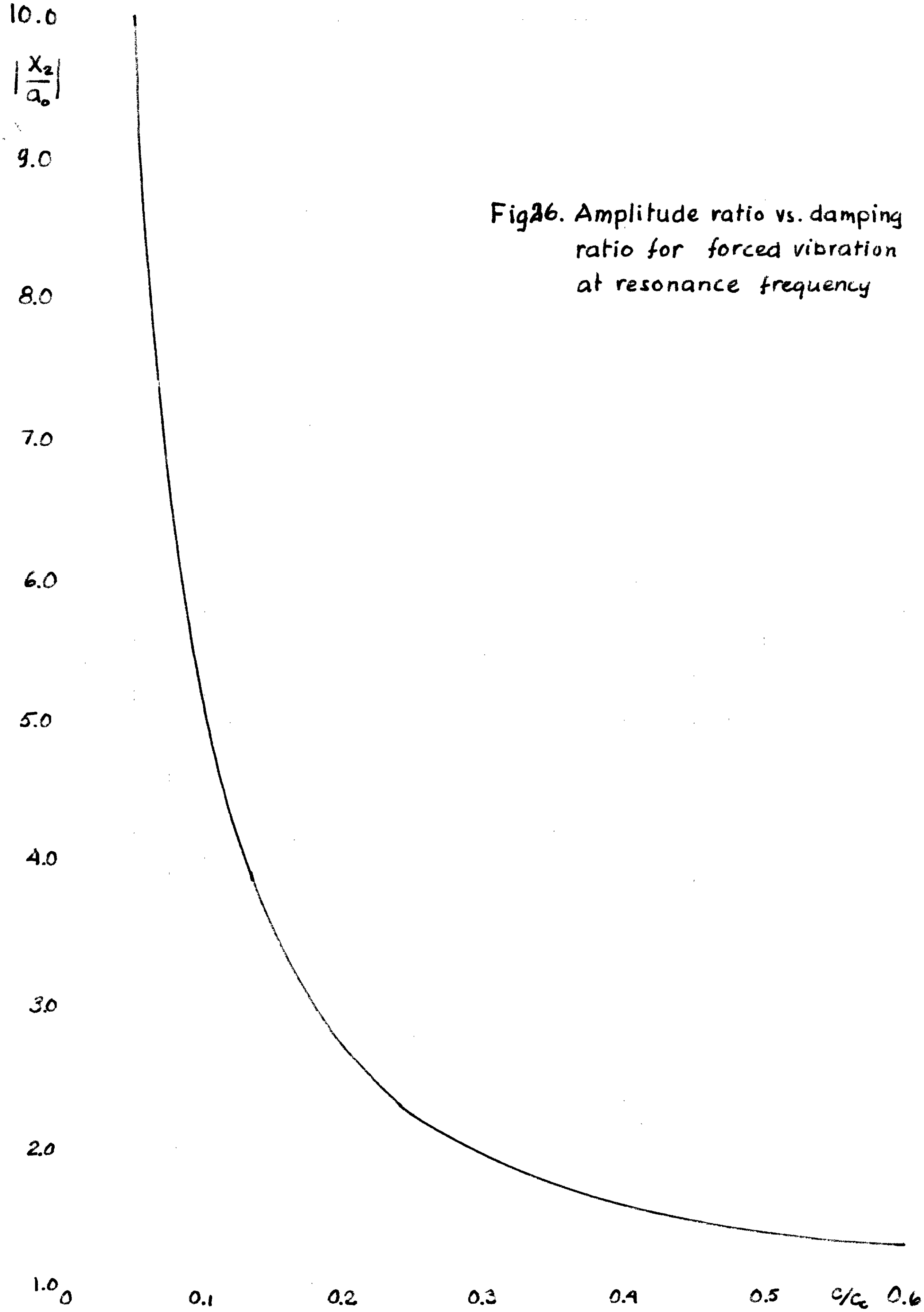
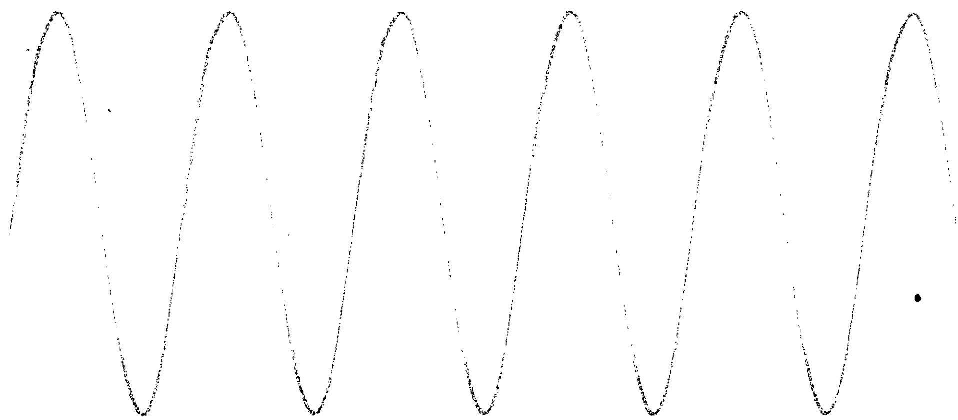
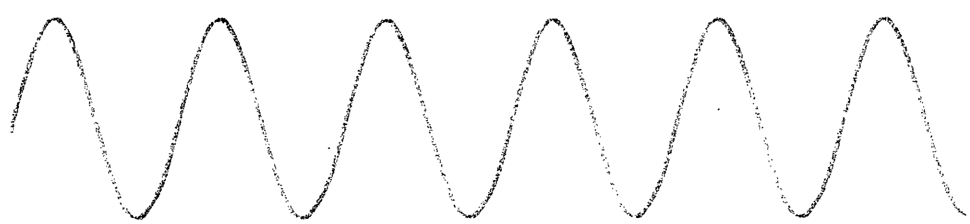


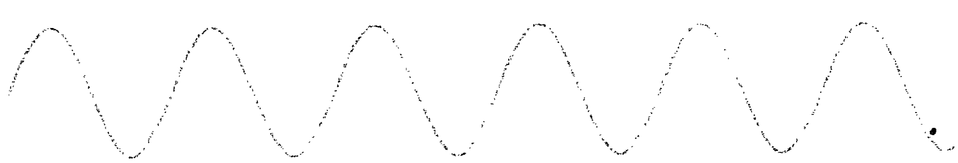
Fig 26. Amplitude ratio vs. damping ratio for forced vibration at resonance frequency



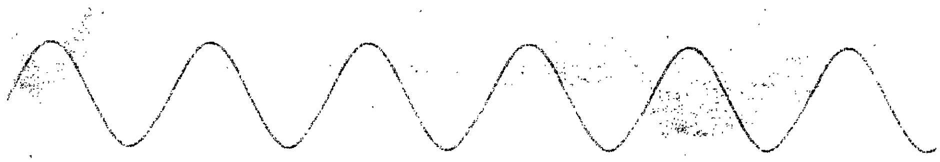
Run 13. I = 0.25 A, C/C_c = 0.08



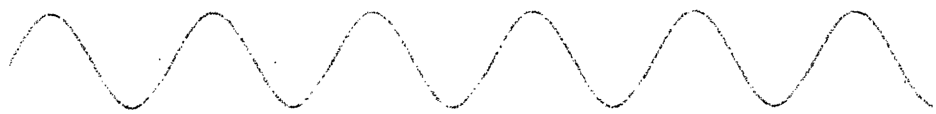
Run 14. I = 0.5 A, C/C_c = 0.175



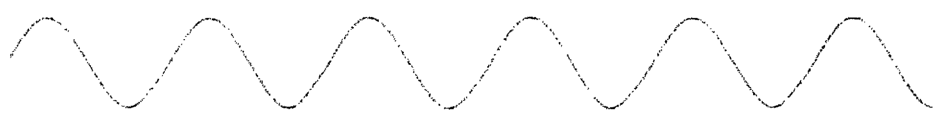
Run 15. I = 0.75 A, C/C_c = 0.28



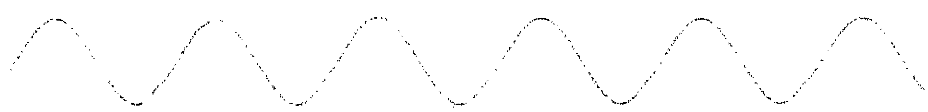
Run 16. I = 1.0 A, C/C_c = 0.35



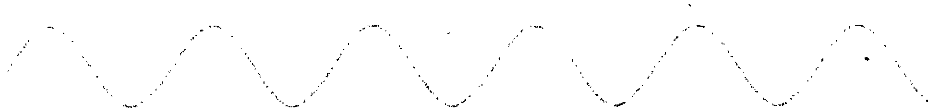
Run 17. I = 1.3 A, C/C_c = 0.40



Run 18. I = 1.7 A, C/C_c = 0.45



Run 19. I = 2.4 A, C/C_c = 0.50



Run 20. I = 3.4 A, C/C_c= 0.55

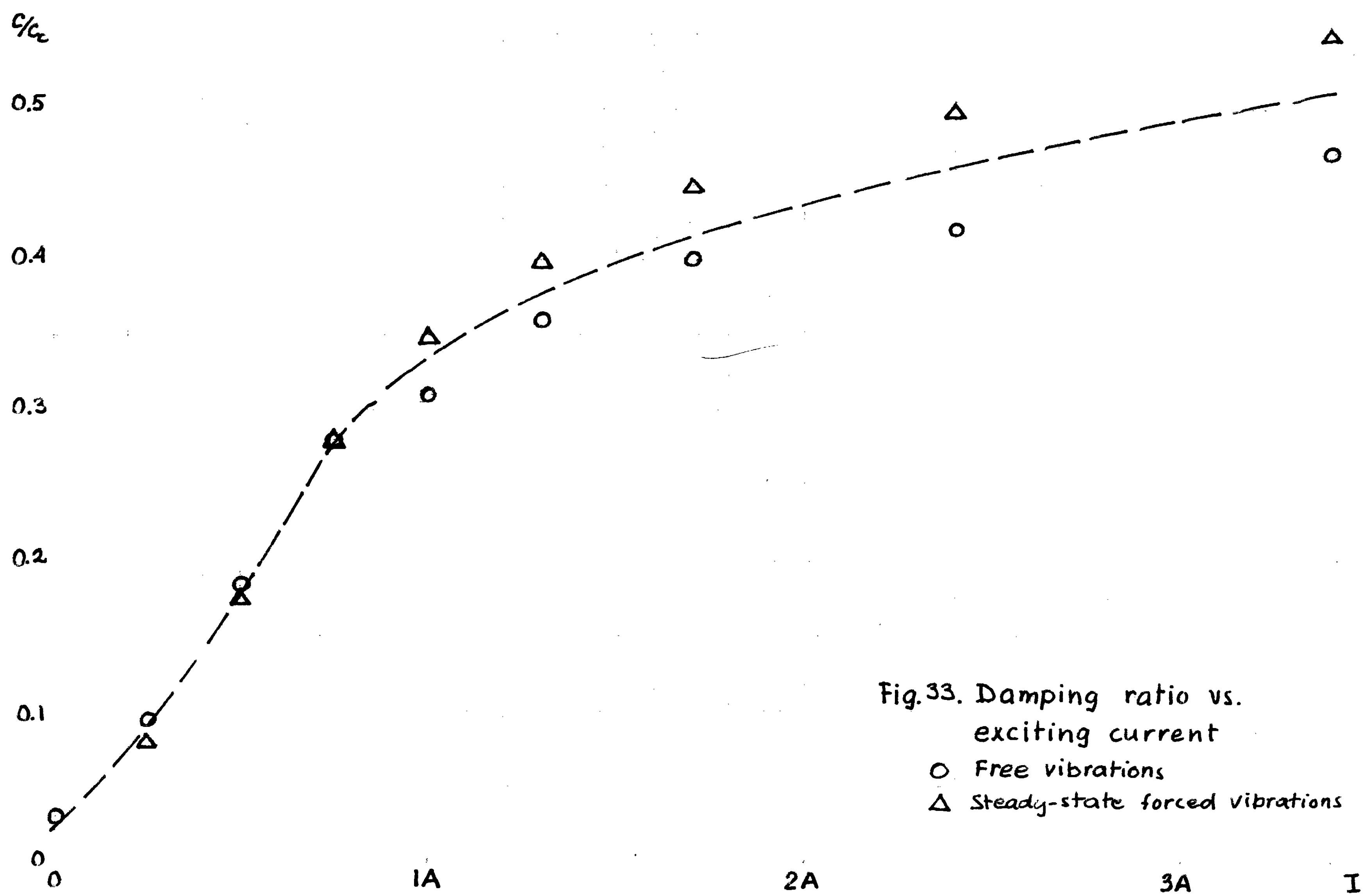


Fig. 33. Damping ratio vs. exciting current

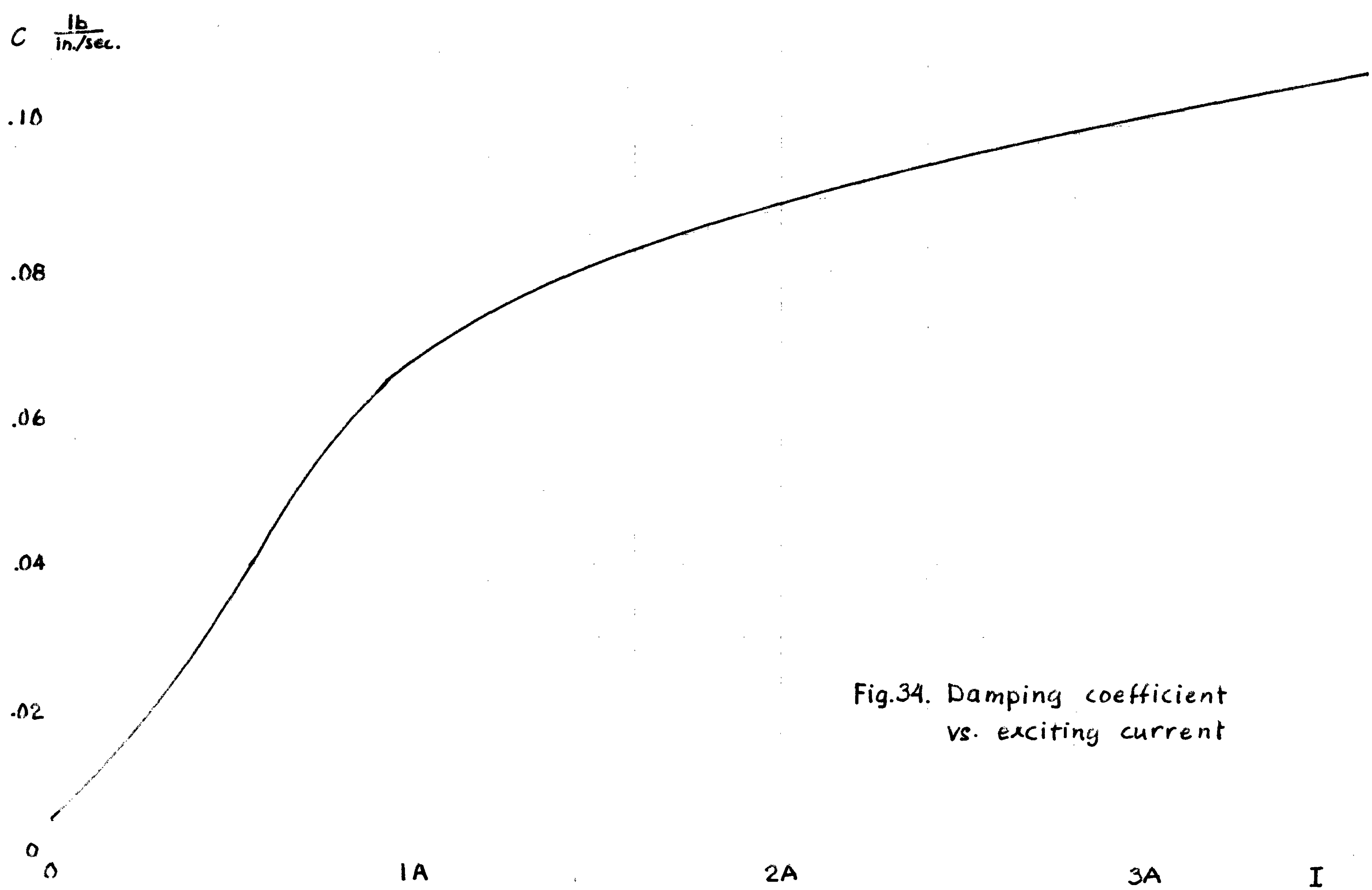


Fig.34. Damping coefficient vs. exciting current

VII. CONCLUSION

If the experimental curve for damping coefficient is compared to the theoretical one (Fig.13) it can be seen that the both curves have same shape. Because of the remaining magnetism the actual curve is moved a little to the left. Although the critical damping was not obtained as calculated, the range of damping is large enough for most purposes.

Vibrational systems may be represented also by electrical analogues because the electrical and the mechanical systems have same form in their differential equations [21]. However, an electrical analogue would not be so clear and descriptive as the analogue designed. With the recording device it shows the motion of the body directly and can be conveniently used for study on problems of motor-vehicle vibration as well as demonstration purposes.

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