Three-dimensional Solitary Waves in Dispersive Wave Systems

by

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February 1998

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

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Abstract

Fully localized three-dimensional solitary waves, commonly referred to as ‘lumps’, have received far less attention than two-dimensional solitary waves in dispersive wave systems. Prior studies have focused in the long-wave limit, where lumps exist if the long-wave speed is a minimum of the phase speed and are described by the Kadomtsev–Petviashvili (KP) equation. In the water-wave problem, in particular, lumps of the KP type are possible only in the strong-surface-tension regime (Bond number, \( B > 1/3 \)), a condition that limits the water depth to a few mm.

In the present thesis, a new class of lumps is found that is possible under less restrictive physical conditions. Rather than long waves, these lumps bifurcate from infinitesimal sinusoidal waves of finite wavenumber at an extremum of the phase speed. As the group and phase velocities are equal there, small-amplitude lumps resemble fully localized wavepackets with envelope and crests moving at the same speed, and the wave envelope along with the induced mean-flow component are governed by a coupled Davey–Stewartson equation system of elliptic–elliptic type. The lump profiles feature algebraically decaying tails at infinity owing to this mean flow.

In the case of water waves, lumps of the wavepacket type are possible when both gravity and surface tension are present on water of finite or infinite depth for \( B < 1/3 \). The asymptotic analysis of these lumps in the vicinity of their bifurcation point at the minimum gravity–capillary phase speed, is in agreement with recent fully numerical computations by Parau, Cooker & Vanden-Broeck (2005) as well as a formal existence proof by Groves & Sun (2005). A linear stability analysis of the gravity–capillary solitary waves that also bifurcate at the minimum gravity–capillary phase speed, reveals that they are always unstable to transverse perturbations, suggesting a mechanism for the generation of lumps.

This generation mechanism is explored in the context of the two-dimensional Benjamin (2-DB) equation, a generalization to two horizontal spatial dimensions of the model equation derived by Benjamin (1992) for uni-directional, small-amplitude, long interfacial waves in a two-fluid system with strong interfacial tension. The 2-DB equation admits solitary waves and lumps of the wavepacket type analogous to those bifurcating at the minimum gravity–capillary phase speed in the water-wave problem. Based on unsteady numerical simulations, it is demonstrated that the transverse instability of solitary waves of the 2-DB equation results in the formation of lumps, which propagate stably and are thus expected to be the asymptotic states of the initial-value problem for fully localized initial conditions.

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Acknowledgments

From my early ages, I have been always wondering what it would be like if I become a PhD. Choosing mathematics as my undergraduate major, I thought I could reach the status by studying something that is everlasting. However, since I realized that everything is transient in time, describing phenomena and predicting the future by modeling a system in terms of mathematical languages sounded much more appealing to me, which is in the realm of applied mathematics. Fortunately, I was able to get in MIT, the most prestigious place in the world to pursue my goal to be a PhD in applied mathematics.

During my graduate years at MIT, it took a significant amount of trial and error to get out of instability full of uncertainty. I am now finishing my PhD, but a right track of my research is yet to come. I am still being disciplined to prove that I am a person who can truly communicate with other people and can show a good commitment where I belong to. It should be accompanied with my achievement as well.

I would like to express an hearty appreciation to my thesis supervisor, Professor Triantaphyllos R. Akylas in mechanical engineering, who led me to do a research topic I have been enjoying to do so much. Professor Rodolfo R. Rosales, my academic adviser in mathematics, has been supportive in following up my research work during every phase of my PhD years. I thank to Doctor David C. Calvo, a former student of my thesis supervisor, who provided me his numerical codes and gave me practical advices.

Ms. Linda Okun, graduate administrator in the graduate mathematics office, always gave me warm and kind guidance whenever I needed to make a important decision. Doctors Jaehyuk Choi and Yong-Il Shin have made enormous efforts to keep me being poised from uncontrollable disturbances during my MIT life. Doctor Kevin T. Chu and other colleagues have been helpful in broadening the scope of my knowledge by exchanging valuable ideas. It should be dedicated to my Grandpa as well, who has been the source of my courage, energy and, faith in my entire life. I am so pleased that they all should be gratefully acknowledged at the end of my PhD study.

Finally, Professors Mark Ablowitz and Victor Shrira should be acknowledged for helpful discussions with my thesis supervisor for a part of my research. I wish to thank to the Department of Mathematics for providing various forms of financial supports, an excellent environment for performing novel research, and teaching opportunities. This work has been partially supported by the Air Force Office of Scientific Research, Air Force Materials Commands, USAF, under Grant Number FA9950-04-1-0125 and by the National Science Foundation Grant Number DMS-0305940.
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Chapter 1

Introduction

1.1 Dispersive wave systems

Quantitative modeling of physical phenomena is one of the major goals of physical applied mathematicians. The starting point of mathematical modeling is to describe the dynamics of physical quantities that arise from various physical systems motivated by field or laboratory observations. Physical quantities are usually functions of time and spatial variables. When they propagate in space as time elapses, they are referred to as waves. Waves arise in many physical contexts, each of which have their own underlying governing principles. These principles are expressed in terms of wave equations. Therefore, wave dynamics can be understood by solving those governing equations.

Waves are categorized into one of two main classes, hyperbolic and dispersive. Hyperbolic waves are classified from the characteristic form of equations; there should exist characteristic directions and corresponding characteristic speeds of waves. The propagation of fronts, shocks, and discontinuities are typical examples of hyperbolic-wave. On the other hand, dispersive waves are classified according to the form of solutions. In a small-amplitude limit, they should be a linear superposition of sinusoidal traveling waves, \( e^{i(k \cdot x - \omega(k)t)} \), for different wavenumbers \( k \) and the corresponding frequencies \( \omega(k) \). (Here \( x = (x, y) \) and \( k = (k, m) \) where \( k \) is the \( x \)-directional wavenumber and \( m \) is the \( y \)-directional wavenumber) More precisely, if \( \phi(x, t) \) denotes a small-amplitude dispersive wave, then it should be written as a Fourier integral representation as follows:

\[
\phi(x, t) = \frac{1}{2\pi} \int \hat{\phi}(k, t)e^{i(k \cdot x - \omega(k)t)} dk, \tag{1.1}
\]

where the dependence of the frequency \( \omega(k) \) on the wavenumber \( k \) is called dispersion relation. Dispersive waves usually feature smooth profiles, otherwise the Fourier integral representation is not an efficient way because very large wavenumber components in an original wave profile
become too significant to be ignored. There are rich tool-kits available regarding the Fourier analysis, which are useful for the mathematical descriptions of dispersive waves.

Among these kinds of waves, we are particularly going to focus on special kinds of dispersive waves in this thesis. In most dispersive wave systems, different wavenumber components propagate with different speeds each other. The phase speed \( c_p \) of each wave component is defined as \( \omega /|k|^2(k, m) \) and the group speed \( c_g \) is defined as \( (\partial \omega /\partial k, \partial \omega /\partial m) \). If \( \omega \) is not linear in \( k \) so \( c_p \) and \( c_g \) are not constant in \( k \), then localized initial wave profiles will disperse out, so that the wave systems are literally dispersive.

### 1.2 Solitary waves

Although most initial wave profiles seem to disperse out in dispersive wave systems, steady coherent wave structures are possible to exist when nonlinearity is involved into the systems. Nonlinearity becomes crucial in wave systems when wave amplitudes are large and different wave components are coupled each other. From the balance between nonlinearity and dispersion, it is possible to have steady waves of a permanent form, which are called solitary waves.

Solitary waves are unidirectional waves with a constant speed. They are also regarded as particle-like and localized-energy states and feature remarkable stability. Sometimes they are called solitons if they can survive under strong external disturbances or mutual interactions. Because they are found only in nonlinear regimes, the linear superposition principle between two solitary waves is no longer valid, so that nontrivial phase shifts occur when they interact each other.

The first discovery of surface solitary water waves was reported by J. Scott Russell (1834). Later, Korteweg and de Vries (KdV) (1895) formulated the celebrated KdV equation for the dynamics of long surface waves in shallow water, and the KdV equation possesses one-parameter family of exponentially decaying solitary wave solutions:

\[
\begin{align*}
    u_t + 6uu_x + u_{xxx} &= 0, \quad (1.2a) \\
    u(x, t) &= \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{\frac{c}{2}}}{2} (x - ct - x_0) \right). \quad (1.2b)
\end{align*}
\]

When Zabusky and Kruskal (1965) discovered the remarkable stability property of these KdV solitary wave solutions (1.2b) from their numerical study of the KdV equation (1.2a), the physical importance of such waves became much clearer than ever. It turns out that a lot of other dispersive wave systems such as stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, and so on, can be described by the KdV equation (1.2a). (For more historical remarks, see §1.1 in Ablowitz & Segur 1991)

These days, there are a lot of promising potential applications of solitary waves in vari-
ous physical scientific disciplines such as fiber optics communication, all-optical switching in optical chips, qubits in quantum computation, nonlinear localized modes in photonic crystals, biophysics, and Bose–Einstein condensation (See Abdullaev & Konotop, 2003).

From the mathematical perspective, there have been many efforts to develop a unified analytical method to solve initial value problems as well as to find steady solitary wave solutions for various nonlinear dispersive wave equations. The most pioneering work regarding to this has been done by Ablowitz, Kaup, Newell, and Segur (1973, 1974), who developed a solution method to solve a general class of nonlinear dispersive wave equations including the KdV equation, the Nonlinear Schrödinger (NLS) equation, the Sine–Gordon equation, and their hierarchies. The main idea of the method is to introduce nonlinear transformations, called inverse scattering transformations, which render original equations to a decoupled linear system. If such transformations exist, nonlinear dispersive wave equations are called integrable. It is believed that unsteady nonlinear partial differential equations (PDEs) are integrable if and only if they allow soliton solutions that are special kinds of solitary waves.

However, most nonlinear dispersive wave equations are not integrable or so are believed. Although systems are not integrable, solitary waves, which are not solitons, may exist. Gravity–capillary solitary waves and Benjamin solitary waves are such examples. In two dimensions, their profiles and stability properties have been thoroughly investigated. In order to find the solitary wave solutions of non-integrable systems, an appropriate generation mechanism of such waves has to be considered.

It is a key observation that most solitary waves are parameterized by the wave speed $c$. They approach small-amplitude or weakly nonlinear waves as the wave speed is getting close to a certain limit value $c_0$, which is thought of as the bifurcation point of solitary waves. There are two kinds of generation mechanisms or bifurcation scenarios: the first kind is bifurcation from a long wave and the second kind is bifurcation from a sinusoidal wave of nonzero wavenumber. Each generation mechanism is schematically illustrated in Figure 1. Actually, the KdV solitary waves bifurcate from the first scenario and the two-dimensional (2-D) gravity–capillary solitary waves follow the second scenario.

1.3 A new generation mechanism of ‘lumps’

Whereas the generation mechanisms of 2-D solitary waves have been comprehensively studied, the generation mechanism of fully localized three-dimensional solitary waves, commonly referred to as lumps, have received far less attention. Prior studies have focused in the long-wave limit and the Kdomptsev–Petviashvili I (KP-I) equation is the only equation which allows the lump-
Figure 1-1: Two generation mechanisms (bifurcation scenarios) of solitary waves: (a) bifurcation from a long wave (zero wavenumber), (b) bifurcation from a sinusoidal wave with a nonzero wavenumber.

type solitary waves of surface elevation $\eta(x, y, t)$:

$$
(\eta_t + (\eta^2)_x + (B - \frac{1}{3}) \eta_{xxx})_x - \eta_{yy} = 0,
$$

(1.3)

where $B = T/(\rho gh^2)$, is greater than 1/3 ($T$ is the surface tension coefficient, $g$ is the gravitational acceleration, and $h$ is the water depth).

The KP-I equation (1.3) is a three-dimensional analogue of the KdV equation under strong surface tension and is known to have lumps with algebraically decaying tails. However, it is mathematically possible to have lump solutions to the KP-I equation (1.3), but such lumps are physically unrealistic to be observed on the surface of water because the condition $B > 1/3$ requires too shallow water depth $h \ll 5mm$ for water with $T = 70g/s^2$. Under these conditions, viscosity is not negligible violating the underlying assumption of the water-wave problem.

On the other hand, $B < 1/3$ is a less restrictive condition in the water-wave problem; it is the weak-surface-tension and non-shallow water regime. Therefore, it is one of our main purposes in this study to explore a new generation mechanism of lumps other than the KP-I type lumps that bifurcate from the long-wave limit. It is performed by generalizing the second bifurcation scenario for 2-D solitary waves.

In Figure 3, the phase speed $c_p$ and the group speed $c_g$ for the water-wave problem (Figure 2) are plotted with respect to $\kappa = \sqrt{k^2 + m^2}$. The Bond number 1/3 is the critical number which separates two physical parameter regimes with the different qualitative behaviors of the dispersion relation. When $B < 1/3$, the phase speed $c_p$ attains the global minimum at a nonzero wavenumber $\kappa_0 h \neq 0$. At the extremum point, $c_g$ coincides with $c_p$ because

$$
\frac{\partial c_p}{\partial \kappa} = \frac{c_g - c_p}{\kappa} = 0 \iff c_p = c_g.
$$

(1.4)
\[ \nabla^2 \Phi(x, y, z) = 0 \text{ in } -h < z < \eta(x, y) \]

\[ \frac{\partial \Phi}{\partial z} = 0 \text{ on } z = -h \]

Figure 1-2: The physical setting for the water-wave problem. Fluid is bounded below by a rigid and flat bottom of depth \( \sqrt{3T/\rho g} < h < \infty \) for \( B = T/\rho gh^2 < 1/3 \). The bulk of the flow is governed by the Laplace equation. On the free surface on \( z = \eta(x, y) \), the kinematic condition and the dynamic condition are applied. On the bottom of a finite depth, the slip boundary condition is applied or the boundary layer thinkness is regarded much smaller than \( h \). For infinite depth, the flow velocity converges to zero as \( z \to \infty \).

When \( B > 1/3 \), however, both phase and group speed are monotonic in wavenumber so that they are equal to each other only at zero wavenumber. The point (ii) in Figure 3 is actually known to be the bifurcation point for 2-D gravity–capillary solitary waves. The intuitive reason why it plays a role of the generation point for solitary waves is because the carrier speed is the same as the speed of the envelope at the commont point of the phase and group speed so that a wavepacket behaves like a solitary wave.

### 1.4 Overview

In Chapter 2, we show that the point (ii) in Figure 3 is also the bifurcation point for new kinds of lumps. This is because the envelope \( A \) and mean flow \( A_0 \) of a three-dimensional wavepacket formed at this point satisfy the steady \textit{elliptic} Davey–Stewartson equation system which allow lump type solutions when \( B < 1/3 \):

\[ -cA_t + \lambda A_{\xi\xi} + \mu |A|^2 A = 0, \]  \hspace{1cm} (1.5)

and by the Davey–Stewartson equations (DS) in two-dimensions

\[ -cA_t + \lambda A_{\xi\xi} + \delta A_{\eta\eta} + \mu |A|^2 A + A\Phi_{\xi}, \]  \hspace{1cm} (1.6a)

\[ \alpha \Phi_{\xi\xi} + \Phi_{\eta\eta} = -\beta (|A|^2)_{\xi}. \]  \hspace{1cm} (1.6b)
Figure 1-3: The phase speed \( c_p (-) \) and the group speed \( c_g (-) \) with respect to \( \kappa = \sqrt{k^2 + l^2} \) based on the dispersion relation for the water-wave problem, \( \omega^2 = \left(\frac{g}{h}\right) \cdot \left[(\kappa h) + B(\kappa h)^3 \right] \cdot \tanh(\kappa h) \). (a) \( B < 1/3 \); (b) \( B > 1/3 \). Note that \( c_0 < \sqrt{gh} \) and \( \lambda = (1/2) \cdot \text{(the slope of} c_g \text{ at (ii))} > 0 \).

The coefficients are defined as

\[
\begin{align*}
\lambda &= \frac{1}{2} \frac{\partial c_g}{\partial k}, \quad \delta = \frac{c_g}{2k}, \quad \mu = \frac{\omega k^2}{16S^2} (8C^2S^2 + 9 - 2T^2) + \frac{\omega}{8C^2S^2} \frac{(2\omega C^2 + kc_g)^2}{gh - c_g^2}, \\
\alpha &= 1 - \frac{c_g^2}{gh}, \quad \beta = \frac{\omega}{8C^2S^2} (2\omega C^2 + kc_g)^2, \\
\end{align*}
\]

(1.7a)

where \( C = \cosh(\kappa h) \), \( S = \sinh(\kappa h) \) and \( T = \tanh(\kappa h) \). All of the coefficients in front of the second derivatives become positive because the minimum phase speed \( c_0 \) is less than \( \sqrt{gh} \) and the slope of the group speed with respect to the wavenumber at the common point of the phase and group speed is positive.

The existence of lumps is closely related to the stability of 2-D solitary waves under transverse disturbances to the dominant propagation direction of waves. In Chapter 3, we provide a functional criterion for the transverse instability of 2-D solitary waves under long-wave transverse disturbances by expressing the leading order instability growth rate in terms of solitary wave solutions.

We assert that the generation mechanism of lumps and the transverse instability of 2-D solitary waves introduced above are universal in other dispersive wave systems. In Chapter 4, we show that interfacial gravity-capillary lumps of Benjamin type also follow a similar bifurcation scenario as that of surface gravity-capillary lumps. In Chapter 5, we provide a similar functional criterion to what is shown in Chapter 3 for the transverse instability of 2-D
interfacial gravity–capillary solitary waves of Benjamin type.

Finally, in appendix, we supplement detailed analytical derivations and numerical methods that are used in this study.

In Appendix A, we present an intuitive idea, which is called a spatial dynamics, to show the existence of radial symmetric envelope NLS lumps. We treat the problem to determine the envelope profiles as an initial value problem of a harmonic oscillator with a time-dependent damping. The spatial variable \( r \) is regarded as a time-like variable. In this way, we are able to prove that there are a countably infinite number of radial symmetric fully localized NLS modes.

In Appendix B, we provide the detailed derivation of the adjoint boundary value problem for the water-wave equations when the surface tension term is included in the dynamic boundary condition.

Appendix C deals with spectral numerical implementations, which are particularly designed for this study in order to investigate localized and coherent solutions of nonlinear PDEs spread over an unbounded domain so that we are able to compute algebraically decaying wave profiles with good accuracy and efficiency.
Chapter 2

Gravity–capillary lumps

2.1 Introduction

As is briefly described in Chapter 1, two distinct kinds of solitary waves are known to exist on the surface of water, assuming two-dimensional (plane) disturbances. The first is found in shallow water and, in the weakly nonlinear limit, is governed by the celebrated Korteweg–de Vries (KdV) equation (see, for example, Whitham 1974, §13.11); the other is possible in water of finite or infinite depth but only if surface tension is present.

The latter class of solitary waves is closely connected with the fact that the phase speed features a minimum at a finite wavenumber when both gravity and surface tension are included: as the phase speed is equal to the group speed there, this minimum is the bifurcation point of gravity–capillary solitary waves in the form of wavepackets, with crests moving at the same speed as the wave envelope (Akylas 1993, Longuet-Higgins 1993). In fact, the long-water-wave speed, at which solitary waves bifurcate in shallow water, is a local maximum (minimum) of the phase speed if the Bond number $B = T/(\rho g h^2)$ ($T$ is the coefficient of surface tension, $\rho$ is the fluid density, $g$ is the gravitational acceleration and $h$ the water depth) is less (greater) than 1/3; both kinds of solitary water waves thus bifurcate at extrema of the phase speed.

While solitary waves of the KdV type have been known for over a century, gravity–capillary solitary waves of the wavepacket type were discovered relatively recently. Longuet-Higgins (1989) first presented numerical evidence of gravity–capillary solitary waves in deep water, followed by a number of related analytical and computational studies (see Dias & Kharif 1999 for a comprehensive review). It turns out that two symmetric solution branches, one corresponding to elevation and the other to depression waves, bifurcate from a linear sinusoidal wavetrain at the minimum phase speed. In the small-amplitude limit, close to bifurcation, these branches are governed by the nonlinear Schrödinger (NLS) equation, their difference being merely a shift of the wave crests by half a wavelength relative to the peak of the wave envelope. In addition, there is an infinity of other symmetric and asymmetric solution branches that bifurcate at finite
amplitude below the minimum phase speed; these may be interpreted as multi-packet solitary waves and are beyond the reach of the NLS equation, although they can be captured by a more refined perturbation approach (Yang & Akylas 1997).

The theory of three-dimensional solitary waves, commonly referred to as ‘lumps’, that are locally confined in all directions, is not as well developed. Most of prior work centres around the Kadomtsev–Petviashvili (KP) equation, an extension of the KdV equation that allows for weak three-dimensional effects in the propagation of weakly nonlinear waves in shallow water. According to the KP equation, KdV solitary waves are stable (unstable) to transverse perturbations when the Bond number is less (greater) than 1/3. In the regime where instability is present, the KP equation (in which case is usually called KP-I equation) admits lump solutions with algebraically decaying tails (see, for example, Ablowitz & Segur 1979). Similar lumps were found numerically by Berger & Milewski (2000) based on the Benney–Luke equations with surface tension, a more general system of weakly nonlinear shallow-water equations than the KP-I. Like their KP-I counterparts, these lumps are possible only if the Bond number is greater than 1/3, a condition that restricts the water depth to less than a few mm, so neglecting viscous effects cannot be justified.

In the present study, it is pointed out that, when surface tension is present, lumps of the wavepacket type can be found for $B < 1/3$ in water of finite or infinite depth. In direct analogy with two-dimensional gravity–capillary solitary waves, these lumps again bifurcate at the minimum gravity–capillary phase speed and, close to bifurcation, they can be approximated as three-dimensional locally confined wavepackets with envelope moving at the same speed as the wave crests. In water of finite depth, the wave envelope and the induced mean flow are strongly coupled and are governed by an elliptic–elliptic Davey–Stewartson equation system. It turns out that the mean flow decays algebraically and so do the tails of lumps, in contrast to two-dimensional solitary waves in water of finite depth that decay exponentially at infinity. In deep water, on the other hand, the induced mean flow is relatively weak and the wave envelope is governed by an elliptic two-dimensional NLS equation to leading order; however, the induced mean flow, which again decays algebraically, prevails at the lump tails, as for two-dimensional deep-water solitary waves (Akylas, Dias & Grimshaw 1998).

While the present chapter was in its final stages of preparation, we became aware of two as yet unpublished computational studies of gravity–capillary lumps, which are complementary to our weakly nonlinear analysis. Based on the full gravity–capillary water-wave equations, Parau, Vanden-Broeck & Cooker (2005) numerically computed two branches of symmetric elevation and depression lumps that bifurcate at the minimum phase speed; close to the bifurcation point, these lumps resemble three-dimensional wavepackets. Retaining only quadratic nonlinear terms in the governing equations, Milewski (2005) used a numerical continuation procedure to compute gravity–capillary lumps in the form of locally confined wavepackets in water of large
depth, starting from shallow-water lumps of the KP-I type.

A rigorous existence proof of fully localised three-dimensional solitary-wave solutions of the gravity–capillary water-wave problem was devised by Groves & Sun (2005).

2.2 Expansion near the bifurcation point

Consider the classical problem of waves on the surface of water of depth $h$ under the action of both gravity and surface tension. For the purpose of discussing waves of permanent form moving with speed $c$, we introduce dimensionless variables employing $T/(\rho c^2)$ as lengthscale and $T/(\rho c^3)$ as timescale, where $\rho$ denotes the fluid density and $T$ the coefficient of surface tension. The phase speed of linear sinusoidal gravity–capillary waves with wavenumber $k$ is thus normalized to unity, and the dispersion relation takes the form

$$G(k; \alpha, H) \equiv k(\alpha + k^2)\tanh kH - k^2 = 0, \quad (2.1)$$

where

$$\alpha = \frac{gT}{\rho c^4}, \quad H = \frac{h\rho c^2}{T}, \quad (2.2)$$

$g$ being the acceleration of gravity. This introduces two flow parameters, the speed parameter $\alpha$ and the inverse Weber number $H$; the Bond number,

$$B = \frac{T}{\rho gh^2}, \quad (2.3)$$

which is independent of the wave speed, is expressed in terms of $\alpha$ and $H$ via

$$B = \frac{1}{\alpha H^2}. \quad (2.4)$$

Lumps bifurcate from linear sinusoidal waves with wavenumber $k = k_0$ corresponding to the minimum gravity–capillary phase speed and, hence, to a double root of the dispersion relation (2.1):

$$G|_0 = 0, \quad \frac{\partial G}{\partial k}|_0 = 0. \quad (2.5)$$

In general, for a given value of one of the two independent flow parameters, conditions (2.5) specify the wavenumber $k_0$ and the value of the other parameter at the bifurcation point. Here, for convenience, we treat $H$ as a free parameter so, combining (2.1) and (2.5), $k_0$ and $\alpha_0$ are
determined from

\[ k_0 \tanh k_0 H + \frac{k_0 H}{\sinh 2k_0 H} - \frac{1}{2} = 0, \]  

Equation (2.6a) has real roots \( \pm k_0 \) only if \( 3 \leq H < \infty \), and the Bond number (2.3) has to be less than \( 1/3 \) for bifurcation of lumps to be possible. In particular, \( H \to 3 \) corresponds to the long-wave limit

\[ k_0 \to 0, \quad \alpha_0 \to \frac{1}{3}, \]  

while, in the deep-water limit \( H \to \infty \),

\[ k_0 \to \frac{1}{2}, \quad \alpha_0 \to \frac{1}{4}. \]  

In order to remain locally confined, a lump must travel at a speed less than the minimum gravity-capillary phase speed. Therefore \( \alpha > \alpha_0 \), and, in the neighbourhood of the bifurcation point, we write

\[ \alpha = \alpha_0 + \epsilon^2, \]  

\( \epsilon \) being a small parameter (\( 0 < \epsilon \ll 1 \)).

Close to the bifurcation point, moreover, lumps are in the form of small-amplitude wavepackets with crests moving at the same speed (equal to 1 in the present normalization) as the wave envelope. The velocity potential \( \phi(\xi, y, z) \) and the free-surface elevation \( z = \eta(\xi, y) \) of a lump propagating along the \( x \)-direction (\( y \) and \( z \) denoting the transverse and vertical directions, respectively) then are expanded as follows:

\[ \phi = \epsilon A_0(z, X, Y) + \epsilon \{ A_1(z, X, Y)e^{i\theta_0} + \text{c.c.} \} + \epsilon^2 \{ A_2(z, X, Y)e^{2i\theta_0} + \text{c.c.} \} + \ldots, \]  

\[ \eta = \epsilon \{ S_1(X, Y)e^{i\theta_0} + \text{c.c.} \} + \epsilon^2 S_0(X, Y) + \epsilon^2 \{ S_2(X, Y)e^{2i\theta_0} + \text{c.c.} \} + \ldots, \]  

where \( \xi = x - t, \, \theta_0 = k_0 \xi \) and c.c. denotes the complex conjugate. The carrier wavevector \( \kappa_0 = (k_0, 0) \) has magnitude \( |\kappa_0| = k_0 \) as obtained from (2.6) and points in the \( x \)-direction, the crests thus being perpendicular to the propagation direction. The above expansions also assume that the amplitudes \( A_0, A_1, A_2, \ldots \) and \( S_0, S_1, S_2, \ldots \), which depend on the ‘stretched’ variables \( (X, Y) = \epsilon(\xi, y) \), remain locally confined in both horizontal directions \( \xi \) and \( y \).

The procedure for determining the amplitudes of the various harmonics in expansions (2.10) and (2.11) closely parallels that followed by Benney & Roskes (1969) and Davey & Stewartson.
briefly, upon substituting (2.10) into laplace's equation for \( \phi \),

\[
\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad (-H < z < \eta),
\]

and imposing the bottom boundary condition

\[
\phi_z = 0 \quad (z = -H),
\]

it follows that

\[
A_1 = a_1 \frac{\cosh k_0(z + H)}{\cosh k_0 H} - \epsilon \left\{ \frac{\partial a_1}{\partial X} (z + H) \frac{\sinh k_0(z + H)}{\cosh k_0 H} \right\} - \epsilon^2 \left\{ \frac{1}{2} \frac{\partial^2 a_1}{\partial X^2} (z + H) \frac{\cosh k_0(z + H)}{\cosh k_0 H} + \frac{1}{2k_0} \frac{\partial^2 a_1}{\partial Y^2} (z + H) \frac{\sinh k_0(z + H)}{\cosh k_0 H} \right\} + \cdots,
\]

\[
A_0 = a_0 - \epsilon^2 \left\{ \frac{1}{2} (z + H)^2 \left( \frac{\partial^2 a_0}{\partial X^2} + \frac{\partial^2 a_0}{\partial Y^2} \right) \right\} + \cdots,
\]

\[
A_2 = a_2 \frac{\cosh 2k_0(z + H)}{\cosh 2k_0 H} + \cdots
\]

the next task is to satisfy the free-surface boundary conditions

\[
\phi_x + \eta_x + \phi_y \eta_y = \phi_x \eta_x + \phi_y \eta_y \quad (z = \eta),
\]

\[
\alpha \eta - \phi_x + \frac{1}{2} (\phi_x^2 + \phi_y^2 + \phi_z^2) = \eta_x (1 + \eta_x^2) + \eta_y (1 + \eta_y^2) - 2 \eta_x \eta_y \eta_x \eta_y \quad (z = \eta).
\]

substituting expansions (2.10) and (2.11) in (2.15) and (2.16), making use of (2.14), and collecting mean terms, correct to \( O(\epsilon^3) \), yields

\[
- \frac{\partial S_0}{\partial X} + H \left( \frac{\partial^2 a_0}{\partial X^2} + \frac{\partial^2 a_0}{\partial Y^2} \right) + 2(\alpha_0 + k_0^2) \frac{\partial}{\partial X} |S_1|^2 = 0,
\]

\[
\alpha_0 S_0 - \frac{\partial a_0}{\partial X} + ((\alpha_0 + k_0^2)^2 - k_0^2)|S_1|^2 = 0.
\]

eliminating \( S_0 \) from equations (2.17), it follows that \( a_0(X, Y) \) satisfies

\[
Q^2 \frac{\partial^2 a_0}{\partial X^2} + \frac{\partial^2 a_0}{\partial Y^2} = \lambda \frac{\partial}{\partial X} |S_1|^2,
\]
where

\[ \lambda = k_0^2 - (\alpha_0 + k_0^2)(3\alpha_0 + k_0^2) \quad Q^2 = \frac{\alpha_0 H - 1}{\alpha_0 H}. \]  

(2.19)

According to (2.2), \( \alpha_0 H = gh/c_0^2 \), \( (gh)^{1/2} \) being the long-wave speed and \( \alpha_0 \) the minimum gravity–capillary phase speed; hence \( \alpha_0 H > 1 \), and equation (2.18), which governs the \( O(\epsilon^4) \) mean-flow component induced by the modulations of the packet, is of the elliptic type. As it turns out, this mean flow controls the behaviour of the tails of a lump (see §3.3).

By a similar procedure, substituting the expansions (2.10) and (2.11), along with (2.14), in the free-surface conditions (2.15) and (2.16), and collecting second-harmonic terms, one finds that \( e_2 \) and \( S_2 \) satisfy

\[ S_2 - i \tanh 2k_0 H e_2 = (\alpha_0 + k_0^2)S_1, \]

(2.20a)

\[ (\alpha_0 + 4k_0^2)S_2 - 2ik_0 e_2 = \frac{1}{2} (3\alpha_0^2 - (\alpha_0 + k_0^2)^2) S_1^2. \]

(2.20b)

This equation system can be readily solved for \( e_2 \) and \( S_2 \) in terms of \( S_1^2 \).

Finally, by collecting primary-harmonic terms in the free-surface conditions (2.15) and (2.16), and consistently eliminating \( e_2 \), \( S_2 \) and \( e_1 \) correct to \( O(\epsilon^3) \), one may derive an amplitude equation for \( S_1(X,Y) \). In view of conditions (2.5) at the bifurcation point, all \( O(\epsilon) \) and \( O(\epsilon^2) \) terms cancel out, and the resulting equation takes the form

\[ -S_1 + \beta \frac{\partial^2 S_1}{\partial X^2} + \gamma \frac{\partial^2 S_1}{\partial Y^2} = \delta|S_1|^2 S_1 + \zeta \frac{\partial \alpha_0}{\partial X} S_1, \]

(2.21)

where \( \beta \), \( \gamma \), \( \delta \) and \( \zeta \) are certain real coefficients. The equation system (2.18) and (2.21) for the primary-harmonic envelope and the induced mean flow is a steady version of the so-called Davey–Stewartson equations, derived in Davey & Stewartson (1974) following Benney & Roskes (1969).

Obtaining expressions for the coefficients in equation (2.21), especially those multiplying the nonlinear terms, involves a considerable amount of algebra. Here, in the interest of brevity, the coefficients \( \beta \) and \( \gamma \) of the linear terms on the left-hand side of (2.21) are deduced from the linear dispersion relation, making use of the fact that a lump comprises plane waves \( \kappa = (k, m) \) that are steady in the reference frame of the lump. Hence,

\[ D(k, m; \alpha) \equiv \kappa(\alpha + \kappa^2) \tanh \kappa H - k^2 = 0, \]

(2.22)

where \( \kappa = |\kappa| \) and, at the carrier wavevector \( \kappa_0 = (k_0, 0) \),

\[ D|_0 = 0, \quad \frac{\partial D}{\partial k}|_0 = \frac{\partial D}{\partial m}|_0 = 0, \]

(2.23)
in view of (2.5). Expanding (2.22) in the vicinity of the bifurcation point,

\[ \alpha = \alpha_0 + \epsilon^2, \quad k = k_0 + \epsilon \Delta k, \quad m = \epsilon \Delta m, \]

the \( O(1) \) and \( O(\epsilon) \) terms vanish by (2.23) so, correct to \( O(\epsilon^2) \),

\[ \frac{\partial D}{\partial \alpha} \bigg|_0 + \frac{1}{2} \frac{\partial^2 D}{\partial k^2} \bigg|_0 \Delta k^2 + \frac{1}{2} \frac{\partial^2 D}{\partial m^2} \bigg|_0 \Delta m^2 = 0. \] (2.24)

Equation (2.24) is entirely equivalent to the left-hand side of (2.21) if one substitutes \( \exp \{ \text{i}(\Delta k X + \Delta m Y) \} \), so

\[ \beta = \frac{1}{2} \frac{\partial^2 D}{\partial k^2} \bigg|_0 = \frac{\alpha_0 + k_0^2}{k_0} \frac{\partial^2 \omega}{\partial k^2} \bigg|_0, \quad \gamma = \frac{1}{2} \frac{\partial^2 D}{\partial m^2} \bigg|_0 = \frac{\alpha_0 + k_0^2}{k_0} \frac{\partial^2 \omega}{\partial m^2} \bigg|_0, \] (2.25)

where

\[ \omega(\kappa) = \{ \kappa (\alpha + \kappa^2 \tanh \kappa H) \}^{\frac{1}{2}}. \]

The coefficients \( \delta \) and \( \zeta \) of the nonlinear terms on the right-hand side of (2.21) now can be read off from Djordjevic & Redekopp (1977) or Ablowitz & Segur (1979). After converting to the nondimensional variables used here, evaluating these coefficients at the minimum gravity-capillary phase speed yields

\[ \delta = \frac{1}{2} (\alpha_0 + k_0^2)^3 \left\{ \frac{(1 - \sigma^2)(9 - \sigma^2)\alpha_0 + k_0^2(3 - \sigma^2)(7 - \sigma^2)}{\alpha_0 \sigma^2 - k_0^2(3 - \sigma^2)} \right\} \]

\[ + 8\sigma^2 - \frac{2}{\alpha_0} (1 - \sigma^2)^2 (\alpha_0 + k_0^2) - \frac{3k_0^2 \sigma^2}{\alpha_0 + k_0^2} \} \] (2.26a)

\[ \zeta = 2(\alpha_0 + k_0^2) \left\{ 1 + \frac{(\alpha_0 + k_0^2)^2 - k_0^2}{2\alpha_0 (\alpha_0 + k_0^2)} \right\}, \] (2.26b)

where

\[ \sigma^2 = \tanh^2 k_0 H = k_0^2 / (\alpha_0 + k_0^2)^2. \]

It is easy to check that the coefficient \( \gamma > 0 \) and, as noted by Dias & Haragus-Courcelle (2000), \( \beta > 0 \) as well. Equations (2.18) and (2.21), which govern the primary-harmonic envelope and the induced mean flow, therefore, are both of the elliptic type. In this instance, as remarked
by Ablowitz & Segur (1979), the Davey–Stewartson equations are not integrable by inverse scattering transforms.

In the special case that transverse modulations are absent ($\partial/\partial Y = 0$), equation (2.18) can be readily solved for the induced mean flow:

$$\frac{\partial a_0}{\partial X} = \frac{\lambda}{Q^2} |S_1|^2,$$

(2.27)

and, upon substituting in (2.21), it is found that $S_1$ satisfies a steady form of the NLS equation:

$$-S_1 + \beta \frac{\partial^2 S_1}{\partial X^2} + \nu |S_1|^2 S_1 = 0,$$

(2.28)

where

$$\nu = -\delta + \frac{\zeta}{1 - \alpha_0 H} \left\{k_0^2 - (\alpha_0 + k_0^2)(3\alpha_0 + k_0^2)\right\}.$$

(2.29)

It turns out that $\nu > 0$ so (2.28) admits the envelope-soliton solution

$$S_1 = \left(\frac{2}{\nu}\right)^{1/2} \text{sech} \left\{\frac{X}{\beta^{1/2}}\right\},$$

(2.30)

which, combined with (2.9) and (2.11), furnishes the well known two-dimensional solution branches of solitary waves of elevation (+) and depression (−):

$$\eta = \pm \eta_0 \text{sech} \left\{\left(\frac{\alpha - \alpha_0}{\beta}\right)^{1/2}(x - t)\right\} \cos k_0(x - t) + \cdots,$$

(2.31)

where, to leading order in $\alpha - \alpha_0$, the peak amplitude $\eta_0$ is given by

$$\eta_0 = 2\left(\frac{2}{\nu}\right)^{1/2}(\alpha - \alpha_0)^{1/2} + \cdots.$$

(2.32)

In the deep-water limit $H \to \infty$, in particular, making use of (2.8) and (2.26a), (2.29) yields $\nu \to 11/32$ so, according to (2.32), the peak amplitude of the bifurcating solitary-wave solutions is given by

$$\eta_0 = \frac{16}{(11)^{1/2}}(\alpha - \alpha_0)^{1/2} + \cdots,$$

(2.33)


When both $X$- and $Y$- modulations are present, the primary-harmonic envelope and the induced mean flow are coupled, and no analytical locally confined solutions of equations (2.18) and (2.21) are known. Nevertheless, as discussed in §3.4, it is possible to compute such solutions numerically and thereby determine the branches of lumps bifurcating at $\alpha = \alpha_0$. 
2.3 Behaviour at the tails of a lump

According to (2.30), when no transverse modulations are present, the wave envelope decays exponentially and so does the induced mean flow (2.27). In water of finite depth, therefore, two-dimensional gravity-capillary solitary wavepackets feature oscillatory tails with exponentially decaying amplitude; only in deep water, where the induced mean flow decays algebraically, these tails are algebraic (Akylas et al. 1998). The tails of lumps behave quite differently, however, because the induced mean flow turns out to decay algebraically at infinity, irrespective of the water depth.

Specifically, returning to the elliptic mean-flow equation (2.18), taking the Fourier transform in $X$ and $Y$ yields

$$\frac{\partial a_0}{\partial X} =\lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l^2 F\{|S_1|^2\} \exp\{i(lX + mY)\} \frac{\exp\{i(lX + mY)\}}{Q^2 l^2 + m^2} \, dl \, dm,$$

$$(2.34)$$

$F\{|S_1|^2\}$ being the Fourier transform of $|S_1|^2$:

$$F\{|S_1|^2\} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_1|^2 \exp\{-i(lX + mY)\} \, dX \, dY.$$

Therefore, in the far field $X^2 + Y^2 \rightarrow \infty$,

$$\frac{\partial a_0}{\partial X} \sim \frac{\lambda}{4\pi^2} I_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l^2 \exp\{i(lX + mY)\} \frac{\exp\{i(lX + mY)\}}{Q^2 l^2 + m^2} \, dl \, dm,$$

$$(2.35)$$

where

$$I_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_1|^2 \, dX \, dY,$$

and, upon evaluating the double integral in (2.35),

$$\frac{\partial a_0}{\partial X} \sim -\frac{\lambda I_0}{2\pi Q} \frac{\partial}{\partial Y} \left\{ \frac{Y}{X^2 + Q^2 Y^2} \right\}.$$

$$(2.36)$$

This confirms that the induced mean flow decays algebraically at infinity and, as a result, controls the behaviour at the tails of a lump; the envelope $S_1$ of the primary harmonic (as well as the higher-harmonic envelopes) decays exponentially there according to the elliptic equation (2.21), and is thus overwhelmed by the mean-flow component.

For the same reason, the free-surface elevation at the tails of a lump is dominated by the
mean flow,

\[ \eta \sim \frac{\epsilon^2}{a_0} \frac{\partial a_0}{\partial X} \quad (X^2 + Y^2 \to \infty), \tag{2.37} \]

and decays algebraically as well:

\[ \eta \sim -\frac{\epsilon^2}{2\pi a_0} \frac{\lambda}{Q} \frac{\partial}{\partial Y} \left\{ \frac{Y}{X^2 + Q^2 Y^2} \right\}. \tag{2.38} \]

### 2.4 Lump solutions

As already remarked, no analytical locally confined solution of equations (2.18) and (2.21) is available. In the deep-water limit \( H \to \infty \), where the coupling of the envelope with the induced mean flow is weak, (2.21) reduces to a steady two-dimensional NLS equation of the elliptic type, for which Strauss (1977) provided a mathematical proof that a non-trivial locally confined solution is possible. A discussion of lumps in deep water, including the effect of the induced mean flow, is deferred to §2.5. Here, as suggested in Papanicolaou et al. (1994), this limit is used as the starting point for computing locally confined solutions of the coupled equations (2.18) and (2.21), by continuation in the parameter \( H \).

We begin by computing locally confined solutions of (2.21) when \( H \to \infty \). In this limit, it follows from (2.25) and (2.26) that

\[ \beta \to 1, \quad \gamma \to 2, \quad \delta \to -\frac{11}{32}. \tag{2.39} \]

Moreover, since \( \lambda \to 0 \) according to (2.19), the forcing term on the right-hand side of the mean-flow equation (2.18) vanishes; so \( a_0 \to 0 \) and, as noted above, the coupling with the induced mean flow may be neglected in equation (2.21).

With the re-scaling \( \tilde{Y} = Y/\sqrt{2} \) and \( \tilde{S}_1 = (11/32)^{1/2} S_1 \), we then seek localised solutions of (2.21) which satisfy the radial NLS equation

\[ \frac{d^2 \tilde{S}_1}{dR^2} + \frac{1}{R} \frac{d \tilde{S}_1}{dR} - \tilde{S}_1 + \tilde{S}_1^3 = 0, \tag{2.40} \]

where \( R^2 = X^2 + \tilde{Y}^2 \), subject to the boundary conditions

\[ \frac{d \tilde{S}_1}{dR} = 0 \quad (R = 0), \tag{2.41a} \]

\[ \tilde{S}_1 \to 0 \quad (R \to \infty). \tag{2.41b} \]

The latter condition ensures that the disturbance remains locally confined, as required for a
Figure 2-1: First three modes of the boundary-value problem (2.40)-(2.41) that governs localized solutions of the envelope equation (2.21) in the deep-water limit $H \to \infty$. —: ground state; --- : second mode; ⋯ : third mode.

The boundary-value problem (2.40)-(2.41) was studied in Chiao, Garmire & Townes (1964), who found numerically a profile $\hat{S}_1(R)$ that decays monotonically in $0 < R < \infty$. Apart from this 'ground state', we also computed other, oscillatory profiles as shown in Figure 2-1, and it appears that there exists a countably infinite set of such modes. In the following, however, only the ground state will be considered.

Starting at large depth with the localized solution of (2.21) corresponding to the ground state of (2.40)-(2.41), solutions to the coupled system (2.18) and (2.21) at finite depth were obtained via numerical continuation by gradually decreasing $H$. The differential equations were discretized using a pseudospectral method in terms of Chebyshev polynomials, combined with a transformation that maps the $XY$-plane into a bounded rectangular domain. The resulting nonlinear algebraic equations were solved by Newton's method in one quarter of the domain, exploiting symmetry. Details of numerical implementation can be found in Appendix A.

Figure 2-1 and 2-2 illustrate the computed profiles of the envelope $S_1(X, Y)$ and the induced mean flow $\partial a_0/\partial X$ for $H = 3.5$, $H = 5$ and $H = 25$, the latter value corresponding essentially to deep water. As expected, the induced mean flow becomes stronger as $H$ is decreased and always prevails at infinity since it decays algebraically, while $S_1$ decays exponentially, there. As a check of the numerical computations, it was verified that the decay of the mean flow at infinity is consistent with the asymptotic expression (2.36) derived earlier.

Figure 2-4 is a plot of the envelope maximum, $S_1(0, 0)$, as $H$ is varied. For comparison, returning to (2.30), the maximum of the envelope of a two-dimensional solitary wave, $(2/\nu)^{1/2}$, is also shown on the same graph. It is seen that the peak amplitude of a lump, like that of a plane solitary wave, increases as $H$ is increased. However, for a given value of $H$, the peak
Figure 2-2: Representative profiles of the lump primary-harmonic envelope $S_1(X, Y)$ (left column) and the induced mean flow $\partial a_0/\partial X$ (right column), as obtained from the equation system (2.18) and (2.21) for various values of $H$. (a) $H = 3.5$; (b) $H = 5$; (c) $H = 25$. 
Figure 2-3: Representative profiles of the lump primary-harmonic envelope $S_1(X, Y)$ and the induced mean flow $\partial a_0/\partial X$, as obtained from the equation system (2.18) and (2.21) for various values of $H$. — : $X$-cross-section for $Y = 0$ ; --- : $Y$-cross-section for $X = 0$. (a) $H = 3.5$; (b) $H = 5$; (c) $H = 25$. 
amplitude of a lump,

$$\eta_0 = 2S_1(0, 0)(\alpha - \alpha_0)^{1/2} + \cdots,$$

(2.42)

exceeds that of its two-dimensional counterpart, as given by (2.33), roughly by a factor of 1.5–2 depending on $H$.

\section*{2.5 Deep water}

We now return to the case of deep water and discuss the bifurcation of gravity-capillary lumps in this limit. As is evident from (2.14a), the perturbation expansions (2.10) and (2.11) break down when $H \to \infty$ because, as it turns out, the induced mean flow is relatively weak, of $O(\epsilon^3)$ rather than $O(\epsilon^2)$ as assumed in (2.10) and (2.11). This disordering is also hinted by the fact that, as noted in §4, in the limit $H \to \infty$, the coupling of the primary harmonic with the induced mean flow vanishes according to (2.18), so $a_0 \to 0$.

The way to rectify the situation is now well known. As suggested by Roskes (1969), expansions (2.10) and (2.11) need to be modified such that the mean-flow terms enter at $O(\epsilon^3)$; furthermore, since the vertical coordinate becomes unbounded in deep water, it is necessary to introduce the additional stretched coordinate $Z = \epsilon z$, the velocity potential $\phi$ now being a function of both $z$ and $Z$.

With these modifications, carrying out the perturbation analysis to $O(\epsilon^3)$ confirms that the primary-harmonic envelope $S_1(X, Y)$ satisfies precisely the steady two-dimensional NLS equation obtained by applying the formal limit $H \to \infty$ to equation (2.21), and the coefficients

Figure 2-4: Peak amplitude of the primary-harmonic envelope $S_1$ as $H$ is varied. — : three-dimensional envelope; --- : two-dimensional envelope. The corresponding asymptotic values in the deep-water limit $H \to \infty$ are indicated by dotted lines.
\( \beta, \gamma, \) and \( \delta \) take the limiting values (2.39). To leading order, therefore, the coupling with the induced mean flow is negligible and the envelope of a deep-water lump is governed by the boundary-value problem (2.39)–(2.41) discussed earlier. Based on the ground state plotted in Figure 1, the peak amplitude of a lump in deep water, to leading order in \( \alpha - \alpha_0 \), is given by (2.42) with \( S_1(0,0) = 3.76 \); this is the three-dimensional counterpart of the well known expression (2.33) for two-dimensional solitary waves in deep water.

Although the effect of the induced mean flow in deep water is of higher order, the behaviour of the tails of a lump is controlled by the mean flow. As explained in Akylas et al. (1998) for two-dimensional solitary waves in deep water, where a similar nonuniformity arises, the reason is that the wave envelope decays exponentially at infinity while the mean flow does so algebraically and eventually dominates at the tails. Accordingly, in order to discuss the behaviour at the tails of a lump in deep water, it is necessary to carry the perturbation analysis to \( O(\epsilon^4) \).

Hogan (1985) derived a fourth-order envelope equation for three-dimensional modulations of gravity–capillary wavepackets in deep water, and we shall adapt his analysis to the case of lumps. Allowing for the fact that the carrier oscillations and the envelope move with speed 1 in the nondimensional variables used here, and evaluating the coefficients of his evolution equation at the bifurcation point, it is found that the primary-harmonic envelope \( S_1(X, Y) \) is governed by

\[
-S_1 + \frac{\partial^2 S_1}{\partial X^2} + 2 \frac{\partial^2 S_1}{\partial Y^2} + \frac{11}{32} |S_1|^2 S_1 + i \epsilon \left( -2 \frac{\partial^2 S_1}{\partial X \partial Y^2} + \frac{3}{4} |S_1|^2 \frac{\partial S_1}{\partial X} \right) - \epsilon S_1 \frac{\partial \phi}{\partial X} \bigg|_{z=0} = 0, \quad (2.43)
\]

while the mean-flow potential \( \epsilon^2 \phi(X, Y, Z) \) (that replaces \( \epsilon A_0(z, X, Y) \) in (2.10)) satisfies Laplace's equation

\[
\phi_{XX} + \phi_{YY} + \phi_{ZZ} = 0 \quad (0 > Z > -\infty), \quad (2.44)
\]

subject to the boundary conditions

\[
\phi_Z = \frac{\partial}{\partial X} |S_1|^2 \quad (Z = 0), \quad (2.45a)
\]

\[
\phi \to 0 \quad (Z \to -\infty). \quad (2.45b)
\]

To leading order, as expected, \( S_1 \) satisfies the elliptic two-dimensional NLS equation that is obtained from the Davey–Stewartson system (2.18) and (2.21) in the deep-water limit \( H \to \infty \). It is easy to check that the higher-order modulation terms in (2.43) affect only the phase of \( S_1 \), resulting in \( O(\epsilon^2) \) corrections to the carrier wavenumber, while the coupling with the induced mean flow produces an \( O(\epsilon^2) \) correction to the wave amplitude.

The boundary-value problem (2.44)–(2.45) for the mean-flow potential \( \phi \) can be readily
solved by taking the Fourier transform in $X$ and $Y$, and the associated free-surface elevation $\tilde{\eta}$ is given by

$$\tilde{\eta} = -\frac{\epsilon^3}{\alpha_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I^2 \mathcal{F}\{|S_1|^2\} \frac{\exp\{i(lX + mY)\}}{(l^2 + m^2)^{1/2}} dl \, dm,$$

(2.46)

$\mathcal{F}\{|S_1|^2\}$ denoting the Fourier transform of $|S_1|^2$ as in (2.34).

Therefore, in the far field $X^2 + Y^2 \to \infty$,

$$\tilde{\eta} \sim -\frac{\epsilon^3}{4\pi^2 \alpha_0} I_0 \frac{\partial^2}{\partial X^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{i(lX + mY)\}}{(l^2 + m^2)^{1/2}} dl \, dm,$$

(2.47)

where $I_0$ is defined as in (2.35). Upon evaluating the double integral in (2.47) and recalling that $\alpha_0 = 1/4$ in deep water, it is finally found that

$$\tilde{\eta} \sim \frac{2\epsilon^3}{\pi} I_0 \frac{\partial^2}{\partial X^2} \left\{ \frac{1}{(X^2 + Y^2)^{1/2}} \right\}.$$  

(2.48)

Comparing (2.48) with the analogous expression (2.38) in water of finite depth, it is seen that the mean flow accompanying a lump in deep water is weaker and decays faster than its finite-depth counterpart. Nevertheless, since $\tilde{\eta}$ decays algebraically, it eventually prevails at the tails of a lump.

2.6 Discussion

Based on small-amplitude expansions, we have pointed out that gravity-capillary lumps are possible for $B < 1/3$ on water of finite or infinite depth. Like two-dimensional gravity-capillary solitary waves, these lumps bifurcate at the minimum gravity-capillary phase speed and, in the small-amplitude limit, close to the bifurcation point, take the form of fully localised wavepackets with envelope and crests moving at the same speed, slightly below the minimum gravity-capillary phase speed. Moreover, two symmetric lump-solution branches, one corresponding to elevation and the other to depression waves, bifurcate from infinitesimal sinusoidal wavetrains, as is the case of two-dimensional solitary waves. The only essential difference of lumps from their two-dimensional counterparts is that the induced mean flow always decays algebraically at infinity and so do the tails of lumps, irrespective of the water depth. This is in contrast to the tails of two-dimensional solitary waves which decay algebraically only in deep water.

Since gravity-capillary lumps travel at speeds less than the minimum phase speed, there is no possibility of resonance with other parts of the water-wave spectrum, precluding the formation of small-amplitude oscillations at infinity, similar to those found at the tails of KdV solitary waves when the Bond number is less than 1/3. This is consistent with the recent numerical work of Parau et al. (2005), who computed locally confined gravity-capillary lumps...
based on the full water-wave equations, and the rigorous existence proof of Groves & Sun (2005).

It is known that the bifurcation diagram of two-dimensional gravity-capillary solitary waves is quite complicated, as, in addition to the two symmetric solution branches that bifurcate at zero amplitude, there is an infinity of other, symmetric and asymmetric, branches that bifurcate at finite amplitude (see, for example, Champneys & Toland 1993). It is likely that analogous lump-solution branches could be found but, in order to capture these branches, the asymptotic approach taken here must be refined to account for exponentially small terms, as was done in Yang & Akylas (1997) for two-dimensional solitary waves.

From a physical standpoint, however, a more important question, that remains open, concerns the stability of the lumps found here. While, in analogy with the stability properties of two-dimensional solitary waves (Calvo & Akylas 2002), one might expect, close to the bifurcation point, the depression lump-solution branch to be stable and the elevation branch to be unstable, there is a property of three-dimensional wavepackets that has no counterpart in two dimensions: the elliptic-elliptic Davey–Stewartson equations, which govern the evolution of the wavepacket envelope and the induced mean flow in the small-amplitude limit, predict the formation of a focusing singularity in finite time, if the initial amplitude is above a certain threshold, the lump solution being at the borderline between stability and instability (Ablowitz & Segur 1979, Papanicolaou et al. 1994). The role that this type of nonlinear modulational instability may play in the propagation of lumps is not known. On the other hand, lumps resemble wavepackets in the small-amplitude limit only, so the focusing singularity of the envelope may not be relevant away from the bifurcation point.

Another related issue of physical interest is how lumps may arise from more general initial conditions. In shallow water, KP-I lumps are intimately connected with the instability of KdV solitary waves to transverse perturbations; whether a similar connection exists between gravity-capillary lumps and plane solitary waves in water of finite or infinite depth is not known, although two-dimensional gravity-capillary solitary wavepackets in fact are unstable to transverse modulations in the small-amplitude limit (Zakharov & Rubenchik 1974, Saffman & Yuen 1978).

To address the questions raised above, it would require solving the unsteady water-wave equations and, for this purpose, one would have to resort to fully numerical simulations.

On the other hand, we have been able to make some progress towards settling these issues in the context of a relatively simple model equation. Specifically, we studied a generalization, that allows for variations in two spatial dimensions, of the equation proposed by Benjamin (1992) for the propagation in one spatial dimension of long weakly nonlinear interfacial gravity-capillary waves in the strong-surface-tension regime, when both the KdV and the Benjamin–Davis–Ono (BDO) dispersive terms are equally important. In certain limits, this two-dimensional Benjamin equation admits lumps of the wavepacket type, similar to those found on water of finite
depth, as well as lumps of the KP-I type. Out of the two symmetric branches of wavepacket lumps that bifurcate at the extremum of the phase speed, the elevation branch can be continued towards the KP-I lumps. Moreover, elevation lumps appear to be stable and emerge from the instability of elevation solitary waves to transverse perturbations. Detailed results will be reported in Chapter 4 (Kim & Akylas 2006).
Chapter 3

Transverse instability of gravity–capillary solitary waves

3.1 Introduction

The stability of plane solitary waves to perturbations transverse to the direction of propagation was examined first by Kadomtsev & Petviashvili (1970) on the basis of a generalization of the classical Korteweg–de Vries (KdV) equation allowing for weak transverse variations. This model equation, now known as the Kadomtsev–Petviashvili (KP) equation, predicts that KdV gravity–capillary solitary waves on shallow water are unstable to transverse perturbations if surface tension is strong enough:

\[ T > \frac{1}{3} \rho gh^2, \quad (3.1) \]

where \( T \) denotes the coefficient of surface tension, \( h \) the water depth, \( \rho \) the fluid density and \( g \) the gravitational acceleration. When this condition is satisfied, moreover, the KP equation admits fully localized solitary-wave solutions, commonly referred to as ‘lumps’, which thus become the asymptotic states of the initial-value problem in two spatial dimensions (Ablowitz & Segur 1979).

In more recent work, Bridges (2001) obtained a condition for transverse instability to long-wave perturbations of solitary-wave solutions of Hamiltonian partial differential equations formulated as multi-symplectic systems. When applied to the water-wave problem with gravity and possibly surface tension present, this condition implies instability if

\[ \frac{\partial I}{\partial V} < 0, \quad (3.2) \]

where \( I \) is a certain quantity related to the total horizontal linear momentum, or impulse, of the solitary wave and \( V \) denotes the wave speed. (Bridges (2001) refers to \( I \) as the impulse but this ignores the circulation of the solitary wave which is non-zero in general; see §3.2 below.)
In the case of pure gravity solitary waves, it was shown by Longuet-Higgins (1974) that
\[ V \frac{\partial I}{\partial V} = \frac{\partial \mathcal{E}}{\partial V}, \] \( \mathcal{E} \) being the total energy of the solitary wave, so an alternative form of the instability condition (3.2) is
\[ \frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} < 0. \] (3.3)
Moreover, \( \mathcal{E} \) is known to be an increasing function of \( V \) in the weakly nonlinear limit, where the KdV equation applies, and also for finite-amplitude waves with steepness (amplitude-to-depth ratio) below \( \epsilon = 0.781 \). Hence, according to (3.3), transverse instability of gravity solitary waves first arises when the wave steepness exceeds this critical value which happens to also mark the onset of superharmonic instability to longitudinal perturbations (Tanaka 1986).

It is worth noting that (3.2) and (3.3) apply in the long-wave-disturbance limit and provide sufficient, but not necessary, conditions for instability. This is consistent with recent work by Kataoka & Tsutahara (2004), who re-visited, using perturbation expansions, the eigenvalue problem governing the transverse instability of gravity solitary waves of the KdV type to long-wave disturbances. While their leading-order solution recovers condition (3.3), at the next order they find that instability in fact sets in at a somewhat lower wave steepness, \( \epsilon = 0.713 \), than the critical value 0.781 furnished by (3.3); according to the refined stability criterion, therefore, transverse instability arises prior to longitudinal instability.

The present chapter is concerned with the transverse instability of gravity-capillary solitary waves. Rather than the classical KdV solitary waves on shallow water, attention is focused on solitary waves of the wavepacket type that are possible on water of finite or infinite depth in the presence of both gravity and surface tension, and have been studied extensively in recent years (see Dias & Kharif (1999) for a review). Solitary waves of this kind bifurcate from linear sinusoidal wavetrains at the minimum gravity-capillary phase speed and, in the small-amplitude limit, resemble wavepackets whose wave envelope and crests travel at the same speed (Akylas 1993, Longuet-Higgins 1993). Out of the two symmetric solitary-wave solution branches that bifurcate at the minimum phase speed, the depression branch is stable to longitudinal perturbations (Calvo & Akylas 2002), so it is natural to inquire into its stability with respect to transverse perturbations.

It is straightforward to show (see §3.2) that the form (3.3), in terms of the total energy \( \mathcal{E} \), of the transverse-instability condition (3.2) obtained by Bridges (2001) remains valid in the case of gravity-capillary solitary waves as well. This suggests that the solitary waves of interest here, which exist below the minimum gravity-capillary phase speed, are unstable to transverse perturbations, as \( \mathcal{E} \) is expected to increase when the wave speed \( V \) is decreased. In fact, for a portion of the depression solitary-wave solution branch in deep water for which computations of \( \mathcal{E} \) are available (Longuet-Higgins 1989), it is clear that \( \frac{\partial \mathcal{E}}{\partial V} < 0 \), implying instability according to (3.3).

Here we examine in a systematic way the stability of gravity-capillary solitary waves of
the wavepacket type to long-wave transverse perturbations and compute the instability growth rate, using the expansion procedure of Kataoka & Tsutahara (2004) with the added effect of surface tension. The entire branch of depression solitary waves on water of finite or infinite depth, while stable to longitudinal perturbations, turns out to be transversely unstable, and the instability growth rate increases with $\mathcal{E}$.

In analogy with transversely-unstable KdV solitary waves that give rise to KP lumps in the high-surface-tension regime (3.1), it is likely that the instability discussed here also results in the formation of gravity-capillary lumps, but of the type recently found (see Chapter 2; Parau, Vanden-Broeck & Cooker (2005)).

### 3.2 Preliminaries

It is customary in studies of gravity-capillary solitary waves of the wavepacket type (e.g., Akylas (1993); Calvo & Akylas (2002); Dias, Menasce & Vanden-Broeck (1996); Vanden-Broeck & Dias (1992)) to introduce non-dimensional variables such that the wave speed $c$ is normalized to 1 and trace solitary-wave solution branches in terms of

$$\alpha = \frac{g T}{\rho c^4}$$  \hspace{1cm} (3.4)

and an additional parameter, such as

$$H = \frac{h \rho c^2}{T}$$  \hspace{1cm} (3.5)

that involves the water depth. For the purpose of discussing the transverse instability of these solitary waves, however, it proves more convenient to use as characteristic length and time scale, respectively, $(T/\rho g)^{1/2}$ and $(T/\rho g^3)^{1/4}$, which do not depend on $c$; solitary-wave solution branches are thus traced via the dimensionless wave speed

$$V = c \left( \frac{\rho}{gT} \right)^{1/4} = \alpha^{-\frac{1}{4}},$$  \hspace{1cm} (3.6)

and the water-depth parameter

$$D = h \left( \frac{\rho g}{T} \right)^{1/2} = H \alpha^{1/2}.$$  \hspace{1cm} (3.7)

Gravity-capillary solitary waves of the wavepacket type arise when the minimum phase speed occurs at a finite wavenumber (Akylas (1993); Longuet-Higgins (1993)), and this is possible when condition (3.1) is not met, implying $D > \sqrt{3}$.

As remarked earlier, the total energy $\mathcal{E}(V, D)$ of a solitary wave plays an important part in the stability analysis. In terms of the velocity potential $\tilde{\phi}(\theta, z; V, D)$ and the free-surface elevation $z = \tilde{\eta}(\theta; V, D)$, where $\theta = x - Vt$, associated with a solitary wave travelling along $x$...
with speed $V$ on water of depth $D$ ($-D < z < \bar{z}$), $\mathcal{E}$ is given by

$$\mathcal{E} = T + \mathcal{V}_G + \mathcal{V}_T, \quad (3.8)$$

where

$$T = \frac{1}{2} \int_{-\infty}^{\infty} \text{d}\theta \int_{-D}^{\bar{z}} \left( \dot{\phi}_\theta^2 + \dot{\phi}_z^2 \right) \text{d}z \quad (3.9)$$

denotes the kinetic energy,

$$\mathcal{V}_G = \frac{1}{2} \int_{-\infty}^{\infty} \bar{\eta}^2 \text{d}\theta \quad (3.10)$$

the gravitational potential energy and

$$\mathcal{V}_T = \int_{-\infty}^{\infty} \left\{ (1 + \bar{\eta}_\theta^2)^{1/2} - 1 \right\} \text{d}\theta \quad (3.11)$$

the potential energy due to surface tension.

In addition, the total horizontal momentum, or impulse, $\mathcal{I}(V, D)$ of the solitary wave is given by

$$\mathcal{I} = \int_{-\infty}^{\infty} \text{d}\theta \int_{-D}^{\bar{z}} \ddot{\phi}_\theta \text{d}z. \quad (3.12)$$

From (3.12), upon integrating by parts in $\theta$, one then has

$$\mathcal{I} = I + D C, \quad (3.13)$$

where

$$C = \bar{\phi}|_{\theta=\infty} - \bar{\phi}|_{\theta=-\infty} \quad (3.14)$$

is the circulation of the solitary wave (Longuet-Higgins 1974) and

$$I = - \int_{-\infty}^{\infty} \text{d}\theta \bar{\eta}_\theta \ddot{\phi}|_{z=\bar{z}} \quad (3.15)$$

is the quantity that enters the instability condition (3.2) derived by Bridges (2001). Note that, in water of finite depth, $C \neq 0$ so $I$ is distinct from the true impulse in general.

Differentiating expression (3.15) for $I$ with respect to $V$ and combining the result with (3.13), it follows that

$$\frac{\partial I}{\partial V} = \frac{\partial \mathcal{I}}{\partial V} - D \frac{\partial C}{\partial V} = \int_{-\infty}^{\infty} \text{d}\theta \left( \ddot{\eta}_\nu \ddot{\phi}_\theta - \ddot{\eta}_\theta \ddot{\phi}_\nu \right) |_{z=\bar{z}}. \quad (3.16)$$

For pure gravity solitary waves, it is known from Longuet-Higgins (1974) that

$$\frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} = \frac{\partial \mathcal{I}}{\partial V} - D \frac{\partial C}{\partial V}, \quad (3.17)$$
so condition (3.2) can be replaced by (3.3), as indicated earlier. This remains true, however, when surface tension is also present, since it can be readily verified by differentiating (3.8) with respect to $V$ (see Appendix B) that

$$\frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} = \int_{-\infty}^{\infty} d\theta \left( \eta V \phi_{\theta} - \eta V \phi_{V} \right)_{z=\tilde{\eta}}. \quad (3.18)$$

Hence, according to (3.16),

$$\frac{\partial I}{\partial V} = \frac{1}{V} \frac{\partial \mathcal{E}}{\partial V} \quad (3.19)$$

so condition (3.2) is entirely equivalent to (3.3) in the case of gravity–capillary solitary waves as well.

### 3.3 Long-wave stability analysis

To diagnose the stability to long transverse perturbations of gravity–capillary solitary waves of the wavepacket type, one may appeal directly to condition (3.2), or equivalently condition (3.3) in view of (3.19). In addition, it is possible to deduce the associated instability growth rate in the long-wave limit from the theory of Bridges (2001). For the latter purpose, rather than specializing the general formalism to the problem at hand, we find it more instructive to directly tackle the stability eigenvalue problem via a long-wave expansion procedure analogous to the one followed by Kataoka & Tsutahara (2004) for pure gravity solitary waves. A similar approach will be taken in Chapter 4 for the transverse instability of solitary waves of the Benjamin equation.

Briefly, assuming that infinitesimal perturbations with wavenumber $\mu$ in the transverse ($y$-) direction and growth rate $\lambda$ are present in the free-surface elevation $\eta$ and the potential $\phi$,

$$\eta = \tilde{\eta} + \hat{\eta}(\theta) e^{i\mu y + \lambda t}, \quad \phi = \tilde{\phi} + \hat{\phi}(\theta, z) e^{i\mu y + \lambda t}, \quad (3.20)$$

we linearize the governing equations about the underlying solitary-wave state. The perturbation eigenfunctions $\hat{\phi}$ and $\hat{\eta}$ then satisfy the Helmholtz equation

$$\hat{\phi}_{\theta \theta} + \hat{\phi}_{zz} = \mu^2 \hat{\phi} \quad (-D < z < \tilde{\eta}), \quad (3.21)$$

subject to the bottom condition

$$\hat{\phi}_z = 0 \quad (z = -D) \quad (3.22)$$

and the following two conditions on the free surface of the solitary wave:

$$\mathcal{L}_1(\hat{\phi}, \hat{\eta}) = -\lambda \hat{\eta} \quad (z = \tilde{\eta}), \quad (3.23)$$
\[ \mathcal{L}_2(\dot{\phi}, \dot{\eta}) = -\lambda \dot{\phi} - \mu^2 \frac{\dot{\eta}}{(1 + \dot{\eta}^2 \theta)^{1/2}} \quad (z = \eta), \]  

(3.24)

where

\[ \mathcal{L}_1(\dot{\phi}, \dot{\eta}) \equiv \left( -\frac{\partial}{\partial z} + \dot{\eta} \frac{\partial}{\partial \theta} \right) \dot{\phi} + \left\{ \frac{d\phi}{d\theta} + \left( -V + \dot{\phi}_\theta \right) \dot{\eta} \right\}, \]  

(3.25)

\[ \mathcal{L}_2(\dot{\phi}, \dot{\eta}) \equiv (V + \ddot{\phi}_\theta) \frac{d\dot{\phi}}{d\theta} + \left\{ \left( -V + \ddot{\phi}_\theta \right) \ddot{\phi}_\theta + \ddot{\phi}_z \ddot{\phi}_{zz} + 1 \right\} \dot{\eta} - \left\{ \frac{\ddot{\eta}_\theta}{(1 + \dot{\eta}^2 \theta)^{3/2}} \right\}. \]  

(3.26)

The boundary conditions (3.23) and (3.24) result from linearizing the kinematic and dynamic free-surface conditions, respectively. Compared with Kataoka & Tsutahara (2004), the additional effects of surface tension in the dynamic condition are reflected in the last term in (3.26) and the \( O(\mu^2) \) term in (3.24).

In the long-wave limit \( (\mu \ll 1) \), the eigenvalue problem (3.21)-(3.24) is solved by expanding \( \dot{\phi}, \dot{\eta} \) and the growth rate \( \lambda \), which acts as the eigenvalue, in powers of \( \mu \):

\[ \dot{\phi} = \dot{\phi}^{(0)} + \mu \dot{\phi}^{(1)} + \mu^2 \dot{\phi}^{(2)} + \cdots, \quad \dot{\eta} = \dot{\eta}^{(0)} + \mu \dot{\eta}^{(1)} + \mu^2 \dot{\eta}^{(2)} + \cdots, \]  

(3.27)

\[ \lambda = \mu \lambda_1 + \mu^2 \lambda_2 + \cdots. \]  

(3.28)

As in Kataoka & Tsutahara (2004), the leading-order solution is readily shown to be

\[ \dot{\phi}^{(0)} = \ddot{\phi}_\theta, \quad \dot{\eta}^{(0)} = \ddot{\eta}_\theta. \]  

(3.29)

Proceeding to \( O(\mu) \), \( \dot{\phi}^{(1)} \) and \( \dot{\eta}^{(1)} \) then satisfy the forced problem

\[ \ddot{\phi}^{(1)} + \ddot{\phi}_z^{(1)} = R \quad (-D < z < \eta), \]  

(3.30)

\[ \ddot{\phi}^{(1)} = 0 \quad (z = -D), \]  

(3.31)

\[ \mathcal{L}_1(\dot{\phi}^{(1)}, \dot{\eta}^{(1)}) = r_1 \quad (z = \eta), \]  

(3.32)

\[ \mathcal{L}_2(\dot{\phi}^{(1)}, \dot{\eta}^{(1)}) = r_2 \quad (z = \eta), \]  

(3.33)

where

\[ R = 0, \quad r_1 = -\lambda_1 \ddot{\eta}_\theta, \quad r_2 = -\lambda_1 \ddot{\phi}_\theta. \]  

(3.34)

Since (3.29) is a well-behaved solution of the corresponding homogeneous problem, the forcing terms (3.34) must satisfy a certain condition for the inhomogeneous boundary-value problem (3.30)-(3.33) to have a solution that behaves acceptably as \( \theta \to \pm\infty \). This solvability condition can be derived by forming the inner product of the forced equation (3.30) with the well-behaved solution of the adjoint boundary-value problem, and then making use of Green’s identity along with the boundary conditions (3.31)-(3.33) and the properties of the adjoint
solution. As it parallels closely an analogous derivation in Kataoka & Tsutahara (2004), here we shall provide the details in Appendix B and simply state the solvability condition:

$$\int_{-\infty}^{\infty} d\theta \int_{-D}^{\eta} \tilde{\phi}_\theta R \, dz + \int_{-\infty}^{\infty} d\theta \left( r_1 \tilde{\phi}_\theta - r_2 \tilde{\eta}_\theta \right) \big|_{z=\tilde{\eta}} = 0. \quad (3.35)$$

For the forcing terms (3.34), this condition is trivially met and the well-behaved solution of the problem (3.30)--(3.33) turns out to be

$$\hat{\phi}^{(1)} = -\lambda_1 \tilde{\phi}_\nu, \quad \hat{\eta}^{(1)} = -\lambda_1 \tilde{\eta}_\nu. \quad (3.36)$$

At the next order, $\hat{\phi}^{(2)}$ and $\hat{\eta}^{(2)}$ are governed again by a forced problem of the form (3.30)--(3.33) but with forcing terms

$$R = \tilde{\phi}_\theta, \quad r_1 = -\lambda_2 \tilde{\eta}_\theta + \lambda_1^2 \tilde{\eta}_\nu, \quad r_2 = -\lambda_2 \tilde{\phi}_\theta + \lambda_1^2 \tilde{\phi}_\nu - \frac{\tilde{\eta}_\theta}{(1 + \tilde{\eta}_\theta^2)^{1/2}}. \quad (3.37)$$

Inserting the forcing terms (3.37) in the solvability condition (3.35) now yields a non-trivial result,

$$\lambda_1^2 \int_{-\infty}^{\infty} d\theta \left( \tilde{\eta}_\nu \tilde{\phi}_\theta - \tilde{\eta}_\theta \tilde{\phi}_\nu \right) \big|_{z=\tilde{\eta}} + \int_{-\infty}^{\infty} d\theta \int_{-D}^{\eta} \tilde{\phi}^2_\theta \, dz + \int_{-\infty}^{\infty} \frac{\tilde{\eta}_\theta^2}{(1 + \tilde{\eta}_\theta^2)^{1/2}} \, d\theta = 0, \quad (3.38)$$

which, taking into account (3.18), furnishes the following expression for the growth rate of long transverse perturbations to leading order in $\mu$:

$$\lambda_1^2 = -\left( \int_{-\infty}^{\infty} d\theta \int_{-D}^{\eta} \tilde{\phi}_\theta^2 \, dz + \int_{-\infty}^{\infty} \frac{\tilde{\eta}_\theta^2}{(1 + \tilde{\eta}_\theta^2)^{1/2}} \, d\theta \right) / \frac{\partial \mathcal{E}}{\partial V}. \quad (3.39)$$

This confirms that (3.3) is a sufficient condition for transverse instability of solitary waves in the presence of gravity and surface tension.

As remarked earlier, in the case of pure gravity solitary waves, the instability condition (3.3) is first met when the wave steepness exceeds a certain critical value (Tanaka 1986); in order to gain information regarding the transverse instability of less steep solitary waves, it is necessary to carry the long-wave expansions (3.27) and (3.28) to higher order, as was done in Kataoka & Tsutahara (2004). Fortunately, in the problem of interest here, as verified below, the leading-order result (3.39) predicts instability ($\lambda_1^2 > 0$) for the entire solution branch of depression gravity–capillary solitary waves, so no higher-order analysis is needed.
3.4 Results

We now proceed to verify that $\partial \mathcal{E} / \partial V < 0$ for gravity–capillary solitary waves of depression and compute the associated instability growth rate. To this end, we shall make use of small-amplitude expansions in the vicinity of the bifurcation point, slightly below the minimum phase speed, and fully numerical computations in the finite-amplitude regime.

In terms of the non-dimensional variables used here, for a given value of the water-depth parameter $D > \sqrt{3}$, the minimum gravity–capillary phase speed $V = V_0$ and the corresponding wavenumber $k_0$ are determined from the equation system

$$V_0^2 = \frac{1}{k_0} \tanh k_0 D,$$

$$\frac{1 - k_0^2}{k_0} \tanh k_0 D - \frac{D(1 + k_0^2)}{\cosh^2 k_0 D} = 0.$$  

In the deep-water limit ($D \to \infty$), in particular,

$$V_0 \to \sqrt{2}, \quad k_0 \to 1.$$  

For $V$ slightly less than $V_0$, the bifurcating solitary-wave solution branches can be computed using perturbation expansions in terms of $\delta = V_0 - V$ ($0 < \delta \ll 1$), and details are given by Akylas (1993) and Longuet-Higgins (1989). Specifically, in the case of deep water, the leading-order results for the depression solitary-wave branch are

$$\bar{\eta} = -\frac{8}{2^{1/4} \sqrt{11}} \delta^{1/2} \text{sech}(2^{3/4} \delta^{1/2} \theta) \cos \theta + \cdots,$$  

$$\bar{\phi} = -\frac{2^{1/4} \delta^{1/2} e^{\pi} \text{sech}(2^{3/4} \delta^{1/2} \theta)}{\sqrt{11}} \cos \theta + \cdots.$$  

Inserting (3.43) and (3.44) in (3.9)–(3.11), we find that

$$T \sim \frac{32}{2^{1/4} 11} \delta^{1/2},$$  

$$\mathcal{V}_G \sim \frac{16}{2^{1/4} 11} \delta^{1/2},$$  

$$\mathcal{V}_T \sim \frac{16}{2^{1/4} 11} \delta^{1/2};$$  

hence,

$$\mathcal{E} \sim \frac{64}{2^{1/4} 11} \delta^{1/2}.$$  

As expected, $\mathcal{E}$ increases as $V$ is decreased below $V_0$ ($\delta > 0$), so $\partial \mathcal{E} / \partial V < 0$, and small-amplitude solitary waves on deep water are unstable to transverse perturbations. Similarly,
from (3.39), making use of (3.43) and (3.44), it follows that

$$\lambda_1 \sim 2^{3/4} \delta^{1/2}. \tag{3.49}$$

Therefore, combining (3.49) and (3.28), the instability growth rate of long-wave transverse perturbations to small-amplitude depression solitary waves, travelling with speed $V$ slightly below the minimum phase speed $V_0 = \sqrt{2}$ in deep water, is given by

$$\lambda \sim 2^{3/4}(V - V_0)^{1/2} \mu, \tag{3.50}$$

$\mu \ll 1$ being the perturbation wavenumber.

Figure 3-1 compares the asymptotic estimates (3.38) and (3.39) against values of $E$ and $A_1$, obtained from (3.8)-(3.11) and (3.39) using numerically-computed solitary-wave profiles close to the bifurcation point $V_0 = \sqrt{2}$ in deep water. Not unexpectedly, the agreement is good only in the immediate vicinity of the bifurcation point. Numerical computations, nevertheless, indicate that $\partial E/\partial V$ continues to be negative in the finite-amplitude regime, confirming the presence of instability, as illustrated in Figure 3-2 for $D = 2.5$ and $D \to \infty$. Moreover, the instability growth rate increases rapidly with the solitary-wave steepness, and the instability is stronger ($\lambda_1$ is larger) in the deep-water limit ($D \to \infty$), at least for long-wave perturbations ($\mu \ll 1$).

Solitary-wave profiles and the corresponding values of $E$ and $\lambda_1$ were computed via a boundary-integral method, similar to the one described by Dias & Kharif (1999), combined with Chebyshev spectral collocation (see §3.5 and Appendix C for details). Table 3-1 illustrates the convergence of $E$ as the number of grid points $N$ is varied, for different values of the wave speed $V$ in finite depth ($D = 2.5$) and in deep water ($D \to \infty$). Our computations of $E$ also are consistent with those presented by Longuet-Higgins (1989) for a portion of the depression solitary-wave branch in deep water. The results shown in Figure 3-1 were computed using $N = 2048$, while those shown in Figure 3-2 using $N = 512$.

### 3.5 Numerical method

Since the energy $E$ and the instability growth rate $\lambda_1$ are computed by using the plane solitary waves solutions, $\tilde{\eta}$ and $\tilde{\phi}$, we need to reproduce the numerical results only for the depression branch by Dias, Menasce and Vanden-Broeck (1996). We use the Cauchy integral formula for the two-dimensional Laplace's equation based on the nondimensionalization employed by them, regarding $x - \tilde{\phi} + iy$ as an analytic function of $\Phi + i\Psi$, where $\Phi$ is the potential function and $\Psi$ is the stream function. By taking $X(\Phi, \Psi = 0) = x_\phi(\Phi, \Psi = 0) - 1 = \tilde{\phi}_\phi(\Phi, \Psi = 0)$ and $Y(\Phi, \Psi = 0) = \tilde{\eta}(\Phi) = y(\Phi, \Psi = 0)$, we have a boundary integral formulation for the
two-dimensional water-wave equations as follows:

\[
X(\phi) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{Y_{\phi'}}{\Phi' - \Phi} \, d\Phi' - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2DX + (\Phi' - \Phi)Y_{\phi'}}{(\Phi' - \Phi)^2 + 4D^2} \, d\Phi' \quad (\psi = 0), \tag{3.51a}
\]

\[
-\frac{1}{2} \left( \frac{(X + 1)^2 + Y_\phi^2 - 1}{(X + 1)^2 + Y_\phi^2} \right) + \alpha Y + \left( \frac{Y_\phi X_{\phi} - Y_{\phi\phi}(X + 1)}{((X + 1)^2 + Y_\phi^2)^3/2} \right) = 0 \quad (\psi = 0). \tag{3.51b}
\]

The solitary wave solutions, \( \eta \) and \( \phi \), are obtained via the following scaling from \( \tilde{\eta} \) and \( \tilde{\phi} \):

\[
\tilde{\eta} = \alpha^{1/2} \phi, \quad \tilde{\phi} = \alpha^{1/4} \phi. \tag{3.52}
\]

Via the Cauchy integral formula, the computational domain becomes a infinite strip defined by \( (\Phi, \Psi) \in [-\infty, \infty] \times [-2D, 0] \). This extended domain is obtained by reflecting the original domain in \( (x, y) \) with respect to the flat bottom because the solution is symmetric with respect to \( y = -D \) due to the slip boundary condition, \( \partial \Psi / \partial y = 0 \) and \( \Psi = -D \) on \( y = -D \). Since \( X(\Phi, \Psi = 0) \) and \( Y(\Phi, \Psi = 0) \) are defined only on the boundary, we need to solve the Laplace’s equation for \( X(\Phi, \Psi) \) and \( Y(\Phi, \Psi) \) over the whole computational domain in order to calculate the energy \( E \) and the instability growth rate \( \lambda_1 \). Then, \( E \) and \( \lambda_1 \) are given in terms of \( X \) and \( Y \) as follows:

\[
T = \frac{\alpha^{1/2}}{2} \int_{-\infty}^{\infty} \int_{-D}^{0} \frac{(X^2 + X + Y_\phi^2)^2 + Y_\phi^2}{(X + 1)^2 + Y_\phi^2} \, d\Phi \, d\Psi, \tag{3.53a}
\]

\[
\nu_T = \frac{\alpha^{2/2}}{2} \int_{-\infty}^{\infty} (X + 1)Y_\phi^2 \bigg|_{\Psi = 0} \, d\Phi, \tag{3.53b}
\]

\[
\nu_C = \alpha^{1/2} \int_{-\infty}^{\infty} \frac{Y_\phi^2}{\sqrt{(X + 1)^2 + Y_\phi^2 + X + 1}} \bigg|_{\Psi = 0} \, d\Phi, \tag{3.53c}
\]

\[
\int_{-\infty}^{\infty} \int_{-D}^{0} \frac{\hat{\eta}_\phi^2}{(1 + \hat{\eta}_\phi^2)^{1/2}} \, d\theta \, dz = \alpha^{1/2} \int_{-\infty}^{\infty} \int_{-D}^{0} \frac{(X^2 + X + Y_\phi^2)^2}{(X + 1)^2 + Y_\phi^2} \, d\Phi \, d\Psi, \tag{3.53d}
\]

\[
\int_{-\infty}^{\infty} \left( \frac{\hat{\eta}_\phi^2}{(1 + \hat{\eta}_\phi^2)^{1/2}} \right)^{1/2} \, d\theta = \int_{-\infty}^{\infty} \frac{Y_\phi^2}{\sqrt{(X + 1)^2 + Y_\phi^2}} \bigg|_{\Psi = 0} \, d\Phi. \tag{3.53e}
\]

A Chebyshev spectral collocation method is used for the numerical computation. \( \Phi \) is discretized as \( \Phi_i = L \cot (\pi i / N) \) \( (1 \leq i \leq N - 1) \) and \( \Psi \) is discretized as \( \Psi_j = L \cos (\pi j / M) \) \( (1 \leq j \leq M - 1) \) for finite depth and \( \Psi_j = L \tan^2 (\pi j / 2M) \) \( 1 \leq j \leq M - 1 \) for infinite depth.
Figure 3-1: Comparison between computed results (+) and the asymptotic estimates (3.48)–(3.49) (· · ·) for gravity-capillary solitary waves of speed $V$ near the bifurcation point $V_0 = \sqrt{2}$ in deep water. (a) energy; (b) instability growth rate $\lambda_1$.

Figure 3-2: Energy $\mathcal{E}$ and instability growth rate $\lambda_1$ of depression gravity-capillary solitary waves as functions of wave speed $V$. Left column: finite depth ($D = 2.5$, $V_0 = 1.402$). Right column: infinite depth ($D \to \infty$, $V_0 = \sqrt{2}$). (a) energy; (b) instability growth rate.
Table 3.1: Convergence of energy $\mathcal{E}$ as the number of grid points $N$ is varied for different values of the wave speed $V$ in finite depth ($D = 2.5$) and in deep water ($D \to \infty$).

<table>
<thead>
<tr>
<th>$D = 2.5$</th>
<th>$V = 1.0$</th>
<th>1.2</th>
<th>1.4</th>
<th>$D \to \infty$</th>
<th>$V = 1.0$</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td></td>
<td></td>
<td></td>
<td>$N$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>512</td>
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<td>0.0230997</td>
<td>3.12468</td>
<td>1.93688</td>
<td>0.284594</td>
<td></td>
</tr>
<tr>
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<td>1.46579</td>
<td>0.0232001</td>
<td>3.12650</td>
<td>1.94051</td>
<td>0.285844</td>
<td></td>
</tr>
<tr>
<td>1536</td>
<td>2.95269</td>
<td>1.46615</td>
<td>0.0232135</td>
<td>3.12711</td>
<td>1.94171</td>
<td>0.286144</td>
<td></td>
</tr>
<tr>
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<td>2.95288</td>
<td>1.46642</td>
<td>0.0232221</td>
<td>3.12741</td>
<td>1.94230</td>
<td>0.286265</td>
<td></td>
</tr>
<tr>
<td>$\infty$</td>
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<td>0.0232252</td>
<td>3.12757</td>
<td>1.94268</td>
<td>0.286317</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3-3: Semilog plots for the residual (res = $\mathcal{E}_\infty - \mathcal{E}_N$) as $N$ increases, which show exponential convergence of the spectral collocation method, for different values of wave speed $V = 1.0$ (—), $V = 1.2$ (--) and $V = 1.4$ (---). (a) Finite depth ($D = 2.5$), (b) Infinite depth ($D = \infty$).
Chapter 4

Interfacial gravity–capillary lumps of the two-dimensional Benjamin model equation

4.1 Introduction

In contrast to plane solitary waves, which are ubiquitous, fully localized solitary waves arise under rather special flow conditions. This apparently accounts for the fact that such so-called lumps have received far less attention than plane solitary waves.

The majority of prior work deals with lumps in the weakly nonlinear long-wave limit and is based on the Kadomtsev–Petviashvili (KP) equation, an extension of the Korteweg–de Vries (KdV) equation that allows for weak spatial variations transverse to the propagation direction (see, for example, Akylas 1994). While the KdV equation predicts that plane solitary waves are always possible, the KP equation admits lump solutions only if the linear-long-wave speed happens to be a minimum of the phase speed, and this property is met under very restricted flow conditions. In the classical water-wave problem, in particular, lumps of the KP type are possible only in the high-surface-tension regime (Bond number greater than 1/3), which requires the fluid depth to be less than a few mm (see, for example, Ablowitz & Segur 1979).

In Chapter 2 (Kim & Akylas 2005), however, it was pointed out that gravity–capillary lumps of a different kind in fact can be found on water of finite or infinite depth for Bond number less than 1/3. Rather than the long-wave speed, these lumps bifurcate from a linear sinusoidal wavetrain with finite wavenumber at the minimum of the gravity–capillary phase speed, and are fully localized counterparts of the plane solitary waves that are known to bifurcate there (Dias & Iooss 1993, Akylas 1993, Longuet-Higgins 1993). Based on weakly nonlinear expansions close to the bifurcation point, it was shown in Chapter 2 that small-amplitude lumps take the form of locally confined modulated wavepackets with carrier oscillations that are stationary relative
to the envelope. In water of finite depth, in particular, the wave envelope and the induced mean flow are governed by an elliptic-elliptic Davey–Stewartson equation system, and lumps always feature algebraically decaying tails owing to this mean flow. Although in Chapter 1 the discussion focused on gravity–capillary water waves, a similar local analysis would apply in general close to an extremum of the phase speed at a finite wavenumber, suggesting that lumps of the same type could be found in other instances as well.

In closely related numerical work, Parau, Vanden-Broeck & Cooker (2005) recently computed gravity–capillary lumps in deep water that, indeed, resemble fully localised wavepackets in the small-amplitude limit close to the minimum phase speed. Also, based on a weakly non-linear model, Milewski (2005) was able to connect, via numerical continuation, lumps of the KP equation in shallow water to lumps of the wavepacket type near the minimum gravity–capillary phase speed in water of finite depth. The asymptotic and numerical results, moreover, are supported by a rigorous existence proof of fully localized gravity–capillary solitary waves, recently devised by Groves & Sun (2005).

These findings have firmly established that free-surface gravity–capillary lumps are possible under quite general flow conditions. On the other hand, the stability of the computed steady-solution branches, as well as the feasibility of obtaining lumps from general locally confined initial conditions, remain unexplored. These issues are of paramount importance in assessing the relevance of the new class of lumps from a physical viewpoint.

In regard to the question of stability, it is worth noting that, in the small-amplitude limit where lumps behave like wavepackets, the elliptic–elliptic Davey–Stewartson equations predict focusing of the wavepacket envelope at a finite time, for initial conditions above a certain threshold (Ablowitz & Segur 1979; Papanicolaou et al. 1994). Given that lumps may be viewed as modulated wavepackets only in the vicinity of the bifurcation point, however, it is not clear to what extent they are affected by this nonlinear-focusing instability.

In the present paper, we make a first step towards settling some of these open questions. For simplicity, rather than the full water-wave equations, we shall work with a model equation for interfacial gravity–capillary waves; namely, we consider an extension to two spatial dimensions of the evolution equation derived by Benjamin (1992) for weakly nonlinear long waves on the interface of a two-fluid system, in the case that the upper layer is bounded by a rigid lid and lies on top of an infinitely deep fluid.

In this flow configuration, when interfacial tension is assumed to be large and the two fluid densities are nearly equal, the KdV and the Benjamin–Davis–Ono (BDO) dispersive terms for long interfacial waves, become equally important. As a result, the phase speed of the Benjamin equation features a minimum at a finite wavenumber, which is the bifurcation point of plane solitary waves akin to the free-surface solitary waves bifurcating at the minimum gravity–capillary phase speed (Akylas, Dias & Grimshaw 1998; Albert, Bona & Restrepo 1999; Calvo
Accounting for transverse spatial variations, the resulting long-wave equation combines the KdV and BDO dispersive terms with the transverse spatial term of the KP equation. This two-dimensional Benjamin (2-DB) equation admits lumps, which again bifurcate at the minimum phase speed and in the small-amplitude limit are analogous to the free-surface lumps on water of finite depth found in Chapter 2. Numerical continuation in the finite-amplitude regime reveals that lumps of elevation are directly connected to lumps of the KP type in the KdV limit, while lumps of depression apparently undergo successive limit-point bifurcations associated with the appearance of multi-modal lumps.

According to the 2-DB equation, plane solitary waves are unstable to transverse perturbations. Numerical simulations indicate that this instability results in the formation of elevation lumps, which appear to propagate stably, thus assuming the role of asymptotic states of the initial-value problem in two spatial dimensions.

Unfortunately, it becomes prohibitively expensive to extend our simulations very close to the bifurcation point, as lumps decay slowly at infinity in this limit, and we have not been able to study in any systematic way the effect of the nonlinear focusing predicted by the Davey–Stewartson equations on the propagation of lumps. Nevertheless, we have seen no evidence of this type of instability in the computations discussed here.

While the 2-DB equation is formally valid in a very specific flow regime, it is likely to remain reliable qualitatively for a wide range of flow conditions, as is the case for the Benjamin equation (Calvo & Akylas 2003).

### 4.2 Preliminaries

Consider a fluid layer of depth \( h \) and density \( \rho_2 \) that is bounded above by a rigid lid and lies on top of an infinitely deep fluid of density \( \rho_1 > \rho_2 \).

The Benjamin equation governs the propagation of straight-crested, uni-directional, weakly nonlinear, long waves on the interface of this two-fluid system, ignoring the effects of viscosity and assuming that interfacial tension is large and the fluid densities are nearly equal (Benjamin 1992). Under these flow conditions, the 2-DB equation is an extension of the Benjamin equation that allows for weak spatial variations transverse to the propagation direction, and can be derived by a standard weakly nonlinear long-wave expansion (see §4.6). Here, however, in the interest of brevity, we sketch a heuristic derivation based on the linear dispersion relation.

In dimensionless variables, using \( h \) as lengthscale and \( (gh)^{1/2} \) as velocity scale, the linear dispersion relation of interfacial waves with wavenumber \( \kappa \) and frequency \( \omega \) has the following
expansion in the long-wave limit ($\kappa \to 0$):

$$\omega = \pm c_0 \kappa \left( 1 - \frac{\kappa^2}{R} + \frac{W^2}{2} \kappa^2 + \cdots \right),$$

where

$$R = \frac{\rho_1}{\rho_2}, \quad W = \frac{T}{\delta \rho g h^2}, \quad c_0 = \frac{1 - R}{R},$$

(4.2)

$T$ being the interfacial tension, $g$ the gravitational acceleration and $\delta \rho = \rho_1 - \rho_2$ the density difference of the two fluids.

According to (4.1), to leading order, long waves are non-dispersive and propagate with speed $\pm c_0$; the first effects of dispersion normally derive from the term proportional to $\kappa|\kappa|$ in (4.1) which translates into a dispersive term of the BDO type in the corresponding long-wave evolution equation. In the flow regime considered by Benjamin (1992), $W \gg 1$, however, the next-order correction, proportional to $\kappa^3$, becomes equally important, in which case a KdV dispersive term comes into play as well; in this instance, the propagation of weakly nonlinear long interfacial waves in one spatial dimension is then governed by the Benjamin equation.

To allow for weak transverse variations, returning to (4.1), we use the fact that uni-directional nearly plane waves with wavenumber

$$\kappa = \left( k^2 + m^2 \right)^{1/2} = k + \frac{m^2}{2k} + \cdots,$$

(4.3)

$k$ being the wavenumber component along the propagation ($x$-, say) direction and $m \ll k$ the wavenumber component in the transverse ($z$-, say) direction, obey the dispersion relation

$$\omega = c_0 k - \frac{c_0}{2R} k|k| + \frac{c_0}{2} W k^3 + \frac{c_0}{2} \frac{m^2}{k},$$

(4.4)

including the leading-order dispersive and transverse-variation effects. Based on (4.4), as each term corresponds to a specific linear operator in the real domain,

$$\omega \leftrightarrow \frac{i}{\partial t}, \quad k \leftrightarrow -\frac{i}{\partial x}, \quad m \leftrightarrow -\frac{i}{\partial z}, \quad |k| \leftrightarrow -\frac{\partial}{\partial x} \mathcal{H},$$

(4.5)

one can read off the linear terms of the 2-DB equation; upon adding the familiar KdV-type quadratic nonlinear term, the full evolution equation for the interfacial elevation $y = \eta(x, z, t)$ in a reference frame moving with the long-wave speed $c_0$,

$$x \to x - c_0 t,$$

(4.6)
then turns out to be

\[
\left( \eta_t + \frac{3}{2} c_0 \eta_{xx} + \frac{c_0}{2 R} \mathcal{H}(\eta_{xx}) - \frac{c_0}{2} W \eta_{xxx} \right)_x + \frac{c_0}{2} \eta_{xx} = 0,
\]

(4.7)

where \( \mathcal{H} \) stands for the Hilbert transform with respect to \( x \):

\[
\mathcal{H}(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi.
\]

As expected, the 2-DB equation (4.7) combines the KdV and BDO dispersive terms with the transverse-variation term of the KP equation. In the absence of the BDO term, since the KdV dispersive term and the transverse-variation term have opposite signs, (4.7) reduces to the so-called KP-I equation that governs shallow-water waves in the high-surface-tension regime (Bond number greater than 1/3). In the other extreme, when the KdV dispersive term is ignored, (4.7) reduces to the two-dimensional BDO equation derived by Ablowitz & Segur (1980) for internal waves in stratified fluids of large depth.

Locally confined solutions of the 2-DB equation (4.7) are subject to the constraint

\[
\int_{-\infty}^{\infty} \eta(x, z, t) dx = 0,
\]

(4.9)

that also applies to the KP and the two-dimensional BDO equations. As explained in Katsis & Akylas (1987), this constraint derives from considering nearly plane wave disturbances with weak transverse variations: it is clear from (4.3) that Fourier components with \( k = 0 \) and \( m \neq 0 \) violate this assumption, and the constraint (4.9) simply ensures that no such wave components are present in the disturbance. For detailed discussions of this constraint in connection with the initial-value problem of the KP equation, see Grimshaw & Melville (1989) and Ablowitz & Wang (1997).

It is also worth noting that, in contrast to the full interfacial-wave equations, the 2-DB equation is not isotropic, as it describes nearly uni-directional waves, \( x \) thus being a preferred direction. This might seem to suggest that possible lump solutions of this model equation depend on the direction of propagation, casting doubt on their physical relevance. As discussed below, however, oblique lump solutions can in fact be mapped to lumps propagating along the \( x \)-direction via a coordinate and speed transformation. Moreover, within the approximations inherent in the derivation of the 2-DB equation, this transformation can be interpreted as merely a rotation of axes.

Specifically, according to (4.7), the profile of an oblique lump \( \eta(\xi, \zeta) \) propagating with speed \( V \), say, at an angle \( \phi \) to the \( x \)-direction,

\[
\xi = x - V t \cos \phi, \quad \zeta = z - V t \sin \phi,
\]

(4.10)
must satisfy
\begin{equation}
( - V \cos \phi \eta_x - V \sin \phi \eta_z + \frac{3}{2} c_0 \eta \eta_x + \frac{c_0}{2R} \mathcal{H}(\eta \xi) - \frac{c_0}{2} W \eta \xi \xi \xi \xi \xi + \frac{c_0}{2} \eta \xi \xi = 0. \end{equation}

(4.11)

By introducing the coordinate transformation
\begin{equation}
\bar{\xi} = \xi + \frac{V}{c_0} \sin \phi \zeta, \quad \bar{\zeta} = \zeta, \end{equation}

(4.12)

however, (4.11) can be reduced to the equation satisfied by a non-oblique lump:
\begin{equation}
( - \bar{V} \eta + \frac{3}{4} c_0 \eta^2 + \frac{c_0}{2R} \mathcal{H}(\eta \xi) - \frac{c_0}{2} W \eta \xi \xi \xi \xi \xi + \frac{c_0}{2} \eta \xi \xi \xi \xi \xi = 0, \end{equation}

(4.13)

with speed
\begin{equation}
\bar{V} = V \cos \phi + \frac{V^2}{2c_0} \sin^2 \phi. \end{equation}

(4.14)

Hence, any oblique lump solution can be related to a non-oblique lump solution having a different speed given by (4.14).

To interpret the speed transformation (4.14), note that a lump solution of the 2-DB equation (4.7) propagating along the x-axis with speed $\bar{V}$, say, in fact would move with speed $c_0 + \bar{V}$ in still fluids, in view of the change of reference frame (4.6) which is equivalent to superposing a steady stream $-c_0$ in the x-direction; furthermore, $\bar{V}/c_0 \ll 1$, since dispersive and nonlinear effects are taken to be weak in deriving the 2-DB equation. Suppose now that the same lump is slightly rotated so that, in still fluids, it propagates at an angle $\alpha \ll 1$ to the x-axis again with speed $c_0 + \bar{V}$. When viewed from the moving reference frame (4.6), however, the x- and z- velocity components of this lump would be
\begin{equation}
V_x \approx \bar{V} - \frac{c_0}{2} \alpha^2, \quad V_z \approx \alpha c_0. \end{equation}

(4.15)

Hence, relative to the moving reference frame, the lump appears to propagate at an angle $\phi$ to the x-axis with a speed $V$, where
\begin{equation}
V \cos \phi = \bar{V} - \frac{c_0}{2} \alpha^2, \quad V \sin \phi = \alpha c_0. \end{equation}

(4.16)

Note that, according to (4.16), $\bar{V}$ is related to $V$ precisely as in (4.14) and, moreover, in the coordinate transformation (4.12), $V \sin \phi/c_0$ is equal to the rotation angle $\alpha$. This confirms that oblique-lump solutions of the 2-DB equation correspond to slightly rotated non-oblique lumps, and in the following attention is focussed solely on lumps propagating along x.

In preparation for the ensuing analysis, we re-scale variables according to
\begin{equation}
(x', z') = \frac{1}{W^{1/2}}(x, z), \quad t' = -\frac{c_0}{2W^{1/2}t}, \quad \eta' = \frac{3}{2} \eta. \end{equation}

(4.17)
so, after dropping the primes, the 2-DB equation (4.7) takes the normalized form

\[(\eta_t + (\eta^2)_{xx} - 2\gamma \mathcal{H} \{\eta_{xxz} \} x - \eta_{zzz})_z - \eta_{xxx} = 0, \quad (4.18)\]

where

\[\gamma = \frac{1}{2RW^{1/2}}. \quad (4.19)\]

Ignoring transverse variations in (4.18) recovers the Benjamin equation in the form considered in Akylas et al. (1998).

### 4.3 Steady lumps

For the purpose of computing steady lumps, we shall adhere to the convention adopted in I and normalize the wave speed to 1. In the reference frame moving with a lump, \(\xi = x - t\), the profile \(\eta(\xi, z)\) then satisfies a steady version of the 2-DB equation (4.18):

\[(\eta_t + \eta^2 - 2\gamma \mathcal{H} \{\eta_{\xi} \} + \eta_{\xi \xi})_{\xi \xi} - \eta_{zz} = 0, \quad (4.20)\]

and lump-solution branches are traced by varying the parameter \(\gamma\).

#### 4.3.1 Bifurcation of lumps

As already remarked, in general, lumps bifurcate from infinitesimal-amplitude sinusoidal wave-trains at specific wavenumber \(k_0\) where the phase speed attains an extremum and is thus equal to the group speed. In the normalization used here, it is easy to check that the phase speed of the 2-DB equation (4.18) has a maximum equal to 1 at \(k_0 = 1\), for \(\gamma = \gamma_0 = 1\). (In view of the time reversal in (4.17), this maximum corresponds to a minimum of the phase speed of the unscaled equation (4.7).)

Close to this bifurcation point, small-amplitude lumps may be viewed as locally confined wavepackets with carrier oscillations that are stationary relative to the wavepacket envelope, and can be described by a multiple-scale expansion. Here we shall only outline the salient features of this weakly nonlinear analysis, as it closely parallels the one presented in I for free-surface lumps.

In the vicinity of \(\gamma_0 = 1\), we write

\[\gamma = 1 - \frac{1}{2} \epsilon^2 \quad (\epsilon \ll 1), \quad (4.21)\]

and expand \(\eta(\xi, z)\) as follows

\[\eta = \frac{1}{2} \epsilon \{A(X, Z)e^{i\xi} + \text{c.c.}\} + \epsilon^2 \{A_2(X, Z)e^{2i\xi} + \text{c.c.}\} + \epsilon^2 A_0(X, Z) + \cdots, \quad (4.22)\]
where \((X, Z) = \epsilon(\xi, z)\) are the ‘stretched’ envelope variables and \(\text{c.c.}\) denotes the complex conjugate.

Substituting expansion (4.22) in equation (4.20) and collecting second-harmonic, mean and primary-harmonic terms up to \(O(\epsilon^3)\), it is found that the envelope \(A\) and the mean term \(A_0\) satisfy the following coupled system:

\[
A_{0,XX} + A_{0,ZZ} = \frac{1}{2}(|A|^2)_{XX},
\]

(4.23)

\[
A_{XX} + A_{ZZ} - A + \frac{1}{2}A^2A^* + 2A_0A = 0.
\]

(4.24)

This system is entirely analogous to the steady elliptic–elliptic Davey–Stewartson equations derived in Chapter 2 for the primary-harmonic envelope and the induced mean flow of small-amplitude gravity–capillary lumps on water of finite depth.

To ensure that lump solutions of the 2-DB equation (4.20) are possible, it is necessary to find locally confined solutions of the equation system (4.23) and (4.24). As no such solutions are known in closed form, this system is solved numerically by a continuation procedure suggested in Papanicolaou et al. (1994). For this purpose, the factor \(\frac{1}{2}\) on the right-hand side of (4.23) is temporarily replaced with \(\nu\), say,

\[
A_{0,XX} + A_{0,ZZ} = \nu(|A|^2)_{XX},
\]

(4.25)

which serves as the continuation parameter in tracking solutions of the equation system (4.24) and (4.25).

From (4.25), \(A_0 \to 0\) as \(\nu \to 0\) so, in this limit, the coupling of \(A\) to \(A_0\) vanishes, and (4.24) reduces to a steady two-dimensional NLS equation:

\[
A_{XX} + A_{ZZ} - A + \frac{1}{2}A^2A^* = 0.
\]

(4.26)

As discussed in Chapter 2, (4.26) admits radially symmetric locally confined solutions, \(A(r)\), where \(r^2 = X^2 + Z^2\), that satisfy the boundary-value problem

\[
\frac{d^2A}{dr^2} + \frac{1}{r}\frac{dA}{dr} - A + \frac{1}{2}A^3 = 0 \quad (0 < r < \infty),
\]

(4.27)

\[
\frac{dA}{dr} = 0 \quad (r = 0),
\]

(4.28a)

\[
A \to 0 \quad (r \to \infty).
\]

(4.28b)

Working now with the equation system (4.24) and (4.25), by exploiting the fact that \(A\) and
Figure 4-1: Locally confined solution of the Davey–Stewartson equations (4.23) and (4.24). (a) Primary-harmonic envelope $A$; (b) Mean term $A_0$. $X$-cross-section (−); $Z$-cross-section (−−).

$A_0$ uncouple in the limit $\nu \to 0$, a locally confined solution of the Davey–Stewartson equations (4.23) and (4.24), corresponding to a lump, was obtained as follows: starting at $\nu = 0$ with $A_0 = 0$ and the known ground-state solution of the problem (4.27)–(4.28) that decays monotonically to zero, we computed locally confined solutions of (4.24) and (4.25) for $\nu > 0$ by increasing $\nu$ incrementally until the desired value of $\nu = \frac{1}{2}$ was reached. As in Chapter 2, the equations (4.24) and (4.25) were discretized by a pseudospectral method combined with mapping the $(X,Z)$-plane into a bounded square domain. For each value of $\nu$, the corresponding nonlinear algebraic equations were solved by Newton’s method in only one quarter of the domain, taking advantage of symmetry, and using the known solution at the previous value of $\nu$ as first guess (see Appendix C for details).

Figure 4-1 shows $X$- and $Z$-cross-sections of the computed profiles $A(X,Z)$ and $A_0(X,Z)$. The maximum of $A$, which corresponds to the lump peak amplitude, occurs at the origin $X = Z = 0$. Also, it is seen that the mean term $A_0$ decays at infinity more slowly than the primary-harmonic envelope $A$ and, as it turns out, controls the behaviour at the tails of lumps.

Specifically, from (4.24), $A$ decays exponentially, $A \propto \exp(-r)$ as $r \to \infty$. On the other
hand, upon taking the Fourier transform with respect to \(X\) and \(Z\), it follows from (4.23) that

\[
A_0 \sim -\frac{I_0}{4\pi} \frac{\partial}{\partial Z} \left( \frac{Z}{X^2 + Z^2} \right) \quad (r \to \infty),
\]  

(4.29)

where

\[
I_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A|^2 dX dZ.
\]  

(4.30)

According to (4.29), the mean term \(A_0\) decays algebraically at infinity and, taking into account (4.22), so do the lump tails:

\[
\eta \sim -\epsilon^2 \frac{I_0}{4\pi} \frac{\partial}{\partial Z} \left( \frac{Z}{X^2 + Z^2} \right).
\]  

(4.31)

Finally, inserting the computed locally confined solution of the Davey–Stewartson equations (4.23) and (4.24) in the expansion (4.22), it follows that, for values of \(\gamma\) slightly below the bifurcation point \(\gamma_0 = 1\), the peak amplitude \(\eta_0\) of a lump is given by

\[
\eta_0 = \pm \sqrt{2} (1 - \gamma)^{1/2} A(0, 0) + \cdots,
\]  

(4.32)

where \(A(0, 0) = 2.15\), \(+\)\((-)\) corresponding to elevation (depression) lumps.

### 4.3.2 Finite-amplitude lumps

We now turn our attention to finite-amplitude lump solutions of the steady 2-DB equation (4.20). Recall that elevation and depression solitary waves of the Benjamin equation also bifurcate at \(\gamma = 1\), and the corresponding solution branches are described close to the bifurcation point by an expression similar to (4.32), the only difference being that \(A(0, 0) = 2.15\) is replaced by \(2\sqrt{2}/3 = 1.63\) (Akylas et al. 1998). As discussed below, lump solutions in fact follow a bifurcation scenario very similar to that of plane solitary waves, in the finite-amplitude regime as well.

For the purpose of tracking lump-solution branches in the finite-amplitude regime, we used numerical continuation in the parameter \(\gamma\), employing the weakly nonlinear results of § 4.3.1 as a first approximation at the starting point, close to \(\gamma = 1\). Following a numerical procedure similar to that used earlier for the Davey–Stewartson equations, the steady 2-DB equation (4.20) was discretized by a pseudospectral approximation combined with mapping of the \((\xi, z)\)-plane into a finite square domain. Locally confined solutions of the 2-DB equation obey the constraint (4.9) so this also had to be imposed on the numerical solution, furnishing an additional equation. The resulting system of nonlinear algebraic equations was solved by Newton’s method. Details of implementation of the numerical procedure, including the resolution used in each of the runs and convergence checks, are given in Appendix C.

We begin by comparing the numerically computed profiles of lumps against the weakly
nonlinear results of §4.3.1. Figure 4-2 shows $\xi$- and $z$-cross-sections of elevation and depression lumps for two values of $\gamma = 0.975$ and $\gamma = 0.995$ which, according to (4.21), correspond to $\varepsilon^2 = 0.05$ and $\varepsilon^2 = 0.01$, respectively. The weakly nonlinear profiles were obtained correct to $O(\varepsilon^2)$, by inserting in expansion (4.22) the locally confined solution of the Davey–Stewartson equations for $A(X,Z)$ and $A_0(X,Z)$ computed in §4.3.1. As expected, the agreement between the numerical and asymptotic results improves as $\varepsilon^2$ is decreased. At the same time, however, it becomes apparent that the weakly nonlinear theory is quantitatively accurate only very close to the bifurcation point.

We next examine the behaviour of the elevation solution branch in the finite-amplitude regime. Figure 4-3 summarizes the results in a bifurcation diagram, displaying the lump peak amplitude, $\eta_0 \equiv \eta(\xi = 0, z = 0)$, as a function of $\gamma$. As $\gamma$ is decreased from 1, $\eta_0$ increases more rapidly than the weakly nonlinear expression (4.32) would suggest, and the lump profile becomes more localised with fewer oscillations along $\xi$ (Figure 4-4(a,b)). Finally, at $\gamma = 0$, the familiar lump solution of the KP-I equation is recovered (Figure 4-4(c)).

Even though $\gamma > 0$ in the flow configuration of interest here, the elevation solution branch can be readily continued for negative values of $\gamma$; the peak amplitude $\eta_0$ continues to increases monotonically and eventually approaches a finite limiting value $\eta_0 \approx 15$ as $\gamma \to -\infty$. In this limit, the BDO dispersive term overwhelms the KdV dispersive term, and the 2-DB equation (4.18) reduces to the two-dimensional BDO equation derived by Ablowitz & Segur (1980) for internal waves in deep fluids, but with the BDO and transverse-variation terms having opposite signs in which case plane BDO solitary waves are unstable to transverse perturbations. This suggests that the two-dimensional BDO equation, like the KP-I equation, admits lump solutions when plane solitary waves happen to be transversely unstable.

Based on the bifurcation diagram in Figure 4-3, elevation lumps behave in a manner entirely analogous to plane elevation solitary waves of the Benjamin equation. As indicated in Figure 4-3, the latter are directly connected to the KdV solitary wave at $\gamma = 0$ (Albert et al. 1999; Calvo & Akylas 2003) and, furthermore, they approach the BDO solitary wave as $\gamma \to -\infty$. Quantitatively, however, lumps feature significantly higher peak amplitudes than plane solitary waves.

Turning now to the other lump-solution branch that bifurcates at $\gamma = 1$, the bifurcation diagram corresponding to depression lumps is shown in Figure 4-5. Again, the value of the profile at the origin, $\eta_0 \equiv \eta(\xi = 0, z = 0)$, which for depression lumps is negative, is plotted as a function of the parameter $\gamma$. Here, $\eta_0$ does not coincide, in general, with the peak lump amplitude because finite-amplitude depression lumps feature a shallow middle trough and relatively tall side crests (Figure 4-6(a,b)). For $\gamma \approx 0.89$, in fact, a limit point is reached at which the solution branch turns towards increasing values of $\gamma$ and the middle trough of the lump profile develops a dimple (Figure 4-6(c,d)); increasing $\gamma$ further, the lump profile decreases in amplitude and

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Figure 4-2: Comparison of numerically computed lumps (−) near the bifurcation point against the wavepackets constructed from the weakly nonlinear theory (−−). Left column: ξ-cross-section, right column: z-cross-section. (a),(b): Elevation branch, (a) $2(1 - \gamma) = \epsilon^2 = 0.05$; (b) $\epsilon^2 = 0.01$. (c),(d): Depression branch, (c) $\epsilon^2 = 0.05$; (d) $\epsilon^2 = 0.01$. 

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looks more like two lumps of elevation pieced together in the middle. This behaviour is very similar to that of depression solitary waves of the Benjamin equation (Calvo & Akylas 2003), and the corresponding solution branch is also plotted in Figure 4-5 for comparison.

Although it becomes prohibitively expensive to carry the numerical continuation past the stage shown in Figure 4-5, we expect the branch of depression lumps to turn back towards decreasing values of $\gamma$ at another turning point very close to 1, and to keep wrapping around, each time getting close to, but never reaching, 1. Such a bifurcation scenario takes place for free-surface gravity–capillary solitary waves of elevation on deep water slightly below the minimum phase speed and the plethora of solitary waves that arises can be interpreted as multi-modal solitary waves (Dias, Menasce & Vanden-Broeck 1996, §4.3). The numerical evidence presented in Figure 4-5, although far from complete, hints that the 2-DB equation also admits multi-modal lumps, but here it is the depression branch that behaves like the elevation branch of free-surface solitary waves.

4.3.3 Convergence of the numerical method

The accuracy of the numerical procedure was tested by monitoring the convergence of the computed lump profiles as the resolution was increased. As a typical example, Table 1 illustrates
Figure 4-4: Representative lump profiles for the elevation branch. (a) $\gamma = 0.975$; (b) $\gamma = 0.925$; (c) $\gamma = 0$; (d) $\gamma = -5$. $\xi$-cross-section (---), $z$-cross-section (----).
the convergence of the peak amplitude $\eta_0$ of elevation lumps for three values of $\gamma$. As expected, higher resolution is required for computing lumps close to the bifurcation point $\gamma = 1$, where the wave profiles resemble wavepackets with several oscillations. The results reported in Figures 4-3-4-6 were computed using the resolution $N = 128$ and $M = 64$ for which $\eta_0$ is converged to four significant figures. Unfortunately, even the higher resolution $N = 128$ and $M = 128$, however, is not adequate for capturing the second turning point of the depression-lump solution branch, that is expected to occur very close to $\gamma = 1$ (see Figure 4-5).

### 4.4 Transverse instability

As remarked earlier, in the KdV limit ($\gamma = 0$), the 2-DB equation (4.18) reduces to the so-called KP-I equation, which predicts that plane KdV solitary waves are unstable to transverse perturbations; moreover, the KP-I equation admits lump solutions that become the asymptotic states of the initial-value problem in the presence of transverse variations (Ablowitz & Segur 1979).

The stability of solitary waves of the Benjamin equation to one-dimensional perturbations was explored in Calvo & Akylas (2003). Out of the two solution branches that bifurcate at $\gamma = 1$, the elevation branch turns out to be stable (as was also found by Benjamin (1992))
Figure 4-6: Representative lump profiles for the depression branch. (a) $\gamma = 0.975$; (b) $\gamma = 0.925$; (c) $\gamma = 0.925$; (d) $\gamma = 0.975$. $\xi$-cross-section (−), $z$-cross-section (−−).
Table 4.1: Convergence of peak amplitude $\eta_0$ of elevation lumps as the number of grid points is increased for three values of the parameter $\gamma$.

<table>
<thead>
<tr>
<th>$N \times M$</th>
<th>$\gamma = 0.5$</th>
<th>$\gamma = 0.75$</th>
<th>$\gamma = 0.975$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128 x 128</td>
<td>2.3761</td>
<td>1.4465</td>
<td>0.4224</td>
</tr>
<tr>
<td>128 x 96</td>
<td>2.3761</td>
<td>1.4465</td>
<td>0.4224</td>
</tr>
<tr>
<td>128 x 64</td>
<td>2.3761</td>
<td>1.4465</td>
<td>0.4223</td>
</tr>
<tr>
<td>96 x 64</td>
<td>2.3765</td>
<td>1.4468</td>
<td>0.4230</td>
</tr>
</tbody>
</table>

while the depression branch is unstable until the limit point at $\gamma \approx 0.795$ is reached (see Figure 4-5), where an exchange of stabilities is to be expected. While a comprehensive study of the initial-value problem is lacking at present — the Benjamin equation is not integrable — sample numerical solutions suggest that elevation solitary waves can emerge from locally confined initial conditions (Calvo & Akylas 2003).

For the 2-DB equation, the fact that lumps and plane solitary waves of elevation co-exist (see Figure 4-3) would suggest that the latter are unstable to transverse perturbations, as in the analogous case of the KP-I equation noted above. We now proceed to verify this claim based on a perturbation analysis for long transverse disturbances, similar to the one used in Ablowitz & Segur (1980).

Returning to the 2-DB equation (4.18), on the assumption that transverse variations are long, $\eta$ depends on the stretched coordinate $\hat{Z} = \mu z$, where $\mu \ll 1$:

$$\left( \eta_t + (\eta^2)_x - 2\gamma \mathcal{H}\{\eta_{xx}\} + \eta_{xxx} \right)_x - \mu^2 \eta_{\hat{Z}\hat{Z}} = 0. \quad (4.33)$$

The leading-order disturbance,

$$\eta = \eta^{(0)}(\xi - \psi(\hat{Z}, \hat{T}); \gamma), \quad (4.34)$$

is assumed to be a plane solitary wave of the Benjamin equation propagating along $x$ with unit speed ($\xi = x - t$) so $\eta^{(0)}$ satisfies

$$-\eta^{(0)} + \eta^{(0)^2} - 2\gamma \mathcal{H}\{\eta^{(0)}_\xi\} + \eta^{(0)}_{\xi\xi} = 0; \quad (4.35)$$

the presence of transverse variations in $\eta^{(0)}$ is reflected in the modulated phase $\psi$ that depends on $\hat{Z}$ and the slow time $\hat{T} = \mu t$. The goal of the perturbation stability analysis is to ascertain whether such modulations grow in time.
To this end, upon expanding $\eta$ in powers of $\mu$,

$$\eta = \eta^{(0)} + \mu \eta^{(1)} + \mu^2 \eta^{(2)} + \cdots,$$  \hspace{1cm} (4.36)

and substituting in (4.33), it is found that the $O(1)$ equation is already satisfied in view of (4.35). To $O(\mu)$, then, $\eta^{(1)}$ satisfies the forced problem

$$-\eta^{(1)}_{\xi} + 2(\eta^{(0)} \eta^{(1)})_{\xi} - 2\gamma H\{\eta^{(1)}_{\xi\xi}\} + \eta^{(1)}_{\xi\xi\xi} = -\psi T\eta^{(0)}_{\xi},$$  \hspace{1cm} (4.37)

or, in short,

$$\mathcal{L}\eta^{(1)} = \mathcal{R}^{(1)},$$  \hspace{1cm} (4.38)

where $\mathcal{L}$ denotes the linear operator on the left-hand side and $\mathcal{R}^{(1)}$ the forcing term on the right-hand side of (4.37).

Invoking now the standard solvability argument, for the forced problem (4.38) to have a locally confined solution, $\mathcal{R}^{(1)}$ must be orthogonal to $\eta^{(0)}$,

$$\int_{-\infty}^{\infty} \mathcal{R}^{(1)} \eta^{(0)} d\xi = 0,$$  \hspace{1cm} (4.39)

since $\eta^{(0)}$ is a proper homogeneous solution of the adjoint problem:

$$\mathcal{L}^{+} \eta^{(0)} = \eta^{(0)}_{\xi} - 2\eta^{(0)} \eta^{(0)}_{\xi} + 2\gamma H\{\eta^{(0)}_{\xi\xi}\} - \eta^{(0)}_{\xi\xi\xi} = 0,$$  \hspace{1cm} (4.40)

in view of (4.35). This solvability condition is trivially met at this order because $\mathcal{R}^{(1)}$ and $\eta^{(0)}$ have opposite parities, and we may readily solve for $\eta^{(1)}$:

$$\eta^{(1)} = \psi T\left(\eta^{(0)} + \frac{1}{2} \xi \eta^{(0)}_{\xi} - \frac{1}{2} \gamma \frac{\partial \eta^{(0)}}{\partial \gamma}\right).$$  \hspace{1cm} (4.41)

Proceeding next to $O(\mu^2)$, it is found that $\eta^{(2)}$ satisfies a forced problem of the form (4.38) with $\mathcal{R}^{(1)}$ replaced by

$$\mathcal{R}^{(2)} = -(\eta^{(1)}_{\xi})^2_{\xi} - \eta^{(1)}_{\xi} + \int_{-\infty}^{\xi} \eta^{(0)}_{\xi\xi\xi} d\xi.$$  \hspace{1cm} (4.42)

Again, for this problem to have a locally confined solution, $\mathcal{R}^{(2)}$ must be orthogonal to $\eta^{(0)}$ and, since $\mathcal{R}^{(2)}$ now is not odd in $\xi$, this solvability condition translates into an evolution equation for $\psi(\hat{Z}, \hat{T})$:

$$\frac{1}{4} \left\{ 3 \int_{-\infty}^{\infty} \eta^{(0)}_{\xi}^2 d\xi - \gamma \frac{\partial}{\partial \gamma} \int_{-\infty}^{\infty} \eta^{(0)}_{\xi}^2 d\xi \right\} \psi \hat{T} \hat{T} + \int_{-\infty}^{\infty} \eta^{(0)}_{\xi}^2 d\xi \psi \hat{Z} \hat{Z} = 0.$$  \hspace{1cm} (4.43)

Based on (4.43), therefore, the plane solitary wave is unstable to long transverse modulations.
if the coefficient of $\psi_{x_T}$ above happens to be positive:

$$3 \int_{-\infty}^{\infty} \eta^{(0)}^2 d\xi - \gamma \frac{\partial}{\partial \gamma} \int_{-\infty}^{\infty} \eta^{(0)}^2 d\xi > 0.$$  \hspace{1cm} (4.44)

It can be verified that if, rather than $\gamma$, one chooses the wave speed $c$, to trace solitary-wave solution branches of the Benjamin equation, so $\eta^{(0)} = \eta^{(0)}(\theta; c)$ with $\theta = x - ct$, (4.44) is equivalent to

$$\frac{\partial}{\partial c} \int_{-\infty}^{\infty} \eta^{(0)}^2 d\theta > 0,$$  \hspace{1cm} (4.45)

consistent with the transverse-instability condition obtained by Bridges (2001) for solitary-wave solutions of Hamiltonian systems.

In the KdV limit ($\gamma = 0$), (4.44) clearly implies instability, recovering the familiar result for the KP-I equation. In the small-amplitude limit near their bifurcation point ($\gamma \to 1$), on the other hand, solitary waves of the Benjamin equation can be approximated as (Akylas et al. 1998)

$$\eta = \pm \frac{2\epsilon}{\sqrt{3}} \text{sech}(\epsilon \xi) \cos \xi + \cdots,$$  \hspace{1cm} (4.46)

where $\epsilon^2 = 2(1 - \gamma)$; to leading order in $\epsilon$, making use of (4.46), therefore,

$$-\gamma \frac{\partial}{\partial \gamma} \int_{-\infty}^{\infty} \eta^{(0)}^2 d\xi = \frac{2}{\epsilon} \int_{-\infty}^{\infty} \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial \epsilon} d\xi = \frac{8}{3} \int_{-\infty}^{\infty} \text{sech}^2 \epsilon \xi \cos^2 \xi d\xi > 0,$$  \hspace{1cm} (4.47)

so the instability condition (4.44) is met, thus confirming that plane solitary waves are unstable to transverse modulations in this limit as well.

For elevation solitary waves, furthermore, we have checked by numerical means that (4.44) is satisfied, and hence the instability persists, for $\gamma$ in the whole range of interest, $0 \leq \gamma < 1$. The evolution of unstable perturbations in the finite-amplitude regime is discussed below.

### 4.5 Numerical simulations

#### 4.5.1 Transverse instability and formation of lumps

Here we report on unsteady numerical simulations of the 2-DB equation (4.18), in an effort to understand the role that lumps play in the evolution of disturbances in two spatial dimensions.

The numerical method of solution is analogous to the one used in Feng, Kawahara & Mitsui (1999) for the KP-I equation. It employs fast Fourier transform (FFT) in $x$ and $z$ combined with leap-frog time stepping:

$$\frac{\hat{\eta}^{n+1} - \hat{\eta}^{n-1}}{2\Delta t} + i l \mathcal{F} \{(\mathcal{F}^{-1} \{\hat{\eta}^{n}\})^2\} + \frac{i}{3} \omega (\hat{\eta}^{n+1} + \hat{\eta}^{n} + \hat{\eta}^{n-1}) = 0,$$  \hspace{1cm} (4.48)
where \( \hat{\eta}^n = \mathcal{F}\{\eta^n\} \) denotes the Fourier transform in \( x \) and \( z \) at the \( n \)-th time step and \( \omega = 2\gamma k|k| - k^3 - m^2/k \) is the linear dispersion relation of (4.18). In this semi-implicit scheme, two FFTs are needed per time step and, as shown in Appendix C, \( \Delta t \) is restricted by a stability condition of the form \( \Delta t \leq O(\Delta x) \). Moreover, the computational domain must be large enough to avoid reflections from the boundaries.

We now return to the transverse instability of plane solitary waves found in § 4.4 and follow the unstable disturbances in the finite-amplitude regime. For this purpose, we choose as initial condition

\[
\eta(x, z, t = 0) = a\tilde{\eta}(ax; \gamma),
\]

where \( \tilde{\eta}(x; \gamma) \) is the profile of a plane solitary wave with unit speed, and

\[
a(z) = 1 + 0.1 \cos \frac{\pi z}{40}
\]

is an amplitude function that imposes a periodic perturbation in the transverse direction.

Note that the initial condition (4.49) is such that the ‘mass’ per unit \( z \),

\[
\mathcal{M} = \int_{-\infty}^{\infty} \eta \, dx,
\]

is uniform along \( z \), as required by the 2-DB equation (4.18) for disturbances that are locally confined in \( x \) and periodic in \( z \). According to (4.18), \( \mathcal{M} \) in fact is independent of \( t \) as well, and this constraint is satisfied exactly by the spectral numerical scheme used here.

Figure 4-7 shows the initial condition (4.49) corresponding to a plane solitary wave of elevation for \( \gamma = 0.85 \) and two snapshots of the disturbance at later times. These computations were carried out using 1600 Fourier modes in the \( x \)-direction and 400 modes in the \( z \)-direction with \( \Delta x = \Delta z = 0.1 \) and \( \Delta t = 0.05 \). As a result of the transverse instability of the plane solitary wave, a fully localized disturbance resembling a lump of elevation emerges and propagates ahead of the rest of the disturbance with speed \( c \approx 1.21 \). The profile of the elevation lump having this speed can be readily obtained, via re-scaling

\[
x \rightarrow \frac{x}{\sqrt{c}}, \quad z \rightarrow \frac{z}{c}, \quad \eta \rightarrow c\eta, \quad \gamma \rightarrow \frac{\gamma}{\sqrt{c}},
\]

from the steady lump solutions with unit speed computed in §4.3.2, and is compared in Figure 4-8 against the upstream-propagating disturbance found in the unsteady computation at \( t = 120 \). There is very good agreement between these two profiles, confirming that the localized disturbance arising from the transverse instability of the plane solitary wave is indeed an elevation lump. We also carried out computations with the same initial condition (4.49)–(4.50) but for a solitary wave of larger amplitude \( (\gamma = 0.75) \), in which case two elevation lumps were shed upstream.
Figure 4-7: Evolution of plane solitary wave of elevation with unit speed (for $\gamma = 0.85$) in the presence of transverse perturbation. (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. An elevation lump with speed $c \approx 1.21$ emerges and propagates ahead of the rest of the disturbance.
Figure 4-8: Comparison of the upstream-propagating disturbance (-) at $t = 120$, that emerges from the transverse instability of a plane solitary wave $\gamma_0 = 0.85$, against the computed steady elevation lump (--) propagating with same speed $c \approx 1.21$. (a) $x$-cross-section, (b) $z$-cross-section.

Similar numerical results are reported in Figure 4-9 ($\gamma = 0.5$) and in Figure 4-10 ($\gamma = 0$). Especially, Figure 4-10 exhibits the formation of a KP-I lump from a KdV solitary wave with a long wave transverse perturbation. This is the first numerical observation which shows that the KdV–KP-I solitary wave pair are dynamically connected each other in the same physical condition and that the KP-I lump is a more stable state than the KdV plane solitary wave.

These simulations suggest that the transverse instability of elevation solitary waves results in the formation of elevation lumps which appear to be stable. A comprehensive numerical study of the initial-value problem of the 2-DB equation for locally confined initial conditions, including the interaction of two lumps, is currently under way. Some of those numerical results on dynamics of lumps are presented in the following subsections.

### 4.5.2 Stability of lumps

Since we know that lumps form as a result of the three-dimensional instability of the plane solitary waves, a natural question that follows is whether these lumps are stable or not. We observe the long-time behaviors of the finite amplitude lumps by performing the unsteady numerical simulations of the 2-DB equation.

From Figure 4-11 to 4-16, the dynamics of lumps are exhibited for $\gamma = 0.95$ and 0.975 by using the numerical method used in the previous subsection with the same resolution. It is interesting to notice that they feature quite analogous behaviors to that of plane solitary waves in terms of stability. The lumps of the primary elevation branch are stable while as the lumps of the primary depression branch are unstable. It is also common that stability changes at the turning point of the bifurcation diagram. Following the depression branch, a turning point is met at $\gamma = 0.893$ (Figure 4-5) and then the second elevation branch arises, of which lumps are
Figure 4-9: Evolution of plane solitary wave of elevation with unit speed (for $\gamma = 0.5$) in the presence of transverse perturbation. (a) $t = 0$, (b) $t = 75$, (c) $t = 150$. An elevation lump with $\max \eta(x, z; t) \approx 4.57$ and $\min \eta(x, z; t) \approx -0.72$ emerges and propagates ahead of the rest of the disturbance.
Figure 4-10: Evolution of a KdV solitary wave with unit speed (for $\gamma = 0$) in the presence of transverse perturbation. (a) $t = 0$, (b) $t = 25$, (c) $t = 50$. A KP-I lump emerges and propagates ahead of the rest of the disturbance.
turn out to be all stable.

These provide a good motivation for the linear stability analysis of the lumps. We actually have to solve a multi-dimensional linear eigenvalue problem to answer the question, which is under consideration.

4.5.3 Lump interactions

Solitary wave interactions show another interesting nonlinear aspect such as nontrivial phase shift. According to our unsteady numerical results listed in Figures from 4-17 to 4-20, the larger amplitude lumps undergo nearly elastic collisions whereas the smaller amplitude ones do not when they interact each other. That is, if the amplitude is larger, then the lumps are less affected by the collision, so that the phase shift is less. For the smaller amplitude lumps, they are even dispersed out in some cases and experience more phase shift. The nonintegrability of the system is thought of as the main reason of these inelastic collisions. More detailed quantitative analysis for these nonlinear effects will be considered in a later work.

4.5.4 Oblique lumps

Oblique lumps are uniquely distinctive in the study of multi-dimensional solitary waves. They are the solitary waves which are not symmetric with respect to the axis of the propagation direction but still possess the rotational symmetry by \( \pi/2 \). Their relation to the straight lumps has been discussed in §4.2.

As a starting point, it is worthwhile to see what oblique lumps look like near the bifurcation point. In the weakly nonlinear limit, small-amplitude lumps of the 2-DB equation (4.18) can be viewed as locally confined wavepackets with carrier oscillations that are stationary relative to the wavepacket envelope. That is, it is possible to construct lumps that, in general, propagate obliquely to the carrier wavevector.

Specifically, at the bifurcation point \( \gamma = \gamma_0 \), say, of oblique lumps propagating at an angle \( \phi \) to the \( x \)-direction, the following conditions must be met. First, according to the linear dispersion relation of the 2-DB equation (4.18),

\[
\omega = 2 \gamma |k| k - k^3 - \frac{m^2}{k}. \tag{4.53}
\]

For the carrier oscillations to be stationary in the reference frame of the lump, the projection of the group velocity onto the direction of the phase propagation coincides with the phase velocity:

\[
(c_g - c_p) \cdot c_p = 0. \tag{4.54}
\]
Figure 4-11: Evolution of a lump of the primary elevation branch for $\gamma = 0.95$ moving with a unit speed, (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. The initial shape of the lump is maintained for a long enough time, so it is stable.
Figure 4-12: Evolution of a lump of the primary elevation branch for $\gamma = 0.975$ moving with a unit speed, (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. It is stable as well.
Figure 4-13: Evolution of a lump of the primary depression branch for $\gamma = 0.95$ moving with a unit speed. (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. Instability arises and the lump is split into two stable single-hump elevation lumps in the end.
Figure 4-14: Evolution of a lump of the primary depression branch for $\gamma = 0.975$ moving with a unit speed. (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. Instability is developed and the double-hump is collapsed into a single elevation hump, with linear parasitic waves getting spread out behind.
Figure 4-15: Evolution of a double-hump elevation lump of the second elevation branch for $\gamma = 0.95$ moving with a unit speed. (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. It is stable.
Figure 4-16: Evolution of a double-hump lump of the second elevation branch for $\gamma = 0.975$. (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. For the relatively smaller amplitude of the lump that that of $\gamma = 0.95$, it is more affected by numerical noise with the same resolution. However, the double-hump is well-maintained in the long time evolution.
Figure 4-17: Lump interaction for $\gamma = 0.5$ for $t = 0, 20, 40, 60, 80, \text{ and } 100$. The taller lump speed = 1. The shorter front lump is obtained from the lump at $\gamma = 0.6$ via an appropriate scaling so that the speed is $(0.5/0.6)^2 \approx 0.69$. Two amplitudes are comparable each other, so two lumps seem to symmetrically separate with respect to the $x$-axis when they collide, but the taller lump eventually passes through the other one.
Figure 4-18: Lump interaction for $\gamma = 0.5$ at $t = 0, 20, 40, 60, 80$ and 100. The taller lump speed = 1. The shorter front lump is obtained from the lump at $\gamma = 0.65$ via an appropriate scaling so that the speed is $(0.5/0.65)^2 \approx 0.59$. The amplitude difference is fairly large, so the taller lump immediately passes through the shorter one.
Figure 4-19: Lump interaction for $\gamma = 0.5$ at $t = 0, 20, 40, 60, 80$ and $100$. The taller lump speed $= 1$. The shorter front lump is obtained from the lump at $\gamma = 0.75$ via an appropriate scaling so that the speed is $(0.5/0.75)^2 \approx 0.44$. The taller lump passes through the shorter one, and then a linear wave is left behind after collision.
Figure 4-20: Lump interaction for $\gamma = 0.75$ at $t = 0, 20, 40, 60, 80$ and 100. The bigger lump speed $= 1$. The smaller front lump is obtained from the lump at $\gamma = 0.925$ via an appropriate scaling so that the speed is $(0.75/0.925)^2 \approx 0.66$. The bigger lump passes through the smaller one, which is scattered out into linear waves left behind.
Figure 4-21: Schematic view for the propagation of a small-amplitude oblique lump at an angle \( \phi \) to the \( x \)-direction. The carrier wave vector \( \kappa_0 \) is orthogonal to the (dotted) phase lines. The phase velocity \( c_p \) is along \( \kappa_0 \) and is inclined at an angle \( -\alpha \) to the \( x \)-direction, where \( \tan \alpha = \frac{1}{2} \sin \phi \). The lump propagates with the group velocity \( c_g \). Since the carrier oscillation remains stationary relative to the wave envelope, the magnitude of the phase speed must equal the projection of \( c_g \) on \( c_g \).

The phase velocity \( c_p \) and the normalised group velocity \( c_g \) of the wavepacket lump are given as follows:

\[
 c_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m} \right) |_{(k_0, m_0)} = (\cos \phi, \sin \phi), \quad c_p = \frac{\omega}{\kappa^2} (k_0, m_0), \quad \kappa = |\kappa|.
\]

From (4.33)-(4.55), we obtain

\[
 \gamma_0 = |k_0| = \sqrt{\cos \phi - \frac{1}{4} \sin^2 \phi}, \quad m_0 = -\frac{1}{2} k_0 \sin \phi.
\]

Since \( \cos \phi - \frac{1}{4} \sin^2 \phi > 0 \), the propagation angle \( \phi \) of the oblique lump relative to the \( x \)-direction, therefore, must satisfy

\[
 0 \leq \phi < \phi_{\text{max}} = \cos^{-1}(\frac{-2 + \sqrt{5}}{2}) = 76.34^\circ.
\]

A schematic diagram for the motion of the oblique wavepacket lump is provided in Figure 4-21.

Along with the result of the previous subsection, oblique lumps that bifurcate from elevation wavepackets should be stable because a domain transformation does not affect the stability of solitary waves. In the previous studies by Rubenchik (1974), Yuen (1978) and Akylas (1993), the envelope plane solitary waves are unstable under any three-dimensional perturbations or oblique carrier oscillations. Therefore, we conjecture that instability driven by oblique perturbations are going to be naturally developed into oblique lumps in the end, and it is justified from the unsteady simulation result in Figure 4-22.

Finally, Figure 4-22 shows an oblique interaction between two lumps. The straight lump
Figure 4-22: Instability of a plane solitary wave under oblique perturbation and a formation of an oblique lump. (a) $t = 0$, (b) $t = 50$, (c) $t = 100$. 
move with the normalized speed and the oblique lump moves with the speed of $2\sqrt{2}$ and the angle of $\pi/4$ from the $x$-axis. Their cores exactly collide at the origin at $t = 10$, and then they are separated each other to recover their original shapes with some non-trivial phase shifts. This is similar to the numerical result for the KP-I lump interactions by Feng et. al. More detailed quantitative results to measure the amount of phase shifts for various lump interactions is being under consideration. It is standard in the study of solitary waves to have more phase shifts for slower interactions.

### 4.6 Derivation of the 2-DB equation by a weakly nonlinear long-wave expansion

In §4.2, we provide a heuristic derivation of the 2-DB equation. In this section, we derive the 2-DB equation by using a weakly nonlinear long-wave expansion directly from the interfacial wave problem in a two-fluid system.

In a dimensional form, the interfacial wave problem is written as follows:

\[
\phi_{1xx} + \phi_{1yy} + \phi_{1zz} = 0, \quad (-\infty < y < \eta),
\]

\[
\phi_{2xx} + \phi_{2yy} + \phi_{2zz} = 0, \quad (\eta < y < h),
\]

\[
\rho_1 [\phi_{1tt} + \frac{1}{2} (\phi_{1x}^2 + \phi_{1y}^2 + \phi_{1z}^2) + g\eta] - \rho_2 [\phi_{2tt} + \frac{1}{2} (\phi_{2x}^2 + \phi_{2y}^2 + \phi_{2z}^2) + g\eta]
- T \eta_{xx} (1 + \eta_z^2) + \eta_{zz} (1 + \eta_x^2) - 2 \eta_{xz} \eta_x \eta_z = 0, \quad (y = \eta),
\]

\[
\eta_t + \phi_{1x} \eta_x + \phi_{1y} \eta_y = \phi_{1y}, \quad (y = \eta),
\]

\[
\eta_t + \phi_{2x} \eta_x + \phi_{2y} \eta_y = \phi_{2y}, \quad (y = \eta).
\]

where $\phi_{1ax} \to c$ as $|x| \to \infty$, $\phi_{1y} \to 0$ as $y \to -\infty$, and $\phi_{2y} = 0$ at $y = h$.

We assume that $x$ is the dominant wave propagation direction, $z$ is the transverse direction to $x$, and $y$ is the vertical direction. In addition, $a$ is the wave-amplitude scale, $h$ is the thickness of the upper layer, $l$ is the $x$-directional wavelength scale, and $L$ is the $z$-directional wavelength scale. And then, we define three non-dimensional parameters $\epsilon$, $\mu$, and $\nu$:

\[
\epsilon = \frac{a}{h}, \quad \mu = \frac{h}{l}, \quad \nu = \frac{l}{L},
\]

which are assumed to be infinitesimal.
Figure 4-23: Lump interaction when $\gamma = 0.5$, $\phi_1 = 0$ and $\phi_2 = \pi/4$. $dx = dz = 0.1$, $dt = 0.05$. (a) $t = 0$, (b) $t = 10$, (c) $t = 20$ (d) $t = 30$. 
Considering the following non-dimensionalization,

\[ \eta \to a\eta, \quad x \to lx, \quad y \to lY \) (lower layer), \quad y \to hy \) (upper layer), \quad z \to Lz, \quad \phi_{1,2} \to cl\phi_{1,2}, \quad t \to \frac{t}{c}. \]

the non-dimensionalized governing equations and boundary conditions read as:

\[ \phi_{1xx} + \phi_{1yy} + \nu^2 \phi_{1zz} = 0, \quad (-\infty < Y < \epsilon\eta), \]
\[ \mu^2 \phi_{2xx} + \phi_{2yy} + \nu^2 \phi_{2zz} = 0, \quad (\epsilon\eta < y < 1), \]
\[ \epsilon \frac{1 - R}{F^2} \eta + \left[ \phi_{1x} + \frac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2 + \nu^2 \phi_{1z}^2) \right] - R\left[ \phi_{2x} + \frac{1}{2}(\phi_{2x}^2 + \nu^2 \phi_{2y}^2 + \nu^2 \phi_{2z}^2) + g\eta \right] \]
\[ - \epsilon^2 W_0 \eta_{xx} \left( 1 + \epsilon^2 \mu^2 \nu^2 \eta_x^2 \right) \]
\[ + \nu^2 \eta_{zz} \left( 1 + \epsilon^2 \mu^2 \nu^2 \eta_z^2 \right) - 2\epsilon^2 \mu^2 \nu^2 \eta_{xz} \eta_{x} \eta_{z} = 0, \quad (y = \eta), \]
\[ \epsilon \mu \eta_t + \phi_{1x} \eta_z + \nu^2 \phi_{1z} \eta_z = \phi_{1y}, \quad (Y = \epsilon\mu\eta), \]
\[ \epsilon \mu^2 \eta_t + \phi_{2x} \eta_z + \phi_{2z} \eta_z = \phi_{2y}, \quad (Y = \epsilon\eta), \]

with \[ \phi_{1,2x} \to 1 \text{ as } |x| \to \infty, \quad \phi_{1Y} \to 0 \text{ as } y \to -\infty, \text{ and } \phi_{2y} = 0 \text{ at } y = 1, \text{ where} \]
\[ R = \frac{\rho_2}{\rho_1}, \quad W_0 = \frac{T}{\rho_1 hc^2}, \quad F^2 = \frac{c^2}{gh}. \]

We introduce a slow time variable \( \tau = \delta t \) where \( \delta \ll 1 \), which should be determined later. In addition, a stretching coordinate \( X = x - \delta t \) is introduced so that the waves of our interest are considered to be slowly changing disturbances to the unidirectional waves of the normalized wave speed in the original \( x \)-direction. Under the description of these new variables \( (X, \tau) \), the time derivatives are expressed as follows:

\[ \phi_{1,2t} = -\phi_{1,2x} + \delta \phi_{1,2\tau}, \]
\[ \eta_t = -\eta_X + \delta \eta_{\tau}. \]

First, we solve (4.61a) in the lower fluid layer. Since \( \nu \ll 1 \), the equation can be regarded a perturbed Laplace equation with Neumann boundary conditions prescribed. Therefore, \( \phi_1 \) is approximated by the Green’s function representation:

\[ \phi_1(X, Y, Z; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-\phi_{1X} \eta_{X'} + \phi_{1Y}) \left[ \log((X' - X)^2 + Y^2) + O(\epsilon\mu) \right] dX'. \] (4.64)

Assuming \( \phi_{1X} \ll \phi_1 \), the linear terms are dominant over the nonlinear terms in (4.61d,e). From (4.62b), thus, \( \phi_{1Y} = -\epsilon \mu \eta_X + O(\epsilon^2 \mu^2, \epsilon \mu \nu^2) \) at \( Y = \epsilon \mu \eta \), and the above expression can
be rephrased by
\begin{equation}
\phi_1(X, Y; Z, \tau) = -\frac{\varepsilon \mu}{2\pi} \int_{-\infty}^{\infty} \eta X' (X') \log \{(X' - X)^2 + O(\varepsilon^2 \mu^2)\} dX'.
\end{equation}

Therefore,
\begin{align}
\phi_{1X} &= \frac{\varepsilon \mu}{\pi} \int_{-\infty}^{\infty} \eta X' \frac{X - X'}{(X - X')^2 + (\varepsilon \mu)^2} dX' = \frac{\varepsilon \mu}{\pi} \int_{-\infty}^{\infty} \frac{\eta X'}{X - X'} dX' + O(\varepsilon^2 \mu^2) \\
&= -\varepsilon \mu \mathcal{H}\{\eta X\} + O(\varepsilon^2 \mu^2),
\end{align}
\begin{align}
\phi_{1XX} &= -\frac{\varepsilon \mu}{\pi} \int_{-\infty}^{\infty} \frac{\eta X'}{(X - X')^2} dX' + O(\varepsilon^2 \mu^2) = -\frac{\varepsilon \mu}{\pi} \int_{-\infty}^{\infty} \frac{\eta X' X'}{X' - X} dX' + O(\varepsilon^2 \mu^2) \\
&= -\varepsilon \mu \mathcal{H}\{\eta XX\} + O(\varepsilon^2 \mu^2).
\end{align}

Now, we solve (4.61b) in the upper fluid layer. Since \( \mu = h/l \ll 1 \), we perform Taylor's expansion in \( \mu \) for \( \phi_2(X, y, z; \tau) \) as follows:
\begin{align}
\phi_2(X, y, z; \tau) &= f - \frac{1}{2} \mu^2 (1 - y)^2 f_{xx} - \frac{1}{2} \mu^2 \nu^2 (1 - y)^2 f_{zz} + \frac{1}{24} \mu^4 (1 - y)^4 f_{xxxx} + \cdots, \\
\phi_{2x}(X, y = \varepsilon \eta, z; \tau) &= f_x - \frac{\mu^2}{2} f_{xxx} + \cdots, \\
\phi_{2xx}(X, y = \varepsilon \eta, z; \tau) &= f_{xx} - \frac{\mu^2}{2} f_{xxxx} + \cdots, \\
\phi_{2\tau x}(X, y = \varepsilon \eta, z; \tau) &= f_{x\tau} - \frac{\mu^2}{2} f_{xxxx}\tau + \cdots, \\
\phi_{2y}(X, y = \varepsilon \eta, z; \tau) &= \mu^2 (1 - \varepsilon \eta) f_{xx} + \mu^2 \nu^2 (1 - \varepsilon \eta) f_{zz} - \frac{\mu^4}{6} f_{xxxx} + \cdots, \\
\phi_{2z}(X, y = \varepsilon \eta, z; \tau) &= f_z - \frac{\mu^2}{2} f_{xxz} + \cdots.
\end{align}

Substituting (4.63b) and (4.67e) into (4.61e), we obtain:
\begin{equation}
\varepsilon \mu^2 (-\eta_X + \delta_{\eta\tau}) + \cdots = \mu^2 (1 - \varepsilon \eta) f_{xx} + \mu^2 \nu^2 (1 - \varepsilon \eta) f_{zz} + \cdots,
\end{equation}
deducing that
\begin{equation}
f_X = -\varepsilon \eta + O(\varepsilon^2).
\end{equation}
Balancing \( \mu^2 \nu^2 f_{xz} \) (transverse disturbance) and \( \mu^2 \varepsilon \eta f_{XX} \) (weak-nonlinearity),
\begin{equation}
\varepsilon = \nu^2.
\end{equation}
with respect to \( X \). Then, up to \( O(\epsilon^2) \),

\[
\epsilon \frac{1 - R}{F^2} \eta_X - \phi_{1XX} - R \{ - \phi_{2XX} + \epsilon \phi_{2x} + \frac{1}{2} (\phi_{2x})^2 \} - \epsilon^3 W_0 \eta_{XXX} = 0. 
\]

(4.71)

where \( W = O(1/\epsilon) \). From the following expansions, we collect the terms of the same order in (4.71):

\[
\phi_{1XX} = -\phi_{1X} + \epsilon \mu H \{ \eta_{XX} \} + \cdots, \quad (4.72a)
\]

\[
\phi_{2XX} = -\phi_{2X} + \epsilon \phi_{2x} = f_{XX} + \delta f_{x} + \cdots = -\eta_X - \delta \eta_x, \quad (4.72b)
\]

\[
(\phi_{2x})_X = (f_{XX})_X + \cdots = ((\epsilon \eta)^2)_X + \cdots. \quad (4.72c)
\]

Then, the coefficient of \( \eta_X \) is \( \epsilon[(1 - R)/F^2 - R] \). We assume

\[
\frac{1 - R}{R} \frac{1}{F^2} = 1 + \epsilon, \quad (4.73)
\]

so that the leading order terms in (4.61c) are \( O(\epsilon^2) \). Physically, this means that the wave speed is above the minimum value \( \sqrt{gh(1 - R)/R} \). In (4.68), \( \phi_{1X} \sim \epsilon \mu H \{ \eta_{XX} \} \) (gravity effect in deep water), \( \phi_{2X} \sim (f_{XX})_X \sim ((\epsilon \eta)^2)_X \) (weak nonlinearity), and \( \delta \eta_x \) (time variance of \( \eta \)) should be balanced each other. Therefore,

\[
\epsilon = \mu = \delta. \quad (4.74)
\]

Then, from (4.68), (4.72), and (4.73), \( O(\epsilon^2) \) terms in (4.71) read as follows:

\[
R(\epsilon \eta_X + f_{XX})_X + \left[ \epsilon^2 \mu \{ \eta_{XX} \} - \epsilon R f_{x} - \frac{1}{2} R(f_{XX})_X - \epsilon^3 W_0 \eta_{XXX} \right]_X = 0. 
\]

(4.75)

Similarly, \( O(\epsilon^2) \) terms (4.68) are collected by

\[
(\epsilon \eta_X + f_{XX})_X + \epsilon(-\eta_x - f_{XX} - \eta f_{XX} + f_{zz})_X = 0. \quad (4.76)
\]

Substituting (4.69) into (4.75) and (4.76), (4.75) and (4.76) are compared each other:

\[
\frac{1}{R} [\mu \{ \eta_{XX} \} + R \eta_x - \frac{1}{2} R(\eta^2)_X - \epsilon W_0 \eta_{XXX}]_X = (-\eta_x + \eta f_{XX} + \eta x)_X - \eta_{zz}. \quad (4.77)
\]

Finally, we obtain

\[
\{ \eta_x - \frac{3}{4} (\eta^2)_X + \frac{1}{2 R} \mu \{ \eta_{XX} \} - \epsilon \frac{W_0}{2R} \eta_{XXX} \}_X + \frac{1}{2} \eta_{zz} = 0. \quad (4.78)
\]

We note that

\[
\frac{W_0}{R} \frac{W_0 F^2}{1 - R} = \frac{T}{\delta \rho g h^2} \equiv W \gg 1. \quad (4.79)
\]
Chapter 5

Conclusions

5.1 Summary

We have mainly discussed three-dimensional (3-D) nonlinear dispersive wave phenomena. Specifically: (i) generation mechanism of fully localized three-dimensional solitary waves (lumps), and (ii) transverse instability of plane solitary waves and its relation to the formation of lumps.

Prior studies have mostly focused on solitary waves in the long-wave limit, where lumps are described by the Kadomptsev–Petviashvili I (KP-I) equation. In the water-wave problem, however, the KP-I Lumps are possible only in water of a few mm depth with the presence of strong surface tension (Bond number, $B > 1/3$).

We have explored the generation mechanism of lumps in less restrictive physical conditions. Rather than long waves, these lumps bifurcate from infinitesimal sinusoidal waves of nonzero finite wavenumber, where the group and phase velocities are equal ($c_g = c_p$). In the small-amplitude limit, lumps resemble fully localized wavepackets with a wave envelope and an induced mean flow. In the water-wave problem, these wavepacket lumps exist in the weak-surface-tension regime ($B < 1/3$). Then, the wave envelope and mean flow satisfy the elliptic Davey–Stewartson equation system (EDS), and the carrier wavenumber is $\kappa_0 h$ where $h$ is the fluid depth.

We have computed their numerical profiles, which agree with fully numerical computations as well as a formal existence proof. Owing to the mean-flow component, the lumps feature algebraically decaying tails at infinity, so the far-field information of the lumps is not negligible. In order to resolve the solution profile accurately as well as to reduce the computational cost, we have used a spectral collocation method with a stretching transformation over non-uniform grids and have taken advantage of the solution symmetry. The method exhibits exponential convergence as the number of grids increases.

In the same physical setting where the lumps exist, we have shown that the 2-D solitary waves that bifurcate below the minimum phase speed are unstable under long-wave transverse...
perturbations. We have found an expression for the leading order instability growth rate $\lambda_1$ in terms of the solitary wave surface elevation $\eta$ and flow potential $\phi$ from the corresponding linearized eigenvalue problem as follows:

$$\lambda_1^2 = -\left( \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \theta^2} \right) \left( \int_{-\infty}^{\infty} \frac{\partial^2 \eta \partial^2 \phi}{\partial \theta^2} \right) + \int_{-\infty}^{\infty} \frac{\eta^2}{(1 + \eta^2)^{1/2}} \frac{1}{V \partial V},$$

(5.1)

where $E$ is the total energy of solitary waves and $V$ is the wave speed. We have numerically verified that $\partial E/\partial V < 0$. Intuitively, this is because $E > 0$ for $0 < V < c_0$ where $E = 0$ at the bifurcation point, which is consistent with the previous result by Bridges (2001). Physically, $\partial E/\partial V$ can be regarded as momentum, so plane solitary waves are transversely unstable when they have negative momentum.

The above generation mechanism of lumps and the transverse instability of 2-D solitary waves can be applied to interfacial wave motion in a two-fluid-layer system with a thin upper fluid layer immersed on a heavier deep fluid in the presence of a very strong interfacial tension. These waves are referred as of "Benjamin" type.

We have derived the 2-D Benjamin equation (2-DB) for the 3-D interfacial motion of fluids,

$$\left( \eta + \eta^2 \right)_{z} - 2\gamma \mathcal{H}(\eta_{zz}) + \eta_{zzz} = \sigma^2 \eta_{zzz},$$

(5.2)

for $0 \leq \gamma = (H/2R) \cdot \sqrt{\delta p g / T} < 1$ where $R$ is the density ratio, $T$ is the interfacial tension coefficient, $\delta p$ is the density difference, $g$ is the gravitational acceleration, and $H$ is the upper-fluid thickness. The 2-DB equation generalizes the 1-DB equation ($0 < \gamma < 1, \sigma^2 = 0$) with a positive transverse dispersion ($\sigma^2 = 1$) as well as the KP-I equation ($\gamma = 0, \sigma^2 = 1$) with the Hilbert transform $\mathcal{H}(\cdot)$.

We have computed a one-parameter family of lump solutions numerically. We have treated the singularity of the Hilbert transformation in the 2-DB equation with spectral accuracy. We have performed unsteady numerical simulations by using the fast Fourier transform (FFT) and demonstrated that 2-D solitary waves with the elevation peak, which are longitudinally stable, are transversely unstable and result in the formation of lumps. Under oblique disturbances, 2-D plane solitary waves develop into oblique lumps.

5.2 Further questions

5.2.1 Stability of lumps/Transverse instability of the interfacial solitary waves

The EDS wave envelope and mean flow develop a focusing singularity in finite time, so the wavepacket lumps should be unstable. However, we have observed stable finite-amplitude lumps of the 2-DB equation. Therefore, we expect that there should be a borderline between stability
and instability. A *multi-dimensional stability analysis* should be considered.

For the interfacial solitary waves in two-layer fluids, an analogous expression to (1) for the instability growth rate under long-wave transverse perturbations can be found. The dependence of the total energy on the wave speed can be computed both analytically near the bifurcation point and numerically in the fully nonlinear regime.

### 5.2.2 Near-integrability for the 2-DB equation

The 2-DB equation admits coherent steady state solutions, so the equation can be regarded as a perturbation of an *integrable system* — the KP-I equation near $\gamma = 0$. For the initial value problems, therefore, a *perturbation theory* can be developed from the already known *inverse scattering transform*. On the other hand, the 2-DB lumps feature a *pitchfork bifurcation* near $\gamma = 0$ and infinitely many turning points in the bifurcation diagram. In addition, it has been numerically demonstrated that the 2-DB lump collisions are *near-elastic*. These facts may indicate that the parameter $\gamma$ triggers the onset of *incoherent phenomena* or the transition to *chaos* as it varies away from the end points.

### 5.2.3 Spectrally accurate boundary integral method on a non-flat unbounded domain/Direct Jacobian method

Interesting dynamical behaviors of lumps such as stability/instability and their mutual interactions would be better understood by performing unsteady numerical simulations. For the purpose, the boundary integral formulation for the 3-D Laplace's equation can be employed over a non-flat unbounded boundary, combined with a spectral method. For steady computations for the numerical profiles of lumps, numerical continuation is used by using the direct Jacobian method, which has been demonstrated to accurately capture the turning points in the bifurcation diagram. At those points, Jacobian becomes singular and longitudinal-stability exchange occurs. We can implement a parallel algorithm for the numerical continuation according to the subspace numerical basis to save computation time.
Appendix A

Existence of radial symmetric envelope NLS lumps

We discuss the existence of radial symmetric NLS lumps, which have been used for the initial guess to obtain the numerical lump solutions of the elliptic-elliptic DS equation system in Chapter 2 and 4. The NLS equation actually comes from the leading order approximation of the water-wave equations in the weakly nonlinear limit with the mean flow absent, so it is a good approximation to the DS system.

A.1 Spatial dynamics

Strauss (1977) provided a rigorous mathematical proof that there is a nontrivial locally confined solution of the generalized 2D NLS equation. Papanicolaou (1994) mentioned that DS solutions can be continued from the radially symmetric NLS solutions in their previous study. Sulem, Patera, Weinstein tried to resolve singular NLS solutions numerically. The computation of a radially symmetric regular envelope NLS modes is found in the study of an optical research (Chiao, Garmire & Townes 1964). Here, we present a simple existence argument for all possible locally confined radially symmetric envelope NLS solutions, which is called spatial dynamics. It turns out that they exist only in a discrete sense.

The radial envelope NLS equation is written as

\[ A_{rr} = -\frac{1}{r} A_r + (A - \frac{1}{2} A^3) \]  \hspace{1cm} (A.1)

where \( r = \sqrt{x^2 + z^2} \).

\( r \) goes from 0 to \( \infty \). So, we may regard \( r \) as a time-like variable because it is always positive, so that the equation is interpreted as one-dimensional dynamics. In other words, \( A_{rr} \) is acceleration, \( -\frac{1}{r} A_r \) is a damping force and \( A - A^3 \) is the conservative force in a field whose
Figure A-1: Double well potential for the radially symmetric NLS equation. Regular and locally confined solutions exist only for a discrete set of initial heights, whose energy is exactly exhausted at the unstable fixed point, \((A, V) = (0, 0)\). Dotted: the double well potential, \(V = -\frac{1}{2}A^2 + \frac{1}{8}A^4\). Solid: the first radial NLS mode, \(A_{r=0} = 3.12\). Dash-Dot: the second radial NLS mode, \(A_{r=0} = 4.71\). Arrows denote the direction along which 'spatial dynamics' proceeds.

The potential \(V = -\int (A - \frac{1}{2}A^3)dA = -\frac{1}{2}A^2 + \frac{1}{8}A^4\). This is a double-well potential which has two stable troughs at \(A = \pm \sqrt{2}\) and one unstable crest at \(A = 0\). As \(r \to \infty\), \(A(r)\) approaches one of the two stable troughs because the damping term dissipates the initial energy.

There is also a rare possibility, however, that \(A(r)\) reaches to the unstable crest at \(A = 0\) in the limit of \(r \to \infty\) for a properly chosen initial energy. There should exist an initial height \(A_{r=0}\) such that the initial energy is exactly exhausted when it reaches the crest \(A = 0\) as \(r \to \infty\). With the condition of zero initial speed \(A_r|_{r=0} = 0\), this is the profile of \(A(r)\) we want. Countably many such initial heights \(A_{r=0}\) exist depending on how many troughs it passes through during the dynamics. We compute those profiles by the second-order forward Euler scheme. A discrete set of the initial height of \(A\) are computed by using the bisection algorithm.
Appendix B

Solvability condition for the stability eigenvalue problem of the water-wave equations

We derive the solvability condition (3.35) for the stability eigenvalue problem of the water-wave equations, which has been discussed in Chapter 3.

B.1 Linearized eigenvalue problem

First, we consider a linearized eigenvalue problem for a small perturbation \( \hat{\eta}(\theta) \) and \( \hat{\phi}(\theta, z) \).
It is obtained by perturbing the original water-wave equations to the basic solitary-wave state \( \hat{\eta}(\theta; V, D) \) and \( \hat{\phi}(\theta, z; V, D) \), where \( \theta = x - Vt \).

\[
\eta = \hat{\eta}(\theta; V, D) + \hat{\eta}(\theta)e^{i\beta y + \lambda t}, \quad \phi = \hat{\phi}(\theta, z; V, D) + \hat{\phi}(\theta, z)e^{i\beta y + \lambda t}. \quad (B.1)
\]

Then, \( \hat{\eta} \) and \( \hat{\phi} \) satisfy (after linearization):

\[
\Delta \hat{\phi} = \beta^2 \hat{\phi} \quad (-D < z < \hat{\eta}) \quad (\Delta \equiv \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}) \quad (B.2a)
\]

\[
\hat{\phi}_z = 0 \quad (z = -D). \quad (B.2b)
\]
Perturbing the kinematic boundary condition at \( z = \bar{\eta}(\theta) \), we obtain

\[
\eta_t \to \lambda \dot{\eta} - V \dot{\eta}_\theta, \tag{B.3a}
\]
\[
\phi_x|_{z=\bar{\eta}+\delta \eta} = \phi_x|_{\eta} + \delta \eta \tilde{\phi}_{xz}|_{\eta} + \cdots = \tilde{\phi}_x|_{\eta} + \delta \eta \tilde{\phi}_{\theta z}|_{\eta} + \cdots \\
\to \lambda \dot{\phi}_x|_{\eta} - \tilde{\eta}_\theta \tilde{\phi}_{\theta z}, \tag{B.3b}
\]
\[
\phi_x \tilde{\eta}_x |_{\eta+\delta \eta} = \phi_x \tilde{\eta}_x |_{\eta} + \tilde{\phi}_{xz} \delta \eta + \cdots \\
\to \tilde{\phi}_{\theta z} \tilde{\eta}_\theta + \tilde{\phi}_x \tilde{\eta}_\theta + \tilde{\phi}_{\theta z} \tilde{\eta}_\theta \dot{\eta} + \cdots. \tag{B.3c}
\]

Hence, the linearized kinematic boundary condition for small perturbations \( \dot{\eta} \) and \( \dot{\phi} \) becomes

\[
\mathcal{L}_1(\dot{\phi}, \dot{\eta}) = -\lambda \dot{\eta} \quad (z = \bar{\eta}(\theta)), \tag{B.4}
\]

with

\[
\mathcal{L}_1(\dot{\phi}, \dot{\eta}) \equiv \left( -\frac{\partial}{\partial z} + \tilde{\eta}_\theta \frac{\partial}{\partial \theta} \right) \dot{\phi} + \left\{ \frac{d\tilde{\phi}_\theta}{d\theta} + (-V + \tilde{\phi}_\theta) \frac{\partial}{\partial \theta} \right\}, \tag{B.5}
\]

where

\[
\frac{d}{d\theta} \equiv \frac{\partial}{\partial \theta} + \tilde{\eta}_\theta \frac{\partial}{\partial z}, \tag{B.6}
\]

which is the directional derivative along the surface of water \( z = \bar{\eta}(\theta) \).

Now, perturbing the dynamic boundary condition at \( z = \bar{\eta}(\theta) \), the followings are obtained:

\[
\dot{\phi}_t |_{\eta+\delta \eta} = \phi_t |_{\eta} + \phi_{tz} |_{\eta} \delta \eta + \cdots \to \lambda \dot{\phi} - V \dot{\phi}_\theta + \tilde{\eta}(\bar{V}(\tilde{\phi}_x z)), \tag{B.7a}
\]
\[
\dot{\phi}_x^2 |_{\eta+\delta \eta} = \phi_x^2 |_{\eta} + 2 \phi_x \phi_{xz} |_{\eta} \delta \eta + \cdots \to 2(\tilde{\phi}_x \dot{\phi}_\theta + \tilde{\phi}_x \tilde{\phi}_x \dot{\eta} \tilde{\eta}_\theta) + \cdots, \tag{B.7b}
\]
\[
\dot{\phi}_x^2 |_{\eta+\delta \eta} = \phi_x^2 |_{\eta} + 2 \phi_x \phi_{xz} |_{\eta} \delta \eta + \cdots \to 2(\tilde{\phi}_x \dot{\phi}_\theta + \tilde{\phi}_x \tilde{\phi}_x \dot{\eta} \tilde{\eta}_\theta) + \cdots, \tag{B.7c}
\]
\[
(1 + \eta_x^2 + \eta_y^2)^{-3/2} = \frac{1}{(1 + \eta_x^2)^{3/2}} \left\{ 1 - 3 \tilde{\eta}_x \delta \eta_x + \cdots \right\}, \tag{B.7d}
\]

\[
\frac{\eta_{xx}(1 + \eta_x^2) + \eta_{yy}(1 + \eta_y^2) - 2 \eta_{xy} \eta_{xz} \eta_{yz}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} \to \frac{\dot{\eta}_{\theta \theta}}{(1 + \eta_\theta^2)^{3/2}} - \frac{3 \tilde{\eta}_{\theta \theta} \tilde{\eta}_x \tilde{\eta}_\theta - \beta^2 \tilde{\eta}}{(1 + \eta_\theta^2)^{3/2}}. \tag{B.7e}
\]

Hence, the linearized dynamic boundary condition for \( \dot{\eta} \) and \( \dot{\phi} \) becomes

\[
\mathcal{L}_2(\dot{\phi}, \dot{\eta}) = -\lambda \dot{\phi} - \beta^2 \frac{\dot{\eta}}{(1 + \eta_\theta^2)^{1/2}} \quad (z = \bar{\eta}(\theta)), \tag{B.8}
\]

with

\[
\mathcal{L}_2(\dot{\phi}, \dot{\eta}) \equiv (-V + \tilde{\phi}_\theta) \left( \frac{\partial}{\partial \theta} + \tilde{\eta}_\theta \frac{\partial}{\partial z} \right) \dot{\phi} + \left\{ (-V + \tilde{\phi}_\theta) \dot{\phi}_{xz} + \tilde{\phi}_x \tilde{\phi}_xz + 1 \right\} \dot{\eta} - \frac{\partial}{\partial \theta} \left\{ \frac{\dot{\eta}}{(1 + \eta_\theta^2)^{3/2}} \right\}. \tag{B.9}
\]
B.2 Solving the eigenvalue problem perturbatively for $\beta \ll 1$

The eigenvalue problem is solved by expanding $\hat{\phi}$, $\hat{\eta}$ and the eigenvalue $\lambda$ in small $\beta \ll 1$ as in (3.27)–(3.28). Then, the leading-order potential $\hat{\phi}^{(0)}$ and surface elevation $\hat{\eta}^{(0)}$ satisfy:

\begin{align}
\Delta \hat{\phi}^{(0)} &= 0 \quad (-D < z < \hat{\eta}), \\
\hat{\phi}^{(0)} &= 0 \quad (z = -D),
\end{align}

(B.10a) \quad (B.10b)

where $\mathcal{L}_1(\hat{\phi}^{(0)}, \hat{\eta}^{(0)}) = \mathcal{L}_2(\hat{\phi}^{(0)}, \hat{\eta}^{(0)}) = 0$ at $z = \hat{\eta}$. From the basic solitary-wave solution,

\begin{align}
\frac{d}{d\theta}\{\text{Kinematic B.C.}\} &= 0 \rightarrow \frac{d}{d\theta}\{-V\hat{\eta}_\theta + \hat{\phi}_\theta\hat{\eta}_\theta - \hat{\phi}_z\}|_{z=\hat{\eta}} = 0, \\
\rightarrow \mathcal{L}_1(\hat{\phi}_\theta, \hat{\eta}_\theta) &= 0 \quad (z = \hat{\eta}), \\
\frac{d}{d\theta}\{\text{Dynamic B.C.}\} &= 0 \rightarrow \frac{d}{d\theta}\{-V\hat{\phi}_\theta + \hat{\eta} + \frac{1}{2}(\hat{\phi}_z^2 + \hat{\phi}_x^2) - \frac{\hat{\eta}_\theta\theta}{(1 + \hat{\eta}_\theta^2)^{3/2}}\}|_{z=\hat{\eta}} = 0, \\
\rightarrow \mathcal{L}_2(\hat{\phi}_\theta, \hat{\eta}_\theta) &= 0 \quad (z = \hat{\eta}).
\end{align}

(B.11a) \quad (B.11b)

Therefore,

\begin{align}
\hat{\phi}^{(0)} &= \hat{\phi}_\theta, \quad \hat{\eta}^{(0)} = \hat{\eta}_\theta. \quad \text{(B.12)}
\end{align}

Now, $\hat{\phi}^{(1)}$ and $\hat{\eta}^{(1)}$ satisfy the following forced boundary value problem:

\begin{align}
\Delta \hat{\phi}^{(1)} &= 0 \quad (-D < z < \hat{\eta}), \\
\hat{\phi}^{(1)} &= 0 \quad (z = -D),
\end{align}

(B.13a) \quad (B.13b)

with a linearized boundary condition at $z = \hat{\eta}$,

\begin{align}
\mathcal{L}_1(\hat{\phi}^{(1)}, \hat{\eta}^{(1)}) &= -\lambda_1\hat{\eta}^{(0)} = -\lambda_1\hat{\eta}_\theta, \\
\mathcal{L}_2(\hat{\phi}^{(1)}, \hat{\eta}^{(1)}) &= -\lambda_1\hat{\phi}^{(0)} = -\lambda_1\hat{\phi}_\theta.
\end{align}

(B.14a) \quad (B.14b)

From the basic solitary-wave solution,

\begin{align}
\frac{\partial}{\partial V}\{\text{Kinematic B.C.}\} &\rightarrow \hat{\eta}_\theta - V\hat{\eta}_V\hat{\theta} + \hat{\phi}_\theta\hat{\eta}_V\hat{\theta} + \hat{\eta}_\theta(\hat{\phi}_V\hat{\theta} + \hat{\phi}_z\hat{\eta}_V) = \hat{\phi}_z\hat{V} + \hat{\phi}_{zz}\hat{\eta}_V, \\
\rightarrow \mathcal{L}_1(\hat{\phi}_V, \hat{\eta}_V) &= \hat{\eta}_\theta, \\
\frac{\partial}{\partial V}\{\text{Dynamic B.C.}\} &\rightarrow -\hat{\phi}_\theta + \hat{\eta}_V + (-V + \hat{\phi}_\theta)(\hat{\phi}_V\hat{\theta} + \hat{\phi}_z\hat{\eta}_V) + \hat{\phi}_z(\hat{\phi}_z\hat{V} + \hat{\phi}_{zz}\hat{\eta}_V) = \frac{\partial}{\partial\theta}\left\{\frac{\hat{\eta}_V\hat{V}}{(1 + \hat{\eta}_\theta^2)^{3/2}}\right\}, \\
\rightarrow \mathcal{L}_2(\hat{\phi}_V, \hat{\eta}_V) &= \hat{\phi}_\theta.
\end{align}

(B.15a) \quad (B.15b)
Therefore,
\[ \phi^{(1)} = -\lambda_1 \bar{\phi}_V, \quad \eta^{(1)} = -\lambda_1 \bar{\eta}_V. \]  

(B.16)

Successively, \( \phi^{(2)} \) and \( \eta^{(2)} \) satisfy the following forced boundary value problem:
\[
\Delta \phi^{(2)} = \bar{\phi}_\theta \quad (-D < z < \bar{\eta}), \quad \phi^{(2)} = 0 \quad (z = -D),
\]

with a linearized boundary condition at \( z = \bar{\eta} \),
\[
\mathcal{L}_1(\phi^{(2)}, \eta^{(2)}) = -\lambda_1 \bar{\eta}^{(1)} - \lambda_2 \bar{\eta}^{(0)} = -\lambda_2 \bar{\eta}_\theta + \lambda_1^2 \bar{\eta}_V = r_1, \quad \mathcal{L}_2(\phi^{(2)}, \eta^{(2)}) = -\lambda_1 \bar{\phi}^{(1)} - \lambda_2 \bar{\phi}^{(0)} - \frac{\bar{\eta}_\theta}{(1 + \bar{\eta}_\theta^{1/2})} = -\lambda_2 \bar{\phi}_\theta + \lambda_1^2 \bar{\phi}_V - \frac{\bar{\eta}_\theta}{(1 + \bar{\eta}_\theta^{1/2})} = r_2. \]

(B.18a)

(B.18b)

For this problem to have a solution that does not grow exponentially as \( \theta \to \pm \infty \), the forcing terms must satisfy a solvability condition which can be deduced by considering the corresponding adjoining boundary value problem.

### B.3 Adjoint boundary value problem

Since the Laplacian is self-adjoint, the adjoint boundary value problem to (B.17)–(B.18) is defined as follows:
\[
\Delta \psi^{(2)} = 0 \quad (-D < z < \bar{\eta}), \quad \psi^{(2)} = 0 \quad (z = -D),
\]

with the adjoint boundary conditions at \( z = \bar{\eta} \)
\[
\mathcal{L}_1^+(\psi(\theta, \bar{\eta}(\theta)), \zeta(\theta)) = 0, \quad \mathcal{L}_2^+(\psi(\theta, \bar{\eta}(\theta)), \zeta(\theta)) = 0.
\]

(B.20a)

(B.20b)

such that
\[
I = \int_{-\infty}^{\infty} d\theta \{ \psi(\theta, \bar{\eta}(\theta)) \mathcal{L}_1(\bar{\phi}, \bar{\eta}) + \zeta(\theta) \mathcal{L}_2(\bar{\phi}, \bar{\eta}) \} = 0.
\]

(B.21)
In order to find the adjoining boundary condition (B.20), we need to rephrase each term of (B.21) by using Green’s identity and integration by part. From the following expressions,

\[
\int_{-\infty}^{\infty} \psi \left( -\frac{\partial}{\partial z} + \tilde{\eta}_\theta \frac{\partial}{\partial \theta} \right) \phi \, d\theta = \int_{-\infty}^{\infty} \phi \left( -\frac{\partial}{\partial z} + \tilde{\eta}_\theta \frac{\partial}{\partial \theta} \right) \psi \, d\theta,
\]

\[
\int_{-\infty}^{\infty} \left\{ \frac{d\tilde{\phi}_\theta}{d\theta} + (-V + \tilde{\phi}_\theta) \frac{d}{d\theta} \right\} \tilde{\eta} \, d\theta = -\int_{-\infty}^{\infty} \left( -V + \tilde{\phi}_\theta \right) \tilde{\eta} \frac{d\psi}{d\theta} \, d\theta,
\]

\[
\int_{-\infty}^{\infty} \xi \frac{d}{d\theta} \left\{ \frac{\tilde{\eta}_\theta}{(1 + \tilde{\eta}^2_\theta)^{3/2}} \right\} \, d\theta = \int_{-\infty}^{\infty} \tilde{\eta} \frac{d}{d\theta} \left\{ \frac{\xi}{(1 + \tilde{\eta}^2_\theta)^{3/2}} \right\} \, d\theta,
\]

terms are collected to get an equivalent expression for \( I \):

\[
0 = I = \int_{-\infty}^{\infty} \phi \left( -\frac{\partial}{\partial z} + \tilde{\eta}_\theta \frac{\partial}{\partial \theta} \right) \psi \, d\theta
- \int_{-\infty}^{\infty} \left( -V + \tilde{\phi}_\theta \right) \tilde{\eta} \frac{d\psi}{d\theta} \, d\theta - \int_{-\infty}^{\infty} \left( -V + \tilde{\phi}_\theta \right) \tilde{\eta} \frac{d\phi}{d\theta} \frac{d\xi}{d\theta} \, d\theta - \int_{-\infty}^{\infty} \phi \frac{d\tilde{\phi}_\theta}{d\theta} \, d\theta
- \int_{-\infty}^{\infty} \xi \frac{d}{d\theta} \left\{ \frac{\tilde{\eta}_\theta}{(1 + \tilde{\eta}^2_\theta)^{3/2}} \right\} \, d\theta.
\]

It is followed by

\[
\int_{-\infty}^{\infty} \frac{d}{d\theta} \phi \frac{d\psi}{d\theta} \, d\theta
= -\int_{-\infty}^{\infty} \phi \left\{ \left( -\frac{\partial}{\partial z} + \tilde{\eta}_\theta \frac{\partial}{\partial \theta} \right) \psi - \left\{ \frac{d\tilde{\phi}_\theta}{d\theta} + (-V + \tilde{\phi}_\theta) \frac{d}{d\theta} \right\} \xi \right\} \, d\theta
- \int_{-\infty}^{\infty} \tilde{\eta} \left\{ (-V + \tilde{\phi}_\theta) \frac{d\psi}{d\theta} - \left( (-V + \tilde{\phi}_\theta) \tilde{\phi}_\theta z + \tilde{\phi}_z \tilde{\phi}_{zz} + 1 \right) \tilde{\eta} \frac{d\theta}{d\theta} \right\} \, d\theta.
\]

Therefore, the adjoint boundary conditions on the free surface \( z = \tilde{\eta} \) should be:

\[
\mathcal{L}_1^+ (\psi, \xi) = 0 : \left( -\frac{\partial}{\partial z} + \tilde{\eta}_\theta \frac{\partial}{\partial \theta} \right) \psi - \left\{ \frac{d\tilde{\phi}_\theta}{d\theta} + (-V + \tilde{\phi}_\theta) \frac{d}{d\theta} \right\} \xi = 0,
\]

\[
\mathcal{L}_2^+ (\psi, \xi) = 0 : (-V + \tilde{\phi}_\theta) \frac{d\psi}{d\theta} - \left( (-V + \tilde{\phi}_\theta) \tilde{\phi}_\theta z + \tilde{\phi}_z \tilde{\phi}_{zz} + 1 \right) \xi + \frac{d}{d\theta} \left\{ \frac{\xi}{(1 + \tilde{\eta}^2_\theta)^{3/2}} \right\} = 0,
\]

such that

\[
I \equiv \int_{-\infty}^{\infty} d\theta \left\{ \psi (\theta, \tilde{\eta}(\theta)) \mathcal{L}_1 (\phi, \tilde{\eta}) + \xi (\theta) \mathcal{L}_2 (\phi, \tilde{\eta}) \right\} = \int_{-\infty}^{\infty} \frac{d}{d\theta} \left( \phi \Delta \psi - \psi \Delta \phi \right) = 0.
\]
Comparing (B.25) with (B.5) and (B.9), we easily deduce that

$$ \mathcal{L}_{1,2}^{\pm}(\psi, \zeta) = \mathcal{L}_{1,2}(-\psi, -\zeta). \quad (B.27) $$

From (B.10) and (B.12), it is followed that the solution of the adjoining boundary value problem (B.19) and (B.25) is given by

$$ \hat{\psi} = \hat{\phi}_\theta, \quad \hat{\zeta} = -\hat{\eta}_\theta. \quad (B.28) $$

Substituting (B.17), (B.19) and (B.28) into (B.26), we obtain the solvability condition (3.35) as follows:

**Solvability condition**

$$ \int_{-\infty}^{\infty} d\theta \int_{-D}^{\eta} \tilde{\phi}_\theta R \, dz + \int_{-\infty}^{\infty} d\theta \left( r_1 \tilde{\phi}_\theta - r_2 \tilde{\eta}_\theta \right) \bigg|_{z=\eta} = 0. \quad (B.29) $$

Taking into account (B.18), the above solvability condition yields the following non-trivial result for the leading-order instability growth rate $\lambda_1$:

$$ \lambda_1^2 \int_{-\infty}^{\infty} d\theta \left( \tilde{\eta}_V \tilde{\phi}_\theta - \tilde{\eta}_\theta \tilde{\phi}_V \right) \bigg|_{z=\eta} + \int_{-\infty}^{\infty} d\theta \int_{-D}^{\eta} \tilde{\phi}_\theta^2 \, dz + \int_{-\infty}^{\infty} \frac{\tilde{\eta}_\theta^2}{(1 + \tilde{\eta}_\theta^2)^{1/2}} \, d\theta = 0. \quad (B.30) $$

Finally, we want to verify (3.39) by proving (3.18). For pure gravity solitary waves, it is known from Longuet-Higgins (1974) that

$$ \frac{1}{V} \frac{\partial E}{\partial V} = \frac{\partial T}{\partial V} - D \frac{\partial \mathcal{G}}{\partial V} = \int_{-\infty}^{\infty} d\theta \left( \tilde{\eta}_V \tilde{\phi}_\theta - \tilde{\eta}_\theta \tilde{\phi}_V \right) \bigg|_{z=\eta}. \quad (B.31) $$

However, our consideration includes surface tension in the dynamic boundary condition at $z = \tilde{\eta}$, and it is shown that the same expression as the above is still valid. The solitary wave energy $\mathcal{E}$ is defined as follows:

$$ \mathcal{E} = T + \mathcal{G} + \mathcal{V}_T, \quad (B.32) $$

where

$$ T = \frac{1}{2} \int_{-\infty}^{\infty} d\theta \int_{-D}^{\eta} \left( \tilde{\phi}_\theta^2 + \tilde{\phi}_\theta^2 \right) \, dz \quad (B.33) $$

denotes the kinetic energy,

$$ \mathcal{G} = \frac{1}{2} \int_{-\infty}^{\infty} \tilde{\eta}_\theta^2 \, d\theta \quad (B.34) $$

de the gravitational potential energy and

$$ \mathcal{V}_T = \int_{-\infty}^{\infty} \{ (1 + \tilde{\eta}_\theta^2)^{1/2} - 1 \} \, d\theta \quad (B.35) $$

denotes the potential energy due to surface tension.
By differentiating $T$ with respect to $V$, we obtain

$$
\frac{\partial T}{\partial V} = \frac{1}{2} \int_{-\infty}^{\infty} \tilde{\eta}_V (\tilde{\phi}_\theta^2 + \tilde{\phi}_z^2) |_{\tilde{\eta}} d\theta + \int_{-\infty}^{\infty} d\theta \int_{-D}^{\tilde{\eta}} (\tilde{\phi}_\theta \tilde{\phi}_\theta V + \tilde{\phi}_z \tilde{\phi}_z V) dz. \tag{B.36}
$$

From the dynamic boundary condition at $z = \tilde{\eta}$,

$$
\frac{1}{2} \tilde{\eta}_V (\tilde{\phi}_\theta^2 + \tilde{\phi}_z^2) |_{\tilde{\eta}} = \frac{\tilde{\eta}_\theta}{(1 + \eta_\theta)^{3/2}} + V \tilde{\phi}_\theta - \tilde{\eta}, \tag{B.37}
$$

we obtain

$$
\frac{\partial T}{\partial V} = V \int_{-\infty}^{\infty} \tilde{\eta}_V \tilde{\phi}_\theta d\theta - \int_{-\infty}^{\infty} \tilde{\eta}_V d\theta + \int_{-\infty}^{\infty} \tilde{\eta}_V \frac{\tilde{\eta}_\theta}{(1 + \eta_\theta)^{3/2}} d\theta + \int_{-\infty}^{\infty} d\theta \int_{-D}^{\tilde{\eta}} (\tilde{\phi}_\theta \tilde{\phi}_\theta V + \tilde{\phi}_z \tilde{\phi}_z V) dz. \tag{B.38}
$$

Using Green’s identity,

$$
\int_{-\infty}^{\infty} d\theta \int_{-D}^{\tilde{\eta}} (\tilde{\phi}_\theta \tilde{\phi}_\theta V + \tilde{\phi}_z \tilde{\phi}_z V) dz = \int_{-\infty}^{\infty} d\theta \tilde{\phi}_V (\tilde{\phi}_z - \tilde{\phi}_\theta \tilde{\eta}) |_{\tilde{\eta}}. \tag{B.39}
$$

From the kinematic boundary condition $\tilde{\phi}_z = -V \tilde{\eta}_\theta + \tilde{\phi}_\theta \tilde{\eta}_\theta$ at $z = \tilde{\eta}$,

$$
\int_{-\infty}^{\infty} d\theta \int_{-D}^{\tilde{\eta}} (\tilde{\phi}_\theta \tilde{\phi}_\theta V + \tilde{\phi}_z \tilde{\phi}_z V) dz = -V \int_{-\infty}^{\infty} d\theta \tilde{\phi}_V \tilde{\eta}_\theta. \tag{B.40}
$$

Differentiating $\mathcal{V}_G$ with respect to $V$,

$$
\frac{\partial \mathcal{V}_G}{\partial V} = \int_{-\infty}^{\infty} \tilde{\eta}_V \tilde{\eta} d\theta, \tag{B.41}
$$

and, differentiating $\mathcal{V}_T$ with respect to $V$,

$$
\frac{\partial \mathcal{V}_T}{\partial V} = \int_{-\infty}^{\infty} \tilde{\eta}_V \frac{\tilde{\eta}_\theta}{(1 + \eta_\theta)^{1/2}} d\theta - \int_{-\infty}^{\infty} \tilde{\eta}_V \frac{\tilde{\eta}_\theta}{(1 + \eta_\theta)^{3/2}}. \tag{B.42}
$$

Therefore, from (B.38)–(B.42), we obtain

$$
\frac{\partial \mathcal{E}}{\partial V} = \frac{\partial T}{\partial V} + \frac{\partial \mathcal{V}_G}{\partial V} + \frac{\partial \mathcal{V}_T}{\partial V} = V \int_{-\infty}^{\infty} (\tilde{\eta}_V \tilde{\phi}_\theta - \tilde{\eta}_\theta \tilde{\phi}_\theta V) |_{z = \eta} d\theta. \tag{B.43}
$$
Appendix C

Spectral and pseudospectral methods for the computation of lumps

Here we discuss the details of spectrally accurate numerical implementations used for both the steady and unsteady computations of lumps solutions to the elliptic-elliptic DS system in Chapter 2 and to the 2-DB equation in Chapter 4. The spatial variables \((x, y)\) used here correspond to \((\xi, z)\) in Chapter 2 and \((X, Y)\) in Chapter 4.

C.1 Computation of steady finite-amplitude lumps by using the spectral collocation method

The numerical method for the steady computation of finite-amplitude lumps shows an outstanding computational performance. Above all, it does not lose any far-field information for slowly decaying solution profiles at all whereas typical domain truncation implementations result in huge amount of error inherently. Also, it features exponential convergence as the grid number increases as is reported in Chapter 2 and 3.

C.1.1 Pseudospectral differentiation matrix via the stretching domain transformation with an algebraic mapping

Typically, the profiles of finite-amplitude lumps are such that, while most of the interesting behaviour is localized near the centre, they require a rather large computational domain owing to the algebraic decay at infinity. For this reason, it was decided to use a domain transformation \((x, y) \rightarrow (p, q)\) that maps the \((x, y)\)-plane \((-\infty < x < \infty, -\infty < y < \infty)\) into a square
Figure C-1: An example of mesh discretization (A.3) for (a) the original domain in \((x, y)\) and (b) the computational domain in \((p, q)\) via the algebraic stretching transform defined in (A.2). The stretching coefficient \(L = 1\), \(N = 32\) and \(M = 16\).

\((-1 \leq p < 1, -1 \leq q \leq 1):\)

\[
x = \frac{Lp}{(1 - p^2)^{1/2}}, \quad y = \frac{Lq}{(1 - q^2)^{1/2}},
\]

\(L\) being a scaling factor that was set to \(L = 15\). Thus, differentiation with respect to \(\xi\) and \(\zeta\) are expressed in terms of \(p\) and \(q\) as follows:

\[
\frac{\partial}{\partial x} = \frac{L}{(1 - p^2)^{3/2}} \frac{\partial}{\partial p}, \quad \frac{\partial}{\partial y} = \frac{L}{(1 - q^2)^{3/2}} \frac{\partial}{\partial q}.
\]

(C.2)

In addition, an uneven collocation mesh, that is clustered near the centre of the wave profile and is relatively coarse in the far field, was used:

\[-1 = p_{2N} < p_{2N-1} < \ldots < p_1 < p_0 = 1, \quad -1 = q_{2M} < q_{2M-1} < \ldots < q_1 < q_0 = 1,\]

(C.3)

where \(p_n = \cos \theta_{1,n}\) and \(q_m = \cos \theta_{2,m}\) with \(\theta_{1,n} = n\pi/2N\) and \(\theta_{2,m} = m\pi/2M\) \((0 \leq n \leq 2N, 0 \leq m \leq 2M)\). Under this discretization, the original domain in \((x, y)\) and the computational domain in \((p, q)\) are illustrated in figure A-1.

As basis functions, we choose

\[
f_n(p) = \frac{1}{a_n} \prod_{k=0}^{2N} (p - p_k), \quad a_n = \prod_{k=0}^{2N} (p_n - p_k),
\]

(C.4)

which satisfy \(f_n(p_k) = \delta_{nk}\).
The derivative \( \partial / \partial p \) then can be represented as a \((2N + 1) \times (2N + 1)\) matrix

\[
D_{ij} = \frac{1}{a_j} \prod_{k=0}^{2N} (p_i - p_k) = \frac{a_i}{a_j(p_i - p_j)} \quad (i \neq j), \quad D_{jj} = \sum_{k=0, k \neq j}^{2N} (p_j - p_k)^{-1}, \quad (C.5)
\]

and likewise \( \partial / \partial q \) corresponds to a similar \((2M + 1) \times (2M + 1)\) differentiation matrix.

**C.1.2 Higher order differentiation matrices**

The \( n \)-th derivative with respect to the computational variables \((p, q)\) is simply defined by taking the \( n \)-th power of the differentiation matrix \( D \). The \( n \)-th derivative with respect to the original variables \((x, y)\) are obtained by the chain rule:

\[
\frac{\partial^n p}{\partial x^n} = \left( \frac{dp}{dx} \right)^n \frac{\partial^2 p}{\partial p^2}, \quad (C.6)
\]

\[
\frac{\partial^3 p}{\partial x^3} = \frac{d^3 p}{dx^3} \frac{\partial}{\partial p} + 3 \frac{d^2 p}{dx^2} \frac{dp}{dx} \frac{\partial^2 p}{\partial p^2} + \left( \frac{dp}{dx} \right)^3 \frac{\partial^3 p}{\partial p^3}, \quad (C.7)
\]

\[
\frac{\partial^4 p}{\partial x^4} = \frac{d^4 p}{dx^4} \frac{\partial}{\partial p} + 4 \left( \frac{d^3 p}{dx^3} \frac{dp}{dx} + 3 \left( \frac{d^2 p}{dx^2} \right)^2 \right) \frac{\partial^2 p}{\partial p^2} + 6 \left( \frac{dp}{dx} \right)^2 \frac{d^2 p}{dx^2} \frac{\partial^3 p}{\partial p^3} + \left( \frac{dp}{dx} \right)^4 \frac{\partial^4 p}{\partial p^4}, \quad (C.8)
\]

where

\[
\frac{dp}{dx} = \frac{1}{L} (1 - p^2)^{3/2}, \quad \frac{d^2 p}{dx^2} = \frac{-3}{L^2} (1 - p^2)^2, \quad (C.9)
\]

\[
\frac{d^3 p}{dx^3} = \frac{3}{L^3} (5p^2 - 1)^2 (1 - p^2)^{5/2}, \quad \frac{d^4 p}{dx^4} = \frac{-60}{L^4} (1 - p^2)^3 + \frac{105}{L^4} p(1 - p^2)^4. \quad (C.10)
\]

Higher order differentiations with respect to \( y \) has the analogous expressions in terms of \( q \).

**C.1.3 Reduced differentiation matrices for symmetric lumps**

Since lump solutions decay to zero at infinity, there are no end-point contributions in any kind of matrix operations. Thus, the first and the last rows and columns can be dropped off from the differentiation matrices defined above.

In addition, owing to the symmetry properties of lumps \( \eta(\xi, z) = \eta(\pm \xi, \pm z) \), it is enough to work on only the first-quadrant of the computational domain in order to obtain the full wave profile. It implies that we can discard the latter half rows of the differentiation matrices and fold the remaining upper half rows in half to reduce the size of the matrices to \((N \times N)\) as follows:

\[
D_{ij} = D_{i+1,j+1} + D_{i+1,2N+1-j} \quad (1 \leq i, j \leq N) \quad (C.11)
\]

Similarly, we get reduced \((M \times M)\) \( q \)-differentiation matrices. The reduced discretization for
the first quadrant of the domain is shown in figure A-2. This reduced discretization exploiting the symmetry of lumps in turn cuts to 1/16 the storage memory required for calculating the Jacobian matrix in Newton’s iterations.

C.1.4 Spectrally accurate numerical implementation of the Hilbert transform

We now describe a spectrally accurate numerical implementation of a nonlocal singular integral, the Hilbert transform $\mathcal{H}\{\eta\}$, over the uneven collocation mesh defined above.

For an even grid of mesh, the Fourier transform method by Weideman and James () is regarded as the most reliable implementation because it is based on the spectral representation of the Hilbert transform:

$$\mathcal{H}\{\eta\}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta(y)}{y-x} dy = \mathcal{F}^{-1}\left\{ \frac{1}{|l|} \mathcal{F}\{\eta\}(l) \right\},$$

where $\mathcal{F}\{\eta\}(l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(x)e^{-i \alpha x} dx$. This expression is still very useful when we use the Fast Fourier Transform (FFT) method for unsteady numerical simulations.

For the steady computation of lumps over uneven grids, however, we newly introduce a spectrally accurate quadrature method, which is rather straightforward. The goal is to construct a linear matrix operator for the corresponding discretized Hilbert transform.

In general, the quadrature method takes an approximated sum $I \approx \sum_{j=1}^{2N-1} w_j \tilde{f}(x_j)$ in order to compute an integral $I \equiv \int_{a}^{b} f(x) dx$, where a positive weight function $w(x)$ is determined by a particular set of collocation points $x_j$ and a representative value $\tilde{f}(x_j)$ of the integrand $f$ is usually taken as $f(x_j)$ itself if $f$ is regular. For the clustered grids of our interest here, the
corresponding weight function is evaluated as follows ([13], 2001).

\[ w(x_j) = w_j = \pi \frac{x_j^2 + L^2}{2NL}, \quad x_j = L \cot \frac{j\pi}{2N} \quad (1 \leq j \leq 2N - 1). \]  

(C.13)

For each \( x_n \), the representative value of the integrand \( \tilde{f}(x_j; x_n) = f(x_j; x_n) + g(x_j; x_n) = (\eta(x_j) - \eta(x_n))/(x_j - x_n) \) when \( j \neq n \) because \( g(x_j; x_n) \) has zero principal value of the integral. For \( j = n \), the integrand becomes singular, but \( \tilde{f}(x_j; x_n) \) can be well approximated by the first derivative of \( \eta(x) \) at \( (x_n) \). Therefore, the numerical Hilbert transform is evaluated as

\[ \mathcal{H}\{\eta\}(x_j) = \frac{1}{\pi} \left( w_j \eta(x_j) + \sum_{i \neq j}^{2N-1} w_i \frac{\eta(x_i) - \eta(x_j)}{x_i - x_j} \right) = \frac{1}{\pi} \left( w_j \eta(x_j) + \sum_{i \neq j}^{N} w_i \frac{\eta(x_i)}{x_i - x_j} \right) \]  

(C.14)

In the matrix form,

\[ \mathcal{H} = \frac{1}{\pi} \left( T \cdot \text{diag}(w) + \text{diag}(w) \cdot D_x \right), \quad T_{ij} = \frac{1}{x_i - x_j} \quad (1 \leq i, j \leq 2N - 1). \]  

(C.15)

For an even function, there are an analogous expression to the reduced differentiation matrix, exploiting symmetry:

\[ \mathcal{H}_N\{\eta\} = \frac{1}{\pi} \left( T_N \cdot \text{diag}(\bar{w}) + \text{diag}(\bar{w}) \cdot D_N \right), \]  

(C.16)

\[ T_{N,ij} = \frac{2x_j}{x_j^2 - x_i^2} \quad (1 \leq i, j \leq N), \quad \bar{w}_i = w_i + w_{2N-i} \quad (1 \leq i \leq N). \]  

(C.17)

C.1.5 Numerical continuation by using the direct Jacobian method

The main idea to perform numerical continuation is to solve a nonlinear system from an already known initial guess by using the Newton-Raphson's iterations.

In order to use the Newton-Raphson’s method, we need compute the Jacobian matrix of the given discretized nonlinear algebraic equations. Here we use the direct Jacobian method, in which the Jacobian of the discretized system is exactly derived in terms of another algebraic expression of the original variables. Two examples are presented to find a symmetric solution branch of the nonlinear PDE system, which have been discussed in the main chapters.

In the following subsections, \( D_N \) is the \( N \times N \) differentiation matrix in \( x \), \( D_M \) is the \( N \times N \) differentiation matrix in \( y \), \( H_N \) is the \( N \times N \) discrete Hilbert transform and \( I_N \) is the \( N \times N \) identity matrix. It is useful to define two kinds of matrix multiplications. \( A \cdot B \) is a usual matrix multiplication where \( A \) is an \( N \times M \) matrix and \( B \) is an \( M \times P \) matrix. \( A \cdot B \) is a term-by-term multiplication of two matrices of the same size. \( \text{diag} \) is the operation which generates a diagonal matrix from a vector.

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The DS system

The discretized DS system, discussed in Chapter 2 and Chapter 4, consists of two nonlinear functionals as follows:

\[
F_{DS,1} = (-I_N + \beta D_N^2) \cdot A + \gamma A \cdot D_M^2 - \delta A \cdot \zeta A_0 \cdot A = 0, \tag{C.18a}
\]

\[
F_{DS,2} = Q^2 D_N^2 \cdot A_0 + A_0 \cdot D_M^2 - \lambda D_N^2 \cdot A = 0, \tag{C.18b}
\]

where \(A\) and \(A_0\) are \((N \times M)\) matrices. Then, the Jacobian \(J_{DS}\) is an \(NM \times NM\) matrix:

\[
J_{DS} = \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix} \tag{C.19}
\]

\[
J_{11} = I_M \otimes (-I_N + \beta D_N^2) + \gamma D_M^2 \otimes I_N - \text{diag}(3\delta A \cdot \zeta A_0),
J_{12} = -\zeta \text{ diag}(A), \tag{C.20}
\]

\[
J_{21} = -2\lambda(I_M \otimes D_N^2) \cdot \text{diag}(A), \quad J_{22} = Q^2 I_M \otimes D_N^2 + D_M^2 \otimes I_N. \tag{C.21}
\]

\(\beta, \gamma, \delta, \zeta, \lambda\) and \(Q\) are constants, so they are all considered as continuation parameters. In Chapter 2 and 4, \(\zeta\) is the only continuation parameter while the others are fixed. The solutions are updated by the routine Newton-Raphson’s iteration:

\[
\begin{bmatrix}
A \\
A_0
\end{bmatrix} \rightarrow
\begin{bmatrix}
A \\
A_0
\end{bmatrix} - J^{-1}_{DS} \cdot \begin{bmatrix}
\bar{F}_{DS,1} \\
\bar{F}_{DS,2}
\end{bmatrix}. \tag{C.22}
\]

The 2-DB equation

The discretized 2-DB equation, discussed in Chapter 4, is as follows:

\[
F_{2-DB} = (-D_N^2 + 2\gamma D_N^3 \cdot \mathcal{H}_N + D_N^4) \cdot \eta + D_N^2 \cdot \eta^2 - \eta \cdot D_M^2 = 0. \tag{C.23}
\]

Then, the corresponding \(N \times N\) Jacobian matrix is obtained as

\[
J_{2-DB} = I_M \otimes (-D_N^2 + 2\gamma D_N^3 \cdot \mathcal{H}_N + D_N^4) + 2(I_2 \otimes D_N^2) \cdot \text{diag}(\eta) - D_M^2 \otimes I_N. \tag{C.24}
\]

The solution is updated by varying the continuation parameter \(\gamma\).

\[
\eta \rightarrow \eta - J_{2-DB}^{-1} \cdot \bar{F}_{2-DB}. \tag{C.25}
\]

One important issue here is whether those Jacobians are nonsingular or not. If we investigate the ranks of the Jacobians, both \(J_{Ben}\) and \(J_{DS}\) have exactly one deficient ranks to be nonsingular. This is because those equations allows one-parameter family of multiple solutions if we do not impose any constraints. From the original equations, we can easily figure out that
the Benajmin equation requires the following constraint.

$$\int_{-\infty}^{\infty} \eta(x, z) dx = 0 \quad \Rightarrow \quad w^T \cdot \eta = 0. \quad (C.26)$$

This issue has been discussed in Katsis and Akylas on the KP-I equation. They used an integral equation to avoid this degeneracy, which is consistent to the way we deal with here. Ablowitz also provided the detailed conditions to guarantee the uniqueness of the solution, according to which the constraint we use can be more generalized.

For the DS equations, a similar constraint is imposed.

$$\int_{-\infty}^{\infty} A_0(x, z) dx = 0 \quad \Rightarrow \quad w^T \cdot A_0 = 0. \quad (C.27)$$

Then, the modified Jacobians with the integral constraints become nonsingular.

### C.2 Fourier spectral method for unsteady numerical simulation

We now discuss the unsteady computation for the initial value problem of the 2-DB Benjamin equation in Chapter 4. The Fourier spectral method is used because Fast Fourier transform (FFT) can be used.

#### C.2.1 Spectrally accurate interpolation via the Fourier transform

In order to apply the Fourier spectral method, we need to interpolate wave profiles over an even grid. We introduce a spectrally accurate interpolation using the Fourier transform. We define $f(u, v) = f(e^{i\theta_1}, e^{i\theta_2}) = \eta(\cot \theta_1, \cot \theta_2) = \eta(x, z)$, considering an even extension of $\eta$. Then,

$$\mathcal{F}\{\eta(\cot \theta_1, \cot \theta_2)\}(l, m) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i\theta_1}, e^{i\theta_2}) e^{-i(l\theta_1 + m\theta_2)} d\theta_1 d\theta_2. \quad (C.28)$$

Once we get the Fourier coefficients $\mathcal{F}\{\eta(\cot \theta_1, \cot \theta_2)\}(l, m)$, it is automatic to evaluate $\eta$ at any desired coordinate in terms of the Fourier series.

$$\eta(x, z) = -\frac{1}{\alpha^2} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{F}\{\eta(\cot \theta_1, \cot \theta_2)\}(l, m) e^{i(l\theta_1 + m\theta_2)} \quad (C.29)$$

$$= -\frac{1}{\alpha^2} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{F}\{\eta(\cot \theta_1, \cot \theta_2)\}(l, m) \cos^l \theta_1 \cos^m \theta_2,$$

where

$$\theta_1 = \cot^{-1} \left( \frac{x}{L} \right), \quad \theta_2 = \cot^{-1} \left( \frac{z}{L} \right). \quad (C.30)$$

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In order to get an oblique wave discussed in Chapter 4, the following substitution is used.

\[
\theta_1 = \cot^{-1} \left( \frac{\alpha x - \frac{1}{2} \delta \sin \phi}{L} \right), \quad \theta_2 = \cot^{-1} \left( \frac{\delta}{L} \right), \quad \alpha = \sqrt{\cos \phi - \frac{1}{4} \sin^2 \phi}, \quad (C.31)
\]

for any oblique angle \( \phi \), as before.

### C.2.2 Numerical stability analysis for the third-order Crank–Nicolson scheme

We use an implicit method for time integration to enhance numerical stability in unsteady computation of partial differential equations, it only involves with a simple matrix multiplication, while the finite difference counterpart usually requires a lot more expensive linear solver inversion every time step.

Once we have \( \eta(x, z) \) over uniform grids, then the Discrete Fourier transform is applied. We concern about the time integration of \( \hat{\eta}(l, m) = \mathcal{F}\{\eta\}(l, m) \), and the third-order Crank–Nicolson method is used:

\[
\frac{\hat{\eta}^{n+1} - \hat{\eta}^{n-1}}{2\Delta t} + i\mathcal{F}\{(\mathcal{F}^{-1}\{\eta^n\})^2\} - \frac{i}{3}\omega(\hat{\eta}^{n+1} + \hat{\eta}^n + \hat{\eta}^{n-1}) = 0, \quad (C.32)
\]

where \( \omega = -2\gamma l|l| + l^3 + m^2/l \) is the linear dispersion relation. The characteristic polynomial of is given by

\[
P(z) = b_1 z^2 - b_2 z + b_1, \quad (C.33)
\]

\[
b_1 = 1 - i\frac{2\Delta t\omega}{3}, \quad b_2 = i2\Delta t(l|\eta|_{\text{max}} - \frac{\omega}{3}). \quad (C.34)
\]

The recurrence formula based on \( P(z) \) is derived as

\[
\hat{\eta}^n = \frac{1}{\beta_1 - \beta_2} \left\{ \beta_1^2(\hat{\eta}^1 - \beta_2\hat{\eta}^0) - \beta_2^2(\hat{\eta}^1 - \beta_1\hat{\eta}^0) \right\}, \quad (C.35)
\]

where \( \beta_{1,2} = (-b_2 \pm \sqrt{b_2^2 + 4|b_1|^2})/2b_1 \) are two distinct zeros of the characteristic polynomial \( P(z) \). In order to avoid instability, it is required that both \( |\beta_1| \leq 1 \) and \( |\beta_2| \leq 1 \). Since \( |\beta_1\beta_2| = |b_1/b_1| = 1, |\beta_1| = |\beta_2| = 1 \) follow. Therefore, \( b_2^2 + 4|b_1|^2 \) should always be positive, such that \( \beta_1 = \beta_2 \).

\[
\frac{b_2^2 + 4|b_1|^2}{4} = -\Delta t^2(\delta|\eta|_{\text{max}} - \frac{\omega}{3})^2 + (1 + \frac{4}{9}\omega^2\Delta t^2) \quad (C.36)
\]

\[
= \Delta t^2 \left\{ \frac{1}{3}(\omega + l|\eta|_{\text{max}})^2 - \frac{4}{3}l^2|\eta|_{\text{max}}^2 \right\} + 1 > 0. \quad (C.37)
\]
A sufficient condition to satisfy the above inequality is

\[ \Delta t < \frac{\sqrt{3}}{2} \frac{1}{l_{max} \eta_{max}} = \frac{\sqrt{3}}{2\pi} \frac{\Delta x}{|\eta|_{max}}. \]  

(C.38)
Bibliography


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