Branching From $K$ To $M$ For Split Classical Groups

by

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Abstract

We provide two algorithms to solve branching from $K$ to $M$ for the real split reductive
group of type $A_n$, one inductive and one related to semistandard Young tableaux. The
results extend to branching from $K_e$ to $M \cap K_e$ for the real split reductive groups of
type $B_n$ and $D_n$.

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1 Introduction

Principal series representations play an important and well-developed role in representation theory of reductive algebraic groups. Perhaps much of this importance, certainly much of this development, stems from the relationship between these representations and representations of simpler, better understood classes of groups. The underlying spaces for principal series representations depend only on the restrictions of these representations to compact groups, and the structures of these representations as modules for reductive groups depend on the structures of modules for parabolic subgroups of these reductive groups. In the absence of theory necessary to speak freely of general representations of reductive groups, those categories of representations to which the current state of theory does apply rise somewhat in prominence. Those categories rise both for lack of any more inventive mathematics and, justifiably, in the hope studying such categories will lead to more inventive mathematics.

In order better to understand the decomposition of principal series representations into irreducible components, we turn to the study of branching laws. Specifically, for any real reductive group, we consider branching from $K$, the maximal compact subgroup of that group, to $M$, the intersection of a chosen maximal torus with $K$. Frobenius reciprocity relates branching to the decomposition of principal series. We may gain significant information further narrowing our project to the consideration of branching from the identity component $K_e$ of $K$ to the intersection of $M$ with $K_e$. Our results deal only with branching from $K_e$ to $M \cap K_e$ for the split real reductive groups of classical type.

Kostant’s multiplicity formula solves the problem of branching from real compact connected reductive groups to connected closed subgroups, so this formula seems a natural point of departure for our project. Two problems arise. In general groups $M$ and $M \cap K_e$ tend toward disconnectedness of a nature severe enough to make very difficult or even impossible the task of massaging the situation into compliance with the conditions Kostant’s formula requires. On the other hand, the classical
groups lend themselves to induction, and, making up somewhat for the subgroup’s stubborn disconnectedness, $M$ is finite and abelian for the classical groups. We take, for example, branching from $SO(n, \mathbb{R})$ to $M$ where $M$ is the intersection of some maximal torus for $SL(n, \mathbb{R})$ with $SO(n, \mathbb{R})$. Kostant’s formula solves branching from $SO(n, \mathbb{R})$ to $SO(n-2, \mathbb{R}) \times SO(2, \mathbb{R})$, and for $i$ equal to $n-2$ and 2, we could hope to understand branching from $SO(i, \mathbb{R})$ to $M(SO(i))$ inductively where $M(SO(i))$ is the intersection of some maximal torus for $SL(i, \mathbb{R})$ with $SO(i, \mathbb{R})$. Use of this process would put us in good stead to solve branching from $SO(n, \mathbb{R})$ to $M$. The restriction map from the set of representations for $SO(n, \mathbb{R})$ to the set of representations for $M(SO(n-2)) \times M(SO(2))$ factors through the set of representations for $M$. We would have only to investigate this factorization. The second problem with utilizing Kostant’s formula now presents itself. The combinatorics behind Kostant’s formula, involving as it does Weyl groups and partition functions, can become complicated quite quickly. Kostant’s formula provides a sure solution to a very nonspecific class of problems. The formula need not take advantage of simplifying assumptions pertaining to any specific subclass of these problems. Although Kostant’s formula serves as a foundation for cited results critical to our approach, we eschew the known territory involving direct use of the formula, favoring instead a different tack.

In section 2, we give an overview of the structure theory for split real reductive algebraic groups and we provide definitions central to the discussion of our project. We recall a result from the study of algebraic groups affirming the uniqueness up to isomorphism of any split reductive algebraic group with a particular root datum. Up to isomorphism, then, there exists only one real split reductive group of each classical type. We then discuss briefly the principal series representations and Frobenius reciprocity, clearing the way for our focus on branching from $K_e$ to $M \cap K_e$.

We determine the real split reductive group of type $A_n$ in section 3. Applying
the structure theory from section 2, we determine $K$ and $M$. We go on to classify the irreducible representations of $M$, associating the isomorphism class of each representation to some subset of $\{1, \ldots, n\}$. Finally, we state and prove a lemma equating the multiplicities in any irreducible representation for $K$ of any two irreducible representations for $M$ corresponding to subsets of $\{1, \ldots, n\}$ having the same cardinality.

Section 4 contains a thorough description of the group $K = SO(n, \mathbb{R})$ for the real split reductive group of type $A_n$. We study the Lie algebra for $K$, establishing notation and conventions such as a choice for a Cartan subalgebra and a choice for a set of positive roots. We establish a correspondence between the irreducible representations for any real reductive Lie group and the analytically integral forms on the Cartan subalgebra related to highest weight representations for the Lie algebra. Determining explicitly the set of analytically integral forms for $K$, we then state Murnaghan’s theorem describing the branching law from $SO(n, \mathbb{R})$ to $SO(n-1, \mathbb{R})$. Making use of Murnaghan’s result, we establish an inductive algorithm to solve branching from $K$ to $M$.

The algorithm from section 4 is fairly simple, moreso probably than the algorithm a direct application of Kostant’s theorem might have produced. On the other hand, even this relatively simple recursion appears computationally costly when applied to branching from $K$ to $M$ for the real split reductive group of type $A_n$ when $n$ is large. We search for a computationally simpler algorithm, one more intimately related to the structure of representations for $K = SO(n, \mathbb{R})$, hoping the methods we encounter might have applications to more general cases. We look in section 5 at the relationship between semistandard Young tableaux and representations for $K$. Semistandard Young tableaux serve as a sort of common language for the expression of results concerning representations, especially for matrix groups. These tableaux appear in the
study of combinatorics and algebraic geometry, so representation-theoretic results making use of semistandard Young tableaux stand to benefit from and to benefit theory related to these two fields. We define a subset of the set of semistandard Young tableaux, referring to this subset as the set of admissible semistandard Young tableaux. We categorize admissible semistandard Young tableaux according to depth and to the length of each row, and we subdivide the set of admissible Young tableaux into types corresponding to the highest weights of $K$ representations. Finally, we develop a process for altering, or decorating, these admissible Young tableaux in such a way that the number of decorated admissible semistandard Young tableaux of type corresponding to some highest weight for $K$ equates to the dimension of the representation for $K$ of that highest weight.

The correspondence between admissible semistandard Young tableaux and representations for $K = SO(n, \mathbb{R})$ from section 5 identifies each decorated admissible semistandard Young tableaux of type corresponding to some highest weight for $K$ with a line in the representation of that highest weight. In section 6, we study how the action of $M$ for the real split reductive group of type $A_n$ on this representation affects this set of lines. We use calculations within the universal enveloping algebra for $K$ and the tenets of highest weight theory to study this action. The lines corresponding to decorated admissible semistandard Young tableaux do not, in general, span one-dimensional representations for $M$, but they do span one-dimensional representations for a large subgroup of $M$. The remainder of $M$ acts on these lines in such a way as to make possible the identification of a basis for the highest weight representation comprising vectors each of which spans a one-dimensional representation for all of $M$. By studying the action of $M$ on these vectors, we determine to which isomorphism class belongs the irreducible representation for $M$ spanned by any one of these vectors. Moreover, using the connection between these vectors and the lines
corresponding to decorated admissible tableaux, we establish a bijection between the decorated admissible tableaux and the isomorphism classes of irreducible representations for $M$ such that this bijection determines the multiplicity of any irreducible representation for $M$ within the highest weight representation for $K$.

In section 7, we show how to extend our solution for branching from $K$ to $M$ for the real split reductive group of type $A_n$ to solutions for branching from $K_e$ to $M \cap K_e$ for the real split reductive groups of type $B_n$ and $D_n$. We do not mention the group of type $C_n$, as branching from $K$ to $M$ for this group has a solution in terms of classical theory.

This paper concerns results only about branching for split classical groups, but the methods used in this paper suggest possible approaches to the pursuit of branching theorems is other cases. For instance, an understanding of the branching law from $SP(n, \mathbb{R})$ to $SP(n - 1, \mathbb{R})$ in terms of semistandard Young tableaux could help to determine a branching law from $K$ to $M$ for the covering groups of type $B_n$ and $D_n$. The linear real covering group of type $A_n$ coincides with the real split group, so this paper does comment indirectly on branching for the covering groups of classical type. Calculations within universal enveloping algebras such as those we use to study the action of $M$ on vectors in highest weight representations of $K$ for the split real reductive group of type $A_n$ could help to solve branching from $K$ to $M$ for a broad array of cases. Specifically, the split groups of exceptional type might benefit from careful analysis in terms of universal enveloping algebras and the tenets of highest weight theory.
2 Some Structure Theory for Split Real Reductive Algebraic Groups

The results of this paper focus entirely on the classical groups, but the motivation behind these results stems from somewhat more general theory. We deal first with affine algebraic groups as a context for introducing the notion of a split group. By restricting our attention in the following sections to split groups, we will allow for the exploration of representation theoretic results through the use of root data. Subsequent to our discussion of affine algebraic groups, we develop notation for the discussion of real reductive Lie groups. We show how well-worn theory applying to semisimple Lie groups extends to the reductive case and reduces certain questions concerning principal series representations for real reductive Lie groups to questions concerning branching laws for compact groups.

Letting $k$ be any field, we start with the definition of an affine algebraic $k$-group.

**Definition 2.1** An affine algebraic $k$-group is a functor $G'$ from the category of $k$-algebras to the category of groups such that there exists a finitely generated $k$ algebra $k[G']$ with the property $G'(R) \cong \text{Hom}_{k-alg}(k[G'], R)$ for any $k$-algebra $R$.

We call $k[G']$ the coordinate ring for $G'$.

We may take $k[G'] = k[X_1 \ldots X_n]/I(G')$ where $I(G')$ is some radical ideal in $k[X_1 \ldots X_n]$ and $n$ is some natural number. In this case, the group $G'(k[G'])$ has the structure of a variety, and we can identify $G'$ with $G'(k[G'])$ in order to bridge the gap between the definition for an affine algebraic group given above with the perhaps more familiar definition given in such texts as [2].

We will consider reductive algebraic groups in the entirety of the sequel.

**Definition 2.2** An algebraic group is reductive if it has a trivial unipotent radical.
We now turn to the notion of a split algebraic group. A \textit{split torus} for $G'$ is an algebraic subgroup isomorphic to a direct product of copies of $G_m$ where $G_m$ is the usual multiplicative algebraic group defined over $k$. Considering $G'$ as an affine variety, we may extend scalars for $G'$ to an algebraically closed field as follows. If $ar{k}$ is the algebraic closure of $k$, then we define $G'_\bar{k}$ to be the affine algebraic $\bar{k}$-group with coordinate ring $\bar{k} \otimes_k k[G']$. Since any $\bar{k}$-algebra is a $k$ algebra via the inclusion $k \hookrightarrow \bar{k}$, we see

$$G'_\bar{k}(R) = \text{Hom}_{\bar{k}-\text{alg}}(\bar{k} \otimes_k k[G'], R)$$

is well-defined for any $\bar{k}$-algebra $R$. (In fact, as outlined on [7] p.15, we can extend scalars in this manner to any separable extension of $k$.) Using this extension of scalars, we can define a \textit{torus} for $G'$ to be an algebraic subgroup $T$ of $G'$ such that the extension $T_{\bar{k}}$ is a split torus for $G'_{\bar{k}}$. By [7] 11.31, an algebraic group is reductive if and only any connected normal abelian subgroup is also a torus. Thus, $G'$ is a reductive group if and only if $G'_k$ is a reductive group. A \textit{maximal} torus for $G'$ is a torus $T$ such that $T_{\bar{k}}$ is properly contained in no torus for $G'_{\bar{k}}$.

\textbf{Definition 2.3} A \textit{reductive algebraic group} is split if it contains a split maximal torus.

We restrict our attention to split reductive algebraic groups primarily because the study of representations over such groups reduces to the study of root data. The following definition is [7] 17.1.

\textbf{Definition 2.4} A \textit{root datum} is a quadruple $\Psi = (\Delta, \Delta^\vee, X, X^\vee)$ where $X$ and $X^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality via a pairing $\langle , \rangle: X \times X^\vee \rightarrow \mathbb{Z}$, $\Delta$ and $\Delta^\vee$ are in bijection via a map sending $\alpha$ in $\Delta$ to $\alpha^\vee$ in $\Delta^\vee$ and $(\Delta \times \Delta^\vee)$ is a finite subset of $X \times X^\vee$. In addition, the following properties must hold.

1. We have $\langle \alpha, \alpha^\vee \rangle = 2$. 
2. If \( s_\alpha : X \to X \) is the homomorphism given by

\[
  s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha
\]

for \( x \) in \( X \) and \( \alpha \) in \( \Delta \), we have \( s_\alpha(\Delta) = \Delta \).

3. The group of automorphisms of \( X \) generated by \( \{ s_\alpha \mid \alpha \in \Delta \} \) is finite.

To each split, reductive, affine algebraic group \( G' \) together with a choice \( T \) of a split torus for \( G' \), we associate a root datum \( \Psi(G', T) \). We define \( \Psi(G', T) \) through these four identifications:

\[
  \Delta = \Delta(g', T),
\]

\[
  \Delta^\vee = \{ \text{coroots for } \Delta \},
\]

\[
  X = \text{Hom}(T, G_m), \text{ and}
\]

\[
  X^\vee = \text{Hom}(G_m, T),
\]

where \( g' \) is the Lie algebra for \( G' \) and \( \Delta(g', T) \) are the roots of \( g' \) with respect to \( T \). [7] 17.20 shows the root datum \( \Psi(G', T) \) actually determines \( (G', T) \) up to isomorphism.

If \( A = \bar{k}[X] \) is any affine \( \bar{k} \)-algebra, we follow [8] 1.3.7 to define a \( k \)-structure on \( X \) as a \( k \)-subalgebra \( A_0 \) of \( A \) of finite type over \( k \) such that the map

\[
  k \otimes_k A_0 \to A
\]

given by multiplication of the left and right coordinates is an isomorphism. If \( \tilde{G}' \) is any affine algebraic \( \bar{k} \)-group with coordinate ring \( \bar{k}[X] \), and if \( k[X] \) is a \( k \)-structure for \( \bar{k}[X] \), we define the affine algebraic \( k \)-group \( \tilde{G}'_k \) of \( k \)-rational points for our \( k \)-structure to be the affine algebraic \( k \) group with coordinate ring \( k[X] \). In the sequel, we consider the groups of real points for complex matrix groups, meaning the subgroups
of matrices with entries in \( \mathbb{R} \). Such subgroups are, in fact, \( \mathbb{R} \)-groups of \( \mathbb{R} \)-rational points for complex affine algebraic \( \mathbb{C} \)-groups defined with respect to an obvious \( \mathbb{R} \) structure.

If \( G' \) is a reductive affine algebraic \( \bar{k} \)-group, we wish to comment on the unipotent radical of \( \tilde{G}'_k \). From the definition given above, we know there exists an injective map

\[
\phi: \bar{k}[\tilde{G}'] \to \bar{k} \otimes_k k[\tilde{G}'_k],
\]

thus, we get a surjective homomorphism

\[
\Phi: (\tilde{G}'_k)_{\bar{k}} \to \tilde{G}'.
\]

By [8] 12.4.3, we know the kernel of \( \Phi \) contains no non-trivial, closed, normal sub-
groups of \( (\tilde{G}'_k)_{\bar{k}} \). As a result, if \( (\tilde{G}'_k) \) has a non-toral, normal, abelian subgroup \( N \),
the image of \( N \) under \( \Phi \) is also a non-toral, normal, abelian subgroup. Hence, if \( \tilde{G} \) is
reductive, \( \tilde{G}'_k \) is also reductive.

We now confine our focus from general affine algebraic groups to real affine algebraic groups and encounter the Lie structure serving as a basic framework for this paper. Suppose \( \tilde{G}' \) is a connected, complex, reductive affine algebraic group. Then
the group \( \tilde{G}'_\mathbb{R} \) of \( \mathbb{R} \)-rational points for \( \tilde{G}' \) is a real, reductive affine algebraic group. This group has additional structure. The following definition comes from [5] p.446.

**Definition 2.5** A real reductive Lie group is a 4-tuple \((G, K, \theta, B)\) consisting of a
real Lie group \( G \), a compact subgroup \( K \) of \( G \), a Lie algebra involution for the Lie
algebra \( \mathfrak{g}_0 \) of \( G \), and an \( \text{Ad}(G) \)-invariant, \( \theta \)-invariant, bilinear form \( B \) on \( \mathfrak{g}_0 \) such that

(i) \( \mathfrak{g}_0 \) is a reductive Lie algebra,
(ii) the decomposition of $\mathfrak{g}_0$ into $+1$ and $-1$ eigenspaces under $\theta$ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where $\mathfrak{k}_0$ is the Lie algebra for $K$.

(iii) $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthonormal under $B$, and $B$ is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$.

(iv) multiplication, as a map from $K \times \exp \mathfrak{p}_0$ into $G$, is a diffeomorphism onto, and

(v) every automorphism $\text{Ad}(g)$ of $\mathfrak{g} = (\mathfrak{g}_0)^\mathbb{C}$ is given by some $x$ in $\text{Int} \mathfrak{g}$.

According to [6] p. 245, the group $\tilde{G}^\mathbb{R}$ is a real, reductive Lie group. Henceforth, we shall refer to $\tilde{G}^\mathbb{R}$ as $G$.

For $k$ in $K$ and $X$ in $\mathfrak{p}_0$, we can define $\Theta : G \to G$ as

$$\Theta(k \exp X) = k \exp(-X).$$

By [5] 7.21, we know $\Theta$ is an automorphism of $G$ with differential $\theta$. We refer to $\Theta$ as the global Cartan involution for $G$. From Definition 2.5, iv, we conclude $K$ is the subgroup $G^\Theta$ of $G$ fixed by $\Theta$. Again by Definition 2.5, iv, $K$ is a maximal compact subgroup of $G$.

From Definition 2.5, i, we know $\mathfrak{g}_0 = Z_\mathfrak{g} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ where $Z_\mathfrak{g}$ denotes the center of $\mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is the derived subalgebra of $\mathfrak{g}_0$. In particular, $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple. We consider the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. If $V$ is the underlying space for some representation $\pi$ of $\mathfrak{g}$, then, for any $\alpha$ in $\mathfrak{g}^*$, we let $V_\alpha$ be

$$\{ v \in V \mid \text{ for every } X \in \mathfrak{g} \text{ there exists an } n \in \mathbb{N} \text{ with } ((\pi(X)v - \alpha(X) \cdot 1)^n = 0) \}.$$ 

If $V_\alpha \neq 0$, we call $\alpha$ a weight, and we refer to $V_\alpha$ as the generalized weight space of weight $\alpha$. We define a subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ to be a Cartan subalgebra if the complex-
ifcation \( h \) of \( h_0 \) is precisely the generalized weight space of weight 0 for the adjoint action \( \text{ad}_g h \) of \( h \) on \( g \). Using the theory of semisimple Lie algebras defined over algebraically closed fields, we can form a set of roots \( \Delta([g, g], h \cap [g, g]) \). We extend these roots to all of \( h \) by defining \( \alpha(H) = 0 \) for any \( \alpha \) in \( \Delta([g, g], h \cap [g, g]) \) and any \( H \) in \( Z_g \). This extended set of roots, \( \Delta(g, h) \), allows for a decomposition of \( g \) into root spaces

\[
g = h \oplus \sum_{\alpha \in \Delta(g, h)} g_{\alpha}
\]

where \( g_{\alpha} = \{ X \in g \mid \text{ad}(H)X = \alpha(H)X \} \). If \( h_0 \) is any Cartan subalgebra for \( g_0 \), there exists an element of \( \phi \) of \( \text{Int}(g_0) \) such that \( \phi(h_0) \) is a \( \theta \)-stable Cartan subalgebra. ([5] p.457)

The Cartan subgroup corresponding to a Cartan subalgebra \( h_0 \) of \( g_0 \) is \( \text{Cent}_G(h_0) \), the centralizer in \( G \) of \( h_0 \) via the adjoint action. We choose \( h_0 \) to be \( \theta \)-stable, so the Cartan subgroup \( H \) of \( h_0 \) is \( \Theta \)-stable. ([5] p.487)

**Theorem 2.6** If \( G \) is the group of \( \mathbb{R} \)-rational points of a connected, affine algebraic \( \mathbb{C} \)-group, let \( H \) be a subgroup. The following are equivalent.

(i) The subgroup \( H \) is a Cartan subgroup.

(ii) The subgroup \( H \) is a maximal torus.

*Proof* omitted

We choose \( H \), a \( \Theta \)-stable Cartan subgroup of \( G \). There exists natural action \( \text{Ad}_C(H) \) on \( g \) realized as an extension of \( \text{Ad}_C(h)(X \otimes z) = \text{Ad}(h)(X) \otimes z \) for \( h \) in \( H \) and \( (X \otimes z) \) a simple tensor in \( g \). Since \( H \) centralizes \( h_0 \), and since the root system \( \Delta(g, h) \) depends only on the action of \( h \), we deduce \( \text{Ad}_C(H) \) preserves each weight space \( g_{\alpha} \). Since each weight space has dimension one over \( \mathbb{C} \), the adjoint action of \( H \)
on any given weight space must correspond to a homomorphism from $H$ to $\mathbb{C}$. As a result, to each $\alpha$ in $\Delta(g, h)$ we may assign a character

$$\alpha : H \to \mathbb{C}^\times.$$ 

(We note $\Delta(g_0, H) = \Delta(g, h)$.)

We next define $\Theta \alpha$ for any $\alpha$ in $\Delta(g, h)$ to be the character for $H$ given by

$$\Theta \alpha(h) = \alpha(\Theta(h))$$

for every $h$ in $H$.

**Lemma 2.7** The character $\Theta \alpha$ is among the characters in $\Delta(g, h)$. In other words, $\Theta$ permutes the set $\Delta(g, h)$.

**Proof** For any $X$ in $g_\alpha$, we know

$$\text{Ad}(h)(X) = \alpha(h)X,$$

and

$$\text{Ad}(\Theta h)(\theta X) = \alpha(h)\theta X.$$ 

Letting $h'$ in $H$ be such that $h = \Theta(h')$, we have

$$\text{Ad}(h')(\theta X) = \alpha(\Theta h')(\theta X),$$

so

$$\text{Ad}(h')(\theta X) = \Theta \alpha(h') (\theta X).$$

Since $\theta$ stabilizes $h_0$, we know $\theta$ maps $g_\alpha$ to some other root space $g_\beta$. We may conclude $\beta = \Theta \alpha$. 

Supposing $G$ is split, we choose a $\Theta$-stable, split Cartan subgroup $H$ for $G$. In this case, $H$ is a product of copies of $\mathbb{R}^\times$, and $h_0$ is a product of copies of the additive group $\mathbb{R}$. With $H$ thus chosen, the root space decomposition from Equation 1 for $g$
over \( \mathbb{C} \) restricts to a root space decomposition

\[
g_0 = \mathfrak{h}_0 \oplus \sum_{\alpha \in \Delta(g,\mathfrak{h})} (\mathfrak{g}_\alpha)_0
\]

over \( \mathbb{R} \). For each root \( \alpha \) in \( \Delta(g,\mathfrak{h}) \), we see the corresponding character for \( H \) takes the form

\[
\alpha: H \to \mathbb{R}^\times \subseteq \mathbb{C}^\times.
\]

**Theorem 2.8** If \( G \) is a split real reductive Lie group with global Cartan involution \( \Theta \) and \( H \) is a split \( \Theta \) stable Cartan subgroup for \( G \), then any root \( \alpha \) in \( \Delta(g,\mathfrak{h}) \) has the property \( \Theta \alpha = -\alpha \).

**Proof** By Lemma 2.7, it suffices to show \( \text{ad}(Y)(\theta X) = -\alpha(Y)(X) \) for any \( X \) in \( \mathfrak{g}_\alpha \) and any \( Y \) in \( \mathfrak{h} \). Considering the case \( H = \mathbb{R}^\times \), we notice \( \Theta(h) = h^{-1} \) for each \( h \) in \( H \). Indeed, with \( \Theta \) thus defined, \( \Theta \) is a Cartan involution for the real reductive group \((H,K_H,\theta_H,B_H)\): here \( K_H = \{1,-1\} \), and \( B_H \) is ordinary multiplication. We see \( \mathfrak{h}_0 \cap \mathfrak{t}_0 = 0 \) where

\[
\mathfrak{h}_0 = \mathfrak{h}_0 \cap \mathfrak{t}_0 \oplus \mathfrak{h}_0 \cap \mathfrak{p}(H)_0
\]
is a decomposition of the Lie algebra \( \mathfrak{h}_0 \) for \( H \) taking the form of the decomposition in Definition 2.5, ii. Indeed, \( \mathfrak{h}_0 \cap \mathfrak{t}_0 \) is the Lie algebra for a finite group. Hence, \( \theta_H \) is multiplication by \(-1\). We notice \( \mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{h}_0) \). By [5] 7.25, \( \theta_H \) is the restriction of \( \theta \) to \( Z_{\mathfrak{g}_0}(\mathfrak{h}_0) \). For any \( X \) in \( \mathfrak{g}_\alpha \) and any \( Y \) in \( \mathfrak{h} \), we have

\[
[Y,\theta X] = \theta[\theta Y,X] = \theta(-\alpha(Y)X) = -\alpha(Y)(\theta X).
\]

Since \( H \) is a product of copies of \( \mathbb{R}^\times \), the result follows. \([\blacksquare]\)
We define \(a_0\) to be \(\mathfrak{h}_0 \cap \mathfrak{p}_0\). Additionally, we assign a notion of positivity to the root system \(\Delta(\mathfrak{g}, \mathfrak{h})\), and we define \(n_0\) to be \(\bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha)_0\).

**Theorem 2.9 (Iwasawa Decomposition)** The Lie algebra \(\mathfrak{g}_0\) has a decomposition into the direct sum \(\mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0\). We let \(A\) and \(N\) be the analytic subgroups of \(G\) with Lie algebras \(\mathfrak{a}_0\) and \(\mathfrak{n}_0\) respectively. Then the map \(K \times A \times N \to G\) given by \((k, a, n) \mapsto kan\) is a diffeomorphism. The groups \(A\) and \(N\) are simply connected.

**Proof** [5]6.43 proves the first remark when \(\mathfrak{g}_0\) is semisimple. If we replace [5]6.40(c) with Theorem 2.8, the arguments translate directly to the split real reductive case. The second statement amounts to [5]7.31. We need only note the first statement implies our definition for \(n_0\) matches the one used in [5]7.31 precisely. Since \(A\) is simply connected, exponentiation

\[
\exp : \mathfrak{a}_0 \to A
\]

is a diffeomorphism. For any \(a\) in \(A\), we refer to the preimage of \(a\) as \(\log a\).

**Remark 2.10** [5]7.31 proves Theorem 2.9 for the more general real reductive case. This proof relies on the notion of restricted roots, and that notion proves unnecessary to our discourse.

We define the closed subgroup \(M\) of \(K\) to be \(Z_K(\mathfrak{a}_0)\). Being closed in \(K\), the subgroup \(M\) is compact, and, since \(AN\) is closed in \(G\), the Borel subgroup \(MAN\) is closed in \(G\). (Setting \(B = MAN\), we refer to \(MAN\) as the Langlands decomposition for \(B\) with respect to \(\theta\). We refer to \(M\) as the Langlands subgroup of \(K\).) By Theorem 2.9, we know \(G = KMAN\). For any real reductive Lie group \(L\), we denote by \(\hat{L}\) the set of representations for \(L\). Since \(M\) is compact abelian, any representation \(\delta\) in \(\hat{M}\) is a character

\[
\delta : M \to \mathbb{C}^\times.
\]
As exponentiation gives a diffeomorphism between $A$ and $a_0$, any representation $\nu$ in $\hat{A}$ differentiates to a representation $\nu'$ for $a_0$. The Lie algebra $a_0$ is a maximal abelian subalgebra for $p_0$, so any representation $\nu$ in $\hat{A}$ is one dimensional and corresponds to a character

$$\nu: A \rightarrow \mathbb{R}^\times.$$ 

We denote by $\rho$ the half-sum of positive roots

$$\frac{1}{2} \sum_{\Delta(p,h)^+} \alpha.$$ 

From [4] chapter 7, section 1, we develop a notion of induced representations. [4] handles induced representations for semisimple groups, and we extend the notion to real reductive groups consisting of the real points for semisimple complex groups. To each pair $(\delta, \nu)$, if $V^\delta$ is the underlying space for the irreducible representation $\delta$ in $\hat{M}$ and $\nu$ is in $\hat{A}$, we associate an induced representation for $G$

$$\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$$

defined as follows. If

$$W: = \{ F: G \rightarrow V^\delta \text{continuous} \mid F(x\text{man}) = e^{-(\nu'+\rho)\log a^\delta(m)^{-1}F(x)} \},$$

we provide $W$ with the $L^2$ norm over $K$:

$$\|F\|^2 = \int_K |F(k)|^2.$$ 

The underlying space for $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$ is the completion $\hat{W}$ of $W$ under the $L^2$ norm, so $W$ is dense in $\hat{W}$. We may define an action of $G$ on $\hat{W}$ via continuous
extension of the action

\[ g F(x) = F(g^{-1} x) \]
on \( W \). We refer to

\[ \{ \text{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes 1) \mid \delta \in \hat{M}, \nu \in \hat{A} \} \]
as the principal series representations for \( G \). (In the expression \( \text{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes 1), 1 \) refers to the trivial representation of \( N \).) Clearly, for any \( F \) in \( W \), the restriction \( F|_K \) completely determines \( F \). Also, since \( G = KMAN \), the representations \( \text{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes 1) \) correspond one-to-one with the representations \( \text{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes 1)|_K \). This observation allows us to study principal series representations using theory concerning representations of compact groups. For the real split reductive group of type \( A_n \), the maximal compact subgroup \( K \) is connected. For types \( B_n \) and \( D_n \), the subgroup \( K \) is not connected. According to [5] 7.33, \( M \) meets every component of \( K \). We denote the identity component of \( K \) by \( K_e \) and we consider the closed subgroup \( M \cap K_e \) of \( K_e \). Since \( M \) meets every component of \( K \), we can write any element of \( K \) as \( k_e m \) for some \( k_e \) in \( K_e \) and some \( m \) in \( M \). As such, the restriction \( F|_{K_e} \) completely determines \( F \) for any \( F \) in \( W \). Furthermore, \( G = K_e MAN \), so the representations \( \text{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes 1) \) correspond one-to-one with the representations \( \text{Ind}_{MAN}^{G}(\delta \otimes \nu \otimes 1)|_{K_e} \). Our attentions restrict further to the study of representations over compact connected groups.

The theory of representations over compact groups allows for its own notion of induced representations. We use the description provided in [1] chapter 3, section 6. If \( M \cap K_e \) in \( K_e \) is any closed subgroup of the compact connected group \( K_e \), and if \( \delta \) is a representation for \( M \cap K_e \) with underlying space \( V^\delta \), we define \( \text{Ind}_{M \cap K_e}^{K_e}(\delta) \) as follows. We set

\[ W: = \{ F: K_e \to V^\delta \text{continuous} \mid F(km) = \delta(m)^{-1}F(k) \text{ for } k \in K_e \text{ and } m \in M \cap K_e \}, \]
and we endow $W$ with an action of $K_e$ via

$$kF(x) = F(k^{-1}x)$$

for $k \in K_e$.

Evidently,

$$\text{Ind}_{M \cap K_e}^{G} \delta \otimes \nu' \otimes 1|_{K_e} = \text{Ind}_{M \cap K_e}^{K_e} \delta.$$  

**Theorem 2.11 (Frobenius Reciprocity)** If $\delta$ is an $M \cap K_e$ module and $\mu$ is a $K_e$ module, there is a canonical isomorphism

$$\text{Hom}_{K_e}(\mu, \text{Ind}_{M \cap K_e}^{K_e} \delta) \cong \text{Hom}_{M \cap K_e}(\mu|_{M \cap K_e}, \delta).$$

**Proof** [1] 6.2 proves this theorem for general compact groups $K_e$ with closed subgroups $M \cap K_e$. If we consider an irreducible $K_e$ module $\mu$ and an irreducible $M \cap K_e$ module $\delta$, then 2.11 tells us the multiplicity of $\mu$ in $\text{Ind}_{M \cap K_e}^{K_e} \delta$ is exactly the multiplicity of $\delta$ in $\mu|_{M \cap K_e}$.

With this structure theory in mind, studying the branching law over $M \cap K_e$ for irreducible representations of $K_e$ becomes a meaningful project. Any information shedding light on this branching law sheds light on the decomposing restrictions of principal series representations for real reductive groups into irreducible representations.

### 3 The Split Real Group of Type $A_n$

Eventually, we will comment on principal series representations for every split real classical group through studying these representations for the split real group of type $A_n$. As usual, we define $GL(n, \mathbb{R})$ to be the set of $n$ by $n$ matrices with nonzero
determinant and with entries in \( \mathbb{R} \) while we define the group \( SL(n, \mathbb{R}) \) to be

\[
SL(n, \mathbb{R}) = \{ g \in M(n, \mathbb{R}) \mid \det g = 1 \}.
\]

As the differential of the determinant morphism \( \det: GL(n, \mathbb{R}) \to GL(n, \mathbb{R}) \) is trace, we see \( SL(n, \mathbb{R}) \) has Lie algebra \( \mathfrak{sl}(n, \mathbb{R}) \) where

\[
\mathfrak{sl}(n, \mathbb{R}) = \{ X \in M(n, \mathbb{R}) \mid \text{tr} X = 0 \}.
\]

The algebra \( \mathfrak{sl}(n, \mathbb{R}) \) has as a spanning set

\[
\{ X_{i,j}, Y_{i,j}, H_{i,j} \mid i < j \leq n \}
\]  

where the entries for the basis vectors in the \( i \)th and \( j \)th rows and columns take the form

\[
X_{i,j} = i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_{i,j} = i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_{i,j} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and all other entries take the value 0.

We know \( \mathfrak{sl}(n, \mathbb{R}) \) is semisimple, hence this algebra has a nondegenerate Killing form

\[
B(X, Y) = \text{tr}(\text{ad} X \text{ ad} Y).
\]

Considering the involution \( \theta \) on \( \mathfrak{sl}(n, \mathbb{R}) \) given by taking the negative transpose of each element, we wish to see this involution is a Cartan involution for \( B \). We know \( B \) is symmetric because \( \text{tr}(xy) = \text{tr}(yx) \) for any endomorphisms \( x \) and \( y \) of a finite-
dimensional vector space. Certainly, \( \theta \) respects multiplication, as

\[
\]

For any automorphism \( \phi \) of an arbitrary semisimple, finite-dimensional Lie algebra \( g \) over \( \mathbb{C} \) or \( \mathbb{R} \), we note

\[
[\phi X, Y] = \phi[X, \phi^{-1} Y] = \phi(\text{ad } X) \phi^{-1} Y
\]

for any \( X \) and \( Y \) in \( g \) ([5] 1.118) As a result, the arbitrary automorphism \( \phi \) holds the Killing form \( B_g \) for \( g \) invariant:

\[
B_g(\phi X, \phi Y) = \text{tr}(\text{ad}(\phi X) \text{ ad}(\phi Y))
\]

\[
= \text{tr}(\phi(\text{ad } X) \phi^{-1} \phi(\text{ad } Y) \phi^{-1})
\]

\[
= \text{tr}(\text{ad } X \text{ ad } Y)
\]

\[
= B_g(X, Y).
\]

([5] 1.119) If we define the inner product \( \langle \cdot, \cdot \rangle \): \( \mathfrak{sl}(n, \mathbb{R}) \to \mathbb{R} \) given by \( \langle X, Y \rangle = \text{tr}(X \iota Y) \), we notice

\[
\langle (\text{ad } Y)X, Z \rangle = \langle X, (\text{ad } \iota Y)Z \rangle.
\]

Indeed, the equations

\[
\iota \iota[Y, Z] = -\iota[Z, \iota Y] = -[Y, \iota Z] \quad \text{and}
\]

\[
\text{tr}(X, -[\iota Y, Z]) = \text{tr}(-[X, \iota Y], \iota Z) = \text{tr}([Y, X], \iota Z)
\]

hold, so we conclude

\[
\text{tr}([Y, X], \iota Z) = \text{tr}(X, \iota \iota[Y, Z]).
\]
As a result, we see
\[ \text{ad}(\,\!^t Y) = \,^t (\text{ad} Y). \]

Clearly, \( \mathfrak{sl}(n, \mathbb{R}) \) has a decomposition
\[ \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \]

where \( \mathfrak{k}_0 \) is the +1 and \( \mathfrak{p}_0 \) is the −1 eigenspace for \( \theta \). In fact, if \( X \) is in \( \mathfrak{sl}(n, \mathbb{R}) \), we can write \( X \) as a sum of unique elements from \( \mathfrak{k}_0 \) and \( \mathfrak{p}_0 \) in the following manner:
\[
X = \frac{1}{2}(X - \,^t X) + \frac{1}{2}(X + \,^t X).
\]

Now the form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{sl}(n, \mathbb{R}) \) has the property
\[
\langle X, X \rangle = -B(X, \theta X) = B(X, \,^t X)
= \text{tr}((\text{ad} X)(\text{ad} \,^t X)) = \text{tr}((\text{ad} X)(\,^t \text{ad} X)) \geq 0.
\]

([5] p.355) Since \( \langle \cdot, \cdot \rangle \) is positive definite, \( B \) is positive definite on \( \mathfrak{p}_0 \) and negative definite on \( \mathfrak{k}_0 \). By [5] 6.31, the analytic subgroup \( K \) of \( SL(n, \mathbb{R}) \) having \( \mathfrak{k}_0 \) as its Lie algebra is compact, and the mapping \( K \times \mathfrak{p}_0 \to SL(n, \mathbb{R}) \) given by \( (k, X) \to k \exp X \) is a diffeomorphism. As \( SL(n, \mathbb{R}) \) is semisimple and connected, \( \text{Ad}(SL(n, \mathbb{R})) = \text{Int}(\mathfrak{g}_0) \subset \text{Int}(\mathfrak{g}) \). We have shown \( (SL(n, \mathbb{R}), K, \theta, B) \) satisfies Definition 2.5.

From our definition for \( \theta \), we know
\[
\mathfrak{k}_0 = \{ X \in \mathfrak{sl}(n, \mathbb{R}) \mid X + \,^t X = 0, \text{ and } \text{tr} X = 0 \}.
\]

Hence, \( \mathfrak{k}_0 = \mathfrak{so}(n, \mathbb{R}) \). The analytic subgroup of \( SL(n, \mathbb{R}) \) having \( \mathfrak{k}_0 \) as its Lie algebra
is
\[ K = SO(n, \mathbb{R}) = \{ g \in GL(n, \mathbb{R}) \mid g^t g = 1, \text{ and } \det g = 1 \}. \]

The group \( K \) is connected, so \( K = K_e \). We examine the maximal abelian subalgebra \( \mathfrak{h}_0 \) for \( \mathfrak{s}(n, \mathbb{R}) \) spanned by the elements \( \{ H_{i,j} \mid 1 \leq i < j \leq n \} \). Clearly, \( \mathfrak{h}_0 = \mathfrak{s}(n, \mathbb{R}) \cap D(n, \mathbb{R}) \) where \( D(n, \mathbb{R}) \) denotes the diagonal matrices in \( \mathfrak{g}(n, \mathbb{R}) \). The diagonal entries in the elements of \( \mathfrak{h}_0 \) have only one restriction, namely, the sum of these entries must equal 0. The Cartan subgroup \( H \) of \( SL(n, \mathbb{R}) \) having \( \mathfrak{h}_0 \) as its Lie algebra has the defining property

\[ H = \{ g \in D(n, \mathbb{R}) \mid \prod_{1 \leq i \leq n} g_i = 1 \} \]

where \( g_i \) is the \( i^{th} \) diagonal entry of \( g \). The group \( H \) is an \( n-1 \)-dimensional split torus for \( SL(n, \mathbb{R}) \). Moreover, \( \Theta \) stabilizes \( H \) where \( \Theta \), the global Cartan involution for \( SL(n, \mathbb{R}) \) must act as \( \Theta g = (g^t)^{-1} \) by our definition for \( \Theta \). The group \( M = H \cap K \) is the group of self-inverse diagonal matrices with determinant 1. Thus, \( M \) has the description

\[ M = \{ g \in D(n, \mathbb{R}) \mid g_{i,i} = \pm 1, \text{ and } \prod_{1 \leq i \leq n} g_{i,i} = 1 \}. \]

Being a finite (hence compact) group, \( M \) has an easily manageable set of irreducible representations \( \hat{M} \). Since \( M \) is abelian, any (complex) irreducible representation \( V \) has dimension 1. We suppose the representation \( V \) has character \( \chi_V : M \to \mathbb{C} \). Now \( \chi_V(m) = \operatorname{tr}(l_m) \) where \( l_m \) is the endomorphism of \( V \) given by the action of \( m \) in \( M \). Every element of \( M \) is self inverse, so \( \chi_V(m) = \pm 1 \) for any \( m \) in \( M \). In particular, \( \chi_V(m) \) is a real number for any \( m \) in \( M \), so any real irreducible representation for \( M \) has dimension 1. There exist only finitely many such characters for \( M \), hence, up to isomorphism, there exist only finitely many irreducible representations. For any
subset $S$ of $\{1, 2, \ldots, n\}$, we define an irreducible character $\chi_S$ for $M$ by

$$\chi_S(g) = \prod_{i \in S} g_i i.$$  

**Lemma 3.1** Every homomorphism $\chi: M \to \pm 1$ is isomorphic to $\chi_S$ for some subset $S$ of $\{1, 2, \ldots, n\}$.

**Proof** For $n = 1$, the statement is obvious. We proceed by induction. For any $j$ with $1 \leq j < n$, we define $m^j$ to be the element of $M$ with $-1$ in the $j^{th}$ and $j + 1^{st}$ diagonal entries and $1$ in all other diagonal entries. Then $\{m^j \mid 1 \leq j < n\}$ generate $M$. (Clearly the set of all elements of $M$ with exactly two entries equal to $-1$ generates all of $M$. If $m^{i,j}$ is such an element of $M$ with $i^{th}$ and $j^{th}$ diagonal entries equal to $-1$ where $i < j$, we see $m^{i,j} = m^i \cdot m^{i+1} \ldots \cdot m^{j-1}$.) If $M(n-1)$ is the subgroup of $M$ with $n, n^{th}$ entries equal to $1$, then, by induction, there exists a subset $S'$ of $\{1, 2, \ldots, n-1\}$ such that $\chi|_{M(n-1)} = \chi_{S'}$. If $\chi(m^{n-1}) = -1$, then let

$$S = \begin{cases} S' & \text{if } n - 1 \in S', \\ S' \cup n & \text{otherwise.} \end{cases}$$

If $\chi(m^{n-1}) = 1$, then let

$$S = \begin{cases} S' \cup n & \text{if } n - 1 \in S', \\ S' & \text{otherwise.} \end{cases}$$

The character $\chi_S$ agrees with $\chi$ on a generating set for $M$, so the two homomorphisms must agree on all of $M$.

In fact, we can describe completely $\hat{M}$. We use the notation $[i/j]$ for $i$ and $j$ in $\mathbb{Z}$ to denote the integer closest to $i/j$ with absolute value less than or equal to $i/j$. To
indicate the cardinality of a set $S$, we write $|S|$. We define the set $F_n$ as follows:

$$F_n: = \{ S \subset \{1, 2, \ldots, n\} \mid |S| \leq [n - 1/2] \}$$

$$\cup \{ S \subset \{1, 2, \ldots, n\} \mid |S| = n/2, \text{ and } 1 \in S \}.$$

**Theorem 3.2** The irreducible characters \\

$$\{ \chi_S \mid S \in F_n \}.$$ \\

represent each of the isomorphism classes of representations in $\hat{M}$.

**Proof** By Lemma 3.1, we know every irreducible character for $M$ is $\chi_S$ for some subset $S$ of $\{1, 2, \ldots, n\}$. We know $M \subset SL(n, \mathbb{R})$, so, for each $m$ in $M$ and each choice for $S$, we have

$$\prod_{j \in S} m_j = \prod_{i \in S^C} m_i,$$

where $S^C$ is the complement of $S$ in $\{1, 2, \ldots, n\}$. As such, we conclude $\chi_S = \chi_{S^C}$ for any subset $S$. Taking complements, we find any irreducible character $\chi$ for $M$ is $\chi_S$ for some subset $S$ in the set $F_n$. It remains to see $\chi_S \neq \chi_{S'}$ if $S \neq S'$ and each of $S$ and $S'$ are elements of the set $F_n$. Clearly, there exists an integer $j$ with $1 \leq j \leq n$ such that $j$ is neither an element of $S$ nor an element of $S'$. Also, the symmetric difference $S \oplus S'$ of $S$ and $S'$ must contain an integer $i$ with $1 \leq i \leq n$. Swapping the roles of $S$ and $S'$ if necessary, we suppose $i$ is an element of $S$ but not an element of $S'$. Using notation from the previous lemma, we consider the element $m^{i,j}$ of $M$. The equations $\chi_S(m^{i,j}) = -1$ and $\chi_{S'}(m^{i,j}) = 1$ both hold, so $\chi_S \neq \chi_{S'}$. 

In much of the sequel, we will examine irreducible representations of $SO(n, \mathbb{R})$, attempting to decompose such representations into sums of irreducible representations $\chi_S$ for $M$. (Here, we see representations of $SO(n, \mathbb{R})$ as representations of $M$ via
inclusion: $M \hookrightarrow SO(n, \mathbb{R})$. The following result will prove useful. If $p$ is any permutation on $n$ elements, we define the $n \times n$ permutation matrix $g_p$ corresponding to $p$ as the result of permuting the rows of the identity matrix via $p$. If $p$ is an even permutation, the permutation matrix corresponding to $p$ has determinant 1 whereas the permutation matrix corresponding to an odd permutation has determinant $-1$.

**Theorem 3.3** If $S$ and $S'$ are two elements of the set $F_n$ having the same cardinality, then the multiplicity of $\chi_S$ in $\delta|_M$ is equal to the multiplicity of $\chi_{S'}$ in $\delta|_M$ for any irreducible representation $\delta$ of $SO(n, \mathbb{R})$.

**Proof** We consider $S \ominus S'$. Choosing some bijection $\phi$ between $S \cap (S \ominus S')$ and $S' \ominus (S \ominus S')$, we define the permutation $t_i$ of the set $\{1, 2, \ldots, n\}$ to be the transposition of $i$ and $\phi(i)$. The permutation $p$ given by

$$p = \prod_{i \in S \cap (S \ominus S')} t_i$$

maps $S$ to $S'$.

First we suppose $p$ is an even permutation, so $g_p$ is an element of $SO(n, \mathbb{R})$. If $V^S$ is the underlying space for a copy of $\chi_S$ in $\delta|_M$, we wish to see $g_p^{-1}V^S$ is a copy of $\chi_{S'}$. We choose some $v$ in $V^S$. By the definition of $g_p$, conjugation of any element $m$ of $M$ by $g_p$ permutes the diagonal entries of $m$ via the permutation $p$. We know $g_v$ generates a copy of $\chi_{S'}$ for some element $S'$ of $F_n$. The equations

$$(g_p m g_p^{-1}) v = (g_p m)(g_p^{-1} v)$$

$$= g_p (\chi_{S'}(m)) (g_p^{-1} v)$$

$$= (\chi_{S'}(m)) g_p (g_p^{-1} v)$$

$$= (\chi_{S'}(m)) v$$
certainly hold. For any \( m \) in \( M \), if \( \chi_{S'}(m) = \epsilon \), then \((g_p m g_p^{-1})v = \epsilon v\) because the permutation \( p \) maps \( S \) to \( S' \). We conclude \( S^* = S' \).

Furthermore, if \( V_1^S, \ldots, V_r^S \) are the underlying spaces for linearly independent copies of \( \chi_S \) in \( \delta|_M \), then \( g_p^{-1}V_1^S, \ldots, g_p^{-1}V_r^S \) are underlying spaces for linearly independent copies of \( \chi_{S'} \). We get an injective map \( \lambda \) from the set of copies of \( \chi_S \) in \( \delta|_M \) to the set of copies of \( \chi_{S'} \delta|_M \). Reversing the roles of \( S \) and \( S' \) in the arguments above yields an injective map in the reverse direction, so \( \lambda \) is a bijection.

We now suppose \( p \) is an odd permutation. Then \( g_p \) is not an element of \( SO(n, \mathbb{R}) \), but

\[
\tilde{g}_p = \begin{pmatrix}
-1 & & \\
& 1 & \\
& & \ddots \\
& & & 1
\end{pmatrix}
\]

is an element of \( SO(n, \mathbb{R}) \). It is easy to see conjugation of any element \( m \) in \( M \) by \( \tilde{g}_p \) permutes the diagonal entries of \( m \) via the permutation \( p \), so we may apply the arguments above replacing \( g_p \) with \( \tilde{g}_p \).

\section{An Inductive Method to Determine Branching}

\textbf{From} \( SO(n, \mathbb{R}) \) \textbf{to} \( M \)

Using highest weight theory and a thorough understanding of the branching law from \( so(n, \mathbb{C}) \) to \( so(n - 1, \mathbb{C}) \), it is possible to determine theoretically the branching law for \( SO(n, \mathbb{R}) \) over \( M \).

To begin this section, we establish some conventions regarding the root system for \( so(n, \mathbb{C}) \). These conventions come verbatim from [5] chapter II, section 1. For \( n = 2k + 1 \), we choose a maximal abelian subalgebra \( \mathfrak{h} \) to be the algebra of matrices
such that
\[
H = \begin{pmatrix}
(0 & \text{i}h_1 \\
-\text{i}h_1 & 0
\end{pmatrix}
\begin{pmatrix}
(0 & \text{i}h_2 \\
-\text{i}h_2 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
(0 & \text{i}h_k \\
-\text{i}h_k & 0
\end{pmatrix}
0
\end{pmatrix}.
\] (3)

For \( n = 2k \), we choose \( \mathfrak{h} \) to be the algebra of matrices \( H \) such that
\[
H = \begin{pmatrix}
(0 & \text{i}h_1 \\
-\text{i}h_1 & 0
\end{pmatrix}
\begin{pmatrix}
(0 & \text{i}h_2 \\
-\text{i}h_2 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
(0 & \text{i}h_k \\
-\text{i}h_k & 0
\end{pmatrix}
0
\end{pmatrix}.
\] (4)

In both cases, \( h_j \) takes an arbitrary value in \( \mathbb{C} \) for each \( j \) such that \( 1 < j < k \). Taking either maximal torus \( H \) described above, we make the following definitions:

\[
\mathfrak{h}_0 : = \{ H \in \mathfrak{h} \mid h_j \in \mathbb{R} \text{ for } 1 \leq j \leq k \}, \text{ and}
\]
\[
e_j : = \text{the element of } H^* \text{ such that } e_j(H) = h_j \text{ for } 1 \leq j \leq k.
\]
Furthermore, for $n = 2k + 1$, we define
\[ \Delta: = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq k \} \cup \{ \pm e_l \mid 1 \leq l \leq k \}, \]
whereas for $n = 2k$, we define
\[ \Delta: = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq k \}. \]

We let $\alpha$ in $\mathfrak{h}^*$ be $\pm e_i \pm e_j$ for $i$ and $j$ such that $1 \leq i < j \leq k$, and we define each entry of $E_\alpha$ to be 0 except for the entries with indices among the $i^{th}$ and $j^{th}$ pairs of indices. (Those entries in either of the $2i - 1^{th}, 2i^{th}, 2j - 1^{th},$ or $2j^{th}$ columns and in either of the $2i - 1^{th}, 2i^{th}, 2j - 1^{th},$ or $2j^{th}$ rows.) The remaining sixteen entries take values according to the following equation:

\[ E_\alpha = \begin{pmatrix} 0 & X_\alpha \\ -i X_\alpha & 0 \end{pmatrix} \tag{5} \]

where
\[ X_{e_i-e_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad X_{e_i+e_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \]
\[ X_{-e_i+e_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \text{ and } X_{-e_i-e_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \]

If $n = 2k + 1$, we also consider $\alpha = \pm e_l$ for $l$ such that $1 \leq l \leq k$. With $\alpha$ as described, we let each entry of $E_\alpha$ equal 0 with the exception of the two entries in the $n^{th}$ row and the $2l - 1^{st}$ or $2l^{th}$ column and the two entries in the $n^{th}$ column and the $2l - 1^{st}$ or $2l^{th}$ row. These four entries take values as described by the following
equations:

\[
E_{e_l} = \begin{pmatrix} 2l - 1 & 0 & 0 \\ 2l & 0 & -i \\ 2k + 1 & -1 & i \end{pmatrix}, \quad \text{and}
\]

\[
E_{-e_l} = \begin{pmatrix} 2l - 1 & 0 & 0 \\ 2l & 0 & i \\ 2k + 1 & -1 & -i \end{pmatrix}.
\]

For each choice of \( \alpha \) in \( \Delta \), the matrix \( E_\alpha \) spans a space of weight \( \alpha \) for the adjoint action of \( \mathfrak{h} \) on \( \mathfrak{so}(n, \mathbb{C}) \). In particular, each \( \alpha \) in \( \Delta \) is a root. Moreover,

\[
\mathfrak{so}(n, \mathbb{C}) = \mathfrak{h} \oplus_{\alpha \in \Delta} \mathbb{C}E_\alpha,
\]

so \( \Delta \) describes the full set of roots for \( n = 2k + 1 \) and for \( n = 2k \). In accordance with the theory of abstract root systems, we choose sets of positive roots within the sets of roots. For \( n = 2k + 1 \), we define

\[
\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq k\} \cup \{e_l \mid 1 \leq l \leq k\},
\]

and for \( n = 2k \), we define

\[
\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq k\}.
\]

We briefly discuss integral forms for a compact connected Lie group \( G \) with max-
imal torus $H$. Writing $\mathfrak{h}_0 \subset \mathfrak{g}_0$ for the Lie algebras of $H \subset G$, we refer to the complexifications of these lie algebras as $\mathfrak{h} \subset \mathfrak{g}$. We write $\Delta$ to denote the root system $\Delta(\mathfrak{g}, \mathfrak{h})$, and we refer to one particular choice of positive roots as $\Delta^+$. For complex Cartan subalgebra $\mathfrak{h}$ of a reductive Lie algebra $\mathfrak{g}$, we suppose $\mathfrak{g}$ has root system $\Delta$ with respect to $\mathfrak{h}$, and we select a set $\Delta^+$ of positive roots. If the compact connected Lie group $G$ has $\mathfrak{g}_0$ as its Lie algebra, and if the complexification of $\mathfrak{g}_0$ is $\mathfrak{g}$, we formulate the subsequent definitions. An element $\lambda$ of $\mathfrak{h}^*$ in the subspace of $\mathfrak{h}^*$ spanned by $\Delta$ is dominant if $\langle \lambda, \alpha \rangle \geq 0$ for each $\alpha$ in $\Delta^+$. Here, the pairing $\langle , \rangle$ takes its definition from the identity

$$\langle \alpha, \beta \rangle = \frac{2\langle \alpha, \beta \rangle}{(\beta, \beta)}$$

for any $\alpha$ and $\beta$ in $\Delta$ where $(\ , \ )$ is the given positive definite symmetric bilinear form on the space spanned by $\Delta$. The form $\lambda$ is algebraically integral if $\langle \lambda, \alpha \rangle$ is an integer for each $\alpha$ in $\Delta$. If there exists a multiplicative character $\xi_\lambda$ of $H$ such that $\xi_\lambda(\exp X) = e^{\lambda(X)}$ for each $X$ in $\mathfrak{h}_0$, the form $\lambda$ is analytically integral. According to [5] 4.58, a form is analytically integral if and only if whenever $\exp(X) = 1$ for some $X$ in $\mathfrak{h}_0$, we have $\lambda(X) = 2\pi ia$ for some $a$ in $\mathbb{Z}$. Any analytically integral form is, in particular, algebraically integral by [5] 4.59.

The following points come from [5] p.254. Being compact, $G$ is a reductive group, and we get a root decomposition for $\mathfrak{g}$ of the form given by Equation 1. Now $\text{Ad}(H)$ acts by orthogonal transformations on $\mathfrak{g}_0$ with respect to the symmetric bilinear form $B$ in Definition 2.5. Extending $B$ to a Hermitian inner product on $\mathfrak{g}$, and extending $\text{Ad}(H)$ complex linearly to $\mathfrak{g}$, we see $\text{Ad} H$ is a commuting family of unitary operators, hence simultaneously diagonalizable. Since $\text{Ad}: H \to \text{Aut}(\mathfrak{h}_0)$ has differential $\text{ad}: \mathfrak{h}_0 \to \text{End}(\mathfrak{h}_0)$, we see the weight space decomposition for $\mathfrak{g}$ agrees with the diagonalization of $\text{Ad} H$. Thus, each root space for $\mathfrak{h}$ is also a root space for $H$, and for
each root \( \alpha \), we get a multiplicative character \( \xi_\alpha \) of \( H \) such that

\[
\text{Ad}(X)(Y) = \xi_\alpha(\exp X)(Y) = e^{\alpha(X)}(Y)
\]

for each \( X \) in \( h_0 \) and each \( Y \) in \( g_\alpha \).

Algebraically integral, dominant forms in \( h^* \) are in one-to-one correspondence with irreducible, finite dimensional representations of \( g \), where we associate to any algebraically integral form \( \lambda \) the unique irreducible representation of highest weight \( \lambda \). ([3] section 21) If we start with a finite-dimensional irreducible representation \( \Phi \) for \( G \), we get a finite-dimensional irreducible representation \( \phi \) for \( g_0 \) by differentiating \( \Phi \). We can extend \( \phi \) to \( g \) complex linearly. The result is a finite-dimensional irreducible representation of a reductive complex Lie algebra corresponding to some dominant algebraically integral form \( \lambda \) in \( h^* \). Since this representation comes from a representation \( \Phi \) of \( G \), we know the weight \( \lambda \) is actually analytically integral. On the other hand, if we start with any finite-dimensional irreducible representation for \( g \) corresponding to an algebraically integral form, we can restrict to a representation for \( g_0 \). As long as this algebraically integral form is analytically integral for \( G \), [5] 5.110 proves there exists a finite-dimensional, irreducible representation \( \Phi \) of \( G \) with differential \( \phi_{g_0} \). These processes establish a one-to-one correspondence between analytically integral dominant forms in \( h \) and finite-dimensional irreducible representations for \( G \).

Returning to the language we have established to describe \( SO(n, \mathbb{R}) \), we wish to identify all possible analytically integral forms in \( h \). The matrix

\[
Y = \begin{pmatrix}
0 & \text{i}h \\
-\text{i}h & 0
\end{pmatrix}
\]
exponentiates to
\[ \exp Y = \begin{pmatrix} \cos(ih) & \sin(ih) \\ -\sin(ih) & \cos(ih) \end{pmatrix}, \]
so \( \exp Y = 1 \) if and only if \( h = 2\pi i l \) for some \( l \) in \( \mathbb{Z} \). We see \( X \) in \( \mathfrak{h} \) exponentiates to 1 if and only if for each \( j \) such that \( 1 \leq j \leq k \), we have \( h_j = 2\pi i l \) for some \( l \) in \( \mathbb{Z} \) where the complex numbers \( h_1, \ldots, h_k \) define \( X \) as described in Equations 3 and 4.

Choosing some arbitrary form \( a_1 e_1 + \cdots + a_k e_k \) in \( \mathfrak{h}^* \) where \( a_j \) is complex for each \( j \) with \( 1 \leq j \leq k \), we see
\[ \lambda(X) = a_1 h_1 + \cdots + a_k h_k. \]
If \( X \) in \( \mathfrak{h} \) exponentiates to 1, we see \( \lambda(X) = 2\pi i l \) for some \( l \) in \( \mathbb{Z} \) if and only if \( a_j \) is in \( \mathbb{Z} \) for each \( j \) with \( 1 \leq j \leq k \). Hence, the set of analytically integral weights for \( SO(n, \mathbb{R}) \) is
\[ \{a_1 e_1 + \cdots + a_k e_k \mid a_i \in \mathbb{Z} \text{ for } 1 \leq i \leq k\}. \]

Determining which of these analytically integral forms satisfies the definition of dominance proves easier still. We need only test whether a form \( \lambda \) satisfies \( \langle \lambda, \alpha \rangle \geq 0 \) for each \( \alpha \) in \( \Delta^+ \), hence it suffices to test whether \( \langle \lambda, \alpha \rangle \geq 0 \) for each \( \alpha \) in \( \Delta^+ \) using the positive definite, symmetric form \( (, ) \). Referring to Equations 7 and 8, we see an analytically integral form \( \lambda = a_1 e_1 + \cdots + a_k e_k \) is dominant if
\[ a_1 \geq \cdots \geq a_k \geq 0 \quad \text{for } n = 2k + 1, \text{ or if } \]
\[ a_1 \geq \cdots \geq a_{k-1} \geq |a_k| \quad \text{for } n = 2k. \]
We realize the embedding \( \iota_{n-1} : SO(n - 1, \mathbb{R}) \hookrightarrow SO(n, \mathbb{R}) \) via the mapping
\[
A \mapsto \begin{pmatrix}
0 \\
A \\
0 \\
0 \ldots 0 \ 1
\end{pmatrix}.
\]

When no confusion will result, we drop the subscript and write \( \iota \) for \( \iota_{n-1} \) For any irreducible representation \( \Phi \) of \( SO(n, \mathbb{R}) \), we can ask for an understanding of the decomposition of \( \Phi|_{SO(n-1, \mathbb{R})} \) into irreducible representations for \( SO(n - 1, \mathbb{R}) \). This branching law has a relatively simple and complete description.

**Theorem 4.1 (Murnaghan)** For \( n = 2k + 1 \), the irreducible, finite-dimensional representation of \( SO(n, \mathbb{R}) \) with highest weight \( a_1 e_1 + \cdots + a_k e_k \) decomposes under restriction to \( SO(n-1, \mathbb{R}) \) into a sum of representations with highest weight \( c_1 e_1 + \cdots + c_k e_k \) such that
\[
a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{k-1} \geq c_{k-1} \geq a_k \geq |c_k|,
\]
each such representation having multiplicity one in the decomposition.

For \( n = 2k \), the representation with highest weight \( a_1 e_1 + \cdots + a_k e_k \) decomposes into a sum of representations with highest weight \( c_1 e_1 + \cdots + c_{k-1} e_{k-1} \) such that
\[
a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{k-1} \geq c_{k-1} \geq |a_k|,
\]
each such representation having multiplicity one in the decomposition.


We now consider the irreducible character \( \chi_{n,S} \) of \( M(SO(n)) \), the Langlands subgroup of \( SO(n, \mathbb{R}) \), corresponding to the subset \( S \) of \( \{1, 2, \ldots, n\} \). By our choice for
the embedding $\iota: SO(n-1, \mathbb{R}) \hookrightarrow SO(n, \mathbb{R})$, we see

$$\chi_{n,S}|_{M(SO(n-1))} = \chi_{n-1,S\setminus\{n\}}.$$  (13)

If $F$ is any reductive algebraic group and $E$ is any closed subgroup of $F$, we refer to the multiplicity of an irreducible representation $\kappa$ of $E$ in the restriction of an irreducible representation $\mu$ of $F$ to the subgroup $E$ by $m(\kappa, \mu)$. We consider the closed subgroup $\iota(SO(n-1, \mathbb{R})) \subset SO(n, \mathbb{R})$. For any analytically integral dominant form $\lambda$ corresponding to an irreducible representation $\mu_\lambda$ of $SO(n, \mathbb{R})$, we define $A_\lambda$ to be the set of analytically integral dominant forms $\gamma$ such that $m(\mu_\gamma, \mu_\lambda) \neq 0$ where $\mu_\gamma$ is the irreducible representation of $SO(n-1, \mathbb{R})$ corresponding to the form $\gamma$. Theorem 4.1 describes $A_\lambda$ entirely for any analytically integral dominant form $\lambda$, and, for any $\gamma$ in $A_\lambda$, we know $m(\mu_\gamma, \mu_\lambda) = 1$.

If $\mu(n)_\lambda$ is the irreducible representation for $SO(n, \mathbb{R})$ with highest weight $\lambda$, the ruminations in section 2 motivate our attempting to ascertain the value $m(\chi_{n,S}, \mu_\lambda)$ for each character $\chi_{n,S}$ of $M(SO(n))$. To finish this section, we give an inductive process to ascertain these very multiplicities. The base case for such an inductive process arising from the case $n = 1$ is trivial since the group $SO(1, \mathbb{R})$ is trivial. For any algebraic group $G$, we refer to the category of representations for $G$ as $\text{Rep}(G)$.

**Lemma 4.2** The equation

$$\sum_{\gamma \in A_\lambda} m(\chi_{n-1,S\setminus\{n\}}, \mu_\gamma) = m(\chi_{n,S\setminus\{n\}}, \mu_\lambda)$$  (14)

holds for any subset $S$ of $\{1, 2, \ldots, n-1\}$.
Proof If each map in the diagram

\[
\begin{align*}
\text{Rep } SO(n, \mathbb{R}) & \to \text{Rep } SO(n - 1, \mathbb{R}) \\
\downarrow & \downarrow \\
\text{Rep } M(SO(n)) & \to \text{Rep } M(SO(n - 1))
\end{align*}
\]

represents restriction, then the diagram commutes by the definition of the restriction morphism. As such, we may calculate \( m(\chi_{n-1, \mathcal{S}}, \mu_{\lambda}) \) either to be

\[
\sum_{\gamma \in A_{\lambda}} m(\mu_{\gamma}, \mu_{\lambda}) \cdot m(\chi_{n-1, \mathcal{S}}, \mu_{\gamma})
\]

or to be

\[
\sum_{\chi_{n,S} \text{ with } \mathcal{S} \subseteq \{1, 2, \ldots, n\}} m(\chi_{n,S}, \mu_{\lambda}) \cdot m(\chi_{n-1, \mathcal{S}}, \chi_{n,S}).
\]

Yet \( m(\mu_{\gamma}, \mu_{\lambda}) = 1 \) for each \( \gamma \) in \( A_{\lambda} \), so the quantity * is

\[
\sum_{\gamma \in A_{\lambda}} m(\chi_{n-1, \mathcal{S}}, \mu_{\gamma}).
\]

By Equation 13, we know

\[
\chi_{n, \mathcal{S}|M(SO(n-1))} = \chi_{n-1, \mathcal{S}} = \chi_{n,(\mathcal{S}\cup\{n\})|M(SO(n-1))}.
\]

Using Equation 13 together with Theorem 3.2, we conclude \( \chi_{n, \mathcal{S}} \) and \( \chi_{n,(\mathcal{S}\cup\{n\})} \) are the only characters of \( M(SO(n)) \) up to isomorphism with restriction to \( M(SO(n - 1)) \) equal to \( \chi_{n-1, \mathcal{S}} \). Hence \( m(\chi_{n-1, \mathcal{S}}, \chi_{n,S}) \) is nonzero if and only if \( \chi_{n,S} = \chi_{n, \mathcal{S}} \) or \( \chi_{n,S} = \chi_{n,(\mathcal{S}\cup\{n\})} \), in which case, \( m(\chi_{n-1, \mathcal{S}}, \chi_{n,S}) = 1 \). The quantity ** is

\[
m(\chi_{n, \mathcal{S}}, \mu_{\lambda}) + m(\chi_{n,(\mathcal{S}\cup\{n\})}, \mu_{\lambda})
\]
For the following lemma, we suppose \( k = n/2 \) and \( \lambda = a_1 e_1 + \cdots + a_k e_k \) is the highest weight for some irreducible representation \( \Phi \) of \( SO(n, \mathbb{R}) \).

**Lemma 4.3** If \( n \) is even and if \( m(\chi_S, \Phi) \) is nonzero, then \( |S| \) has the same parity as \( a_1 + \cdots + a_k \).

**Proof** Since \( n \) is even, the scalar matrix \( m_{-1} \) with all diagonal entries equal to \(-1\) is an element of \( M(SO(n)) \). Moreover, \( m_{-1} \) is a central element. We choose some highest weight vector \( v \) in \( V^\Phi \) for the representation \( \Phi \). Then \( v \) generates \( \Phi \) in the sense that any vector for the representation \( \Phi \) takes the form \( c(gv) \) for some scalar \( c \) and some element \( g \) in \( SO(n, \mathbb{R}) \). We consider the action of \( m_{-1} \) on \( v \). The dominant weight \( \lambda \) is an analytic integral form, so there exists a character \( \xi_\lambda \) of the maximal torus \( H \) in \( SO(n, \mathbb{R}) \) such that \( \xi_\lambda(\exp X) = e^{\lambda(X)} \) for any \( X \) in \( h_0 \). We define \( M_{-1} \) in \( h_0 \) by letting \( h_j = \pi \) for each \( 1 \leq j \leq k \). Then \( \exp(M_{-1}) = m_{-1} \). We see \( \lambda(M_{-1}) = a_1 + \cdots + a_k \). Hence, according to the discussion preceding this proof,

\[
m_{-1}v = (\xi_\lambda(m_{-1}))v = (-1)^{(a_1+\cdots+a_k)}v.
\]

If we let \( \epsilon = (-1)^{(a_1+\cdots+a_k)} \), we see

\[
m_{-1}(c(gv)) = c(g(m_{-1}v)) = \epsilon(c(gv))
\]

for any scalar \( c \) and any element \( g \) in \( SO(n, \mathbb{R}) \). Hence, viewing \( \Phi(m_{-1}) \) as an element of \( GL(V^\Phi) \), we interpret \( \Phi(m_{-1}) \) as the diagonal matrix with each diagonal entry equal to \( \epsilon \). By Theorem 3.2, each irreducible subrepresentation of \( \Phi|M \) has character \( \chi_S \) for some subset \( S \) of \( \{1, 2, \ldots, k\} \). Choosing \( S \) such that \( \chi_S \) is the character for some such subrepresentation with underlying space \( V^S \subset V^\Phi \), we denote the trace of
\( \Phi(m_1) \) restricted to \( V^S \) by \( \text{tr}_S(m_{-1}) \). Then

\[ \chi_S(m_{-1}) = \text{tr}_S(m_{-1}) = \epsilon. \]

As \( \chi_S(m_{-1}) \) is exactly \((-1)^{|S|}\), we see \(|S|\) has the same parity as \(a_1 + \cdots + a_k\). \( \blacksquare \)

We have broached all the information necessary to prove the existence of an inductive method for the branching law from \( SO(n, \mathbb{R}) \) to \( M \).

**Proposition 4.4** For any \( n \) in \( \mathbb{N} \) and any analytically integral dominant form \( \lambda \) corresponding to an irreducible representation \( \mu_\lambda \) for \( SO(n, \mathbb{R}) \), Equation 14 allows us to determine \( m(\chi_{n,S}, \mu_\lambda) \) for any subset \( S \) of \( \{1, 2, \ldots, n\} \) provided we already know \( m(\chi_{n-1,S}, \mu_\gamma) \) for any subset \( \hat{S} \) of \( \{1, 2, \ldots, n-1\} \) and any analytically integral dominant form \( \gamma \) corresponding to an irreducible representation \( \mu_\gamma \) for \( SO(n-1, \mathbb{R}) \).

**Proof** By Theorem 3.2, it suffices to show Equation 14 determines \( m(\chi_{n,S}, \mu_\lambda) \) for each \( S \) in \( F_n \), where we recall

\[ F_n : = \{ S \subset \{1, 2, \ldots, n\} \mid |S| \leq [n - 1/2] \} \]

\[ \cup \{ S \subset \{1, 2, \ldots, n\} \mid |S| = n/2, \text{ and } 1 \in S \}. \]

Furthermore, by Theorem 3.3, we need only determine \( m(\chi_{n,S}, \mu_\lambda) \) when \( S = S_i \) for \( S_i = \{1, 2, \ldots, i\} \) and for \( i \) such that \( 0 \leq i \leq [n/2] \). Using Theorem 4.1, we may determine completely the set \( A_\lambda \), so, by our inductive hypothesis, we may assume we know the value of the left-hand side of Equation 14 for any subset \( \hat{S} \) of \( \{1, 2, \ldots, n-1\} \).

We split the proof into two cases. First, we suppose \( n = 2k \). We write \( \lambda = \)}
For any $i$ such that $1 \leq i \leq k$, Lemma 4.3 allows us to see

$$m(\chi_{n,S_{i-1}}, \mu_\lambda) = 0,$$

and

$$m(\chi_{n,(S_{i-1} \cup \{n\})}, \mu_\lambda) = \sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1,S_{i-1}}, \mu_\gamma)$$

if the parity of $i - 1$ does not match the parity of $a_1 + \cdots + a_k$. If the parity of $i - 1$ matches the parity of $a_1 + \cdots + a_k$,

$$m(\chi_{n,S_{i-1}}, \mu_\lambda) = \sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1,S_{i-1}}, \mu_\gamma),$$

and

$$m(\chi_{n,(S_{i-1} \cup \{n\})}, \mu_\lambda) = 0.$$

Using Theorem 3.3 again, we see

$$m(\chi_{n,(S_{i-1} \cup \{n\})}, \mu_\lambda) = m(\chi_{n,S_i}, \mu_\lambda).$$

Hence, we can determine $m(\chi_{n,S_i}, \mu_\lambda)$ for any $i$ such that $0 \leq i \leq k$.

Now, we suppose $n = 2k + 1$. Theorem 3.2 tells us

$$\chi_{n,S_k \cup \{n\}} = \chi_{n,(S_k \cup \{n\})^C},$$

and $(S_k \cup \{n\})^C$ has cardinality $k$, as does $S_k$. By Theorem 3.3, we surmise

$$m(\chi_{n,S_k}, \mu_\lambda) = m(\chi_{n,(S_k \cup \{n\}), \mu_\lambda}).$$

As a result

$$\sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1,S_k}, \mu_\gamma) = \frac{2}{m(\chi_{n,S_k}, \mu_\lambda)}.$$
such that $0 \leq i < k$, Equation 14 tells us

$$
\sum_{\gamma \in A_\lambda} m(\chi_{n-1,s_i}, \mu_\gamma) = m(\chi_{n,s_i}, \mu_\lambda) + m(\chi_{n,(s_i \cup \{n\})}, \mu_\lambda).
$$

Turning once again to Theorem 3.3, we see

$$
m(\chi_{n,(s_i \cup \{n\})}, \mu_\lambda) = m(\chi_{n,s_{i+1}}, \mu_\lambda),
$$

and our inductive hypothesis allows us to assume we have determined the value $m(\chi_{n,s_{i+1}}, \mu_\lambda)$. As such, Equation * allows us to determine $m(\chi_{n,s_i}, \mu_\lambda)$. 

Practically speaking, this inductive understanding of the branching law from $SO(n, \mathbb{R})$ to $M$ makes actually calculating multiplicities a daunting task. Relying on this algorithm means computing multiplicities for the branching law from $SO(j, \mathbb{R})$ to $M$ for each $j$ such that $1 \leq j < n$. We would like to establish a more manageable algorithm for computing these multiplicities. Tableaux will serve as a tool to help us meet that objective.

5 Young Tableaux and $SO(n, \mathbb{R})$

Supposing $\beta$ is some subset of $\mathbb{Z}^2$, we will refer to a \textit{tableau with shape} $\beta$ as a map $\Omega: \beta \rightarrow \mathbb{Z}$. Young tableaux have visual representations gotten by considering $\mathbb{Z}$ as a lattice in the plane $\mathbb{R}^2$ and dividing the plane into unit boxes. To each $(i, j)$ in the subset $\beta$, we associate the unit box in $\mathbb{R}^2$ having $(i, j)$ in the lower right-hand corner. We then fill the unit box corresponding to each point $(i, j)$ in $\beta$ with the
integer \( \Omega(i, j) \). For instance, if

\[
\beta = \{(i, -1) \in \mathbb{Z}^2 \mid 1 \leq i \leq 4\}
\cup \{(i, -2) \in \mathbb{Z}^2 \mid 1 \leq i \leq 3\}
\cup \{(i, -3) \in \mathbb{Z}^2 \mid 1 \leq i \leq 1\},
\]

we may define a tableau with shape \( \beta \) by allowing the following diagram to illustrate the value of \( \Omega(i, j) \) for any point \((i, j)\) in \( \beta \):

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 3 \\
3 \\
\end{array}
\]

(15)

We assign to each \( n \)-tuple of natural numbers, \( a = (a_1, \ldots, a_n) \), a subset \( \beta_a \) of \( \mathbb{Z}^2 \) by letting \( \beta_a \) be the set of pairs \((i, -j)\) with \( 1 \leq j \leq n \) and with \( 0 \leq i \leq a_j \). To simplify notation, we will say a tableau with shape \( \beta_a \) has shape \( a \).

A partition in \( n \) parts of a natural number \( m \) is an \( n \)-tuple of natural numbers \( (a_1, \ldots, a_n) \) such that

\[
a_1 + a_2 + \cdots + a_n = m, \text{ and } a_1 \geq a_2 \geq \cdots \geq a_n.
\]

(We allow 0 to be a natural number.) We refer to \( a_i \) for \( 1 \leq i \leq n \) as a part of \( a \). If a tableau has shape \( a \) for some partition \( a \), we refer to the tableau as a Young tableau.

A Young tableau \( \Omega \) is semistandard if \( 1 \leq \Omega(i, j) \leq q \) for some natural number \( q \), and if \( \Omega(i, j) < \Omega(i, j - 1) \) for each \((i, j)\) in \( \beta_a \) with \((i, j - 1)\) in \( \beta_a \) while \( \Omega(i, j) \leq \Omega(i + 1, j) \) for each \((i, j)\) in \( \beta_a \) with \((i + 1, j)\) in \( \beta_a \). We refer to \( q \) as the bound for \( \Omega \). Justified by this definition, we say a semistandard Young tableau increases weakly along its rows and increases strictly down its columns. If a tableau \( \Omega \) has shape \( a \) for some partition
a with \( n \) parts, we say the \( j^{th} \) row of \( \Omega \) has length \( a_j \). We refer to \( m \), the number of nonzero parts of \( a \) and, hence, the number of rows in \( \Omega \), as the depth of \( \Omega \). Tableau 15 is semistandard. The following two tableau are not semistandard. The tableau on the left-hand side does not increase weakly along its first row, and the tableau on the right-hand side does not increase strictly along its second and third columns:

<table>
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<tr>
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<td>3</td>
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<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Furthermore, the length of the second row for the tableau on the left-hand side is 2 whereas the length of the second row for the tableau on the right-hand side is 3. Each tableau has depth 3.

As a point of reference, we illustrate the use of Young-diagramatic methods in studying irreducible, finite-dimensional representations of \( U(n) \) where

\[
U(n) = \{ g \in GL(n, \mathbb{C}) \mid g \cdot g^* = 1 \}.
\]

This compact (hence reductive) group has as its Lie algebra

\[
u_0(n) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^* = 0 \}.
\]

The complexification,

\[
u(n) = \nu_0(n) \oplus i \nu_0(n),
\]

is simply \( \mathfrak{gl}(n, \mathbb{C}) \). In order to study finite-dimensional representations of \( U(n) \), the commentary in section 4 allows for us to restrict our attention to analytically integral dominant weights for \( U(n) \) with respect to the root system for \( \mathfrak{gl}(n, \mathbb{C}) \) formed after
choosing some Cartan subalgebra. For a Cartan subalgebra $\mathfrak{h}$, we choose

$$\mathfrak{h} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X \text{ is diagonal}\}.$$

We define $e_i$ in $\mathfrak{h}^*$ for each $i$ such that $1 \leq i \leq n$ by

$$e_i \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = h_i.$$

Furthermore, we define the set

$$\Delta: = \{e_i - e_j \mid i \neq j, 1 \leq i \leq n, \text{ and } 1 \leq j \leq n\}.$$

If we choose $E_{ij}$ to be the matrix in $\mathfrak{gl}(n, \mathbb{C})$ with 1 in the $i, j^{th}$ entry and 0 in all other entries, we see $E_{ij}$ spans a space in $\mathfrak{gl}(n, \mathbb{C})$ with weight $e_i - e_j$ for the adjoint action $\text{ad}_g \mathfrak{h}$. Furthermore,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij},$$

so $\Delta$ is the set of roots for $\mathfrak{g}$ with respect to our choice $\mathfrak{h}$ of Cartan subalgebra. We choose a notion of positivity for our root system by setting

$$\Delta^+: = \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

With these definitions in place, calculations similar to those in section 4 show the analytically integral dominant weights for $U(n)$ are exactly

$$\{a_1 e_1 + \cdots + a_n e_n \mid (a_1, \ldots, a_n) \in \mathbb{Z}^n, \text{ and } a_1 \geq a_2 \geq \cdots \geq a_n\}. \quad (16)$$
We realize the embedding $\iota_{n-1}: U(n-1) \hookrightarrow U(n)$ via the mapping

$$A \mapsto \begin{pmatrix} 0 \\ \vdots \\ A \\ 0 \end{pmatrix}.$$ 

When no confusion will result, we drop the subscript and write $\iota$ for $\iota_{n-1}$.

**Theorem 5.1 (Weyl)** The irreducible, finite-dimensional representation for $U(n)$ with highest weight $a_1 e_1 + \cdots + a_n e_n$ for $(a_1, \ldots, a_n)$ in $\mathbb{Z}^n$ decomposes under restriction to $U(n-1)$ into a sum of representations with highest weight $c_1 e_1 + \cdots + c_{n-1} e_{n-1}$ such that

$$a_1 \geq c_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq a_n,$$

(17)

each such representation having multiplicity one in the decomposition.


The structure of semistandard Young tableaux relates directly to valuable information regarding the irreducible, finite-dimensional representations of $U(n)$. To begin to realize this relationship, we reproduce the following well-known result. We let $\alpha$ be a partition with $n$ parts $(a_1, \ldots, a_n)$ for the natural number $a_1 + \cdots + a_n$ (so $(a_1, \ldots, a_n)$ is in $\mathbb{N}^n$).

**Theorem 5.2** The number of semistandard Young tableaux with shape $\alpha$ and bound $n$ is the dimension of the irreducible representation of $U(n)$ with highest weight $a_1 e_1 + \cdots + a_n e_n$.

**Proof** We prove this result by induction, the case $n = 1$ being trivial. If $\Omega_\alpha$ is a semistandard Young tableau having shape $\alpha$, then we define the Young tableau
\( \pi(\Omega_a) \) to be the tableau given by removing every occurrence of the number \( n \) in the visual representation of \( \Omega_a \). If \( \Omega_a(i, j) = n \), then, since semistandard Young tableaux increase strictly down their columns, we know \((i, j - 1)\) is not an element of \( \beta_a \). Also, we know \( \Omega_a(i, -n) = n \) if \((i, -n)\) is an element of \( \beta_a \). Hence, \( \pi(\Omega_a) \) has depth at most \( n - 1 \), and, for each \( j \) such that \( 1 \leq j \leq n - 1 \), we see the \( j^{th} \) row of \( \pi(\Omega_a) \) has length \( c_j \) for some \( c_j \) such that \( a_j \geq c_j \geq a_{j+1} \). Since \( \Omega_a \) increases weakly along its rows and strictly down its columns, \( \pi(\Omega_a) \) increases weakly along its rows and strictly along its columns as well. We have shown the mapping \( \Omega_a \mapsto \pi(\Omega_a) \) gives a well-defined map \( \pi \) from the set of semistandard Young tableaux having shape \( a \) to the set of semistandard Young tableaux having shape \( c \) for some partition

\[
c = (c_1, \ldots, c_{n-1}) \text{ such that } a_1 \geq c_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq a_n.
\]

We now denote \( \pi(\Omega_a) \) by \( \Omega_c \). The map \( \pi \) is clearly injective: if \( \Omega_a \) and \( \Omega_a' \) both map to \( \Omega_c \) then \( \Omega_a(i, j) = \Omega_a'(i, j) \) if \( \Omega_a(i, j) < n \) and \((i, j)\) is an element of \( \beta_a \). This equation shows \( \Omega_a(i, j) = \Omega_a'(i, j) \) for each \((i, j)\) in \( \beta_a \) since both \( \Omega_a \) and \( \Omega_a' \) have the same shape \( a \) and map to the same tableau under \( \pi \). Any semistandard Young tableau \( \tilde{\Omega}_c \) with shape \( c \) for some partition \( c \) satisfying Equation * has a completion to a semistandard Young tableau \( \Omega_a \) of shape \( a \) by allowing \( \Omega_a(i, j) = n \) for each \((i, j)\) such that \(-1 \geq j \geq -(n - 1) \) and \( c_{-j} \leq i \leq a_{-j} \) or \( j = -n \) and \( 0 \leq i \leq a_n \). Then

\[
\pi(\Omega_a) = \Omega_c = \tilde{\Omega}_c,
\]

and we have shown \( \pi \) is surjective.

By our inductive hypothesis, the number of semistandard Young tableaux having shape \( c \) is the dimension of the irreducible representation for \( U(n - 1) \) with highest weight \( c_1e_1 + \cdots + c_{n-1}e_{n-1} \). Since the map \( \pi \) is bijective, we know the number of
semistandard Young tableaux having shape $a$ is the sum of the dimensions of the representations for $U(n - 1)$ with highest weight $c_1 e_1 + \cdots + c_{n-1} e_{n-1}$ such that the $n$-tuple of coefficients $c = (c_1, \ldots, c_n)$ satisfies Equation *. Finally, by Theorem 5.1, we see the sum of the dimensions of the representations for $U(n - 1)$ with highest weight $c_1 e_1 + \cdots + c_{n-1} e_{n-1}$ such that the $n$-tuple of coefficients $c = (c_1, \ldots, c_n)$ satisfies Equation * is exactly the dimension of the representation for $U(n)$ with highest weight $a_1 e_1 + \cdots + a_n e_n$.

The proof of Theorem 5.2 establishes a correlation between the irreducible, finite-dimensional representations of $U(n)$ and semistandard Young tableaux. The semistandard Young tableaux make plain the relationship between the dimensions of irreducible representations of $U(n)$ and the dimensions of irreducible representations of $U(n - 1)$. We wish to find an analogous picture for irreducible, finite-dimensional representations of $SO(n, \mathbb{R})$, a structure encoding information about these representations in such a way as to tie this information to corresponding facts about representations of $SO(n - 1, \mathbb{R})$.

**Definition 5.3** A semistandard Young tableau $\Omega$ is admissible for $SO(n, \mathbb{R})$ if the following conditions hold:

(i) $\Omega$ is a semistandard Young tableau having shape $a$ for some partition $a = (a_1, \ldots, a_n)$, and

(ii) $\Omega(i, j) \geq 2j$ for each $j$ such that $1 \leq j \leq n$ and each $i$ such that $1 \leq i \leq a_j$.

We consider two semistandard Young tableaux:

\[
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
5 & 5 & 6 \\
6 & 6 & 7 \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
5 & 5 & 6 \\
6 & 6 & 7 \\
7 \\
\end{array}
\]
The tableau on the left-hand side is admissible with shape \((5, 3, 3, 0, 0, 0, 0)\). The tableau on the right-hand side is admissible for no shape as its 4\(^{th}\) row contains a number smaller than 8. In fact, it is easy to see an admissible semistandard Young tableau must have a shape \(a = (a_1, \ldots, a_n)\) such that \(a_j = 0\) for all \(j > \lfloor n/2 \rfloor\).

If \(a\) is a partition \((a_1, \ldots, a_n)\) such that \(a_j = 0\) for \(j > \lfloor n/2 \rfloor\), we refer to the set of admissible semistandard Young tableaux having shape \(a\) as \(\Psi_a\). We define

\[
\Psi_a := \left( \Psi_a \times \{\epsilon_1, \ldots, \epsilon_{\lfloor n/2 \rfloor} \mid \epsilon_i = \pm 1 \text{ for } 1 \leq i \leq \lfloor n/2 \rfloor \} \right) / \sim
\]

where

\[
(\Omega_a, (\epsilon_1, \ldots, \epsilon_{\lfloor n/2 \rfloor})) \sim (\Omega'_a, (\epsilon'_1, \ldots, \epsilon'_{\lfloor n/2 \rfloor}))
\]

if and only if \(\Omega_a = \Omega'_a\) and \(\epsilon_j = \epsilon'_j\) for each \(j\) such that \(1 \leq j \leq \lfloor n/2 \rfloor\) and \(\Omega_a(1, -j) = 2j\). In other words, \(\Psi_a\) is the set of admissible semistandard Young tableau having shape \(a\) decorated with a choice of parity \(\epsilon_j\) for each row \(j\) with \(1 \leq j \leq \lfloor n/2 \rfloor\) such that every occurrence of the number \(2j\) on row \(j\) has parity \(\epsilon_j\). Henceforth, we identify the pair \((\Omega_a, (\epsilon_1, \ldots, \epsilon_{\lfloor n/2 \rfloor}))\) with its equivalence class in \(\Psi_a\) when context allows for no confusion. We refer to elements of the set \(\Psi_a\) as decorated admissible semistandard Young tableaux. If \(\Omega_a = (\Omega_a, (\epsilon_1, \ldots, \epsilon_{\lfloor n/2 \rfloor}))\) is an element of \(\Psi_a\), we refer to \(\Omega_a\) as the admissible semistandard Young tableau associated to \(\Omega_a\).

If we let \(\Omega\) be the admissible semistandard Young tableau

\[
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
5 & 5 & 6 & \\
6 & 6 & 7 & \\
\end{array}
\]
then $\Omega$ is associated to four decorated tableaux, namely

\begin{align*}
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
5 & 5 & 6 \\
6 & 6 & 7 \\
\end{array}
\begin{array}{cccc}
-2 & -2 & 3 & 3 \\
5 & 5 & 6 \\
-6 & -6 & 7 \\
\end{array},
\end{align*}

\begin{align*}
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
5 & 5 & 6 \\
6 & 6 & 7 \\
\end{array}
\begin{array}{cccc}
-2 & -2 & 3 & 3 \\
5 & 5 & 6 \\
-6 & -6 & 7 \\
\end{array},
\end{align*}

and

The pair $(\Omega, (1, -1, 1))$ is equivalent to the pair $(\Omega, (1, 1, 1))$.

We select an irreducible, finite-dimensional representation $\mu_\lambda$ for $SO(n, \mathbb{R})$ with highest weight $\lambda = a_1 e_1 + \cdots + a_{[n/2]} e_{[n/2]}$, and we let $a$ be the partition $(|a_1|, \ldots, |a_n|)$ such that $a_j = 0$ for each $j$ such that $j > [n/2]$. We define a map $\pi_n$ following exactly the steps used to define the map $\pi$ from Theorem 5.2 such that $\pi_n(\Omega_a)$ is a semistandard Young tableau for any $\Omega_a$ in $\Psi_a$. In fact, $\pi_n(\Omega_a)$ is admissible. This tableau has Property ii from Definition 5.3 because $\Omega_a$ has Property ii. Since $a_j = 0$ for each $j > [n/2]$, the proof of Theorem 5.2 shows $\pi_n(\Omega_a)$ has shape $c = (c_1, \ldots, c_{n-1})$ where

\begin{align*}
a_1 &\geq c_1 \geq a_2 \geq \cdots \geq c_{[n/2]-1} \geq a_{[n/2]} \geq c_{[n/2]}, \quad \text{and} \\
c_j &= 0 \text{ for each } j > [n-1/2]. 
\end{align*}

Equation 19 comes from the equality $[n-1/2] = [n/2]$ if $n$ is odd and, if $n$ is even, from the fact $\Omega_a(i, -j) = 2j$ if $j = n/2$ and $1 \leq i \leq a_j$. Whenever possible, we drop the subscript and write $\pi$ for $\pi_n$. 


We make the following definition: for even \( n \),

\[
\Psi_\lambda = \left\{ (\Omega_a, (\epsilon_1, \ldots, \epsilon_{[n/2]}) : (\Omega_a, (\epsilon_1, \ldots, \epsilon_{[n/2]}), a_{[n/2]} = a_{[n/2]} \right\},
\]

and for odd \( n \), we have the equality \( \Psi_\lambda = \Psi_a \). We will refer to the set \( \Psi_\lambda \) as the set of decorated admissible tableaux of type \( \lambda \).

**Theorem 5.4** The dimension of the irreducible representation \( \mu_\lambda \) is \( |\Psi_\lambda| \).

**Proof** Just as we relied on Theorem 5.1 for giving structure to our proof of Theorem 5.2, we now rely on Theorem 4.1. We proceed by induction, the case \( n = 1 \) being trivial. Given an equivalence class

\[
\Omega_\lambda = (\Omega_a, (\epsilon_1, \ldots, \epsilon_{[n/2]}))
\]
in \( \Psi_\lambda \), we define \( \pi_n(\Omega_\lambda) \) to be \( (\pi_n(\Omega_a), (\epsilon_1, \ldots, \epsilon_{[n-1/2]}) \). As with the map \( \pi_n \), we drop the subscript from \( \pi_n \) and write simply \( \pi \) whenever doing so will not cause confusion. According to the arguments immediately preceding this theorem, \( \pi(\Omega_a) \) is an element of \( \Phi_c \) for some partition \( c = (c_1, \ldots, c_{n-1}) \) satisfying Equation 18 with \( c_j = 0 \) for each \( j \) such that \( j \geq n/2 \) when \( n \) is even. When \( n \) is odd, \( \pi(\Omega_a) \) is an element of \( \Phi_c \) for some partition \( c = (c_1, \ldots, c_{n-1}) \) satisfying Equations 18 and 19. In fact, \( \pi(\Omega_a) \) is an element of \( \Phi_\gamma \) where \( \gamma \) is the highest weight

\[
c_1 \epsilon_1 + \cdots + c_{[n-1/2]} \epsilon_{[n-1/2]}
\]
for \( SO(n-1, \mathbb{R}) \) when \( n \) is even and where \( \gamma \) is the highest weight

\[
c_1 \epsilon_1 + \cdots + c_{[n-1/2]-1} \epsilon_{[n-1/2]-1} + \epsilon_{[n-1/2]} (c_{[n-1/2]} \epsilon_{[n-1/2]})
\]
for $SO(n-1, \mathbb{R})$ when $n$ is odd. From Theorem 4.1, we see each such integral form $\gamma$ has the property $m(\mu_\gamma, \mu_\lambda) = 1$. Thus, we get a well-defined map

$$\pi: \Psi_\lambda \rightarrow \bigcup_{\gamma \in A_\lambda} \Psi_\gamma$$

where, as in section 4, $A_\lambda$ is the set of analytically integral forms $\gamma$ for $SO(n-1, \mathbb{R})$ such that $m(\mu_\gamma, \mu_\lambda) \neq 0$. Given two elements,

$$(\Omega_a, (\epsilon_1, \ldots, \epsilon_{[n/2]})) \text{ and } (\Omega'_a, (\epsilon'_1, \ldots, \epsilon'_{[n/2]}))$$

of $\Psi_\lambda$, we suppose $\pi$ maps each element to the same element of $\Psi_\gamma$ for some integral form $\gamma$. Then $\pi(\Omega_a) = \pi(\Omega'_a)$, and, since $\pi$ is injective, $\Omega_a = \Omega'_a$. We know $\epsilon_{[n/2]} = \epsilon'_{[n/2]} = \epsilon$ where $\epsilon|a_{[n/2]}| = a_{[n/2]}$. Furthermore, for $j$ such that $1 \leq j < [n/2]$, if $\epsilon_j \neq \epsilon'_j$, we may assume $\Omega_a(1, -j) > 2j$. Hence, $\pi$ is injective. We let $A_a$ be the set of partitions $c = (c_1, \ldots, c_{n-1})$ such that

$$a_1 \geq c_1 \geq a_2 \geq \cdots \geq c_{n-1} \geq a_n.$$ 

From the proof of Theorem 5.2, we know

$$\pi: \Psi_a \rightarrow \{\Psi_c \mid c \in A_a\}$$

is surjective. By Theorem 4.1, we know

$$\{\{\Omega_c, (\epsilon_1, \ldots, \epsilon_{[n-1/2]}) \mid c \in A_a\} = \{\Psi_\gamma \mid \gamma \in A_\lambda\}.$$ 

Hence, $\pi$ is surjective.

By our inductive hypothesis, we know $|\Psi_\gamma| = \dim \mu_\gamma$ for each integral form $\gamma$ of
By Theorem 4.1,

\[ \text{Dim } \mu_\lambda = \sum \text{Dim } \mu_\gamma. \]

Since \( \tilde{\pi} \) is a bijection, we derive the equation

\[ |\Psi_\lambda| = \sum |\Psi_\gamma|. \]

The theorem follows readily.

We make use of the map \( \tilde{\pi} \) from Theorem 5.4 throughout the remainder of the paper.

6 Decorated Admissible Tableaux and Branching

Over \( M \)

In section 5, we established a connection between the irreducible representations of \( SO(n, \mathbb{R}) \) having finite dimension and the set of decorated admissible tableaux. Namely, we showed the representation with highest weight \( \lambda \) has dimension equal to the cardinality of the set of decorated admissible tableaux of type \( \lambda \). In proving this result, we saw each decorated admissible tableau of type \( \lambda \) corresponds to a subspace of the representation with highest weight \( \lambda \), itself an irreducible representation under restriction to \( SO(n - 1, \mathbb{R}) \) via the embedding \( \nu: SO(n - 1, \mathbb{R}) \rightarrow SO(n, \mathbb{R}) \) defined in section 4. Applied recursively to Theorem 4.1, this reasoning allows us to associate to each decorated admissible tableau of type \( \lambda \) a unique line within the representation of \( SO(n, \mathbb{R}) \), itself an irreducible representation under restriction to \( SO(2, \mathbb{R}) \) via the
embedding

\[ SO(2, \mathbb{R}) \hookrightarrow SO(3, \mathbb{R}) \hookrightarrow \cdots \hookrightarrow SO(n, \mathbb{R}). \]

We may realize these lines somewhat more explicitly as follows. As in section 4, we denote by \( \mu_\lambda \) the irreducible representation corresponding to \( \lambda \), and we denote by \( m(\mu_\gamma, \mu_\lambda) \) the multiplicity of the \( SO(n - 1, \mathbb{R}) \)-representation \( \mu_\gamma \) in the restriction of \( \mu_\lambda \) to \( SO(n - 1, \mathbb{R}) \). In the interest of developing notation for this section, we select a highest weight \( \nu_n(\lambda) \) for \( \mu_\lambda \). To each decorated admissible tableau \( \Omega \) of type \( \lambda \), we apply the map \( \pi_n \) to arrive at a decorated admissible tableau of type \( \gamma_{n-1} \) for some highest weight \( \gamma_{n-1} \) of \( SO(n - 1, \mathbb{R}) \) such that \( m(\mu_{\gamma_{n-1}}, \mu_\lambda) = 1 \). We select a highest weight vector \( \nu_{n-1}(\gamma_{n-1}) \) for the irreducible \( SO(n - 1, \mathbb{R}) \)-representation \( \mu_{\gamma_{n-1}} \subset \mu_\lambda \). We repeat this process, applying \( \pi_{n-1} \) to \( \pi \Omega \) and replacing \( \lambda \) with \( \gamma_{n-1} \) to arrive at a highest weight \( \gamma_{n-2} \) for \( SO(n - 2, \mathbb{R}) \) and to select a highest weight vector \( \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) \) of highest weight \( \gamma_{n-2} \) for \( \mu_{\gamma_{n-2}} \). Recursively, we continue in this manner until we have arrived at a highest weight \( \gamma_2 \) and selected a highest weight vector \( \nu_2(\gamma_{n-1}, \ldots, \gamma_2) \) of highest weight \( \gamma_2 \) for \( \mu_{\gamma_2} \). We denote \( \nu_2(\gamma_{n-1}, \ldots, \gamma_2) \) by \( \nu_2(\Omega) \). The vector \( \nu_2(\Omega) \) is unique only up to multiplication by a scalar. As in section 4, we use the symbol \( \mathcal{A}_\lambda \) to refer to the set of analytically integral forms \( \gamma \) for \( SO(n - 1, \mathbb{R}) \) such that \( m(\mu_\gamma, \mu_\lambda) \neq 0 \). According to Theorem 4.1,

\[ \mu_\lambda|_{SO(n-1, \mathbb{R})} = \bigoplus_{\gamma \in \mathcal{A}_\lambda} \mu_\gamma. \]

Theorem 5.4 proves \( \nu_2(\Omega) \neq \nu_2(\Omega') \) if \( \Omega \neq \Omega' \) and

\[ \mu_\lambda|_{SO(2, \mathbb{R})} = \bigoplus_{\Omega \in \mathcal{A}_\lambda} C \nu_2(\Omega). \quad (20) \]

In this section, we determine exactly how the decomposition in Equation 20 allows us to find the multiplicities of irreducible representations for \( M \) in irreducible represen-
tations for $SO(n, \mathbb{R})$ without first having to determine the multiplicities of irreducible representations for $M$ in irreducible representations for $SO(n-1, \mathbb{R})$.

Ultimately, we want to provide a decomposition

$$
\mu_\lambda|_M = \bigoplus_{1 \leq i \leq \text{Dim } \mu_\lambda} \mathbb{C} \nu(i)
$$

and to understand the action of $M$ on $\nu(i)$ for each $i$ such that $1 \leq i \leq \text{Dim } \mu_\lambda$. We cannot assume $\mathbb{C} \nu_2(\Omega)$ is a representation of $M$ for every element $\Omega$ in $\Psi_\lambda$. Nonetheless, we adopt as a provisional goal our understanding the manner in which $M$ acts on the vectors in set

$$
\mathcal{L} = \{ \nu_2(\Omega) \mid \Omega \in \Psi_\lambda \}
$$

via the representation of $M$ on $\mu_\lambda$ obtained by restriction. In order to gain a first foothold on a path toward this understanding, we delve into the details of the recursion defining the vectors in $\mathcal{L}$.

Supposing $n = 2k + 1$, we recall our notation from section 4, Equations 5 and 6, for the root vectors $E_\alpha$ in $\mathfrak{so}(n, \mathbb{C})$ where $\alpha$ takes values from among the positive roots

$$
\Delta^+(\mathfrak{so}(n, \mathbb{C}), \mathfrak{h}) = \{ e_i \pm e_j, e_l \mid 1 \leq i < j \leq k, \text{ and } 1 \leq l \leq k \}
$$

chosen with respect to our designation $\mathfrak{h}$ of a Cartan subalgebra for $\mathfrak{so}(n, \mathbb{C})$. To simplify notation, we refer to $\Delta^+(\mathfrak{so}(n, \mathbb{C}))$ as $\Delta^+$, just as in section 4. We impose an ordering on $\mathcal{N}_+: = \{ E_\alpha \mid \alpha \in \Delta^+ \}$ such that

$$
E_{e_k} < E_{e_{k-1}} < \cdots < E_{e_1} < E_{e_i \pm e_j}
$$

for any $i$ and $j$ such that $1 \leq i < j \leq k$. The set $\mathcal{N}_+$ comprises an ordered basis for $\mathfrak{n}_+$, the Lie subalgebra of $\mathfrak{so}(n, \mathbb{C})$ spanned by the set $\mathcal{N}_+$ of positive root vectors. Next,
we take the union $\mathcal{B}$ of a basis for $\mathfrak{h}$ and the set of negative root vectors. We extend $\mathcal{N}_+$ to an ordered basis $\mathcal{N}_+ \cup \mathcal{B}$ for $\mathfrak{so}(n, \mathbb{C})$ by choosing an arbitrary ordering for $\mathcal{B}$ and stipulating $E_\alpha < b$ for any $b$ in $\mathcal{B}$ and any $\alpha$ in $\Delta^+$. This ordering determines a Poincaré-Birkhoff-Witt or PBW basis for the universal enveloping algebra $\mathcal{U}(\mathfrak{so}(n, \mathbb{C}))$ of $\mathfrak{so}(n, \mathbb{C})$. We let $\mathfrak{b}$ be the Borel subalgebra with basis $\mathcal{B}$. Now we specify a decorated admissible tableau, $\tilde{\Omega}$ of type $\lambda$, and we define $\nu_{n-1}(\gamma_{n-1})$ as above. We write

$$\lambda = \lambda_1 e_1 + \cdots + \lambda_k e_k,$$

and

$$\gamma_{n-1} = (\gamma_{n-1})_1 e_1 + \cdots + (\gamma_{n-1})_k e_k,$$

and we recall $n = 2k + 1$.

**Lemma 6.1** For some $k$-tuple of natural numbers $(a_k, a_{k-1}, \ldots, a_1)$ and some scalar $d$, We have

$$E_{e_k}^{a_k} E_{e_{k-1}}^{a_{k-1}} \cdots E_{e_1}^{a_1} \nu_{n-1}(\gamma_{n-1}) = d\nu_n(\lambda)$$

for some scalar $d$ and where $(\lambda_j - (\gamma_{n-1})_j) = a_j$ for each $j$ such that $1 \leq j \leq k$.

**Proof** Since $\nu_n(\lambda)$ is a highest weight for $\mu_\lambda$, we know $E_1 E_2 \cdots E_p \nu_{n-1}(\gamma_{n-1}) = c\nu_n(\lambda)$ for some natural number $p$, some scalar $C$, and some $E_1, E_2, \ldots E_p$ such that $E_i$ is an element of $\mathcal{N}_+$ for each $i$ such that $1 \leq i \leq p$. By the Poincaré-Birkhoff-Witt theorem ([3] p.92), we know $\mathcal{U}(\mathfrak{so}(n, \mathbb{C}))$ is a free $\mathfrak{n}_+$-module with basis

$$\{1\} \cup \bigcup_{j \in \mathbb{N}} \{b_1 b_2 \cdots b_j \mid b_i \in \mathcal{B} \text{ for } 1 \leq i \leq j \text{ and } b_1 \leq b_2 \leq \cdots \leq b_j\}.$$ 

Hence, we can express $E_1 E_2 \cdots E_p$ as

$$\sum_{i=1}^{m} c(i) E_{i_1} E_{i_2} \cdots E_{i_{\rho(i)}}$$
for some natural number $m$ and an $m$-tuple of scalars $(c(1), \ldots, c(m))$ where for $1 \leq i \leq m$ and $1 \leq j \leq p(i)$ the expression $E_{ij}$ is an element of $\mathcal{N}_+$ and where $E_{ij} \leq E_{ij+1}$ if $j < p(i)$. We have determined

$$
\sum_{i=1}^{m} c(i) E_{i_1} E_{i_2} \cdots E_{i_{p(i)}} \nu_{n-1}(\gamma_{n-1}) = c\nu_n(\lambda). \tag{21}
$$

For any $i$ and $j$ such that $1 \leq i < j \leq k$, we know $E_{e_i e_j}$ is the image of a positive root vector for $\mathfrak{so}(n-1, \mathbb{C})$ under the differential of $\iota$ mapping $\mathfrak{so}(n-1, \mathbb{C})$ to $\mathfrak{so}(n, \mathbb{C})$, and $\nu_{n-1}(\gamma_{n-1})$ is a highest weight for the irreducible representation $\mu_{\gamma_{n-1}}$ of $\mathfrak{so}(n-1, \mathbb{C})$. Hence, in the sum from Equation 21, we may exclude all terms such that $E_{i_{p(i)}} = E_{e_i e_j}$ for some $i$ and $j$ such that $1 \leq i < j \leq k$. According to our ordering for $\mathcal{N}_+$ we may express the sum of the remaining terms as

$$
\sum_{i=1}^{m} c(i) E_{e_i, k} E_{e_{k-1}} \cdots E_{e_1} \nu_{n-1}(\gamma_{n-1})
$$

where, for each $i$ such that $1 \leq i \leq m$, we know $a_{i,j}$ is a natural number for each $j$ with $1 \leq j \leq k$. Furthermore,

$$
E_{e_i, k} E_{e_{k-1}} \cdots E_{e_1} \nu_{n-1}(\gamma_{n-1})
$$

has weight $\gamma_{n-1} + a_{i,k} e_k + a_{i,k-1} e_{k-1} \cdots + a_{i,1} e_1$, so we may remove from Equation 21 any term such that $\gamma_{n-1} + a_{i,k} e_k + a_{i,k-1} e_{k-1} \cdots + a_{i,1} e_1$ is not $\lambda$. We arrive at the equation

$$
c(i)(E_{e_i, k} E_{e_{k-1}} \cdots E_{e_1} \nu_{n-1}(\gamma_{n-1})) = c\nu_n(\lambda)
$$

for some $i$ such that $1 \leq i \leq m$ and some nonzero scalar $c(i)$. Letting $d = c/c(i)$ and defining $a_{i,j} = a_j$ for each $j$ with $1 \leq j \leq k$ gives the desired result.

We must venture one step further into the recursive process used to define the
vectors in \( \mathcal{L} \). Considering again the terminology in Equation 5, and recalling \( n = 2k + 1 \) is odd, we enumerate the positive roots of \( \mathfrak{so}(n - 1, \mathbb{C}) \):

\[
\Delta^+(\mathfrak{so}(n - 1, \mathbb{C}), \mathfrak{h}) = \{ e_i \pm e_j \mid 1 \leq i < j \leq k \}.
\]

We choose a somewhat less intuitive basis for the Lie subalgebra \( \mathfrak{n}_+(\mathfrak{so}(n - 1)) \) of \( \mathfrak{so}(n - 1, \mathbb{C}) \) than the basis \( \mathcal{N}_+ \), defining

\[
\mathcal{N}_+(\mathfrak{so}(n - 1)) = \{ E_{e_i \pm e_j} \mid 1 \leq i < j < k \}
\]

\[
\cup \{ E_{e_i + e_k} + E_{e_i - e_k} \mid 1 \leq i < k \}
\]

\[
\cup \{ E_{e_i + e_k} - E_{e_i - e_k} \mid 1 \leq i < k \}.
\]

We order \( \mathcal{N}_+(\mathfrak{so}(n - 1)) \) such that

\[
E_{e_i + e_k} - E_{e_i - e_k} < E_{e_j + e_k} + E_{e_j - e_k} < E_{e_i \pm e_p}
\]

for any \( i, j, l, \) and \( p \) such that \( 1 \leq i, j, l, p < k \) and \( l < p \), while \( (E_{e_i + e_k} - E_{e_i - e_k}) < (E_{e_i - e_k} - E_{e_i - e_k}) \) for any \( i \) with \( 1 \leq i < k \). Next, we define \( \mathcal{B} \) to be the union of a basis for \( \mathfrak{h} \) and the negative root vectors. We extend \( \mathcal{N}_+(\mathfrak{so}(n - 1)) \) to a basis \( \mathcal{N}_+(\mathfrak{so}(n - 1)) \cup \mathcal{B}(\mathfrak{so}(n - 1)) \) for \( \mathfrak{so}(n - 1, \mathbb{C}) \) and impose an ordering on this basis such that \( v < b \) for any \( v \) in \( \mathcal{N}_+(\mathfrak{so}(n - 1)) \) and any \( b \) in \( \mathcal{B}(\mathfrak{so}(n - 1)) \). This ordering determines a PBW basis for the universal enveloping algebra \( \mathfrak{U}(\mathfrak{so}(n - 1, \mathbb{C})) \). We let \( \mathfrak{b}(\mathfrak{so}(n - 1)) \) be the Borel subalgebra of \( \mathfrak{so}(n - 1, \mathbb{C}) \).

**Lemma 6.2** We have

\[
(E_{e_{k-1} + e_k} - E_{e_{k-1} - e_k})^{a_{k-1}}(E_{e_{k-2} + e_k} - E_{e_{k-2} - e_k})^{a_{k-2}} \ldots
\]

\[
\ldots(E_{e_1 + e_k} - E_{e_1 - e_k})^{a_1}v_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = dv_{n-1}(\gamma_{n-1})
\]
for some scalar $d$ and where $(\gamma_{n-1})_j - (\gamma_{n-2})_j = a_j$ for each $j$ such that $1 \leq j \leq k-1$.

**Proof** The reasoning used to prove this lemma follows more or less precisely the reasoning used to prove Lemma 6.1. Since $\nu_{n-1}(\gamma_{n-1})$ is a highest weight for $m\gamma_{n-1}$, we know

$$E_1 E_2 \cdots E_p \nu_{n-2}(\gamma_{n-2}, \gamma_{n-1}) = c \nu_{n-1}(\gamma_{n-1})$$

for some natural number $p$, some scalar $c$, and some $E_1, E_2, \ldots, E_p$ such that $E_i$ is an element of $\mathcal{N}_+(\mathfrak{so}(n-1))$ for each $i$ such that $1 \leq i \leq p$. By the Poincaré-Birkhoff-Witt theorem ([3] p.92), we know $\mathfrak{U}(\mathfrak{so}(n-1, \mathbb{C}))$ is a free $\mathfrak{n}_+(\mathfrak{so}(n-1))$-module with basis

$$\{1\} \cup \bigcup_{j \geq 1, j \in \mathbb{N}} \{b_1 b_2 \cdots b_j \mid b_i \in \mathcal{B} \text{ for } 1 \leq i \leq j \text{ and } b_1 \leq b_2 \leq \cdots \leq b_j\}.$$

Hence, we can express $E_1 E_2 \cdots E_p$ as

$$\sum_{i=1}^{m} c(i) E_{i,1}^{a_{i,1}} E_{i,2}^{a_{i,2}} \cdots E_{i,p(i)}^{a_{i,p(i)}}$$

(22)

for some natural number $m$ and an $m$-tuple of scalars $(c(1), \ldots, c(m))$ where for $1 \leq i \leq m$ and $1 \leq j \leq p(i)$ the expression $E_{i,j}$ is an element of $\mathcal{N}_+$ and where $E_{i,j} \leq E_{i,j+1}$ if $j < p(i)$. Here, $a_{i,j}$ is some natural number for each $i$ and $j$ with $1 \leq i \leq m$ and $1 \leq j \leq k-1$. For $i$ and $j$ with $1 \leq i < j < k$, the positive root vector $E_{e_i \pm e_j}$ for $\mathfrak{so}(n-1, \mathbb{C})$ is the image of a positive root vector for $\mathfrak{so}(n-3, \mathbb{C})$ under the differential of $\iota_{n-1} \circ \iota_{n-2}$ mapping $\mathfrak{so}(n-3, \mathbb{C})$ to $\mathfrak{so}(n-1, \mathbb{C})$. Somewhat less obviously, for $i$ such that $1 \leq i < k$, the vector $E_{e_i + e_k} + E_{e_i - e_k}$ in $\mathcal{N}_+$ is the image of a vector in the span of the positive root spaces for $\mathfrak{so}(n-2, \mathbb{C})$ under the differential of $\iota$ mapping $\mathfrak{so}(n-2, \mathbb{C})$ to $\mathfrak{so}(n-1, \mathbb{C})$. In fact, we see $E_{e_i + e_k} + E_{e_i - e_k}$ is the image of $2E_{e_i}(\mathfrak{so}(n-2))$ where $E_{e_i}(\mathfrak{so}(n-2))$ is the positive root vector for $\mathfrak{so}(n-2, \mathbb{C})$ corresponding to the root $e_i$ as described in Equation 6. Since $\nu_{n-2}(\gamma_{n-1}, \gamma_{n-2})$ is a
highest weight vector for $\mathfrak{so}(n-2, \mathbb{C})$, we know

$$E_{e_i, e_j} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = E_{e_i + e_k} + E_{e_i - e_k} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = 0$$

for $i$ and $j$ such that $1 \leq i < j < k$. As such, in the sum from Expression 22, we may exclude any term such that $E_{i_p(i)} > (E_{e_1 + e_k} - E_{e_1 - e_k})$. We have shown

$$\sum_{i=1}^{m} c(i)(E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{i,k-1}}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{i,k-2}} \cdots$$

$$\cdots (E_{e_1+e_k} - E_{e_1-e_k})^{a_{i,1}} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = d\nu_{n-1}(\gamma_{n-1}).$$

For any $i$ such that $1 \leq i \leq m$, we see

$$(E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{i,k-1}}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{i,k-2}} \cdots$$

$$\cdots (E_{e_1+e_k} - E_{e_1-e_k})^{a_{i,1}} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2})$$

is the sum of weight vectors, each one having weight

$$\gamma_{n-2} + \sum_{j=1}^{k-1} a_{i,j}(e_i) + z(e_k)$$

for some integer $z$. For only one $k - 1$-tuple $(a'_{k-1}, a'_{k-2}, \ldots, a'_{1})$ does

$$\gamma_{n-2} + \sum_{j=1}^{k-1} a'_j(e_j) + z(e_k) = \gamma_{n-1}$$

for some integer $z$. Hence, we may further exclude from the sum in Expression 22 any term such that $a_{i,j} \neq a'_j$ for each $j$ such that $1 \leq j \leq k - 1$. There exists exactly
one $i$ with $1 \leq i \leq m$ such that

\[
c(i)(E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_i,k-1}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_i,k-2} \ldots
\]

\[
\ldots (E_{e_1+e_k} - E_{e_1-e_k})^{a_i,1} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = c \nu_{n-1}(\gamma_{n-1})
\]

Setting $a'_j = a_{i,j}$ for each $j$ with $1 \leq j \leq k - 1$, and letting $c/c(i) = d$, we arrive at the desired result.

Lemmas 6.1 and 6.2 give us a description of the vectors in $\mathcal{L}$ precise enough to make calculating the action of $M$ on these vectors a tractable problem for odd $n$. When $n$ is even, we will rely on Lemma 4.3. We will use these lemmas within an inductive framework built to accommodate either parity for $n$. From section 3, we recall our notation $m^j$ for the element of $M$ having $j^{th}$ and $j+1^{st}$ diagonal elements equal to $-1$ and all other diagonal elements equal to 1. According to the arguments in Lemma 3.1, $\{m^j \mid 1 \leq j < n - 1\}$ generates $M$, so we focus on the action of elements from this set in determining the action of $M$ on each of the elements in $\mathcal{L}$. If $n = 2$, we understand completely the action of $m^1$ on $\nu_n(\lambda) = \nu_2(\bar{\lambda})$, and $\mathcal{L} = \{\nu_n(\lambda)\}$. By our definition for $\nu_2(\bar{\lambda})$, we notice $\nu_2(\bar{\lambda}) = c\nu_2(\bar{\pi}\bar{\lambda})$ for some scalar $c$ where $\bar{\pi}$ is the map defined in Theorem 5.4 and $\bar{\pi}\bar{\lambda}$ is a decorated semistandard Young tableau admissible for $SO(n-1, \mathbb{R})$. We may, of course, choose $\nu_2(\bar{\pi}\bar{\lambda})$ such that $\nu_2(\bar{\lambda}) = \nu_2(\bar{\pi}\bar{\lambda})$. For $j$ such that $1 \leq j < n - 1$, the element $m^j$ of $M = M(SO(n)) \subset SO(n, \mathbb{R})$ is the image of an element $m^j_{SO(n-1)}$ in $M(SO(n-1)) \subset SO(n-1, \mathbb{R})$ under the map $\iota$. The action of $m^j_{SO(n-1)}$ on the vector $\nu_2(\bar{\pi}\bar{\lambda})$ in the representation $\mu_{n-1} |_{M(SO(n-1))}$ is exactly the action of $m^j$ on the vector $\nu_2(\bar{\lambda})$ in the representation $\mu_{n-1} |_{M}$ whenever $1 \leq j < n - 1$. Hence, in the context of any inductive proof, the burden will shift to discovering the action of $m^{n-1}$ on each of the vectors in $\mathcal{L}$.
With an eye toward making use of Lemmas 6.1 and 6.2, we calculate

\[ m^{n-1}(E_{e_k}^{a_k})\nu, \]

\[ m^{n-1}(E_{e_{k-1}}^{a_{k-1}} \cdots E_{e_1}^{a_1})\nu, \text{ and} \]

\[ m^{n-1} ((E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})a_{k-1}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})a_{k-2} \cdots \]

\[ \cdots (E_{e_1+e_k} - E_{e_1-e_k})a_1)\nu \]

where \( \nu \) is any vector in a representation for \( SO(n, \mathbb{R}) \). If \( E \) is any element of \( \mathfrak{so}(n, \mathbb{C}) \) and \( \nu \) is some vector in an \( SO(n, \mathbb{R}) \)-representation, we know \( m^{n-1}(E\nu) = (\text{Ad}(m^{n-1})E)m^{n-1}\nu \). We know \( \text{Ad}(m^{n-1})E \) is conjugation of the matrix \( E \) by the matrix \( m^{n-1} \). Conjugation of \( E_{e_k} \) by \( m_{n-1} \) is \( -E_{-e_k} \), and, for \( j \) such that \( 1 \leq j < k \), conjugation of \( E_{e_j} \) by \( m^{n-1} \) is \( -E_{e_j} \). For any \( j \) such that \( 1 \leq j \leq k-1 \), conjugation of \( E_{e_{j+e_k}} \) by \( m^{n-1} \) is \( E_{e_j-e_k} \) and conjugation of \( E_{e_j+e_k} \) by \( m^{n-1} \) is \( E_{e_j+e_k} \). We see Equation 23 is

\[ ((-1)^{a_k}E_{-e_k})m^{n-1}\nu, \]

while Equation 24 is

\[ ((-1)^{a_k-1+a_{k-1}+\cdots+a_1}E_{e_{k-1}} \cdots E_{e_1})m^{n-1}\nu. \]

Also, Equation 24 is

\[ (-1)^{a_{k-1}+a_{k-1}+\cdots+a_1}((E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})a_{k-1}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})a_{k-2} \cdots \]

\[ \cdots (E_{e_1+e_k} - E_{e_1-e_k})a_1)m^{n-1}\nu. \]

For the most part, describing the action of \( m^{n-1} \) reduces to performing computations in \( \mathfrak{so}(3, \mathbb{C}) \) and \( SO(3, \mathbb{R}) \). In order to study representations of \( SO(3, \mathbb{R}) \), we turn to the simply connected covering group for \( SO(3, \mathbb{R}) \) and study explicit descriptions
of representations for that covering group. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and these basis elements obey the relations

$$[Z, X] = 2X, \quad [Z, Y] = -2Y, \quad \text{and} \quad [X, Y] = Z.$$  

We determine a map $\phi: \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{so}(3, \mathbb{C})$ by defining $\phi$ on basis vectors for $\mathfrak{sl}(2, \mathbb{C})$ as follows:

$$X \mapsto E_{e_1}, \quad Y \mapsto -E_{-e_1}, \quad \text{and} \quad Z \mapsto H: = \begin{pmatrix} 0 & 2i & 0 \\ -2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Clearly, $E_{e_1}, E_{-e_1}$, and $H$ form a basis for $\mathfrak{so}(3, \mathbb{C})$, and a quick calculation shows

$$[E_{e_1}, -E_{-e_1}] = H, \quad [Y, E_{e_1}] = 2E_{e_1}, \quad \text{and} \quad [Y, -E_{-e_1}] = 2E_{e_1},$$

so we see $\mathfrak{sl}(2, \mathbb{C})$ is isomorphic to $\mathfrak{so}(3, \mathbb{C})$. Now $SL(2, \mathbb{C})$ is simply connected, hence a covering group for $SO(3, \mathbb{C})$, and there exists a unique surjective homomorphism of Lie groups $\Gamma$ such that the diagram

$$\begin{array}{ccc}
\mathfrak{sl}(2, \mathbb{C}) & \xrightarrow{\phi} & \mathfrak{so}(3, \mathbb{C}) \\
\exp \downarrow & & \downarrow \exp \\
SL(2, \mathbb{C}) & \xrightarrow{\Gamma} & SO(3, \mathbb{C})
\end{array}$$
commutes. We construct $\Gamma$ explicitly. First, we choose a new basis

$$
X' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
$$

for $\mathfrak{sl}(2, \mathbb{C})$, and we let $\langle , \rangle$ be the symmetric inner product on $\mathfrak{sl}(2, \mathbb{C})$ given by $\langle A, B \rangle = \text{tr}(AB)$. With respect to this inner product, the basis $X', Y', Z'$ is orthogonal, and each basis vector has length $-2$. The adjoint action of $SL(2, \mathbb{C})$ on $\mathfrak{sl}(2, \mathbb{C})$ is orthogonal with respect to this inner product since conjugation leaves trace invariant. We get a map

$$
\Gamma': SL(2, \mathbb{C}) \to O(3, \mathbb{C}) \subset \text{Aut}(\mathfrak{sl}(2, \mathbb{C})),
$$
determined completely by the adjoint action of $SL(2, \mathbb{C})$ on the basis vectors $X', Y', Z'$. Since $SL(2, \mathbb{C})$ is connected, $\Gamma'(SL(2, \mathbb{C}))$ is connected, and $1_{SL(2, \mathbb{C})} \mapsto 1_{O(3, \mathbb{C})}$, so $\Gamma'(SL(2, \mathbb{C})) \subset SO(3, \mathbb{C})$. In writing $\Gamma'(g)$ as a matrix for any $g$ in $SL(2, \mathbb{C})$, we need to order $X', Y', Z'$ in such a way as to ensure agreement between the Cartan subalgebra $\mathfrak{h}(\mathfrak{sl}(2))$ for $\mathfrak{sl}(2, \mathbb{C})$ spanned by $Z'$ and our choice for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(3, \mathbb{C})$. We notice

$$
\text{Ad}(\exp Z')Z' = Z', \quad \text{Ad}(\exp Z')X' = \left( \frac{1}{2}e^{2i}(X' + iY') \right) + \left( \frac{1}{2}e^{-2i}(X' - iY') \right), \quad \text{and}
$$

$$
\text{Ad}(\exp Z')Y' = \left( \frac{1}{2}e^{-2i}(Y' + iX') \right) + \left( \frac{1}{2}e^{2i}(Y' - iX') \right).
$$

Hence, we determine the matrix form of $\Gamma'(g)$ for any $g$ in $SL(2, \mathbb{C})$ such that, if $\text{Ad}(g)(X') = a_1X' + b_1Y' + c_1Z'$, $\text{Ad}(g)(Y') = a_2X' + b_2Y' + c_2Z'$, and $\text{Ad}(g)(Z') = \ldots$. 

(29)
\[ a_3 X' + b_3 Y' + c_3 Z', \]
then
\[ \Gamma'(g) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \]

By Equations 29, \( \Gamma' \) maps \( \exp Z' \) to the torus for \( SO(3, \mathbb{C}) \) corresponding to our choice for \( \mathfrak{g} \). By calculating \( \Gamma' \) on \( \exp X' \), \( \exp Y' \), and \( \exp Z' \), we see \( \Gamma' \) is the surjective map \( \Gamma \). The irreducible representations of \( SO(3, \mathbb{R}) \) correspond exactly to the dominant analytically integral forms of \( \mathfrak{so}(3, \mathbb{C}) \). Since \( SL(2, \mathbb{C}) \) is simply connected, any dominant weight for \( \mathfrak{sl}(2, \mathbb{C}) \) is analytically integral for \( SL(2, \mathbb{C}) \). Every irreducible representation for \( SO(3, \mathbb{C}) \) descends from a representation of \( SL(2, \mathbb{C}) \) via the map \( \Gamma \). (This fact follows from the arguments in [5] 5.110.) We denote by \( I_n \) the \( n \times n \) diagonal matrix with each diagonal entry equal to 1. The kernel of \( \Gamma \) is \( \{ \pm 1_{SL(2, \mathbb{C})} \} \), so \( \text{SO}(3, \mathbb{C}) \) is the subset of \( SL(2, \mathbb{C}) \) consisting of all representations such that \(-I_2\) acts as the identity. We take a description of \( SL(2, \mathbb{C}) \) from [1] p. 117. We define \( V_a \) to be the space of homogeneous polynomials of degree \( a \) in two variables, \( z_1 \) and \( z_2 \). The space \( V_a \) has basis
\[ \{ x_j = \binom{m}{j} z_1^{a-j} z_2^j \mid 0 \leq j \leq a \}, \]
and \( SL(2, \mathbb{C}) \) acts on any polynomial \( P(z_1, z_2) \) in \( V_a \) by
\[ \begin{pmatrix} b & c \\ d & e \end{pmatrix} \cdot P(z_1, z_2) = P(bz_1 + dz_2, cz_1 + ez_2) \quad \text{for} \quad \begin{pmatrix} b & c \\ d & e \end{pmatrix} \in SL(2, \mathbb{C}). \]

Any irreducible representation for \( SL(2, \mathbb{C}) \) is, in particular, a representation for \( \mathfrak{sl}(2, \mathbb{C}) \). We calculate the action of the basis \( X, Y, \) and \( Z \) for \( \mathfrak{sl}(2, \mathbb{C}) \) on \( V_a \) as in [1]
We see $x_0$ is a highest weight of weight $a$. Clearly, $V_a$ is isomorphic to the representation of highest weight $a/2e_1$ for $so(3, \mathbb{C})$. For any $j$ such that $0 \leq j \leq a$, we know

$$-I_2 \cdot \binom{a}{j} z_1^{a-j} z_2^j = \binom{a}{-j} (-1)^a z_1^{a-j} z_2^j.$$ 

Hence, the representation $V_a$ descends to $SO(3, \mathbb{C})$ if and only if $a$ is even. The element $m^2$ of $M(SO(3, \mathbb{C}))$ has preimage $\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}$ under $\Gamma$ because

$$\text{Ad} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) X' = X', \quad \text{Ad} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) Y' = -Y', \quad \text{and} \quad \text{Ad} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) Z' = -Z'.$$

Since $SO(3, \mathbb{R})$, the group of real points in $SO(3, \mathbb{C})$, is compact, we know $SO(3, \mathbb{R})$ is exactly the set of restrictions of $SO(3, \mathbb{C})$ to $SO(3, \mathbb{R})$.

For the following two theorems, we choose $\lambda$ to be an analytically integral form for $SO(3, \mathbb{R})$, and by $M$ we mean $M(SO(3, \mathbb{R}))$. We fix some $\bar{\Omega}$ in $\bar{\Psi}_\lambda$, and we get $\nu_2(\bar{\Omega}) = \nu_2(\gamma_2)$ in $\mathcal{L}$.

**Theorem 6.3** If $\gamma_2 = 0$ and $\lambda = a(e_1)$ for some natural number $a$, then

$$m^2(\nu_2(\bar{\Omega})) = (-1)^a(\nu_2(\bar{\Omega})).$$

**Proof** We work with the $sl(2, \mathbb{C})$ representation $V_{2a}$ isomorphic to the representation of $so(3, \mathbb{C})$ of highest weight $a(e_1)$. Choosing $x_0 = \nu_3(\lambda)$, we find $\nu_2(\bar{\Omega})$ is some scalar multiple $c$ of the basis vector $x_a$ in $V_{2a}$ with weight 0. Now $m^2 x_0 = (-z_2)^{2a} = z_2^{2a} = \cdots$. 

10.7 to find

$$Z x_j = (a - 2j)x_j, \quad X x_j = (a - j + 1)x_{j-1}, \quad \text{and} \quad Y x_j = -(j + 1)x_{j+1}. \quad (30)$$
$x_{2a}$. Using the action of $\mathfrak{sl}(2,\mathbb{C})$ on $V_{2a}$ given by Equations 30, we find

$$E_{e_1}^a x_m = E_{-e_1}^a x_0 = \left( \sum_{i=1}^a i \right) x_a, \text{ so }$$

$$E_{e_1}^a m^2 \nu_3(\lambda) = E_{-e_1}^a \nu_3(\lambda)$$

(31)

From Lemma 6.1 we know we can choose the scalar $c$ such that $\nu_3(\lambda) = E_{e_1}^a \nu_2(\Omega)$. The following equation holds:

$$E_{e_1}^a m^2 \nu_3(\lambda) = E_{e_1}^a m^2 E_{e_1}^a \nu_2(\Omega)$$

$$= E_{e_1}^a (-1)^a E_{-e_1}^a m^2 \nu_2(\Omega) \text{ (by Equation 26)}$$

$$= (-1)^a E_{e_1}^a E_{-e_1}^a m^2 \nu_2(\Omega)$$

$$= E_{-e_1}^a E_{e_1}^a \nu_2(\Omega) \text{ (by Equation 31)}.$$

We want to show

$$E_{e_1}^a E_{e_1}^a \nu_2(\Omega) = E_{e_1}^a E_{-e_1}^a \nu_2(\Omega).$$

(**)  

Considering the expression

$$E_{e_1 e_1} E_{e_2 e_1} \cdots E_{e_{2a} e_1} \nu_2(\Omega),$$

(32)

where, for each $i$ such that $1 \leq i \leq 2a$, we have $\epsilon_i = \pm 1$ and $\epsilon_i = 1$ for exactly half of the natural numbers $i$ with $1 \leq i \leq 2a$, we attempt to find the highest value for $j$ such that $\epsilon_j = -1$ and $\epsilon_{j+1} = 1$. If no such $j$ exists, Expression 32 is $E_{e_1}^a E_{-e_1}^a \nu_2(\Omega)$. Otherwise, Expression 32 is

$$E_{e_1 e_1} \cdots E_{e_{j+1} e_1} E_{e_j e_1} \cdots E_{e_{2a} e_1} \nu_2(\Omega)$$

$$+ E_{e_1 e_1} \cdots [E_{e_j e_1}, E_{e_{j+1} e_1}] \cdots E_{e_{2a} e_1} \nu_2(\Omega).$$
We induct on $2a - 1 - j$ to show the second term in Expression 33 is 0. If $j = 2a - 1$, the second term in Expression 33 is 0 since $\nu_2(\bar{\Omega})$ has weight 0 and $[E_{e_j e_1}, E_{e_{j+1} e_1}]$ is in the subspace of $\mathfrak{sl}(2, \mathbb{C})$ spanned by $Z$. Otherwise, the second term in Expression 33 is

$$E_{e_1 e_1} \cdots E_{e_{j+2} e_1} [E_{e_j e_1}, E_{e_{j+1} e_1}] E_{e_{j+3} e_1} \cdots E_{e_{2a} e_1} \nu_2(\bar{\Omega}) + E_{e_1 e_1} \cdots E_{e_{j-1}} [E_{e_j e_1}, E_{e_{j+1} e_1}] E_{e_{j+3} e_1} \cdots E_{e_{2a} e_1} \nu_2(\bar{\Omega}).$$

The second term in Expression 33 has weight not equal to 0, so, since Expression 32 has weight 0, the second term in Expression 33 must be 0. By inductive hypothesis, the first term in Expression 33 is 0. Hence, the second term in Expression 33 is 0, and we may freely interchange $E_{e_l e_1}$ and $E_{e_{l+1} e_1}$ in Expression 32 for any $l$ such that $1 \leq l < 2a$. Thus, Equation ** holds.

To conclude, we combine Equations * and ** to see

$$(-1)^a E_{e_1}^a E_{-e_1}^a m^2 \nu_2(\bar{\Omega}) = E_{e_1}^a E_{-e_1}^a \nu_2(\bar{\Omega}),$$

so $m^2 \nu_2(\bar{\Omega}) = (-1)^a \nu_2(\bar{\Omega})$, as desired.

For the next theorem, we specify $\Omega$, the admissible semistandard Young tableau associated to $\bar{\Omega} = (\Omega, \epsilon)$, where we use the term “associated” in the sense described by section 5

**Theorem 6.4** If $\lambda = a(e_1)$ and $\nu_2(\Omega, \epsilon)$ has weight $b(e_i)$ with $1 \leq b \leq a$, we may choose $\nu_2(\Omega, \epsilon)$ and $\nu_2(\Omega, -\epsilon)$ such that $m^2 \nu_2(\Omega, \epsilon) = \nu_2(\Omega, -\epsilon)$.

**Proof** As in Theorem 6.3, we choose $\nu_3(\lambda) = x_0$, identifying $\mu_\lambda$ with the $SL(2, \mathbb{R})$ representation $V_{2a}$. We know $m^2 \nu_3(\lambda) = x_m$. Also, by Lemma 6.1, we may choose
\[ \nu_2(\Omega, \epsilon) \text{ such that } \nu_3(\lambda) = E_{\epsilon_1}^{a-b} \nu_2(\Omega, \epsilon). \] Hence, we may use Equation 26 to see

\[ (-1)^{a-b} E_{-\epsilon_1}^{a-b} m^2 \nu_2(\Omega, \epsilon) = x_m. \]

This equation shows the weight of \( m^2 \nu_2(\Omega, \epsilon) \) is \(-2b\) (working even still within the representation \( V_{2a} \)). There exists only one weight space of weight \(-2b\) in \( V_{2a} \), and \( \nu_2(\Omega, -\epsilon) \) must have weight \(-2b\), so we can choose \( m^2 \nu_2(\Omega, \epsilon) = \nu_2(\Omega, -\epsilon) \).

Beyond the case \( n = 3 \), a complete understanding of the action of \( M \) on \( \mathcal{L} \) proves somewhat too complicated to pursue all at once. To continue, we narrow our focus to a small portion of the larger project. For each \( \Omega \) in \( \Psi_\lambda \), we consider a subset \( \mathcal{L}_\Omega \subset \mathcal{L} \) defined as

\[ \mathcal{L}_\Omega: = \{ \nu_2(\tilde{\Omega}) \in \mathcal{L} \mid \Omega \text{ is associated to } \tilde{\Omega} \}. \]

In addition, we consider a subset \( M_\Omega \subset \{ m^j \mid 1 \leq j < n \} \) defined as

\[ M_\Omega: = \{ m^j \mid j \in \mathcal{C}_\Omega \}, \]

where \( j \) is an element of \( \mathcal{C}_\Omega \subset \{ 1, \ldots, n-1 \} \) if and only if \( j \) is even and \( \Omega(1, -j/2) = j \). If \( |\mathcal{C}_\Omega| = l \), then \( |\mathcal{L}_\Omega| = 2^l \). We fix some \( \tilde{\Omega} = (\Omega, (\epsilon_1, \ldots, \epsilon_{n/2})) \).

**Theorem 6.5** For each \( j \) in \( \mathcal{C}_\Omega \), the element \( m^j \) in \( M_\Omega \) acts on \( \nu_2(\Omega, (\epsilon_1, \ldots, \epsilon_{n/2}) \) according to the equation

\[ m^j \nu_2(\Omega, (\epsilon_1, \ldots, \epsilon_{n/2})) = \nu_2(\Omega, (\epsilon_1, \ldots, \epsilon_{j/2-1}, -\epsilon_{j/2}, \epsilon_{j/2+1}, \ldots, \epsilon_{n/2}) \).

for some choice of elements in the set \( \mathcal{L}_\Omega \).

**Proof** For \( n = 2 \) the statement of the theorem is empty, and for \( n = 3 \) the statement amounts to Theorem 6.4. We suppose \( \mathcal{C}_\Omega \) is nonempty, as, otherwise, the statement of the theorem is empty. We prove the result for general \( n \) by induction. If \( j < n - 1 \),
our inductive hypothesis shows the preimage of $m^j$ under $\iota$ acts on $\nu_2(\pi(\Omega))$ according to the equation

$$m^j \nu_2(\pi(\Omega), (\epsilon_1, \ldots, \epsilon_{n-1/2}))$$

$$= \nu_2(\pi(\Omega), (\epsilon_1, \ldots, \epsilon_j/2-1, -\epsilon_j/2, \epsilon_j/2+1, \ldots, \epsilon_{n-1/2})) .$$

Hence, $m^j$ acts on $\nu_2(\Omega)$ as desired.

It remains to prove the statement for $j = n - 1$, in which case $n = 2k + 1$ is odd. In this case, we consider the subgroup $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ of $SO(n, \mathbb{R})$ where we identify $SO(n-3, \mathbb{R})$ with the upper left $n-3$ by $n-3$ block diagonal part of $SO(n, \mathbb{R})$ and $SO(3, \mathbb{R})$ with the lower right 3 by 3 block diagonal part of $SO(n, \mathbb{R})$. We focus on the action of the subgroup $SO(3, \mathbb{R})$ of $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ on $\nu_n(\lambda)$ where we identify $SO(3, \mathbb{R})$ with the image of $SO(3, \mathbb{R})$ under the inclusion $i: SO(3, \mathbb{R}) \to SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ given by the mapping $g \mapsto (1_{SO(n-3, \mathbb{R})}, g)$. We examine the irreducible $SO(3, \mathbb{R})$-representation $\tilde{\mu}_\lambda$ generated by $\nu_n(\lambda)$. By Lemma 6.1, $\nu_n(\lambda) = E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$. The weight vector $E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ is a highest weight vector for $SO(2, \mathbb{R})$ where we identify $SO(2, \mathbb{R})$ with the image of $SO(2, \mathbb{R})$ under the usual inclusion $i$. Moreover, since $\Omega(1, -j/2) = j$, we know $E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ has weight $b(e_1)$ not equal to 0. By Theorem 6.4, the element $m^2(SO(3))$ of $M(SO(3))$ maps $E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ to a highest weight vector for $SO(2, \mathbb{R})$ of weight $-b(e_1)$. Now, $m^2(SO(3))$ maps under $i$ to the element $(1, m^2(SO(3)))$ and $(1, m^2(SO(3)))$ maps to $m^j$ under the natural inclusion of $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ in $SO(n, \mathbb{R})$. We now consider the entirety of the $SO(n, \mathbb{R})$ representation $\mu_\lambda$. In this representation, $E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ has weight $b_1(e_1) + \cdots + b_k(e_k)$ where $b_k = b$. According to our understanding of the action of $m^2(SO(3))$ on $\tilde{\mu}_\lambda$, the element $m^j$ of $M$ maps $E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ to a vector of weight $b_1(e_1) + \cdots + b_{k-1}(e_{k-1}) - b_k(e_k)$. By our
definition for $\nu_2(\Omega)$, and by Lemmas 6.1 and 6.2, we know
\[
E_{k+1}^{a_1} \cdots E_{k+1}^{a_k-1} \nu_{n-1}(\gamma_{n-1}) = E_{k+1}^{a_1} \cdots E_{k+1}^{a_k-1} \\
\cdots \cdots (E_{k+1}^{a_1} - E_{k+1}^{a_2} + E_{k+1}^{a_3} - \cdots - E_{k+1}^{a_k}) U \nu_2(\Omega)
\]

for some $U$ in the universal enveloping algebra of the image of $(so(n-2, C))$, the complexification of the Lie algebra for $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, under the differential of $\iota_{n-1} \circ \iota_{n-2}$. We know $m^j$ acts as the identity on $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, so, combining Equations 27 and 28, we see
\[
m^j(E_{k+1}^{a_1} \cdots E_{k+1}^{a_k-1} \nu_{n-1}(\gamma_{n-1}) = \pm E_{k+1}^{a_1} \cdots E_{k+1}^{a_k-1} \\
\cdots \cdots (E_{k+1}^{a_1} + E_{k+1}^{a_2} - E_{k+1}^{a_3} + \cdots + E_{k+1}^{a_k}) U m^j-1 \nu_2(\Omega)
\]

Considering the weight of $m^j(E_{k+1}^{a_1} \cdots E_{k+1}^{a_k-1} \nu_{n-1}(\gamma_{n-1})$, we know
\[
m^j-1 \nu_2(\Omega, (\epsilon_1, \ldots, \epsilon_k)) = \pm \nu_2(\Omega, (\epsilon_1, \ldots, -\epsilon_k)).
\]

Suppose $G$ is a finite abelian group generated by $g_1, g_2, \ldots, g_n$ where $g_i^2 = 1$ for each $i$ such that $1 \leq i \leq n$. For any subset $S$ of $\{1, \ldots, n\}$, we define $g_S$ to be $\prod_{i \in S} g_i$. We consider $V$, a $2^n$-dimensional representation of $G$ with basis
\[
\beta: = \{gS\nu \mid S \subset \{1, \ldots, n\}\}
\]
for some $\nu$ in $V$.

**Lemma 6.6** The representation $V$ decomposes into the sum $\bigoplus_{S \subset \{1, \ldots, n\}} V_S$ where, if $\nu_S$ is an element of $V_S$, we have $g_i\nu_S = \nu_S$ when $i$ is an element of $S$ and $g_i\nu_S = -\nu_S$.
when \( n \) is an element of \( S^C \).

**Proof** The representation \( V \) is the regular representation of \( G \), so the lemma is a special case of the Peter-Weyl theorem. We produce a simple proof so as to keep this section self-contained. We induct on \( n \). If \( n = 1 \), we have \( \{\nu, g_1\nu\} \) as a basis of \( V \). Then \( \{\nu + g_1\nu, \nu - g_1\nu\} \) is also a basis of \( V \). Moreover, \( g_1(\nu + g_1\nu) = g_1\nu + \nu \), and \( g_1(\nu - g_1\nu) = g_1\nu - \nu \). Hence, \( \nu + g_1\nu \) generates the irreducible representation \( V_0 \), and \( \nu - g_1\nu \) generates the irreducible representation \( V_1 \).

In general, \( V \) has \( \{g_S\nu \mid n \not\in S\} \cup \{g_S\nu \mid n \not\in S\} \) as a basis. We consider the basis \( \beta(n) \cup \beta(n)' \) where \( \beta(n) = \{g_S\nu + g_{S\setminus\{n\}}\nu \mid n \in S\} \) and \( \beta(n)' = \{g_S\nu - g_{S\setminus\{n\}}\nu \mid n \in S\} \). If \( \tilde{G} \) is the subgroup of \( G \) generated by \( \{g_1, \ldots, g_{n-1}\} \), then consider the representations \( V(n) \) and \( V(n)' \) of \( \tilde{G} \) generated by \( \beta(n) \) and \( \beta(n)' \) respectively. For any subset \( S_0 \) of \( \{1, \ldots, n\} \) such that \( n \) is an element of \( S \), we let \( \tilde{S}_0 \) be the set \( S_0 \setminus \{n\} \). Then \( g_{\tilde{S}_0}(g_n\nu + \nu) = g_{\tilde{S}_0}\nu + g_{\tilde{S}_0 \setminus \{n\}}\nu \). Similarly, \( g_{\tilde{S}_0}(g_n\nu - \nu) = g_{\tilde{S}_0}\nu - g_{\tilde{S}_0 \setminus \{n\}}\nu \). Hence, both \( V(n) \) and \( V(n)' \) are \( 2^{n-1} \)-dimensional representations of \( \tilde{G} \), and

\[
\beta(n) = \left\{ g_{\tilde{S}}(g_n\nu + \nu) \mid \tilde{S} \subset \{1, \ldots, n-1\} \right\}, \quad \text{whereas} \quad \\
\beta(n)' = \left\{ g_{\tilde{S}}(g_n\nu - \nu) \mid \tilde{S} \subset \{1, \ldots, n-1\} \right\}.
\]

We may apply our inductive hypothesis to \( V(n) \) and \( V(n)' \) to show

\[
V(n) = \oplus_{\tilde{S}} \subset \{1, \ldots, n-1\} V(n)_{\tilde{S}}, \quad \text{and} \quad \\
V(n)' = \oplus_{\tilde{S}} \subset \{1, \ldots, n-1\} V(n)'_{\tilde{S}}
\]

where \( g_i\nu_{\tilde{S}} = \nu_{\tilde{S}} \) when \( i \) is an element of \( \tilde{S} \) and \( g_i\nu_{\tilde{S}} = -\nu_{\tilde{S}} \) when \( n \) is an element of \( S^C \) for any \( \nu_{\tilde{S}} \) in \( V(n)_{\tilde{S}} \cup V(n)'_{\tilde{S}} \). If we let \( V_{S_0} = V(n)_{\tilde{S}_0} \) and \( V_{S_0 \setminus \{n\}} = V(n)'_{\tilde{S}_0} \) for any
subset $S_0$ of $\{1, \ldots, n\}$, the decomposition

$$V = \oplus_{S \subset \{1, \ldots, n\}} V_S$$

has the desired property. \hfill \blacksquare

**Remark 6.7** We suppose $C_\Omega \subset \{1, \ldots, n-1\} = \{i_1, \ldots, i_l\}$, and we fix a bijection $\theta: \{1, \ldots, 2^l\} \to \mathcal{E}$ where $\mathcal{E}$ is the set of subsets of $C_\Omega$. Applying Lemma 6.6 to Theorem 6.5, we can, for each $\Omega$ in $\Psi_{\lambda}$, find a basis $\{\nu(\Omega)_1, \ldots, \nu(\Omega)_{2^l}\}$ for the subspace of $\mu_\lambda$ spanned by $L_\Omega$ with the following property. For each $j$ and each $p$ such that $1 \leq j \leq l$ and $1 \leq p \leq 2^l$, we have

$$m_j^{\nu(\Omega)} = \begin{cases} \nu(\Omega)_p & i_j \in \theta(p), \\ -\nu(\Omega)_p & \text{otherwise}. \end{cases}$$

Fixing any $\Omega$ in $\Psi_{\lambda}$, we must now consider the action of $m_j^{\nu_2(\Omega)}$ on each vector $\nu_2(\Omega)$ where $\Omega$ is associated to $\bar{\Omega}$ and for each $j$ such that $j$ is an element of the complement $C_{\bar{\Omega}}^c$ of $C_\Omega$ in $\{1, \ldots, n-1\}$. We start with the even elements of $C_{\bar{\Omega}}^c$.

**Theorem 6.8** If $j$ is an even element of $C_{\bar{\Omega}}^c$, then

$$m_j^{\nu_2(\Omega)} = \epsilon(\nu_2(\Omega))$$

where $\epsilon = (-1)^{a_1 + \cdots + a_{[j+1/2]}}$ for $\gamma_j+1 - \gamma_{j-1} = a_1(e_1) + \cdots + a_{[j+1/2]}(e_{[j+1/2]})$.

**Proof** It suffices to consider the action of $m_j^{\nu_2(SO(j + 1))}$ on the weight vector

$$\nu_2(\pi_{j+2} \circ \pi_{j+3} \circ \cdots \circ \pi_n(\bar{\Omega}))$$
in the representation $\mu_{\gamma_{j+1}}$ for

$$\iota_{n-1} \circ \iota_{n-2} \circ \cdots \circ \iota_{j+1}(SO(j+1, \mathbb{R})),$$

so we may as well assume $j = n - 1$. We also write $n = 2k+1$. As in Theorem 6.5, we identify $SO(3, \mathbb{R})$ with the image of $SO(3, \mathbb{R})$ under the map $\tilde{i}$ followed by the natural inclusion of $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ in $SO(n, \mathbb{R})$. We study the irreducible $SO(3, \mathbb{R})$ representation $\tilde{\mu}_\lambda$ generated by $\nu_n(\lambda)$. By Lemma 6.1, $\nu_n(\lambda) = E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$.

The weight vector $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ is a highest weight vector for $SO(2, \mathbb{R})$ where we identify $SO(2, \mathbb{R})$ with the image of $SO(2, \mathbb{R})$ under the usual inclusion $\iota$.

Since $j$ is an element of $C^2_{\Omega}$, we know $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ has weight equal to 0. By Theorem 6.3,

$$m^2(SO(3))(E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}) = (-1)^{a_k} E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}). \tag{34}$$

Now, $m^2(SO(3))$ maps under $\tilde{i}$ to the element $(1, m^2(SO(3)))$, and $(1, m^2(SO(3)))$ maps to $m^j$ under the natural inclusion of $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ in $SO(n, \mathbb{R})$. We consider the entirety of the $SO(n, \mathbb{R})$ representation $\mu_{\lambda}$. By our definition for $\nu_2(\tilde{\Omega})$, and by Lemmas 6.1 and 6.2, we know

$$E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}) = E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \cdots \cdots (E_{c_k-e_k} - E_{c_k-e_k})^{b_{k-1}} \cdots (E_{c_1+e_k} - E_{c_1-e_k})^{b_1} U \nu_2(\tilde{\Omega})$$

for some $U$ in the universal enveloping algebra of the image of $(\mathfrak{so}(n-2, \mathbb{C}))$, the complexification of the Lie algebra for $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, under the differential of $\iota_{n-1} \circ \iota_{n-2}$. We know $m^j$ acts as the identity on $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, so,
combining Equations 27 and 28, we see

\[ m^j(E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}) = (-1)^{a_{k-1}+\cdots+a_1+b_{k-1}+\cdots+b_1} E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \cdots \] \[ \cdots (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{b_{k-1}} \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{b_1} U m^{j-1} \nu_2(\bar{\Omega}). \] Combining Equations 34 and 35, we see

\[ m^j \nu_2(\bar{\Omega}) = (-1)^{a_{k-1}+\cdots+a_1+b_{k-1}+\cdots+b_1} \nu_2(\bar{\Omega}). \] Clearly, \( \lambda - \gamma_{n-2} = b_1 + a_1(e_1) + \cdots + b_{k-1}a_{k-1}(e_{k-1}) + a_k(e_k) \).

By Theorem 6.8, \( m^j \) acts by a scalar on the space spanned by \( L_{\Omega} \) for any even \( j \) in \( C_{\Omega} \). Combining this observation with remark 6.7, we see the vector \( \nu(\Omega)_p \) for any \( p \) with \( 1 \leq p \leq 2^j \) spans a representation for \( M_{\text{even}} \), the subgroup of \( M \) generated by \( \{m^j \mid j \text{ is even}\} \). We want to show, for each \( \Omega \) in \( \Psi_{\lambda} \) and each \( p \) such that \( 1 \leq p \leq 2^j \), the vector \( \nu(\Omega)_p \) spans a representation for all of \( M \), and we want eventually to describe this action completely. We will do so by showing \( m^j \) acts by a scalar on the space spanned by \( L_{\Omega} \) for any odd \( j \) such that \( 1 \leq j \leq n - 1 \). For even \( n \), we do already know the action of one element of \( M \) on any vector in \( \mu_{\lambda} \). Namely, we know the action of \( -I_n \) on any vector in \( \nu_n(\lambda) \) by the arguments in Lemma 4.3. The element \( -I_n \) acts by \( \pm 1 \) as determined by the weight of \( \nu_n(\lambda) \): if \( \lambda = a_1(e_1) + \cdots + a_{n/2}(e_{n/2}) \), then \( -I_n v = (-1)^{a_1+\cdots+a_{n/2}} v \) for any vector \( v \) in \( \mu_{\lambda} \). We define

\[ \epsilon(\lambda) = (-1)^{a_1+\cdots+a_{n/2}}, \] and we consider any two decorated admissible tableaux \( \bar{\Omega} \) and \( \bar{\Omega}' \) such that \( \Omega \) is associated to both \( \bar{\Omega} \) and \( \bar{\Omega}' \).

**Theorem 6.9** For any odd \( j \) with \( 1 \leq j \leq n - 1 \), the element \( m^j \) of \( M \) acts on \( \nu_2(\bar{\Omega}) \)
by $\epsilon(\Omega)_j$ where

$$\epsilon(\Omega)_j = \epsilon(\gamma_{j+1})\epsilon(\gamma_{j-1}).$$

Moreover, for each odd $j$ with $1 \leq j \leq n - 1$, the element $m^j$ acts on $\nu_2(\Omega')$ by $\epsilon(\Omega)_j$ as well.

**Proof** For any odd $j$ such that $1 \leq j \leq n - 1$, we may calculate the action of $m^j$ on $\nu_2(\Omega)$ by determining the action of $m^j(SO(j + 1))$ on the weight vector

$$\nu_2(\tilde{\pi}_{j+2} \circ \tilde{\pi}_{j+3} \circ \cdots \circ \tilde{\pi}_n(\tilde{\Omega}))$$

in the representation $\mu_{\gamma_{j+1}}$ for

$$\iota_{n-1} \circ \iota_{n-2} \circ \cdots \circ \iota_{j+1}(SO(j + 1, \mathbb{R})),$$

so we may as well assume $j = n - 1$. Now,

$$-I_n(\iota_{n-1} \circ \iota_{n-2}(-I_{n-2})) = m^j,$$

so the first statement follows.

As for the second statement, we compare $\gamma_{j-1}$, defined with respect to $\tilde{\Omega}$, and $\gamma'_{j-1}$, defined with respect to $\tilde{\Omega}'$. The weights $\gamma_{j-1}$ and $\gamma'_{j-1}$ can differ only in the sign of the coefficients for $e_{j-1/2}$. According to the definitions for $\epsilon(\gamma_{j-1})$ and $\epsilon(\gamma'_{j-1})$, we have

$$\epsilon(\gamma_{j-1}) = \epsilon(\gamma'_{j-1}).$$

The second statement follows.

**Remark 6.10** Theorem 6.9 provides a simple way to calculate the action of $m^j$ on $\nu_2(\Omega)$ for any odd $j$ and any $\Omega$ in $\Psi_\lambda$. If $j = n - 1$, then, for $\lambda - \gamma_{n-2} = c_1(e_1) +$
\[ m^j(\nu_2(\Omega)) = (-1)^{c_1 + \cdots + c_{j+1/2}}. \]

Likewise, if \( 1 \leq j < n - 1 \), then, for \( \gamma_{j+1} - \gamma_j \equiv c_1(e_1) + \cdots + c_{j+1/2}(e_{j+1/2}) \), we have

\[ m^j(\nu_2(\Omega)) = (-1)^{c_1 + \cdots + c_{j+1/2}}. \]

We have shown

\[ \mu_\lambda|_M = \bigoplus_{\Omega \in \Psi_\lambda} \bigotimes_{p=1}^{2^j} \mathbb{C}\nu(\Omega)_p, \]

and we have described entirely the action of \( \{m^j \mid 1 \leq j \leq n - 1\} \), hence the action of \( M \), on each vector \( \nu(\Omega)_p \) with \( \Omega \in \Psi_\lambda \) and with \( p \) such that \( 1 \leq p \leq 2^j \). Using this information, we now introduce a map \( \xi_n : \Psi \to F_n \) where Equation 3 defines the set \( F_n \).

We define \( \xi_n \) inductively as follows. For \( n = 2 \), we have \( \Omega \mapsto \{2\} \) if \( m^1(\nu(\Omega)) = 1\nu(\Omega) \), and \( \Omega \mapsto \emptyset \) otherwise. For general \( n \), we may assume \( \xi_{n-1}(\pi\Omega) = \hat{S} \) where \( \hat{S} \) is some element of \( S_{n-1} \). For even \( n \), if \( m^{n-1}\nu(\Omega) = -\nu(\Omega) \), and if \( n - 1 \) is an element of \( \hat{S} \), then we define \( \xi_n \) such that \( \xi_n(\Omega) = \hat{S} \). If \( n - 1 \) is not an element of \( \hat{S} \), we define \( \xi_n \) such that \( \xi_n(\Omega) = \hat{S} \cup \{n\} \). On the other hand, if \( m^{n-1}\nu(\Omega) = \nu(\Omega) \), we define \( \xi_n \) such that \( \xi_n(\Omega) = \hat{S} \cup \{n\} \) when \( n - 1 \) is an element of \( \hat{S} \) and such that \( \xi_n(\Omega) = \hat{S} \) when \( n - 1 \) is not an element of \( \hat{S} \). Now we suppose \( n \) is odd. We define \( S' \) to be \( \hat{S} \cup \{n\} \) if \( \hat{S} \cup \{n\} \) is an element of \( S_n \). Otherwise, we define \( S' \) to be \( \{1, \ldots, n\} \setminus \hat{S} \cup \{n\} \). If \( n - 1 \) is an element of \( C_{\Omega} \), we define \( \xi_n \) such that \( \xi_n(\Omega) = \hat{S} \) if \( m^{n-1}\nu(\Omega) = -\nu(\Omega) \) and \( n - 1 \) is an element of \( \hat{S} \) or if \( m^{n-1}\nu(\Omega) = \nu(\Omega) \) and \( n - 1 \) is not an element of \( \hat{S} \). We define \( \xi_n \) such that \( \xi_n(\Omega) = S' \) if \( m^{n-1}\nu(\Omega) = -\nu(\Omega) \) and \( n - 1 \) is not an element of \( \hat{S} \) or if \( m^{n-1}\nu(\Omega) = \nu(\Omega) \) and \( n - 1 \) is an element of \( \hat{S} \). When \( n \) is odd and \( n - 1 \) is an element of \( C_\Omega \), we make an arbitrary choice. If \( \Omega = (\Omega, (e_1, \ldots, e_{[n/2]})) \), we define \( \xi_n \) such that \( \xi_n(\Omega) = S' \) if \( e_{[n/2]} = -1 \) and \( n - 1 \) is not an element of \( \hat{S} \) or if \( e_{[n/2]} = 1 \).
and \( n - 1 \) is an element of \( \hat{S} \). we define \( \xi_n \) such that \( \xi_n(\hat{\Omega}) = \hat{S} \) if \( \epsilon_{[n/2]} = -1 \) and \( n - 1 \) is an element of \( \hat{S} \) or if \( \epsilon_{[n/2]} = 1 \) and \( n - 1 \) is not an element of \( \hat{S} \).

**Proposition 6.11** The order of the preimage of \( S \) under the map \( \xi_n \) is exactly \( m(\chi_n, s, \mu_\lambda) \).

*Proof* We recall the Definition 33 of \( M_\Omega \). For each \( \Omega \) in \( \Psi_\lambda \) and for each \( j \) in \( C_{\Omega}^c \), we have shown \( m^j \) acts uniformly on every vector in \( \mathcal{L}_\Omega \). Hence, for any \( i \) and \( q \) such that \( 1 \leq i < q \leq 2^l \), the \( M_\Omega \)-representation spanned by \( \nu(\Omega)_i \) is isomorphic to the \( M_\Omega \)-representation spanned by \( \nu(\Omega)_q \). We write \( \nu(\Omega, M_\Omega) \) to denote this representation. If \( \hat{\Omega} \mapsto S \), then we conclude \( \chi_{n,S,M_\Omega} \) is isomorphic to \( \nu(\Omega, M_\Omega) \) from the definition of \( \xi_n \). For any \( l \)-tuple \((\epsilon_{h_1}, \ldots, \epsilon_{h_l})\) indexed by the elements of \( C_{\Omega} \) where, for each \( i \) such that \( 1 \leq i \leq l \), we know \( \epsilon_i = \pm 1 \), the following equation holds:

\[
|\{p \in \{1, \ldots, 2^l\} \mid m^j \nu(\Omega)_p = \epsilon_j \nu(\Omega)_p \text{ for each } j \in C_{\Omega}\}| = 1.
\]

We know

\[
\xi_n(\Omega, (\epsilon_1, \ldots, \epsilon_{[n/2]})) = S
\]

where \( \nu(\Omega)_p \) spans the \( M \)-representation \( \chi_{n,S} \) and \( p \) with \( 1 \leq p \leq 2^l \) is such that \( m^j \nu(\Omega)_p = \epsilon_j \nu(\Omega)_p \) for each \( j \) in \( C_{\Omega} \). There exists only one \([n/2]\)-tuple \((\epsilon_1', \ldots, \epsilon_{[n/2]}')\) such that \( \Omega \) is associated to the element

\[
(\Omega, (\epsilon_1', \ldots, \epsilon_{[n/2]}'))
\]

and \( \epsilon_j = \epsilon'_j \) for each \( j \) in \( C_{\Omega} \). The statement of the proposition follows.

The method given by proposition 6.11 to determine \( m(\chi_n, s, \mu_\lambda) \) allows for computation with much less overhead than the method given in section 4, especially for large \( n \). Although proposition 6.11 does call for induction in determining the action
of $M$ on a chosen decomposition for $\mu_\lambda$ into irreducible representations for $M$, this proposition then allows for the calculation of $m(\chi_{n,S}, \mu_\lambda)$ for any $S$ in $F_n$ without having to resort to induction.

We give a synopsis of the results in this section, focusing on the visual representation of decorated admissible semistandard Young tableaux. The set $\tilde{\Psi}_\lambda$ corresponds to a basis

$$\{\nu_2(\bar{\omega}) \mid \bar{\omega} \in \tilde{\Psi}\}$$

for the underlying space of the representation $\mu_\lambda$. Using this correspondence, we see each tableau in $\tilde{\Psi}$ as a basis vector. The set $C_\Omega$ comprises all even natural numbers $2k \leq n - 1$ such that the number $2k$ actually appears in row $k$ of the admissible semistandard Young tableau $\Omega$. The cardinality of $C_\Omega$ is $l$, so $l \leq [n/2]$. According to Theorem 6.8, if $j$ is an even natural number such that $1 \leq j \leq n - 1$ and such that the number $j$ does not appear in the $j/2^{th}$ row of $\bar{\Omega}$, then $m^j$ acts on $\bar{\Omega}$ by $-1^q$ where $q$ is the quantity of the numbers $j + 1$ and $j$ in $\bar{\Omega}$. According to Theorem 6.9, if $j$ is an odd natural number such that $1 \leq j \leq n - 1$, then $m^j$ acts on $\bar{\Omega}$ by $-1^q$ where $q$ is the quantity of the numbers $\pm j + 1$ and $j$ in $\bar{\Omega}$. On the other hand, by Theorem 6.5, if $j$ is an element of $C_\Omega$, then $m^j$ maps $\bar{\Omega}$ to $\bar{\Omega}_j$ where every occurrence of $\pm j$ in row $j/2$ of $\bar{\Omega}$ is $\mp j$ in row $j/2$ of $\bar{\Omega}$ and all other entries are equal. The set $L_\Omega$ comprises each decorated tableau $\bar{\Omega}$ such that taking the absolute values of every number in $\bar{\Omega}$ yields $\Omega$. The order of $L_\Omega$ is $2^l$, one element for each choice of sign given to the numbers $j$ in row $j/2$ of $\Omega$ for every $j$ in $C_\Omega$. We define $M_\Omega$ to be the subgroup of $M$ generated by $\{m^j \mid j \in C_\Omega\}$. The group $M_\Omega$ has $2^l$ distinct irreducible representations, one for each choice of the action for $m^j$ where $j$ is an element of $C_\Omega$. (The element $m^j$ must act by $\pm 1$.) By the Peter-Weyl theorem, the subspace spanned by $L_\Omega$ has a basis such that each of the $2^l$ basis vectors spans a unique irreducible $M_\Omega$-representation.
Since \( \cup_{\Omega \in \Psi} C_\Omega \) spans the underlying space for \( \mu_\lambda \), the Peter-Weyl theorem gives a basis for this underlying space such that each basis vector spans an irreducible representation for \( M \). Moreover, we understand completely the action of \( M \) on this basis because \( \{m^j \mid 1 \leq j \leq n - 1\} \) generates \( M \).

This information allows for us to find \( m(\chi_{n,s}, \mu_\lambda) \) for any \( S \) in \( F_n \). We express this information in terms of an algorithm for associating each \( \tilde{\Omega} \) in \( \tilde{\Psi}_\lambda \) with some subset \( S \) in \( F_n \) such that the number of decorated tableaux associated to \( S \) is exactly \( m(\chi_{n,s}, \mu_\lambda) \). The algorithm used to associate each \( \tilde{\Omega} \) with some subset \( S \) is our map \( \xi_n \), and Proposition 6.11 proves our map \( \xi_n \) does in fact map exactly \( m(\chi_{n,s}, \mu_\lambda) \)-many decorated tableaux to the subset \( S \). To more cleanly express the definition of the map \( \xi_n \), we introduce a map

\[
*: \{1, \ldots, n - 1\} \times \tilde{\Psi} \rightarrow \pm 1.
\]

We define \( *(j, \tilde{\Omega}) = \epsilon \) where \( m^j(\tilde{\Omega}) = \epsilon(\tilde{\Omega}) \) for \( j \) in \( C_\Omega \). For \( j \) in \( C_\Omega \), we define \( *(j, \tilde{\Omega}) = \epsilon \) where \( \epsilon = -1^q \) for \( q \) equal to the quantity of the numbers \( j + 1 \) and \( \pm j \) in \( \tilde{\Omega} \) if each occurrence of the number \( \pm j \) in row \( j/2 \) of \( \tilde{\Omega} \) is positive and where \( \epsilon = -(-1^q) \) if each occurrence of the number \( \pm j \) in row \( j/2 \) of \( \tilde{\Omega} \) is negative. In terms of tableaux, the map \( \xi_n \) acts recursively as follows. If \( n = 2 \), then \( \tilde{\Psi}_\lambda \) contains only one element, a tableau having one row such that each entry is equal to 2 or such that each entry is equal to \(-2\). In this case, \( \xi_2(\tilde{\Omega}) = \{1\} \) if the number of entries is odd, while \( \xi_2(\tilde{\Omega}) = \emptyset \) if the number of entries is even. For \( n > 2 \), we suppose \( \xi_{n-1}(\tilde{\pi}(\tilde{\Omega})) = \tilde{S} \) for some element \( \tilde{S} \) of \( F_{n-1} \). If \( *(n-1, \tilde{\Omega}) = -1 \) and \( \tilde{S} \) contains \( \{n-1\} \), then we define \( S' \) to be \( \tilde{S} \). If \( *(n-1, \tilde{\Omega}) = -1 \) and \( \tilde{S} \) does not contain
\{n - 1\}, then we define \(S'\) to be \(\tilde{S} \cup \{n\}\). If \(*_{(n - 1, \bar{\Omega})} = 1\) and \(\tilde{S}\) contains \(\{n - 1\}\), then we define \(S'\) to be \(\tilde{S} \cup \{n\}\). If \(*_{(n - 1, \bar{\Omega})} = 1\) and \(\tilde{S}\) does not contain \(\{n - 1\}\), then we define \(S'\) to be \(\tilde{S}\). Either \(S'\) is an element of \(F_n\) or the complement of \(S'\) in \(\{1, \ldots, n\}\) is an element of \(F_n\). If we define \(S\) to be the element of \(F_n\) equal either to \(S'\) or the complement of \(S'\) in \(\{1, \ldots, n\}\), then \(\xi_{n}(\bar{\Omega}) = S\).

7 Application to Split Real Reductive Groups of Type \(B_n\) and \(D_n\)

We have given a neat description of the branching law from \(K\) to \(M\) for \(SL(n, \mathbb{R})\), the split real group of type \(A_n\). As far as possible, we would like to use that result to determine similarly complete descriptions of the branching laws from \(K_e\) to \(M \cap K_e\) for the split real groups of type \(B_n\) and \(D_n\). We start by identifying these groups.

We consider the group

\[
SO(n, n) = \{g \in SL(2n, \mathbb{R}) \mid {}^t gI_{n,n}g = I_{n,n}\}
\]

where \(I_{n,n}\) is the \(2n\) by \(2n\) diagonal matrix with \(j^{th}\) diagonal entry equal to 1 for \(j\) such that \(1 \leq j \leq n\) and \(j^{th}\) diagonal entry equal to \(-1\) for \(j\) such that \(n + 1 \leq j \leq 2n\). Differentiating the defining relations for \(SO(n, n)\), we see \(SO(n, n)\) has Lie algebra

\[
\mathfrak{so}(n, n) = \{X \in sl(2n, \mathbb{R}) \mid {}^t X I_{n,n} + I_{n,n}X = 0\}.
\]

Any two quadratic forms over \(\mathbb{C}\) are equivalent. In particular, \(I_{n,n}\) is equivalent to \(I_{2n}\) via the transformation \({}^t gI_{n,n}g\) where \(g\) is the diagonal \(2n\) by \(2n\) matrix with \(j^{th}\) diagonal entry equal to 1 for \(j\) such that \(1 \leq i \leq n\) and \(j^{th}\) diagonal entry equal to \(i\) for \(j\) such that \(n + 1 \leq j \leq 2n\). We see the complexification of \(\mathfrak{so}(n, n)\) is \(\mathfrak{so}(2n, \mathbb{C})\).
hence $SO(n,n)$ is a real group of type $D_n$. We identify a split torus for $SO(n,n)$ by changing coordinates. We use as a change of basis matrix $(1/\sqrt{2})b$ where $b$ is the $2n$ by $2n$ matrix with upper left $n$ by $n$ block diagonal submatrix equal to $I_n$, with $n$ by $n$ submatrix formed by each entry $(i,j)$ where $i \geq n + 1$ and $j \leq n$ equal to $I_n$, with $n$ by $n$ submatrix formed by each entry $(i,j)$ where $i \leq n$ and $j \geq n + 1$ equal to $-I_n$, and with lower right $n$ by $n$ block diagonal submatrix equal to $I_n$. As usual, we define

$$O(p) = \{ g \in M(n, \mathbb{R}) \mid {}^tgg = 1 \}.$$ 

The change of basis matrix $(1/\sqrt{2})b$ is an element of $O(2n)$. Using the new set of coordinates defined by $(1/\sqrt{2})b$, we may write

$$SO(n,n) = \{ g \in SL(2n, \mathbb{R}) \mid {}^tgJ_{n,n}g = J_{n,n} \}$$

where $J_{n,n}$ is the $2n$ by $2n$ matrix with $n$ by $n$ submatrix formed by each entry $(i,j)$ where $i \geq n + 1$ and $j \leq n$ equal to $I_n$, with $n$ by $n$ submatrix formed by each entry $(i,j)$ where $i \leq n$ and $j \geq n + 1$ equal to $I_n$, and with all other entries equal to 0. Written thus, $SO(n,n)$ has a subgroup $H$ consisting of all matrices

$$\begin{pmatrix}
t_1 \\
\cdots \\
t_n \\
t_1^{-1} \\
\cdots \\
t_n^{-1}
\end{pmatrix}$$

where $t_i$ is an element of the group of units $\mathbb{R}^\times$ in $\mathbb{R}$ for each $i$ with $1 \leq i \leq n$. Clearly $H$ is an $n$-dimensional split torus for $SO(n,n)$. This torus is maximal since
each Cartan subalgebra of \( \mathfrak{so}(2n, \mathbb{C}) \) is \( n \)-dimensional. For the remainder of our discussion concerning \( SO(n, n) \), we revert to our original set of coordinates. According to [5] 1.144, \( SO(n, n) \) has maximal compact subgroup \( K = S(O(n) \times O(n)) \) where \( S(O(n) \times O(n)) \) is the subgroup of matrices in \( O(n) \times O(n) \) having determinant equal to 1. (Here we see \( O(n) \times O(n) \) as the group of block diagonal matrices with the upper left \( n \) by \( n \) block diagonal equal to an element of \( O(n) \) and with the lower right \( n \) by \( n \) block diagonal equal to an element of \( O(n) \).) We know \( M = K \cap H \) consists of all matrices

\[
\begin{pmatrix}
\varepsilon_1 \\
\vdots \\
\varepsilon_n \\
\varepsilon_1 \\
\vdots \\
\varepsilon_n
\end{pmatrix}
\]

where, for each \( i \) such that \( 1 \leq i \leq n \), the number \( \varepsilon_i \) is \( \pm 1 \).

If we consider any pair \((k_1, k_2)\) such that \( k_1 \) and \( k_2 \) are elements of \( O(n) \), then \((k_1, k_2)\) is an element of \( K \) if and only if \( \det k_1 = \det k_2 = \varepsilon \) where \( \varepsilon = \pm 1 \). We see \( K \) has two connected components:

\[
\{(k_1, k_2) \in O(n) \times O(n) \mid \det k_1 = \det k_2 = 1\}, \quad \text{and} \quad (36)
\]

\[
\{(k_1, k_2) \in O(n) \times O(n) \mid \det k_1 = \det k_2 = -1\}.
\]

Component 36 is the identity component \( K_e \) of \( K \). Clearly, \( K_e \) is \( SO(n, \mathbb{R}) \times SO(n, \mathbb{R}) \). The closed subgroup \( M \cap K_e \) of \( K_e \) is simply the image of \( M(SO(n)) \) under the diagonal map \( \delta: SO(n, \mathbb{R}) \to SO(n, \mathbb{R}) \times SO(n, \mathbb{R}) \) where \( M(SO(n)) \) is the Langlands subgroup for \( SO(n) \). We know \( M \cap K_e \) is a subgroup of \( M(SO(n)) \times M(SO(n)) \). The map \( \delta \) isomorphically identifies \( M(SO(n)) \) with \( M \cap K_e \), and Theorem 3.2 describes
the set $M(SO(n))$ as $\{\chi_{n,S} | S \in F_n\}$ where $F_n$ takes its definition from Equation 3.

We can determine a branching law from $K_e$ to $M \cap K_e$ using proposition 6.11 so long as we can determine a branching law from $M(SO(n)) \times M(SO(n))$ to $M \cap K_e$. For any subset $S$ of $\{1, \ldots, n\}$ we refer to the complement of $S$ in $\{1, \ldots, n\}$ by $S^c$. We choose an irreducible representation $\chi_{n,S'} \otimes \chi_{n,S''}$ of $M(SO(n)) \times M(SO(n))$ and an irreducible representation $\chi_{n,S}$ of $M \cap K_e$.

**Theorem 7.1** We have $m(\chi_{n,S}, \chi_{n,S'} \otimes \chi_{n,S''}) = 1$ if and only if $S = S' \ominus S''$ or $S = (S' \ominus S'')^c$. Otherwise, $m(\chi_{n,S}, \chi_{n,S'} \otimes \chi_{n,S''}) = 0$.

**Proof** Each irreducible representation for $M(SO(n)) \times M(SO(n))$ has dimension equal to 1, so $(\chi_{n,S'} \otimes \chi_{n,S''})|_{M \cap K_e}$ is itself irreducible. We need only find the subset $S$ in $F_n$ such that $\chi_{n,S} = (\chi_{n,S'} \otimes \chi_{n,S''})|_{M \cap K_e}$. With $m^j$ defined as in Lemma 3.1, we consider the action of $(m^j, m^j)$ on some vector $(v', v'')$ in $(\chi_{n,S'} \otimes \chi_{n,S''})$ for each $j$ such that $1 \leq j < n$. We have

$$(m^j, m^j)(v', v'') = (m^j(v'), m^j(v'')),$$

so $(m^j, m^j)(v', v'') = -1(v', v'')$ if

$$\{j, j + 1\} \cup S' = \{j, j + 1\} \cup S' = \emptyset \text{ and } \{j, j + 1\} \cup S'' = \{j\} \text{ or } \{j, j + 1\} \cup S'' = \{j + 1\}.$$

Also, $(m^j, m^j)(v', v'') = -1(v', v'')$ if

$$\{j, j + 1\} \cup S' = \{j\} \text{ or } \{j, j + 1\} \cup S' = \{j + 1\} \text{ and } \{j, j + 1\} \cup S'' = \{j, j + 1\} \text{ or } \{j, j + 1\} \cup S'' = \emptyset.$$

Otherwise, $(m^j, m^j)(v', v'') = (v', v'')$. From this description of the action of $M \cap K_e$ on $(\chi_{n,S'} \otimes \chi_{n,S''})$, we know $\chi_{n,S} = (\chi_{n,S'} \otimes \chi_{n,S''})|_{M \cap K_e}$ if and only if $S = S' \ominus S''$ or
We now choose an irreducible representation $\chi_{n,S}$ for $M \cap K_e$ and an irreducible representation $\mu_\chi \otimes \mu_{\lambda''}$.

**Proposition 7.2** The branching law from $K_e$ to $M \cap K_e$ for the split real reductive group of type $D_n$ determines $m(\chi_{n,S}, \mu_\chi \otimes \mu_{\lambda''})$ to be

$$\sum_{S', S'' \in F_n} \left[ m(\chi_{n,S'}, \mu_\chi)m(\chi_{n,S''}, \mu_{\lambda''}) \right] + m(\chi_{n,S''}, \mu_\chi)m(\chi_{n,S'}, \mu_{\lambda''})$$

$$\sum_{S', S'' \in F_n} \left[ m(\chi_{n,S'}, \mu_{\lambda''})m(\chi_{n,S''}, \mu_\chi) \right] + m(\chi_{n,S''}, \mu_{\lambda''})m(\chi_{n,S'}, \mu_\chi)$$

**Proof** This result follows directly from proposition 6.11 and Theorem 7.1.

A branching law for the split real reductive group of type $B_n$ develops along very similar lines. We focus on the group

$$SO(n + 1, n) = \{ g \in SL(2n + 1, \mathbb{R}) \mid {}^tgI_{n+1,n}g = I_{n+1,n} \}$$

where $I_{n+1,n}$ is the $2n + 1$ by $2n + 1$ diagonal matrix with $i^{th}$ diagonal entry equal to 1 for $i$ such that $1 \leq i \leq n + 1$ and with $i^{th}$ diagonal entry equal to $-1$ for $i$ such that $n + 2 \leq i \leq 2n + 1$. Differentiating the defining relations for $SO(n + 1, n)$, we see $SO(n + 1, n)$ has Lie algebra

$$\mathfrak{so}(n + 1, n) = \{ X \in \mathfrak{sl}(2n + 1, \mathbb{R}) \mid {}^tXI_{n+1,n} + I_{n+1,n}X = 0 \}.$$
$I_{2n+1}$ via the transformation $^t g J_{n,n} g$ where $g$ is the diagonal $2n + 1$ by $2n + 1$ matrix with $j^{th}$ diagonal entry equal to 1 for $j$ such that $1 \leq i \leq n + 1$ and with $j^{th}$ diagonal entry equal to $i$ for $j$ such that $n + 2 \leq j \leq 2n + 1$. We see the complexification of $\mathfrak{so}(n + 1, n)$ is $\mathfrak{so}(2n + 1, \mathbb{C})$, hence $SO(n + 1, n)$ is a real group of type $B_n$. We identify a split torus for $SO(n + 1, n)$ by changing coordinates. We use as a change of basis matrix $\bar{b}$ where the lower right $2n$ by $2n$ block diagonal submatrix of $\bar{b}$ is $(1/\sqrt{2})b$, where the $1^{st}$ diagonal entry of $\bar{b}$ is 1, and where all other entries of $\bar{b}$ are equal to 0. The change of basis matrix $\bar{b}$ is an element of $O(2n + 1)$. Using the new set of coordinates defined by $\bar{b}$, we may write

$$SO(n, n + 1) = \{ g \in SL(2n, \mathbb{R}) \mid ^t g J_{1,n,n} g = J_{1,n,n} \}$$

where $J_{1,n,n}$ is the $2n + 1$ by $2n + 1$ matrix with lower right $2n$ by $2n$ block diagonal submatrix equal to $J_{n,n}$, with $1^{st}$ diagonal entry equal to 1, and with all other entries equal to 0. Written thus, $SO(n + 1, n)$ has a subgroup $H$ consisting of all matrices

The group $SO(n, n + 1)$ has a subgroup $H$ consisting of all matrices

$$
\begin{pmatrix}
1 \\
t_1 \\
& \cdots \\
& & t_n \\
& & & \cdots \\
& & & & t_1^{-1} \\
& & & & & \cdots \\
& & & & & & t_n^{-1}
\end{pmatrix}
$$

where $t_i$ is an element of the group of units $\mathbb{R}^\times$ in $\mathbb{R}$ for each $i$ with $1 \leq i \leq n$. Clearly $H$ is an $n$-dimensional split torus for $SO(n + 1, n)$. This torus is maximal
since each Cartan subalgebra of $\mathfrak{so}(2n + 1, \mathbb{C})$ is $n$-dimensional. For the remainder of our discussion concerning $SO(n + 1, n)$, we revert to our original set of coordinates. According to [5] 1.144, $SO(n + 1, n)$ has maximal compact subgroup

$$K = S(O(n + 1) \times O(n))$$

where $S(O(n + 1) \times O(n))$ is the subgroup of matrices in $O(n + 1) \times O(n)$ having determinant equal to 1. (Here we see $O(n + 1) \times O(n)$ as the group of block diagonal matrices with the upper left $n + 1$ by $n + 1$ block diagonal equal to an element of $O(n + 1)$ and with the lower right $n$ by $n$ block diagonal equal to an element of $O(n)$.) We know $M = K \cap H$ consists of all matrices

$$
\begin{pmatrix}
1 \\
\epsilon_1 \\
& \ddots \\
& & \epsilon_n \\
& & & \epsilon_1 \\
& & & & \ddots \\
& & & & & \epsilon_n
\end{pmatrix}
$$

where, for each $i$ such that $1 \leq i \leq n$, the number $\epsilon_i$ is $\pm 1$.

If we consider any pair $(k_1, k_2)$ such that $k_1$ is an element of $O(n + 1)$ and $k_2$ is an element of $O(n)$, then $(k_1, k_2)$ is an element of $K$ if and only if $\det k_1 = \det k_2 = \epsilon$ where $\epsilon = \pm 1$. We see $K$ has two connected components:

$$
\{(k_1, k_2) \in O(n + 1) \times O(n) \mid \det k_1 = \det k_2 = 1\}, \text{ and}
\{(k_1, k_2) \in O(n + 1) \times O(n) \mid \det k_1 = \det k_2 = -1\}.
$$

(37)
Component 37 is the identity component $K_e$ of $K$. Clearly, $K_e$ is $SO(n + 1, \mathbb{R}) \times SO(n, \mathbb{R})$. The closed subgroup $M \cap K_e$ of $K_e$ is simply the image of $M(SO(n))$ under the diagonal map $\delta: SO(n, \mathbb{R}) \rightarrow SO(n, \mathbb{R}) \times SO(n, \mathbb{R})$ followed by the map $\iota': SO(2n, \mathbb{R}) \rightarrow SO(2n + 1, \mathbb{R})$ where $M(SO(n))$ is the Langlands subgroup for $SO(n)$ and where $\iota'$ is the embedding

$$A \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 \\ \vdots \\ 0 \\ \end{pmatrix}.$$

We know $M \cap K_e$ is a subgroup of $M(SO(n)) \times M(SO(n)) \subset M(SO(n + 1)) \times M(SO(n))$. The map $\delta$ isomorphically identifies $M(SO(n))$ with $M \cap K_e$, and Theorem 3.2 describes the set $M(SO(n))$ as $\{ \chi_{n,S} | S \in F_n \}$ where $F_n$ takes its definition from Equation 3.

We can determine a branching law from $K_e$ to $M \cap K_e$ using proposition 6.11, so long as we can determine a branching law from $M(SO(n + 1)) \times M(SO(n))$ to $M \cap K_e$. We choose an irreducible representation $\chi_{n,S'} \otimes \chi_{n,S'}$ of $M(SO(n + 1)) \times M(SO(n))$ and an irreducible representation $\chi_{n,S}$ of $M \cap K_e$. For any subset $S''$ of $\{1, \ldots, n+1\}$ we denote by $\tilde{S}''$ the subset $S'' \setminus \{n+1\}$.

**Theorem 7.3** We have $m(\chi_{n,S}, \chi_{n,S''} \otimes \chi_{n,S'}) = 1$ if and only if $S = S' \cup \tilde{S}''$ or $S = (S' \cup \tilde{S}'').c$. Otherwise, $m(\chi_{n,S}, \chi_{n,S''} \otimes \chi_{n,S'}) = 0$.

**Proof** Branching from $M(SO(n + 1)) \times M(SO(n))$ to $M(SO(n)) \times M(SO(n))$, we find

$$m(\chi_{n,S''} \otimes \chi_{n,S'}, \chi_{n+1,S''} \otimes \chi_{n,S'}) = 1$$

where we realize $M(SO(n)) \times M(SO(n))$ as a subgroup of $M(SO(n+1)) \times M(SO(n))$.
via the embedding \( i' \) and where \( \chi_{n+1,S''} \otimes \chi_{n,S'} \) is any irreducible representation of \( M(SO(n+1)) \times M(SO(n)) \). Since any irreducible representation of \( M(SO(n+1)) \times M(SO(n)) \) has dimension equal to 1, any irreducible representation of \( M(SO(n)) \times M(SO(n)) \) not isomorphic to \( \chi_{n,S''} \otimes \chi_{n,S'} \) has multiplicity in \( \chi_{n+1,S''} \otimes \chi_{n,S'} \) equal to 0. Combining this result with the branching law from \( M(SO(n)) \times M(SO(n)) \) to \( M \cap K_e \) given in Theorem 7.1, the statement of this theorem follows.

We choose an irreducible representation \( \mu_{\lambda''} \otimes \mu_{\lambda'} \) of \( SO(n+1, \mathbb{R}) \times SO(n, \mathbb{R}) \) and an irreducible representation \( \chi_{n,S} \) of \( M \cap K_e \).

**Proposition 7.4** The branching law from \( K_e \) to \( M \cap K_e \) for the split real reductive group of type \( B_n \) determines \( m(\chi_{n,S}, \mu_{\lambda''} \otimes \mu_{\lambda'}) \) to be

\[
\sum_{S' \in F_n, S'' \in F_{n+1}, S' \oplus S'' = S} [m(\chi_{n,S'}, \mu_{\lambda'}) m(\chi_{n+1,S''}, \mu_{\lambda''})] + m(\chi_{n,S''}, \mu_{\lambda'}) m(\chi_{n+1,S'}, \mu_{\lambda''})] \]

\[
\sum_{S', S'' \in F_n, (S' \oplus S'') \cap S = S} [m(\chi_{n,S'}, \mu_{\lambda'}) m(\chi_{n+1,S''}, \mu_{\lambda''})] + m(\chi_{n,S''}, \mu_{\lambda'}) m(\chi_{n+1,S'}, \mu_{\lambda''})] \]

**Proof** This result follows directly from proposition 6.11 and Theorem 7.3.

For the split classical groups, then, we have determined to extensive detail the branching law from \( K_e \) to \( M \cap K_e \) with respect to these classical groups. We have done so in a manner making the multiplicities of irreducible representations for \( M \cap K_e \) in irreducible representations for \( K_e \) fairly easy to calculate. All the benefits these branching laws yield for the study of principal series representations attatch.
Bibliography


