Universal Polynomials in Lambda rings and the K-theory of the infinite loop space tmf

by

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ABSTRACT

The algebraic structure of the K-theory of a topological space is described by the more general notion of a lambda ring. We show how computations in a lambda ring are facilitated by the use of Adams operations, which are ring homomorphisms, and apply this principle to understand the algebraic structure.

In a torsion free ring the Adams operations completely determine the lambda ring. This principle can be used to determine the K-theory of an infinite loop space functorially in terms of the K-theory of the corresponding spectrum. In particular we obtain a description of the K-theory of the infinite loop space tmf in terms of Katz's ring of divided congruences of modular forms. At primes greater than 3 we can also relate this to a Hecke algebra.

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Contents

1 Introduction

The main goal of this thesis is to describe the K-cohomology of the infinite loop space $\Omega_0^\infty tmf$ with p-adic coefficients. The algebraic structure of the K-theory of a space is what is called a λ-ring. This structure can be concretely described if we consider the example of the K-theory of a finite CW-complex X. In this case the group K^0X is obtained by considering the monoid of vector bundles on X under the operation of direct sum, and introducing additive inverses. Specifically we take the free abelian group generated by the monoid and quotient by terms of the form $E + F - E \oplus F$. A basic theorem, the splitting principle, says that any element of K^0X can be represented by (the pullback of) a sum of line bundles. There are various operations on K^0X corresponding to the various constructions that can be performed on vector bundles. For example it becomes a ring under the operation of tensor product. There is also the operation of taking the n^{th} exterior power. On a sum of line bundles $E = \sum_{i=1}^m E_i$ we have

$$\lambda^n E = E^{\wedge n} = \frac{1}{n!} \sum_{\sigma \in S_m} E_{\sigma(1)} \otimes \cdots \otimes E_{\sigma(n)}$$

which, upon identifying tensor products in different orders, is the n^{th} elementary symmetric function in the E_i (provided $n \leqslant m$).

The operations λ^n are not ring homomorphisms (at least for $n > 1$). The additive behaviour is relatively simple to describe but the multiplicative behaviour is more complicated. However the theory of symmetric functions allows us to describe the action of λ^n on a product. Specifically, it is a classical theorem that any symmetric polynomial can be written as a polynomial in the elementary symmetric functions. If we consider a product $E \otimes F = (\sum E_i) \otimes (\sum F_j)$ then the n^{th} exterior power will be symmetric under permutations of the E_i and under permutations of the F_j. It follows that $\lambda^n(E \otimes F)$ will be a polynomial in the $\lambda^k(E)$ and $\lambda^l(F)$ for $k, l \leqslant n$. This polynomial is independent of the number of summands, so long as there at least k, and is denoted P_n.

Similarly we can consider what happens if we take $\lambda^m(\lambda^n E)$. The result will be the m^{th} elementary symmetric function in the summands of the n^{th} elementary symmetric function. This will still be symmetric in the original summands and so is a polynomial in $\lambda^k E$ for $k \leqslant mn$. Again this is independent of the number of summands provided that there are at least mn, and the resulting polynomial is denoted $P_{m,n}$.

In section 2 we will calculate some of these polynomials P_n and $P_{m,n}$, giving formulae when it is reasonable to write them down, and giving an effective method to compute them in general. The key idea is that to understand these operations that are not ring homomorphisms, we need to relate them to operations which are. These operations are called the Adams operations, which can be defined topologically, but are more easily described algebraically. In our example of K^0X, the Adams operation ψ^k can be defined as the operation which sends a sum of line bundles to the sum of their k^{th} (tensorial) power. In the theory of symmetric functions, the Adams operations are the power sums. The power sums form a rational basis for the ring of symmetric functions. This gives us a way to compute the polynomials P_n and $P_{m,n}$, and is also the starting point for the theory that we will use to describe $K^*(\Omega_0^\infty tmf; \mathbb{Z}_p)$.

The formula for the elementary symmetric functions in terms of power sums gives us the statement that in a torsion free ring, the Adams operations determine the λ operations. The torsion free condition reflects the need to cancel coefficients. Bousfield generalized this theorem to describe $\mathbb{Z}/2$-graded λ-rings, which are an algebraic model of K^*X. The torsion free condition can be removed if we introduce p-local, or p-adic coefficients, at the expense of replacing the Adams operation ψ^p (which is a ring homomorphism) with a non-additive operation θ^p. The point is that once we introduce the coefficients, we only need to worry about p-torsion. The operation θ^p is related to ψ^p by the formula

$$\psi^p(x) = x^p + p\theta^p(x).$$

When we try to write λ^p in terms of the Adams operations, we have a formula $\psi^p = p\lambda^p + \cdots$ and so θ^p is exactly what we want to get by cancelling the p. If we have θ^p in advance then we do not need to worry about p-torsion.

Removing the torsion free requirement allows us to describe the p-adic K-theory of an infinite loop space in terms of the p-adic K-theory of the corresponding spectrum. The idea is that with p-adic coefficients, we have stable Adams operations ψ^k for k prime to p. Therefore if we introduce the operation θ^p freely we should obtain a λ-ring isomorphic to the K-theory of the infinite loop space. In section 3 we give a detailed explanation of this theory which we will apply in later sections.

We will apply this theory to the infinite loop space of the spectrum tmf. The property of tmf that we will use is that for an even periodic spectrum R, $R \wedge tmf$ can be identified with the stack of elliptic curves E together with an isomorphism of formal groups $\hat{E} \cong G_R$. In section 4 we use this property to describe the K-homology of tmf in terms of Katz's ring of divided congruences. We can relate K-cohomology to K-homology using the Pontryagin duality functor and so this gives us a description of the K-cohomology of tmf. We expand upon this description in section 5 relating the K-cohomology to the Hecke algebra corresponding to the ring of divided congruences. We then use Bousfield's theorem to obtain a description of the K-theory of $\Omega_0^\infty tmf$.

2 λ-rings

We begin with the definition of an abstract λ-ring.

Definition 2.1 *A λ-structure on a commutative unital ring R is a sequence of maps λ^n for $n \geqslant 0$ which satisfy the following conditions:*

1. $\lambda^0(r) = 1$ *for all* $r \in R$
2. λ^1 *is the identity map*
3. $\lambda^n(1) = 0$ *for* $n > 1$
4. $\lambda^n(r+s) = \sum_{k=0}^{n} \lambda^k(r)\lambda^{n-k}(s)$ *for all* $r, s \in R$
5. $\lambda^n(rs) = P_n(\lambda^1(r), ..., \lambda^n(r); \lambda^1(s), ..., \lambda^n(s))$ *for all* $r, s \in R$
6. $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), ..., \lambda^{mn}(r))$ *for all* $r \in R$

where P_n and $P_{m,n}$ are certain universal polynomials with integer coefficients.

We can think of any λ-ring as a power series ring as follows. We make $1 + R[[t]]$ into a λ-ring by defining addition to be multiplication of power series, multiplication by the formula

$$(1 + \sum_{n=1}^{\infty} r_n t^n)(1 + \sum_{n=1}^{\infty} s_n t^n) = 1 + \sum_{n=1}^{\infty} P_n(r_1, ..., r_n; s_1, ..., s_n)t^n$$

and the λ operations by the formula

$$\lambda^n(1 + \sum_{m=1}^{\infty} r_m t^m) = 1 + \sum_{m=1}^{\infty} P_{m,n}(r_1, ..., r_{mn})t^m).$$

The fact that this does define a λ-ring structure on $1 + R[[t]]$ is a consequence of certain identities among the universal polynomials (see [Tal69]). The conditions in the definition combine to show that the total λ-operation $\lambda_t(r) = \sum_{n=1}^{\infty} \lambda^n(r)t^n$ defines an injective λ-homomorphism from R into $1 + R[[t]]$. That is to say it is an identity preserving ring monomorphism which commutes with the λ-operations. In particular the injectivity of the map follows from condition 2.

2.1 The polynomials $P_{m,n}$ and P_n

The universal polynomials $P_{m,n}$ and P_n that occur in the definition of a λ-ring are somewhat mysterious. The polynomial $P_{m,n}(e_1, ..., e_{mn})$ can be described ([Tal69]) as the coefficient of t^m in the product $\prod_{1\leqslant i_1<\cdots<i_n\leqslant mn}(1+x_{i_1}\cdots x_{i_n}t)$, where e_i is the i^{th} elementary symmetric function in the indeterminates x_j. Similarly if f_i is the i^{th} elementary symmetric function in indeterminates y_j then $P_n(e_1, ..., e_n; f_1, ..., f_n)$ is the coefficient of t^n in $\prod_{i,j}(1 + x_i y_j t)$. These descriptions arise when we consider the indeterminates x_i, y_i as universal line bundles and the operation of taking exterior powers as λ-operations. Specifically we write an element of R as (a pullback of) a sum of line bundles and then the k^{th} exterior power of this element is the k^{th} elementary symmetric function in the line bundles in the sum. An interesting problem is to give a more explicit formula for these polynomials, directly in terms of the e_i and f_i. For example since λ^1 is the identity we find that $P_{1,n}(e_1, ..., e_n) = P_{n,1}(e_1, ..., e_n) = e_n$. In general however it is not true that $P_{m,n} = P_{n,m}$ as λ-operations do not commute. Below we give formulae in the next two simplest cases. In section 2.4 we will discuss how to compute the formula in general and also how to compute the polynomials P_n.

We begin with the following simple observation about binomial coefficients.

Lemma 2.1 *For all m and p we have*

$$\binom{m-1}{p} = \sum_{k=0}^{p}(-1)^k\binom{m}{p-k}.$$

Proof

We prove this by induction on p, noting that for $p = 0$ the formula reads $1 = 1$. Now for $p > 0$ the binomial recurrence formula and induction give us

$$\begin{aligned}\binom{m-1}{p} &= \binom{m}{p} - \binom{m-1}{p-1}\\ &= \binom{m}{p} - \left(\sum_{k=0}^{p-1}(-1)^k\binom{m}{p-1-k}\right)\\ &= \binom{m}{p} - \left(\sum_{k=1}^{p}(-1)^{k+1}\binom{m}{p-k}\right)\\ &= \sum_{k=0}^{p}(-1)^k\binom{m}{p-k}.\end{aligned}$$

□

Proposition 2.1 *With the convention that $e_0 = 1$ we have for all n that*

$$P_{2,n}(e_1, ..., e_{2n}) = \sum_{k=1}^{n} (-1)^{k+1} e_{n-k} e_{n+k}.$$

Proof

We can represent a symmetric polynomial by looking at single monomial of each type in the polynomial. For example $x_1^2 x_2 x_3$ and $x_1 x_3^2 x_4$ have the same type. In the product $e_{n-k}e_{n+k}$ and $P_{2,n}$ each monomial that occurs has the same type as $x_1^2 \cdots x_i^2 x_{i+1} \cdots x_{2n-i}$ for some i. To prove the proposition we will compare the coefficient of this expression that occurs on each side of the equation. The coefficient of $x_1^2 \cdots x_i^2 x_{i+1} \cdots x_{2n-i}$ in $e_{n-k}e_{n+k}$ is zero if $n - k < i$ and $\binom{2(n-i)}{n-k-i}$ otherwise. To see this we simply note that the terms in e_{n-k} which can (and do) give this monomial must contain the indeterminates x_1 through x_i but then the remaining $n - k - i$ indeterminates can be any of the $2(n - i)$ indeterminates x_{i+1} through x_{2n-i}. The polynomial $P_{2,n}$ is obtained by taking the sum of all possible products of 2 distinct terms $x_I = x_{i_1} \cdots x_{i_n}$ and $x_J = x_{j_1} \cdots x_{j_n}$. Each index set I such that $\{1, 2, ..., i\} \subseteq I$ has a complementary set J such that $x_I x_J = x_1^2 \cdots x_i^2 x_{i+1} \cdots x_{2n-i}$. There are $\binom{2(n-i)}{n-i}$ ways to choose a suitable index set I. However this would give us the term $x_I x_J$ twice (once when we choose I and again when we choose J) so the coefficient of $x_1^2 \cdots x_i^2 x_{i+1} \cdots x_{2n-i}$ in P_{2n} is $\frac{1}{2}\binom{2(n-i)}{n-i} = \binom{2(n-i)-1}{n-i-1}$. Therefore to prove the proposition it is sufficient to show that

$$\binom{2(n-i)-1}{n-i-1} = \sum_{k=1}^{n-i} (-1)^{k+1} \binom{2(n-i)}{n-k-i} = \sum_{k=0}^{n-i-1} (-1)^k \binom{2(n-i)}{n-k-i-1}.$$

This follows from the lemma with $m = 2(n - i)$ and $p = n - i - 1$. □

An alternative way to express the final calculation above is that

$$2\sum_{k=1}^{n-i} (-1)^{k+1} \binom{2(n-i)}{n-k-i} = (1-1)^{2(n-i)} + \binom{2(n-i)}{n-i}.$$

The point here is that extending the sum to include k from $-(n - i)$ to -1 gives the same terms repeated, then adding $k = 0$ gives exactly the binomial expansion of $(1 - 1)^{2(n-i)}$.

The case $m = 3$ is also sufficiently tractable for us to compute.

Proposition 2.2 *Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ be the cube root of unity. Then for all n*

$$P_{3,n}(e_1, ..., e_{3n}) = \sum_{l \leqslant j \leqslant k,\, l+j+k=3n} c_{j-l,k-l}^{n-l} e_l e_j e_k$$

where $c^t_{r,s} = 2\Re(\omega^r)$ *unless* $r = 0, t, 3t/2$. *If* $t > 0$ *then for* $r = 0, 3t/2$ *the value of* $c^t_{r,s}$ *is 1 while for* $r = t$ *it is* $2\Re(\omega^r) + (-1)^{t+1}$. *If* $t = r = 0$ *the coefficient is zero.*

An alternative description of the coefficient of $e_l e_j e_k$ is that it is 1 if two of l, j, k are equal and $2\Re(\omega^{l-j})$ otherwise, except when $j = n$ when we must add $(-1)^{n-l+1}$ if $l < n$, and have zero for e_n^3. We defer the proof of this proposition for the moment as a different approach will turn out to be much simpler, and indeed give us a more useable formula in general. This formula was discovered using similar reasoning to the $m = 2$ case and then a Maple program to compute the coefficients. A key point in the calculation is that we can obtain the n^{th} polynomial from the $(n-1)^{th}$ by replacing every monomial $e_l e_j e_k$ by $e_{l+1} e_{j+1} e_{k+1}$ and adding $\sum_{k=0}^{\lfloor \frac{3n}{2} \rfloor} c^n_{k,3n-k} e_k e_{3n-k}$. Indeed this description is easily seen to be equivalent to the one given above and the missing terms in $P_{3,1}$ give the exceptions in the general case. The calculation reduces to a triangular linear system for the last $3n/2$ coefficients which is actually reasonable to solve. In the appendix we give details of this calculation which involves many relations between binomial coefficients.

2.2 Adams operations

The difficulty in calculating the universal polynomials is in large part due to the fact that the λ-operations are not ring homomorphisms. Indeed they are are not even group homomorphisms. If we think of our indeterminates as line bundles and elements of R as formal sums of line bundles then instead of taking the elementary symmetric functions to get the λ operations we can take the power sums. This gives us the Adams operations ψ^n. To explain this let us recall some facts about symmetric functions from [Mac79]. With indeterminates $x_1, x_2, ...$ we call the expression

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

the n^{th} elementary symmetric function. Given a partition $\mu = (\mu_1, \mu_2, ...)$ we define $e_\mu = e_{\mu_1} e_{\mu_2} ...$. Then the e_μ as μ runs over all partitions form a $\mathbb{Z}$ basis for the ring of symmetric functions Λ. Equivalently $\Lambda = \mathbb{Z}[e_1, e_2, ...]$.

Similarly the expression

$$p_n = \sum x_i^n$$

is called the n^{th} power sum. With p_μ defined similarly to e_μ we obtain a $\mathbb{Q}$ basis for $\Lambda \otimes \mathbb{Q}$. The power sums and elementary symmetric functions are related as follows. For a partition μ let $m_i = m_i(\mu)$ be the number of times that i occurs in the partition. Let $l(\mu)$ be the length of the partition (that is the number of non-zero parts) and $|\mu|$ the sum of the parts. Define $z_\mu = \prod_{i\geqslant 1} i^{m_i} m_i!$ and $u_\mu = \frac{l(\mu)!}{\prod_{i\geqslant 1} m_i!}$. Then

$$e_n = \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_\mu^{-1} p_\mu$$

$$p_n = n \sum_{|\mu|=n} \frac{(-1)^{n-l(\mu)}}{l(\mu)} u_\mu e_\mu = \nu_n(e_1, ..., e_n).$$

If we think of the indeterminates x_i as universal line bundles and the exterior powers as λ operations then e_n corresponds to λ^n and $\psi^n(x_i) = x_i^n$. More precisely we have the following theorem. ([Mac79])

Theorem 2.1 *If R is a λ-ring then for every $x \in R$ there is a unique λ-homomorphism*

$$\phi_x : \Lambda \to R$$

which maps e_1 to x. This homomorphism maps e_n to $\lambda^n(x)$ and p_n to $\psi^n(x)$.

Proof
Once we define $\phi_x(e_1) = x$ then requiring ϕ_x to be a λ-homomorphism forces $\phi_x(e_n) = \phi_x(\lambda^n(e_1)) = \lambda^n(x)$. This and the ring homomorphism requirement defines ϕ_x uniquely on Λ. The latter statement is the definition of the Adams operation. □

Another way to express this result is that Λ is the free λ-ring in the single variable e_1. This theorem means that any formula in Λ is valid in any λ-ring, with appropriate translation. We can apply this to perform calculations for the Adams operations. Firstly we find that the Adams operation can be expressed as the Newton polynomial $\nu_n(\lambda^1, ..., \lambda^n)$. This is the polynomial above which expresses the power sums in terms of the elementary symmetric functions. This polynomial can be calculated recursively from the relation

$$\psi^n = \lambda^1\psi^{n-1} - \lambda^2\psi^{n-2} + \cdots + (-1)^n\lambda^{n-1}\psi^1 + (-1)^{n+1}n\lambda^n.$$

To prove this we note that in terms of the indeterminates x_i this reads

$$\sum x_i^n = \sum x_i \sum x_i^{n-1} - \sum x_{i_1}x_{i_2} \sum x_i^{n-2} + \cdots$$

In the right hand side the j^{th} term has monomials of lengths j and $j+1$, for $j < n$. The longer terms from the j^{th} term exactly cancel the shorter ones from the $(j+1)^{th}$. For $j = n$ we only have terms of length n which are cancelled by the terms from the $(n-1)^{th}$ term. After all the cancellation all that remains on the right hand side is the terms of length 1 which is exactly the left hand side.

Next we have the equations

$$\psi^n(\sum x_i + \sum y_j) = \sum x_i^n + \sum y_j^n = \psi^n(\sum x_i) + \psi^n(\sum y_j)$$

and

$$\psi^n(\sum x_i \sum y_j) = \sum (x_i y_j)^n = \psi^n(\sum x_i)\psi^n(\sum y_j)$$

which show that ψ^n is a ring homomorphism. Similarly

$$\psi^m(\psi^n(\sum x_i)) = \psi^m(\sum x_i^n) = \sum x_i^{mn} = \psi^{mn}(\sum x_i)$$

shows that $\psi^m\psi^n = \psi^{mn}$. Also if p is a prime then the congruence $(\sum x_i)^p \equiv \sum x_i^p \mod p$ shows that $\psi^p(x) \equiv x^p \mod p$.

This last property allows us to define a new operation θ^p on R by the formula

$$\theta^p(x) = \frac{\psi^p(x) - x^p}{p}.$$

One can write down formulae for $\theta^p(x+y)$ and $\theta^p(xy)$ just using the fact that ψ_p is a ring homomorphism. It also follows that $\theta^p(1) = 0$ and $\psi^k\theta^p = \theta^p\psi^k$ for all k. Conversely if we have an operation θ^p which satisfies these formulae then the operation ψ^p defined by the usual equation is a ring homomorphism.

2.3 The operation θ^p

The Adams operations make most calculations in a λ-ring much simpler. In this section we apply this principle to prove that if we have operations λ^k for k prime to p then by introducing the operation θ^p with its usual formal properties we obtain the λ^k with k a multiple of p. To compute $\theta^p(\lambda^n)$ we first compute $\psi^p(\lambda^n)$ using the basis of power sums for the ring of symmetric polynomials. The advantage of using the power sums instead of the elementary symmetric functions is that the Adams operations are described by the formula $\psi^p(p_n) = p_{pn}$, which follows easily from the fact that ψ^p is a ring homomorphism. Combining this with the two inversion formulae gives us the formula that we want. The formula for $\psi^p(\lambda^n)$ will then allow us to prove Proposition 2.2.

Lemma 2.2 *Modulo decomposables we have*

$$\theta^p(\lambda^n) = (-1)^{(p-1)n}\lambda^{pn}.$$

Proof
Working mod decomposables we have $p_n = (-1)^{n-1}\lambda_n$ and so applying ψ^p we obtain

$$(-1)^{n-1} n\psi^p(\lambda_n) = p_{pn} = (-1)^{pn-1} pn\lambda_{pn}.$$

Therefore $(-1)^{n-1} pn\theta^p(\lambda^n) = (-1)^{pn-1} pn\lambda^{pn}$ which gives the result in a torsion free ring. However this is a universal formula so it must hold in general. □

For $p = 3$ we can be more specific.

Proposition 2.3 *Let $\omega = e^{2\pi i/3}$ and define $c_{r,s}$ to be $2\Re(\omega^r)$ if $r \neq 0, s$ and 1 when $r = 0, s$. Then*

$$\theta^3(\lambda^n) = \sum_{i+j+k=3n,\, i\leqslant j\leqslant k} c_{j-i,k-i}\lambda^i\lambda^j\lambda^k.$$

Proof
To compute θ^3 we first compute the Adams operation ψ^3. We have two different formulae that we can use. The first comes from the Newton formula:

$$\psi^3(\lambda^n) = (\lambda^n)^3 + 3(\lambda^3(\lambda^n) - \lambda^1(\lambda^n)\lambda^2(\lambda^n)).$$

The second formula comes from the formulae for elementary symmetric functions and power sums:

$$\begin{aligned}
\psi^3(\lambda^n) &= \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_\mu^{-1} p_{3\mu} \\
&= \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_\mu^{-1} \prod_{i>0} \left(3i \sum_{|\nu|=3i} (-1)^{3i-l(\nu)} \frac{u_\nu}{l(\nu)} \lambda^\nu \right)^{m_i(\lambda)}.
\end{aligned}$$

The first formula allows us to simplify the calculation of the second, based on our knowledge of the universal polynomials $P_{m,n}$. Firstly it tells us that $\theta^p(\lambda^n)$ has degree 3. Secondly from the remarks in the appendix we see that all the monomials of degree 3 are of the form $c_{l,j,k}\lambda^{l+1}\lambda^{j+1}\lambda^{k+1}$ where $c_{l,j,k}\lambda^l\lambda^j\lambda^k$ is a monomial in $\psi^3(\lambda^{n-1})$. We should notice that the formula for the coefficients in the theorem is consistent with this. The conclusion that we draw from this is that in the second formula we only need to consider the terms which give us monomials of degree 2 or less. The higher degree monomials must either cancel out or be known. This means that in the second formula we can restrict our partitions to have length no more than 2.

To perform this calculation let us first assume that n is odd. The partitions μ are of the form $(k, n-k)$ or simply (n). From the definition we calculate that

$z_{(k,n-k)} = k(n-k)$ and $z_{(n)} = n$. Therefore from the formula for elementary symmetric functions in terms of power sums we have

$$\psi^3(\lambda^n) = \frac{(-1)^{n-1}}{n} p_{3n} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^n}{k(n-k)} p_{3k} p_{3(n-k)}$$

ignoring terms of higher degree. Similarly from the formula for power sums in terms of elementary symmetric functions we find that in p_m the terms of degree 2 are

$$m \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^m \lambda^k \lambda^{m-k} + \frac{m(\lambda^{m/2})^2}{2}$$

where the last term is zero for m odd. We also have the degree 1 term $m(-1)^{m-1}\lambda^m$. Putting these expression together we find that in $\psi^3(\lambda^n)$ the terms of degree less than 3 are

$$\frac{(-1)^{n-1}}{n}(3n(-1)^{3n-1}\lambda^{3n} + 3n(-1)^{3n} \sum_{k=1}^{\lfloor \frac{3n}{2} \rfloor} \lambda^k \lambda^{3n-k}) +$$

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^n}{k(n-k)} 3k(-1)^{3k-1} 3(n-k)(-1)^{3(n-k)-1} \lambda^{3k} \lambda^{3(n-k)}$$

$$= 3\lambda^{3n} - 3 \sum_{k=1}^{\lfloor \frac{3n}{2} \rfloor} \lambda^k \lambda^{3n-k} + 9 \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \lambda^{3k} \lambda^{3(n-k)}.$$

If we divide by 3 we see that the coefficient of $\lambda^k \lambda^{3n-k}$ in $\theta^3(\lambda^n)$ is 1 for $k = 0$, -1 for $k > 0$ but not a multiple of 3, and $-1 + 3 = 2$ for $k > 0$ a multiple of 3. In other words it is $c_{k,3n-k}$. This completes the proof for n odd.

For n even the only difference is that the term $p^2_{3n/2}$ comes with a coefficient $1/2$ which means that the coefficient $c_{3n/2,3n/2}$ of $(\lambda^{3n/2})^2$ is not 2 but 1 as claimed. □

As a corollary to this theorem we can prove proposition 2.2.

Proof of proposition 2.2

From the Newton formula and the calculation of $\lambda^2(\lambda^n)$ we have

$$\begin{aligned} \lambda^3(\lambda^n) - \theta^3(\lambda^n) &= \lambda^1(\lambda^n)\lambda^2(\lambda^n) \\ &= \lambda^n (\sum_{k=1}^{n} (-1)^{k+1} \lambda^{n-k} \lambda^{n+k}) \\ &= (-1)^{n+1} \lambda^n (\lambda^{2n}) + \cdots \end{aligned}$$

where we ignore the terms of degree 3. If we compare the formulae for the coefficients in $P_{3,n}$ and $\theta^3(\lambda^n)$ we see that the difference is indeed just the $(-1)^{n+1}\lambda^n(\lambda^{2n})$ that we have here. □

It is interesting to compare this proof to the combinatorial calculation given in the appendix. The description of the coefficients as real parts of powers of a cube root of unity, which is so crucial to the latter calculation, bears no obvious relation to how they arise here. Indeed in the above we only need to deal with real numbers.

2.4 The universal polynomials revisited

The Adams operations give us a much more effective way to compute the universal polynomials $P_{m,n}$ and P_n. Let us first consider the polynomial P_n which gives $\lambda^n(rs)$ in terms of $\lambda^i(r)$ and $\lambda^i(s)$ for $i \leqslant n$. To compute this we look at $\psi^n(rs)$. On the one hand this is $\nu_n(\lambda^1(rs), ..., \lambda^n(rs))$ while on the other it is $\psi^n(r)\psi^n(s) = \nu_n(\lambda^1(r), ..., \lambda^n(r))\nu_n(\lambda^1(s), ..., \lambda^n(s))$. Setting these two expressions equal and rearranging allows us to write $\lambda^n(rs)$ in terms of $\lambda^i(r)$, $\lambda^i(s)$, and $\lambda^j(rs)$ for $j < n$. This gives a recursive algorithm to compute P_n for n as large as we please. In the appendix we tabulate values of P_n for $n \leqslant 10$. A general formula would be difficult to write down but from the first few cases we might conjecture that the sum of the coefficients is always zero, for $n > 1$. This indeed is the case as we now show.

Lemma 2.3 *The sum of the coefficients in the n^{th} Newton Polynomial ν_n is $(-1)^{n+1}$.*

Proof
We can calculate the sum of the coefficients of $\nu_n(e_1, ..., e_n)$ by evaluating $\sum_{i=1}^n x_i^n$ at values of the indeterminates x_i such that the e_i all take the value 1. This means that $-x_i$ should be a root of $1 + x + x^2 + \cdots + x^n$ for each i. These roots, which we denote ω_i, are the $(n+1)^{th}$ roots of unity, except for 1. Then

$$\sum x_i^n = (-1)^n \sum \omega_i^n = (-1)^n \sum w_i^{-1} = (-1)^n \sum w_i = (-1)^{n+1}$$

since $1 + \sum w_i = 0$. □

Theorem 2.2 *The sum of the coefficients in P_n is zero for $n > 1$ and 1 for $n = 1$.*

Proof
The case $n = 1$ holds since $P_1 = e_1 f_1$. Now suppose that the result holds for all $k < n$ for some $n > 2$. The equality

$$\nu_n(\lambda^1(rs), ..., \lambda^n(rs)) = \nu_n(\lambda^1(r), ..., \lambda^n(r))\nu_n(\lambda^1(s), ..., \lambda^n(s))$$

and the fact that the sum of the coefficients of a product of polynomials is the product of the sums shows that the sum of the coefficients of the polynomial $\nu_n(\lambda^1(rs), ..., \lambda^n(rs))$ is $((-1)^{n+1})^2 = 1$ for each n. However we also have

$$\nu_n(\lambda^1(rs), ..., \lambda^n(rs)) = \lambda^1(rs)\nu_{n-1}(\lambda^1(rs), ..., \lambda^{n-1}(rs))$$
$$-\lambda^2(rs)\nu_{n-2}(\lambda^1(rs), ..., \lambda^{n-2}(rs)) + \cdots + (-1)^{n+1}n\lambda^n(rs).$$

By induction when we sum coefficients this reads

$$1 = 1 * 1 - 0 * 1 + 0 * 1 + \cdots + (-1)^{n+1}n * x$$

where x is the unknown sum of coefficients in

$$\lambda^n(rs) = P_n(\lambda^1(r), ..., \lambda^n(r); \lambda^1(s), ..., \lambda^n(s)).$$

Solving this gives $x = 0$ as claimed. □

We now turn to the polynomials $P_{m,n}$. In calculating $P_{3,n}$ we used the best method available to us. That was firstly to calculate $\psi^m(\lambda^n)$ using the inversion formulae and then rearrange the Newton formula to calculate $\lambda^m(\lambda^n)$ recursively. We actually fixed m but it is easier to fix n instead as this means you are calculating what you will need for the next value of m. Again we have tabulated some values in the appendix and can calculate the sum of the coefficients. In the calculations below we abuse notation and make no distinction between an expression and the sum of the coefficients in that expression.

Lemma 2.4 *The sum of the coefficients in $\psi^m(\lambda^n)$ is 1 if m is odd and $(-1)^n$ if m is even.*

Proof
We have

$$e_n = \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_\mu^{-1} p_\mu$$

so

$$\psi^m(e_n) = \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_\mu^{-1} p_{m\mu}.$$

By the previous lemma $p_n = (-1)^{n+1}$ so if m is even $p_{m\mu} = (-1)^{l(\mu)}$. Therefore for m even

$$\psi^m(e_n) = (-1)^n \sum_{|\mu|=n} z_\mu^{-1}.$$

If m is odd then $p_{m\mu} = (-1)^{l(\mu)-k}$ where k is the number of odd parts of μ. Note that $n-k$ is even, since mod 2, it is the sum of the even parts. Therefore

$$\psi^m(e_n) = \sum_{|\mu|=n} (-1)^{n-2l(\mu)+k} z_\mu^{-1} = \sum_{|\mu|=n} z_\mu^{-1}$$

since $n - 2l(\mu) + k$ is even. To prove the lemma it remains to show that $\sum_{|\mu|=n} z_\mu^{-1} = 1$. This is a classical result that goes back to Cauchy, though perhaps a more accessible proof can be found in [Dwy38] □

Theorem 2.3 *The sum of the coefficients in $P_{m,n}$ is 1 if n is odd or $m = 1$ and 0 for n even and $m > 1$.*

Proof
We prove this by induction on m, the case $m = 1$ holding since $\lambda^1(\lambda^n) = \lambda^n$. To prove the induction step we use the newton formula again:

$$\psi^m(\lambda^n) = \lambda^1(\lambda^n)\psi^{m-1}(\lambda^n) - \lambda^2(\lambda^n)\psi^{m-2}(\lambda^n) + \cdots + (-1)^{m+1} m\lambda^m(\lambda^n).$$

In n is even this reads

$$1 = \lambda^1(\lambda^n) - \lambda^2(\lambda^n) + \cdots + (-1)^{m+1} m\lambda^m(\lambda^n)$$

by the lemma so by induction we find that $\lambda^m(\lambda^n) = 0$ for $m > 1$. If n is odd it reads

$$1 = -\lambda^1(\lambda^n) - \lambda^2(\lambda^n) - \cdots + m\lambda^m(\lambda^n)$$

if m is odd and

$$-1 = \lambda^1(\lambda^n) + \lambda^2(\lambda^n) + \cdots - m\lambda^m(\lambda^n)$$

if m is even. In either case the equation gives $\lambda^m(\lambda^n) = 1$ as required. □

2.5 K-Theory examples

The definition of a λ-ring is set up so that for a CW-complex X, the ring K^0X has a λ-ring structure with λ^k induced by the operation of taking the k^{th} exterior power of a vector bundle. In this section we give explicit computations in the cases $X = S^{2n}$ and $X = BU$ to illustrate the operations λ^k and θ^p.

$\mathbf{K^0S^{2n}}$

Let us recall from [Hat01] the ring structure on K^0S^{2n}. As an Abelian group it is freely generated by 1, the trivial line bundle, and the external product

$\alpha = (H-1) \star \cdots \star (H-1)$ of n copies of the generator $H-1$ of K^0S^2. Here H is the canonical line bundle over $S^2 = \mathbb{CP}^1$. The multiplication is trivial since $(H-1)^2 = 0$. If m denotes a trivial vector bundle of rank m then the k^{th} exterior power of m is a trivial vector bundle of rank $\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$. Therefore on the first summand we have $\lambda^k(m) = \binom{m}{k}$, where the given formula makes sense for all values of m. To calculate the Adams operations we note that on a sum of line bundles $L_1 + \cdots + L_m$, ψ^k acts by taking it to the sum $L_1^k + \cdots + L_m^k$. A trivial line bundle raised to a power is still a trivial line bundle so ψ^k is the identity on trivial bundles. Finally this means that $\theta^p(m) = \frac{m-m^p}{p}$.

To compute the λ-operations on α we first have to figure out the Adams operations. The Adams operations are additive so once we know them on vector bundles we can extend them to virtual bundles too. Once we know ψ^k for all k we can define λ^k on virtual bundles using the Newton formula. To determine $\psi^k(\alpha)$ we first compute that on S^2 we have

$$\psi^k(H-1) = H^k - 1 = ((H-1)+1)^k - 1 = k(H-1)$$

where the last equality uses $(H-1)^2 = 0$. Then the equation

$$\psi^k(x \star y) = \psi^k(x) \star \psi^k(y)$$

shows that ψ^k acts by multiplication by k^n on α. Now we can show that $\lambda^k(\alpha) = (-1)^{k+1}k^{n-1}\alpha$. Indeed this is true for $k=1$ and if it is true for k then the Newton formula shows that $k^n\alpha = (-1)^{k+1}k\lambda^k(\alpha)$ mod terms of degree 2. The latter all vanish by the induction hypothesis since $\alpha^2 = 0$ and so dividing by k gives the result. The formula for $\psi^p(\alpha)$ also gives that $\theta^p(\alpha) = p^{n-1}\alpha$.

This calculation illustrates two important points. The first is that if we want to determine a λ-ring structure then we may well have to find the Adams operations first. The second is that if we can do this, then so long as the ring is torsion free (to allow us to cancel the k) we will have determined the λ-operations. This latter statement is the content of the following theorem of Wilkerson.

Theorem 2.4 *If R is a torsion free unital commutative ring equipped with ring endomorphisms $\psi^k : R \to R$ for $k \geqslant 1$ such that*

1. $\psi^1 = Id$
2. $\psi^k\psi^m = \psi^{km}$
3. $\psi^p(r) \equiv r^p \mod pR$ *for p prime*

then R has a unique λ-ring structure with the given ψ^k as Adams operations.

$\mathbf{K^0BU}$

The space BU is a topological direct limit of spaces $BU(n)$. The space $BU(n)$ is the classifying space for complex vector bundles of rank n. In can be constructed as the complex Grassmannian $G_n(\mathbb{C}^\infty)$ of all n-planes in $\mathbb{C}^\infty$. Atiyah and Hirzebruch proved that as a ring $K^0BU(n)$ can be computed as the completed representation ring $\hat{R}(U(n))$ which itself is the ring of invariants in $\hat{R}(T(n))$ under the action of the Weyl group, where $T(n)$ is the maximal torus of diagonal matrices. Since $\hat{R}(T(n)) = \mathbb{Z}[[z_1, ..., z_n]]$ and the Weyl group is the full symmetric group permuting the variables we find that $\hat{R}(U(n))$ is also a power series ring in n variables, which are the elementary symmetric polynomials in the z_i. Taking a limit we find that K^0BU is the power series ring in infinitely many variables.

The proof of the theorem gives a useful description of the variables in $K^0BU(n)$. The cohomology of $G_n(\mathbb{C}^\infty)$ is shown in [Hat01] to be a polynomial ring in the Chern classes c_1,...,c_n of the universal bundles over $G_n(\mathbb{C}^\infty)$. The operation of taking the i^{th} Chern class of a vector bundle extends to operations $c_i : \tilde{K}^0(X) \to H^{2i}(X)$ which behave additively much the same as λ-operations. As before we consider the ring homomorphism that results by looking at power sums instead of elementary symmetric functions and we obtain the Chern character $ch = \sum_k ch^k = \sum_k \nu_k(c_1, ..., c_k)/k!$. Formally we can say that if L is a line bundle then $ch(L) = e^{c_1(L)}$. The Chern character gives a map to the direct product of the individual cohomology groups rather than the direct sum, which in the case of $G_n(\mathbb{C}^\infty)$ means power series in the c_i rather than polynomials. The key theorem is that if X is a CW-complex with torsion free homology then

$$ch : \tilde{K}^0(X) \to H^\star(X) \otimes \mathbb{Q}$$

is injective. In the case of $X = BU$ we have one element $i \in \tilde{K}^0(BU)$ given by the identity map of BU and using the exterior power operations we obtain a family of elements $\lambda^n(i)$.

Theorem 2.5 *The elements $ch(\lambda^n(i)) \in H^\star(BU) \otimes \mathbb{Q}$ form a set of polynomial generators and so $K^0(BU) = \mathbb{Z}[[\lambda^1(i), \lambda^2(i)...]]$.*

Proof
We give a sketch and refer to [Hir61] for details. The idea is to introduce the total γ-operation defined by $\gamma_t(x) = \lambda_{t/1-t}(x)$. The γ-operations γ^i, defined by $\gamma_t(x) = \sum \gamma^i(x)t^i$, give another set of generators for the polynomials in the λ^i, so it is sufficient to prove that the Chern characters of the γ^i are a set of generators. If we have $\lambda_t(x_i) = (1 + x_i t)$ then $ch(\lambda_t(x_i)) = (1 + e^{x_i}t)$ and so if we introduce new indeterminates $z_i = x_i - 1$ we have

$$ch(\gamma_t(z_i)) = (1 + (e^{x_i} - 1)t).$$

Now $e^{x_i} - 1 = x_i +$ decomposables so any polynomial in the x_i can be written as a polynomial in $e^{x_i} - 1$ plus higher terms. To prove the theorem we start with a maximal torus $T(n)$ in $U(n)$ and consider the product B_{2k} of n copies of $\mathbb{CP}^k$. The cohomology of this product is a polynomial ring in n variables x_i modulo terms of degree $k+1$ and using the Chern character we find that K^0B_{2k} is the same truncated polynomial ring in the z_i. Taking an inverse limit over k we obtain that $K^0BT(n)$ is a power series ring in n variables. Next a naturality argument gives us $K^0BU(n)$ and another inverse limit gets us to $K^\star BU$ □

The power series ring K^0BU is a completed version of the free λ-ring in one variable. In fact it is the inverse limit over n of the quotient by the ideal generated by terms of degree at least n. This means that the actions of λ^k and θ^p are given by the universal formulae described above.

The γ-operations introduced in the proof of theorem 2.5 give rise to the so called γ-filtration $I = \Gamma^1(I) \supseteq \Gamma^2(I) \supseteq \cdots$ of a λ-ideal I. Here $\Gamma^n(I)$ is the ideal generated by products $\gamma^{i_1}(x_1)\cdots\gamma^{i_k}(x_k)$ where $i_1 + \cdots + i_k \geqslant n$ and $x_i \in I$. I is called γ-nilpotent if it is nilpotent and $\gamma^i(x) = 0$ for $x \in I$ and i sufficiently large. We remark that in the above calculation the augmentation ideal of $K^\star B_{2k}$ is γ-nilpotent, the operations γ^i being zero for $i > n$, and any polynomial with zero constant term raised to the $(k+1)^{th}$ power being zero.

3 Bousfield's Theorem

In the previous section we saw how in a torsion free unital commutative ring, a λ-ring structure is equivalent to a set of Adams operations. We also saw how in K-theory the operation of taking the k^{th} power of a line bundle gives rise to Adams operations and hence a λ-ring structure on the K-theory of a space. A natural question to ask is whether the K-theory of a spectrum E is also a λ-ring. The answer is generally no because the Adams operation ψ^k only defines a stable operation if k is a unit in the coefficient ring. If we were to work over the p-adics for p a prime then we would have all the Adams operations save for those which are a multiple of p, plus we wouldn't have to worry about the ring being torsion free. In this case by formally introducing ψ^p, or indeed as we saw previously even θ^p, we would expect to obtain a λ-ring. We would also expect this λ-ring to be closely related to $K^0(\Omega^\infty E;\mathbb{Z}_p)$. This is the content of Bousfield's theorem. The object of this section is to understand the theorem in detail so that we may apply it to the spectrum tmf and give explicit details.

There are two main issues in stating the correct theorem. The first is determining the formal algebraic properties that characterise $K^\star(X;\mathbb{Z}_p)$ and $K^\star(E;\mathbb{Z}_p)$ for a space X and spectrum E. This involves the notions (in the language of

Bousfield) of $\mathbb{Z}/2$-graded p-adic λ-rings and $\mathbb{Z}/2$-graded p-adic ψ-modules. The second issue is establishing which properties will be required so that the introduction of the θ^p operation to $K^*(E;\mathbb{Z}_p)$ gives a unique (p-adic) λ-ring structure. The first of these conditions is that $K^*(E;\mathbb{Z}_p)$ should be torsion free, due to this being the class of spectra that can be constructed starting from $E = K$. The second of these conditions is more subtle and relates to the γ-filtration of the augmentation ideal $(\tilde{K}^0, K^1)$. The operations ψ^k must satisfy the relations $\psi^k(y) = ky \mod \Gamma^2(\tilde{K}^0(E;\mathbb{Z}_p))$ for $y \in \tilde{K}^0(E;\mathbb{Z}_p)$ and $\psi^k(x) = x \mod \Gamma^2(K^1(E;\mathbb{Z}_p))$ for $x \in K^1(E;\mathbb{Z}_p)$. If $\tilde{K}^*(E;\mathbb{Z}_p)/\Gamma^2(\tilde{K}^*(E;\mathbb{Z}_p)) = 0$ then these conditions are automatic and the operation ψ^p, or equivalently θ^p, can be introduced freely to give $K^*(\Omega^\infty E;\mathbb{Z}_p)$. Otherwise these conditions must be included as part of the data when we introduce θ^p. The space $\tilde{K}^*(E;\mathbb{Z}_p)/\Gamma^2(\tilde{K}^*(E;\mathbb{Z}_p))$ can be identified as the first and second stable cohomology groups of E which gives a more useable criterion for $K^*(\Omega^\infty E;\mathbb{Z}_p)$ to be a free object.

3.1 $\mathbb{Z}/2$-graded p-adic λ-rings and $\mathbb{Z}/2$-graded p-adic ψ-modules.

If X is a connected CW-complex then by considering the finite subcomplexes of X and regarding $\mathbb{Z}_p$ as a (topological inverse) limit of finite p-torsion groups we can write $K^*(X;\mathbb{Z}_p)$ as a limit of rings R with the following properties:

1. R is an augmented, $\mathbb{Z}/2$-graded commutative ring.
2. R^0 is a λ-ring and R^1 has endomorphisms ψ^n for $n \geqslant 1$ such that $\psi^1 = \mathrm{Id}$, $\psi^j\psi^k = \psi^{jk}$ and $\psi^k(xy) = \psi^k(x)\psi^k(y)$ for $x \in R^0$ and $y \in R^1$. Furthermore $\lambda^k(xy) = \sum(-1)^{n-j}\psi^{i_1}(x)\psi^{i_1}(y)\cdots\psi^{i_j}(x)\psi^{i_j}(y)$ for $x, y \in R^1$ where the sum is over all strictly increasing partitions $(i_1, ..., i_j)$ of k.
3. The augmentation ideal $\tilde{R} = (\tilde{R}^0, R^1)$ is finite p-torsion and γ-nilpotent.

An inverse limit of such rings R is called a $\mathbb{Z}/2$-graded p-adic λ-ring.

A ring R satisfying properties 1,2 and 3 also has the following properties:

4. $\Gamma^2(\tilde{R}) = \{x \in \tilde{R} | (\theta^p)^n x = 0 \text{ for some } n > 0\}$.
5. $\theta^p(y) = y \mod \Gamma^2(\tilde{R}^0)$ for $y \in \tilde{R}^0$.
6. $\psi^k y \equiv ky \mod \Gamma^2(\tilde{R}^0)$ for $y \in \tilde{R}^0$.
7. $\psi^k x \equiv x \mod \Gamma^2(R^1)$ for $x \in R^1$.

8. The Adams operations $\psi^k : \tilde{R} \to \tilde{R}$ satisfy $\psi^k = \psi^{k+p^r}$ for $k > 0$ and some r.

9. $\psi^k(xy) = k\psi^k(x)\psi^k(y)$ for $x, y \in R^1$.

There are several remarks to be made here. Firstly R^1 gives rise to a λ-ring $\mathbb{Z} \oplus R^1$ with trivial multiplication, where $(-1)^{k+1}\psi^k$ corresponds to λ^k. Then θ^p is identified with ψ^p so that $\Gamma^2(R^1) = \{x \in R^1 | (\psi^p)^n x = 0 \text{ for some } n > 0\}$ and property 7 is consistent with property 5. We note that to obtain the λ-ring corresponding to the group R^1 we apply the functor which is the left adjoint to the forgetful functor from the category of λ-rings to the category of abelian groups with endomorphisms ψ^n. The maps in the latter category are homomorphisms that commute with the ψ^n. Secondly the operations ψ^k on R^1 can be thought of as ψ^k/k as in [Bou96] p17. In this case the formula in property 9 becomes $\psi^k(xy) = \frac{1}{k}\psi^k(x)\psi^k(y)$ for $x, y \in R^1$ which is consistent with [Goe04]. We note that this formula comes from the fact that the Adams operations commute with the Bott periodicity isomorphism only up to multiplying by k. Finally we remark that the formula for $\lambda^n(xy)$ with $x, y \in R^1$ is seen to be equivalent to property 9 when we consider the formula $e_n = \sum_{|\mu|=n}(-1)^{n-l(\mu)}z_\mu^{-1}p_\mu$ and remark that since squares are zero in R^1, we only need to look at the partitions μ with no multiplicities greater than 1.

If we ignore the operations ψ^k for k prime to p then we arrive at the definition of a $\mathbb{Z}/2$-graded p-adic θ^p-ring. Specifically it is an inverse limit of rings R over $\mathbb{Z}_p$ with operations $\theta^p : R^0 \to R^0$ and $\psi^p : R^1 \to R^1$, such that ψ^p defined by the usual formula is a ring homomorphism on R^0, and ψ^p on R^1 is a group homomorphism. With the convention $\psi^p = \theta^p$ on R^1 in 4 and 5 as remarked above, R must satisfy properties 1, 3 and 5. We don't have the γ-operations so property 4 is used here as a definition and γ-nilpotent reduces to just nilpotent. Furthermore the operations must be consistent with properties 2 and 9, which means we have the formulae $\psi^p(xy) = \psi^p(x)\psi^p(y)$ for $x \in R^0$ and $y \in R^1$ and $\theta^p(xy) = \psi^p(x)\psi^p(y)$ for $x, y \in R^1$.

Bousfield proves that if we take a $\mathbb{Z}/2$-graded p-adic θ^p-ring and endow it with Adams operations ψ^k for k prime to p that satisfy properties 7,8 and 9 above, together with the usual properties described in section 2.2, then the ring has a unique structure as a $\mathbb{Z}/2$-graded p-adic λ-ring with the given Adams operations. We remark that property 2 is straightforward so the difficulty here is in proving that γ^i vanishes on the augmentation ideal for sufficiently large i, so that $\tilde{R}$ is γ-nilpotent.

This theorem can be viewed the other way around too. If we consider $K^*(E; \mathbb{Z}_p)$ for E a spectrum, then additively we have a limit of $\mathbb{Z}/2$-graded finite abelian p-groups which are equipped with Adams operations ψ^k for k prime to p. This means that ψ^k satisfy property 8 plus the conditions $\psi^1 = \text{Id}$ and $\psi^j\psi^k = \psi^{jk}$.

A space which satisfies these properties is called a $\mathbb{Z}/2$-graded p-adic ψ-module. These properties mean that the action of $\mathbb{Z}^+ \setminus p\mathbb{Z}$ extends to a continuous action of the p-adic units $\mathbb{Z}_p^\times$. We can make such a space into a $\mathbb{Z}/2$-graded p-adic θ^p-ring by simply applying a functor which forces conditions 1,3 and 5 to be satisfied and conditions 7 and 9 will naturally be satisfied too. Bousfield's theorem is that when we apply this functor to $K^\star(E;\mathbb{Z}_p)$, at least when it is torsion free, then we will get $K^\star(\Omega_0^\infty E;\mathbb{Z}_p)$, where the subscript 0 denotes the baseloop component of the loopspace.

3.2 The free θ^p-ring functors

Bousfield constructs ($\mathbb{Z}/2$-graded p-adic) θ^p-rings out of p-profinite abelian groups using the adjoint functor method for various forgetful functors. To motivate the definitions let us consider the case when our abelian group has Adams operations, so that when we make it into a θ^p-ring it becomes a λ-ring. In a λ-ring R the ideal $\Gamma^2(\tilde{R})$ is a λ-ideal so we can consider a decomposition

$$R = \mathbb{Z} + \Gamma^2(\tilde{R}) + \tilde{R}/\Gamma^2(\tilde{R})$$

which is preserved by the λ-operations. The summands are also closed under addition and multiplication but the sum is not necessarily direct. In the p-adic case this decomposition is a limit over α of decompositions

$$R_\alpha = \mathbb{Z} + \Gamma^2(\tilde{R_\alpha}) + \tilde{R_\alpha}/\Gamma^2(\tilde{R_\alpha})$$

where $\Gamma^2(\tilde{R_\alpha}) = \{x \in \tilde{R} | (\theta^p)^n x = 0 \text{ for some } n > 0\}$ and θ^p acts as the identity on $\tilde{R_\alpha}/\Gamma^2(\tilde{R_\alpha})$. This means that these decompositions can be described without reference to the Adams operations ψ^k for k prime to p. If we forget the operational structure but remember the decomposition then we should be able to recover the original λ-ring from this. Encoding the decomposition as the map $\tilde{R} \to \tilde{R}/\Gamma^2(\tilde{R})$ gives us the functor W. This can be built up out of functors $\bar{T}$ and J that are left adjoint to functors which remember $\tilde{R}$ and $\tilde{R}/\Gamma^2(\tilde{R})$ respectively. $\bar{T}$ is the simplest way to introduce θ^p freely but it forgets the decomposition of $\tilde{R}$ so it doesn't give us the λ-ring that we want unless $\Gamma^2(\tilde{R}) = \tilde{R}$. Restricting (the right adjoint of) $\bar{T}$ to rings which satisfy this condition gives us the functor T, which is the special case $W(M \to 0)$. Similarly if it happened that $\Gamma^2(\tilde{R}) = 0$ then the functor J would give us what we want and so J is the special case $W(0 \subset H)$.

We recall the definitions of $\bar{T}$, J, W and T from [Bou96]

$\bar{T}$ is left adjoint to the functor sending a $\mathbb{Z}/2$-graded p-adic θ^p-ring R to the $\mathbb{Z}/2$-graded p-profinite abelian group $\tilde{R}$.

J is left adjoint to the functor sending a $\mathbb{Z}/2$-graded p-adic θ^p-ring R to the $\mathbb{Z}/2$-graded p-profinite abelian group $\tilde{R}/\Gamma^2\tilde{R}$. Here $\Gamma^2\tilde{R}$ is defined as the limit of $\Gamma^2\tilde{R}_\alpha$ where $R = \lim R_\alpha$.

W is left adjoint to the functor sending a $\mathbb{Z}/2$-graded p-adic θ^p-ring R to the map of $\mathbb{Z}/2$-graded p-profinite abelian groups $\tilde{R} \to \tilde{R}/\Gamma^2\tilde{R}$.

T is left adjoint to the functor sending a connective $\mathbb{Z}/2$-graded p-adic θ^p-ring R to the $\mathbb{Z}/2$-graded p-profinite abelian group $\tilde{R}$. Here connective means that $\tilde{R}/\Gamma^2\tilde{R} = 0$.

To evaluate these functors on specific groups we can use the fact that left adjoints preserve cokernels to reduce ourselves to dealing with free groups. The functors J and T are simple to describe. In the case of T we note that a connective $\mathbb{Z}/2$-graded p-adic θ^p-ring R can be written as $R = \mathbb{Z}_p \oplus \Gamma^2\tilde{R}$ and the adjunction relation is

$$\mathrm{Hom}_{\theta^p}(TH, \mathbb{Z}_p \oplus \Gamma^2\tilde{R}) = \mathrm{Hom}_{Ab}(H, \Gamma^2\tilde{R}).$$

In the case where H consists of $\mathbb{Z}_p$ on an even generator x and an odd generator y this formula tells us that

$$T(\mathbb{Z}_p(x), \mathbb{Z}_p(y)) = \mathbb{Z}_p[[x, \theta^p x, (\theta^p)^2 x, \ldots]]\hat{\otimes}\Lambda_p[[y, \theta^p y, (\theta^p)^2 y, \ldots]].$$

To find J we use the identifications ([Bou96]) $\tilde{R}^0/\Gamma^2\tilde{R}^0 \cong \{u \in \tilde{R}^0 | \theta^p u = 0\}$ and $R^1/\Gamma^2 R^1 \cong \{u \in R^1 | \psi^p u = u\}$ and obtain

$$J(H) = \mathbb{Z}_p[[H^0]]\hat{\otimes}\Lambda_p[[H^1]]$$

where θ^p acts by zero on H^0 and ψ^p is the identity on H^1. Note that we can view this functor as starting with T then introducing by the relations that exist in $\tilde{R}/\Gamma^2\tilde{R}$.

Next if we write down what the definitions entail we find that $W(M \to H) = \bar{T}(M)\hat{\otimes}_{J(M)}J(H)$ and so we can compute W from $\bar{T}$ and J. Specifically, the action of θ^p on a tensor product is given by the formula

$$\theta^p(r \otimes s) = \theta^p(r) \otimes s^p + r^p \otimes \theta^p(s) + p\theta^p(r) \otimes \theta^p(s)$$

which extends the action on the two factors. In this case the tensor product is over $J(M)$ which means that we introduce relations given by equalizing the actions on the two factors. The action of $J(M)$ on $J(H)$ is given by the natural map $J(M \to H)$. The action of $J(M)$ on $\bar{T}(M)$ is also given by a map $J(M) \to \bar{T}(M)$. M is naturally a subset of $\bar{T}(M)$ and this map extends the inclusion to $J(M)$, whose elements are formal $\mathbb{Z}_p$ combinations of elements of M. Another way to describe this map is as the adjoint of the map obtained by composing the adjoint of the identity of $\bar{T}(M)$ (which is the inclusion of M in $\widetilde{\bar{T}(M)}$) with the projection to $\widetilde{\bar{T}(M)}/\Gamma^2\widetilde{\bar{T}(M)}$.

This just leaves us needing to describe $\bar{T}$. We can describe $\bar{T}(\mathbb{Z}_p, 0)$ as follows. Let $T_p(\mathbb{Z}/p^m)$ be the cokernel of the θ^p-endomorphism of $\mathbb{Z}_p[x, \theta^p x, (\theta^p)^2 x, ...]$ which sends x to $p^m x$. Since $\theta^p(p^m x) = p^m \theta^p x$ mod decomposables, this is a polynomial ring with coefficients in $\mathbb{Z}/p^m$, though with a more complicated θ^p operation. Let $I_n \subset T_p(\mathbb{Z}/p^m)$ denote the θ^p-ideal generated by $(\widetilde{T_p(\mathbb{Z}/p^m)})^n$ together with $(\theta^p)^n(\theta^p x - x)$ for $x \in T_p(\tilde{\mathbb{Z}}/p^m)$. Then $T_p(\mathbb{Z}/p^m)/I_n$ is a p-adic θ^p-ring and we have

$$\bar{T}(\mathbb{Z}_p, 0) = \lim_m \lim_n T_p(\mathbb{Z}/p^m)/I_n.$$

As a ring this is a power series ring in infinitely many generators as with T, but the θ^p operation is more complicated.

The functor $\bar{T}$ is not the easiest to to understand so a reasonable question to ask, given the decomposition of R, is whether we can build W out of J and the functor T_0 which is left adjoint the functor sending R to $\Gamma^2 \tilde{R}$. In fact since $\Gamma^2 \tilde{R} = \{x \in \tilde{R} | (\theta^p)^n x = 0 \text{ for some } n > 0\}$ in the finite case, we see that θ^p-maps preserve these spaces and so $T_0(M)$ is simply the connective θ^p ring $T(M)$ viewed as a non-connective ring. Intuitively the functor sending $\tilde{R} \to \tilde{R}/\Gamma^2\tilde{R}$ to $T_0(\Gamma^2\tilde{R})\hat{\otimes}J(\tilde{R}/\Gamma^2\tilde{R})$ will be W provided that we know how to reconstruct $\tilde{R}$ from $\Gamma^2\tilde{R}$ and $\tilde{R}/\Gamma^2\tilde{R}$. Specifically we have the following proposition.

Proposition 3.1 *Let C be the category of $\mathbb{Z}/2$-graded p-adic θ^p-rings R together with a splitting $\tilde{R} = \Gamma^2(\tilde{R}) \oplus \tilde{R}/\Gamma^2(\tilde{R})$ that is preserved by the morphisms. If W_0 is the left adjoint to the functor sending $R \in C$ to the split surjection $\tilde{R} \to \tilde{R}/\Gamma^2(\tilde{R})$ then $W_0(M \oplus H \to H) = T_0(M)\hat{\otimes}J(H)$.*

Proof

This is just an exercise in definition chasing. □

This proposition is a generalisation of the case $H = 0$ which gives the functor T. However the naturality condition seems unlikely to be satisfied which limits its usefulness.

3.3 $K^*(\Omega_0^\infty E; \mathbb{Z}_p)$

The free θ^p-ring functors above are defined on the category of $\mathbb{Z}/2$-graded p-profinite groups. When we come to describe $K^*(\Omega_0^\infty E; \mathbb{Z}_p)$ we will apply them to $\mathbb{Z}/2$-graded p-adic ψ-modules to obtain $\mathbb{Z}/2$-graded p-adic λ-rings. More precisely let λ-ring, ψ-mod, θ^p-ring and Ab denote respectively the categories of $\mathbb{Z}/2$-graded p-adic λ-rings, $\mathbb{Z}/2$-graded p-adic ψ-modules, $\mathbb{Z}/2$-graded p-adic θ^p-rings and $\mathbb{Z}/2$-graded p-profinite abelian groups. Then we have a commutative diagram of forgetful functors

$$\begin{array}{ccc} \lambda\text{-ring} & \longrightarrow & \psi\text{-mod} \\ \downarrow & & \downarrow \\ \theta^p\text{-ring} & \longrightarrow & Ab \end{array}$$

where the horizontal arrows are the various functors defined above and the vertical arrows keep the underlying space but forget the operational structure. These diagrams correspond to a commutative diagram of adjoints

$$\begin{array}{ccc} \lambda\text{-ring} & \longleftarrow & \psi\text{-mod} \\ \uparrow I & & \uparrow I \\ \theta^p\text{-ring} & \longleftarrow & Ab \end{array}$$

where the functor I introduces the operations ψ^k for k prime to p, and is in fact the same functor in both vertical arrows. The functors referred to below are the top arrows in this diagram, but in fact they are the same functors as the bottom arrows defined in the previous section. This is because the ψ^k commute with θ^p and so introducing θ^p after the ψ^k is the same operation as introducing θ^p before the ψ^k.

We are now in a position to state Bousfield's theorem. Let E be a spectrum with 0-connected section $E\langle 0\rangle$ and consider the Atiyah-Hirzebruch spectral sequence for $K^*(E\langle 0\rangle;\mathbb{Z}_p)$. The picture for E_2 is shown below.

t					
2	0	H^1	H^2	H^3	
1	0				
0	0	H^1	H^2	H^3	
-1	0				
	0	1			s

Here every group to the left of the picture is zero since we take the zero connected section of E. This means that there are no differentials that hit the groups $H^1(E\langle 0\rangle;\mathbb{Z}_p)$ and $H^2(E\langle 0\rangle;\mathbb{Z}_p)$ and so there is an edge map

$$K^*(E;\mathbb{Z}_p)_H : K^*(E;\mathbb{Z}_p) \cong K^*(E\langle 0\rangle;\mathbb{Z}_p) \longrightarrow \{H^2(E\langle 0\rangle;\mathbb{Z}_p), H^1(E\langle 0\rangle;\mathbb{Z}_p)\}.$$

We remark that this is a quotient map and so is surjective.

Theorem 3.1 *If E is a spectrum with $K^\star(E;\mathbb{Z}_p)$ torsion free then*

$$K^\star(\Omega_0^\infty E;\mathbb{Z}_p)\cong W(K^\star(E;\mathbb{Z}_p)_H)$$

as $\mathbb{Z}/2$-graded p-adic λ^p-rings. In particular if E is also 0-connected with $H^2(E;\mathbb{Z}_p)=H^1(E;\mathbb{Z}_p)=0$ then $K^\star(\Omega^\infty E;\mathbb{Z}_p)\cong T(K^\star(E;\mathbb{Z}_p))$.

Both statements in the theorem are consequences of the following proposition. (The former requiring a bar spectral sequence argument while the latter being trivial.)

Proposition 3.2 *If E is a spectrum with $K^\star(E;\mathbb{Z}_p)$ torsion free then*

$$K^\star(\Omega^\infty E\langle 2\rangle;\mathbb{Z}_p)\cong T(K^\star(E;\mathbb{Z}_p)).$$

Proof

The proof of this statement is to check that it works for $E=K$ and $E=\Sigma K$ and then build up the class of spectra with torsion free K-theory from these. Let us consider the case of $E=K$. According to [Tor77] $K^1(K;\mathbb{Z}_p)=0$ and $K^0(K;\mathbb{Z}_p)$ is the completed group ring $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Here $\mathbb{Z}_p^\times$ is the group of units in $\mathbb{Z}_p$ and $\alpha\in\mathbb{Z}_p^\times$ corresponds to the stable Adams operation ψ^α. This means that as a $\mathbb{Z}/2$-graded p-adic ψ-module $K^\star(K;\mathbb{Z}_p)$ is a free module on the single even generator x corresponding to the identity map. When we introduce θ^p by applying T we therefore obtain the free $\mathbb{Z}/2$-graded p-adic λ-rings on an even generator, which is $\mathbb{Z}_p[[x,\lambda^2x,\lambda^3x,...]]$. The 2-connected cover of K has zero space BSU, which is the 2-connected cover of BU. The other spaces in $K\langle 2\rangle$ are given by the homotopy equivalences $\Omega BSU\simeq SU$, $\Omega SU\simeq BU$, $\Omega BU\simeq U$ and $\Omega U\simeq\mathbb{Z}\times BU$. The calculation of $K^\star BSU$ is similar to that of $K^\star BU$ is section 2.5. The cohomology of BSU is again a polynomial ring in infinitely many chern classes, with c_1 omitted. The conclusion is that $K^0(BSU;\mathbb{Z}_p)\cong\mathbb{Z}_p[[y,\lambda^2y,\lambda^3y,...]]$, where y corresponds to the covering $BSU\to BU$. Therefore we see that in this case $K^\star(BSU;\mathbb{Z}_p)\cong T(K^\star(K;\mathbb{Z}_p))$ which is what we want. For $E=\Sigma K$ the indices are shifted so now $K^\star(\Sigma K;\mathbb{Z}_p)$ is a free module on the single odd generator x. Now we get that $T(K^\star(\Sigma K;\mathbb{Z}_p))$ is the free $\mathbb{Z}/2$-graded p-adic λ-rings on an odd generator, which is the completed exterior algebra $\Lambda_p[[x,\lambda^2x,\lambda^3x,...]]$. Shifting indices in the 2-connected cover we have the zero space SU. The calculation of $K^\star U$ can be found in [Ati67] and with suitable modifications for SU, we have that $K^1(SU;\mathbb{Z}_p)$ is also a completed exterior algebra on infinitely many generators. □

3.4 Examples

In the previous section we saw how Bousfield's theorem applies to the the spectra $K\langle 2\rangle$ and $\Sigma K\langle 2\rangle$. Now we look at some other examples to help us to understand the theorem further.

Given a spectrum E we will want to compute the stable cohomology groups $H^2(E\langle 0\rangle;\mathbb{Z}_p)$ and $H^1(E\langle 0\rangle;\mathbb{Z}_p)$ of the 0-connected cover of E. If we know the homotopy groups $\pi_1 E$ and $\pi_2 E$ then we can use the following theorem.

Theorem 3.2 *If E is a CW-spectrum and G is any coefficient group then*

$$H_1(E\langle 0\rangle) \cong \pi_1 E$$

$$H_2(E\langle 0\rangle) \cong \operatorname{coker}(\times\eta : \pi_1 E \to \pi_2 E)$$

and so

$$H^1(E\langle 0\rangle; G) \cong \operatorname{Hom}(\pi_1 E, G)$$

$$H^2(E\langle 0\rangle; G) \cong \operatorname{Hom}(H_2(E\langle 0\rangle), G) \oplus \operatorname{Ext}(\pi_1 E, G).$$

Proof

The formula for $H_1(E\langle 0\rangle)$ follows from the Hurewicz isomorphism. To compute $H_2(E\langle 0\rangle)$ consider the cofibration

$$\begin{array}{ccc} E\langle 1\rangle & & \\ \downarrow & & \\ E\langle 0\rangle & \to & \Sigma H\pi_1 E \end{array}$$

where $H\pi_1 E$ is the Eilenberg-Maclane spectrum. The stable homology groups $H_{n+1}(K(\Pi, n))$ and $H_{n+2}(K(\Pi, n))$ are computed in [Mac54] as 0 and $\Pi \otimes \mathbb{Z}/2$ respectively. Therefore part of the long exact homology sequence for the fibration is

$$\pi_1 E \otimes \mathbb{Z}/2 \stackrel{\times\eta}{\to} \pi_2 E \to H_2(E\langle 0\rangle) \to H_1(H\pi_1 E) = 0$$

where the second group is given by the Hurewicz isomorphism for $E\langle 1\rangle$ and η is the non-zero element of π_1^s. This sequence gives the formula for $H_2(E\langle 0\rangle)$. The formulae for the cohomology groups follow from the universal coefficient theorem. □

We remark that in the preceding proof we could equally well have used the long exact sequence in cohomology. The stable cohomology groups we would need are $H^n(K(\Pi, n); G) = \operatorname{Hom}(\Pi, G)$, $H^{n+1}(K(\Pi, n); G) = \operatorname{Ext}(\Pi, G)$ and $H^{n+2}(K(\Pi, n); G) = \operatorname{Hom}(\Pi \otimes \mathbb{Z}/2, G)$. The cohomology long exact sequence then gives us isomorphisms $H^1(E\langle 0\rangle; G) \cong \operatorname{Hom}(\pi_1 E, G)$ and $H^2(E\langle 0\rangle; G) \cong$

$\ker\left(\mathrm{Hom}(\pi_2 E, G) \overset{(\times\eta)^*}{\to} \mathrm{Hom}(\pi_1 E \otimes \mathbb{Z}/2, G)\right) \oplus \mathrm{Ext}(\pi_1 E, G)$ which amounts to the same as above.

We can use this theorem to look at some examples. Let us begin with the spectrum K. This has homotopy groups $\pi_{2n} = \mathbb{Z}$ and $\pi_{2n+1} = 0$. This means that $H^1(K\langle 0\rangle; \mathbb{Z}_p) = 0$ and $H^2(K\langle 0\rangle; \mathbb{Z}_p) = \mathbb{Z}_p$. The space $\Omega^\infty K$ is $\mathbb{Z} \times BU$ so the connected component of the base loop is just BU. We saw that $K^0 BU$ is the free $\mathbb{Z}/2$-graded p-adic λ-ring on an even generator, while $K^0(K; \mathbb{Z}_p) \cong \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Therefore we have the formula

$$W(\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p) \cong \mathbb{Z}_p[[x, \lambda^2(x), ...]].$$

A second example is the sphere spectrum S^0. The K-cohomology of this spectrum is simply the coefficient ring $\mathbb{Z}_p$ in degree zero. The groups $\pi_1 S^0$ and $\pi_2 S^0$ are well known to be $\mathbb{Z}/2$ generated by η and η^2 respectively. From the formulae in theorem 3.2 this gives us $H_1 S^0\langle 0\rangle = \mathbb{Z}/2$ and $H_2 S^0\langle 0\rangle = 0$ and so we have $H^1(S^0\langle 0\rangle; \mathbb{Z}_p) = 0$ and $H^2(S^0\langle 0\rangle; \mathbb{Z}_p) = \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}_p)$. This last group is zero for $p > 2$ and $\mathbb{Z}/2$ for $p = 2$. The space $\Omega_0^\infty S^0$ is what is usually called QS_0^0. The integral cohomology of this space was calculated by [Pri70] as the cohomology of the classifying space of the infinite symmetric group, BS_∞. This isomorphism is induced by a map so we can apply the Atiyah-Hirzebruch spectral sequence to deduce that the K-cohomology of QS_0^0 is also isomorphic to the K-cohomology of BS_∞. This latter ring can be calculated using Atiyah's theorem that for a finite group G, $K^\star BG$ is isomorphic to the completed representation ring $\hat{R(G)}$. Atiyah proves that the dual of the representation ring of S_n is the ring of symmetric functions of degree n in n variables. Taking a direct sum over n gives the polynomial ring $\mathbb{Z}[c_1, c_2, ...]$ in the elementary symmetric functions. Dualising and completing we have the free $\mathbb{Z}/2$-graded p-adic θ^p-ring on an even generator. Therefore in this example the theorem gives us the formulae

$$W(\mathbb{Z}_p \to 0) = T(\mathbb{Z}_p) \cong \mathbb{Z}_p[[x, \theta^p x, ...]]$$

for $p > 2$ and

$$W(\mathbb{Z}_2 \to \mathbb{Z}/2) \cong \mathbb{Z}_2[[x, \theta^2 x, ...]].$$

Note that these rings, as in the example above, are power series rings in infinitely many generators. The difference is in the grading of the generators: the operation λ^i increases the grading by i while θ^p increases it by p.

More generally we could consider the Moore spectrum MA for an Abelian group A. The cohomology of this spectrum (by definition) consists of A in dimension zero only. Therefore the Atiyah-Hirzebruch spectral sequence is trivial and gives us that $K^0 MA = A$ also. This means that to apply Bousfield's theorem we have to assume that $A \otimes \mathbb{Z}_p$ is a torsion free group, or equivalently that A has no p-torsion. The homotopy groups of MA for torsion free A can be found using the following lemma.

Lemma 3.1 *If A is a torsion free abelian group then $\pi_n MA \cong A \otimes \pi_n S^0$.*

Proof
Let

$$0 \to F_1 \xrightarrow{k} F_2 \twoheadrightarrow A$$

be a free resolution of A, which occurs as the homology of a cofibration sequence

$$\bigvee S^0 \to \bigvee S^0 \to MA.$$

In fact we would construct MA as the cofibre of a map between a wedge of spheres like this. The long exact sequence of homotopy groups has terms

$$\to F_1 \otimes \pi_n S^0 \xrightarrow{k\otimes 1} F_2 \otimes \pi_n S^0 \to \pi_n MA \to$$

and so can be broken up into short exact sequences

$$0 \to \operatorname{coker}(k \otimes 1) \to \pi_n MA \to \ker(k \otimes 1) \to 0.$$

To identify the terms in this sequence we can tensor our free resolution of A with $\pi_k S^0$ and obtain an exact sequence

$$F_1 \otimes \pi_k S^0 \xrightarrow{k\otimes 1} F_2 \otimes \pi_k S^0 \to A \otimes \pi_k S^0 \to 0.$$

This tells us that the cokernel in the short exact sequence above is $A \otimes \pi_n S^0$ and the kernel (by definition of Tor) is $\operatorname{Tor}(A, \pi_{n-1} S^0)$. Since we are assuming that A is torsion free this Tor term vanishes and the short exact sequences become isomorphisms $\pi_n MA \cong A \otimes \pi_n S^0$. $\square$

From this lemma and the previous proposition we deduce the isomorphisms $H^1(MA\langle 0\rangle; \mathbb{Z}_p) \cong \operatorname{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Z}_p) = 0$ and $H^2(MA\langle 0\rangle; \mathbb{Z}_p) \cong \operatorname{Ext}(A \otimes \mathbb{Z}/2, \mathbb{Z}_p)$ for torsion free A. In fact these results still hold under the weaker assumption that A has no p-torsion, or indeed with no assumptions on A.

Lemma 3.2 *If A is an Abelian group then*

$$H^1(MA\langle 0\rangle; \mathbb{Z}_p) \cong \operatorname{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Z}_p) = 0$$

and

$$H^2(MA\langle 0\rangle; \mathbb{Z}_p) \cong \operatorname{Ext}(A \otimes \mathbb{Z}/2, \mathbb{Z}_p) \cong \operatorname{Hom}(A, \mathbb{Z}/2 \otimes \mathbb{Z}_p).$$

Proof
The result for H^1 is clear since the tor term vanishes when $n = 1$. For $n = 2$ we find that $H_2MA\langle 0\rangle \cong \text{coker}(\times\eta) \cong \pi_2 MA/(A \otimes \mathbb{Z}/2) \cong \text{Tor}(\mathbb{Z}/2, A) \cong {}_2(A)$ so that the Hom term in H^2MA vanishes here too. The final isomorphism can be obtained using the following result from [CE99]:

If C is torsion free and B is a torsion module then the sequence

$$0 \to C \to \mathbb{Q} \otimes C \to \mathbb{Q}/\mathbb{Z} \otimes C \to 0$$

is exact and so

$$\text{Hom}(B, \mathbb{Q}/\mathbb{Z} \otimes C) \cong \text{Ext}(B, C).$$

In our case B is the torsion module $A \otimes \mathbb{Z}/2$ and C is the p-adics. Therefore we have isomorphisms $\text{Ext}(A \otimes \mathbb{Z}/2, \mathbb{Z}_p) \cong \text{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}_p) \cong \text{Hom}(A, \text{Hom}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}_p)) \cong \text{Hom}(A, {}_2(\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_p) \cong \text{Hom}(A, \mathbb{Z}/2 \otimes \mathbb{Z}_p)$.
□

We now state our conclusion in the following proposition.

Proposition 3.3 *Let p be a prime and let A be a abelian group that has no p-torsion. If $p > 2$ or $p = 2$ and $A = 2A$ then $K^0(\Omega_0^\infty MA; \mathbb{Z}_p)$ is the free p-adic θ^p-ring on $A \otimes \mathbb{Z}_p$. If $A/2A \neq 0$ then $K^0(\Omega_0^\infty MA; \mathbb{Z}_2) \cong W(A \otimes \mathbb{Z}_2 \to \text{Hom}(A/2A, \mathbb{Z}/2))$.*

Proof
The proposition follows from Bousfield's theorem and the isomorphisms

$$\begin{aligned}\text{Hom}(A, \mathbb{Z}/2 \otimes \mathbb{Z}_2) &\cong \text{Hom}(A, \mathbb{Z}/2) \cong \text{Hom}(A, \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2)) \\ &\cong \text{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Z}/2)\end{aligned}$$

for $p = 2$. (For $p > 2$ the first term is of course zero.) □

In the subsequent sections we will apply Bousfield's theorem to the example $E = tmf$ to obtain a description of $K^*(\Omega_0^\infty tmf; \mathbb{Z}_p)$. For the moment let us record the computation of $H^1(tmf\langle 0\rangle; \mathbb{Z}_p)$ and $H^2(tmf\langle 0\rangle; \mathbb{Z}_p)$.

Lemma 3.3 *For all primes p*

$$H^1(tmf\langle 0\rangle; \mathbb{Z}_p) = 0.$$

For $p > 2$, $H^2(tmf\langle 0\rangle; \mathbb{Z}_p) = 0$ while

$$H^2(tmf\langle 0\rangle; \mathbb{Z}_2) \cong \mathbb{Z}/2.$$

Proof
The homotopy groups of tmf agree with those of the sphere spectrum in small dimensions so the result is the same as for S^0. □

This result means that at odd primes $K^*(\Omega_0^\infty tmf; \mathbb{Z}_p)$ is a free θ^p-ring.

4 The K-homology of tmf

4.1 K-cohomology and K-homology

In the previous section we described Bousfield's theorem which gives the $\mathbb{Z}/2$-graded p-adic λ-ring $K^*(\Omega_0^\infty E; \mathbb{Z}_p)$ functorially in terms of $K^*(E; \mathbb{Z}_p)$ when the latter is torsion free. Our next objective is to apply the theorem to the case $E = tmf$. To describe the K-cohomology of the spectrum tmf it will be convenient first to consider K-homology. To this end we now describe the relation between K-homology and K-cohomology with various coefficients. Let $\mathbb{Z}_{p^\infty}$ denote the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. Also for a locally compact Hausdorff abelian group A let $A^\#$ be the Pontryagin dual of continuous characters of A into S^1. (Note that if A is finite then such characters must map into $\mathbb{Q}/\mathbb{Z}$.) The relevant result is the following (corollary 2.3 of [Bou99])

Proposition 4.1 *For a space or spectrum X there are natural isomorphisms*

$$K^n(X; \mathbb{Z}_p) \cong K_n(X; \mathbb{Z}_{p^\infty})^\#$$

$$K^n(X; \mathbb{Z}/p^i) \cong K_n(X; \mathbb{Z}/p^i)^\#$$

for $i, n \in \mathbb{Z}$ with $i > 0$.

Proof

To prove this statement we simply need to check that the expressions on the right hand side define a cohomology theory in X and that the coefficients agree (naturally). The first statement follows from the fact that the Pontryagin duality functor is exact and transforms direct sums into direct products. To see exactness it suffices to show that $S^1 = \mathbb{R}/\mathbb{Z}$ is injective. To prove the injectivity of $\mathbb{R}/\mathbb{Z}$ we use Baer's criterion, which says that for a ring R, an R module M is injective if and only if every R module homomorphism from an ideal of R into M, extends to all of R. In our case $R = \mathbb{Z}$ so an ideal of R is of the form $n\mathbb{Z}$ for some n. If $f : n\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a homomorphism then we can find some $x \in \mathbb{R}/\mathbb{Z}$ such that $nx = f(n)$. We can extend f to all of $\mathbb{Z}$ by defining $f(1) = x$. To see the preservation of limits we simply note that the Pontryagin duality functor is representable and hence continuous. It remains to check that the coefficients agree. In the first case this means we want an isomorphism $\mathbb{Z}_p \cong \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})$. The map sending a p-adic number α to the map multiplication by α gives us such an isomorphism. In fact since this map is induced by the multiplication map of the spectra $K\mathbb{Z}_p$ and $K\mathbb{Z}_{p^\infty}$ this gives us the naturality that we want. The argument in the second case is the same. □

As a corollary to this proposition we find that if $K^n(X;\mathbb{Z}_p)$ is torsion free then $K_n(X;\mathbb{Z}/p) \cong (K^n(X;\mathbb{Z}_p)/pK^n(X;\mathbb{Z}_p))^{\#}$. In addition to the functorial description of $K^*(\Omega_0^\infty E;\mathbb{Z}_p)$, Bousfield proves that is torsion free when $K^*(E;\mathbb{Z}_p)$ and so there is the following statement in terms of mod p K-homology.

Theorem 4.1 *If E is a spectrum with $K^*(E;\mathbb{Z}_p)$ torsion free then there is a $\mathbb{Z}/2$ graded Hopf algebra isomorphism*

$$K_*(\Omega_0^\infty E;\mathbb{Z}/p) \cong (W(K^*(E;\mathbb{Z}_p)_H)/pW(K^*(E;\mathbb{Z}_p)_H))^{\#}.$$

The K-homology groups of an infinite loopspace, with mod p coefficients, are an attractive object to study, as they are a commutative cocommutative Hopf algebra. These objects can be understood quite well. For example in the case where E is 2-connected we find that $K_*(\Omega_0^\infty E;\mathbb{Z}/p)$ is a tensor product of Witt and exterior Hopf algebras.

4.2 The ring of divided congruences

According to proposition 4.1 we can recover the K-cohomology of a spectrum with $\mathbb{Z}_p$ coefficients from the K-homology with coefficients in $\mathbb{Z}_{p^\infty}$. We will see below that we can describe the K-homology of tmf in terms of Katz ring of divided congruences. We describe this ring now. We recall some definitions from [Gou88] Let N be an integer prime to p and A be a p-adic ring (meaning that A is a $\mathbb{Z}_p$-algebra which is complete and Hausdorff in the p-adic topology). If E is an elliptic curve over A then a level N structure is an inclusion of the cyclic group scheme of order N over A, μ_N, into E. Let $M(A,k,N)$ denote the space of classical holomorphic modular forms of weight k and level N over a p-adic ring A. This A-module can be viewed as containing functions f of isomorphism classes of triples (E,ω,ι), consisting of an elliptic curve E over A, a nowhere vanishing differential ω, and a level N structure ι. These functions are natural with respect to A and are homogeneous of degree $-k$ in ω, meaning that for any $\lambda \in A^\times$, $f(E,\lambda\omega,\iota) = \lambda^{-k}f(E,\omega,\iota)$. The q-expansion of a modular form f is defined to be the value of f on the triple $(\mathrm{Tate}(q),\omega_{can},\iota_{can})$. Tate$(q)$ is the elliptic curve with canonical Weierstrass equation

$$y^2 + xy = x^3 + h_4x + h_6$$

where $h_4 = -5S_3$ and $h_6 = -\frac{5S_3+7S_5}{12}$ for $S_k = \sum_{n\geqslant 1}\frac{n^k}{1-q^n}$. The canonical differential is $dx/(2y+x)$ and $\iota_{can}(\xi) = \xi$ "mod $q^{\mathbb{Z}}$". We refer to [Kat76] for full details.

If $p \geqslant 5$ then the Tate curve can be given by the equation

$$y^2 = 4x^3 - \frac{E_4x}{12} + \frac{E_6}{216} := 4x^3 - g_2x + g_3$$

with canonical differential dx/y. In this case $M(A,k,N)$ consists of the homogeneous polynomials of degree k in the Eisenstein series E_4 and E_6 (which have degrees 4 and 6 respectively) with coefficients in A (or equivalently polynomials in g_2 and g_3). The Eisenstein series have q-expansions

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

and

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

where $\sigma_k(n)$ is the sum of the k^{th} powers of the divisors of n, which gives an explicit way to compute q-expansions in general.

Now let us suppose that B is a p-adically complete discrete valuation ring, and that K is its field of fractions. We remark that this means that B is the topological inverse limit of the discrete rings B/p^nB, and that the localization, B_p, of B at its maximal ideal can be viewed as the ring of K-endomorphisms of K/B_p. More pertinently though is that this condition assures us that any elliptic curve over B, with ordinary fibres, admits a trivialization. The module of divided congruences of weight less than or equal to k is defined by

$$D_k(B,N) = \{f \in \bigoplus_{j=0}^{k} M(K,j,N) | f(q) \in B[[q]]\}.$$

The ring $D(B,N)$ of divided congruences of modular forms defined over B is the direct limit over k of the modules $D_k(B,N)$. The point of this is that the individual summands in f need not have q-expansions with coefficients in B, but that the denominators are cancelled in the whole sum.

Katz showed that the ring of divided congruences (or rather its p-adic completion) is, in some sense, the coordinate ring of the moduli problem of 'elliptic curves over p-adic rings together with isomorphisms of their formal groups with the multiplicative group $\mathbb{G}_m$'. More precisely, for a p-adic ring A we consider the set of isomorphism classes of triples (E, ϕ, ι) where E is an elliptic curve over A, $\phi : \hat{E} \to \hat{\mathbb{G}}_m$ is an isomorphism of formal groups over A, and ι is a level N structure which composes with ϕ to give the canonical inclusion. This set has the structure of an algebraic stack $\mathcal{M}$. The functor which assigns this set to a p-adic ring A is representable by a p-adic ring $W = W(\mathbb{Z}_p, N)$ (which should be thought of as the coordinate ring of the stack). This means that for any p-adic ring A, the set of (p-adically) continuous homomorphisms from W to A (or equivalently the stack morphisms from Spec$A$ to $\mathcal{M}$) can be naturally identified with $\mathcal{M}$. The elements of W are called generalized p-adic modular functions. Restricting attention to B-algebras, for a p-adic ring

B, we obtain in the same way the representing space $W(B, N) = W\hat{\otimes} B$ of generalized p-adic modular functions over B.

Given a triple (E, ϕ, ι) and a generalized p-adic modular function f we can apply the homomorphism corresponding to (E, ϕ, ι) to f to obtain a value $f(E, \phi, \iota) \in A$. (Note that we regard ourseves as evaluating f on the triple.) The q-expansion $f(q)$ of f is defined in this setting by evaluating f on the triple consisting of the Tate curve over $\widehat{B((q))}$ with its canonical trivialisation and level N structure. The key facts are that the q-expansion map $W(B, N) \to \widehat{B((q))}$ is injective with flat (over B) cokernel, and that the smallest ring that f is defined over is that generated by the coefficients of $f(q)$. These facts are called the q-expansion principle. The function f is called holomorphic if its q-expansion lies in $B[[q]]$. The ring of holomorphic generalized p-adic modular functions over B is denoted $V = V(B, N)$. Clearly the q-expansion principle remains true if we replace W by V.

We can regard classical modular forms $\tilde{f}$ as generalized p-adic modular functions by identifying $\tilde{f}$ with the function whose value on the triple (E, ϕ, ι) is given by $\tilde{f}(E, \phi^* \left(\frac{dt}{1+t}\right), \iota)$, where $dt/(1+t)$ is the canonical differential on $\mathbb{G}_m$. With this identification, Katz proves the following theorem:

Theorem 4.2 *If B is a p-adically complete discrete valuation ring then there is an injection $D(B, N) \to V(B, N)$. Moreover the image is dense.*

Proof
The map is constructed using the q-expansion principle. Given $\tilde{f} \in D(B, N)$ the fact that B is a DVR implies that $\tilde{f}(q)$ is torsion in $\widehat{B((q))}/V$ and so by flatness, there is some $f \in V$ with the same q-expansion as $\tilde{f}$. The proof that the image is dense consists of explicitly constructing certain divided congruences d_n which form a sequence of Artin-Schreier generators for V. We refer to [Gou88] for details. □

The second statement in this theorem refers to the p-adic topology. It follows that the p-adic completion $\lim D(B, N)/p^n D(B, N)$ of $D(B, N)$ is equal to V.

It remains to explain in what sense V is the coordinate ring of a moduli problem. $V = V(\mathbb{Z}_p, N)$ can be constructed as a limit of rings $V_{n,m}$, which in each case are the coordinate ring of an affine scheme. For $M > 0$ this scheme is $\mathcal{M}(Np^m) \otimes \mathbb{Z}/p^n$, where $\mathcal{M}(Np^m)$ is obtained by adding cusps to the moduli space of elliptic curves with an arithmetic level Np^m structure. For $m = 0$ we have to delete the supersingular points in $\mathcal{M}(N)$ to obtain the affine scheme of ordinary points, and $V_{n,0}$ is the coordinate ring of $\mathcal{M}^{ord}(N) \otimes \mathbb{Z}/p^n$. Then

$$V = \varprojlim_n \varinjlim_m V_{n,m}.$$

As above we take the completed tensor product over $\mathbb{Z}_p$ with B to obtain $V(B, N)$.

An alternative approach, which we shall use below, is that described in [Kat75] The ring V_B is defined to be the representing object for the functor $M^{\text{triv}}(A)$ of isomorphism classes of trivialized elliptic curves over a B-algebra A. M^{triv} is constructed from the stack of trivialized elliptic curves by passing to isomorphism classes. The q-expansion map is defined by evaluation on the Tate curve with its canonical trivialization and using this to define $D = D_B$, the q-expansion principle and theorem 4.2 still hold. This approach is preferable because in the above we need $N \geqslant 3$ for $\mathcal{M}^{ord}(N)$ to be representable by a scheme. To recover V_B from $V(B, N)$ we need to take the ring of invariants under the action of $GL(2, \mathbb{Z}/N)$.

4.3 Formulae for the generators of D

The generators d_n for the ring of divided congruences constructed by Katz in his proof of theorem 4.2 are not uniquely determined. For $p \geqslant 5$ Katz gives a formula for a particular choice of d_n in terms of the canonical differential. Specifically if we set $t = x/y$ and expand $\omega = dx/y$ in terms of t we obtain

$$\omega = \sum_{n \geqslant 1} a_n t^{n-1} dt$$

for some $a_n \in \mathbb{Z}[g_2, g_3]$. The d_n can be described by saying that a_{p^n} is the n^{th} Witt polynomial in the d_i. This means that

$$a_{p^n} = \sum_{i=0}^{n} p^i d_i^{p^{n-i}}.$$

Given a_{p^n} for all n we can construct d_n inductively from this formula by rearranging and dividing by p^n. In appendix B we carry out the computation of a_n for $n \leqslant 125$, expanding and correcting the formulae given by Katz. This gives us the following explicit formulae for certain d_n.

p = 5 :

The coefficients of t^4, t^{24} and t^{124} are $16g_2$,$1081344(240g_2^3g_3^2 - 7g_2^6 - 60g_3^4)$ and

$$\begin{gathered}
99556996932177919076296818688 g_2 (147267354446784 00 g_3^8 g_2^{18} \\
+290884183154073600 g_3^{12} g_2^{12} - 105705234414024960 g_3^{10} g_2^{15} \\
+43119208 g_2^{30} - 6096167686963200 g_3^{18} g_2^3 \\
+8882987201003520 0 g_3^{16} g_2^6 - 288697084032614400 g_3^{14} g_2^9 \\
+37563035811840 g_3^{20} + 14430842538840 g_3^4 g_2^{24} \\
-780963243278400 g_3^6 g_2^{21} - 72682814985 g_3^2 g_2^{27})
\end{gathered}$$

respectively. Therefore we have

$$d_0 = a_1 = -2$$

$$d_1 = \frac{a_5 - d_0^5}{5} = \frac{16g_2 + 32}{5}$$

$$d_2 = \frac{a_{25} - 5d_1^5 - d_0^{25}}{25}$$

$$= \frac{1081344(240g_2^3g_3^2 - 7g_2^6 - 60g_3^4) - \left(\frac{16g_2+32}{5}\right)^5 + 1048576}{25}$$

$$= -\frac{7569408}{25}g_2^6 + \frac{51904512}{5}g_2^3g_3^2 - \frac{12976128}{5}g_3^4 -$$

$$\frac{1048576}{15625}g_2^5 - \frac{2097152}{3125}g_2^4 - \frac{8388608}{3125}g_2^3$$

$$-\frac{16777216}{3125}g_2^2 - \frac{16777216}{3125}g_2 + \frac{20937965568}{15625}$$

and d_3 can be written down from this, but is too cumbersome to print. Note that in Katz original paper $a_1 = -2$ was erroneously replaced by 1 in the formula for d_n. Given these formulae and the q-expansions for g_2 and g_3 we can check that q-expansions of these forms do indeed lie in $\mathbb{Z}[1/6][q]$. For example $\frac{16g_2+32}{5}$ has q-expansion

$$\frac{1}{5}\left(\frac{16}{12}(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n) + 32\right)$$

$$= \frac{1}{5}\left(\frac{400}{12} + 320\sum_{n=1}^{\infty}\sigma_3(n)q^n\right) = \frac{20}{3} + 64\sum_{n=1}^{\infty}\sigma_3(n)q^n.$$

$\mathbf{p = 7}$:

From our calculations in appendix B we have

$$a_7 = 96g_3$$

$$a_{49} = 997103501312(6336g_3^8 + 236544g_2^3g_3^6 - 221760g_3^4g_2^6 + 18480g_3^2g_2^9 - 91g_2^{12})$$

and so for $p = 7$ we find

$$d_0 = -2$$

$$d_1 = \frac{96}{7}g_3 + \frac{128}{7}$$

$$d_2 = -\frac{6317647784312832}{49}g_3^8 + \frac{33694121516335104}{7}g_2^3g_3^6$$

$$-\frac{31588238921564160}{7}g_3^4g_2^6+\frac{2632353243463680}{7}g_3^2g_2-\frac{12962345517056}{7}g_2^{12}$$
$$-\frac{75144747810816}{5764801}g_3^7-\frac{100192997081088}{823543}g_3^6-\frac{400771988324352}{823543}g_3^5$$
$$-\frac{890604418498560}{823543}g_3^4-\frac{1187472557998080}{823543}g_3^3-\frac{949978046398464}{823543}g_3^2.$$

As before we can check that these forms are defined over $\mathbb{Z}[1/6]$. For example the q-expansion of d_1 is

$$\frac{1}{7}(\frac{-96}{216}(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n+128)$$
$$=\frac{1}{7}(\frac{1148}{9}+224\sum_{n=1}^{\infty}\sigma_5(n)q^n)$$
$$=\frac{164}{9}+32\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

$\mathbf{p=11}$:

We give one more calculation, for $p=11$. Again from appendix B we have

$$a_{11}=-2560g_2g_3$$

$$a_{121}=-272699558638918420400406398026907648g_2^{30}$$
$$+72196520290575139055781872468913449808691200 0g_3^{14}g_2^9$$
$$+3844345961196332445610095594047817331507200g_3^6g_2^{21}$$
$$-15959230801074504422857045493128236273500160 0g_3^{16}g_2^6$$
$$-644493293494679262940516026060957611458560 00g_3^8g_2^{18}$$
$$+4006906876126920217595893922024696464382361 60g_3^{10}g_2^{15}$$
$$+641899680284545175378865580417425290035200 0g_3^{18}g_2^3$$
$$-78349442143946992777379665639561494528000g_3^4g_2^{24}$$
$$-920779105372600353033399865919816611764633600g_3^{12}g_2^{12}$$
$$-1013525810975597645335050916448566247424 0g_3^{20}$$
$$+4285781450609066271583161197377722777 60g_3^2g_2^{27}.$$

Therefore

$$d_0=-2$$
$$d_1=-\frac{2560}{11}g_3g_2+\frac{2048}{11}$$

$$
\begin{aligned}
d_2 = {} & \frac{349485996472393858691826872186165211955200}{11} g_3^6 g_2^{21} \\
& + \frac{58354516389504106852624143674311390003200 0}{11} g_3^{18} g_2^3 \\
& + \frac{656332002641592173234380658808304089169920 00}{11} g_3^{14} g_2^9 \\
& - \frac{920779105372600353033399865919816611764633600}{121} g_3^{12} g_2^{12} \\
& - \frac{921387100886906950304591742225969315840}{11} g_3^{20} \\
& - \frac{712267655854063570703451505814195404800 0}{11} g_3^4 g_2^{24} \\
& + \frac{364264261466083656145081265638608769489305 60}{11} g_3^{10} g_2^{15} \\
& - \frac{585902994086072057218650932782688737689600 0}{11} g_3^8 g_2^{18} \\
& + \frac{389616495509915115598469199761611161 60}{11} g_3^2 g_2^{27} \\
& - \frac{145083916373404585662336777210256693395456 00}{11} g_3^{16} g_2^6 \\
& - \frac{247908689671744018545823998206279 68}{11} g_2^{30} \\
& + \frac{689535018337607465415713573579381846040379392 00}{3138428376721} \\
& + \frac{309485009821345068724781056000000000000}{3138428376721} g_3^{11} g_2^{11} \\
& - \frac{247588007857076054979824844800000000000}{285311670611} g_3^{10} g_2^{10} \\
& + \frac{990352031428304219919299379200000000000}{285311670611} g_3^9 g_2^9 \\
& - \frac{237684487542793012780631851008000000000}{285311670611} g_3^8 g_2^8 \\
& + \frac{380295180068468820449010961612800000000}{285311670611} g_3^7 g_2^7 \\
& - \frac{425930601676685078902892277006336000000}{285311670611} g_3^6 g_2^6 \\
& + \frac{340744481341348063122313821605068800000}{285311670611} g_3^5 g_2^5 \\
& - \frac{194711132195056036069893612345753600000}{285311670611} g_3^4 g_2^4
\end{aligned}
$$

$$+\frac{7788445287802241442795744493830144000 0}{285311670611}g_2^3g_3^3$$

$$-\frac{20769187434139310514121985316880384000}{285311670611}g_3^2g_2^2$$

$$+\frac{3323069989462289682259517650700861440}{285311670611}g_3g_2.$$

Let us try to show that the q-expansion of $-\frac{2560}{11}g_3g_2+\frac{2048}{11}$ has coefficients in $\mathbb{Z}[1/6]$. If we multiply by 11 then the constant term is $\frac{2560}{2592}+2048=\frac{165968}{3^4}$. Since $165968=11\times 15088$ this coefficient is okay. For $n\geqslant 1$ the coefficient of q^n is $\frac{80}{81}(240\sigma_3(n)-504\sigma_5(n)-240\cdot 504\sum_{k=1}^{n-1}\sigma_3(k)\sigma_5(n-k))$ so it is necessary to show that the term inside the parentheses is divisible by 11. Reducing mod 11, then dividing by 2 this becomes $\sigma_5(n)-\sigma_3(n)-2\sum_{k=1}^{n-1}\sigma_3(k)\sigma_5(n-k)$ which should be divisible by 11. A direct proof of this congruence has proved to be elusive, which demonstrates how remarkable it is that every coefficient in these q-expansions lies in $\mathbb{Z}[1/6]$. In the course of searching for a proof we did however uncover the following fact.

Proposition 4.2 *Half the number of integer solutions to the equation*

$$x^2+xy+3y^2=n$$

is equal to $\sigma_5(n)$ mod 11.

Proof

The form $x^2+xy+3y^2$ has discriminant $1-4\cdot 3=-11$. The number of equivalence classes of quadratic forms of discriminant -11 is equal to the number of triples (a,b,c) of integers satisfying $b^2-4ac=-11$ and either $-a<b\leqslant a<c$ or $0\leqslant b\leqslant a=c$. (See [Ken82] p323) In this case we see that $b^2-4ac\leqslant b^2-4b^2=-3b^2$ so we must have $b=\pm1$. In either case the inequalities force $a=1$ and $c=3$. However for $b=-1$ we contradict $b\geqslant 0$ and $b>-1$ so we must have $b=1$. It follows that $x^2+xy+3y^2$ represents the unique class of forms of discriminant -11. A theorem due to Dirichlet ([Ken82] p307) states in this case that for $n>0$ prime to -11, the number $s(n)$ of solutions to $x^2+xy+3y^2=n$ is given by

$$s(n)=2\sum_{d|n}\left(\frac{-11}{n}\right)$$

where $\left(\frac{-11}{n}\right)$ is Kronecker's symbol. Specifically $\left(\frac{-11}{n}\right)$ is zero if 11 divides n, 1 if n is a quadratic residue mod 11 and -1 if n is a quadratic non-residue mod 11. Both this expression and $\sigma_5(n)$ are multiplicative so we can assume that n is a prime power. If $n=p^r$ where p is not 11 then $\sigma_5(n)=\sum_{s=0}^{r}p^{5s}$. If p^s is a quadratic residue mod 11 then p^{5s} is a tenth power so is 1 mod 11. Otherwise $p^{5s} = -1 \bmod 11$. It follows that $\sigma_5(n) \mod 11 = s(n)/2$ in this case.

If $n = 11^r$ we claim that there are exactly 2 solutions to $x^2 + xy + 3y^2 = n$ and so the formula holds in this case too. For $n = 11$ it is easy to see that the only solutions (x, y) are $(-1, 2)$ and $(1, -2)$ so this is okay. For higher powers of 11 we will show that both x and y are divisible by 11 and so by dividing by 121 and using induction on r we will be done. (We should note that the only solutions with $n = 1$ are $(1, 0)$and $(-1, 0)$.) To prove that x and y are both divisible by 11 we use the standard algebraic reduction that says for a solution (x, y) to $x^2 + xy + 3y^2 = n$ we obtain a solution (X, Y) to $X^2 + 11Y^2 = 44n$ where $X = -11y$ and $Y = 2x + y$. If 11 divides n then 121 divides $11Y^2$ and so 11 divides Y. Now for $r > 1$ we have 11^3 divides X^2 and so 11 divides y. This and the fact that 11 divides $2x + y$ allow us to conclude that 11 divides x. □

The same proof will show that $\sigma_3(n)$ mod 7 is half the number of integer solutions to $x^2 + xy + 2y^2 = n$. Indeed whenever we have a prime p of the form $4k+3$ ($k > 0$) for which $x^2+xy+(k+1)y^2$ represents the unique class of forms of discriminant $-p$, we will find that $\sigma_{2k+1}(n) \bmod p$ is half the number of solutions to $x^2 + xy + (k+1)y^2 = n$. Note that $p = 3$ doesn't work because the automorphism group of $x^2 + xy + y^2$ has 6 elements rather than 2. The next primes that work are $p = 19$, $p = 43$ and $p = 67$. A necessary condition for $x^2+xy+(k+1)y^2$ to represent the unique class of forms of discriminant $-p$ for larger primes is that both $k+1$ and $k+3$ must be prime, for if either were composite, the factors would give different values for a and c that represent a different class of forms of discriminant $-p$. The only part of the proof which is not clearly the same is that $(-1, 2)$ and $(1, -2)$ are the only solutions for $n = p$. To see this we change the equation to $(2x + y)^2 + py^2 = 4p$. It follows that $|y|$ is at most 2. The values $y = 1, 0, -1$ are easily ruled out if $p > 3$ since neither $3p$ nor $4p$ is a square. This just leaves us with the two stated solutions.

Higher Primes

The formulae for a_p in appendix B for larger primes p will give many more congruences mod p between $\sigma_3(m)$ and $\sigma_5(n)$ for various m and n. These formulae will be quite large for any primes bigger than 11, and so are perhaps most useful simply stated as congruences for the q-expansions of various polynomials in g_2 and g_3.

4.4 D as $K_0 tmf$.

In section 4.2 we defined the ring of divided congruences and explained how its p-adic completion can be viewed as the coordinate ring of a certain moduli problem. Next we show that this ring is precisely $K_0 tmf$. We give two related but philosophically different approaches to this calculation, based on two different characterisations of tmf. The first constructs tmf directly as a homotopy inverse limit and then uses various homotopy theory results to calculate $K_0 tmf$ as an inverse limit. The second approach is to describe $K \wedge tmf$ directly as a stack of trivialized elliptic curves, as considered above, and then to prove that this stack is in fact a scheme. This approach is more direct, and in fact will be used to complete the calculation from the first approach, but the first approach has the advantage of being described using only homotopy theory.

The following lemma is the key algebraic step.

Lemma 4.1 *If J is an elliptic curve and $T + O(T^5)$ is a coordinate near infinity. Then there exists a unique pair (x', y') of Weierstrass coordinates for J such that $x'/y' \equiv T \mod T^5$.*

Proof

If we choose an arbitrary pair (x, y) of Weierstrass coordinates then any such coordinates (x', y') are related to (x, y) by a transformation of the form $x' = \lambda^{-2}x + r$, $y' = \lambda^{-3}y + \lambda^{-2}sx + t$. We can expand x and y as Laurent series in T to get $x = T^{-2}(u_0 + u_1 T + u_2 T^2 + \cdots)$ and $y = T^{-3}(v_0 + v_1 T + v_2 T^2 + \cdots)$ where u_0 and v_0 are invertible and $u_0^3 = v_0^2$. Substituting these power series into x' and y' and expanding x'/y' as a power series in T (with the help of Maple) gives

$$
\begin{aligned}
x'/y' = {} & \frac{l u_0}{v_0} T - \frac{l(-u_1 v_0 + u_0 v_1 + l s u_0^2)}{v_0^2} T^2 \\
& + \frac{\lambda(-u_0 v_0 v_2 - 2u_0 v_0 \lambda s u_1 + u_2 v_0^2 + r v_0^2 \lambda^2 - u_1 v_0 v_1 + u_0 v_1^2 + 2u_0^2 v_1 \lambda s + \lambda^2 s^2 u_0^3)}{v_0^3} T^3 \\
& - \lambda(2u_0 v_0^2 s u_2 \lambda + u_0 v_0^2 v_3 + u_0 v_0^2 \lambda^3 t - u_3 v_0^3 + u_1 v_0^2 v_2 + u_1^2 v_0^2 \lambda s - 2v_0 u_0 v_1 v_2 \\
& - 4v_0 u_0 v_1 \lambda s u_1 - 2v_0 \lambda s u_0^2 v_2 - 3v_0 \lambda^2 s^2 u_0^2 u_1 + u_2 v_0^2 v_1 + r v_0^2 \lambda^2 v_1 + r v_0^2 \lambda^3 s u_0 \\
& - u_1 v_0 v_1^2 + u_0 v_1^3 + 3u_0^2 v_1^2 \lambda s + 3u_0^3 v_1 \lambda^2 s^2 + \lambda^3 s^3 u_0^4) T^4 / v_0^4 + O(T^5)
\end{aligned}
$$

We have to solve for λ, r, s and t such that this expression is $T \bmod T^5$. This gives us 4 equations in 4 variables (from the coefficients of T, T^2, T^3, T^4) with Jacobian matrix

$$
\begin{pmatrix}
\frac{u_0}{v_0} & \star & \star & \star \\
0 & \frac{\lambda^2 u_0^2}{v_0^2} & \star & \star \\
0 & 0 & \frac{\lambda^3}{v_0} & \star \\
0 & 0 & 0 & -\frac{u_0 \lambda^4}{v_0^2}
\end{pmatrix}.
$$

The first equation gives $\lambda = \frac{v_0}{u_0}$ and so the Jacobian is invertible. The inverse function theorem therefore tells us that there is a unique solution to the equations. In fact with some more algebra we find that this solution is

$$\begin{aligned} \lambda &= \frac{v_0}{u_0} \\ s &= \frac{v_1}{v_0} - \frac{u_1}{u_0} \\ r &= \frac{(-u_3 u_0 v_0 + v_2 u_0^2 - u_1 v_1 u_0 + u_1^2 v_0) u_0}{v_0^3} \\ t &= \frac{(-u_3 u_0 v_0 + v_3 u_0^2 - u_2 v_1 u_0 + u_2 u_1 v_0) u_0}{v_0^3}. \end{aligned}$$

□

The spectrum *tmf* is constructed to have the mapping properties of a universal elliptic cohomology theory. Specifically, we can construct a functorial diagram of Landweber exact elliptic cohomology theories, E_i, each of which is an E_∞ spectrum satisfying further properties from obstruction theory. This means in particular that each E_i is complex orientable, and the formal group associated to this orientation comes with an isomorphism to the formal completion of an elliptic curve E over $E_i^\star$. Furthermore on finite CW-complexes X we have $E_i^\star(X) = E_i^\star \otimes_{MU^\star} MU^\star(X)$, where the $MU^\star$ action on $E_i^\star$ comes from the complex orientation. For any spectrum R, there is a map from $R \wedge tmf$ to the homotopy limit $\mathrm{Holim}(R \wedge E_i)$. The first result from homotopy theory is that this map is a homotopy equivalence. A proof of this statement would involve the nilpotence theorems of [Smi88] but it is certainly reasonable given the relation of E_i to MU.

This equivalence means that for any R, there is a spectral sequence

$$\lim\nolimits_{\leftarrow}^{s} \pi_t(R \wedge E_i) \Longrightarrow \pi_{t-s}(R \wedge tmf).$$

The second result that will allow us to calculate $\pi_\star(K \wedge tmf) = K_\star tmf$ is the following.

Proposition 4.3 *If R is complex orientable then the derived functors $\lim_{\leftarrow}^{s}$ are zero for $s > 0$. In particular,*

$$\pi_\star(R \wedge tmf) = \lim_{\leftarrow} R_\star E_i.$$

Proof

We may assume that R is even and periodic, so that we can associate to it a formal group $\hat{G}_R$. This is sufficient because it is true for the periodic version of MU, which we denote $MU\{u^{\pm 1}\}$, and we can extend to other R by naturality. In this case $R \wedge tmf$ can be identified with the stack of elliptic curves E together with an isomorphism of $\hat{E}$ with $\hat{G_R}$. (For $R = K$ this would be the stack of trivialized elliptic curves.) If J is the universal trivialized elliptic curve over W then this stack is naturally identified with the stack of isomorphisms $Iso(\hat{J}, \hat{G_R})$ (since such isomorphisms classify such E). We can think of the $R \wedge E_i$ as forming an etale cover of this stack. Then by definition, the s^{th} cohomology group of the stack is the $\lim^s_{\leftarrow}$ that we want. A basic fact about the category of stacks is that it contains the category of schemes as a subcategory, and that the s^{th} cohomology of a scheme is zero for $s > 0$. Therefore it remains to show that $Iso(\hat{J}, \hat{G_R})$ is a scheme. If we choose a coordinate on $\hat{G_R}$ then pulling back that coordinate defines a map from $Iso(\hat{J}, \hat{G_R})$ to the stack $\mathcal{A}$ of coordinates on $\hat{J}_R$. $\mathcal{A}$ can be identified with $MU\{u^{\pm 1}\} \wedge tmf$. We claim that $\mathcal{A}$ is an affine scheme and so this map identifies $Iso(\hat{J}, \hat{G_R})$ as a closed subscheme. This follows from the lemma: If we let b_n denote the coefficients of T^n for $n \geqslant 5$ then

$$\mathcal{A} = \mathrm{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][u^{\pm 1}][b_5, b_6, ...].$$

The details of this can be found in [Rez]. Note that an element of π_{2n} is identified with an element of π_0 by multiplying by u^{-n}. □

Corollary 4.1 *For any p-adic ring B, the ring $K_0(tmf; B)$ can be identified with the p-adic completion of ring of divided congruences D_B, that is V_B.*

Proof

The previous proposition gives us two different ways to see this statement. The direct approach is to observe that $\mathrm{Spec}\,\pi_0(K \wedge tmf) = Iso(\hat{J}_B, \hat{G_m}) = \mathrm{Spec}\, V_B$ Under this identification, forms of weight $2i$ must be multiplied by v^{-1} (v being the Bott class) to put them in π_0. The alternative method is to calculate $K_\star E_i$ directly as in [Lau99] and use the statement of the proposition to obtain $K_\star tmf$ as a limit. □

Corollary 4.2 *There is an isomorphism of $\mathbb{Z}_p$-modules*

$$K^0(tmf; \mathbb{Z}_p) \cong \lim_{\leftarrow} \hom (V_{\mathbb{Z}/p^i}, \mathbb{Z}/p^i)$$

for any prime p.

Proof

This follows from proposition 4.1 and the previous corollary. One simply needs to note that the continuous character must map p^i-torsion to p^i-torsion. □

In section 5 we will expand upon this description and then apply Bousfield's theorem to obtain $K^*(\Omega_0^\infty tmf; \mathbb{Z}_p)$.

4.5 Diamond operators

The ring of divided congruences has a natural action of $\mathbb{Z}_p^\times$ by what are called diamond operators. The p-adic K-homology of a spectrum also has an action of $\mathbb{Z}_p^\times$ given by the stable Adams operations. In this section we define each of these actions and show that they correspond under the identifications of the previous sections.

The diamond operators act on generalized p-adic modular functions through varying the trivialization. Specifically, given a generalized p-adic modular function f and $\alpha \in \mathbb{Z}_p^\times$ we define $\langle\alpha\rangle f$ by the equation

$$\langle\alpha\rangle f(E,\phi) = f(E,\alpha^{-1}\phi).$$

This action preserves the ring V and so gives rise to an action on D. If $\langle\alpha\rangle f = \alpha^k f$ then f is said to have weight k. Under the identification of classical modular forms as generalized p-adic modular functions, a form of weight k becomes a function of weight k. It follows that if $f = \sum f_i \in D$ where f_i has weight i then $\langle\alpha\rangle f = \sum \alpha^i f_i$.

The stable Adams operations are natural transformations of K-homology. For an integer k they can be defined axiomatically as follows. From the work of Miller [Mil89] it follows that natural transformations of a Landweber exact homology theory are determined by their action on coefficients and on line bundles. As we noted in section 3, the operation ψ^k acts by sending the Bott class v to kv, while it acts on line bundles by raising them to the k^th power. We remark that Miller phrased this in the language of formal group laws. In this language ψ^k is described as the ring endomorphism of $K_\star = \mathbb{Z}[v^{\pm1}]$ determined by sending v to kv, together with the strict isomorphism

$$\frac{1}{k}[k] : G_m \to \psi^k G_m$$

of the multiplicative formal group law. To see this we just have to note that the multiplicative formal group law $x + y - xy$ corresponds to the tensor product of the line bundles $(1-x)$ and $(1-y)$. Of course this definition requires us to be able to divide by k so over the integers we only obtain two operations. Working with p-adic K-theory we obtain ψ^k for any k prime to p. We can then extend this to any p-adic unit by continuity.

Given this description of the stable Adams operations, we can easily prove the result that we want.

Proposition 4.4 *Under the identification of $K_0 tmf$ with the completion of D, the Adams operation $\psi^k \wedge 1$ corresponds to the diamond operator $\langle k^{-1}\rangle$.*

Proof
We showed above that if $f = \sum f_i \in D$ where f_i has weight i then $\langle \alpha \rangle f = \sum \alpha^i f_i$. Thus $\langle k^{-1} \rangle f_i = k^{-i} f_i$. Our identification of $K_0 tmf$ with D requires us to multiply f_i by v^{-i} to put it in degree zero. Therefore we have $(\psi^k \wedge 1)(v^{-i} f_i) = (\psi^k(v^{-i}) f_i = k^{-i} v^{-i} f_i$, and so $\psi^k \wedge 1$ also acts by multiplying f_i by k^{-i}. □

In terms of the representing spectra, if we think of $K \wedge tmf$ as the stack of trivializations of the universal elliptic curve, then the diamond operator $\langle k^{-1} \rangle$, which act by multiplying the trivialization by k is represented by a map of the spectrum K. This map is precisely the Adams operation ψ^k.

5 The K-cohomology of tmf and $\Omega_0^\infty tmf$

5.1 Hecke Algebras and duality

In section 4 we showed that there is an isomorphism of $\mathbb{Z}_p$-modules

$$K^\star(tmf; \mathbb{Z}_p) \cong \varprojlim \hom \left(V_{\mathbb{Z}/p^i}, \mathbb{Z}/p^i\right)$$

for any prime p. There is a duality between the ring of generalized p-adic modular functions and the ring of Hecke operators acting on V. Given the formula above we can expect there to be a relation between the K-cohomology of tmf and this Hecke algebra. We will use this relation below to obtain a description of the K-cohomology of $\Omega_0^\infty tmf$.

For the moment let us assume that $p \geqslant 5$ and that l is an integer prime to p. We recall the definition of Hecke algebras from [Gou88] The Hecke operator T_l acts on the ring V of generalized p-adic modular functions as follows. If (E, ϕ) is a trivialized elliptic curve then for every subgroup H of order l, the projection $\pi : E \to E/H$ induces an isomorphism on formal groups. Therefore we obtain a trivialized elliptic curve $(E/H, \phi\pi^{-1})$. If $f \in V$ then $T_l f$ is defined by the formula

$$T_l f(E, \phi) = \frac{1}{l} \sum f(E/H, \phi\pi^{-1})$$

where the sum is over all subgroups of order l. If f has q-expansion $f(q) = \sum a_n q^n$ then

$$T_l f(q) = \sum a_{nl} q^n + \frac{1}{l} (\langle l \rangle f)(q^l).$$

For $l = p$ it can be shown [Gou88] that there is a linear operator U_p on V which on q-expansions sends $\sum a_n q^n$ to $\sum a_{np} q^n$. If we denote U_p by T_p then we obtain T_l for all l according to the following formulae:

1. T_1 is the identity.

2. If l and m are coprime then $T_{lm} = T_l T_m$.

3. If l is a prime distinct from p and $k \geqslant 2$ then

$$T_{l^k} = T_l T_{l^{k-1}} - \frac{1}{l}\langle l \rangle T_{l^{k-2}}$$

4. For any $k > 0$, $T_{p^k} = T_p^k$

It follows from these formulae that any T_l can be written as a polynomial in the T_q for q a prime dividing l. Furthermore the T_q commute with each other. It is clear from the modular definitions that the Hecke operators commute with the diamond operators as well.

Definition 5.1 *If $V' \subset V$ is a subring that is preserved by the Hecke and diamond operators then the Hecke algebra $T_{V'}$ of V' is defined to be the completion of the commutative subalgebra of the space of $\mathbb{Z}_p$-linear endomorphisms of V' generated by the T_l for l prime to p, U_p and the diamond operators. The restricted Hecke algebra $T_{V'}^{\star}$ is obtained by omitting the U_p operator.*

As two particular cases we have $T = T_V$ and $T_{D'}$, where D' is the ring of divided congruences of forms of positive weight. To see that D' is preserved by the Hecke operators we can look at the effect on q-expansions. The only concern would be the term $\frac{1}{l}$, but the effect of the diamond operator $\langle l \rangle$ on a form of weight i is to multiply by l^i, so if we restrict to forms of positive weight this will cancel the l in the denominator.

Now let us consider the ring V_B for a p-adic ring B. For any V' as above there is a bilinear form

$$\begin{array}{rcl} T_{V'} \times V' & \longrightarrow & B \\ (T, f) & \longmapsto & a_1(Tf) \end{array}$$

where $a_1(Tf)$ is the coefficient of q in the q-expansion of Tf. In particular if T is the Hecke operator T_l and f has q-expansion $f(q) = \sum a_n q^n$ then $(T_l, f) = a_l$. Varying l we should be able to determine all of the coefficients (except the constant term) in the q-expansion of f and hence determine f itself up to an additive constant. This means, for example, that if we restrict attention to space V_B^{par} of forms with zero constant term (parabolic forms) then we obtain an injection

$$V_B^{par} \hookrightarrow \hom(T_{V_B^{par}}, B)$$

where the right hand side consists of continuous (with respect to the p-adic topology) homomorphisms of B-modules.

There are several results which strengthen this observation; we mention two.

Proposition 5.1 *Let B be a p-adic ring. Then the map $f \mapsto (\cdot, f)$ induces an isomorphism of $\mathbb{Z}_p$-modules*

$$V_B^{par} \cong \hom\left(T_{V_{\mathbb{Z}_p}^{par}}, B\right)$$

(continuous $\mathbb{Z}_p$-module homomorphisms). Moreover f corresponds to an algebra homomorphism if and only if it is an eigenform for the Hecke and diamond operators and $a_1(f) = 1$.

Proof
We refer to [Gou88] for details. The condition $a_1(f) = 1$ corresponds to the condition that ring homomorphisms must respect multiplicative units. Notice that the ring B occurs in the right hand side only as the target space, and not as part of the Hecke algebra. □

Proposition 5.2 *Let V be a p-adic ring and define $V_B^1 = (V_B + \mathbb{Q}_p)/\mathbb{Q}_p$. Then the map $f \mapsto (\cdot, f)$ induces an isomorphism of $\mathbb{Z}_p$-modules*

$$V_B^1 \cong \hom\left(T_{V_{\mathbb{Z}_p}}, B\right)$$

(continuous $\mathbb{Z}_p$-module homomorphisms). Moreover f corresponds to an algebra homomorphism if and only if it is an eigenform for the Hecke and diamond operators and $a_1(f) = 1$. For $B = \mathbb{Z}_p$ there is also an isomorphism

$$T_{V_{\mathbb{Z}_p}} \cong \hom\left(V_{\mathbb{Z}_p}^1, \mathbb{Z}_p\right)$$

induced by the same pairing.

Proof
See [Gou88]. We note that V_B^1 has a dense subset consisting of sums of forms, whose q-expansion lies in $\mathbb{Q}_p + qB[[q]]$. □

We are interested in the inverse limit over i of the spaces $\hom_{cts}(V_{\mathbb{Z}/p^i}, \mathbb{Z}/p^i)$. We remark that these spaces consist of module homomorphisms. If we consider the subspace of algebra homomorphisms then this space would be exactly the set of isomorphism classes of trivialized elliptic curves over $\mathbb{Z}/p^i$, by definition of V. Also we note that this inverse limit does make sense, since V_B is contravariant as a functor of $\mathbb{Z}_p$-algebras B.

Now let p be any prime and let $\rho_i : \mathbb{Z}_p \to \mathbb{Z}/p^i$ denote the reduction mod p^i map. The map induced by ρ_i on spaces of generalized modular functions is denoted $\rho_i^\star$. For each i there is a map

$$\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p) \stackrel{(\rho_i)_\star}{\to} \hom(V_{\mathbb{Z}_p}, \mathbb{Z}/p^i) \stackrel{(\rho_i^\star)^\star}{\to} \hom(V_{\mathbb{Z}/p^i}, \mathbb{Z}/p^i)$$

which give rise to a map

$$\rho : \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p) \to \underleftarrow{\lim} \hom(V_{\mathbb{Z}/p^i}, \mathbb{Z}/p^i).$$

The final statement in proposition 5.2 also shows that the inclusion of $V_{\mathbb{Z}_p}$ in $V^1_{\mathbb{Z}_p}$ induces a map

$$\phi : T_{V_{\mathbb{Z}_p}} \to \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$$

when $p \geqslant 5$.

Proposition 5.3 *The map* $\rho : \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p) \to \lim_{\leftarrow} \hom(V_{\mathbb{Z}/p^i}, \mathbb{Z}/p^i)$ *is an isomorphism, while* $\phi : T_{V_{\mathbb{Z}_p}} \to \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$ *is injective but not surjective.*

Proof
The map ϕ is not surjective as any map from $V_{\mathbb{Z}_p}$ to $\mathbb{Z}_p$ which does not vanish on the constants does not extend to $V^1_{\mathbb{Z}_p}$. It is injective because a map from $V^1_{\mathbb{Z}_p}$ to $\mathbb{Z}_p$ must vanish on the constants so is determined by its restriction to $V_{\mathbb{Z}_p}$. To prove that ρ is an isomorphism we first note that $V_{\mathbb{Z}_p}$ is the direct limit of the $V_{\mathbb{Z}/p^i}$ as an algebra, but we can regard the same direct system as a diagram of $\mathbb{Z}_p$-modules and obtain the same object as a direct limit of modules. Given this the statement follows by some definition chasing. Specifically an element of the direct limit consists of a family of maps from $V_{\mathbb{Z}/p^i}$ to $\mathbb{Z}/p^i$. These maps factor through each $\mathbb{Z}/p^k$ for $k > i$ and so factor through $\mathbb{Z}_p$. But such a family of maps is precisely a map from the direct limit of the $V_{\mathbb{Z}/p^i}$ to $\mathbb{Z}_p$. □

Corollary 5.1 *There is an isomorphism*

$$K^\star(tmf, \mathbb{Z}_p) \cong \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$$

for all primes p and a split inclusion

$$T_{V_{\mathbb{Z}_p}} \hookrightarrow K^\star(tmf, \mathbb{Z}_p)$$

for every prime $p \geqslant 5$.

Proof
This is mostly just a restatement of proposition 5.3. To obtain a splitting of ϕ we note that elements of $T_{V_{\mathbb{Z}_p}}$ can be viewed as elements of $\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$ which vanish on the constants. Such an element can never be a multiple of a map which doesn't vanish on constants, and so we can construct a right inverse to ϕ by extending the identity map to zero on functions which do not vanish on constants. □

5.2 $K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p)$ for $p > 2$

From corollary 5.1, our calculation of the first and second cohomology groups of tmf and Bousfield's theorem, we have the following statement.

Theorem 5.1 *For $p \geqslant 3$ there is an isomorphism*

$$K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p) \cong T(\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p))$$

while for $p = 2$

$$K^\star(\Omega_0^\infty tmf; \mathbb{Z}_2) \cong W(\hom(V_{\mathbb{Z}_2}, \mathbb{Z}_2) \to \mathbb{Z}/2).$$

Let us recall the definitions of T and W to make this statement explicit. T is left adjoint to the functor sending a connective $\mathbb{Z}/2$-graded p-adic θ^p-ring R to the $\mathbb{Z}/2$-graded p-profinite abelian group $\tilde{R}$. Therefore for $p > 2$ and any connective $\mathbb{Z}/2$-graded p-adic θ^p-ring R, there is an isomorphism

$$\hom_{\theta^p}(K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p), R) \cong \hom_{Ab}(\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p), \tilde{R}).$$

Similarly W is left adjoint to the functor sending a $\mathbb{Z}/2$-graded 2-adic θ^2-ring R to the map $\tilde{R} \to \tilde{R}/\Gamma^2\tilde{R}$. Therefore for any R there is an isomorphism

$$\hom_{\theta^2}(K^\star(\Omega_0^\infty tmf; \mathbb{Z}_2), R) \cong \hom_{Ab}(\hom(V_{\mathbb{Z}_2}, \mathbb{Z}_2) \to \mathbb{Z}/2, \tilde{R} \to \tilde{R}/\Gamma^2\tilde{R}).$$

Let us consider a particular example for $p > 2$. If Y is a connected CW-complex with $\tilde{H}^i(Y; \mathbb{Z}_p) = 0$ for $i < 3$ then $X = K^*(Y; \mathbb{Z}_p)$ is a connective $\mathbb{Z}/2$-graded p-adic θ^p-ring. In particular $Y = S^{2n}$ for $n > 1$ satisfies these hypotheses and according to [Hat01] $\tilde{K}^*(S^{2n}) = \mathbb{Z}$ generated by $(H-1) * \cdots * (H-1)$. It follows that the reduced $p-$adic K-theory of S^{2n} is a single copy of the p-adics and there is an isomorphism

$$\hom_{\theta^p}(K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p), K^\star(S^{2n}; \mathbb{Z}_p)) \cong \hom_{Ab}(\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p), \mathbb{Z}_p).$$

There is a natural map

$$K^\star : \pi_{2n}\Omega_0^\infty tmf \to \hom_{\theta^p}(K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p), K^\star(S^{2n}; \mathbb{Z}_p))$$

given by applying K-theory to a representing map from S^{2n} to $\Omega_0^\infty tmf$. There is also a map

$$e : V_{\mathbb{Z}_p} \to \hom_{Ab}(\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p), \mathbb{Z}_p)$$

sending $v \in V_{\mathbb{Z}_p}$ to the map which sends a map to its value at v.

Lemma 5.1 *The evaluation map e is injective but not surjective.*

Proof
If $e(v) = 0$ then $f(v) = 0$ for all $f \in \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$. However if $v \neq 0$ then since $V_{\mathbb{Z}_p}$ is torsion free, there is a map sending αv to α and the rest of $V_{\mathbb{Z}_p}$ to zero. Thus if $e(v) = 0$ then $v = 0$ and e is injective. It is not surjective because the space of group homomorphisms is much larger than the space of $\mathbb{Z}_p$-module homomorphisms. For any copy of $\mathbb{Z}_p$ in $V_{\mathbb{Z}_p}$ we obtain a copy of the completed group ring $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ in the space of group homomorphisms. □

We remark that the copies of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ correspond to the action of the Adams operations ψ^k for k prime to p. The θ^p operation on the other hand is introduced freely on the left hand side.

A natural question to ask is whether any elements of the homotopy groups of $\Omega_0^\infty tmf$ correspond to elements of $V_{\mathbb{Z}_p}$. For $p \geqslant 5$ we have the following theorem.

Theorem 5.2 *With 6 inverted, the homotopy of $\Omega_0^\infty tmf$ is*

$$\pi_\star \Omega_0^\infty tmf \cong \mathbb{Z}[\frac{1}{6}, E_4, E_6].$$

Proof
For details of this computation and some history see [Bau]. □

Corollary 5.2 *For $p \geqslant 5$, the elements $p^i d_i \in V_{\mathbb{Z}_p}$ correspond to elements of the homotopy groups of $\Omega_0^\infty tmf$.*

Proof
From our formula for d_i in terms of a_i in section 4.3 we know that $p^i d_i$ is a polynomial in g_2 and g_3 and so this result follows from the theorem and the equations $12g_2 = E_4$, $216g_3 = E_6$. □

The map $\phi : T_{V_{\mathbb{Z}_p}} \to \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$ from corollary 5.1 gives us another way to look at $K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p)$. Applying the functor T we obtain a map

$$T\phi : T(T_{V_{\mathbb{Z}_p}}) \to K^\star(\Omega_0^\infty tmf; \mathbb{Z}_p).$$

The Hecke algebra $T_{V_{\mathbb{Z}_p}}$ is a module over the ring $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ under the action of the Diamond operators. It follows that each Hecke operator T_l generates a free λ-ring on one generator when we introduce θ^p freely with the functor T, and this gives all of $T(T_{V_{\mathbb{Z}_p}})$.

Let us summarise:

Theorem 5.3 *For $p \geqslant 3$, $K^*(\Omega_0^\infty tmf; \mathbb{Z}_p)$ is the free θ^p ring generated by the group* $\hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$. *The λ-ring structure is given by the Adams operations, which correspond to the diamond operators on V. If $p > 3$ then $K^*(\Omega_0^\infty tmf; \mathbb{Z}_p)$ contains the free θ^p ring generated by the Hecke algebra $T_{V_{\mathbb{Z}_p}}$ as a direct summand.*

Proof
The only part which requires comment is the last statement. The map $\phi : T_{V_{\mathbb{Z}_p}} \to \hom(V_{\mathbb{Z}_p}, \mathbb{Z}_p)$ is a split inclusion of p-profinite abelian groups. It follows that $T\phi$ is a split injection also. □

5.3 $K^*(\Omega_0^\infty tmf; \mathbb{Z}_2)$

For $p = 2$ we found that there is an isomorphism

$$K^*(\Omega_0^\infty tmf; \mathbb{Z}_2) \cong W(\hom(V_{\mathbb{Z}_2}, \mathbb{Z}_2) \to \mathbb{Z}/2).$$

We have some understanding of the functor W from our examples in section 2.4. We know that

$$W(\mathbb{Z}_2 \to \mathbb{Z}/2) \cong \mathbb{Z}_2[[x, \theta^2 x, ...]],$$

and

$$W(\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p) \cong \mathbb{Z}_p[[x, \lambda^2(x), ...]]$$

for all p. We are interested in $V_{\mathbb{Z}_2}$ as a $\mathbb{Z}_2[[\mathbb{Z}_2^\times]]$-module under the diamond operators and so we would like to understand what we get if we apply W to the composition

$$\mathbb{Z}_2[[\mathbb{Z}_2^\times]] \to \mathbb{Z}_2 \to \mathbb{Z}/2.$$

More generally we have the following construction.

Lemma 5.2 *For any composition $M \to H \to K$ of p-profinite abelian groups there is a composition*

$$W(M \to H) \to W(M \to K) \to W(H \to K).$$

Proof
For any R there is an isomorphism between $\hom_{\theta^p} W(M \to H); R)$ and the set of commutative squares

$$\begin{array}{ccc} M & \longrightarrow & \tilde{R} \\ \downarrow & & \downarrow \\ H & \longrightarrow & \tilde{R}/\Gamma^2\tilde{R} \end{array}$$

and similarly for the other maps. Given such a square for $M \to K$ we obtain a square for $M \to H$ since the left hand arrow factors through H. Given a square for $H \to K$ we obtain a square for $M \to H$ by mapping M onto the H in the square. This means that we have a composition

$$\hom_{\theta p}(W(H \to K); R) \to \hom_{\theta p}(W(M \to K); R) \to \hom_{\theta p}(W(M \to H); R)$$

for any R. Now if we take R to be $W(H \to K)$ we obtain a map $W(M \to H) \to W(M \to K)$ as the image of the identity of $W(H \to K)$. Similarly we obtain a map $W(M \to K) \to W(H \to K)$ as the image of the identity of $W(M \to K)$. □

We could also view the maps in the lemma as the result of applying W to the squares

$$\begin{array}{ccc} M & \longrightarrow & M \\ \downarrow & & \downarrow \\ H & \longrightarrow & K \end{array}$$

and

$$\begin{array}{ccc} M & \longrightarrow & H \\ \downarrow & & \downarrow \\ K & \longrightarrow & K \end{array}$$

which are maps in the category of maps. This allows us to deduce the following result.

Proposition 5.4 *The maps*

$$W(\mathbb{Z}_2[[\mathbb{Z}_2^\times]] \to \mathbb{Z}_2) \to W(\mathbb{Z}_2[[\mathbb{Z}_2^\times]] \to \mathbb{Z}/2)$$

and

$$W(\mathbb{Z}_2[[\mathbb{Z}_2^\times]] \to \mathbb{Z}/2) \to W(\mathbb{Z}_2 \to \mathbb{Z}/2)$$

are surjections. The latter is split.

Proof

The functor W is a left adjoint and so is right exact. This means in particular that it takes surjections to surjections. The inclusion of $\mathbb{Z}_2$ in $\mathbb{Z}_2[[\mathbb{Z}_2^\times]]$ gives a square

$$\begin{array}{ccc} \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2[[\mathbb{Z}_2^\times]] \\ \downarrow & & \downarrow \\ \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \end{array}$$

and applying W to this square gives the desired splitting map. $\square$

We could replace $\mathbb{Z}_2[[\mathbb{Z}_2^\times]]$ in the above with $\hom(V_{\mathbb{Z}_2}, \mathbb{Z}_2)$ and reach a similar conclusion. This allows us to say the following.

Theorem 5.4 *The λ-ring $K^\star(\Omega_0^\infty tmf; \mathbb{Z}_2)$ is a quotient of the free $\mathbb{Z}/2$-graded p-adic λ-ring generated by* $\hom(V_{\mathbb{Z}_2}, \mathbb{Z}_2)$. *As a θ^p ring is contains the free θ^p ring on one generator as a direct summand.*

Proof
We only remark that $W(\mathbb{Z}_2 \to \mathbb{Z}/2) \cong \mathbb{Z}_2[[x, \theta^p x, ...]]$ is the direct summand coming from the split surjection from proposition 5.4. $\square$

A Combinatorial proof of the formula for $P_{3,n}$

In this appendix we give some more details on the calculation of the polynomial $P_{3,n} = P_{3,n}(e_1, ..., e_{3n})$ from the definition of a λ-ring. Let us recall the formula for $P_{3,n}$.

Proposition A.1 *Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be the cube root of unity. Then for all n*

$$P_{3,n}(e_1, ..., e_{3n}) = \sum_{l \leqslant j \leqslant k,\, l+j+k=3n} c^{n-l}_{j-l,k-l} e_l e_j e_k$$

where $c^t_{r,s} = 2\Re(\omega^r)$ unless $r = 0, t, 3t/2$. If $t > 0$ then for $r = 0, 3t/2$ the value of $c^t_{r,s}$ is 1 while for $r = t$ it is $2\Re(\omega^r) + (-1)^{t+1}$. If $t = r = 0$ the coefficient is zero.

The definition of $P_{3,n}$ that we use is that it is the coefficient of t^3 in the product $\prod_{1 \leqslant i_1 < \cdots < i_n \leqslant 3n}(1 + x_{i_1} \cdots x_{i_n} t)$, where e_i is the i^{th} elementary symmetric function in the indeterminates x_j. Note that the highest exponent of any x_i that can occur in the coefficient of t^3 is 3 so the degree of $P_{3,n}$ is 3. Let us introduce the notation $x^{r,s,t}$ to represent the monomial

$$x^{r,s,t} = x_1^3 \cdots x_r^3 x_{r+1}^2 \cdots x_{r+s}^2 x_{r+s+1} \cdots x_{r+s+t},$$

which is a typical monomial in the coefficient of t^3. In the polynomial $P_{3,n}$ there are two types of monomial that may occur. There are the degree 3 monomials, which are also exactly those of the type $e_{l+1}e_{j+1}e_{k+1}$ for $e_l e_j e_k$ a monomial that may occur in $P_{3,n-1}$, and there are monomials of the form $e_k e_{3n-k}$. Notice that the coefficient of $x^{r+1,s,t}$ in $e_{l+1}e_{j+1}e_{k+1}$ is the same as the coefficient of $x^{r,s,t}$ in $e_l e_j e_k$. In fact in both cases it is equal to $\sum_p \binom{s}{p}\binom{t}{l-r-p} \sum_q \binom{p}{q}\binom{t-l+r+p}{j-r-p-q}$ where the sum is taken so that all the numbers are non-negative. Here p is the number of the terms from e_l that are chosen to lie between $r+1$ and $r+s$. Once these are chosen the terms from e_j must include the other $s-p$ such numbers, but then may also have q of the original p as well. We can also see that the coefficient of $x^{r+1,s,t}$ is zero in the degree 2 monomials, and that its coefficient in $P_{3,n}$ is equal to that of $x^{r,s,t}$ in $P_{3,n-1}$. Taking these three facts together we conclude that the coefficient of $e_{l+1}e_{j+1}e_{k+1}$ in $P_{3,n}$ is equal to the coefficient of $e_l e_j e_k$ in $P_{3,n-1}$. This means that we only need to determine the coefficients of the monomials $e_k e_{3n-k}$ in $P_{3,n}$. We remark that if we had made this same observation when dealing with $P_{2,n}$ the calculation would have become trivial, as we would only have to determine one coefficient.

We have reduced ourselves to dealing with the monomials $x^{0,i,3n-2i}$. Conveniently these are equal in number to the monomials $e_k e_{3n-k}$ so we expect to obtain an invertible linear system for the unknown coefficients. In fact it will turn out to be triangular. The full system has the form

$$\begin{pmatrix} B & 0 \\ A & L \end{pmatrix} \begin{pmatrix} c \\ v \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

In this equation c is the vector of known coefficients for the monomials $e_l e_j e_k$ with $l > 0$ and $Bc = b$ is precisely the set of equations that we have in the $n-1$ case. The matrices A and L contain the coefficient of $x^{0,i,3n-2i}$ in $e_l e_j e_k$ for $l > 0$ and $l = 0$ respectively. The vector d contains the coefficients of $x^{0,i,3n-2i}$ in $P_{3,n}$ and so the system that we have to solve is $Lv = d - Ac$.

We can give explicit formulae for the entries of d, A, and L. In the formulae below we adopt the convention that $\binom{n}{m} = 0$ if $m > n$ or either term is negative. These values correspond to counting impossible choices so in our context they should be zero. This will allow us to have cleaner formulae without affecting our calculation. For example with this convention the formula

$$\binom{m+n}{l} = \sum_{a=0}^{l} \binom{m}{a} \binom{n}{l-a}$$

holds for any positive values of m and n.

Lemma A.1 *If the* i^{th} *row of* d *corresponds to* $x^{0,i,3n-2i}$ *then it contains the number*

$$\frac{1}{6} \sum_{l=0}^{n} \binom{i}{l} \binom{3n-2i}{n-l} \binom{2(n-i+l)}{n-i+l}$$

if $n \neq i$ *and*

$$\frac{1}{6} \left(\sum_{l=0}^{n} \binom{i}{l} \binom{3n-2i}{n-l} \binom{2(n-i+l)}{n-i+l} - 3 \right)$$

if $n = i$.

Proof

The coefficient of $x^{0,i,3n-2i}$ in $P_{3,n}$ is the number of ways of partitioning the numbers $1, 1, 2, 2, ..., i, i, i+1, i+2, ..., 3n-i$ into three distinct sets of n distinct numbers. Let us suppose that we put l numbers less than $i+1$ in the first set. Choosing the first set involves first picking l from i, then $n-l$ from the other $3n-2i$ so it can be done in $\binom{i}{l}\binom{3n-2i}{n-l}$ ways. The second set must contain all the numbers from 1 to i that weren't in the first as we cannot have two equal numbers in the third. Therefore we just have to pick $n-i+l$ numbers from the set containing the l numbers less than $i+1$ that are in the first and the $3n-2i-(n-l)$ numbers bigger than i that are not in the first. Note that by our convention we do not need to worry about whether all this is possible. The product $\binom{i}{l}\binom{3n-2i}{n-l}\binom{2(n-i+l)}{n-i+l}$ therefore gives the number of ordered partitions. We don't care about the ordering so we have to divide by 6. The final thing that we have to worry about is that our three sets are all distinct. This will be automatic unless two of the sets do not contain any numbers bigger than i. This can only happen if $n = i$ and either $l = 0$ or $l = n$. If $l = 0$ the second and third sets are both $\{1, 2, ..., i\}$ so this should not be counted. If $l = n$ we just have to exclude the 2 cases where either 0 or n of 1 through n are in the second set. This leads us to the expression in the second formula. □

Next we consider the coefficient of $x^{0,i,3n-2i}$ that occurs in $e_l e_j e_k$.

Lemma A.2 *If $l \leqslant j \leqslant k$ then the coefficient of $x^{0,i,3n-2i}$ that occurs in $e_l e_j e_k$ is*

$$\sum_{a=0}^{l} \binom{i}{a}\binom{3n-2i}{l-a}\binom{3n-2i+2a-l}{j-i+a}.$$

Proof

The argument is the similar to the previous lemma. This time however the sets do not need to be distinct and the partitions are ordered by l, j, k so we neither have to divide by 6 nor rule out any cases. □

The previous lemma give us the entries of both A and L from our system of equations. However we should take a closer look at L which has columns that correspond to the monomials $e_k e_{3n-k}$.

Proposition A.2 *Let the $(k+1)^{th}$ column of L correspond to the monomial $e_k e_{3n-k}$. Then L is an upper triangular of dimension $\lfloor\frac{3n}{2}\rfloor$ and in the i^{th} row and j^{th} column it contains $\binom{3n-2i}{j-i}$. Furthermore let v_0 be the vector whose $(k+1)^{th}$ component is $2\Re(e^{2\pi ki/3})$. If n is even we modify v_0 by replacing the 2 in the final component by 1. Then every component of Lv_0 is $(-1)^n$.*

Proof

The formula for the entries of L comes from the previous lemma. To prove the second claim about L we let $\omega = e^{2\pi i/3}$. Consider the i^{th} entry of Lv_0. For n odd it is given by

$$\begin{aligned} 2\Re\sum_{k=i}^{\lfloor\frac{3n}{2}\rfloor}\binom{3n-2i}{k-i}\omega^k &= 2\Re\sum_{k=0}^{\lfloor\frac{3n}{2}\rfloor-i}\binom{3n-2i}{k}\omega^{k+i} \\ &= \omega^i(1+\omega)^{3n-2i} \\ &= \omega^i(-\bar{\omega})^{3n-2i} \\ &= (-1)^{3n}\bar{\omega}^{3n-3i} \\ &= (-1)^n \end{aligned}$$

Here the key observation for the second equality is that we sum over exactly half the terms in the binomial expansion of $(1+\omega)^{3n-2i}$, but we multiply by two to account for the other half. The binomial coefficients and the real parts of the powers of ω match up perfectly so that the real parts of the two halves are equal. It is not true that the imaginary parts are equal though. For n even the calculation is essentially the same. This time instead of being between consecutive occurrences of -1 in the sequence $2,-1,-1,2,-1,-1,2,-1\ldots$ we would have a 2 in the final position. Our modification to v_0 splits this 2 evenly between the two halves and so the second equality still holds. $\square$

With these results in hand we see that the formula for $P_{3,n}$ reduces as in the $n=2$ case to some identities among binomial coefficients. The formula for Lv_0 is our guide as to how to proceed with the calculation. We can generalise it to the case $l>0$ (where we are looking at monomials $e_le_je_k$).

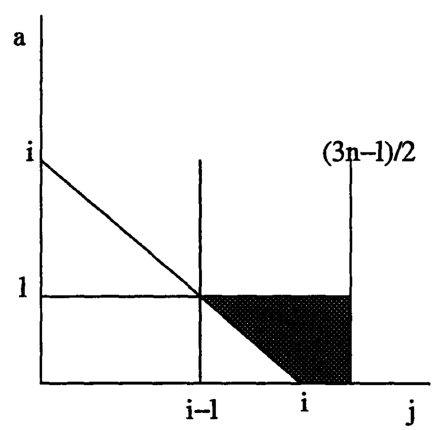
Lemma A.3 *With ω as above we have*

$$\sum_{j=0}^{\lfloor\frac{3n-l}{2}\rfloor}\sum_{a=0}^{l}\binom{i}{a}\binom{3n-2i}{l-a}\binom{3n-2i+2a-l}{j-i+a}2\Re(\omega^{j-l})=(-1)^{n-l}\binom{3n-i}{l}.$$

In this formula when $\omega^{3(n-l)/2}=2$ we replace it by 1 as before.

Proof

First we assume that $l\leqslant i$ and $3n-2i\geqslant l$. We can sketch the region of summation in the (j,a) plane as shown.



From this we can see that we can interchange the order of summation and obtain

$$\sum_{a=0}^{l}\binom{i}{a}\binom{3n-2i}{l-a}\sum_{j=i-a}^{\lfloor\frac{3n-l}{2}\rfloor}\binom{3n-2i+2a-l}{j-i+a}2\Re(\omega^{j-l})$$

$$= \sum_{a=0}^{l} \binom{i}{a} \binom{3n-2i}{l-a} \sum_{k=0}^{\lfloor \frac{3n-l-2i+2a}{2} \rfloor} \binom{3n-2i+2a-l}{k} 2\Re(\omega^{k+i-a-l})$$

$$= \sum_{a=0}^{l} \binom{i}{a} \binom{3n-2i}{l-a} \omega^{i-a-l} (1+\omega)^{3n-2i+2a-l}$$

$$= \sum_{a=0}^{l} \binom{i}{a} \binom{3n-2i}{l-a} (-1)^{3n-l} \omega^{i-a-l} (\bar{\omega})^{3n-2i+2a-l}$$

$$= \sum_{a=0}^{l} \binom{i}{a} \binom{3n-2i}{l-a} (-1)^{n-l} (\bar{\omega})^{3n-3i+3a}$$

$$= (-1)^{n-l} \binom{3n-i}{l}$$

exactly as above. In this proof we use the identity that

$$\binom{m+n}{l} = \sum_{a=0}^{l} \binom{m}{a} \binom{n}{l-a}.$$

To see this one just considers how to distribute l items among $m+n$ positions by putting a in the first m and $l-a$ in the remaining n, for various a. From this point of view we can see that if with our convention on binomial coefficients the above equalities will still hold even if we change our limits on a and j to ensure that all the terms make sense for arbitrary values of l and i. □

This calculation is very similar to that following proposition 2.1. It is reasonable to expect that in calculating $P_{m,n}$ by this technique we would be able to use the trick of taking real parts of powers of a primitive m^{th} root of unity in the same way to complete the calculation.

The expression on the left hand side in this lemma is close to what we have when we multiply the part of A corresponding to each l by a vector like v_0, of appropriate dimension. The difference is that the sum in the lemma includes all the terms corresponding to $e_l e_j e_k$ with $j < l$ as well. These columns of A are present but are in a block corresponding to the smaller value j. The next lemma is a key observation that allows us to rearrange the triple (l, j, k) without affecting the formula above.

Lemma A.4 *Let $l + j + k = 3n$. If p is the difference between 2 of l, j, k then $\Re(\omega^p)$ is independent of which 2 we choose.*

Proof

We simply note that working modulo 3 we have just 0 and ± 1 to worry about and $\Re(\omega) = \Re(\omega^{-1})$ so the order of the pair that we choose does not matter. Then, for example, we have $k = 3n - j - l$ so $j - k = 2j + l - 3n = l - j$ modulo 3. □

Let us now consider how the product $Ac + Lv$ relates to what we obtain by summing the formula in lemma A.3 over l. If we sum over l from 0 to $3n$ and for each l for j from 0 to $\lfloor\frac{3n-l}{2}\rfloor$ we obtain an alternating sum of binomial coefficients which of course is zero. In this sum the sum of the coefficients of terms corresponding to $e_l e_j e_k$ is -3 if $l - j$ is not a multiple of 3. This follows from the previous lemma. If $l - j$ is a multiple of 3 then the total coefficient is 6, 3 or 1 according to whether l, j, k are all distinct, 2 are equal or all are equal respectively. Note that the coefficient 3 when two are equal consists of a 2 from $l, l, 3n - 2l$ and 1 from $3n - 2l, l, l$, which is how we modify our coefficients to make the formula work. Now what about $Ac + Lv$? Our formula for the coefficient of $e_l e_j e_k$ is that it is 1 if two of l, j, k are equal, and $2\Re(\omega^{l-j})$ otherwise, except when $j = n$, in which case we have to add $(-1)^n$ as well. There is also the exception that the coefficient of $e_n e_n e_n$ is zero. Putting this altogether we see that with the coefficients as we claim we have the formula

$$3(Ac + Lv)_i = 3\sum_{l=0}^{n}(-1)^{n-l-1}\sum_{a=0}^{l}\binom{i}{a}\binom{3n-2i}{l-a}\binom{3n-2i+2a-l}{n-i+a}$$
$$+2\sum_{a=0}^{n}\binom{i}{a}\binom{3n-2i}{n-a}\binom{2(n-i+a)}{n-i+a}.$$

In fact by lemma A.4 if we sum of all l and rearrange the terms we arrive at 3 times the sum from $l = 0$ to n minus twice the term with $l = j = k = n$. This sum is zero on the one hand and $3(Ac + Lv)_i$ minus 3 times the $j = n$ terms minus twice the n, n, n term on the other. Taking $3(Ac + Lv)_i$ to the other side gives the formula above.

If we denote the expression on the right hand side as $3x + 2y$ then we see that y is 6 times the expression for the i^{th} component of d (unless $i = n$) so it remains to show the identity $2x = -y$. We remark that if $n > i$ each term in y is even so this is certainly reasonable. For $n = i$ we instead want $2x = -y - 1$.

At the moment the binomial identity that we have to prove is somewhat difficult but it will become clear once we see an alternative expression for the term labelled x. This is supposed to be the sum over l of the coefficients of $x^{0,i,3n-i}$ in $e_l e_n e_{2n-l}$ and was arrived at by first considering the terms from e_l. If we first consider e_n and so reverse n and l in the formula we instead arrive at the expression

$$x = \sum_{l=0}^{n}(-1)^{n-l-1}\sum_{a=0}^{n}\binom{i}{a}\binom{3n-2i}{n-a}\binom{2(n-i+a)}{l-i+a}$$

$$= \sum_{a=0}^{n} \binom{i}{a} \binom{3n-2i}{n-a} \sum_{l=i-a}^{n} (-1)^{n-l-1} \binom{2(n-i+a)}{l-i+a}$$

We observe that $2\sum_{l=i-a}^{n}(-1)^{n-l-1}\binom{2(n-i+a)}{l-i+a} = (1-1)^{2(n-i+a)} - \binom{2(n-i+a)}{n-i+a}$ so if $n > i$ we have $2x = -y$ as required. If $n = i$ then the same holds for all a except $a = 0$ since 0^0 is not well defined. However if we look at the $a = 0$ term we have $2x = -2 = -1 - 1 = -y - 1$ as required. Note that here only the $l = n$ terms contribute. If $i > n$ we should have a similar problem when $a = i - n$. However in this case we still only get a difference from the $l = n$ terms and having $a = i - n$ means that all i of the numbers 1 through i must go in the third set. However this set has only n members since $l = j = n$ so in this case the problem goes away and we have $2x = -y$ as required.

This completes the combinatorial proof of the formula for $P_{3,n}$.

B The canonical differential

In this appendix we calculate the modular forms a_{2i-1} (in the notation of Katz) that occur as the coefficients of the differential on an elliptic curve, as polynomials in g_2 and g_3. Katz gave formulae up to a_{11}, which we will extend to a_{101}, and indeed we will note a correction to the formula for a_{11}.

We recall that in Weierstrass form the curve is given by an equation

$$y^2 = 4x^3 - g_2x - g_3$$

and the canonical differential is $\omega = dx/y$. If we define $t = x/y$ then the modular forms a_i are defined by

$$\omega = \sum a_i t^{i-1} dt.$$

To calculate this series we introduce $s = 1/y$ and divide the Weierstrass equation by y^3 to obtain

$$s = 4t^3 - g_2s^2t - g_3s^3.$$

This formula can be recursively substituted into itself to determine s as a formal power series in t to whatever order of t we require. Indeed if we define $s_0 = 0$ and $s_n = 4t^3 - g_2s_{n-1}^2t - g_3s_{n-1}^3$ for $n > 0$ then we claim that $s_n \equiv s_{n-1} \mod t^{4n-1}$. This is true for $n = 1$ since $s_1 = 4t^3$. Now if $s_n \equiv s_{n-1} \mod t^k$ then $s_n^m \equiv s_{n-1}^m \mod t^{k+3m}$ because the smallest term occurring is t^3. This means that the definitions of s_{n+1} and s_n agree mod t^{k+4} so they are equal for 4 extra terms. The conclusion is that the sequence s_n converges to a unique power series that is s and that $s_n \equiv s \mod t^{4n+3}$.

To determine the differential in terms of t we first calculate that

$$dx/y = dt + tdy/y$$

from the quotient rule. Also $ds = \frac{-1}{y^2}dy$ so $dy/y = -yds = -ds/s$ and we obtain

$$\omega = dx/y = dt - tds/s.$$

Next we differentiate $s = 4t^3 - g_2s^2t - g_3s^3$ to obtain

$$ds = (12t^2 - g_2s^2)dt - (2g_2st + 3g_3s^2)ds$$

and we substitute this in the above, which gives

$$\omega = dt - \frac{t(12t^2 - g_2s^2)}{s(1 - 2g_2st + 3g_3s^2)}dt.$$

We can replace s in this formula by the power series calculated above and then formally invert the denominator to obtain the series that gives the formulae for the a_i. We note that we have to divide out by t^3 before performing the inversion. This means that if we use s_n instead of s we will obtain a formula for ω that is accurate mod t^{4n}.

We carried out this calculation with the aid of Maple and found that the first 50 terms of the series are:

$$
\begin{aligned}
&-2 + 16g_2t^4 + 96g_3t^6 - 192g_2^2t^8 - 2560g_3g_2t^{10} + 2560(g_2^3 - 3g_3^2)t^{12} \\
&+53760g_3g_2^2t^{14} - 7168g_2(5g_2^3 - 48g_3^2)t^{16} - 344064g_3(3g_2^3 - 2g_3^2)t^{18} \\
&+516096g_2^2(g_2^3 - 20g_3^2)t^{20} - 2703360g_3g_2(-7g_2^3 + 16g_3^2)t^{22} \\
&+1081344(240g_2^3g_3^2 - 7g_2^6 - 60g_3^4)t^{24} \\
&-337379328g_2^2g_3(g_2^3 - 5g_3^2)t^{26} + 18743296g_2(6g_2^6 - 315g_2^3g_3^2 + 280g_3^4)t^{28} \\
&+131203072g_3(48g_3^4 - 400g_2^3g_3^2 + 45g_2^6)t^{30} \\
&-562298880g_2^2(3g_2^6 + 448g_3^4 - 224g_2^3g_3^2)t^{32} \\
&-1853882368g_2g_3(336g_3^4 - 770g_2^3g_3^2 + 55g_2^6)t^{34} \\
&+463470592(20160g_3^4g_2^3 - 1344g_3^6 + 55g_2^9 - 5544g_3^2g_2^6)t^{36} \\
&+158506942464g_2^2g_3(224g_3^4 - 224g_2^3g_3^2 + 11g_2^6)t^{38} \\
&-2709520384g_2(109200g_3^4g_2^3 - 26880g_3^6 - 18720g_3^2g_2^6 + 143g_2^9)t^{40} \\
&-325142446080g_3(4704g_3^4g_2^3 - 2548g_3^2g_2^6 + 91g_2^9 - 192g_3^6)t^{42} \\
&+455199424512g_2^2(-10560g_3^6 - 2145g_3^2g_2^6 + 13g_2^9 + 18480g_3^4g_2^3)t^{44} \\
&-5484069257216g_3g_2(1536g_3^6 - 10080g_3^4g_2^3 + 3360g_3^2g_2^6 - 91g_2^9)t^{46} \\
&+997103501312(6336g_3^8 + 236544g_2^3g_3^6 - 221760g_3^4g_2^6 + 18480g_3^2g_2^9 - 91g_2^{12})t^{48} \\
&-239304840314880g_2^2g_3(35g_2^9 - 2640g_3^6 + 7392g_3^4g_2^3 - 1650g_3^2g_2^6)t^{50}
\end{aligned}
$$

Note that the coefficient of t^{2k} is a_{2k+1}. We calculated as far at t^{124} which requires determining the 31^{st} iteration of s. For the purposes of section 4 we include these terms below.

$$
\begin{aligned}
&11730629427200g_2(119g_2^{12} + 82368g_3^8 - 816816g_2^3g_3^6 + 466752g_3^4g_2^6 - 29172g_3^2g_2^9)t^{52} \\
&+182997819064320g_3(-190080g_2^3g_3^6 + 3520g_3^8 + 282744g_3^4g_2^6 \\
&-44880g_3^2g_2^9 + 765g_2^{12})t^{54} \\
&-253381595627520g_2^2(320320g_3^8 - 24752g_3^2g_2^9 + 85g_2^{12} - 1345344g_2^3g_3^6 \\
&+510510g_3^4g_2^6)t^{56} \\
&-1203192138301440g_3g_2(91520g_3^8 - 1304160g_2^3g_3^6 + 1173744g_3^4g_2^6 \\
&-138567g_3^2g_2^9 + 1938g_2^{12})t^{58} \\
&-257826886778880(256256g_3^{10} + 440895g_2^{12}g_3^2 + 42602560g_2^6g_3^6 \\
&-11411400g_2^9g_3^4 - 1292g_2^{15} - 19219200g_2^3g_3^8)t^{60} \\
&+39963167450726400g_2^2g_3(912912g_3^4g_2^6 - 1537536g_2^3g_3^6 + 256256g_3^8 \\
&-82992g_3^2g_2^9 + 969g_2^{12})t^{62} \\
&-5328422326763520g_2(969g_2^{15} - 61501440g_3^6g_2^6 - 383040g_3^2g_2^{12} \\
&+121721600g_3^4g_2^9 + 46126080g_3^8g_2^3 - 2342912g_3^{10})t^{64} \\
&-341019028912865280g_3(2013440g_3^8g_2^3 - 6342336g_3^6g_2^6 - 19968g_3^{10}
\end{aligned}
$$

$$+2642640g_3^4g_2^9 - 190190g_3^2g_2^{12} + 1881g_2^{15})t^{66}$$
$$+1406703494265569280g_2^2(57g_2^{15} + 7468032g_3^8g_2^3 - 6534528g_3^6g_2^6$$
$$-25840g_3^2g_2^{12} - 905216g_3^{10} + 990080g_3^4g_2^9)t^{68}$$
$$+63014385777377280g_2g_3(-22364160g_3^{10} + 589388800g_3^8g_2^3 - 1111418880g_3^6g_2^6$$
$$+340372032g_3^4g_2^9 - 19835200g_3^2g_2^{12} + 168245g_2^{15})t^{70}$$
$$-259471000259788800(-358465536g_2^3g_3^{10} + 4807g_2^{18} + 112720608g_2^{12}g_3^4$$
$$+2715648g_3^{12} - 2477376g_2^{15}g_3^2 - 944706048g_2^9g_3^6 + 1545882624g_2^6g_3^8)t^{72}$$
$$+19749449848345067520g_3g_2^2(7920640g_3^{10} - 87127040g_3^8g_2^3 + 107352960g_3^6g_2^6$$
$$-25049024g_3^4g_2^9 + 1204280g_3^2g_2^{12} - 8855g_2^{15})t^{74}$$
$$+438876663296557056g_2(-14277943680g_3^6g_2^9 + 31925836800g_2^6g_3^8 - 25741485g_3^2g_2^{15}$$
$$+1372879200g_3^4g_2^{12} - 12415603200g_3^{10}g_2^3 + 361181184g_3^{12} + 44275g_2^{18})t^{76}$$
$$+25015969807903752192g_3(115115g_2^{18} + 2924544g_3^{12} - 17892160g_3^2g_2^{15}$$
$$+444050880g_3^4g_2^{12} - 494247936g_3^{10}g_2^3 - 2427110400g_3^6g_2^9$$
$$+2831628800g_2^6g_3^8)t^{78}$$
$$-34232379737131450368g_2^2(555663360g_3^{12} + 8855g_2^{18} - 7945986048g_3^{10}g_2^3$$
$$+13302432000g_2^6g_3^8 - 4514764800g_3^6g_2^9 + 356592000g_3^4g_2^{12}$$
$$-5768400g_3^2g_2^{15})t^{80}$$
$$-959676970408471429120g_2g_3(-1729316160g_3^6g_2^9 + 2766905856g_2^6g_3^8$$
$$+253955520g_3^4g_2^{12} + 49335g_2^{18} - 8691930g_3^2g_2^{15} + 18522112g_3^{12}$$
$$-812657664g_3^{10}g_2^3)t^{82}$$
$$+95967697040847142912(16855121920g_2^3g_3^{12} - 79380480g_3^{14}$$
$$-38516587200g_2^{12}g_3^6 - 125149280256g_2^6g_3^{10} - 35790300g_2^{18}g_3^2$$
$$+145262557440g_2^9g_3^8 + 2539555200g_2^{15}g_3^4 + 49335g_2^{21})t^{84}$$
$$+173317660855769940099072g_2^2g_3(-240787456g_3^{10}g_2^3 + 30232800g_3^4g_2^{12}$$
$$-889200g_3^2g_2^{15} + 532097280g_2^6g_3^8 - 251536896g_3^6g_2^9 + 4485g_2^{18}$$
$$+13230080g_3^{12})t^{86}$$
$$+569187720380196847616g_2(-8438730300g_3^4g_2^{15} + 104504400g_3^2g_2^{18}$$
$$-712903935744g_3^8g_2^9 + 151222046976g_3^6g_2^{12} + 844923183104g_3^{10}g_2^6$$
$$+3492741120g_3^{14} - 192027996160g_3^{12}g_2^3 - 130065g_2^{21})t^{88}$$
$$-50088519393457322590208g_3(-56408625g_3^2g_2^{18} + 2209413024g_3^4g_2^{15}$$
$$-39278453760g_3^{10}g_2^6 + 60008748800g_3^8g_2^9 - 22094130240g_3^6g_2^{12}$$
$$-15876096g_3^{14} + 4167475200g_3^{12}g_2^3 + 254475g_2^{21})t^{90}$$
$$+102453789668435432570880g_2^2(-10015005g_3^2g_2^{18} + 11310g_2^{21} - 2677768192g_3^{14}$$
$$+60919226368g_3^{12}g_2^3 - 173511010816g_3^{10}g_2^6 - 19110478464g_3^6g_2^{12}$$
$$+913368456g_3^4g_2^{15} + 110416097792g_3^8g_2^9)t^{92}$$
$$+79392506545934553841664g_2g_3(2629575g_2^{21} + 1169313333760g_3^8g_2^9$$
$$-343434685440g_3^6g_2^{12} + 28977301584g_3^4g_2^{15} - 1055681003520g_3^{10}g_2^6$$
$$-647927280g_3^2g_2^{18} + 189481205760g_3^{12}g_2^3 - 2794192896g_3^{14})t^{94}$$
$$-20711088664156840132608(15107968139264g_2^6g_3^{12} + 14669567278080g_2^{12}g_3^8$$
$$+87121298880g_2^{18}g_3^4 - 2106399404032g_2^{15}g_3^6 - 29676365987840g_2^9g_3^{10} + 876525g_2^{24}$$
$$-1285328732160g_2^3g_3^{14} - 851561568g_2^{21}g_3^2 + 4016652288g_3^{16})t^{96}$$
$$-4639283860771132189704192g_2^2g_3(736281g_2^{21} + 590155005440g_3^8g_2^9$$

$$-141053531520g_3^6g_2^{12} - 708186006528g_3^{10}g_2^6 + 10164151536g_3^4g_2^{15}$$
$$+196815962112g_3^{12}g_2^3 - 7029141504g_3^{14} - 200608254g_3^2g_2^{18})t^{98}$$
$$+27062489187831604439941112g_2(-111655800g_3^2g_2^{21} - 352633828800g_3^6g_2^{15}$$
$$+12737032000g_3^4g_2^{18} + 2918348928000g_3^8g_2^{12} + 5221647974400g_3^{12}g_2^6$$
$$+9128755200g_3^{16} - 753122304000g_3^{14}g_2^3 + 105183g_2^{24}$$
$$-7419091496960g_3^{10}g_2^9)t^{100}$$
$$19330349419879717457100800g_3(451067904g_3^{16} - 866118176g_3^2g_2^{21}$$
$$-173811499008g_3^{14}g_2^3 + 4013054036992g_3^8g_2^{12} - 6213761089536g_3^{10}g_2^9$$
$$-793790908416g_3^6g_2^{15} + 2888487g_2^{24} + 49368736032g_3^4g_2^{18}$$
$$+2509403516928g_3^{12}g_2^6)t^{102}$$
$$-78867825633109247224971264g_2^2(-1661433446400g_3^{14}g_2^3 + 48865689600g_3^{16}$$
$$+8279070800g_3^4g_2^{18} - 260362502400g_3^6g_2^{15} + 56637g_2^{24} + 2529235737600g_3^8g_2^{12}$$
$$-7919499427840g_3^{10}g_2^9 + 7424530713600g_3^{12}g_2^6 - 65447200g_3^2g_2^{21})t^{104}$$
$$-85239108801901602098831687 68g_3g_2(-35150900g_3^2g_2^{21} + 322191360g_3^{16}$$
$$+2239960800g_3^4g_2^{18} - 485450085120g_3^{10}g_2^9 + 271580467200g_3^{12}g_2^6$$
$$+106981g_2^{24} - 31950643200g_3^{14}g_2^3$$
$$-41343276480g_3^6g_2^{15} + 248948761600g_3^8g_2^{12})t^{106}$$
$$+(-73532997504419514394539844527772925952 00g_3^{14}g_2^6$$
$$-43532555814248358348156401013837791232 0g_3^6g_2^{18}$$
$$+49109537619023032827781967595334061260 80g_3^8g_2^{15}$$
$$+22570545067328767612768480056441411993600g_3^{12}g_2^9$$
$$-18620699680546233280533996046564164894720g_3^{10}g_2^{12}$$
$$-915443479669088258880047862157148160g_3^{18}$$
$$+42018855716811151082594196873013100544 0g_3^{16}g_2^3$$
$$-879328777378136974791600160899072 00g_3^2g_2^{24}$$
$$+12274216864919649910194473970067046400g_3^4g_2^{21}$$
$$+70145885374894117647193175228416g_2^{27})t^{108}$$
$$+885175360635132021795559833 60g_3g_2^2(-1529903298560g_3^{10}g_2^9$$
$$+1142235494400g_3^{12}g_2^6 + 168113g_2^{24} + 4270858592g_3^4g_2^{18}$$
$$-89747074560g_3^6g_2^{15} + 5119262720g_3^{16} - 60288800g_3^2g_2^{21}$$
$$+635708444800g_3^8g_2^{12} - 208865918976g_3^{14}g_2^3)t^{110}$$
$$-194933512852581626421444608 0g_2(-189688230051840g_3^{14}g_2^6$$
$$-4671141995520g_3^6g_2^{18} + 117536852160g_3^4g_2^{21} - 766544688g_3^2g_2^{24}$$
$$-274323429918720g_3^{10}g_2^{12} + 419553481052160g_3^{12}g_2^9 + 18441911255040g_3^{16}g_2^3$$
$$-156370206720g_3^{18} + 565471g_2^{27} + 60620579379360g_3^8g_2^{15})t^{112}$$
$$-595433275622431149796412620 80g_3(4075291g_2^{27} - 18354092630016g_3^{14}g_2^6$$
$$+68827847362560g_3^{12}g_2^9 - 71174251249920g_3^{10}g_2^{12}$$
$$+875393925120g_3^{16}g_2^3 - 2957446566360g_3^6g_2^{18} + 124524065952g_3^4g_2^{21}$$
$$+24376529274240g_3^8g_2^{15} - 1589363490g_3^2g_2^{24} - 1616609280g_3^{18})t^{114}$$
$$+161617603383231312087597711 36g_2^2(-1567485612g_3^2g_2^{24}$$
$$+1738150135787520g_3^{12}g_2^9 - 11641361691960g_3^6g_2^{18} - 3281716838400g_3^{18}$$
$$+262933070400g_3^4g_2^{21} - 919214014118400g_3^{10}g_2^{12} + 172352627647200g_3^8g_2^{15}$$

$$+158999180820480g_3^{16}g_2^3 + 1072445g_2^{27} - 1050530301849600g_3^{14}g_2^6)t^{116}$$
$$+9702726996095044736206304706565g_3g_2(-131398617692160g_3^{10}g_2^{12}$$
$$-1722664944g_3^2g_2^{24} - 34819276800g_3^{18} + 160598310512640g_3^{12}g_2^9$$
$$+4075291g_2^{27} + 37640229026400g_3^8g_2^{15} + 148579851420g_3^4g_2^{21}$$
$$-3962129371200g_3^6g_2^{18} - 59526786908160g_3^{14}g_2^6$$
$$+4837703270400g_3^{16}g_2^3)t^{118}$$
$$+(-2726995586389184204004063980269076 48g_2^{30}$$
$$+72196520290575139055781872468913449808691 2000g_3^{14}g_2^9$$
$$+38443459611963324456100955940478173315072 00g_3^6g_2^{21}$$
$$-15959230801074504422857045493128236273500 1600g_3^{16}g_2^6$$
$$-64449329349467926294051602606095761145856000g_3^8g_2^{18}$$
$$+40069068761269202175958939220246964643823616 0g_3^{10}g_2^{15}$$
$$+641899680284545175378865580417425290035200 0g_3^{18}g_2^3$$
$$-78349442143946992777379665639561494528000g_3^4g_2^{24}$$
$$-92077910537260035303339986591981661176463360 0g_3^{12}g_2^{12}$$
$$-1013525810975597645335050916448566247424 0g_3^{20}$$
$$+42857814506090662715831611973777227776 0g_3^2g_2^{27})t^{120}$$
$$-1224551062265788404638450869862 40g_3g_2^2(-30242538992640g_3^{10}g_2^{12}$$
$$+7338263137920g_3^8g_2^{15} + 45772491448320g_3^{12}g_2^9 - 675892657440g_3^6g_2^{18}$$
$$+22707856080g_3^4g_2^{21} - 22714168688640g_3^{14}g_2^6 + 2877813227520g_3^{16}g_2^3$$
$$-50487951360g_3^{18} - 240179247g_3^2g_2^{24} + 525844g_2^{27})t^{122}$$
$$+99556996932177919076296818688g_2(14726735444678400g_3^8g_2^{18}$$
$$+290884183154073600g_3^{12}g_2^{12} - 105705234414024960g_3^{10}g_2^{15} + 43119208g_2^{30}$$
$$-6096167686963200g_3^{18}g_2^3 + 8882987201003520 0g_3^{16}g_2^6$$
$$-288697084032614400g_3^{14}g_2^9 + 37563035811840g_3^{20}$$
$$+14430842538840g_3^4g_2^{24} - 780963243278400g_3^6g_2^{21}$$
$$-72682814985g_3^2g_2^{27})t^{124}$$

C Tables of P_n and $P_{m,n}$

Table of P_n

n	P_n
1	$e_1 f_1$
2	$e_1^2 f_2 + e_2 f_1^2 - 2e_2 f_2$
3	$e_1^3 f_3 + e_1e_2f_1f_2 - 3e_1e_2f_3 + e_3f_1^3 - 3e_3f_1f_2 + 3e_3f_3$
4	$-2e_1e_3f_2^2 + 2e_4f_2^2 + 4e_4f_1f_3 - 4e_1^2e_2f_4 - 2e_2^2f_1f_3 - 4e_4f_1^2f_2 + 4e_1e_3f_4$
	$+e_1^2e_2f_1f_3 + e_1e_3f_1^2f_2 - e_1e_3f_1f_3 + e_1^4f_4 + e_2^2f_2^2 + 2e_2^2f_4 + e_4f_1^4 - 4e_4f_4$
5	$-5e_5f_2f_3 + 5e_5f_1^2f_3 + 5e_5f_1f_2^2 - 5e_5f_1^3f_2 - 5e_2e_3f_5 + 5e_1^2e_3f_5 + 5e_1e_2^2f_5$
	$-5e_1^3e_2f_5 - 5e_5f_1f_4 - 5e_1e_4f_5 + 5e_5f_5 + e_1^5f_5 + e_5f_1^5 + e_1^3e_2f_1f_4$
	$-3e_1e_2^2f_1f_4 + e_1e_2^2f_2f_3 - e_1^2e_3f_1f_4 - 2e_1^2e_3f_2f_3 + e_1^2e_3f_1^2f_3$
	$+e_2e_3f_1f_2^2 + 5e_2e_3f_1f_4 - e_2e_3f_2f_3 - 2e_2e_3f_1^2f_3 + e_1e_4f_1f_4$
	$+5e_1e_4f_2f_3 - e_1e_4f_1^2f_3 - 3e_1e_4f_1f_2^2 + e_1e_4f_1^3f_2$
6	$e_1^4e_2f_1f_5 + e_1^2e_2^2f_2f_4 - 4e_1^2e_2^2f_1f_5 + e_1^3e_3f_1^2f_4 - 2e_1^3e_3f_2f_4$
	$-e_1^3e_3f_1f_5 - 3e_1e_2e_3f_3^2 + e_1^6f_6 + e_2^3f_3^2 - 2e_2^3f_6 + e_3^2f_2^3 + 3e_3^2f_3^2$
	$+3e_3^2f_6 + e_6f_1^6 - 2e_6f_2^3 + 3e_6f_3^2 - 6e_6f_6 - 2e_2^3f_2f_4 + 2e_2^3f_1f_5 + 3e_3^2f_1^2f_4$
	$-3e_3^2f_2f_4 - 3e_3^2f_1f_5 + 3e_1^2e_4f_3^2 - 2e_2e_4f_2^3 - 3e_2e_4f_3^2 + 2e_1e_5f_2^3$
	$-3e_1e_5f_3^2 + e_1e_2e_3f_1f_2f_3 - 3e_1e_2e_3f_1^2f_4 + 4e_1e_2e_3f_2f_4 + 7e_1e_2e_3f_1f_5$
	$-3e_3^2f_1f_2f_3 + e_1^2e_4f_1^3f_3 - e_1^2e_4f_1^2f_4 + 2e_1^2e_4f_2f_4 + e_1^2e_4f_1f_5$
	$+e_2e_4f_1^2f_2^2 - 2e_2e_4f_1^3f_3 + 2e_2e_4f_1^2f_4 + 2e_2e_4f_2f_4 - 6e_2e_4f_1f_5$
	$+e_1e_5f_1^4f_2 - 4e_1e_5f_1^2f_2^2 - e_1e_5f_1^3f_3 + e_1e_5f_1^2f_4 - 6e_1e_5f_2f_4$
	$-e_1e_5f_1f_5 - 12e_1e_2e_3f_6 - 12e_6f_1f_2f_3 - 3e_1^2e_4f_1f_2f_3 + 4e_2e_4f_1f_2f_3$
	$+7e_1e_5f_1f_2f_3 - 6e_1^4e_2f_6 + 9e_1^2e_2^2f_6 + 6e_1^3e_3f_6 - 6e_1^2e_4f_6 + 6e_2e_4f_6$
	$+6e_1e_5f_6 - 6e_6f_1^4f_2 + 9e_6f_1^2f_2^2 + 6e_6f_1^3f_3 - 6e_6f_1^2f_4 + 6e_6f_2f_4 + 6e_6f_1f_5$
7	$7e_1^4e_3f_7 - 7e_7f_1f_2^3 + 14e_1^3e_2^2f_7 - 7e_7f_1^5f_2 - 7e_7f_1^3f_4 - 7e_1e_6f_7$
	$+7e_2^2e_3f_7 - 7e_1e_2^3f_7 - 7e_1^3e_4f_7 - 7e_2e_5f_7 + 7e_1e_3^2f_7 + 7e_7f_1^4f_3$
	$-7e_3e_4f_7 + 7e_1^2e_5f_7 - 7e_1^5e_2f_7 + 14e_7f_1^3f_2^2 + 7e_7f_2^2f_3 + 7e_7f_1f_3^2 - 7e_7f_3f_4$
	$+e_1^5e_2f_1f_6 + e_1^3e_2^2f_2f_5 - 5e_1^3e_2^2f_1f_6 + e_1e_2^3f_3f_4 - 3e_1e_2^3f_2f_5$
	$+5e_1e_2^3f_1f_6 + e_1^4e_3f_1^2f_5 - 2e_1^4e_3f_2f_5 - e_1^4e_3f_1f_6 + e_2^2e_3f_1f_3^2$
	$-e_2^2e_3f_3f_4 + 2e_2^2e_3f_1^2f_5 + 3e_2^2e_3f_2f_5 - 7e_2^2e_3f_1f_6 + e_1e_3^2f_2^2f_3$
	$-2e_1e_3^2f_1f_3^2 + 5e_1e_3^2f_3f_4 + 4e_1e_3^2f_1^2f_5 - 7e_1e_3^2f_2f_5 - 4e_1e_3^2f_1f_6$
	$+e_1^7f_7 + e_7f_1^7 + 7e_7f_7 + e_1^2e_2e_3f_1f_2f_4 - 3e_1^2e_2e_3f_3f_4 - 4e_1^2e_2e_3f_1^2f_5$
	$+6e_1^2e_2e_3f_2f_5 + 9e_1^2e_2e_3f_1f_6 - 2e_2^2e_3f_1f_2f_4 - e_1e_3^2f_1f_2f_4$
	$+e_1^3e_4f_1^3f_4 + 3e_1^3e_4f_3f_4 - e_1^3e_4f_1^2f_5 + 2e_1^3e_4f_2f_5 + e_1^3e_4f_1f_6$
	$+e_3e_4f_1f_2^3 - e_3e_4f_2^2f_3 + 5e_3e_4f_1f_3^2 + 3e_3e_4f_1^3f_4 - 5e_3e_4f_3f_4$
	$-7e_3e_4f_1^2f_5 + 7e_3e_4f_2f_5 + 7e_3e_4f_1f_6 + e_1^2e_5f_1^4f_3 + 2e_1^2e_5f_2^2f_3$
	$+4e_1^2e_5f_1f_3^2 - e_1^2e_5f_1^3f_4 - 7e_1^2e_5f_3f_4 + e_1^2e_5f_1^2f_5 - 2e_1^2e_5f_2f_5$
	$-e_1^2e_5f_1f_6 + e_2e_5f_1^3f_2^2 - 3e_2e_5f_1f_2^3 - 2e_2e_5f_1^4f_3 + 3e_2e_5f_2^2f_3$
	$-7e_2e_5f_1f_3^2 + 2e_2e_5f_1^3f_4 + 7e_2e_5f_3f_4 - 2e_2e_5f_1^2f_5 - 3e_2e_5f_2f_5$
	$+7e_2e_5f_1f_6 + e_1e_6f_1^5f_2 - 5e_1e_6f_1^3f_2^2 + 5e_1e_6f_1f_2^3 - e_1e_6f_1^4f_3$

$-7e_1e_6f_2^2f_3-4e_1e_6f_1f_3^2+e_1e_6f_1^3f_4+7e_1e_6f_3f_4-e_1e_6f_1^2f_5$
$+7e_1e_6f_2f_5+e_1e_6f_1f_6-21e_1^2e_2e_3f_7+14e_1e_2e_4f_7-21e_7f_1^2f_2f_3$
$+14e_7f_1f_2f_4-3e_1^3e_4f_1f_2f_4+e_1e_2e_4f_1^2f_2f_3-2e_1e_2e_4f_2^2f_3$
$-e_1e_2e_4f_1f_3^2-3e_1e_2e_4f_1^3f_4+8e_1e_2e_4f_1f_2f_4-2e_1e_2e_4f_3f_4$
$+3e_1e_2e_4f_1^2f_5-4e_1e_2e_4f_2f_5-8e_1e_2e_4f_1f_6-3e_3e_4f_1^2f_2f_3$
$-2e_3e_4f_1f_2f_4-4e_1^2e_5f_1^2f_2f_3+3e_1^2e_5f_1f_2f_4+6e_2e_5f_1^2f_2f_3$
$-4e_2e_5f_1f_2f_4+9e_1e_6f_1^2f_2f_3-8e_1e_6f_1f_2f_4+7e_7f_1^2f_5-7e_7f_2f_5-7e_7f_1f_6$
8 $-8e_2e_3^2f_8+8e_8f_2f_6-8e_1^6e_2f_8-8e_1^2e_6f_8+12e_8f_1^2f_3^2+8e_8f_1f_7$
$+20e_1^4e_2^2f_8+20e_8f_1^4f_2^2-16e_8f_1^2f_2^3-8e_2^2e_4f_8+8e_1e_7f_8-8e_8f_1^6f_2$
$+8e_8f_1^3f_5+8e_2e_6f_8-8e_8f_1^2f_6-8e_8f_2^2f_4+8e_8f_1^5f_3-8e_8f_2f_3^2$
$-8e_8f_1^4f_4+8e_8f_3f_5+12e_1^2e_3^2f_8-16e_1^2e_2^3f_8-8e_1^4e_4f_8+8e_1^3e_5f_8$
$+8e_3e_5f_8+8e_1^5e_3f_8-2e_2^4f_3f_5+2e_2^4f_2f_6-2e_2^4f_1f_7+2e_1^2e_3^2f_4^2$
$+4e_2e_3^2f_4^2-4e_2^2e_4f_4^2+2e_4^2f_1^2f_3^2+4e_4^2f_2f_3^2-4e_4^2f_2^2f_4-4e_4^2f_1^3f_5$
$-4e_4^2f_3f_5+4e_4^2f_1^2f_6-4e_4^2f_2f_6-4e_4^2f_1f_7-4e_1^3e_5f_4^2-2e_3e_5f_2^4$
$-4e_3e_5f_4^2+4e_1^2e_6f_4^2+2e_2e_6f_2^4-4e_2e_6f_4^2-2e_1e_7f_2^4-4e_1e_7f_4^2$
$+e_1^6e_2f_1f_7+e_1^4e_2^2f_2f_6-6e_1^4e_2^2f_1f_7+e_1^2e_2^3f_3f_5-4e_1^2e_2^3f_2f_6$
$+9e_1^2e_2^3f_1f_7+e_1^5e_3f_1^2f_6-2e_1^5e_3f_2f_6-e_1^5e_3f_1f_7+e_1^8f_8+e_2^4f_4^2$
$+2e_2^4f_8+e_4^2f_2^4+6e_4^2f_4^2+4e_4^2f_8+e_8f_1^8+2e_8f_2^4+e_1^3e_2e_3f_1f_2f_5$
$-3e_1^3e_2e_3f_3f_5-5e_1^3e_2e_3f_1^2f_6+8e_1^3e_2e_3f_2f_6+11e_1^3e_2e_3f_1f_7$
$-4e_1e_2^2e_3f_4^2+e_1^2e_3^2f_2^2f_4+3e_1^2e_3^2f_3f_5+5e_1^2e_3^2f_1^2f_6-9e_1^2e_3^2f_2f_6$
$-5e_1^2e_3^2f_1f_7+e_2e_3^2f_2f_3^2-2e_2e_3^2f_2^2f_4-7e_2e_3^2f_3f_5-5e_2e_3^2f_1^2f_6$
$+2e_2e_3^2f_2f_6+8e_2e_3^2f_1f_7+e_1^4e_4f_1^3f_5+3e_1^4e_4f_3f_5-e_1^4e_4f_1^2f_6$
$+2e_1^4e_4f_2f_6+e_1^4e_4f_1f_7+4e_1^2e_2e_4f_4^2+e_2^2e_4f_1^2f_3^2-2e_2^2e_4f_2f_3^2$
$+4e_2^2e_4f_2^2f_4+2e_2^2e_4f_1^3f_5+8e_2^2e_4f_3f_5-2e_2^2e_4f_1^2f_6-4e_2^2e_4f_2f_6$
$+8e_2^2e_4f_1f_7-8e_1e_3e_4f_4^2-4e_4^2f_1f_2^2f_3+4e_4^2f_1^2f_2f_4-8e_4^2f_1f_3f_4$
$+8e_4^2f_1f_2f_5+e_1^3e_5f_1^4f_4+2e_1^3e_5f_2^2f_4-e_1^3e_5f_1^3f_5-3e_1^3e_5f_3f_5$
$+e_1^3e_5f_1^2f_6-2e_1^3e_5f_2f_6-e_1^3e_5f_1f_7+e_1e_2^2e_3f_1f_3f_4$
$-3e_1e_2^2e_3f_1f_2f_5+6e_1e_2^2e_3f_3f_5+5e_1e_2^2e_3f_1^2f_6-17e_1e_2^2e_3f_1f_7$
$-2e_1^2e_3^2f_1f_3f_4-e_1^2e_3^2f_1f_2f_5-e_2e_3^2f_1f_3f_4+5e_2e_3^2f_1f_2f_5$
$-3e_1^4e_4f_1f_2f_5+e_1^2e_2e_4f_1^2f_2f_4-2e_1^2e_2e_4f_2^2f_4-e_1^2e_2e_4f_1f_3f_4$
$-4e_1^2e_2e_4f_1^3f_5+11e_1^2e_2e_4f_1f_2f_5-9e_1^2e_2e_4f_3f_5+4e_1^2e_2e_4f_1^2f_6$
$-6e_1^2e_2e_4f_2f_6-10e_1^2e_2e_4f_1f_7-2e_2^2e_4f_1^2f_2f_4-4e_2^2e_4f_1f_2f_5$
$+e_1e_3e_4f_1f_2^2f_3-2e_1e_3e_4f_1^2f_3^2-e_1e_3e_4f_2f_3^2-e_1e_3e_4f_1^2f_2f_4$
$+10e_1e_3e_4f_1f_3f_4+4e_1e_3e_4f_1^3f_5-10e_1e_3e_4f_1f_2f_5+e_1e_3e_4f_3f_5$
$-9e_1e_3e_4f_1^2f_6+16e_1e_3e_4f_2f_6+9e_1e_3e_4f_1f_7-4e_1^3e_5f_1^2f_2f_4$
$+4e_1^3e_5f_1f_3f_4+3e_1^3e_5f_1f_2f_5+e_1e_2e_5f_1^3f_2f_3-3e_1e_2e_5f_1f_2^2f_3$
$-e_1e_2e_5f_1^2f_3^2+5e_1e_2e_5f_2f_3^2-3e_1e_2e_5f_1^4f_4+11e_1e_2e_5f_1^2f_2f_4$
$-4e_1e_2e_5f_2^2f_4-10e_1e_2e_5f_1f_3f_4+8e_1e_2e_5f_4^2+e_3e_5f_1^2f_2^3$
$+3e_3e_5f_1^2f_3^2-7e_3e_5f_2f_3^2+3e_3e_5f_1^4f_4+8e_3e_5f_2^2f_4-3e_3e_5f_1^3f_5$
$+7e_3e_5f_3f_5+8e_3e_5f_1^2f_6-8e_3e_5f_2f_6-8e_3e_5f_1f_7+e_1^2e_6f_1^5f_3$
$+5e_1^2e_6f_1^2f_3^2-5e_1^2e_6f_2f_3^2-e_1^2e_6f_1^4f_4-2e_1^2e_6f_2^2f_4+e_1^2e_6f_1^3f_5$
$+8e_1^2e_6f_3f_5-e_1^2e_6f_1^2f_6+2e_1^2e_6f_2f_6+e_1^2e_6f_1f_7+e_2e_6f_1^4f_2^2$

$-4e_2e_6f_1^2f_2^3 - 2e_2e_6f_1^5f_3 - 9e_2e_6f_1^2f_3^2 + 2e_2e_6f_2f_3^2 + 2e_2e_6f_1^4f_4$
$-4e_2e_6f_2^2f_4 - 2e_2e_6f_1^3f_5 - 8e_2e_6f_3f_5 + 2e_2e_6f_1^2f_6 + 4e_2e_6f_2f_6$
$-8e_2e_6f_1f_7 + e_1e_7f_1^6f_2 - 6e_1e_7f_1^4f_2^2 + 9e_1e_7f_1^2f_2^3 - e_1e_7f_1^5f_3$
$-5e_1e_7f_1^2f_3^2 + 8e_1e_7f_2f_3^2 + e_1e_7f_1^4f_4 + 8e_1e_7f_2^2f_4 - e_1e_7f_1^3f_5$
$-8e_1e_7f_3f_5 + e_1e_7f_1^2f_6 - 8e_1e_7f_2f_6 - e_1e_7f_1f_7 - 32e_1^3e_2e_3f_8$
$+24e_1e_2^2e_3f_8 + 24e_1^2e_2e_4f_8 - 16e_1e_3e_4f_8 - 16e_1e_2e_5f_8 - 32e_8f_1^3f_2f_3$
$+24e_8f_1f_2^2f_3 + 24e_8f_1^2f_2f_4 - 16e_8f_1f_3f_4 - 16e_8f_1f_2f_5 + 3e_1e_2e_5f_1^3f_5$
$-8e_1e_2e_5f_1f_2f_5 + e_1e_2e_5f_3f_5 - 3e_1e_2e_5f_1^2f_6 + 4e_1e_2e_5f_2f_6$
$+9e_1e_2e_5f_1f_7 - 3e_3e_5f_1^3f_2f_3 + 6e_3e_5f_1f_2^2f_3 - 9e_3e_5f_1^2f_2f_4$
$+e_3e_5f_1f_3f_4 + e_3e_5f_1f_2f_5 - 5e_1^2e_6f_1^3f_2f_3 + 5e_1^2e_6f_1f_2^2f_3$
$+4e_1^2e_6f_1^2f_2f_4 - 9e_1^2e_6f_1f_3f_4 - 3e_1^2e_6f_1f_2f_5 + 8e_2e_6f_1^3f_2f_3$
$-6e_2e_6f_1^2f_2f_4 + 16e_2e_6f_1f_3f_4 + 4e_2e_6f_1f_2f_5 + 11e_1e_7f_1^3f_2f_3$
$-17e_1e_7f_1f_2^2f_3 - 10e_1e_7f_1^2f_2f_4 + 9e_1e_7f_1f_3f_4 + 9e_1e_7f_1f_2f_5$
$+4e_8f_4^2 - 8e_8f_8$

9 $-3e_4e_5f_3^3 + 9e_9f_2^2f_5 + 6e_3^3f_3f_6 + 6e_3e_6f_3^3 - 9e_9f_1f_8 + 9e_1^2e_7f_9$
$-30e_1^3e_2^3f_9 - 9e_2e_7f_9 - 30e_9f_1^3f_2^3 - 9e_9f_1^7f_2 + 3e_3^3f_2^2f_5 + 3e_3^3f_1^2f_7$
$-9e_1^5e_4f_9 + 9e_9f_1f_2^4 + 18e_9f_1^3f_3^2 - 9e_9f_2f_7 - 3e_3^3f_2f_7 + 9e_1^4e_5f_9$
$+18e_1^3e_3^2f_9 - 9e_3e_6f_9 + 9e_1e_4^2f_9 - 9e_9f_3f_6 + 9e_1^6e_3f_9 + 3e_1e_4^2f_3^3 - 9e_9f_2^3f_3$
$+9e_2^2e_5f_9 - 3e_3^3f_4f_5 - 9e_9f_1^5f_4 - 9e_4e_5f_9 + 9e_9f_1^4f_5 + 27e_1^5e_2^2f_9$
$+3e_3^3f_1f_4^2 - 3e_3^3f_1f_8 + 9e_9f_1^2f_7 + 9e_9f_1^6f_3 - 9e_1^3e_6f_9 + 9e_1e_2^4f_9$
$-11e_1^2f_1^2e_3e_4f_7 - 9e_1e_8f_9 - 3e_2e_7f_3^3 - 9e_9f_1^3f_6 - 9e_9f_4f_5 + 27e_9f_1^5f_2^2$
$+3e_1^2e_7f_3^3 - 9e_1^7e_2f_9 - 4e_1^2f_1^2e_2e_5f_7 - 11e_1^2f_1^2e_7f_3f_4 + 9e_9f_1f_4^2$
$-9e_2^3e_3f_9 - 3e_1e_8f_3^3 + 9e_1^2f_1^2e_7f_2^2f_3 + 9e_1^2f_1^2e_2^2e_3f_7 - 3e_1^4e_5f_3f_6$
$+3e_2^2e_5f_3^3 - 2e_1^4e_5f_2f_7 + e_1^6f_1^2e_3f_7 - e_1^2f_1^5e_7f_4 - e_1^3f_1^2e_6f_7$
$-e_1^5f_1^2e_4f_7 + 6e_1^3f_1^2e_3^2f_7 + e_1^2f_1^6e_7f_3 + e_1^4f_1^2e_5f_7 + 6e_1^2f_1^3e_7f_3^2$
$+e_1^2f_1^2e_7f_7 + e_1^5f_1^3e_4f_6 + e_1^2f_2^2e_3f_1e_4f_4 - e_1^2e_3f_2f_3e_4f_4$
$+e_2^2f_1^2e_1f_3e_4f_4 - e_2e_3f_1^2f_3e_4f_4 - 2e_1e_2^2f_2f_3e_4f_4 - 2e_2e_3f_1f_2^2e_4f_4$
$+e_2e_3f_2f_3^2e_4f_1 + e_2e_3^2f_2f_3e_1f_4 + 8e_2e_3f_2f_3e_4f_4 - 2e_1^2f_1^2e_3e_4f_3f_4$
$-e_1^2f_1^2e_3e_4f_2f_5 + 5e_1^2f_1^3e_3e_4f_6 + e_1^4f_1^4e_5f_5 - e_1^4f_1^3e_5f_6$
$+e_1^3f_1^5e_6f_4 - e_1^3f_1^4e_6f_5 + e_1^3f_1^3e_6f_6 + e_1^2f_1^4e_7f_5 - e_1^2f_1^3e_7f_6$
$-4e_1^4f_1^2e_5f_2f_5 - e_1^2f_1^2e_2e_5f_3f_4 - 4e_1^2f_1^4e_2e_5f_5 + 15e_1^2f_1^2e_2e_5f_2f_5$
$+4e_1^2f_1^3e_2e_5f_6 + 5e_1^3f_1^2e_6f_3f_4 + 4e_1^3f_1^2e_6f_2f_5 - 6e_1^4f_1^2e_2e_3f_7$
$+5e_1^3f_1^2e_2e_4f_7 - 6e_1^2f_1^4e_7f_2f_3 + 5e_1^2f_1^3e_7f_2f_4 + e_1^3f_1^2e_2e_4f_2f_5$
$-5e_1^3f_1^3e_2e_4f_6 + e_1^2f_1^3e_2e_5f_2f_4 - 5e_1^3f_1^3e_6f_2f_4 - 4e_1^2f_1^2e_7f_2f_5$
$-e_1f_1e_2^2e_4f_4^2 - e_1f_1e_4^2f_2^2f_4 - 5e_1f_1e_4^2f_4^2 + e_1^7f_1e_2f_8 + e_1^3f_1e_6f_8$
$-6e_1f_1^3e_8f_3^2 - e_1f_1^2e_8f_7 - 7e_1^5f_1e_2^2f_8 - 7e_1f_1^5e_8f_2^2 + 14e_1f_1^3e_8f_2^3$
$-e_1^2f_1e_7f_8 + e_1f_1^7e_8f_2 - e_1f_1^4e_8f_5 + e_1f_1^3e_8f_6 - e_1f_1^6e_8f_3$
$+e_1f_1^5e_8f_4 - 6e_1^3f_1e_3^2f_8 + 14e_1^3f_1e_2^3f_8 + e_1^5f_1e_4f_8 - e_1^4f_1e_5f_8$
$-e_1^6f_1e_3f_8 - 5e_1f_1^3e_4^2f_6 + 5e_1f_1^2e_4^2f_7 - 5e_1^3f_1e_6f_4^2 + 5e_1^2f_1e_7f_4^2$
$+5e_1^2f_1e_3e_4f_4^2 + 5e_1f_1^2e_4^2f_3f_4 + 19e_1f_1e_2e_3^2f_8 - 10e_1f_1e_8f_2f_6$
$+19e_1f_1e_2^2e_4f_8 - 10e_1f_1e_2e_6f_8 + 19e_1f_1e_8f_2^2f_4 + 19e_1f_1e_8f_2f_3^2$
$-10e_1f_1e_8f_3f_5 - 10e_1f_1e_3e_5f_8 - 3e_1f_1e_2e_3^2f_4^2 - 3e_1f_1e_4^2f_2f_3^2$

$$
\begin{aligned}
&-6e_1f_1e_4^2f_3f_5+14e_1f_1e_4^2f_2f_6-6e_1f_1e_3e_5f_4^2+14e_1f_1e_2e_6f_4^2\\
&-7e_1f_1e_2^4f_8-5e_1f_1e_4^2f_8-7e_1f_1e_8f_2^4-5e_1f_1e_8f_4^2+e_1f_1e_8f_8\\
&+e_1^4f_1e_2e_3f_2f_6-2e_1^3f_1e_3^2f_3f_5-e_1^3f_1e_3^2f_2f_6+4e_1f_1e_2e_3^2f_3f_5\\
&+7e_1f_1e_2e_3^2f_2f_6-12e_1f_1^2e_2e_3^2f_7-3e_1^5f_1e_4f_2f_6+2e_1f_1e_2^2e_4f_3f_5\\
&+5e_1f_1^3e_2^2e_4f_6-12e_1f_1e_2^2e_4f_2f_6-5e_1f_1^2e_2^2e_4f_7+e_1f_1^2e_4^2f_2f_5\\
&+4e_1^4f_1e_5f_3f_5+3e_1^4f_1e_5f_2f_6+e_1^2f_1e_2^2e_3f_3f_5-e_1^3f_1e_2e_4f_3f_5\\
&+14e_1^3f_1e_2e_4f_2f_6-3e_1f_1^2e_2^2e_4f_2f_5+5e_1^2f_1e_3e_4f_3f_5-13e_1^2f_1e_3e_4f_2f_6\\
&-3e_1^2f_1e_2e_5f_2^2f_4+e_1^2f_1e_2e_5f_4^2-2e_1f_1^3e_3e_5f_3^2+4e_1f_1e_3e_5f_2f_3^2\\
&+2e_1f_1e_3e_5f_2^2f_4+4e_1f_1^4e_3e_5f_5+5e_1f_1e_3e_5f_3f_5-4e_1f_1^3e_3e_5f_6\\
&+10e_1f_1e_3e_5f_2f_6+10e_1f_1^2e_3e_5f_7+5e_1^3f_1e_6f_2^2f_4-4e_1^3f_1e_6f_3f_5\\
&-3e_1^3f_1e_6f_2f_6-e_1f_1^3e_2e_6f_3^2+7e_1f_1e_2e_6f_2f_3^2-3e_1f_1^5e_2e_6f_4\\
&-12e_1f_1e_2e_6f_2^2f_4+3e_1f_1^4e_2e_6f_5+10e_1f_1e_2e_6f_3f_5-3e_1f_1^3e_2e_6f_6\\
&+8e_1f_1e_2e_6f_2f_6+3e_1f_1^2e_2e_6f_7-12e_1^2f_1e_7f_2f_3^2-5e_1^2f_1e_7f_2^2f_4\\
&+10e_1^2f_1e_7f_3f_5+3e_1^2f_1e_7f_2f_6+13e_1^4f_1e_2e_3f_8-30e_1^2f_1e_2^2e_3f_8\\
&-12e_1^3f_1e_2e_4f_8+11e_1^2f_1e_3e_4f_8+11e_1^2f_1e_2e_5f_8+13e_1f_1^4e_8f_2f_3\\
&-30e_1f_1^2e_8f_2^2f_3-12e_1f_1^3e_8f_2f_4+11e_1f_1^2e_8f_3f_4+11e_1f_1^2e_8f_2f_5\\
&-15e_1^2f_1e_2e_5f_3f_5-11e_1^2f_1e_2e_5f_2f_6+e_1f_1^2e_3e_5f_2^2f_3-e_1f_1^3e_3e_5f_2f_4\\
&+5e_1f_1^2e_3e_5f_3f_4-15e_1f_1^2e_3e_5f_2f_5+e_1f_1^4e_2e_6f_2f_3+14e_1f_1^3e_2e_6f_2f_4\\
&-13e_1f_1^2e_2e_6f_3f_4-11e_1f_1^2e_2e_6f_2f_5-e_1e_4^2f_2^2f_5+e_1e_4^2f_2^3f_3\\
&+e_1^3e_3^2f_2^2f_5-3e_1e_3e_5f_3^3-3e_2e_3e_4f_3^3-3e_1e_2e_6f_3^3+11e_1e_4^2f_4f_5\\
&-9e_1e_4^2f_3f_6-9e_1e_4^2f_2f_7+e_1e_2^4f_4f_5-3e_1e_2^4f_3f_6+5e_1e_2^4f_2f_7\\
&-2e_1^3e_2e_4f_2^2f_5-6e_1^2e_2e_5f_2^2f_5+5e_1^2e_2e_5f_2f_3f_4+8e_1e_3e_5f_2^2f_5\\
&-2e_1e_3e_5f_2f_3f_4-2e_1e_3e_5f_2^3f_3+6e_1e_2^2e_4f_2^2f_5-2e_2e_3e_4f_1f_3f_5\\
&-6e_2e_3e_4f_2^2f_5+5e_2e_3e_4f_1^2f_2f_5+4e_2e_3e_4f_1f_2f_6-4e_1e_2e_6f_1^2f_2^2f_3\\
&+4e_1e_2e_6f_2^2f_5+4e_1e_2e_6f_2f_3f_4+2e_1e_2e_6f_2^3f_3-4e_1^2e_2^2e_3f_1f_2f_6\\
&-3e_1e_2e_3^2f_2^2f_5-2e_1e_4^2f_2f_3f_4+4e_1^3e_2e_4f_4f_5-12e_1^3e_2e_4f_3f_6\\
&-8e_1^3e_2e_4f_2f_7+9e_9f_9+e_9f_1^9+3e_9f_3^3+e_1^9f_9+3e_3^3f_9+e_3^3f_3^3+7e_1^2e_2e_5f_4f_5\\
&+9e_1^2e_2e_5f_3f_6+6e_1^2e_2e_5f_2f_7+2e_1e_3e_5f_4f_5-18e_1e_3e_5f_2f_7\\
&-3e_1e_2^2e_4f_4f_5+9e_1e_2^2e_4f_3f_6-e_1e_2^2e_4f_2f_7-2e_2e_3e_4f_1f_4^2\\
&+2e_2e_3e_4f_4f_5-5e_2e_3e_4f_1^3f_6+11e_2e_3e_4f_1^2f_7-4e_2e_3e_4f_2f_7\\
&-18e_2e_3e_4f_1f_8-18e_1e_2e_6f_4f_5-4e_1e_2e_6f_2f_7-13e_1^2e_3e_4f_4f_5\\
&+9e_1^2e_3e_4f_3f_6+20e_1^2e_3e_4f_2f_7-4e_1^2e_2^2e_3f_4f_5+9e_1^2e_2^2e_3f_3f_6\\
&-5e_1^2e_2^2e_3f_2f_7+7e_1e_2e_3^2f_4f_5-18e_1e_2e_3^2f_3f_6+13e_1e_2e_3^2f_2f_7\\
&-3e_1^4e_2e_3f_3f_6+10e_1^4e_2e_3f_2f_7+e_1^5e_2^2f_2f_7+2e_1^3e_3^2f_4f_5+3e_1^3e_3^2f_3f_6\\
&-11e_1^3e_3^2f_2f_7+e_1^3e_2^3f_3f_6-5e_1^3e_2^3f_2f_7+36e_1^3e_2e_4f_9-27e_1^2e_2e_5f_9\\
&+18e_1e_3e_5f_9-27e_1e_2^2e_4f_9+18e_2e_3e_4f_9+18e_1e_2e_6f_9-27e_1^2e_3e_4f_9\\
&+54e_1^2e_2^2e_3f_9-27e_1e_2e_3^2f_9-45e_1^4e_2e_3f_9+2e_1^4e_5f_2^2f_5+2e_2^2e_5f_1^4f_5\\
&+e_4e_5f_2^2f_5-2e_2^3e_3f_1f_3f_5+2e_2^3e_3f_1f_2f_6-3e_2^2e_5f_1f_2f_3^2\\
&+8e_2^2e_5f_1f_3f_5-6e_2^2e_5f_1^2f_2f_5-6e_2^2e_5f_2f_3f_4+4e_2^2e_5f_1f_2f_6\\
&+6e_2^2e_5f_1f_2^2f_4-13e_4e_5f_1^2f_3f_4-4e_4e_5f_1^2f_2^2f_3+7e_4e_5f_1f_2f_3^2\\
&+2e_4e_5f_1f_3f_5-4e_4e_5f_1^4f_5-e_4e_5f_2^3f_3-2e_1^3e_6f_2^2f_5+3e_3e_6f_1^5f_4\\
&-9e_3e_6f_2^2f_5-3e_3e_6f_1^4f_5+3e_3e_6f_2^3f_3+2e_1^2e_7f_2^2f_5-2e_1^2e_7f_2^3f_3
\end{aligned}
$$

$+2e_2e_7f_1^5f_4+5e_2e_7f_2^2f_5-2e_2e_7f_1^4f_5-5e_2e_7f_2^3f_3-2e_2e_7f_1^6f_3$
$-9e_1e_8f_2^2f_5+7e_4e_5f_1^2f_2f_5+2e_4e_5f_2f_3f_4-18e_4e_5f_1f_2f_6-3e_4e_5f_1f_2^2f_4$
$-5e_1^3e_6f_2f_3f_4+9e_3e_6f_1^2f_3f_4+9e_3e_6f_1^2f_2^2f_3-18e_3e_6f_1f_2f_3^2$
$+9e_3e_6f_1^2f_2f_5-3e_3e_6f_1^4f_2f_3+9e_3e_6f_1f_2^2f_4+11e_1^2e_7f_2f_3f_4$
$+20e_2e_7f_1^2f_3f_4-5e_2e_7f_1^2f_2^2f_3+13e_2e_7f_1f_2f_3^2-18e_2e_7f_1f_3f_5$
$+6e_2e_7f_1^2f_2f_5-4e_2e_7f_2f_3f_4+10e_2e_7f_1^4f_2f_3-4e_2e_7f_1f_2f_6$
$-e_2e_7f_1f_2^2f_4-18e_1e_8f_2f_3f_4+9e_1e_8f_2^3f_3-3e_3^3f_1f_3f_5-3e_3^3f_2f_3f_4$
$-3e_3^3f_1f_2f_6-27e_9f_1^2f_3f_4+54e_9f_1^2f_2^2f_3-27e_9f_1f_2f_3^2+18e_9f_1f_3f_5$
$-27e_9f_1^2f_2f_5+18e_9f_2f_3f_4-45e_9f_1^4f_2f_3+18e_9f_1f_2f_6-27e_9f_1f_2^2f_4$
$-2e_1^6e_3f_2f_7+e_2^3e_3f_1f_4^2-e_2^3e_3f_4f_5+3e_2^3e_3f_3f_6-2e_2^3e_3f_1^2f_7$
$-5e_2^3e_3f_2f_7+9e_2^3e_3f_1f_8+3e_1^5e_4f_3f_6+2e_1^5e_4f_2f_7-4e_1^4e_5f_4f_5$
$-e_2^2e_5f_1f_4^2+e_2^2e_5f_1^3f_3^2+e_2^2e_5f_4f_5-2e_2^2e_5f_1^3f_6-9e_2^2e_5f_3f_6$
$+2e_2^2e_5f_1^2f_7+5e_2^2e_5f_2f_7-9e_2^2e_5f_1f_8+11e_4e_5f_1f_4^2+e_4e_5f_1f_2^4$
$+2e_4e_5f_1^3f_3^2-11e_4e_5f_4f_5+9e_4e_5f_1^3f_6+9e_4e_5f_3f_6-9e_4e_5f_1^2f_7$
$+9e_4e_5f_2f_7+9e_4e_5f_1f_8+9e_1^3e_6f_4f_5+3e_1^3e_6f_3f_6+2e_1^3e_6f_2f_7$
$-9e_3e_6f_1f_4^2-3e_3e_6f_1f_2^4+3e_3e_6f_1^3f_3^2+e_3e_6f_1^3f_2^3+9e_3e_6f_4f_5$
$+3e_3e_6f_1^3f_6-9e_3e_6f_3f_6-9e_3e_6f_1^2f_7+9e_3e_6f_2f_7+9e_3e_6f_1f_8$
$-9e_1^2e_7f_4f_5-9e_1^2e_7f_3f_6-2e_1^2e_7f_2f_7-9e_2e_7f_1f_4^2+5e_2e_7f_1f_2^4$
$+e_2e_7f_1^5f_2^2-11e_2e_7f_1^3f_3^2-5e_2e_7f_1^3f_2^3+9e_2e_7f_4f_5+2e_2e_7f_1^3f_6$
$+9e_2e_7f_3f_6-2e_2e_7f_1^2f_7-5e_2e_7f_2f_7+9e_2e_7f_1f_8+9e_1e_8f_4f_5+9e_1e_8f_3f_6$
$+9e_1e_8f_2f_7+36e_9f_1^3f_2f_4-2e_2^2e_5f_1^3f_2f_4+4e_4e_5f_1^3f_2f_4$
$-12e_3e_6f_1^3f_2f_4-8e_2e_7f_1^3f_2f_4$

10 $10e_{10}f_2f_8-2e_1^4f_2^2e_2e_4f_6-10e_2^2e_6f_{10}+15e_1^2e_4^2f_{10}-2e_2^2f_1^4e_6f_2f_4$
$-8e_2^2f_1^3e_6f_2f_5-11e_2f_1^2e_8f_4^2+10e_{10}f_1f_3^3+8e_1e_3f_4e_2^3f_6+2e_1e_3^3f_4f_6$
$-2e_1e_3^3f_4^2f_2+10e_4f_2^3e_6f_4+2e_4f_1^4f_3^2e_6-2e_4f_2^5e_6+4e_4^2f_1f_3e_2f_6$
$-8e_4f_2^2e_1e_5f_6-4e_4f_2^2e_6f_1^3f_3-12e_4f_2^2e_6f_1^2f_4-18e_4f_2^2e_6f_1f_5$
$+28e_1^2f_2e_2e_3^2f_8+2e_2^2f_2^2e_6f_3^2-4e_2^2f_2^3e_6f_4+2e_2f_2e_8f_4^2+6e_2f_2e_8f_8$
$-4e_2^2f_2e_6f_3f_5-4e_2^2f_2e_6f_1f_7+4e_2f_2^3e_1e_7f_4-4e_2f_2^2e_1e_7f_6$
$-10e_1^4e_6f_{10}+4e_1e_3^3f_2^2f_6-8e_2f_2^2e_1e_3e_4f_6+6e_1^3f_1^3e_3e_4f_7$
$+5e_2e_4^2f_5^2-2e_2f_2^5e_8-2e_2^5f_2f_8-15e_1^2e_4^2f_3f_7+e_1^4f_1^3e_6f_7$
$-8e_4f_1^3f_3e_6f_4+12e_4f_1f_3e_6f_2f_4-6e_4f_1^2f_3e_6f_5-4e_1^3e_2^2f_4e_3f_6$
$+e_1^3f_1^6e_7f_4-2e_2^5f_4f_6-6e_4f_1^2f_2e_6f_6+15e_{10}f_2^2f_3^2+8e_1^2f_2^2e_2^2e_4f_6$
$+2e_1^3f_2^2e_7f_6+35e_1^6e_2^2f_{10}+4e_1^4e_2f_4e_4f_6-12e_1^2e_2^2f_4e_4f_6$
$+16e_1^3e_2f_4e_5f_6+e_1^3f_1^3e_2e_5f_2f_5-5e_1^3f_1^4e_2e_5f_6-5e_1^4f_1^3e_6f_2f_5$
$-6e_1^4f_1^3e_2e_4f_7+4e_1e_3f_4^2e_6f_2+e_1^5f_1^4e_5f_6+e_1^4f_1^5e_6f_5-e_1^4f_1^4e_6f_6$
$-8e_1^3e_3f_4e_4f_6+12e_1e_3f_4e_2e_4f_6-6e_1^2e_3f_4e_5f_6-10e_2e_4^2f_{10}+10e_4e_6f_{10}$
$+2e_1^2f_2^3e_2e_6f_4+6e_1^2f_2^2e_2e_6f_6+12e_1^5f_2e_2e_3f_8-12e_1^3f_2e_2^2e_3f_8$
$-10e_1^4f_2e_2e_4f_8+24e_1^3f_2e_3e_4f_8-2e_1^2f_2^3e_3e_5f_4+10e_1^2f_2^2e_3e_5f_6$
$+4e_4f_1^4f_2e_6f_4+16e_4f_1^3f_2e_6f_5+2e_1^4e_3^2f_4f_6+6e_1^3f_1^3e_7f_3f_4$
$-e_1^3f_1^3e_7f_7+5e_2^2e_6f_5^2-5e_4e_6f_5^2-5e_2e_8f_5^2-e_1^5f_1^3e_5f_7-5e_5^2f_1f_3^3$
$+25e_1^2e_2^4f_{10}-22e_1^2f_2e_3e_5f_8+e_1^2f_2e_4^2f_3f_5+2e_1^2f_2e_3e_5f_4^2$
$-6e_1^2f_2e_2e_6f_4^2+9e_1^2f_2e_2^4f_8-3e_1^2f_2e_4^2f_4^2-11e_1^2f_2e_4^2f_8$

$$
\begin{aligned}
&-4e_4f_2^2e_3^2f_6-9e_4f_2^2e_6f_3^2-2e_4f_2^2e_6f_6+5e_5^2f_2^2f_6+8e_1^3f_2e_2e_5f_8\\
&-8e_1^3f_2^2e_2e_5f_6+5e_5^2f_1^2f_8-5e_5^2f_2f_8+2e_2^2f_2e_1e_5f_4^2+4e_2f_2^3e_3e_5f_4\\
&-4e_2f_2^2e_3e_5f_6-5e_1e_9f_5^2+2e_1e_9f_2^5-8e_2^3f_2e_1e_3f_8-28e_2f_2e_1e_3e_4f_8\\
&+2e_2^2f_2e_1e_5f_8-8e_2f_2^3e_8f_1f_3-28e_2f_2e_8f_1f_3f_4+2e_2f_2^2e_8f_1f_5\\
&+4e_2^2f_2^2e_1e_5f_6-5e_1e_3^3f_5^2+2e_4f_1f_3^3e_6-2e_4^2f_1f_3^3e_2+e_1^2e_2^4f_4f_6\\
&+4e_2^2f_1f_3^3e_6+e_4f_1^2f_2^4e_6-8e_2^2f_2e_6f_1f_3f_4+4e_2^2f_2^2e_6f_1f_5\\
&-10e_{10}f_2^2f_6-10e_{10}f_1^6f_4+25e_{10}f_1^4f_3^2+5e_1^2e_8f_5^2-5e_3e_7f_5^2+2e_3e_7f_2^5\\
&+10e_1e_9f_{10}+5e_2^2e_3^2f_5^2+5e_1^4e_6f_5^2-5e_1^3e_7f_5^2-6e_1^2e_2f_4e_6f_6\\
&-6e_2f_1^2e_8f_2f_6-2e_2f_1^5e_8f_5-2e_2^2f_1^2e_6f_8+2e_2f_1^4e_8f_6-2e_2f_1^7e_8f_3\\
&+2e_2f_1^6e_8f_4+e_2^3f_1^2e_4f_4^2+6e_2f_1^3e_4^2f_7-e_2^2f_1^2e_6f_4^2-6e_1^2f_2e_8f_4^2\\
&+2e_1^2f_2e_8f_8+7e_2^2f_1^2e_3^2f_8-13e_2f_1^4e_8f_3^2-2e_2f_1^3e_8f_7+e_2f_1^6e_8f_2^2\\
&-6e_2f_1^4e_8f_2^3+2e_2^3f_1^2e_4f_8+5e_1^3f_1^3e_7f_2f_5-e_1^3f_1^5e_7f_5+e_1^3f_1^4e_7f_6\\
&-6e_1^3f_1^4e_7f_2f_4+10e_2e_4^2f_3f_7+10e_2e_4^2f_1f_9-10e_2e_8f_3f_7-10e_2e_8f_1f_9\\
&+10e_2e_8f_3^2f_4-10e_2e_8f_4f_6-7e_1f_1e_3e_6f_3^3+11e_1f_1e_2e_7f_9+11e_1f_1e_9f_2f_7\\
&-4e_1f_1e_3^3f_2f_7+11e_1f_1e_3e_6f_9+11e_1f_1e_9f_3f_6+31e_1f_1e_9f_2^3f_3\\
&-21e_1f_1e_2^2e_5f_9+5e_1f_1e_3^3f_4f_5+e_1f_1^6e_9f_4-e_1f_1^5e_9f_5-8e_1^6f_1e_2^2f_9\\
&+7e_1f_1^2e_3^3f_8-e_1f_1^3e_9f_7-e_1f_1^7e_9f_3+e_1^4f_1e_6f_9-16e_1^2f_1e_2^4f_9\\
&+e_1^2f_1e_8f_9+e_1f_1^2e_9f_8-e_1^3f_1e_7f_9+20e_1^4f_1e_2^3f_9+20e_1f_1^4e_9f_2^3\\
&+e_1f_1^8e_9f_2+e_1^6f_1e_4f_9-16e_1f_1^2e_9f_2^4-7e_1f_1^4e_9f_3^2-e_1^5f_1e_5f_9\\
&-7e_1^4f_1e_3^2f_9-6e_1^2f_1e_4^2f_9-e_1^7f_1e_3f_9-10e_4e_6f_1^3f_7-10e_4e_6f_3f_7\\
&-10e_4e_6f_1f_9+10e_2^2e_6f_3f_7+10e_2^2e_6f_1f_9+4e_2^3f_2e_4f_3f_5+4e_2^3f_2e_4f_1f_7\\
&+4e_2f_2e_1e_3e_4f_4^2+4e_2f_2e_4^2f_1f_3f_4+2e_2f_2^2e_4^2f_1f_5-3e_2f_1^2e_4^2f_4^2\\
&-6e_2f_1^2e_4^2f_8+9e_2f_1^2e_8f_2^4-2e_2^3f_1^3e_4f_7+e_2^2f_2e_3^2f_8-6e_2f_2^2e_8f_6\\
&+6e_2^3f_2e_4f_8-6e_2^2f_2e_6f_8+4e_2f_1^2e_8f_2^2f_4+28e_2f_1^2e_8f_2f_3^2\\
&-22e_2f_1^2e_8f_3f_5-12e_2f_1^2e_3e_5f_8+2e_2f_1^2e_4^2f_3f_5-6e_2f_1^2e_4^2f_2f_6\\
&+e_2f_1^2e_3e_5f_4^2+2e_2^2f_1^3e_6f_7-9e_2^2f_4e_3^2f_6+4e_1^4f_4e_6f_6\\
&+e_1f_1e_2e_3f_2f_3e_4f_4+3e_1^2f_1^2e_4^2f_4^2-e_1^4f_1^2e_6f_8+7e_1^2f_1^4e_8f_3^2\\
&+e_1^2f_1^3e_8f_7+e_1^3f_1^2e_7f_8+e_1^2f_1^5e_8f_5-e_1^2f_1^4e_8f_6+e_1^2f_1^7e_8f_3\\
&-e_1^2f_1^6e_8f_4+7e_1^4f_1^2e_3^2f_8-60e_1^2e_2e_3^2f_{10}-20e_1e_4e_5f_{10}+100e_1^3e_2^2e_3f_{10}\\
&+50e_1^4e_2e_4f_{10}-20e_2e_3e_5f_{10}-40e_1^3e_3e_4f_{10}-40e_1^3e_2e_5f_{10}-20e_1e_2e_7f_{10}\\
&-60e_1^2e_2^2e_4f_{10}+30e_1^2e_2e_6f_{10}-60e_1^5e_2e_3f_{10}-40e_1e_2^3e_3f_{10}+30e_1^2e_3e_5f_{10}\\
&-20e_1e_3e_6f_{10}+30e_1e_2^2e_5f_{10}-10e_1e_3^3f_2f_8-10e_1e_9f_2f_8+10e_1e_9f_2^2f_6\\
&+10e_2e_8f_{10}-2e_2^2f_1^4e_6f_6+11e_1f_1e_9f_4f_5+31e_1f_1e_2^3e_3f_9+4e_1^5f_1e_5f_3f_6\\
&+3e_1^5f_1e_5f_2f_7+11e_1f_1e_4e_5f_9-4e_1f_1e_2e_7f_3^3+e_1f_1^4e_9f_6-8e_1f_1^6e_9f_2^2\\
&+e_1^8f_1e_2f_9-6e_1f_1^2e_9f_4^2+7e_1^2f_1e_8f_3^3-8e_4f_4e_3^2f_1f_5+3e_4^2f_4e_1^2f_3^2\\
&+2e_4^2f_4e_2f_3^2-4e_1^5f_4e_5f_6-2e_4f_4e_3^2f_6-2e_4f_4e_6f_3^2+14e_4f_4e_6f_6\\
&+3e_4f_4^2e_3^2f_1^2+2e_4f_4^2e_3^2f_2-e_4f_2^2e_1e_5f_3^2+8e_4f_1f_3e_6f_2^3\\
&-4e_4f_1f_3e_6f_6-e_2^2f_4e_3^2f_1f_5+4e_4f_1^4e_6f_6+10e_4f_4e_2^3f_6-4e_2^2f_4e_6f_3^2\\
&-2e_2^2f_4e_6f_6-12e_1f_1^3e_2e_3e_4f_7+21e_1f_1e_2e_3e_4f_2f_7+26e_1f_1^2e_2e_3e_4f_8\\
&+18e_1^2f_1e_2e_6f_4f_5+11e_1^2f_1e_2e_6f_2f_7+5e_1^3f_1e_3e_4f_4f_5+5e_1^3f_1e_3e_4f_3f_6\\
&-16e_1^3f_1e_3e_4f_2f_7+e_1^3f_1e_2^2e_3f_3f_6-5e_1^3f_1e_2^2e_3f_2f_7\\
&-3e_1^2f_1e_2e_3^2f_4f_5+6e_1^2f_1e_2e_3^2f_3f_6+9e_1^2f_1e_2e_3^2f_2f_7
\end{aligned}
$$

$$+e_1^5f_1e_2e_3f_2f_7+20e_1e_9f_2f_3f_5-10e_3e_7f_2f_8+10e_3e_7f_1^2f_8$$
$$+10e_3e_7f_2^2f_6-3e_1^2f_1e_4^2f_2f_3f_4-e_1^4f_1e_2e_4f_3f_6+17e_1^4f_1e_2e_4f_2f_7$$
$$+e_1^3f_1e_2e_5f_4f_5-19e_1^3f_1e_2e_5f_3f_6-14e_1^3f_1e_2e_5f_2f_7-12e_1^2f_1e_3e_5f_4f_5$$
$$+13e_1^2f_1e_3e_5f_2f_7-e_1^2f_1e_2^2e_4f_4f_5+3e_1^2f_1e_2^2e_4f_3f_6$$
$$-23e_1^2f_1e_2^2e_4f_2f_7-3e_1f_1^2e_2e_3e_4f_4^2-3e_1^3f_1e_2e_5f_2^2f_5$$
$$+2e_1^2f_1e_3e_5f_2^2f_5+4e_1^2f_1e_3e_5f_2f_3f_4+4e_1f_1^2e_2e_3e_4f_3f_5$$
$$-3e_1f_1e_2e_3e_4f_2^2f_5+7e_1f_1^2e_2e_3e_4f_2f_6-17e_1^2f_1e_2e_6f_2^2f_5$$
$$+7e_1^2f_1e_2e_6f_2f_3f_4-2e_1^2f_1^3e_3e_5f_3f_4-e_1^2f_1^3e_3e_5f_2f_5$$
$$+e_1^2f_1^4e_2e_6f_2f_4-e_1^2f_1^3e_2e_6f_3f_4+19e_1^2f_1^3e_2e_6f_2f_5-e_1^2f_1e_4^2f_2^2f_5$$
$$-8e_1^2f_1e_4^2f_4f_5-3e_1^2f_1e_4^2f_3f_6+17e_1^2f_1e_4^2f_2f_7+4e_1^2f_1^4e_2e_6f_6$$
$$-15e_1^2f_1^2e_2e_6f_2f_6-4e_1^2f_1^3e_2e_6f_7+9e_1^3f_1^2e_7f_2^2f_4-5e_1^3f_1^2e_7f_3f_5$$
$$-4e_1^3f_1^2e_7f_2f_6-7e_1^5f_1^2e_2e_3f_8+14e_1^3f_1^2e_2^2e_3f_8+6e_1^4f_1^2e_2e_4f_8$$
$$-13e_1^3f_1^2e_3e_4f_8-5e_1^3f_1^2e_2e_5f_8-7e_1^2f_1^5e_8f_2f_3+14e_1^2f_1^3e_8f_2^2f_3$$
$$+6e_1^2f_1^4e_8f_2f_4-13e_1^2f_1^3e_8f_3f_4-5e_1^2f_1^3e_8f_2f_5-e_1^3f_1^2e_2e_5f_3f_5$$
$$+19e_1^3f_1^2e_2e_5f_2f_6+e_1^2f_1^2e_3e_5f_2^2f_4+3e_1^2f_1^2e_3e_5f_3f_5$$
$$+5e_1^2f_1^4e_3e_5f_6-19e_1^2f_1^2e_3e_5f_2f_6-5e_1^2f_1^3e_3e_5f_7+5e_1^4f_1^2e_6f_3f_5$$
$$+4e_1^4f_1^2e_6f_2f_6-4e_1^2f_1^2e_2e_6f_2^2f_4-4e_1^2f_1^5e_2e_6f_5-19e_1^2f_1^2e_2e_6f_3f_5$$
$$-8e_4f_4e_1e_5f_3^2+e_1^2f_1^2e_2^2e_4f_3f_5-4e_1^2f_1^2e_2^2e_4f_2f_6+9e_1^2f_1^3e_2^2e_4f_7$$
$$-4e_1^5f_1^2e_5f_2f_6+e_1^4f_1^2e_2e_4f_2f_6-2e_1^3f_1^2e_3e_4f_3f_5-e_1^3f_1^2e_3e_4f_2f_6$$
$$-21e_1^2f_1^2e_2e_3^2f_8+4e_1^2f_1^2e_8f_2f_6-9e_1^2f_1^2e_2^2e_4f_8+4e_1^2f_1^2e_2e_6f_8$$
$$-9e_1^2f_1^2e_8f_2^2f_4-21e_1^2f_1^2e_8f_2f_3^2+12e_1^2f_1^2e_8f_3f_5+12e_1^2f_1^2e_3e_5f_8$$
$$+2e_1^2f_1^2e_4^2f_3f_5+e_1^2f_1^2e_4^2f_2f_6+2e_1^2f_1^2e_3e_5f_4^2+e_1^2f_1^2e_2e_6f_4^2$$
$$-e_1^6f_1^2e_4f_8+e_1^5f_1^2e_5f_8+e_1^7f_1^2e_3f_8-6e_1^2f_1^3e_4^2f_7-6e_1^3f_1^2e_7f_4^2$$
$$+6e_1^2f_1^2e_4^2f_8+6e_1^2f_1^2e_8f_4^2-e_1^2f_1^2e_8f_8-e_1f_1e_9f_9-7e_1f_1e_9f_3^3$$
$$-7e_1f_1e_3^3f_9-31e_1f_1e_2e_7f_4f_5-3e_1f_1^4e_2e_7f_6-10e_1f_1e_2e_7f_3f_6$$
$$+3e_1f_1^3e_2e_7f_7-8e_1f_1e_2e_7f_2f_7-3e_1f_1^2e_2e_7f_8-11e_1^2f_1e_8f_4f_5$$
$$-11e_1^2f_1e_8f_3f_6-3e_1^2f_1e_8f_2f_7-14e_1f_1^4e_9f_2f_4-e_1f_1^4e_3e_6f_2f_4$$
$$+17e_1f_1^4e_2e_7f_2f_4-4e_1^4f_1e_6f_3f_6-3e_1^4f_1e_6f_2f_7-3e_1f_1^2e_3e_6f_4^2$$
$$-2e_1f_1^4e_3e_6f_3^2-7e_1f_1e_3e_6f_4f_5-4e_1f_1^4e_3e_6f_6-4e_1f_1e_3e_6f_3f_6$$
$$+4e_1f_1^3e_3e_6f_7-10e_1f_1e_3e_6f_2f_7-11e_1f_1^2e_3e_6f_8+11e_1^3f_1e_7f_4f_5$$
$$+4e_1^3f_1e_7f_3f_6+3e_1^3f_1e_7f_2f_7+17e_1f_1^2e_2e_7f_4^2-e_1f_1^4e_2e_7f_3^2$$
$$-e_1f_1^2e_2^2e_5f_4^2-e_1f_1e_2^2e_5f_4f_5+5e_1f_1^4e_2^2e_5f_6+18e_1f_1e_2^2e_5f_3f_6$$
$$-5e_1f_1^3e_2^2e_5f_7+12e_1f_1e_2^2e_5f_2f_7+5e_1f_1^2e_2^2e_5f_8-8e_1f_1^2e_4e_5f_4^2$$
$$+23e_1f_1e_4e_5f_4f_5-5e_1f_1^4e_4e_5f_6-7e_1f_1e_4e_5f_3f_6+11e_1f_1^3e_4e_5f_7$$
$$-31e_1f_1e_4e_5f_2f_7-11e_1f_1^2e_4e_5f_8-5e_1^4f_1e_6f_4f_5+13e_1f_1^3e_9f_3f_4$$
$$-46e_1f_1^3e_9f_2^2f_3+33e_1f_1^2e_9f_2f_3^2-12e_1f_1^2e_9f_3f_5+13e_1f_1^3e_9f_2f_5$$
$$-42e_1f_1e_9f_2f_3f_4+15e_1f_1^5e_9f_2f_3-12e_1f_1^2e_9f_2f_6+33e_1f_1^2e_9f_2^2f_4$$
$$+e_1f_1e_2^3e_3f_4f_5-3e_1f_1e_2^3e_3f_3f_6+5e_1f_1e_2^3e_3f_2f_7-7e_1f_1^2e_2^3e_3f_8$$
$$-3e_1^6f_1e_4f_2f_7-19e_1f_1^3e_3e_6f_2f_5+3e_1f_1^2e_3e_6f_2^2f_4-12e_1^3f_1e_7f_2f_3f_4$$
$$-16e_1f_1^3e_2e_7f_3f_4-5e_1f_1^3e_2e_7f_2^2f_3+9e_1f_1^2e_2e_7f_2f_3^2$$
$$+13e_1f_1^2e_2e_7f_3f_5-14e_1f_1^3e_2e_7f_2f_5+21e_1f_1e_2e_7f_2f_3f_4$$
$$+e_1f_1^5e_2e_7f_2f_3+11e_1f_1^2e_2e_7f_2f_6-23e_1f_1^2e_2e_7f_2^2f_4+26e_1^2f_1e_8f_2f_3f_4$$

$$-7e_1^2f_1e_8f_2^3f_3+4e_1f_1^5e_3e_6f_5-3e_1f_1e_3e_6f_2^3f_3-5e_1^3f_1e_7f_2^2f_5$$
$$-3e_1f_1^6e_2e_7f_4+12e_1f_1e_2e_7f_2^2f_5+3e_1f_1^5e_2e_7f_5+5e_1f_1e_2e_7f_2^3f_3$$
$$+5e_1^2f_1e_8f_2^2f_5+e_1f_1^3e_4e_5f_2f_5+18e_1f_1^2e_4e_5f_2f_6-e_1f_1^2e_4e_5f_2^2f_4$$
$$+5e_1f_1^3e_3e_6f_3f_4+e_1f_1^3e_3e_6f_2^2f_3+6e_1f_1^2e_3e_6f_2f_3^2+2e_1f_1^2e_2^2e_5f_3f_5$$
$$-3e_1f_1^3e_2^2e_5f_2f_5-3e_1f_1e_2^2e_5f_2f_3f_4-17e_1f_1^2e_2^2e_5f_2f_6$$
$$+5e_1f_1^3e_4e_5f_3f_4-3e_1f_1^2e_4e_5f_2f_3^2-12e_1f_1^2e_4e_5f_3f_5+e_1f_1e_4e_5f_2^3f_3$$
$$+5e_1^4f_1e_6f_2^2f_5+18e_1f_1e_3e_6f_2^2f_5-2e_1^4f_1e_3^2f_3f_6-e_1^4f_1e_3^2f_2f_7$$
$$-14e_1^4f_1e_2e_4f_9+13e_1^3f_1e_2e_5f_9-12e_1^2f_1e_3e_5f_9+33e_1^2f_1e_2^2e_4f_9$$
$$-42e_1f_1e_2e_3e_4f_9-12e_1^2f_1e_2e_6f_9+13e_1^3f_1e_3e_4f_9-46e_1^3f_1e_2^2e_3f_9$$
$$+33e_1^2f_1e_2e_3^2f_9+15e_1^5f_1e_2e_3f_9-e_1f_1e_4e_5f_2^2f_5+20e_3e_7f_1f_4f_5$$
$$+5e_1^3f_1^3e_2e_5f_7-4e_1e_3f_4e_6f_6+30e_{10}f_1^2f_3f_5-10e_{10}f_1^2f_8-4e_4f_4e_6f_1f_5$$
$$+4e_4f_4^2e_1e_5f_2-4e_4f_4e_1e_5f_6+9e_4f_4^2e_6f_1^2-14e_4f_4^2e_6f_2+4e_4^2f_4e_2f_1f_5$$
$$-18e_2^2f_4e_1e_5f_6-8e_2^2f_4e_6f_1f_5-4e_4f_1^5e_6f_5+9e_4^2f_4e_1^2f_6-14e_4^2f_4e_2f_6$$
$$-8e_1e_3f_2^2e_6f_6+3e_1^4f_3e_6f_7-2e_2^3f_1^2e_4f_3f_5+2e_2^3f_1^2e_4f_2f_6$$
$$-4e_2^2f_1^2e_6f_2f_3^2+8e_2^2f_1^2e_6f_2^2f_4+10e_2^2f_1^2e_6f_3f_5+6e_2^2f_1^2e_6f_2f_6$$
$$+4e_2f_2e_8f_1f_7+4e_2f_2e_1e_7f_8+8e_2f_2e_8f_1^3f_5+6e_2f_2^3e_8f_4+e_2f_2^2e_8f_3^2$$
$$+e_2^2f_2e_3^2f_4^2-2e_2^3f_2e_4f_4^2+e_2f_2^2e_4^2f_3^2-2e_2f_2^3e_4^2f_4+10e_2f_2^2e_4^2f_6$$
$$+12e_1^3f_3e_2e_5f_7+12e_1^3f_3e_3e_4f_7+3e_1^3f_3^2e_7f_4-3e_1^3f_3e_7f_7-4e_1^2e_2^4f_3f_7$$
$$+12e_1^3f_3e_2^2e_3f_7+e_1^4f_3e_2^3f_7+3e_1^6f_3e_4f_7+3e_1^4f_3e_3^2f_7-3e_1^5f_3e_5f_7$$
$$-3e_1^5e_2f_3e_3f_7-9e_1^2e_2f_3e_6f_7-2e_1e_2^3f_3e_3f_7-15e_1^4e_2f_3e_4f_7$$
$$-9e_1e_2^2f_3e_5f_7-24e_1^2e_2f_3e_3^2f_7+3e_1e_2f_3e_3e_4f_7-7e_1e_2f_3^2e_7f_2^2$$
$$+e_1e_2f_3^2e_7f_4-12e_2f_1^3e_8f_2^2f_3+24e_2f_1^3e_8f_3f_4+2e_2f_1^2e_8f_8$$
$$+5e_1f_1e_4e_5f_3^3-21e_1f_1e_9f_2^2f_5-7e_1f_1e_3^3f_3f_6+5e_1^3f_2e_2f_3e_5f_5$$
$$-e_1e_2f_3e_7f_7+3e_1e_2^2f_3^2e_5f_4-3e_1^2e_2f_3^2e_6f_4+15e_1^2e_2f_3e_6f_2f_5$$
$$+15e_1^2e_2f_3e_6f_1f_6-12e_2f_2e_1e_7f_4^2+2e_2f_2e_4^2f_4^2+2e_2f_2e_4^2f_8$$
$$+2e_2^2f_2^2e_3^2f_6-4e_2^3f_2^2e_4f_6+e_2^2f_1^4e_6f_3^2+2e_2^2f_1^5e_6f_5-3e_1e_2f_3^2e_3e_4f_4$$
$$-15e_1e_2f_3e_3e_4f_1f_6+12e_2f_2e_8f_1^5f_3-10e_2f_2e_8f_1^4f_4+4e_2f_2e_8f_3f_5$$
$$+4e_2f_2e_3e_5f_8-12e_2f_2e_4^2f_3f_5-12e_2f_2e_4^2f_1f_7-12e_2f_2e_3e_5f_4^2$$
$$+10e_2^2f_2e_6f_4^2+5e_1^3f_3e_7f_2f_5-5e_1^4f_3e_6f_2f_5-e_3f_3e_1e_6f_7+6e_3^2f_3e_2^2f_7$$
$$+11e_3^3f_3e_1f_7-11e_3^2f_3e_4f_7+6e_3f_3^2e_7f_2^2+11e_3f_3^3e_7f_1-11e_3f_3^2e_7f_4$$
$$-e_3f_1f_2e_7f_7-24e_3f_1^2f_2e_7f_3^2+3e_3f_1f_2e_7f_3f_4-15e_3f_1^4f_2e_7f_4$$
$$-7e_3^2f_1f_2e_2^2f_7-3e_3f_1f_2e_2e_5f_7-3e_3f_1^5f_2e_7f_3+e_3^2f_1f_2e_4f_7$$
$$-2e_3f_1f_2^3e_7f_3+12e_3f_1^3e_7f_3f_4+12e_3f_1^3e_7f_2^2f_3+3e_3f_1^4e_7f_3^2$$
$$-3e_3f_1^3e_7f_7-4e_3f_1^2f_2^4e_7+5e_5^2f_1^2f_2^2f_4-5e_5^2f_1f_2^3f_3-15e_5^2f_1f_4f_5$$
$$+10e_5^2f_1^2f_3f_5-15e_5^2f_1^2f_2f_6+10e_5^2f_1f_3f_6-5e_5^2f_1^3f_2f_5+10e_5^2f_1f_2^2f_5$$
$$+5e_5^2f_1^2f_2f_3^2+10e_5^2f_1f_2f_7-15e_5^2f_2f_3f_5-5e_5^2f_1^3f_3f_4-e_1^3e_3f_2f_3e_4f_5$$
$$-4e_1^2e_3f_2f_3e_5f_5+e_1^2e_3^2f_2f_3e_2f_5-3e_1e_2f_3e_7f_2f_5+5e_3f_1^3e_2e_5f_7$$
$$+3e_3^2f_1^3e_4f_7+18e_1^2e_2^2f_3e_4f_7+e_1^3f_2^2e_3f_1e_4f_5-2e_1^2e_2^2f_2f_3e_4f_5$$
$$-13e_1e_2^2f_2f_3e_5f_5+9e_2^2f_2^2e_1f_1e_5f_5-e_2e_3f_1^3f_3e_5f_4+e_2e_3f_1^2f_3^2e_5f_2$$
$$-4e_2e_3f_1^2f_3e_5f_5+18e_3f_1^2f_2^2e_7f_4+15e_3f_1^2f_2e_2e_5f_6-15e_3f_1f_2e_1e_6f_3f_4$$
$$+15e_3f_1^2f_2e_1e_6f_6+5e_2f_1^3e_3f_2e_5f_5+3e_3^2f_1f_2^2e_4f_5-3e_3^2f_1^2f_2e_4f_6$$
$$+10e_3^2e_4f_2f_8-10e_3^2e_4f_1^2f_8+12e_3f_1^3e_7f_2f_5-3e_3^2f_1f_2e_4f_3f_4$$

$$
\begin{aligned}
&-5e_3f_1^4e_2e_5f_6 - e_3^3f_3e_1f_2f_5 + 11e_3f_3e_7f_7 + e_3^2f_3^3e_4f_1 - 3e_3f_1^5e_7f_5\\
&+3e_3f_1^4e_7f_6 - e_3^2f_3^2e_4f_4 - 2e_1^2f_2^2e_8f_6 + 2e_1^4f_2e_6f_8 + e_1^6f_2e_2^2f_8\\
&-2e_1^3f_2e_7f_8 + 2e_1^2f_2^3e_8f_4 + 7e_1^2f_2^2e_8f_3^2 + e_2^2f_1^3e_1f_3e_5f_4 - e_3f_3e_2e_5f_7\\
&-9e_3f_3e_1^2e_5f_7 + 3e_3^2f_3e_2^2f_1f_6 + e_3^3f_3^2e_1f_4 + 17e_2e_3f_2f_3e_5f_5\\
&+5e_2e_3f_2f_3e_5f_1f_4 - 2e_2^2e_3^2f_2f_3f_5 + 5e_2e_3f_2f_3e_1e_4f_5 - 2e_2e_3f_2^2f_3^2e_5\\
&-13e_2e_3f_1f_2^2e_5f_5 - 2e_2e_3f_1^2f_2^2e_5f_4 + e_1^6f_1^3e_4f_7 + 4e_1^2f_2e_2^2e_4f_8\\
&-6e_1^2f_2e_2e_6f_8 - 12e_1^2f_2e_8f_3f_5 - 13e_1^4f_2e_3^2f_8 + 2e_1^6f_2e_4f_8 - 2e_1^5f_2e_5f_8\\
&-2e_1^7f_2e_3f_8 + e_1^2f_2^3e_4^2f_4 - e_1^2f_2^2e_4^2f_6 + 6e_1^3f_2e_7f_4^2 - 6e_1^4f_2e_2^3f_8\\
&-10e_1^8e_2f_{10} - 10e_1^6e_4f_{10} + 10e_1^7e_3f_{10} + 10e_1^5e_5f_{10} - 10e_3^2e_4f_{10} + 10e_2^3e_4f_{10}\\
&+10e_{10}f_4f_6 - 10e_{10}f_3^2f_4 + 10e_{10}f_1^7f_3 + 10e_{10}f_1f_9 + 10e_{10}f_3f_7 + 35e_{10}f_1^6f_2^2\\
&-50e_{10}f_1^4f_2^3 + 10e_{10}f_2^3f_4 + 10e_{10}f_1^5f_5 + 10e_{10}f_1^3f_7 - 10e_{10}f_1^8f_2 - 10e_{10}f_1^4f_6\\
&-10e_{10}f_2f_4^2 + 15e_{10}f_1^2f_4^2 + 25e_{10}f_1^2f_2^4 + 25e_1^4e_3^2f_{10} - 50e_1^4e_2^3f_{10} + 15e_2^2e_3^2f_{10}\\
&+10e_1^3e_7f_{10} - 10e_1^2e_8f_{10} + 10e_3e_7f_{10} + 10e_1e_3^3f_{10} + 5e_3^2e_4f_5^2 - 5e_2^3e_4f_5^2\\
&-5e_5^2f_4f_6 + 5e_5^2f_3^2f_4 - 5e_5^2f_1f_9 - 5e_5^2f_3f_7 - 5e_5^2f_2^3f_4 - 5e_5^2f_1^3f_7\\
&+5e_5^2f_2^2f_3^2 + 5e_5^2f_1^4f_6 + 5e_5^2f_2f_4^2 + 5e_5^2f_1^2f_4^2 + 2e_2^5f_1f_9 + 2e_2^5f_3f_7\\
&+60e_{10}f_1f_2f_3f_4 - 10e_2e_8f_1f_3^3 + e_1^4f_2^2e_3^2f_6 + 2e_1^5f_2^2e_5f_6 - 2e_1^4f_2^2e_6f_6\\
&-2e_1^3f_2^3e_7f_4 + 20e_1e_4e_5f_2f_8 + 10e_1e_4e_5f_2f_3f_5 - 10e_4e_6f_2f_8 - 8e_2^2f_1f_3e_6f_6\\
&+10e_1e_9f_2f_4^2 - 15e_1e_9f_2^2f_3^2 - 10e_1e_9f_2^3f_4 - 10e_1e_9f_3f_7 + 10e_1e_9f_3^2f_4\\
&-10e_1e_9f_4f_6 + 30e_{10}f_1^2f_2f_6 - 20e_{10}f_1f_3f_6 - 40e_{10}f_1^3f_2f_5 + 30e_{10}f_1f_2^2f_5\\
&-60e_{10}f_1^2f_2f_3^2 - 20e_{10}f_1f_2f_7 + 100e_{10}f_1^3f_2^2f_3 - 20e_{10}f_2f_3f_5 - 60e_{10}f_1^5f_2f_3\\
&-40e_{10}f_1^3f_3f_4 + 50e_{10}f_1^4f_2f_4 - 40e_{10}f_1f_2^3f_3 - 20e_{10}f_1f_4f_5 - 60e_{10}f_1^2f_2^2f_4\\
&-15e_3e_7f_1^2f_4^2 + 10e_3e_7f_2f_4^2 - 10e_3e_7f_2^3f_4 - 10e_3e_7f_1f_9 - 10e_3e_7f_4f_6 - 2e_2^5f_{10}\\
&-4e_1^2f_2^2e_2e_3^2f_6 - 9e_3f_1f_2^2e_7f_5 - 9e_3f_1^2f_2e_7f_6 - 3e_3^2f_3e_4f_1^2f_5\\
&+e_3^2f_3e_4f_2f_5 + 13e_3^2f_3e_4f_1f_6 - 3e_3f_3^2e_1^2e_5f_4 + 15e_3f_3e_1^2e_5f_1f_6\\
&+10e_2e_3e_5f_1f_4f_5 + 10e_1^2e_8f_3f_7 - 10e_1^2e_8f_3^2f_4 + 10e_1^2e_8f_4f_6 - 10e_1^3e_7f_4f_6\\
&-9e_3f_3e_7f_1^2f_5 - e_3f_3e_7f_2f_5 - e_3f_3e_7f_1f_6 - 19e_3f_3e_1e_6f_2f_5\\
&-e_3f_3^3e_2e_5f_1 + e_3f_3^2e_2e_5f_4 - 19e_3f_3e_2e_5f_1f_6 + 3e_3f_3^2e_1e_6f_2^2\\
&+13e_3f_3^2e_1e_6f_4 + 15e_3f_3e_1e_6f_1^2f_5 + 20e_2e_8f_1f_4f_5 + 20e_2e_8f_1f_3f_6\\
&-5e_1e_2e_3e_4f_5^2 + e_2^5f_5^2 + 20e_1e_3e_6f_2f_8 + 10e_4e_6f_1^2f_8 + 5e_1^2e_2e_3^2f_5^2\\
&-15e_1e_4e_5f_5^2 - 15e_2^2e_3^2f_1f_9 - 5e_5^2f_1f_2f_3f_4 + e_5^2f_2^5 + 5e_5^2f_{10} + e_{10}f_1^{10}\\
&-2e_{10}f_2^5 + 5e_{10}f_5^2 - 10e_{10}f_{10} + 10e_5^2f_5^2 - 10e_2^3e_4f_3f_7 - 10e_2^3e_4f_1f_9\\
&+10e_3^2e_4f_1f_9 + e_3f_1^4e_7f_2^3 + 3e_3f_1^6e_7f_4 - 15e_2e_3e_5f_5^2 + 20e_1e_4e_5f_3f_7\\
&+20e_4e_6f_1f_2f_7 + 20e_4e_6f_2f_3f_5 + e_1^{10}f_{10} + 20e_1e_2e_7f_4f_6 + 5e_1^2e_4^2f_5^2\\
&-5e_1^3e_3e_4f_5^2 - 5e_1^3e_2e_5f_5^2 + 10e_1e_2e_7f_5^2 + 5e_1^2e_2^2e_4f_5^2 - 15e_1^2e_2e_6f_5^2\\
&-5e_1e_2^3e_3f_5^2 + 10e_1^2e_3e_5f_5^2 + 10e_1e_3e_6f_5^2 + 10e_1e_2^2e_5f_5^2 + 20e_2e_3e_5f_1f_9\\
&+20e_2e_3e_5f_4f_6 + 60e_1e_2e_3e_4f_{10}
\end{aligned}
$$

Table of $P_{m,2}$ for $m > 3$

m	$P_{m,2}$
4	$e_1e_3e_4 - 3e_1e_2e_5 + e_1^3e_5 - e_4^2 + e_3e_5 - e_1^2e_6 + e_1e_7 + 2e_2e_6 - e_8$
5	$e_1^4e_6 + e_2e_4^2 + 3e_1e_2e_7 + 3e_1e_3e_6 - 4e_1^2e_2e_6 - 2e_1e_4e_5 - 2e_2e_3e_5$
	$+e_1^2e_3e_5 + e_{10} - e_3e_7 + 2e_5^2 - e_1^3e_7 - 2e_4e_6 + 2e_2^2e_6 + e_1^2e_8 - e_1e_9 - 2e_2e_8$
6	$e_1e_{11} + 3e_3^2e_6 - 3e_1e_2e_3e_6 + e_5e_7 - e_1^4e_8 + e_1e_2e_4e_5 - e_{12} + e_4^3 - e_1^2e_{10}$
	$+2e_2e_{10} + 2e_4e_8 + e_1^3e_9 + e_3e_9 - 2e_6^2 + e_1^5e_7 - 3e_3e_4e_5 - e_1^2e_5^2 + e_2e_5^2$
	$+e_1^3e_3e_6 - e_1^2e_4e_6 + 3e_1e_5e_6 - 5e_1^3e_2e_7 + 5e_1e_2^2e_7 - 2e_2^2e_8 + 4e_1^2e_3e_7$
	$-3e_2e_3e_7 - 3e_1e_4e_7 + 4e_1^2e_2e_8 - 3e_1e_3e_8 - 3e_1e_2e_9$
7	$e_1e_4^2e_5 - e_4e_5^2 + 3e_7^2 + e_2^2e_5^2 - 2e_2^2e_4e_6 + e_1^2e_6^2 + 3e_3e_5e_6 - e_1^3e_5e_6$
	$+3e_1^2e_5e_7 - 4e_3e_4e_7 - e_1^3e_4e_7 + 4e_1e_3^2e_7 + 2e_2^2e_3e_7 + e_1^4e_3e_7 + 5e_1^3e_3e_8$
	$+9e_1^2e_2^2e_8 - 6e_1^4e_2e_8 - 4e_1e_6e_7 - 4e_2e_5e_7 + e_{14} + 3e_2e_3e_9 - 4e_1^2e_3e_9$
	$-5e_1e_2^2e_9 + 5e_1^3e_2e_9 + 3e_1e_5e_8 + 6e_2e_4e_8 - 5e_1^2e_4e_8 + 3e_1e_2e_{11} + 3e_1e_3e_{10}$
	$-4e_1^2e_2e_{10} + 3e_1e_4e_9 - e_3e_{11} + e_1^2e_{12} - 2e_2e_{12} - e_1e_{13} - 2e_2^3e_8 + e_1^6e_8$
	$-2e_1e_3e_5^2 + e_1^2e_2e_4e_6 - e_1e_3e_4e_6 + e_1e_2e_5e_6 - 4e_1^2e_2e_3e_7 + 2e_1e_2e_4e_7$
	$-9e_1e_2e_3e_8 + e_4^2e_6 - e_5e_9 + e_1^4e_{10} + 2e_2^2e_{10} - 2e_4e_{10} - e_1^3e_{11} - 2e_6e_8 - e_1^5e_9$
8	$e_1e_2^2e_5e_6 - 2e_1^2e_3e_5e_6 - e_2e_3e_5e_6 + 3e_4^2e_8 - e_1^4e_5e_7 - 3e_4e_5e_7$
	$+2e_1^3e_6e_7 + e_1^3e_2e_4e_7 - 2e_2^2e_{12} + e_1e_3e_5e_7 - 3e_1e_2e_6e_7 - 6e_2e_4e_{10}$
	$-3e_1e_5e_{10} - 5e_1^3e_2e_{11} + 5e_1e_2^2e_{11} - 3e_2e_3e_{11} - 3e_1e_4e_{11} + 4e_1^2e_2e_{12}$
	$-3e_1e_3e_{12} - 3e_1e_2e_{13} - 17e_1^2e_2e_3e_9 + 12e_1e_2e_4e_9 - 3e_1e_2^2e_4e_7 - e_1^2e_3e_4e_7$
	$+5e_2e_3e_4e_7 + 3e_1^2e_2e_5e_7 + 2e_2e_{14} + e_1e_{15} + e_1e_4e_5e_6 - e_1^2e_2e_6^2$
	$+3e_1e_3e_6^2 - 2e_1^2e_7^2 + e_1^5e_{11} - 3e_8^2 - e_1e_5^3 + e_5^2e_6 + e_1^7e_9 - e_{16} + e_2e_4e_5^2$
	$+2e_4e_{12} - 3e_3e_6e_7 + 2e_2e_7^2 + e_3e_{13} - e_1^2e_{14} + e_1^5e_3e_8 + 5e_1^2e_3^2e_8 - 5e_2e_3^2e_8$
	$-5e_1^3e_2e_3e_8 + 5e_1e_2^2e_3e_8 + e_1^2e_4^2e_6 - 2e_2e_4^2e_6 - 7e_1^5e_2e_9 + 14e_1^3e_2^2e_9$
	$-7e_1e_2^3e_9 + 6e_1^4e_3e_9 + 5e_2^2e_3e_9 + 3e_1e_3^2e_9 + e_2^2e_6^2 + 9e_1e_2e_3e_{10} + 4e_1^2e_3e_{11}$
	$+e_5e_{11} + e_7e_9 - e_1^6e_{10} - 2e_2^2e_5e_7 - e_1^4e_{12} - 6e_1^3e_4e_9 - 3e_3e_4e_9 + 4e_1^2e_5e_9$
	$-3e_2e_5e_9 - 3e_1e_6e_9 + 6e_1^4e_2e_{10} - 9e_1^2e_2^2e_{10} - 5e_1^3e_3e_{10} + 2e_6e_{10} - e_1^4e_4e_8$
	$+2e_1^3e_5e_8 + 5e_3e_5e_8 - 3e_1^2e_6e_8 + 2e_2e_6e_8 + 5e_1e_7e_8 + 3e_1^2e_2e_4e_8 - 8e_1e_3e_4e_8$
	$-4e_1e_2e_5e_8 + 5e_1^2e_4e_{10} + 2e_2^3e_{10} + e_1^3e_{13}$
9	$-2e_6e_{12} - 2e_2^3e_5e_7 - 3e_3^2e_5e_7 - 2e_1e_5^2e_7 + e_1e_2e_4e_5e_6 + 3e_1e_5e_6^2 - 8e_1^6e_2e_{10}$
	$+20e_1^4e_2^2e_{10} - 16e_1^2e_2^3e_{10} + 7e_1^5e_3e_{10} + 7e_1^2e_3^2e_{10} - 3e_2e_3^2e_{10} + 3e_2e_3e_{13}$
	$-6e_2e_7e_9 + e_1^6e_{12} + 3e_5e_6e_7 + e_1^4e_7^2 + 2e_2^2e_7^2 - 2e_4e_7^2 - 3e_3e_4e_5e_6 - e_5^2e_8$
	$+7e_1e_2e_3e_4e_8 - e_1e_3e_7^2 + 17e_1^2e_2e_3e_{11} - 12e_1e_2e_4e_{11} - 9e_1e_2e_3e_{12} + 2e_2^2e_{14}$
	$-e_1e_{17} + 3e_3^2e_6^2 + 6e_2e_4e_{12} + 4e_9^2 + 6e_1^3e_5e_{10} + 6e_2e_6e_{10} - 27e_1^3e_2e_3e_{10}$
	$+19e_1e_2^2e_3e_{10} + 21e_1^2e_2e_4e_{10} - 8e_1e_3e_4e_{10} - e_3e_{15} + e_1^2e_{16} - 2e_2e_{16} + e_1^6e_3e_9$
	$-2e_2^3e_3e_9 - 12e_1e_2e_3^2e_9 + 4e_1^3e_2e_4e_9 - 2e_1e_2^2e_4e_9 - 10e_1^2e_3e_4e_9$
	$+6e_2e_3e_4e_9 - 7e_1^2e_2e_5e_9 + 9e_1e_3e_5e_9 + 6e_1e_2e_6e_9 + 3e_3^3e_9 + e_1^3e_4^2e_7$
	$+3e_3e_4^2e_7 + e_{18} - 2e_8e_{10} - 6e_1^4e_2e_{12} + 9e_1^2e_2^2e_{12} + 5e_1^3e_3e_{12} + e_1^4e_6e_8$
	$+6e_1^3e_3^2e_9 - e_1^5e_4e_9 + 5e_1e_4^2e_9 + 2e_1^4e_5e_9 + 4e_2^2e_5e_9 - 3e_1^3e_6e_9 - 6e_3e_6e_9$
	$+5e_1^2e_7e_9 - 6e_1e_8e_9 + 2e_2^4e_{10} - 2e_1^2e_3e_5e_8 - e_5e_{13} + e_1^4e_{14} + 3e_1e_7e_{10}$
	$+7e_1^5e_2e_{11} - 14e_1^3e_2^2e_{11} + 7e_1e_2^3e_{11} - 6e_1^4e_3e_{11} - 5e_2^2e_3e_{11} - 3e_1e_3^2e_{11}$

$+6e_1^3e_4e_{11} + 3e_3e_4e_{11} - 4e_1^2e_5e_{11} + 3e_2e_5e_{11} + 3e_1e_6e_{11} + 2e_2^3e_4e_8 - 3e_3^2e_4e_8$
$-4e_2e_4^2e_8 - e_1^5e_5e_8 - 8e_2^2e_4e_{10} + 4e_1e_2e_3e_5e_7 - e_1^2e_5^2e_6 - e_2e_5^2e_6$
$+e_2^3e_6^2 - 2e_6^3 + e_1^2e_4e_6^2 - 6e_4e_5e_9 + 3e_1e_3e_{14} - 7e_1^4e_4e_{10} + e_2e_3e_5e_8$
$+2e_1e_4e_5e_8 - e_1^2e_2e_6e_8 - 3e_1e_3e_6e_8 + 5e_1e_2e_7e_8 - 3e_1e_2e_3e_6^2 + e_4^2e_{10}$
$+e_1^4e_2e_4e_8 - 4e_1^2e_2^2e_4e_8 - e_1^3e_3e_4e_8 + 4e_1^3e_2e_5e_8 - 3e_1e_2^2e_5e_8$
$-e_1^7e_{11} + e_3e_5^3 - 2e_2^2e_6e_8 + 6e_4e_6e_8 - 3e_1^3e_7e_8 + 3e_3e_7e_8 + 2e_1^2e_8^2 - e_2e_8^2$
$-2e_4e_{14} - e_1^3e_{15} - 2e_2^3e_{12} - 12e_1e_2e_5e_{10} - 6e_1^4e_2e_3e_9 + 9e_1^2e_2^2e_3e_9 + e_1^8e_{10}$
$-e_7e_{11} - 5e_1^2e_6e_{10} + 3e_3e_5e_{10} + 4e_2e_4e_5e_7 - e_1^3e_2e_6e_7 + e_1e_2^2e_6e_7$
$+5e_1^2e_3e_6e_7 - 3e_2e_3e_6e_7 - 5e_1e_4e_6e_7 - 3e_1^2e_2e_7^2 - 3e_1e_2e_4^2e_7$
$+e_1^2e_2^2e_5e_7 - 2e_1^3e_3e_5e_7 - e_1^5e_{13} - 5e_1^2e_4e_{12} + 3e_1e_5e_{12} + 5e_1^3e_2e_{13}$
$-5e_1e_2^2e_{13} - 4e_1^2e_3e_{13} + 3e_1e_4e_{13} - 4e_1^2e_2e_{14} + 3e_1e_2e_{15}$
10 $-e_1^4e_8^2 + e_1^5e_7e_8 + e_4e_8^2 - 3e_5e_7e_8 + 5e_2^2e_5e_{11} + 4e_1^3e_8e_9 + 2e_2^3e_5e_9$
$-3e_1^3e_2e_7e_8 - e_1^6e_5e_9 - 7e_2^2e_3e_4e_9 - 4e_1e_3^2e_4e_9 + 3e_3^2e_5e_9 + 2e_1e_2^2e_7e_8$
$+e_1^5e_6e_9 - 3e_4e_7e_9 + e_1^5e_2e_4e_9 - 5e_1^3e_2^2e_4e_9 + 2e_2e_3e_7e_8 - 3e_1^4e_7e_9$
$-3e_1^2e_3e_7e_8 + 4e_1e_4e_7e_8 + 3e_2e_9^2 - 3e_1^2e_9^2 - 3e_3e_8e_9 - 4e_4e_5^2e_6 - 5e_1^3e_2e_{15}$
$+5e_1e_2^2e_{15} + 4e_1^2e_3e_{15} - 3e_2e_3e_{15} - 3e_1e_4e_{15} + 4e_1^2e_2e_{16} - 3e_1e_2e_{17} - 3e_1e_5e_{14}$
$+2e_1^2e_2e_8^2 + e_1e_3e_5^2e_6 + e_1^7e_3e_{10} + e_1e_2e_3e_5e_9 + 2e_1e_3e_8^2$
$+19e_1e_3e_2e_4e_{10} + 4e_3e_4^2e_9 - 2e_2^2e_{16} - 3e_1e_3e_{16} + 7e_1^4e_3^2e_{10} + 16e_1^2e_2^3e_{12}$
$-7e_1^5e_3e_{12} - 7e_1^2e_3^2e_{12} + 7e_1e_3^3e_{10} + 4e_3e_4e_5e_8 - e_1^4e_2e_6e_8 + 3e_1^2e_2^2e_6e_8$
$+7e_2^2e_3^2e_{10} - e_1^4e_3e_4e_9 + 4e_1^2e_4e_5e_9 + 3e_2e_4e_5e_9 - 2e_1e_2e_5e_6^2 - e_1^6e_4e_{10}$
$+8e_1e_2^2e_5e_{10} + 14e_1^2e_3e_5e_{10} - 13e_2e_3e_5e_{10} - 13e_1e_4e_5e_{10} - 9e_1^7e_2e_{11} + 2e_4e_{16}$
$+e_5^4 - 2e_2e_4^2e_{10} + 6e_1^2e_4^2e_{10} - 7e_4e_3^2e_{10} - e_1^2e_5e_6e_7 + 5e_2e_5e_6e_7 + 3e_2e_3^2e_{12}$
$+7e_1^4e_4e_{12} - 6e_1^3e_5e_{12} - 3e_3e_5e_{12} + 5e_1^2e_6e_{12} - 3e_1e_7e_{12} + 2e_1^5e_5e_{10} + 8e_2^2e_4e_{12}$
$-4e_{10}^2 - 2e_1^4e_6e_{10} + 6e_5^2e_{10} - 3e_2e_7e_{11} - 3e_1e_8e_{11} + 8e_1^6e_2e_{12} - 20e_1^4e_2^2e_{12}$
$+4e_1^2e_7e_{11} - 5e_1e_2e_4^2e_9 + 4e_4e_6e_{10} + 2e_8e_{12} + 4e_1^3e_7e_{10} - 5e_1^2e_8e_{10} - 4e_4^3e_8$
$-e_1^6e_{14} - 7e_1e_2^3e_3e_{10} - 21e_1^2e_2e_3^2e_{10} + e_1^9e_{11} + 4e_2e_8e_{10} - 6e_1^3e_4e_{13} - e_1^8e_{12}$
$-e_1^2e_3e_6e_9 + 3e_2e_3e_6e_9 + 2e_1e_4e_6e_9 + 7e_1e_9e_{10} - 6e_1^2e_2^2e_5e_9 + e_5e_{15} + 2e_6e_{14}$
$+2e_6e_7^2 - 3e_1^2e_4e_7^2 - 39e_1^4e_2e_3e_{11} + 45e_1^2e_2^2e_3e_{11} + 27e_1^5e_2^2e_{11} + 12e_1e_2e_4e_{13}$
$+e_1^3e_{17} - e_4^2e_{12} + 7e_3e_7e_{10} - 6e_2e_6e_{12} - 17e_1^2e_2e_3e_{13} + 3e_3e_5e_6^2 - 21e_1^2e_2e_4e_{12}$
$+8e_1e_3e_4e_{12} + 12e_1e_2e_5e_{12} + 3e_1^3e_3e_6e_8 - 3e_1^2e_4e_6e_8 - 7e_1^5e_2e_3e_{10}$
$+14e_1^3e_2^2e_3e_{10} + e_1^2e_2e_4e_5e_7 - 30e_1^3e_2^3e_{11} - 25e_1e_2^2e_4e_{11} - 17e_1^2e_3e_4e_{11}$
$+12e_2e_3e_4e_{11} - 20e_1^2e_2e_5e_{11} + 9e_1e_3e_5e_{11} - 3e_1^3e_3e_5e_9 - 3e_1e_5^2e_9$
$+4e_4^2e_5e_7 - 2e_2^4e_{12} + 6e_1^2e_2e_3e_5e_8 + 6e_1e_5e_6e_8 - 4e_1e_6^2e_7 + e_2^3e_7^2$
$+9e_1e_2^4e_{11} - 7e_1^5e_2e_{13} + 14e_1^3e_2^2e_{13} - 7e_1e_2^3e_{13} + 6e_1^4e_3e_{13} + 5e_2^2e_3e_{13}$
$+3e_1e_3^2e_{13} - e_1^2e_2^2e_7^2 + 2e_1^3e_3e_7^2 - e_2e_4e_7^2 + e_1e_5e_7^2 + e_3e_{17}$
$-3e_1^2e_2e_3e_6e_7 + 5e_1^4e_2e_4e_{10} - 5e_1^2e_2^2e_4e_{10} - 7e_1e_2e_8e_9 + 8e_1^6e_3e_{11}$
$+12e_1^3e_3^2e_{11} - 7e_2^3e_3e_{11} - 8e_1^5e_4e_{11} + 5e_1^4e_2e_5e_9 - e_1e_3e_4e_5e_7 - 4e_2^2e_7e_9$
$+e_1^5e_{15} + e_2^2e_4e_6^2 - 3e_1^3e_2e_6e_9 + e_1e_2^2e_6e_9 - e_1^2e_{18} + 2e_2e_{18} - 4e_1^2e_2e_4^2e_8$
$-5e_1e_2e_3e_6e_8 - e_1^3e_5^2e_7 - 3e_3e_5^2e_7 - 3e_3e_4e_{13} + 4e_1^2e_5e_{13} - 3e_2e_5e_{13}$
$-3e_1e_6e_{13} + 6e_1^4e_2e_{14} - 9e_1^2e_2^2e_{14} - 5e_1^3e_3e_{14} + 5e_1^2e_4e_{14} - 6e_2e_4e_{14} + e_1^7e_{13}$
$+9e_1^2e_2e_7e_9 - e_1e_3e_7e_9 - 3e_5e_6e_9 + e_7e_{13} + 12e_1e_2e_6e_{11} + e_1e_2e_4e_5e_8$
$+4e_1e_3e_4^2e_8 + e_1^3e_2^2e_5e_8 - 3e_1e_2^3e_5e_8 - 2e_1^4e_3e_5e_8 + 3e_2^2e_3e_5e_8$

$$
\begin{aligned}
&-7e_1e_3^2e_5e_8 + e_1^4e_4^2e_8 + 2e_2^2e_4^2e_8 + 4e_1e_4^2e_{11} + e_9e_{11} + 2e_1e_2e_4e_6e_7 \\
&+2e_4^2e_6^2 + 2e_1^2e_6^3 - 2e_2e_6^3 + 9e_1e_2e_3e_{14} - 2e_1e_3e_4e_6^2 + 9e_1^2e_2e_3e_4e_9 \\
&+e_1e_{19} - 15e_1e_2e_3^2e_{11} + 32e_1^3e_2e_4e_{11} + 7e_1^4e_5e_{11} - 12e_1^3e_3e_4e_{10} - 9e_1^3e_2e_5e_{10} \\
&-e_{20} - 2e_2^2e_4e_5e_7 + 2e_1e_2e_5^2e_7 + 2e_2^3e_{14} - 4e_2e_5^2e_8 + 5e_1e_2^3e_4e_9 - e_1e_2e_3e_7^2 \\
&-6e_1^3e_6e_{11} - 3e_4e_5e_{11} + 5e_1^2e_2e_6e_{10} - 8e_1e_3e_6e_{10} - 8e_1e_2e_7e_{10} - e_1^4e_{16} \\
&+27e_1^3e_2e_3e_{12} - 19e_1e_2^2e_3e_{12} + e_1e_2^3e_6e_7 - e_2^2e_3e_6e_7 + 5e_1e_3^2e_6e_7 \\
&+e_1^3e_4e_6e_7 - 5e_3e_4e_6e_7 - 2e_2^3e_6e_8 - 3e_3e_6e_{11}
\end{aligned}
$$

Table of $P_{m,3}$ for $m > 3$

m	$P_{m,3}$
4	$-e_1e_{11} + 3e_3^2e_6 - 3e_1e_2e_3e_6 - e_5e_7 + e_2^3e_6 + e_1e_2e_4e_5 + e_{12} + e_4^3$
	$+e_1^2e_{10} - e_2e_{10} - e_2e_4e_6 + 3e_4e_8 - e_1^3e_9 - 3e_3e_4e_5 - e_1^2e_5^2 + e_2e_5^2$
	$+e_1^2e_4e_6 + e_1e_5e_6 - e_1e_2^2e_7 - e_2^2e_8 + 2e_1^2e_3e_7 + e_2e_3e_7 - 2e_1e_4e_7$
	$+e_1^2e_2e_8 - 3e_1e_3e_8 + 2e_1e_2e_9$
5	$2e_1e_2^2e_{10} - 4e_2e_3e_{10} - 3e_1e_4e_{10} - 3e_1^2e_2e_{11} + e_1e_3e_{11} + 2e_1e_2e_{12}$
	$-3e_1e_3e_4e_7 + 5e_1e_2e_5e_7 + e_2e_4^2e_5 + e_1^4e_{11} + e_1e_6e_8 + 2e_1^3e_6^2$
	$-e_2^3e_9 + 2e_1e_3e_5e_6 - 4e_1e_2e_6^2 - 2e_1e_2e_4e_8 + e_2^2e_3e_8 + 3e_1e_4^2e_6$
	$-2e_1e_4e_5^2 + 3e_2^2e_5e_6 + e_1^2e_2^2e_9 - 2e_1^3e_3e_9 + 2e_1^2e_4e_9 + 3e_2e_4e_9$
	$-4e_1e_5e_9 - e_1^3e_2e_{10} + 3e_3e_6^2 - 4e_4e_5e_6 + e_1^2e_{13} - e_2e_{13} + e_{15} + e_1^2e_3e_5^2$
	$-2e_2e_3e_5^2 + e_1e_2e_3e_9 + e_2^4e_7 - 4e_1e_2^2e_3e_7 + 3e_1^2e_2e_4e_7 + e_4^2e_7$
	$-e_1^3e_{12} - e_1^3e_4e_8 - e_7e_8 + e_2^2e_{11} + e_1e_2^2e_4e_6 - 2e_1^2e_3e_4e_6 - e_2e_3e_4e_6$
	$+4e_5e_{10} + 2e_1^2e_3^2e_7 + 4e_2e_3^2e_7 - e_4e_{11} + 3e_1^2e_3e_{10} + e_1e_7^2 + e_2e_6e_7$
	$-2e_1^2e_6e_7 - 4e_3e_5e_7 - 4e_2^2e_4e_7 - e_1e_{14} - e_1e_2^3e_8 - 2e_1^2e_2e_5e_6$
	$-2e_1^3e_5e_7 - 5e_1e_3^2e_8 + 5e_3e_4e_8 + 3e_1^2e_5e_8 - 3e_2e_5e_8 + 3e_1^2e_2e_3e_8 + 2e_5^3$
6	$e_3e_5^3 + e_1^3e_5^3 - e_4^2e_5^2 - 2e_3e_4e_5e_6 - 2e_2^2e_3e_5e_6 - e_1e_3^2e_5e_6$
	$-3e_1^3e_4e_5e_6 + 4e_1^2e_6e_{10} + e_2^2e_4^2e_6 + 4e_1^2e_4e_6^2 - 8e_2e_4e_6^2 - 4e_1e_5e_6^2$
	$-2e_1e_2e_3e_6^2 + e_2^3e_3e_9 - 2e_1^3e_3^2e_9 - 2e_1e_3e_7^2 + e_4^2e_{10} + 4e_2e_5^2e_6$
	$+3e_1e_3e_6e_8 - 5e_1e_2e_7e_8 + e_2^5e_8 - e_1^2e_2^2e_5e_7 - 5e_1^2e_4e_5e_7 - 5e_5e_6e_7$
	$-2e_1^4e_7^2 - e_2^2e_7^2 + e_1e_2e_5e_{10} - 3e_1^2e_2e_3e_{11} - e_1e_2e_4e_{11} + 2e_2^3e_6^2$
	$+4e_1e_2e_3e_5e_7 + 3e_3^2e_6^2 + 6e_6e_{12} - e_1^2e_2^2e_6^2 + e_1^3e_3e_6^2 + e_1e_2^3e_4e_7$
	$-e_2^2e_3e_4e_7 + 8e_1e_2e_3e_{12} + e_3e_4e_{11} - e_1^5e_{13} + 5e_2^2e_3^2e_8 + e_1^4e_6e_8$
	$+2e_2^2e_6e_8 - e_4e_6e_8 + e_3e_7e_8 - e_1^2e_8^2 + e_2e_8^2 + 5e_1^2e_3^2e_{10} - 3e_2e_3^2e_{10}$
	$+e_1^4e_4e_{10} + 2e_2^2e_4e_{10} - 3e_1^3e_5e_{10} + 6e_3e_5e_{10} + e_1^2e_{16} - e_2e_{16} - e_1e_{17}$
	$+5e_1e_3^2e_4e_7 - 3e_1e_2e_4^2e_7 + 4e_1^2e_2^2e_4e_8 - 3e_1^3e_3e_4e_8 - 3e_1^3e_2e_5e_8$
	$-3e_1e_2e_5^3 + 2e_1e_3^2e_{11} + 4e_1^2e_5e_{11} + 2e_2e_5e_{11} - 5e_1e_6e_{11} + e_1^4e_2e_{12}$
	$-3e_1^2e_2^2e_{12} - 3e_1^3e_3e_{12} + 3e_1^2e_4e_{12} - 3e_2e_4e_{12} - 5e_1e_5e_{12} - 2e_1e_3e_4^2e_6$
	$-e_1^3e_{15} + e_2^3e_{12} - 3e_1^2e_2e_3e_4e_7 + e_1e_3e_4e_5^2 - e_2^4e_{10} + e_{18} + e_4e_5e_9$
	$-e_7e_{11} + 6e_1e_2e_4e_5e_6 - 2e_3e_4^2e_7 + e_2^2e_{14} - 3e_1^2e_2e_{14} - e_4e_{14} - e_8e_{10}$
	$+3e_1^3e_7e_8 + e_1e_8e_9 + e_1^2e_2^3e_{10} + 3e_3^3e_9 - 5e_1e_3^3e_8 - 5e_2^3e_4e_8$
	$-5e_2e_6e_{10} + 2e_1e_7e_{10} - e_1^3e_2^2e_{11} + 2e_1e_2^3e_{11} + 2e_1^4e_3e_{11} - 2e_2^2e_3e_{11}$
	$+e_5^2e_8 + 7e_2e_3e_6e_7 + 4e_1e_4e_6e_7 + 5e_1^2e_2e_7^2 + e_2^3e_5e_7 - 3e_3^2e_5e_7$
	$+4e_1e_5^2e_7 - 2e_1e_4^2e_9 + 2e_1^4e_5e_9 - 2e_2^2e_5e_9 - 3e_1^3e_6e_9 - 3e_3e_6e_9$
	$-2e_1^2e_7e_9 + e_2e_7e_9 - 3e_1^3e_2e_4e_9 + e_2e_3e_{13} - 3e_3^2e_{12} - 5e_1e_2^3e_3e_8$
	$+5e_1^2e_2e_3^2e_8 + 3e_1^3e_4^2e_7 - e_5e_{13} + e_1^4e_{14} + 7e_1e_2e_6e_9 - 4e_1e_3e_5e_9$
	$+4e_1^3e_2e_{13} - 3e_1e_2^2e_{13} - 2e_1^2e_3e_{13} + 2e_1e_4e_{13} + e_1e_3e_{14} + 2e_1e_2e_{15}$
	$+e_4e_7^2 + 2e_4^3e_6 + 2e_1e_2^2e_4e_9 + 8e_1^2e_3e_4e_9 + e_2e_3e_4e_9 - 2e_1^2e_2e_5e_9$
	$-3e_1e_2e_3e_4e_8 - 2e_1^3e_4e_{11} - e_1e_2^4e_9 + 2e_3^2e_4e_8 - e_1^2e_4^2e_8 + 6e_2e_4^2e_8$
	$+3e_6^3 + 4e_1^2e_2^2e_3e_9 - 7e_1e_2e_3^2e_9 + e_1^2e_2e_3e_5e_6 - 3e_1^3e_2e_3e_{10}$
	$+2e_1e_2^2e_3e_{10} + e_1^2e_2e_4e_{10} - 9e_1e_3e_4e_{10} + 3e_1^3e_2e_6e_7 - 6e_1e_2^2e_6e_7$

$-5e_1^2e_3e_6e_7 + 6e_1e_2^2e_5e_8 + 6e_1^2e_3e_5e_8 - 9e_2e_3e_5e_8 - 2e_1e_4e_5e_8$
$-4e_1^2e_2e_6e_8$

7 $-e_1^3e_2^3e_{12} - 5e_1^3e_3^2e_{12} - 2e_2^3e_3e_{12} + 2e_1e_2^4e_{12} - 2e_1^5e_3e_{13} - 3e_1^2e_2^3e_{13}$
$+e_1e_8e_{12} + 5e_1^2e_7e_{12} + 5e_2^2e_5e_{12} + 3e_1^4e_5e_{12} - 2e_1e_4^2e_{12} - e_1^5e_4e_{12}$
$+e_2e_6e_{13} + 5e_1^2e_6e_{13} + e_3e_5e_{13} - 4e_1^3e_5e_{13} + 2e_1^4e_4e_{13} - e_2e_3^2e_{13}$
$-4e_1^2e_3^2e_{13} + 3e_1^4e_3e_{14} - 3e_1e_2^3e_{14} + 5e_7^3 + 4e_1^3e_2^2e_{14} - 9e_2e_5e_7^2$
$-5e_1e_7e_{13} - e_2e_{19} + 6e_1e_3^2e_{14} + 4e_1^3e_2e_4^2e_8 + 5e_2^2e_3e_{14} - 3e_3^3e_4e_8$
$-6e_6e_7e_8 - 5e_2e_5e_{14} + 5e_1^2e_5e_{14} - 9e_1e_6e_7^2 - 6e_3e_4e_{14} - 3e_1^3e_4e_{14}$
$-3e_1^2e_4e_{15} + 6e_1^2e_2^2e_{15} + 2e_1^3e_2e_3e_5e_8 - e_2^3e_{15} + 2e_1e_2^2e_3e_4e_9$
$-5e_1^4e_2e_{15} + e_1^2e_5^2e_9 + 5e_2e_5^2e_9 + 4e_2^3e_6e_9 - 2e_1^5e_7e_9 + 6e_1e_4^3e_8$
$+3e_2e_4e_{15} + 2e_1e_4e_{16} + 8e_2^2e_4^2e_9 - e_1e_2^5e_{10} - 3e_1e_2^2e_{16} + 4e_1^3e_2e_{16}$
$+7e_1e_2e_3^2e_4e_8 + 2e_1e_2e_{18} + 3e_1^4e_2e_3e_{12} - 4e_1e_2e_3e_{15} - 3e_1^2e_2e_{17}$
$+e_2^4e_5e_8 + 7e_1^2e_2e_3e_6e_8 - 5e_1^2e_2^2e_3e_{12} - 5e_1^3e_3e_6e_9 + e_1^2e_4e_6e_9$
$-6e_2e_4e_6e_9 - 2e_1^3e_4^2e_{10} + 5e_3e_4^2e_{10} + e_1^4e_{17} + e_4^2e_{13} - 6e_5e_7e_9$
$-4e_1^4e_8e_9 - 4e_2^2e_8e_9 + e_1^3e_9^2 - 6e_4^2e_5e_8 + 11e_1e_2e_3^2e_{12} + 8e_1^2e_3e_5e_{11}$
$+3e_2e_3e_5e_{11} - 4e_1e_4e_5e_{11} - 4e_1^2e_2e_6e_{11} - 4e_1e_3e_6e_{11} + 9e_1e_2e_7e_{11}$
$+3e_1^3e_5^2e_8 - 2e_3e_5^2e_8 + e_1e_2e_3e_4e_5e_6 - 4e_1^3e_2^2e_3e_{11} + 3e_1e_2^3e_3e_{11}$
$+5e_1^2e_2e_3^2e_{11} - 3e_1e_2e_3^2e_6^2 + 15e_1e_2e_4e_7^2 + 6e_2^3e_3^2e_9 - 2e_1^3e_3^3e_9$
$-4e_1^4e_2e_7e_8 + 11e_1^2e_2^2e_7e_8 + 8e_1^3e_3e_7e_8 - e_1e_3e_5e_6^2 - 4e_1e_2^2e_4^2e_8$
$-11e_1^2e_3e_4^2e_8 + e_3e_4e_5e_9 + 3e_1^4e_2e_6e_9 - 10e_1^2e_2^2e_6e_9 - 4e_2^3e_5e_{10}$
$-7e_1e_5^2e_{10} + 9e_1^2e_2^2e_3^2e_9 - 12e_1e_2e_3^3e_9 + 8e_1^2e_3e_4e_{12} - 12e_1^2e_2e_3e_{14}$
$+7e_1e_2e_4e_{14} - 4e_2e_3e_4e_{12} + e_2^2e_{17} + e_1e_3e_{17} - 10e_1e_2e_4e_5e_9$
$+4e_1^2e_2e_4e_5e_8 - e_2^2e_4e_5e_8 + e_{21} - 5e_2e_3e_4e_6^2 - 4e_1^2e_2^2e_3e_4e_8$
$-e_1^5e_2e_{14} + e_1^4e_2^2e_{13} + 7e_1^2e_4e_5e_{10} + 6e_2e_4e_5e_{10} - e_1^3e_2e_6e_{10}$
$+8e_1e_2^2e_6e_{10} + 6e_1^2e_3e_6e_{10} - 16e_2e_3e_6e_{10} - 2e_1e_4e_6e_{10} - 11e_1^2e_2e_7e_{10}$
$-2e_1^2e_3e_{16} - 6e_2e_7e_{12} - 2e_2^3e_3e_5e_7 - 2e_1^3e_3^2e_5e_7 - 4e_2^2e_4e_{13}$
$-4e_1^2e_2e_5e_{12} + 11e_1^3e_2e_7e_9 - 12e_1e_2^2e_7e_9 - 11e_1^2e_3e_7e_9 + 12e_2e_3e_7e_9$
$+6e_1e_4e_7e_9 + 11e_1^2e_2e_8e_9 + 6e_1e_6^2e_8 + 7e_1^2e_3^3e_{10} + 4e_2e_3^3e_{10}$
$+2e_1e_3e_4e_6e_7 + 4e_1e_2e_3^2e_5e_7 - 2e_1e_2e_3e_4e_{11} + 15e_1e_2e_3e_5e_{10}$
$+5e_2^3e_4e_{11} + 6e_3^2e_4e_{11} + 5e_1^2e_4^2e_{11} - 4e_2e_4^2e_{11} - 2e_1^5e_5e_{11} - 11e_1e_3e_5e_{12}$
$-5e_1^3e_2^2e_5e_9 + 11e_1e_2^3e_5e_9 + 6e_1^4e_3e_5e_9 - 13e_2^2e_3e_5e_9 + 6e_1e_3^2e_5e_9$
$+5e_4^2e_6e_7 + 3e_1^2e_6^2e_7 + 6e_2e_6^2e_7 - e_3^2e_5e_{10} + e_1^3e_2e_5^2e_6$
$-3e_1e_2^2e_5^2e_6 - e_1^2e_3e_5^2e_6 + e_2e_3e_5^2e_6 + 4e_1e_4e_5^2e_6 + 10e_1e_3e_7e_{10}$
$-5e_1e_2e_8e_{10} + 7e_2^2e_3e_6e_8 - 8e_1e_3^2e_6e_8 + 6e_1^3e_4e_6e_8 + 2e_1e_2e_5e_{13}$
$+6e_7e_{14} - 6e_1e_6e_{14} + e_1^2e_{19} + e_3^2e_5^3 - 2e_2^3e_7e_8 + 2e_1^4e_3^2e_{11} - 4e_2^2e_3^2e_{11}$
$-3e_1e_3^3e_{11} - e_8e_{13} + 2e_1e_2e_6e_{12} - e_1^2e_3^2e_6e_7 + e_2^4e_{13} + 19e_1e_2e_3e_6e_9$
$+2e_2e_3^2e_6e_7 + 3e_1^4e_4e_6e_7 - 9e_2^2e_4e_6e_7 - 3e_1^3e_5e_6e_7 - 2e_3e_5e_6e_7$
$+5e_1e_2^2e_4e_6^2 - 6e_3^2e_6e_9 - e_1e_5^4 + 6e_3^2e_7e_8 - 6e_1e_4e_8^2 - e_4e_{17}$
$+e_4e_5e_{12} + e_2e_9e_{10} + e_1^2e_2^4e_{11} - 6e_4e_5e_6^2 - 6e_1e_2e_6^3 - e_1^2e_2^3e_5e_8$
$+e_1^2e_4e_5^3 - 2e_2e_4e_5^3 + e_1^2e_2^2e_3e_5e_7 - e_1^5e_{16} - 6e_1e_3e_4^2e_9$
$+3e_1^5e_8^2 - 4e_1^2e_4e_7e_8 + 4e_2e_4e_7e_8 + e_1e_4^2e_6^2 - e_1^4e_5e_6^2 + e_2^2e_5e_6^2$
$+e_1^2e_3e_4e_6^2 - 5e_1e_2e_4e_6e_8 - e_{10}e_{11} - e_1^5e_6e_{10} + 6e_5e_6e_{10} + 3e_1^4e_7e_{10}$

$+5e_2^2e_7e_{10} - 5e_4e_7e_{10} + 3e_1^3e_8e_{10} + e_3e_8e_{10} - 2e_1^2e_9e_{10} - 3e_1e_2e_3e_4^2e_7$
$+e_1e_{10}^2 + e_2^3e_4^2e_7 + 3e_3^2e_4^2e_7 + 3e_1^2e_4^3e_7 - 3e_2e_4^3e_7 - e_5e_{16}$
$+12e_1^2e_2e_3e_4e_{10} + e_1^4e_5^2e_7 + 4e_2^2e_5^2e_7 - 6e_2^4e_4e_9 + e_1^6e_{15}$
$-3e_3^3e_{12} + 5e_1^2e_2^3e_4e_9 - e_1^3e_2e_4e_5e_7 - 7e_1^2e_2e_3e_5e_9 + 3e_1^3e_3e_{15}$
$+e_1^3e_3^2e_6^2 + 3e_3^3e_6^2 - 3e_3^2e_4e_5e_6 - 3e_1^2e_4^2e_5e_6 + 5e_2e_4^2e_5e_6$
$-10e_1^2e_5e_6e_8 - 2e_2e_5e_6e_8 + 8e_3e_4e_6e_8 + 5e_1^3e_2e_3e_{13} + e_5e_8^2$
$-9e_1^3e_2e_3e_4e_9 + 5e_4e_8e_9 + 11e_1^2e_3^2e_4e_9 - e_2e_3^2e_4e_9 + 2e_1^2e_2e_4^2e_9$
$+3e_1^4e_2e_4e_{11} - 5e_1^2e_2^2e_4e_{11} - 8e_1^3e_3e_4e_{11} + 4e_1^3e_2e_5e_{11}$
$-4e_1e_2^2e_5e_{11} - 3e_3^3e_5e_7 + 5e_1^2e_3e_4e_5e_7 - e_2^5e_{11} - 9e_3e_4e_7^2$
$+4e_1^2e_5e_7^2 - 2e_1^2e_3^2e_5e_8 - e_2e_3^2e_5e_8 - 4e_1^4e_4e_5e_8 - 2e_2^2e_6e_{11}$
$+9e_3e_6e_{12} + e_2e_3e_{16} + 3e_3^4e_9 + 5e_4e_5^2e_7 + 2e_1e_2^2e_3e_{13} + e_1^3e_6^3$
$+3e_3e_6^3 - e_1e_{20} - 9e_1e_4^2e_5e_7 - 4e_1^2e_2e_5^2e_7 - e_1e_3e_5^2e_7 + 7e_1e_3e_4e_5e_8$
$+e_1^3e_2^2e_7^2 - 6e_1e_2e_4^2e_{10} + 3e_1^4e_2e_5e_{10} - 3e_1^2e_2^2e_5e_{10} - 12e_1^3e_3e_5e_{10}$
$+e_1^3e_2e_3e_6e_7 + 4e_1^2e_2e_5e_6^2 - e_1e_2^2e_3e_6e_7 + 10e_1e_5e_7e_8 - 11e_1^3e_2e_8^2$
$+7e_1e_2^2e_8^2 + 7e_1^2e_3e_8^2 - 2e_2e_3e_8^2 + e_2^4e_3e_{10} - 3e_1^2e_2e_3e_7^2$
$+e_2e_3e_4e_5e_7 - e_1^3e_{18} - 5e_1^2e_2e_4e_6e_7 - 4e_1^3e_2^2e_4e_{10} + 3e_1e_2^3e_4e_{10}$
$+3e_1^4e_3e_4e_{10} - 2e_2^2e_3e_4e_{10} - 15e_1e_3^2e_4e_{10} - e_1^2e_2e_4e_{13}$
$+e_1e_2^2e_4e_5e_7 + 5e_1^2e_2^3e_3e_{10} - 5e_1^3e_2e_3^2e_{10} - 9e_1e_2^2e_3^2e_{10}$
$+3e_1^3e_2^2e_6e_8 - 7e_1e_2^3e_6e_8 - 4e_1^4e_3e_6e_8 + e_2^6e_9 + e_5^2e_{11} + 3e_1e_3e_4e_{13}$
$-23e_1e_2e_3e_7e_8 - 6e_1e_3e_8e_9 - e_1e_2e_9^2 + e_5^3e_6 + e_1e_2^4e_4e_8$
$-2e_1^3e_2e_4e_6^2 + 3e_1^4e_6e_{11} + e_4e_6e_{11} - 4e_1^3e_7e_{11} - 6e_3e_7e_{11} - 2e_1^2e_8e_{11}$
$+2e_2e_8e_{11} + e_1e_9e_{11} - 2e_1e_2^3e_7^2 + 3e_2^2e_3e_7^2 + 3e_1e_3^2e_7^2 - 5e_1^3e_4e_7^2$
$-6e_1e_2^4e_3e_9 - e_1^2e_2^3e_6e_7 + 2e_2^4e_6e_7 - e_2^3e_3e_4e_8 + 2e_1^3e_3^2e_4e_8$
$+e_1e_5e_{15} - 4e_1^3e_6e_{12} + 5e_1e_2e_5e_6e_7 + 3e_2e_3e_4^2e_8$

8 $-3e_1^4e_9e_{11} - 5e_1^3e_7e_{14} + e_4e_6e_{14} - 6e_5e_8e_{11} + 6e_2^2e_6e_{14} + 4e_1^4e_6e_{14}$
$-3e_4^2e_8^2 - 6e_2e_8e_{14} - 2e_2^2e_9e_{11} + 6e_1^2e_8e_{14} + 4e_1^3e_2^3e_{15} - e_1^5e_2^2e_{15}$
$+e_2e_{11}^2 - e_1^2e_{11}^2 + e_3e_{10}e_{11} + 3e_1^3e_{10}e_{11} + e_1e_9e_{14} - 7e_2^5e_4e_{10}$
$+e_4e_9e_{11} + 4e_3e_7^3 + 6e_1^3e_3^2e_{15} - e_1^3e_7^3 - 17e_1^3e_2^2e_3e_4e_{10} + 13e_2^3e_4^2e_{10}$
$-e_2^3e_{18} + 3e_2^3e_3e_{15} + 4e_1e_3^2e_5e_6^2 - e_4^3e_6^2 - e_5^2e_7^2 + 8e_1^2e_2e_4e_6e_{10}$
$+2e_1^6e_3e_{15} + e_1^2e_2^5e_{12} - 3e_1e_2^4e_{15} - 5e_2^3e_3^2e_{12} - 4e_2e_3^4e_{10}$
$-9e_2e_3e_7e_{12} - 2e_2^2e_5e_{15} + 6e_1^2e_7e_{15} + 6e_3^2e_4^2e_{10} + 2e_1e_4^2e_{15} - 7e_1e_8e_{15}$
$-7e_1^3e_3^3e_{12} + e_2e_7e_{15} - 5e_1^3e_6e_{15} + 12e_1^4e_2e_3e_5e_{10} + e_1^6e_2e_{16}$
$-4e_1e_2e_3e_{18} + e_5e_6e_{13} - 2e_2^2e_7e_{13} + 2e_4e_7e_{13} - 2e_1^2e_9e_{13} + 4e_1^4e_5e_{15}$
$+2e_2^2e_3^2e_5e_9 + 5e_1e_3^3e_5e_9 + 14e_1^3e_2e_8e_{11} - 3e_2^2e_3^2e_7^2 + 6e_6^2e_{12}$
$-8e_1^2e_2e_3^2e_5e_9 - 5e_1^4e_2^2e_{16} - 12e_1e_2^2e_8e_{11} - 9e_2^3e_4e_6e_8$
$-7e_1e_3^3e_6e_8 - 7e_1^3e_5e_8^2 - e_2^4e_3e_4e_9 - e_{10}e_{14} - 6e_1^3e_2^3e_5e_{10}$
$+6e_1^2e_2^3e_{16} + 15e_1^2e_2e_6e_7^2 - 3e_1^5e_3e_4e_{12} + 15e_1e_2^2e_3^2e_{13}$
$+4e_1e_3^3e_7^2 - 2e_2^4e_3e_{13} - e_1^3e_{21} + 2e_1e_2e_{21} + 3e_3^2e_6^3 - 5e_1e_2e_3e_6^3$
$+6e_2^4e_4e_{12} - 7e_3e_5e_{16} + e_2^5e_3e_{11} + 29e_1e_2^2e_3e_6e_{10} + 3e_1e_2e_6e_{15}$
$+e_1^4e_{20} + e_1e_3e_{20} - 10e_1^2e_2e_3e_4e_5e_8 - 3e_1^5e_3e_{16} - 3e_1^4e_2e_3e_7e_8$
$+2e_4^2e_5e_{11} - 4e_1^3e_5^2e_{11} + 6e_3e_5^2e_{11} - e_1e_2^2e_4^2e_{11} + 22e_1^2e_3e_4^2e_{11}$
$+3e_2e_3e_4^2e_{11} - 6e_1^2e_2e_6e_{14} - 14e_3e_5e_8^2 - 14e_1^2e_3e_8e_{11} -$

$e_1^2e_2e_4^2e_5e_7 - e_1^2e_3^3e_6e_7 + 5e_2e_3^3e_6e_7 + 7e_1^3e_2e_4e_7e_8$
$-7e_3^3e_5e_{10} + 6e_1^2e_6e_{16} + 4e_4^3e_{12} + 3e_1^3e_2^2e_3e_6e_8 - 18e_1^2e_6e_7e_9$
$+6e_2e_6e_7e_9 + 5e_1^5e_2e_8e_9 + 7e_1e_2e_4e_5e_{12} + 7e_1^2e_6e_8^2 - 3e_1^5e_2e_7e_{10}$
$-6e_2e_3e_5e_6e_8 - 8e_3^2e_4e_{14} + e_2^5e_{14} + e_1^2e_2^3e_3e_5e_8 + 5e_1^5e_4e_6e_9$
$+3e_1e_4^2e_6e_9 + 8e_3^3e_4e_{11} - 3e_1^5e_3e_6e_{10} - 9e_1^2e_3^2e_{16} - 11e_1e_2e_3^2e_6e_9$
$+6e_2^3e_3e_7e_8 + 3e_1^3e_3^2e_7e_8 - 3e_1^5e_4e_7e_8 + 11e_1e_2e_4e_5e_6^2$
$-14e_2e_6e_8^2 - e_3^2e_4e_5e_9 - 20e_1^2e_4^2e_5e_9 + 5e_2e_4^2e_5e_9 - 8e_1^4e_2e_4e_5e_9$
$-2e_1^2e_5^3e_7 + e_2e_5^3e_7 + 3e_1^4e_4e_{16} - 17e_1e_3e_5^2e_{10} + 2e_1^6e_5e_{13}$
$+3e_2^3e_5e_{13} + 2e_3e_4^2e_{13} + 6e_1^2e_2^4e_3e_{11} + 12e_2e_3e_8e_{11} - e_1^4e_2e_6^3$
$+4e_1^2e_2^2e_6^3 - 6e_1e_7e_{16} + 2e_1^4e_4^2e_{12} - 2e_1^2e_5e_{17} - 7e_2e_6e_{16} - 14e_1e_7e_8^2$
$+10e_1^2e_2^2e_3e_4e_{11} + 12e_1^4e_3e_5e_{12} + 6e_1^5e_2e_{17} + 6e_4^4e_8 - 11e_1^3e_2e_4e_6e_9$
$-2e_1^5e_4e_{15} + 7e_8^3 + e_5^2e_{14} + 8e_1^3e_2e_4e_5e_{10} + 5e_1^4e_3^2e_{14} + 2e_2^2e_3^2e_{14}$
$+8e_1e_3^3e_{14} + e_1^6e_4e_{14} - 12e_1e_3e_6e_{14} + e_1^4e_2e_4e_6e_8 - 10e_2^2e_4^2e_{12}$
$+25e_1^2e_2e_3^2e_4e_{10} + 2e_2^5e_6e_8 - e_1^2e_2^4e_6e_8 - 13e_2e_3e_4e_7e_8$
$+18e_3e_5e_6e_{10} - 20e_1e_2e_4e_8e_9 - 10e_1^3e_2^2e_{17} + 9e_2^2e_5^2e_{10} + 6e_1^2e_2^4e_4e_{10}$
$-3e_1^2e_2^2e_6e_{12} - 12e_1^3e_3e_6e_{12} - e_1e_{23} + 8e_1e_4e_8e_{11} - 7e_4e_5^2e_{10}$
$+3e_1e_3^2e_4e_5e_8 - 8e_1^4e_5e_6e_9 - 5e_2^2e_5e_6e_9 - 7e_4e_5e_6e_9 - e_1e_2e_6^2e_9$
$+7e_2e_3^2e_8^2 + 4e_1^4e_4e_8^2 + 8e_2^2e_4e_8^2 - 4e_1^4e_3e_{17} + 4e_1e_2^3e_{17} + 3e_3^3e_6e_9$
$-2e_2e_3^2e_4e_6^2 + e_2^2e_{20} + 6e_1^3e_2^2e_5e_{12} - 3e_1e_2^3e_5e_{12} + 22e_1e_2e_5e_8^2$
$-4e_3^2e_4e_7^2 + 10e_1e_2^2e_4e_7e_8 - 2e_1^4e_3^2e_6e_8 + 5e_2^2e_3^2e_6e_8$
$+7e_1^2e_2e_9e_{11} + 6e_1e_2^2e_3^2e_5e_8 + 4e_1^2e_2^2e_3e_6e_9 - 9e_1e_2e_3^2e_7e_8$
$+e_1^4e_3e_5e_6^2 + 9e_1e_2^2e_5e_7^2 + 15e_1^2e_2^2e_4e_5e_9 + 3e_1^2e_4^2e_7^2$
$-15e_1e_4e_5e_6e_8 - e_1^2e_2^2e_4e_6e_8 - e_{11}e_{13} - e_1e_3^2e_{17} + 3e_1e_2e_7e_{14}$
$+e_1e_2e_3^3e_{12} + 7e_1^2e_3^2e_5e_{11} - 11e_2e_3^2e_5e_{11} + 7e_1^4e_4e_5e_{11}$
$+4e_2^2e_4e_5e_{11} - 2e_2^2e_3e_{17} - 8e_1^2e_5e_8e_9 + 7e_2e_5e_8e_9 + 14e_1e_6e_8e_9$
$+13e_1^4e_2e_9^2 - 12e_1^2e_2^2e_9^2 + 14e_3e_4e_8e_9 + 3e_1^5e_5^2e_9 - 5e_1^4e_3e_4^2e_9$
$+4e_2^2e_3e_4^2e_9 + e_3e_4e_{17} - 4e_1e_4e_5e_7^2 + 3e_2^4e_6e_{10} - e_1^7e_{17} + 4e_1^3e_4e_{17}$
$-e_2^3e_6^3 - 16e_1^2e_2e_3e_5e_{12} - 7e_1e_2^5e_3e_{10} + e_1e_6e_{17} + 4e_1^2e_3^2e_8^2$
$-5e_1e_2^3e_4^2e_9 - 2e_1e_3e_9e_{11} + 7e_3^2e_5e_{13} - 2e_1e_5^2e_{13} - 3e_1^5e_6e_{13}$
$+e_1^2e_4e_6^3 - 7e_2e_4e_6^3 - 5e_1e_5e_6^3 + 2e_2e_4^2e_7^2 + 4e_1^4e_2^2e_4e_{12}$
$-7e_1^2e_2^3e_4e_{12} + 6e_5^2e_6e_8 + 2e_1^4e_6^2e_8 + 13e_2^2e_6^2e_8 + e_4e_6^2e_8$
$+8e_1e_2e_3e_9^2 - 6e_1e_2^3e_3e_6e_8 - e_1e_2^3e_3e_5e_9 + 9e_1^2e_5^2e_{12}$
$+3e_1^4e_2e_4e_7^2 - 5e_1e_2^4e_7e_8 + 9e_1^2e_4e_5e_{13} - 6e_2e_4e_5e_{13} + 7e_1^3e_2e_6e_{13}$
$-6e_1e_2^2e_6e_{13} + 8e_1^2e_3e_6e_{13} + 3e_2e_3e_6e_{13} - 2e_2e_5^2e_{12} - 2e_1^5e_3^2e_{13}$
$-3e_1^5e_4e_5e_{10} + e_2^2e_4^2e_5e_7 + 7e_1^3e_2e_5e_{14} - 6e_1e_2^2e_5e_{14} + 8e_1^2e_3e_5e_{14}$
$-3e_2e_3e_5e_{14} - 4e_1e_4e_5e_{14} - 7e_1^3e_3e_4e_{14} + 7e_3^2e_5^2e_8 - 7e_1e_5^3e_8$
$-5e_1^4e_2e_{18} - 17e_1^2e_2e_5e_7e_8 + 9e_1e_2^2e_3^2e_4e_9 + 6e_1e_2e_4e_6e_{11}$
$-4e_2^2e_4^3e_8 + 6e_1^2e_2^2e_{18} + 18e_1e_2e_3e_4e_{14} + e_1^5e_6^2e_7 + e_1e_2^3e_4e_5e_8$
$-5e_3e_4e_5e_6^2 + 3e_1^3e_3e_{18} + 14e_1^3e_2^2e_7e_{10} - 13e_1e_2^3e_7e_{10} + 4e_1^4e_3e_7e_{10}$
$-15e_1e_2e_3e_4^2e_{10} - 12e_1^2e_2^3e_6e_{10} + 6e_1e_3e_4e_8^2 + 36e_1e_2e_3^2e_5e_{10}$
$-e_1^2e_2^4e_5e_9 + e_1^2e_3e_5^2e_9 - 3e_2e_3e_5^2e_9 + 11e_1e_4e_5^2e_9$
$+e_1^4e_2e_5e_6e_7 - 5e_1e_2e_{10}e_{11} + 8e_1^2e_3e_4e_5e_{10} + 5e_1^3e_2^2e_4^2e_9$

$$-e_1^3e_2e_3e_5e_{11} - 3e_1^2e_4e_{18} - e_7e_{17} + 3e_2e_4e_{18} - e_2^4e_8^2 + 11e_1e_2e_5e_{16}$$
$$-9e_1e_4e_9e_{10} + 5e_1^2e_2e_{10}^2 - 2e_1e_3e_{10}^2 + 18e_2^2e_3e_7e_{10} + 12e_1e_3^2e_7e_{10}$$
$$-13e_1e_3e_4e_5e_{11} + e_1^2e_3^3e_{13} - 3e_1^3e_2e_6^2e_7 - 6e_1e_2^2e_6^2e_7$$
$$+4e_1^2e_3e_6^2e_7 + 9e_2e_3e_6^2e_7 + 10e_1e_4e_6^2e_7 - 9e_1^3e_2e_3e_4e_{12}$$
$$+e_1^6e_{18} - 2e_1e_2e_5^3e_6 - 4e_1^3e_2e_5e_7^2 + 6e_2^3e_3e_6e_9 + 6e_1^3e_3^2e_6e_9$$
$$+19e_1^2e_3^2e_4e_{12} - e_1^3e_2e_3e_6e_{10} - 2e_1^3e_4e_5e_6^2 - 8e_3^2e_7e_{11}$$
$$-6e_1^4e_4e_7e_9 + 11e_1^3e_5e_7e_9 - 6e_3e_5e_7e_9 + 4e_1^3e_2e_{19} + 9e_1^2e_2e_3e_{17}$$
$$-6e_1e_3e_4e_6e_{10} - e_1^5e_{19} - 4e_1^3e_3e_4e_7^2 - 8e_1^4e_2e_8e_{10} - 3e_1e_2^2e_{19}$$
$$+14e_1^2e_2^2e_8e_{10} + 11e_1^3e_3e_8e_{10} + 5e_1^4e_3e_4e_5e_8 + 5e_1e_2e_4^3e_9$$
$$-9e_2e_5e_7e_{10} + 4e_1e_6e_7e_{10} - 16e_1^2e_2^2e_3e_5e_{10} - 4e_2e_3e_5e_7^2$$
$$-2e_1^2e_2e_4^2e_6^2 + e_2e_3e_{19} - 22e_2e_3^2e_6e_{10} + 16e_1^3e_2e_3e_{16} - 15e_1e_2^2e_3e_{16}$$
$$+5e_1e_2^3e_4e_6e_7 + 2e_1^4e_3^3e_{11} + 5e_2^2e_3^3e_{11} + e_1^3e_2^2e_5^2e_7$$
$$-3e_1e_2^3e_5^2e_7 - 2e_1^4e_3e_5^2e_7 + e_2^2e_3e_5^2e_7 - 5e_1e_3^2e_5^2e_7 - 2e_2^3e_6e_{12}$$
$$-22e_1e_2e_3e_4e_5e_9 + 22e_1e_3e_5e_7e_8 + 7e_1e_3e_5e_6e_9 - 2e_2^4e_3e_5e_8$$
$$+2e_1e_4e_{19} - 5e_1^3e_8e_{13} - 4e_1e_4e_6e_{13} - 6e_1^2e_2e_7e_{13} - 3e_1e_3e_7e_{13}$$
$$+11e_1e_2e_8e_{13} + 4e_1^4e_2^2e_6e_{10} - 14e_1e_3e_6e_7^2 + e_{24} - 3e_3^2e_6e_{12}$$
$$+2e_4^2e_6e_{10} - 6e_1e_2e_4e_5^2e_7 - 3e_1^4e_2^2e_7e_9 + 5e_1^4e_3^2e_4e_{10}$$
$$+7e_2e_3^2e_{16} - 7e_2e_3^3e_{13} - 7e_1e_2e_3e_5e_{13} - 5e_1^3e_2^3e_4e_{11} - e_2^6e_{12}$$
$$+4e_1^4e_7e_{13} + 2e_2^3e_5^2e_8 - 2e_1^4e_6e_7^2 + 13e_1e_2^4e_5e_{10} - 4e_1^2e_2^2e_5e_6e_7$$
$$-10e_3e_4^2e_5e_8 - 9e_1^3e_4^2e_5e_8 + e_4e_5e_{15} + e_1e_3e_5^4 + 9e_1^4e_2e_3e_4e_{11}$$
$$+e_1^2e_{22} - 5e_1^2e_2^3e_3e_4e_9 - 18e_1^3e_2e_9e_{10} + 12e_1e_2^2e_9e_{10} + 11e_1^2e_3e_9e_{10}$$
$$-7e_2e_3e_9e_{10} - 6e_1^2e_3e_4e_7e_8 - 7e_1^4e_2e_3e_{15} + 7e_2^2e_3e_5e_{12}$$
$$-6e_1e_3^2e_5e_{12} - 14e_1^3e_4e_5e_{12} - e_2e_{22} + 7e_2e_4e_5e_6e_7 - 27e_1^2e_2e_3e_7e_{10}$$
$$-6e_1e_4^3e_{11} - 3e_1^5e_7e_{12} - 9e_1^3e_2^2e_3^2e_{11} - 9e_1^2e_2e_3e_4e_{13}$$
$$+4e_1^3e_4^2e_6e_7 + 8e_3e_4^2e_6e_7 + e_4e_{10}^2 + e_2e_9e_{13} + e_1^2e_6^2e_{10}$$
$$-5e_2^2e_6e_7^2 - 3e_1^2e_2e_3^2e_7^2 + 7e_1^3e_2e_7e_{12} + e_1e_2^2e_7e_{12} + e_1^2e_3e_7e_{12}$$
$$+8e_1e_3e_4^2e_5e_7 - 3e_1^3e_3e_5^2e_8 + 15e_1^2e_4e_5^2e_8 - 5e_2e_4e_5^2e_8$$
$$-8e_2^2e_3^2e_4e_{10} - 9e_2^2e_3e_4e_6e_7 + e_2e_6^2e_{10} - 11e_1e_2^3e_3^2e_{11}$$
$$+17e_1^2e_2e_3^3e_{11} - 2e_4e_6e_7^2 + 4e_2^2e_4^2e_6^2 - e_4e_{20} - 11e_1e_2^2e_3e_8^2$$
$$+17e_2e_3e_4e_5e_{10} - 9e_1e_2e_5e_6e_{10} + 19e_1^3e_3e_4e_5e_9 - 21e_1e_2^2e_3e_7e_9$$
$$+6e_1e_7^2e_9 + e_6e_7e_{11} - 3e_1^5e_8e_{11} + 13e_1^3e_2e_5e_6e_8 + 10e_1e_3e_8e_{12}$$
$$-2e_1e_2e_9e_{12} - 13e_1^2e_2e_8e_{12} - 7e_3e_8e_{13} + 3e_2e_3^2e_4e_{12} + 11e_1^2e_2e_4^2e_{12}$$
$$+2e_3e_4e_5e_{12} + 12e_1^4e_3e_6e_{11} - 7e_2^2e_3e_6e_{11} + 16e_1e_3^2e_6e_{11} - 7e_1^3e_4e_6e_{11}$$
$$-5e_3e_4e_6e_{11} + 10e_1^2e_5e_6e_{11} + 4e_2e_5e_6e_{11} - e_1e_2^6e_{11} + 9e_1e_2^3e_3e_4e_{10}$$
$$-11e_1^2e_2e_4e_{16} + e_5e_7e_{12} + 5e_1^4e_2^2e_5e_{11} - 9e_1^2e_2^3e_5e_{11} - 6e_1^5e_3e_5e_{11}$$
$$+e_1^2e_2e_4e_5^2e_6 - 7e_7e_8e_9 - 3e_1^6e_9^2 + 2e_4^2e_7e_9 + 12e_1e_2e_3e_4e_7^2$$
$$+3e_6^4 - 7e_2e_3^3e_5e_8 + e_1^2e_3^2e_4e_6^2 + 5e_1^3e_4^3e_9 - e_2^4e_7e_9 + 13e_1e_3^2e_4^2e_9$$
$$+e_1^3e_2e_3^2e_6e_7 + e_2e_3^2e_5^2e_6 - 2e_2^2e_4e_5^2e_6 - 7e_1e_6^2e_{11} + 2e_1^6e_7e_{11}$$
$$+3e_2^3e_7e_{11} - 5e_1^3e_3e_4e_6e_8 - 2e_1e_2e_3e_5^2e_8 - 24e_1e_2e_3e_8e_{10}$$
$$+8e_1^2e_4e_6e_{12} + 2e_2e_4e_6e_{12} - 14e_1e_5e_6e_{12} + e_1^4e_2^3e_{14} - 3e_1^2e_2^4e_{14}$$
$$+2e_1e_{10}e_{13} - 7e_1^3e_2e_3^3e_{10} - 21e_1e_2^2e_3^3e_{10} + e_1e_2^2e_3e_4e_5e_7$$
$$+4e_1^2e_2e_4^3e_8 - 8e_1e_3e_4^3e_8 - 7e_5e_6^2e_7 - 8e_1e_2^2e_5e_6e_8 + 8e_1^2e_2e_4e_7e_9$$

$$-2e_1^4e_2^2e_8^2+8e_3e_7e_{14}+6e_1^2e_2^3e_8^2+2e_1^5e_3e_8^2-e_1^4e_3e_4e_6e_7$$
$$+4e_1^2e_2e_3^2e_6e_8-13e_1e_3^3e_4e_{10}-e_1^3e_4e_7e_{10}-11e_3e_4e_7e_{10}$$
$$-3e_1^2e_5e_7e_{10}-2e_1^3e_2^2e_4e_6e_7+8e_1^2e_7^2e_8+6e_2e_7^2e_8-6e_1^4e_2e_5e_{13}$$
$$+6e_1^2e_2^2e_5e_{13}-6e_1^3e_3e_5e_{13}+4e_1^4e_8e_{12}-2e_1e_2e_3^2e_{15}+3e_1e_2^3e_3e_7^2$$
$$+3e_1^3e_2^2e_3e_5e_9-4e_2^3e_4e_5e_9+e_1e_{11}e_{12}+e_2e_{10}e_{12}-2e_1^2e_{10}e_{12}$$
$$-5e_1e_2e_7^3-3e_1^3e_2e_3^2e_5e_8+2e_1^3e_9e_{12}-3e_4e_8e_{12}-6e_3e_4^3e_9$$
$$+3e_3e_6e_{15}-3e_1^5e_2e_5e_{12}-7e_1e_2e_4e_{17}+5e_1^3e_2e_3^2e_4e_9+e_1^6e_8e_{10}$$
$$+12e_1e_3e_4e_{16}+4e_2^2e_4e_{16}+5e_1^4e_2e_3^2e_{12}-4e_1e_2e_5e_7e_9+8e_1^2e_2^3e_7e_9$$
$$+2e_1^5e_3e_7e_9-2e_1^2e_3^2e_7e_9+14e_2e_3^2e_7e_9+3e_1^3e_2e_4e_{15}+e_2e_5^2e_6^2$$
$$+4e_4^2e_5^2e_6-9e_1^2e_2e_3e_4^2e_9-18e_2^3e_3e_5e_{10}-17e_1^3e_3^2e_5e_{10}-e_5e_{19}$$
$$+6e_3e_4e_5^2e_7+3e_1^3e_4e_5^2e_7-3e_2^3e_3e_4e_{11}-18e_1^3e_3^2e_4e_{11}$$
$$-9e_1^3e_2e_4^2e_{11}+4e_2^3e_5e_6e_7-8e_3^2e_5e_6e_7+10e_1e_5^2e_6e_7$$
$$-30e_1^2e_2e_3e_6e_{11}-5e_1^2e_3e_4e_6e_9+e_2^7e_{10}-12e_2^2e_4e_6e_{10}+e_2^4e_4^2e_8$$
$$+14e_1^2e_2^3e_3^2e_{10}+e_1e_5e_{18}-3e_1e_2^2e_3^2e_6e_7-18e_1^3e_2^2e_8e_9$$
$$+10e_1e_2^3e_8e_9-11e_1^4e_3e_8e_9-13e_2^2e_3e_8e_9-21e_1e_3^2e_8e_9+6e_1^3e_4e_8e_9$$
$$-2e_2^4e_5e_{11}-e_1^3e_2^2e_4e_5e_8+6e_5e_9e_{10}-2e_1^4e_{10}^2-e_2^2e_{10}^2+3e_2^2e_3e_4e_{13}$$
$$-3e_1e_3^2e_4e_{13}+5e_1e_2e_4^2e_{13}-2e_1^2e_3e_{19}-8e_1^2e_2^2e_4e_7^2$$
$$+5e_1^2e_2^2e_3e_7e_8+2e_2e_5e_{17}+e_2^5e_5e_9+6e_1e_2^2e_4e_{15}+3e_1^3e_2e_3e_7e_9$$
$$+2e_1^3e_2^3e_7e_8+9e_1e_2e_4^2e_5e_8+4e_3^2e_4e_6e_8+8e_1^2e_4^2e_6e_8$$
$$-3e_1^5e_5e_6e_8-7e_1^2e_3^3e_4e_9-4e_2e_3^3e_4e_9-7e_1^2e_3e_4e_{15}+e_1e_2^3e_3e_{14}$$
$$-19e_1^2e_2e_3^2e_{14}-e_1^4e_2e_4e_{14}-e_1^2e_2^2e_4e_{14}+3e_1^4e_2e_4^2e_{10}$$
$$+7e_2^2e_3e_4e_5e_8+6e_1^2e_2e_3e_5^2e_7-e_2^4e_{16}-2e_1^2e_3^2e_4e_5e_7$$
$$+e_1^2e_2e_4e_5e_{11}-3e_1^5e_2e_3e_{14}+8e_1^3e_2^2e_3e_{14}+10e_1e_4^2e_5e_{10}$$
$$-4e_1e_3^2e_4e_6e_7+19e_1e_2e_3e_6e_{12}-4e_1e_4^2e_7e_8+6e_1^4e_5e_7e_8$$
$$-9e_2^2e_5e_7e_8+8e_4e_5e_7e_8-4e_1^3e_6e_7e_8+4e_3e_6e_7e_8+21e_1e_2^2e_3e_5e_{11}$$
$$+e_1e_2^5e_4e_9-3e_2^3e_8e_{10}-e_1^3e_2^4e_{13}-4e_1^2e_2^2e_4^2e_{10}-5e_2e_3e_4e_{15}$$
$$+7e_8e_{16}-4e_1^2e_2e_3e_5e_6^2+3e_1^2e_3^3e_5e_8+15e_1e_2e_3e_4e_6e_8+7e_1^2e_3^4e_{10}$$
$$+7e_2e_3^4e_{10}+4e_1e_5e_7e_{11}-3e_4^2e_{16}-5e_1^3e_5e_{16}+e_5^3e_9+7e_3^2e_8e_{10}$$
$$-6e_1^2e_2e_5e_{15}-5e_1^3e_2^3e_3e_{12}+2e_1e_2^5e_{13}-5e_1^3e_4^2e_{13}-7e_1^3e_2e_5^2e_9$$
$$+4e_1e_2^2e_5^2e_9-3e_2^3e_4e_{14}+e_1^2e_4^2e_{14}-3e_1^5e_5e_{14}-7e_6e_8e_{10}$$
$$+4e_1^4e_2^2e_3e_{13}-7e_1^2e_2^3e_3e_{13}-3e_1^3e_2e_3^2e_{13}+2e_1^3e_2^3e_6e_9$$
$$-4e_1e_2^4e_6e_9-3e_1^5e_2e_4e_{13}+8e_1^3e_2^2e_4e_{13}-7e_1e_2^3e_4e_{13}$$
$$+8e_1^4e_3e_4e_{13}+6e_1^3e_6^2e_9-5e_1^3e_3e_4^2e_{10}+9e_1^2e_2e_5e_6e_9$$
$$+7e_1^2e_2e_3e_4e_6e_7-5e_1^2e_3e_5e_6e_8-6e_1^4e_2e_7e_{11}-e_1^2e_2^2e_7e_{11}$$
$$+e_1^3e_3e_7e_{11}-3e_1^2e_4e_7e_{11}-6e_2e_4e_7e_{11}-16e_1e_3e_4^2e_{12}-8e_1e_3^4e_{11}$$
$$+2e_1^3e_3e_5e_6e_7+e_2^3e_9^2+3e_1e_3e_5e_{15}+4e_1^2e_2^2e_3^2e_{12}+e_1^6e_6e_{12}$$
$$+2e_1^2e_5^2e_6^2+4e_1e_2^4e_3e_{12}+4e_1e_2^4e_4e_{11}+15e_1e_2e_4e_7e_{10}$$
$$-e_2e_3^2e_4e_5e_7+5e_1^5e_9e_{10}+13e_1e_2e_3e_7e_{11}+7e_2^4e_3^2e_{10}$$
$$-14e_1e_2e_4^2e_6e_7+4e_2e_3e_4e_6e_9-6e_3e_6^2e_9-3e_1^3e_2e_3e_8^2$$
$$+e_1^4e_3^2e_7^2+2e_1^2e_3^2e_4^2e_8+4e_2e_3^2e_4^2e_8-12e_1^2e_4e_5e_6e_7$$
$$-9e_1^3e_3e_9^2+8e_1^2e_4e_9^2-e_2e_4e_9^2-6e_1e_5e_9^2+7e_1e_2^2e_4e_6e_9$$
$$-e_1e_3e_4e_7e_9-e_1^3e_2^2e_3e_7^2-21e_1e_2^2e_4e_5e_{10}+e_7^2e_{10}-6e_4^3e_5e_7$$

$-4e_1^4e_2e_3e_6e_9 - 6e_1^2e_4e_8e_{10} + 6e_2e_4e_8e_{10} + 7e_1e_5e_8e_{10} - 3e_1^2e_2e_{20}$
$-e_4e_5^4 + 3e_1e_2e_3e_5e_6e_7 - 10e_1^2e_2e_6^2e_8 + e_1e_3e_6^2e_8 + 4e_1^2e_2e_5^2e_{10}$
$-16e_1e_2e_3^2e_4e_{11} + 45e_1^2e_2e_3e_8e_9 - 4e_1e_2^2e_3e_4^2e_8 - 4e_1e_3e_4e_5^2e_6$
$-7e_1e_2e_5^2e_{11} - 3e_1^5e_2e_6e_{11} + 6e_1^3e_2^2e_6e_{11} + 4e_1e_2^3e_6e_{11}$
$-15e_1^2e_2e_4e_8^2 + 4e_2^2e_8e_{12}$

Table of $P_{m,4}$ for $m > 3$

m	$P_{m,4}$
4	$2e_4e_6^2 + e_1e_3e_6^2 + 3e_1e_4^2e_7 + e_3^3e_7 - e_2e_7^2 + e_4e_5e_7 + e_2^2e_5e_7$
	$-e_2e_3^2e_8 + e_6e_{10} + e_1e_7e_8 + 3e_2e_6e_8 - e_1^2e_6e_8 + 2e_2^2e_4e_8 - 2e_3e_6e_7$
	$+e_2^2e_3e_9 + 3e_1^2e_5e_9 + 4e_3e_4e_9 - e_1e_3^2e_9 - 2e_1e_6e_9 + e_1^3e_{13}$
	$-2e_2e_5e_9 + e_3e_{13} - e_2^3e_{10} - 2e_1e_5e_{10} - 3e_2e_3e_4e_7 - 2e_2e_3e_{11}$
	$-e_1^2e_3e_{11} + e_1e_2^2e_{11} + 2e_1e_3e_{12} + e_1e_4e_{11} + e_2e_4e_{10} - 2e_1e_2e_{13}$
	$-2e_4e_{12} - e_1e_3e_5e_7 + e_1e_2e_6e_7 - e_{16} - e_1^2e_2e_{12} + e_1e_{15} - 3e_1e_2e_4e_9$
	$-e_3^2e_{10} - 4e_4^2e_8 + e_1e_5^3 + e_5e_{11} - e_8^2 + e_3e_5e_8 + e_2^2e_{12} + e_2e_{14} - e_5^2e_6$
	$+e_1e_3e_4e_8 - e_1^2e_{14} + 2e_1e_2e_3e_{10} + e_7e_9 + e_2e_3e_5e_6 - 3e_1e_4e_5e_6$
	$-2e_1e_2e_5e_8 - e_2^2e_6^2$
5	$-3e_1^2e_6e_{12} - 2e_3e_5e_{12} - e_1^3e_5e_{12} + 3e_1e_7e_{12} + e_2e_6e_{12} - e_1^3e_{17}$
	$+2e_1e_6e_{13} - 3e_2e_5e_{13} + 2e_1^2e_5e_{13} + 3e_3e_4e_{13} - 3e_1e_3^2e_{13} + 2e_2^2e_3e_{13}$
	$-e_1e_2^3e_{13} - 2e_1e_4e_{15} - 3e_2e_3e_{15} + 2e_1^2e_3e_{15} + 2e_1e_2^2e_{15} - e_1^3e_2e_{15}$
	$-e_1^3e_3e_{14} - 4e_4^3e_8 + e_1^2e_2^2e_{14} + 4e_1e_2e_5e_{12} - 3e_1^2e_2e_{16} + e_6e_7^2$
	$-4e_1e_2e_3e_7^2 + e_1^4e_{16} - e_1e_{19} + 4e_2e_4^2e_{10} - 2e_4e_6e_{10} - 3e_3e_7e_{10}$
	$+2e_1^2e_2e_3e_{13} + 3e_1e_2e_5^2e_7 - 3e_1e_2e_4e_{13} + e_2^2e_{16} + 2e_1e_2e_{17}$
	$+3e_2^2e_3e_5e_8 - 4e_1e_3^2e_5e_8 + 2e_2^3e_7^2 - 5e_1e_2e_6e_{11} - 3e_2^2e_7e_9$
	$+e_5^4 - 2e_4^2e_{12} + 3e_1e_2^2e_4e_{11} - 4e_1^2e_3e_4e_{11} - 3e_2e_3e_4e_{11} - 3e_1^2e_2e_5e_{11}$
	$+3e_1e_3e_5e_{11} - e_2e_{18} + e_1^2e_5^2e_8 - 4e_2e_5^2e_8 + 5e_3e_4e_5e_8 - e_1e_2e_3e_{14}$
	$+5e_1e_2e_3e_6e_8 + e_1e_3^2e_4e_9 - 5e_1e_2e_4^2e_9 - 2e_2^2e_4e_5e_7 - 2e_5e_7e_8$
	$+e_4e_8^2 + 2e_1e_2e_4e_6e_7 - e_9e_{11} + 3e_1e_8e_{11} - 2e_2^2e_4e_{12} + 2e_2e_3^2e_{12}$
	$-e_1e_3^3e_{10} - 2e_2^3e_4e_{10} - 4e_4e_5^2e_6 + 2e_2^2e_4^2e_8 + 6e_5^2e_{10} + e_3^2e_4e_{10}$
	$-2e_4e_{16} - e_1e_3e_4e_5e_7 + e_3^2e_{14} - 2e_1e_2e_5e_6^2 - 3e_5e_6e_9 + 3e_4e_7e_9$
	$-e_1^3e_8e_9 + e_1^2e_9^2 + e_2e_9^2 - e_2^3e_5e_9 + 4e_3^2e_5e_9 - 3e_1e_5^2e_9$
	$+2e_1e_2^2e_5e_{10} + 3e_1^2e_3e_5e_{10} - 2e_8e_{12} + 5e_2e_5e_6e_7 - e_3e_4^2e_9$
	$-7e_2e_3e_5e_{10} - 8e_1e_4e_5e_{10} - 4e_1^2e_2e_6e_{10} + 4e_1e_3e_6e_{10} + 3e_1e_2e_7e_{10}$
	$-2e_1e_2^2e_7e_8 + e_1^2e_3e_7e_8 + 5e_2e_3e_7e_8 - 3e_1e_4e_7e_8 + e_1^2e_2e_8^2$
	$+4e_4^2e_5e_7 - 3e_1e_5e_{14} - 3e_1e_2^2e_3e_{12} + e_1^2e_2e_4e_{12} + e_1e_3e_4e_{12}$
	$+e_1^2e_3^2e_{12} - 2e_2^3e_6e_8 - 2e_2^2e_3e_6e_7 + 3e_1e_3^2e_6e_7 + 3e_1^2e_4e_7^2$
	$-4e_2e_4e_7^2 + e_2^4e_{12} - e_3e_{17} - 3e_3e_4e_6e_7 - 4e_1^2e_5e_6e_7 + e_1e_2e_3e_4e_{10}$
	$+3e_2^2e_6e_{10} - 2e_1e_6^2e_7 + e_1e_5e_7^2 + 3e_2e_4e_{14} + e_2e_3^2e_5e_7 + e_3^4e_8$
	$-2e_1e_3e_8^2 - e_3^3e_{11} - 4e_1^2e_4e_6e_8 + 4e_2e_4e_6e_8 + 5e_1e_5e_6e_8 - 4e_3^2e_6e_8$
	$-4e_3e_5^2e_7 - 4e_2e_3^2e_4e_8 + 4e_1e_3e_4^2e_8 - e_2^3e_3e_{11} + e_2^2e_3^2e_{10}$
	$+e_1e_3e_5^2e_6 + e_2^2e_4e_6^2 + 2e_1e_3e_{16} - e_7e_{13} + e_1^2e_8e_{10} - 2e_2e_8e_{10}$
	$-3e_1e_9e_{10} + e_3^2e_7^2 + 3e_3e_5e_6^2 - e_6e_{14} + e_1^2e_4e_{14} - 2e_1e_2e_3e_5e_9$
	$+5e_1^2e_4e_5e_9 + 2e_2e_4e_5e_9 + 3e_1e_2^2e_6e_9 - 3e_1^2e_3e_6e_9 - 3e_2e_3e_6e_9$
	$+2e_1e_4e_6e_9 - e_2^3e_{14} + 4e_5e_{15} + 2e_4^2e_6^2 + 2e_1^2e_6^3 - 2e_2e_6^3 - 2e_4e_5e_{11}$
	$+4e_1e_4^2e_{11} + 2e_2^2e_5e_{11} + 4e_1^3e_6e_{11} + 2e_3e_6e_{11} - 4e_1^2e_7e_{11} + 2e_2e_7e_{11}$
	$+2e_{10}^2 + e_1^2e_{18} + e_{20} - 3e_1e_2e_4e_5e_8 - 2e_1e_3e_4e_6^2 + e_1^2e_2e_7e_9$
	$+e_1e_3e_7e_9 - e_2e_3^3e_9 + 2e_1e_2e_3^2e_{11} + 3e_2^2e_3e_4e_9$

6 $e_4e_{10}^2 - e_2^2e_{10}^2 + 4e_1e_5e_{18} + 3e_1e_{11}e_{12} - 2e_5e_9e_{10} + 3e_2e_{10}e_{12} + 3e_2e_4e_{18}$
$-4e_1^3e_2e_{19} + e_3e_{21} - 2e_1e_2e_{21} - 2e_1e_3e_{20} + 3e_1^2e_2e_{20} - 3e_1e_4e_{19}$
$-2e_2e_3e_{19} + 3e_1^2e_3e_{19} + 3e_1e_2^2e_{19} + e_2^4e_3e_{13} - 5e_1e_3e_4e_7e_9$
$-3e_1e_3e_4e_8^2 - 2e_3^3e_6e_9 + 8e_3e_6^2e_9 + 4e_1e_2e_3e_9^2 - 3e_1e_2^3e_6e_{11}$
$+2e_2e_3^3e_{13} + 2e_2^2e_4e_5e_{11} + e_1^2e_3^3e_{13} + 3e_2^2e_7e_{13} - e_2e_3^4e_{10}$
$+e_2^4e_7e_9 - 3e_1e_2e_3e_5^2e_8 + 3e_2^3e_5^2e_8 - 2e_2e_3e_9e_{10} + 3e_1^2e_2^2e_5e_{13}$
$-3e_1^3e_3e_5e_{13} + 8e_1e_2e_8e_{13} - e_3^2e_5^2e_8 - e_7^2e_{10} + e_2^2e_3^3e_{11}$
$-e_1e_2e_3e_4e_{14} + e_2e_{22} + 3e_1e_4e_9e_{10} + 2e_8e_{16} - 7e_1e_2e_3e_4^2e_{10}$
$-3e_7e_8e_9 - e_1e_3^4e_{11} + 10e_3^2e_4e_5e_9 + 2e_1^2e_4^2e_5e_9 + 4e_2e_4^2e_5e_9$
$-2e_2e_9e_{13} - e_1e_2^2e_5^2e_9 - 3e_2^3e_4e_5e_9 + 6e_1e_2e_3^2e_6e_9 + 7e_1e_2e_7e_{14}$
$+2e_1^3e_2e_5e_{14} - 6e_1e_2^2e_5e_{14} + 3e_2^3e_3e_7e_8 + e_2^3e_6^3 + 3e_3^2e_6^3$
$-2e_4^2e_{16} - 14e_1e_2e_3e_6e_{12} - 3e_1^3e_4^2e_{13} + e_1e_5^3e_8 + 5e_3e_4^2e_{13}$
$+7e_1^2e_2e_8e_{12} - 9e_1e_3e_8e_{12} - 7e_1e_2e_9e_{12} - e_1^2e_2e_{10}^2 + 3e_1e_2^3e_8e_9$
$+4e_5e_6e_{13} + e_{11}e_{13} + 5e_4^2e_7e_9 - 2e_1^2e_2e_4e_8^2 + e_2^4e_5e_{11} + e_1^2e_2e_9e_{11}$
$-2e_1e_3e_9e_{11} + e_1e_2e_{10}e_{11} - 8e_1^2e_3e_5e_7^2 + e_5e_{19} - e_1^2e_{22} + e_1e_3^3e_4e_{10}$
$-e_2^5e_{14} + 3e_1e_3e_{10}^2 + 4e_1e_7^2e_9 - 2e_1^2e_4e_9^2 + e_2e_4e_9^2 - e_3^2e_9^2$
$-7e_2e_3e_6e_{13} - 4e_1e_4e_6e_{13} + 4e_1^2e_2^2e_7e_{11} - 4e_1^2e_2e_6e_7^2$
$-2e_1e_2^2e_3e_6e_{10} + 5e_5e_6^2e_7 - 5e_1e_2^2e_5e_6e_8 - 9e_6^2e_{12} + 4e_1^2e_3e_6^2e_7$
$-5e_2e_3e_6^2e_7 + 11e_1e_5e_6e_{12} - 5e_1^3e_2e_7e_{12} + 7e_1e_2^2e_7e_{12} + 8e_1^2e_3e_7e_{12}$
$-7e_2e_3e_7e_{12} - 4e_1e_4e_7e_{12} + e_3^2e_5e_{13} - 2e_2e_3^2e_8^2 + 4e_2^2e_4e_8^2$
$-e_1^2e_3^2e_8^2 + 2e_1e_2^2e_4e_{15} - 4e_1^2e_3e_4e_{15} - 3e_2e_3e_4e_{15} + e_2e_3^3e_5e_8$
$+e_1e_{23} + 5e_1e_2e_5^2e_{11} + 2e_1^2e_3e_8e_{11} + 8e_2e_3e_8e_{11} + 2e_1e_4e_8e_{11}$
$-3e_1e_5^2e_{13} + e_3^3e_4e_{11} - e_3e_5e_7e_9 + 4e_1^2e_6e_7e_9 + 4e_6e_7e_{11}$
$+2e_1^2e_3^2e_7e_9 + 6e_1e_3e_4e_5e_{11} - 2e_1e_3^2e_4e_6e_7 + e_2^3e_4e_7^2 - 2e_3^2e_4e_7^2$
$+3e_1^2e_4^2e_7^2 - 3e_2e_4^2e_7^2 - 2e_1e_2e_3e_5e_{13} - 4e_3^2e_4e_6e_8 - 3e_1^2e_4^2e_6e_8$
$-2e_1^2e_4e_5^2e_8 - 4e_1e_2^2e_3e_7e_9 + e_1e_2^2e_3e_5e_{11} - 2e_2^4e_8^2$
$+4e_1^2e_2e_5e_7e_8 + 2e_1^2e_3^2e_5e_{11} - 8e_2e_3^2e_5e_{11} - 4e_3e_4e_6e_{11}$
$-5e_1^2e_5e_6e_{11} - 7e_2e_5e_6e_{11} + 4e_1e_2e_3e_4e_6e_8 - e_3e_4e_5^2e_7$
$+e_1e_3^2e_5^2e_7 + 5e_1e_6^2e_{11} + e_3^3e_7e_8 + e_3e_5^3e_6 + e_2^4e_{16} - e_1^2e_2^2e_9^2$
$+e_1^3e_3e_9^2 - 3e_2^3e_3e_6e_9 + 11e_1e_2e_6e_7e_8 - 3e_2^2e_4e_7e_9 - 2e_2^3e_7e_{11}$
$+3e_1^2e_2e_4e_7e_9 - e_2^2e_5^2e_{10} + 9e_4e_5^2e_{10} - 3e_1^2e_5^2e_{12} + 6e_1^4e_7e_{13}$
$-2e_2e_5^2e_{12} + e_3^2e_4e_{14} + 2e_1^2e_4^2e_{14} - e_2e_4^2e_{14} + 2e_1^3e_6e_{15} + 8e_1e_2^2e_6e_{13}$
$+6e_1^2e_3e_6e_{13} - 3e_3e_8e_{13} - e_1^2e_2e_3e_5e_{12} - 5e_1^3e_3e_7e_{11} + 2e_1^2e_4e_7e_{11}$
$-3e_1e_2^2e_3e_4e_{12} + 2e_1e_2e_3^2e_4e_{11} - 3e_2^3e_6e_{12} + 7e_3^2e_6e_{12} - 7e_2e_6e_7e_9$
$-5e_1^3e_8e_{13} - 3e_4e_7e_{13} + 4e_1e_2^3e_3e_{14} - 3e_1^2e_2e_3^2e_{14} + 7e_2^2e_3e_5e_{12}$
$+2e_1e_3^2e_5e_{12} + e_1^3e_4e_5e_{12} - 12e_3e_4e_5e_{12} - e_5^2e_7^2 + 7e_1^2e_2e_3e_7e_{10}$
$+4e_2e_4e_5e_6e_7 - 2e_1^2e_2e_5e_6e_9 - 2e_1e_{10}e_{13} + 5e_1e_2^2e_3e_8^2$
$-5e_1e_2^2e_4e_7e_8 - 2e_1^2e_3^2e_4e_{12} + 2e_2e_3^2e_4e_{12} + 2e_1^2e_2e_4^2e_{12}$
$+5e_1e_3e_4^2e_{12} + 4e_2^2e_3^2e_5e_9 - 5e_1e_3^3e_5e_9 + 3e_2^3e_4e_{14} - 2e_1e_2^3e_5e_{12}$
$-2e_{12}^2 - e_1^2e_5^2e_6^2 - e_2e_5^2e_6^2 - 4e_1^2e_7^2e_8 + 4e_2e_7^2e_8 + e_1^2e_3e_4e_5e_{10}$
$-14e_1^2e_2e_7e_{13} + 7e_1e_3e_7e_{13} - 7e_2e_3e_4e_6e_9 + 2e_3^2e_{18} - e_1^4e_{20}$
$-6e_1e_2e_4e_6e_{11} - 2e_2^3e_4^2e_{10} - 9e_1e_3^2e_4e_{13} - e_2^3e_9^2 + 7e_1e_4e_5e_7^2$

$$
\begin{aligned}
&+2e_1e_2e_3^2e_5e_{10}-5e_1^3e_2e_6e_{13}-3e_2^2e_6^2e_8+4e_4e_6^2e_8-e_1^2e_3e_5e_{14}\\
&+7e_2e_3e_5e_{14}-3e_1e_4e_5e_{14}-5e_2e_3^3e_4e_9+e_1^3e_5^2e_{11}+4e_3e_5^2e_{11}\\
&+2e_1e_2e_3^3e_{12}-e_1^3e_5e_7e_9+e_1e_2e_4e_5e_6^2+2e_1e_2e_4e_{17}-e_3^2e_4^2e_{10}\\
&+3e_1^2e_4^3e_{10}+6e_2e_4^3e_{10}-4e_2e_4e_5^2e_8+4e_1^2e_2^2e_6e_{12}+2e_1^3e_3e_6e_{12}\\
&-e_1e_2e_4^2e_{13}-9e_1^2e_4e_6e_{12}+15e_2e_4e_6e_{12}-12e_1e_4^2e_5e_{10}\\
&-2e_1^2e_2e_5^2e_{10}-2e_1e_3e_5^2e_{10}+7e_1^2e_2e_6e_{14}-6e_1e_3e_6e_{14}+e_1e_2^4e_{15}\\
&-6e_1e_2e_3^2e_7e_8+4e_2^2e_3^2e_4e_{10}+3e_1e_2e_4e_8e_9-3e_1^2e_3e_4^2e_{11}\\
&-e_3^4e_{12}-2e_1e_9e_{14}+e_1^3e_6^2e_9+e_1^3e_2^2e_{17}-2e_1e_2^3e_{17}-e_1^4e_3e_{17}\\
&+3e_2^2e_3e_{17}+e_1^3e_4e_{17}-3e_3e_4e_{17}-3e_1^2e_5e_{17}-4e_1^2e_2e_4e_6e_{10}\\
&+e_{10}e_{14}-3e_2^2e_3e_4e_5e_8-6e_1e_2e_3e_{18}+4e_2^2e_6e_7^2-2e_4e_6e_7^2\\
&-5e_1e_2e_7^3+e_8^3-2e_6^4-e_2^2e_3^2e_6e_8-6e_1e_5^2e_6e_7-2e_2^3e_3e_{15}\\
&-e_2e_3e_4^2e_{11}+4e_2^3e_8e_{10}+6e_1e_2e_3e_5e_6e_7+2e_4e_{20}+4e_1e_2^2e_5e_7^2\\
&+2e_6e_9^2+10e_1e_4^2e_6e_9+3e_5e_7e_{12}-e_1^4e_8e_{12}-3e_2^2e_8e_{12}+4e_4e_8e_{12}\\
&+2e_1^3e_9e_{12}-3e_1^2e_3^2e_{16}+2e_2^4e_4e_{12}-2e_2^3e_5e_{13}-e_1^2e_{11}^2\\
&+8e_2e_3e_4e_7e_8-e_1e_3^2e_4e_5e_8-2e_3^3e_{15}+8e_1e_4e_5e_6e_8-3e_1^2e_2^2e_3e_{15}\\
&+6e_1e_2e_3^2e_{15}+e_1^3e_2e_4e_{15}-2e_1^2e_3^2e_6e_{10}+6e_1e_3e_5e_7e_8\\
&+5e_2^2e_3e_4^2e_9+5e_1e_3^2e_4^2e_9-5e_1e_2e_4^3e_9+3e_1^3e_7^3+e_3e_7^3-e_{24}\\
&+e_2^2e_3e_4e_6e_7-3e_3e_4e_5e_6^2+e_5^3e_9-e_1^4e_6e_{14}-e_2^2e_{20}\\
&-2e_1e_2e_3e_4e_7^2-4e_3e_6e_7e_8-3e_1e_2e_3e_6^3+5e_1e_2e_4^2e_5e_8+e_9e_{15}\\
&-4e_1e_2e_5e_8^2+8e_2^2e_3e_6e_{11}-e_1e_3^2e_6e_{11}+6e_1^3e_4e_6e_{11}+4e_4^2e_8^2\\
&+4e_2e_3^2e_7e_9+4e_1^2e_2e_6^2e_8+e_1^3e_3^2e_{15}-3e_1e_2^2e_3^2e_{13}+5e_1e_2^2e_4^2e_{11}\\
&+8e_2^2e_5e_6e_9+11e_2e_3e_5e_6e_8-10e_1e_4^2e_7e_8-5e_1e_4e_6^2e_7\\
&-e_1e_2e_3e_8e_{10}+e_1^2e_2e_3e_6e_{11}-3e_1e_2^3e_7e_{10}+e_2^2e_3e_7e_{10}\\
&-7e_1e_3^2e_7e_{10}-3e_1^3e_4e_7e_{10}+e_1^3e_{21}-3e_2e_4e_5e_{13}+8e_1^2e_4e_5e_{13}\\
&-4e_2^2e_6e_{14}+3e_4e_6e_{14}-5e_1^3e_7e_{14}-2e_3e_7e_{14}+4e_1^2e_8e_{14}-3e_2e_8e_{14}\\
&-5e_3e_4^3e_9+4e_1e_2e_4e_5e_{12}-8e_2^2e_5e_7e_8-3e_4e_5e_7e_8-5e_1^3e_6e_7e_8\\
&+e_3e_4^2e_5e_8-3e_1^2e_6e_{16}-8e_1e_3e_6^2e_8-9e_4e_5e_6e_9+2e_2e_4e_7e_{11}\\
&-4e_1e_5e_7e_{11}+2e_1^3e_2e_8e_{11}-8e_1e_2^2e_8e_{11}-9e_1^2e_2e_4e_5e_{11}\\
&-3e_1e_2e_3e_4e_5e_9+e_1^3e_4e_8e_9-4e_3e_4e_8e_9-3e_1^2e_5e_8e_9+8e_2e_5e_8e_9\\
&-4e_1e_6e_8e_9-2e_1^2e_2^2e_4e_{14}+e_1^3e_3e_4e_{14}+e_7e_{17}-e_1e_2e_5e_7e_9\\
&-5e_1e_2e_6^2e_9+e_1^5e_{19}-3e_2^3e_5e_6e_7-3e_3^2e_5e_6e_7-3e_2^3e_3e_5e_{10}\\
&-4e_1e_3e_5e_6e_9+6e_1^2e_3e_5^2e_9-6e_2e_3e_5^2e_9-3e_1e_4e_5^2e_9+e_1^2e_4e_6^3\\
&+3e_1e_5e_6^3+2e_1e_3^3e_{14}-2e_1e_2e_4e_5^2e_7-3e_2^3e_3e_4e_{11}-3e_5e_8e_{11}\\
&-e_1^4e_9e_{11}+3e_2^2e_9e_{11}+e_1^3e_{10}e_{11}-2e_3e_{10}e_{11}-e_2e_{11}^2+5e_3e_4^2e_6e_7\\
&+e_3^5e_9+8e_1e_2^2e_4e_5e_{10}+e_3^3e_5e_{10}-4e_2^2e_4e_6e_{10}+e_1^3e_5e_6e_{10}\\
&-5e_3e_5e_6e_{10}-6e_2^2e_4^2e_{12}-6e_4^2e_6e_{10}-2e_1^2e_6^2e_{10}+4e_2e_6^2e_{10}\\
&+2e_2^2e_3e_4e_{13}+16e_1e_3e_4e_6e_{10}+3e_1^2e_9e_{13}-e_1^4e_5e_{15}+3e_2^2e_5e_{15}\\
&+2e_4e_5e_{15}+4e_3e_6e_{15}+4e_1^2e_7e_{15}-2e_2e_7e_{15}-3e_1e_2^3e_4e_{13}+2e_2^4e_6e_{10}\\
&+2e_1^2e_5^3e_7+3e_3e_4e_7e_{10}+3e_1^2e_5e_7e_{10}+e_2e_5e_7e_{10}-5e_1e_6e_7e_{10}\\
&-2e_2e_4^2e_6e_8-9e_1^2e_3e_4e_6e_9-e_2^3e_3^2e_{12}+3e_6e_8e_{10}-e_2^2e_3^2e_7^2\\
&+2e_1e_3^3e_7^2+4e_1^2e_6e_8^2-4e_2e_6e_8^2+e_1^3e_5e_8^2-5e_2^2e_3e_8e_9\\
&+5e_1e_3^2e_8e_9+e_1e_3e_5e_{15}-7e_1e_2e_6e_{15}-2e_1^2e_4e_5e_6e_7-e_1^2e_2^3e_{16}
\end{aligned}
$$

$$
\begin{aligned}
&-3e_1e_8e_{15} - e_1e_2e_4^2e_6e_7 + 4e_1e_3e_6e_7^2 + e_3^2e_8e_{10} + 8e_1^2e_2e_3e_4e_{13} \\
&-5e_1^2e_2e_3e_8e_9 + e_1^3e_2e_9e_{10} + e_1e_2^2e_9e_{10} - 4e_1^2e_3e_9e_{10} + 6e_3e_9e_{12} \\
&-2e_1^2e_{10}e_{12} - e_5^2e_{14} + 3e_1e_2e_5e_6e_{10} - 3e_2^2e_3^2e_{14} - 4e_2e_3e_4e_5e_{10} \\
&-2e_1^2e_2^2e_8e_{10} + e_1^3e_3e_8e_{10} + e_1^2e_4e_8e_{10} - 9e_2e_4e_8e_{10} + 2e_1e_5e_8e_{10} \\
&+e_2e_5^3e_7 - 5e_6e_{18} - e_2^3e_{18} + e_1e_3^3e_6e_8 + 3e_1e_2e_4e_7e_{10} - 3e_4e_9e_{11} \\
&+2e_1^3e_2e_3e_{16} - 3e_1^2e_2e_4e_{16} + 8e_1e_3e_4e_{16} + 7e_1^2e_3e_4e_7e_8 \\
&-e_1e_2e_3e_7e_{11} + 6e_1e_2^2e_4e_6e_9 - 2e_2^2e_4e_{16} + 2e_1^3e_5e_{16} - 3e_3e_5e_{16} \\
&+5e_2e_6e_{16} - 3e_1e_7e_{16} - 2e_2e_5e_{17} + 4e_1e_6e_{17} - e_1^4e_2e_{18} + 3e_1^2e_2^2e_{18} \\
&+2e_1^3e_3e_{18} - 2e_1^2e_4e_{18} - 2e_5^2e_6e_8
\end{aligned}
$$

Table of $P_{m,5}$ for $m > 3$

m	$P_{m,5}$
4	$-e_1e_6^2e_7 + 4e_5e_6e_9 + 3e_2^2e_6e_{10} + 4e_1e_4e_5e_{10} - 2e_1e_3e_6e_{10} - e_7e_{13}$
	$+2e_3^2e_5e_9 - 4e_1e_5^2e_9 + e_4^3e_8 - e_1e_{19} - e_2e_8e_{10} - e_6^2e_8 + e_3e_4e_6e_7$
	$+3e_4^2e_{12} - e_3^2e_7^2 + e_1e_2e_5e_{12} + e_6e_7^2 - 2e_1e_7e_{12} + 2e_3e_6e_{11} - 3e_5e_7e_8$
	$+e_1^2e_4e_{14} - 2e_2e_4e_{14} + e_1e_5e_{14} - e_1e_2^2e_{15} + e_1^2e_3e_{15} + 2e_2e_3e_{15}$
	$-2e_1e_4e_{15} + e_1^2e_2e_{16} - 2e_1e_3e_{16} + 2e_1e_2e_{17} + e_2e_4e_5e_9 - 3e_2e_5e_6e_7$
	$+e_2e_3^2e_{12} - e_2^2e_4e_{12} + e_1^2e_6e_{12} - 2e_2e_6e_{12} - e_1e_4^2e_{11} + e_3^2e_4e_{10}$
	$-e_2e_4^2e_{10} + 3e_8e_{12} + 2e_2e_3e_4e_{11} + e_1e_3e_5e_{11} - 2e_1e_2e_6e_{11} - 3e_3e_4e_5e_8$
	$-e_1^2e_9^2 + 2e_1e_2e_4e_{13} - 2e_1e_2e_3e_{14} + 3e_2e_5^2e_8 + e_3^2e_6e_8$
	$+e_2e_3e_7e_8 - 2e_1e_4e_7e_8 - 3e_2e_3e_5e_{10} - e_6e_{14} - 2e_2e_3e_6e_9 + e_1e_4e_6e_9$
	$-e_2^2e_{16} + e_2e_4e_7^2 - 2e_1e_3e_4e_{12} - e_3^3e_{11} + 2e_1e_5e_7^2 + 3e_4e_{16} - 3e_4e_5e_{11}$
	$+e_1^2e_7e_{11} + 2e_2e_7e_{11} - 2e_1e_8e_{11} + e_1e_9e_{10} - e_3e_4^2e_9 - e_4e_6e_{10}$
	$+e_3e_7e_{10} + e_1^2e_8e_{10} + e_3^2e_{14} - e_2^2e_3e_{13} + e_1e_3^2e_{13} - 2e_3e_4e_{13}$
	$-2e_1^2e_5e_{13} + e_2e_5e_{13} + 2e_1e_6e_{13} + e_2^3e_{14} + e_1e_5e_6e_8 + e_2e_6^3 - e_1^3e_{17}$
	$-e_2^2e_7e_9 - 2e_4e_7e_9 + e_2e_9^2 + e_{20} + 3e_1e_3e_7e_9 + e_1e_2e_8e_9 - e_9e_{11}$
	$-e_1e_3e_8^2 - 2e_3e_8e_9 - e_3e_{17} + 3e_4e_8^2 - 2e_1e_2e_7e_{10} - e_2e_4e_6e_8$
	$+e_1^2e_{18} - e_2e_{18}$
5	$-3e_3e_4e_9^2 + e_1^2e_5e_9^2 + 5e_5e_{10}^2 - e_1^3e_4e_{18} + 4e_4e_6^2e_9 - 2e_1e_2e_4e_9^2$
	$+e_2^2e_6^2e_9 - e_2^3e_3e_{16} - 2e_2e_3e_4e_6e_{10} + 3e_2e_4^2e_7e_8 + e_2e_4^2e_5e_{10}$
	$-3e_2e_3e_{20} - 3e_1e_4e_{20} - 4e_2e_3e_4e_8^2 + 5e_1e_3e_6e_7e_8 + 4e_1e_4^2e_6e_{10} - e_1e_8^3$
	$+2e_1^2e_2e_3e_{18} - 5e_5^2e_7e_8 + e_2^2e_4^2e_{13} + 5e_1e_2e_6e_7e_9 + e_1e_2e_6e_8^2$
	$-e_1^3e_2e_{20} + e_2^2e_3^2e_{15} - e_1e_3^3e_{15} + e_3^2e_4^2e_{11} - e_2e_4^3e_{11} - 2e_1^2e_5e_7e_{11}$
	$+4e_1e_3e_6e_{15} + 4e_1e_2e_7e_{15} - e_2e_3^3e_{14} + 3e_2e_4e_9e_{10} - 5e_1e_5e_9e_{10}$
	$+3e_3e_6^2e_{10} - e_1e_3e_4e_{17} + e_2^4e_{17} - e_4^3e_{13} - 4e_1e_3e_6^2e_9 - e_2e_{23}$
	$-4e_5e_6e_{14} + 6e_2e_5e_7e_{11} + 4e_1e_3e_5^2e_{11} - 2e_1^2e_4e_6e_{13} - 3e_2e_3^2e_4e_{13}$
	$+2e_1e_7e_{17} + 2e_1e_4^2e_{16} + 6e_3e_5e_6e_{11} + 2e_1^2e_4e_7e_{12} - 6e_2e_4e_7e_{12}$
	$+2e_1e_5e_7e_{12} + 3e_1^2e_4e_5e_{14} + 5e_2^2e_5e_6e_{10} - 6e_1e_6e_7e_{11} + e_8^2e_9$
	$+2e_2^2e_3e_4e_{14} + 2e_6e_8e_{11} + 4e_1^3e_6e_{16} + 2e_1e_2e_5e_7e_{10} - 3e_1^2e_2e_{21}$
	$+e_4^4e_9 + e_1^2e_3e_7e_{13} - e_1^3e_{10}e_{12} + e_1^2e_3e_{10}^2 - 4e_2e_3e_{10}^2 + e_1e_2^2e_{10}^2$
	$+4e_1e_2e_5^2e_{12} - 5e_4e_5^2e_{11} + 3e_3e_7^2e_8 + 2e_3e_6e_{16} + 2e_1e_2e_3^2e_{16}$
	$-5e_1e_2e_3e_7e_{12} + 2e_2e_7e_{16} - 3e_1^2e_7e_{16} + 3e_3^2e_5e_{14} - 3e_2e_3e_5e_6e_9$
	$-9e_4e_5e_6e_{10} - 3e_3e_4^2e_{14} - 4e_1e_2^2e_8e_{12} + 2e_1^2e_3e_8e_{12} + 2e_1e_3e_4^2e_{13}$
	$+2e_2e_4e_5e_{14} + 3e_1e_6e_8e_{10} + 2e_1e_8e_{16} + e_1^2e_{23} + e_2^2e_8e_{13} - e_{11}e_{14}$
	$+e_2^2e_3e_9^2 - e_6^3e_7 + 3e_3^2e_4e_6e_9 + 6e_1e_5e_6e_{13} - 3e_3e_6e_7e_9 - 3e_1e_3e_9e_{12}$
	$+3e_1e_2e_{10}e_{12} - 3e_1e_2e_6^2e_{10} + e_3^4e_{13} - 3e_4^2e_8e_9 - 2e_2e_3^2e_8e_9$
	$+3e_1^2e_5e_8e_{10} - 6e_2e_5e_8e_{10} + 3e_1e_4e_6e_7^2 + 2e_1e_2^2e_4e_{16} - 4e_2^2e_6e_7e_8$
	$-5e_3e_5^2e_{12} + 2e_3^2e_4e_{15} - 4e_1e_{10}e_{14} + 2e_3e_8e_{14} + 3e_1^2e_9e_{14} + 2e_2e_9e_{14}$
	$-4e_2e_4^2e_6e_9 - 2e_3^3e_5e_{11} + 2e_2e_4e_{19} - 3e_1^2e_3e_4e_{16} - 6e_1e_3e_8e_{13}$
	$-4e_5e_9e_{11} + 4e_2^3e_7e_{12} - 3e_1e_4e_5e_7e_8 - 2e_4e_6e_7e_8 - 3e_2^2e_3e_6e_{12}$
	$+2e_1e_3^2e_6e_{12} - e_2^3e_{19} + 6e_2e_3e_8e_{12} - e_1e_4e_8e_{12} + 3e_1^2e_2e_9e_{12}$

$-4e_1e_3e_5e_8^2+6e_1^2e_6^2e_{11}-3e_2e_6^2e_{11}+e_4^2e_{17}-2e_1e_2e_8e_{14}-e_2e_3e_4e_{16}$
$+6e_1e_2e_5e_{17}+2e_3e_5e_8e_9+e_1e_2e_4e_5e_{13}+2e_1e_3e_{21}+3e_2^2e_{10}e_{11}$
$-3e_1e_2e_5e_8e_9-3e_3e_6e_8^2+e_1^2e_7e_8^2+e_3e_4e_6e_{12}-2e_1^2e_5e_6e_{12}$
$+2e_2e_5e_6e_{12}+e_3^2e_5e_7^2+2e_1e_5^2e_7^2-2e_1e_2e_7^2e_8-5e_2e_3e_5^2e_{10}$
$-e_1e_4e_5^2e_{10}+2e_1e_{11}e_{13}+e_7e_9^2+e_1e_4^2e_5e_{11}-3e_2^2e_7e_{14}-e_1^3e_8e_{14}$
$-e_1e_4^3e_{12}-2e_3^3e_7e_9+5e_1e_3e_4e_8e_9+2e_1^3e_{11}^2-e_6e_9e_{10}-4e_4e_{10}e_{11}$
$+e_3e_{11}^2-3e_1^2e_4e_8e_{11}-4e_3e_7e_{15}+2e_3^2e_5^2e_9-4e_1e_5^3e_9+e_{25}-e_8e_{17}$
$-3e_2^2e_9e_{12}+2e_4e_9e_{12}-3e_1^2e_{11}e_{12}+3e_1e_2e_4e_6e_{12}-e_2e_4e_5e_6e_8$
$-3e_1e_3e_4e_7e_{10}-3e_1e_3^2e_8e_{10}-e_1e_{24}-2e_3^2e_4e_7e_8-2e_2e_4e_8e_{11}$
$-4e_2e_{10}e_{13}+5e_2e_3e_7e_{13}-e_1e_4e_7e_{13}-e_1e_5^2e_{14}-2e_1^2e_2e_5e_{16}$
$+2e_1e_5e_8e_{11}+3e_3^2e_4e_5e_{10}-4e_3e_4^2e_5e_9+2e_1e_3e_5e_{16}+2e_1e_3^2e_4e_{14}$
$-2e_1^2e_5^2e_{13}+e_6^2e_{13}+e_3^2e_{19}-e_9e_{16}+5e_4e_5e_8^2-e_1e_2e_3e_{19}$
$+3e_1^2e_2e_8e_{13}-4e_7e_8e_{10}+e_1^4e_{21}-e_1e_2e_3e_4e_{15}-7e_1e_3e_4e_6e_{11}$
$+5e_4^2e_5e_{12}-e_5^2e_6e_9-3e_1e_2e_3e_5e_{14}-e_2e_5^2e_{13}-4e_2^2e_3e_7e_{11}$
$-6e_1e_2e_6e_{16}+e_1e_2^2e_9e_{11}-3e_1e_2e_4^2e_{14}+2e_2^2e_6e_{15}-2e_4e_6e_{15}$
$-e_1^3e_7e_{15}+2e_1^2e_8e_{15}-2e_2e_8e_{15}-4e_1e_9e_{15}-2e_1^2e_3e_9e_{11}$
$+4e_1e_3e_5e_7e_9-e_2e_3e_9e_{11}-e_1e_2e_4e_{18}+3e_2e_3^2e_5e_{12}+e_1^2e_3^2e_{17}$
$-2e_2e_3e_6e_7^2-4e_2^2e_4e_5e_{12}-e_{12}e_{13}+e_2^2e_3e_5e_{13}+e_2e_4e_6^2e_7$
$+3e_1e_4e_6e_{14}+2e_1^2e_2e_7e_{14}+e_4^2e_6e_{11}-e_7e_{18}+5e_2e_3e_4e_7e_9$
$+e_3e_4^2e_6e_8+3e_3^2e_9e_{10}+4e_2e_4e_5^2e_9+5e_1e_4e_9e_{11}+2e_2e_3^2e_6e_{11}$
$-e_3^3e_6e_{10}+e_1e_2e_4e_8e_{10}+2e_1e_2e_{22}+2e_1e_2^2e_6e_{14}-3e_1^2e_3e_6e_{14}$
$-e_2e_3e_6e_{14}+3e_1^2e_{10}e_{13}+2e_2e_3^2e_{17}-2e_1e_3^2e_5e_{13}+2e_2e_3e_5e_7e_8$
$+e_7^2e_{11}+3e_4^2e_7e_{10}+3e_1e_3^2e_7e_{11}+5e_5^3e_{10}-3e_1^2e_2e_{10}e_{11}-e_3e_{22}$
$+3e_1e_2e_3e_8e_{11}-e_6e_{19}+4e_1e_2e_3e_6e_{13}-3e_1e_6^2e_{12}+e_1e_2^2e_5e_{15}$
$+3e_1^2e_3e_5e_{15}+3e_2e_3^2e_7e_{10}+3e_1e_7^2e_{10}+4e_1e_2e_4e_7e_{11}+2e_3^3e_8^2$
$+2e_2e_{11}e_{12}-2e_1^3e_5e_{17}+e_1e_{12}^2-e_2^3e_6e_{13}-8e_1e_2e_5e_6e_{11}+5e_{10}e_{15}$
$+2e_2e_3e_4^2e_{12}+2e_3e_4e_5e_{13}+3e_1e_3e_{10}e_{11}-4e_3e_5e_{17}+e_2e_7e_8^2$
$+e_2^2e_3e_8e_{10}+e_5e_6^2e_8+2e_4e_8e_{13}-3e_3^2e_8e_{11}+5e_5e_{20}+e_2^2e_4e_8e_9$
$-3e_2e_7^2e_9-e_3e_{10}e_{12}-3e_2^2e_4e_7e_{10}+5e_5^2e_{15}-3e_1^2e_6e_{17}-3e_1e_2e_{11}^2$
$-3e_3^2e_6e_{13}-e_1^3e_{22}+2e_2e_6e_{17}+3e_2^2e_4e_6e_{11}+e_1^2e_4^2e_{15}+2e_2e_4^2e_{15}$
$-e_4e_{21}-2e_1^2e_6e_8e_9+e_2e_6e_8e_9+2e_3e_9e_{13}+3e_2e_5e_9^2-3e_1e_2^2e_7e_{13}$
$+e_1e_4^2e_8^2-e_1e_2^3e_{18}+3e_2^2e_5e_8^2-e_1e_4e_{10}^2+2e_6e_7e_{12}-e_3e_4^3e_{10}$
$+3e_3e_4e_8e_{10}+3e_2e_3e_6^2e_8-4e_1e_4e_6^2e_8-e_3^3e_{16}-4e_1e_4^2e_7e_9$
$-e_2^3e_4e_{15}-2e_3^2e_5e_6e_8+4e_1e_5^2e_6e_8+2e_4e_7e_{14}+5e_1e_4e_5e_6e_9$
$+2e_2^2e_3e_{18}+e_2e_3e_4e_5e_{11}-3e_1e_3^2e_{18}-4e_1e_5e_6^2e_7+2e_1e_3e_5e_6e_{10}$
$+e_1e_6^4+2e_1^2e_2e_4e_{17}-3e_2^2e_4e_{17}+e_2^2e_5e_{16}-5e_3e_5e_7e_{10}-4e_2e_3e_5e_{15}$
$-8e_1e_4e_5e_{15}-3e_1^2e_2e_6e_{15}+2e_3e_4e_{18}+e_2^2e_{21}+2e_2^2e_7^3-e_4e_7^3$
$-2e_2e_4e_6e_{13}+3e_1^2e_5e_{18}-e_3^3e_4e_{12}-4e_5e_7e_{13}+3e_5e_6e_7^2-2e_1e_3e_7^3$
$+2e_1^2e_4e_{19}+2e_1e_2^2e_{20}+2e_1^2e_3e_{20}-2e_1^3e_9e_{13}-e_1^3e_3e_{19}-4e_2e_5e_{18}$
$+e_1^2e_2^2e_{19}-4e_1e_5e_{19}-3e_1e_3e_4e_5e_{12}+3e_2e_6e_7e_{10}+2e_4e_5e_7e_9$
$-3e_1^2e_6e_7e_{10}-3e_1e_2^2e_3e_{17}-4e_2^2e_5e_7e_9+2e_1e_6e_{18}$
$-e_2^3e_9e_{10}-2e_2e_4e_5e_7^2$

Table of $P_{m,6}$ for $m = 4, 5$

m	$P_{m,6}$
4	$2e_1e_3e_8e_{12} + e_2^2e_9e_{11} + e_4e_6e_{14} - e_1e_3^2e_{17} - 2e_2^2e_6e_{14} + e_3^3e_{15}$
	$+2e_5e_8e_{11} + 2e_4e_9e_{11} + e_3e_5e_6e_{10} - e_7^2e_{10} - 3e_1e_5e_6e_{12} - 2e_3e_{10}e_{11}$
	$-2e_3e_7e_{14} - 2e_2e_5e_{17} + e_1e_3e_6e_{14} - e_2e_{11}^2 + 2e_2e_8e_{14} + e_4e_5e_7e_8$
	$-e_1^2e_8e_{14} - 3e_3e_6e_7e_8 + e_3e_5e_8^2 - e_{12}^2 - 3e_1e_6e_{17} - 3e_4e_5e_6e_9$
	$+2e_1e_7e_{16} - 2e_1e_9e_{14} + 2e_1e_2e_3e_{18} + e_1e_7e_8^2 + 2e_3e_4e_{17} + 3e_1^2e_5e_{17}$
	$-e_1^2e_4e_{18} + 2e_2e_6e_8^2 - 2e_2e_3e_8e_{11} + e_2^2e_4e_{16} + 2e_3e_5e_{16} + e_1e_4e_8e_{11}$
	$+2e_1e_2e_7e_{14} - 2e_2e_4e_5e_{13} + 2e_2e_3e_5e_{14} - 2e_1e_4e_5e_{14} - 4e_2e_6^2e_{10}$
	$+e_3e_9e_{12} - e_1^2e_{10}e_{12} - 2e_2e_5e_8e_9 - 3e_1e_6e_8e_9 - 2e_1e_2e_4e_{17}$
	$+e_2^2e_7e_{13} - e_1e_3e_9e_{11} - e_3^2e_4e_{14} + e_2e_4^2e_{14} + 2e_6e_{18} - e_4^3e_{12}$
	$+e_2^2e_{20} + e_1^3e_{21} + e_1e_2e_{10}e_{11} - e_3^2e_8e_{10} - e_4^2e_8^2 - e_1e_7^2e_9 + e_{11}e_{13}$
	$-e_1e_5e_7e_{11} - e_3e_5e_7e_9 + e_2e_6e_7e_9 + 2e_1e_4e_7e_{12} + e_2e_3e_9e_{10}$
	$+3e_6^2e_{12} - 3e_5e_6e_{13} - e_8^3 + 2e_3e_8e_{13} + e_1e_{23} + e_2e_6e_{16} - 2e_1e_2e_9e_{12}$
	$-3e_8e_{16} - e_2^2e_{10}^2 - 2e_5e_9e_{10} + 2e_4e_7e_{13} + 3e_1^2e_9e_{13} - 2e_2e_9e_{13} - 2e_1e_{10}e_{13}$
	$+e_{10}e_{14} - 2e_1e_3e_7e_{13} - e_{24} + e_2e_5e_7e_{10} + 4e_1e_6e_7e_{10} + 2e_2e_4e_{18}$
	$-2e_2e_3e_4e_{15} - e_5^2e_{14} + e_1e_2^2e_{19} - e_1^2e_3e_{19} - 2e_2e_3e_{19} - e_1^2e_2e_{20}$
	$-2e_1e_2e_8e_{13} + e_3e_7^3 - 2e_1e_2e_{21} - 2e_1e_3e_5e_{15} - 2e_2e_3e_7e_{12} - 2e_1e_4e_9e_{10}$
	$+e_3e_4^2e_{13} + e_1e_2e_6e_{15} + e_1e_4^2e_{15} - 3e_4^2e_{16} - e_1^2e_{22} + e_4^2e_5e_{11}$
	$-e_3e_5^2e_{11} + e_7e_{17} + 2e_3e_4e_5e_{12} + e_5e_{19} + e_5^3e_9 + 3e_3e_6^2e_9 + e_5e_7e_{12}$
	$+e_2^2e_5e_{15} - 2e_2e_4e_7e_{11} + 2e_7e_8e_9 - e_2e_3^2e_{16} + e_2e_3e_6e_{13} - e_2^3e_{18}$
	$-3e_4e_{20} - e_4e_5^2e_{10} + 2e_4e_5e_{15} + e_2^2e_3e_{17} - e_3^2e_{18} + e_2^2e_8e_{12} - 5e_4e_8e_{12}$
	$-e_1^2e_7e_{15} - e_2e_4e_9^2 + 3e_1e_5e_9^2 - 2e_3e_4e_7e_{10} + e_4e_{10}^2 + 2e_4^2e_6e_{10}$
	$-2e_2e_7e_{15} + 2e_1e_8e_{15} + e_1e_3e_{10}^2 + e_3e_4e_8e_9 + e_4^2e_7e_9 - e_2e_7^2e_8$
	$+2e_1e_3e_4e_{16} + e_2e_{22} - e_3^2e_5e_{13} + 3e_1e_5^2e_{13} - 3e_3e_4e_6e_{11} + 4e_2e_5e_6e_{11}$
	$+e_3e_{21} - 2e_1e_5e_{18} + e_2e_4e_6e_{12} - 2e_1e_2e_5e_{16} + e_1e_{11}e_{12} + 3e_2e_{10}e_{12}$
	$-e_2e_5^2e_{12} + e_6e_8e_{10} + e_9e_{15} + 3e_3^2e_7e_{11} - 3e_6e_7e_{11} + 3e_2e_4e_8e_{10}$
	$-2e_1e_5e_8e_{10} + 2e_1e_4e_{19} + 2e_1e_3e_{20}$
5	$-e_1e_7e_8e_{14} + 4e_3^2e_9e_{15} + 3e_1e_2e_8^2e_{11} + 2e_3e_6e_{10}e_{11} - 5e_1e_2e_6e_{10}e_{11}$
	$+4e_3e_5e_8e_{14} - e_1e_2e_6^2e_{15} - 2e_{12}e_{18} - e_1e_2e_9e_{18} + 3e_2^2e_6e_9e_{11}$
	$-e_7^3e_9 - e_1e_2e_3e_{24} - e_1^3e_9e_{18} - e_1e_3e_4e_5e_{17} + 5e_1e_2e_5e_{10}e_{12}$
	$-5e_3e_5e_{10}e_{12} - 3e_1e_5^2e_{19} + 3e_3e_9e_{18} - 3e_5e_8e_{17} + e_8e_{11}^2 - e_4^3e_7e_{11}$
	$+2e_1^2e_{10}e_{18} + 3e_3^2e_6e_9^2 - 4e_2e_{10}e_{18} + 2e_3e_5e_{11}^2 - e_3^3e_{10}e_{11} + 3e_6e_7e_{17}$
	$+3e_1e_{11}e_{18} + e_6e_7e_8e_9 + 3e_2e_{12}e_{16} + e_5^4e_{10} + 2e_3^2e_5e_{19} - 2e_2^3e_{10}e_{14}$
	$-e_{13}e_{17} - e_1^2e_7e_{10}e_{11} + 5e_2e_7e_{10}e_{11} - 2e_4^3e_8e_{10} - 7e_2e_4e_5e_7e_{12}$
	$-2e_5e_7e_{18} + e_4^2e_{11}^2 + 3e_1e_4^2e_6e_{15} + 2e_1e_2^2e_9e_{16} + e_3e_5e_7^2e_8$
	$-4e_4e_5^2e_6e_{10} - 6e_4e_5e_{10}e_{11} + 5e_1^2e_6e_{11}^2 + 2e_2^2e_5e_8e_{13} + 3e_1e_6e_7e_8^2$
	$+e_2e_7^4 - e_4e_{26} - e_4^3e_{18} + 4e_1e_3e_5e_{21} + 2e_2^2e_5e_{21} - 2e_3e_4^2e_9e_{10}$
	$-4e_3e_6e_8e_{13} + 2e_1^2e_7e_8e_{13} + 2e_8e_9e_{13} - 2e_2^3e_6e_{18} + 2e_4^2e_7e_{15}$
	$+2e_1e_3e_6e_7e_{13} - 4e_3^2e_4e_8e_{12} + 2e_2e_4e_5^2e_{14} - 4e_2e_6e_{11}^2 + e_1^2e_3^2e_{22}$
	$-2e_3e_4e_7e_8^2 - 6e_1e_5e_9e_{15} - 7e_4e_6e_8e_{12} - 8e_5e_6e_9e_{10} + 4e_3e_4e_{10}e_{13}$

$$+2e_1^2e_3e_7e_{18} + 2e_1^2e_4e_5e_{19} - 4e_1^2e_6e_8e_{14} + e_2e_6e_8e_{14} + 2e_3^2e_{10}e_{14}$$
$$+5e_3e_4e_5e_8e_{10} - 4e_1e_7e_{11}^2 - e_1e_2e_4e_5e_{18} - 6e_1e_2e_3e_7e_{17} + e_3e_4e_6e_{17}$$
$$+2e_1^2e_4e_7e_{17} + 2e_3e_4e_6e_8e_9 - 2e_6e_{12}^2 - 4e_4e_5e_6e_{15} + 12e_5e_{10}e_{15}$$
$$+3e_3e_4^2e_6e_{13} + e_1e_3^2e_6e_{17} - 3e_2e_5^2e_{18} - 7e_1e_2e_{11}e_{16} + e_1^2e_7^2e_{14}$$
$$+2e_1e_2e_4e_6e_{17} - e_2^3e_{11}e_{13} + 4e_1e_2e_4e_7e_{16} - e_2^3e_8e_{16} - e_1e_2e_8e_{19}$$
$$-4e_3e_4e_5e_9^2 + 2e_2e_4^2e_7e_{13} + 4e_2e_4e_6e_8e_{10} + 2e_2e_3e_6e_8e_{11}$$
$$-2e_2^2e_5e_7e_{14} + e_{30} + 2e_1e_2^2e_4e_{21} - 4e_3^2e_5e_6e_{13} + 4e_{10}e_{20} + e_2e_3e_5e_9e_{11}$$
$$-e_1e_3e_4e_{22} + 3e_1e_5^2e_8e_{11} + 5e_3e_7e_8e_{12} - e_2e_4^3e_{16} - 2e_6e_{10}e_{14}$$
$$+5e_4e_5e_9e_{12} + 2e_5e_8^2e_9 - 3e_3e_7e_{10}^2 + e_1^2e_8e_{10}^2 - 3e_2^2e_4e_{22}$$
$$+4e_1e_5e_8e_{16} - 3e_1e_3^2e_{10}e_{13} - 3e_3^2e_4e_7e_{13} + 3e_1e_8e_{10}e_{11} + 8e_5e_6e_7e_{12}$$
$$+6e_5^2e_{20} + 3e_2^2e_4e_6e_{16} + 6e_2^2e_7^2e_{12} - e_1e_2e_5e_9e_{13} - 2e_2^2e_6^2e_{14}$$
$$-e_3e_5e_6e_7e_9 + 2e_1e_7^2e_{15} - e_1e_6^2e_7e_{10} + e_6e_7^2e_{10} - e_2e_3e_4e_{21}$$
$$+2e_1^2e_2e_3e_{23} - 4e_1^2e_8e_9e_{11} + 2e_1^2e_4e_{10}e_{14} - e_2e_8e_9e_{11} + e_2e_5^2e_6e_{12}$$
$$+2e_3^2e_8^3 - 3e_1^2e_7e_{21} + 2e_2e_7e_{21} - e_1^3e_5e_{22} + 2e_1e_8e_{21} + 2e_{10}^3 - e_3e_4^3e_{15}$$
$$-3e_3e_4e_6e_7e_{10} + 4e_4e_6^2e_{14} + e_4^2e_{22} + e_2^2e_4e_8e_{14} - 4e_1^2e_4e_6e_{18}$$
$$+5e_1^3e_6e_{21} - 3e_1e_2e_3e_{12}^2 + 3e_3e_5e_9e_{13} - 6e_2e_4e_7e_{17} - 4e_4e_8e_9^2$$
$$-3e_1e_4e_5e_{10}^2 + 3e_6e_{11}e_{13} + 3e_1e_2^2e_{10}e_{15} - 4e_4e_7^2e_{12} - 4e_1^2e_2e_{10}e_{16}$$
$$+2e_2e_5e_6e_{17} - 3e_2^2e_3e_7e_{16} + 4e_1e_2e_5e_{22} + 2e_1^2e_4e_9e_{15} - 2e_4^2e_5e_8e_9$$
$$-4e_3e_5^2e_7e_{10} + e_7^2e_{16} - 3e_2^2e_7e_8e_{11} + e_1^2e_8^2e_{12} + 2e_3e_7^2e_{13} + e_1^4e_{26}$$
$$+2e_1e_2e_6e_8e_{13} + 3e_1e_2e_4e_{11}e_{12} + e_4^4e_{14} + 2e_2e_3e_4^2e_{17} + e_2^2e_6e_{10}^2$$
$$-3e_3e_5e_{22} - 3e_2^2e_9e_{17} + 2e_4e_9e_{17} + 2e_1^2e_2e_9e_{17} + 3e_4^2e_5e_7e_{10}$$
$$-e_3^3e_4e_{17} - 3e_2e_3e_6e_{19} + 3e_1e_3e_6e_{10}^2 - 2e_2^2e_3e_6e_{17} + e_4^2e_5^2e_{12}$$
$$-4e_1^2e_6e_{22} - 3e_2e_3e_{25} - 3e_2e_3e_5^2e_{15} + 2e_{15}^2 + 5e_2e_4e_{10}e_{14}$$
$$-6e_1e_3e_6e_9e_{11} + 2e_1e_3e_5e_6e_{15} - 4e_2e_4e_6e_9^2 + 3e_2e_6e_{22} - e_2^3e_5e_{19}$$
$$-3e_7e_{10}e_{13} + 2e_2^2e_{11}e_{15} + 4e_3^2e_7e_{17} - e_1^3e_{12}e_{15} - e_8e_{22} - 2e_2^2e_6e_7e_{13}$$
$$-2e_1^2e_3e_9e_{16} + 2e_1e_2e_3^2e_{21} + 2e_4e_7e_{19} + 5e_1e_5e_6e_9^2 + 3e_1e_3e_5e_{10}e_{11}$$
$$+2e_1e_7e_{22} - e_2^3e_3e_{21} + 3e_2^2e_3e_9e_{14} + 2e_1e_3^2e_7e_{16} - 4e_1^2e_4e_8e_{16}$$
$$+2e_1^2e_2e_8e_{18} - 2e_2e_3e_5e_{10}^2 - e_2e_5^3e_{13} - 4e_2e_4e_7^2e_{10} + e_1^2e_4^2e_{20}$$
$$+e_2^2e_4^2e_{18} + 3e_2e_4^2e_8e_{12} + 4e_2e_3e_5e_8e_{12} - e_1e_3e_6^2e_{14} + 2e_1e_{13}e_{16}$$
$$-e_1e_2e_3e_4e_{20} - 4e_3e_5e_7e_{15} + 5e_4e_5e_8e_{13} + e_2e_3e_7e_9^2 - 6e_1e_5e_{10}e_{14}$$
$$-4e_2e_5^2e_8e_{10} - 3e_3^2e_5e_8e_{11} - 3e_1e_3e_7^2e_{12} + 4e_1e_5e_7^2e_{10} - e_1e_7^3e_8$$
$$-4e_1^2e_6e_7e_{15} - 6e_1e_3e_5e_8e_{13} + e_7^2e_8^2 + e_2^4e_{22} - e_1^2e_4e_{11}e_{13}$$
$$-e_1^3e_8e_{19} + e_2^2e_3^2e_{20} - e_1e_4e_7e_{18} + 4e_2e_3e_7e_{18} + 6e_1e_3e_4e_9e_{13}$$
$$-3e_1e_2^2e_7e_{18} + e_4e_6e_7e_{13} + 2e_3^2e_4e_{20} + e_2e_4e_6e_{18} + 2e_1e_5e_6e_{18}$$
$$-3e_5e_{11}e_{14} + e_1^2e_9^2e_{10} + 4e_1e_2e_3e_8e_{16} - e_2e_4e_{11}e_{13} - e_1e_3e_4e_{10}e_{12}$$
$$+3e_5e_7e_9^2 - 2e_4e_6e_{10}^2 + e_2e_9^2e_{10} + e_2^2e_{13}^2 - e_1e_3^3e_{20} + 2e_3e_8e_{19}$$
$$+2e_1^2e_9e_{19} + 2e_2e_9e_{19} + 6e_5e_6e_8e_{11} - 3e_2e_3^2e_8e_{14} - 8e_1e_2e_5e_6e_{16}$$
$$-e_1e_5e_8^3 + 4e_2e_4e_5e_{19} + 2e_3e_4e_5^2e_{13} + e_3e_4e_5e_6e_{12} + e_4e_{13}^2$$
$$+2e_3e_{11}e_{16} - 4e_1^2e_{12}e_{16} - 4e_1^2e_2e_{11}e_{15} + 2e_2e_6e_7e_{15} - e_3e_{27}$$
$$+2e_1^2e_3e_8e_{17} - 3e_2e_4e_5e_6e_{13} + 2e_1e_3e_4^2e_{18} + 3e_1e_2^2e_{11}e_{14}$$
$$+e_1^2e_4e_{12}^2 - 2e_1e_6e_{11}e_{12} + 2e_1e_2^2e_5e_{20} + 2e_2e_4^2e_5e_{15} - 3e_3e_8^2e_{11}$$
$$-e_3e_4e_7e_{16} - 5e_1e_6^2e_{17} - e_1e_2^3e_{23} + e_2^2e_8e_9^2 + 3e_1e_5e_{11}e_{13}$$

$$-3e_1e_2e_7^2e_{13}+e_4^2e_{10}e_{12}+4e_2e_7e_8e_{13}-2e_5e_{12}e_{13}-3e_1e_8^2e_{13}$$
$$+4e_6^2e_8e_{10}+6e_1e_3e_6e_8e_{12}-8e_2e_3e_6e_7e_{12}-3e_2e_3e_6e_9e_{10}$$
$$+e_2e_3^2e_{10}e_{12}+2e_2^2e_3e_4e_{19}-3e_3e_7e_{20}+2e_1^2e_8e_{20}+e_1e_6e_7^2e_9$$
$$-2e_1^2e_5e_7e_{16}-3e_1e_{10}e_{19}-3e_3e_{10}e_{17}+e_3^2e_{24}+6e_1e_4e_6e_7e_{12}$$
$$+5e_1e_2e_{12}e_{15}+2e_1^2e_3e_5e_{20}-4e_2^2e_{12}e_{14}+5e_1^2e_6^2e_{16}-e_2e_6^2e_{16}$$
$$+2e_1e_5e_6e_8e_{10}-2e_3e_4e_5e_7e_{11}-e_3^3e_5e_{16}+5e_2e_3e_7e_8e_{10}$$
$$+2e_1e_4e_6e_9e_{10}-3e_1e_2^2e_{12}e_{13}+2e_2^2e_4e_9e_{13}-e_1e_3e_7e_{19}$$
$$-6e_2e_3e_5e_{20}-3e_2^2e_7e_{19}+2e_2^2e_3e_{23}+4e_4e_7e_9e_{10}-6e_1e_2e_4e_9e_{14}$$
$$+4e_5e_{25}+4e_2e_3e_4e_7e_{14}+3e_2e_5e_7e_{16}-3e_1e_4^2e_7e_{14}-4e_1^2e_3e_{11}e_{14}$$
$$-5e_2e_5e_{10}e_{13}+2e_1e_6e_{10}e_{13}+3e_4e_8^2e_{10}+3e_1e_2e_3e_{11}e_{13}+3e_2e_3e_5e_7e_{13}$$
$$-4e_3^2e_7e_8e_9+3e_1e_4^2e_{10}e_{11}-2e_8e_{10}e_{12}-5e_1e_4e_6^2e_{13}+3e_4e_{12}e_{14}$$
$$+e_1e_3e_7e_9e_{10}-3e_1e_{14}e_{15}-e_3^3e_{21}-3e_2^2e_5e_9e_{12}+3e_1e_4e_5e_9e_{11}$$
$$-e_1e_2e_9^3-6e_1e_4e_5e_{20}+e_2e_3^2e_{11}^2+2e_4^2e_6^2e_{10}+e_4e_5^2e_7e_9$$
$$+e_2e_5^2e_9^2-2e_6e_8^3+e_1e_4e_5e_7e_{13}-e_1^3e_{13}e_{14}-2e_2^2e_4e_{10}e_{12}$$
$$-3e_2e_6e_9e_{13}-e_1e_7e_9e_{13}+e_2^2e_5^2e_{16}-3e_4e_5^2e_{16}-2e_2e_3e_8^2e_9$$
$$-5e_2e_3e_4e_8e_{13}-3e_1e_3^2e_{23}+e_2e_3e_5e_6e_{14}-e_1^3e_4e_{23}+2e_1^2e_3e_{12}e_{13}$$
$$+2e_2e_5e_{11}e_{12}+5e_1e_5^2e_6e_{13}-2e_2e_4e_8^3-e_1e_4^3e_{17}-2e_1e_3e_5e_9e_{12}$$
$$+2e_3e_{13}e_{14}+2e_4^3e_9^2-3e_2e_4^2e_9e_{11}-e_2e_3e_{11}e_{14}+e_4^2e_6e_8^2+2e_7e_9e_{14}$$
$$-4e_1e_6e_7e_{16}-4e_3^2e_6e_8e_{10}+3e_2^2e_{10}e_{16}-3e_2e_4e_5e_8e_{11}+e_2^2e_4e_{11}^2$$
$$+e_2e_{14}^2+3e_1e_9^2e_{11}+3e_1e_4e_{11}e_{14}+e_1^2e_5^2e_{18}+e_1^2e_{14}^2+5e_2e_3e_{12}e_{13}$$
$$-e_2e_5e_6^2e_{11}-e_2^3e_4e_{20}-4e_2e_5e_7^2e_9+2e_2e_6^2e_8^2+2e_1e_4e_5e_6e_{14}$$
$$-5e_1e_5e_6^2e_{12}+6e_5^2e_{10}^2-4e_3^2e_6e_{18}+3e_3e_8e_9e_{10}-3e_1e_4e_7e_9^2$$
$$+3e_3e_5^2e_8e_9+2e_3e_4e_{23}+4e_2e_6^2e_7e_9-5e_6^2e_7e_{11}+3e_4e_5e_7e_{14}$$
$$-3e_1e_2e_4e_{10}e_{13}+3e_1e_5e_7e_{17}+3e_1e_3e_7e_8e_{11}+4e_3^3e_8e_{13}$$
$$-3e_2e_3^2e_4e_{18}+5e_2e_4e_5e_9e_{10}+6e_1e_6e_8e_{15}+2e_2e_3^2e_5e_{17}-e_1e_2e_4e_{23}$$
$$+e_3^4e_{18}+2e_2^2e_3e_5e_{18}+4e_2e_5e_9e_{14}+2e_1e_6e_9e_{14}+4e_2e_5^2e_7e_{11}$$
$$-e_1e_2e_5e_8e_{14}+2e_1^2e_3e_{25}-3e_1e_4e_{25}+6e_1e_2e_6e_7e_{14}-e_1e_3e_4e_7e_{15}$$
$$-3e_1^2e_2e_{26}-4e_1e_3e_4e_{11}^2-3e_3e_6e_7e_{14}-3e_1e_4e_{12}e_{13}+e_1e_3e_8e_9^2$$
$$-e_1^3e_{27}+e_8^2e_{14}+3e_3e_4^2e_8e_{11}-2e_4^2e_6e_7e_9+6e_4e_6e_9e_{11}-2e_2e_3e_9e_{16}$$
$$+3e_1e_2e_5e_7e_{15}+5e_1e_3e_4e_8e_{14}-3e_2^2e_4e_5e_{17}+2e_4^2e_5e_{17}$$
$$+2e_1^2e_2e_{12}e_{14}+2e_1^2e_5e_9e_{14}-3e_2e_5e_{23}-2e_5e_6e_{19}-e_4e_5^3e_{11}$$
$$-3e_2^2e_3e_{11}e_{12}+2e_1^2e_5e_{23}+2e_3^2e_7e_{17}+3e_2^2e_3e_{10}e_{13}+2e_2e_4e_6e_7e_{11}$$
$$-3e_2e_3e_4e_6e_{15}+e_1e_5^2e_7e_{12}+e_4e_8e_{18}+4e_5^3e_{15}+3e_1e_6e_{23}$$
$$+2e_3e_4^2e_7e_{12}-4e_1^2e_2e_6e_{20}+2e_1e_3^2e_4e_{19}+3e_1e_{12}e_{17}-e_1e_{29}$$
$$+2e_1^2e_3e_{10}e_{15}+2e_2^2e_8e_{18}-3e_3^2e_{11}e_{13}+e_2e_3^2e_6e_{16}+2e_1^2e_5e_{10}e_{13}$$
$$-5e_3e_{12}e_{15}-3e_2e_8e_{20}+e_2^2e_{26}-2e_3e_5e_6e_8^2-3e_1e_9e_{20}+4e_3e_4e_5e_{18}$$
$$+e_1^2e_2e_{13}^2-2e_2^2e_6e_8e_{12}+2e_1e_3e_5^2e_{16}+2e_2e_4^2e_{20}-4e_1^2e_6e_{10}e_{12}$$
$$+e_3^2e_7^2e_{10}-e_4^3e_5e_{13}-4e_1^2e_5e_{11}e_{12}+3e_3^2e_5e_7e_{12}-4e_2e_6^3e_{10}$$
$$+6e_1e_3e_6e_{20}-7e_5e_7e_8e_{10}+2e_1e_7e_{10}e_{12}+2e_2e_3^2e_7e_{15}+6e_2e_6e_{10}e_{12}$$
$$+2e_1e_2e_5^2e_{17}-3e_3e_5^2e_{17}-e_1e_2e_{13}e_{14}-3e_1e_2^2e_3e_{22}-e_9e_{21}$$
$$-6e_1e_3e_8e_{18}-6e_1e_4e_{10}e_{15}-2e_4^3e_6e_{12}-e_1^3e_{10}e_{17}-4e_1^2e_{11}e_{17}$$
$$+e_1^2e_2^2e_{24}+2e_2e_{11}e_{17}-e_2e_{28}-3e_1e_3e_{13}^2-2e_1e_3e_5e_7e_{14}-3e_4^2e_9e_{13}$$

$$-7e_3e_4e_9e_{14} - 3e_1e_2e_4^2e_{19} - e_6e_8e_{16} - 4e_2e_3^2e_9e_{13} - e_3^3e_7e_{14}$$
$$-8e_2e_3e_{10}e_{15} + e_3^2e_4e_6e_{14} - 2e_2e_4^2e_6e_{14} - 3e_3e_4^2e_5e_{14} + 4e_1e_4e_9e_{16}$$
$$-3e_7e_8e_{15} + e_3e_5^2e_6e_{11} - 5e_3e_4e_6^2e_{11} - 2e_3e_7e_9e_{11} + 2e_2^2e_3e_8e_{15}$$
$$-2e_4^2e_8e_{14} + 2e_1^2e_2e_4e_{22} - e_1^3e_3e_{24} - 3e_1^2e_2e_5e_{21} + e_5e_7^2e_{11}$$
$$-2e_1e_5e_7e_8e_9 - 3e_3e_4^2e_{19} + 2e_1^2e_7e_9e_{12} + 4e_1e_2e_7e_{20} - 3e_4e_{11}e_{15}$$
$$+3e_3e_6e_{21} - 3e_1e_3e_8^2e_{10} + e_3^2e_5e_9e_{10} - 3e_9e_{10}e_{11} + 2e_2^3e_{12}^2$$
$$-3e_1e_3^2e_8e_{15} + 3e_1e_2e_7e_9e_{11} - 3e_1^2e_3e_4e_{21} + 4e_2e_3e_6^2e_{13}$$
$$+6e_1e_3e_{11}e_{15} - 3e_1e_4e_5^2e_{15} - 2e_1e_2e_3e_9e_{15} + 5e_3^2e_6e_7e_{11} - e_3e_5^3e_{12}$$
$$+3e_2e_5e_7e_8^2 + 3e_1e_4e_8^2e_9 - e_7e_{23} - 2e_2e_8e_{10}^2 - 3e_1e_9e_{10}^2 + 2e_1^2e_4e_{24}$$
$$-2e_6e_9e_{15} + 2e_1e_2e_6e_9e_{12} + 5e_1e_2e_{10}e_{17} + 4e_3e_5e_6^2e_{10} + 2e_6^2e_{18}$$
$$-5e_2e_7e_9e_{12} + 2e_1^2e_2e_7e_{19} - e_2e_3^3e_{19} + 2e_3e_4e_8e_{15} + 3e_7e_{11}e_{12}$$
$$+4e_2e_4e_9e_{15} - 4e_5^2e_8e_{12} + 2e_1e_4^2e_{21} + 2e_2e_4e_{24} + 3e_1e_2^2e_6e_{19}$$
$$-3e_1e_5^2e_9e_{10} + 2e_2e_3^2e_{22} + e_3^2e_4e_{10}^2 - 3e_1e_4e_7e_8e_{10} + 5e_1e_6^3e_{11}$$
$$+e_2e_8^2e_{12} - 3e_2e_{13}e_{15} - 5e_1e_4e_6e_8e_{11} - 3e_3^2e_8e_{16} - 3e_1e_5e_{24}$$
$$+5e_1^3e_{11}e_{16} + e_1^2e_{28} + 3e_2^2e_5e_6e_{15} - e_1e_2e_3e_5e_{19} - 6e_1e_2e_7e_8e_{12}$$
$$+e_3^2e_4^2e_{16} - e_2e_3e_4e_5e_{16} - 3e_5^2e_9e_{11} + 5e_2e_4e_7e_8e_9 - 3e_5e_9e_{16}$$
$$-2e_2^2e_7e_9e_{10} + 4e_2e_4e_6^2e_{12} - e_1e_5^3e_{14} - 4e_1^2e_3e_6e_{19} - e_2^3e_9e_{15}$$
$$-e_2^3e_{24} + 3e_2^2e_6e_{20} - e_1e_2e_5e_{11}^2 - 3e_1e_4^2e_9e_{12} - 2e_2e_4e_{12}^2$$
$$+3e_1e_5e_{12}^2 + 6e_1e_4e_6e_{19} + 4e_2e_3e_8e_{17} + 3e_2e_3e_4e_9e_{12} + 4e_1e_3e_{10}e_{16}$$
$$+2e_1e_4^2e_5e_{16} - 3e_1e_2^2e_8e_{17} - 3e_2e_3e_7^2e_{11} - 8e_1e_2e_6e_{21} - 3e_4e_5e_{21}$$
$$-e_{11}e_{19} - 4e_5^2e_7e_{13} - 4e_2e_6e_7^2e_8 + 3e_1e_4e_7^2e_{11} + 2e_1^2e_{13}e_{15}$$
$$-3e_2e_5e_6e_8e_9 + 3e_1e_3^2e_{11}e_{12} - e_{14}e_{16} + 3e_4^2e_5e_6e_{11} - e_1^3e_2e_{25}$$
$$-6e_4e_6e_{20} - 4e_1e_3e_4e_6e_{16} - e_4^2e_6e_{16} - 2e_1e_3e_9e_{17} + 6e_1e_2e_3e_6e_{18}$$
$$-5e_1e_6^2e_8e_9 + 3e_3e_4e_7^2e_9 + e_3^2e_5^2e_{14} - 3e_2e_7^2e_{14} + 2e_1^2e_5e_8e_{15}$$
$$+e_3^2e_4e_9e_{11} - 9e_1e_5e_6e_7e_{11} + 2e_3^2e_4e_5e_{15} - 3e_1e_3^2e_5e_{18} - e_1^3e_7e_{20}$$
$$-4e_5^2e_6e_{14} + 2e_1e_3e_{26} + 2e_1e_2e_{27} - 6e_2e_5e_8e_{15} + 6e_3e_5e_6e_{16} - 3e_4e_{10}e_{16}$$
$$+2e_1e_2^2e_{25} + 5e_2e_5e_6e_7e_{10} + e_1e_3^2e_9e_{14} - 2e_6e_{24} - 3e_2^2e_4e_7e_{15}$$

Table of $P_{4,n}$ for $n = 7, 8$

n $P_{4,n}$

7 $-2e_4e_{11}e_{13} - 2e_8e_9e_{11} + e_2e_{13}^2 - e_1^2e_{13}^2 - 2e_3e_{12}e_{13} + e_5e_9e_{14} - 2e_1e_4e_{23}$
$+2e_2e_3e_{23} - 3e_5e_7e_{16} + e_3^2e_6e_{16} + e_1^2e_3e_{23} - e_2e_6e_8e_{12} + e_3e_{11}e_{14}$
$-e_1e_2^2e_{23} - e_2e_4^2e_{18} - 2e_1e_3e_{24} - e_2^2e_8e_{16} + e_1^2e_2e_{24} + 2e_1e_2e_{25}$
$+6e_4e_8e_{16} + 2e_1e_2e_5e_{20} + 3e_4e_{24} + e_1^2e_{10}e_{16} + 2e_1e_5e_6e_{16} + e_2e_3e_{11}e_{12}$
$-2e_1e_4e_{11}e_{12} - 3e_1e_7e_8e_{12} - 2e_2e_6e_9e_{11} - 2e_1e_{11}e_{16} + e_7e_{10}e_{11}$
$+3e_2e_6e_{10}^2 - 2e_2e_{10}e_{16} + e_2e_7e_{19} + e_1e_2e_{12}e_{13} - 3e_2e_7e_9e_{10}$
$+2e_1e_8e_9e_{10} + 2e_3e_4e_6e_{15} + e_3^2e_{22} + e_4e_6e_7e_{11} - e_3e_4^2e_{17} - 3e_4e_5e_7e_{12}$
$+e_5^2e_{18} - 3e_4e_7e_8e_9 + e_1e_7e_9e_{11} - 2e_1e_2e_{10}e_{15} - 2e_4e_9e_{15} - 3e_1e_5^2e_{17}$
$+3e_2^2e_6e_{18} + e_4^3e_{16} - 2e_4e_6e_{18} + e_1^2e_{12}e_{14} - e_1e_6^2e_{15} - e_5^3e_{13}$
$+4e_3e_7e_{18} + 2e_2e_3e_4e_{19} - e_6e_{22} - 3e_1e_7e_{20} - 2e_2e_8e_{18} - e_3^2e_{11}^2$
$+e_1^2e_8e_{18} + e_5^2e_8e_{10} + e_3^2e_8e_{14} - e_1e_{27} - 2e_2e_5e_{10}e_{11} - 2e_2e_6e_{20}$
$+2e_1e_9e_{18} - e_2e_4e_{10}e_{12} - e_6e_8e_{14} + e_3^2e_{10}e_{12} + e_2e_4e_7e_{15} - 2e_6e_{10}e_{12}$
$-e_2^2e_3e_{21} + e_1e_3^2e_{21} - 2e_3e_4e_{21} - 3e_1^2e_5e_{21} - e_8e_{10}^2 + e_3e_4e_7e_{14}$
$-e_2^2e_{24} - 3e_4e_7e_{17} - 2e_3e_8e_{17} + e_2e_3^2e_{20} + 2e_5e_{10}e_{13} - e_2^2e_{11}e_{13}$
$+e_1^2e_{26} + e_6^3e_{10} - 2e_2e_4e_6e_{16} + e_1e_4e_{10}e_{13} + 2e_1e_3e_5e_{19} - 3e_1^2e_9e_{17}$
$-e_{13}e_{15} + 2e_1e_2e_8e_{17} - e_3e_{25} - e_9e_{19} + e_4e_8^3 - 2e_3e_5e_8e_{12} + e_2e_3e_7e_{16}$
$+4e_2e_7e_8e_{11} - 3e_1e_6e_7e_{14} - e_1e_3e_{12}^2 + e_5^2e_6e_{12} + e_{28} + 3e_4e_7^2e_{10}$
$+e_4e_5^2e_{14} + 2e_2e_5e_{21} - e_3e_6^2e_{13} + 3e_4e_{12}^2 + 2e_4e_5e_6e_{13} - 2e_3^2e_7e_{15}$
$+2e_1e_4e_5e_{18} + 2e_2e_4e_5e_{17} - 2e_2e_4e_{22} + 2e_1e_5e_{22} - 2e_1e_4e_8e_{15}$
$-e_4^2e_9e_{11} + 4e_6e_7e_{15} - e_{11}e_{17} - e_3^3e_{19} + e_6e_{11}^2 - e_5^2e_9^2 + e_5e_6e_8e_9$
$+2e_5e_6e_{17} + 3e_1e_7^2e_{13} + 3e_3e_5e_9e_{11} + 2e_3e_6e_{19} + e_3^2e_9e_{13} + 2e_6e_9e_{13}$
$+2e_5^2e_7e_{11} - 2e_1e_2e_{11}e_{14} - 2e_4e_5e_{19} + e_2e_5^2e_{16} - 2e_1e_3e_{10}e_{14}$
$+2e_1e_5e_{10}e_{12} - 2e_5e_{11}e_{12} + 3e_4^2e_{20} - 2e_1e_3e_8e_{16} - 2e_3e_5e_{20} - 2e_3e_6e_9e_{10}$
$-2e_2e_5e_6e_{15} - 2e_1e_3e_6e_{18} - 3e_7e_8e_{13} - e_5e_{23} + 3e_8^2e_{12} + e_2^3e_{22}$
$+2e_2e_5e_8e_{13} + e_1e_6e_8e_{13} + e_1^2e_4e_{22} - 2e_3e_4e_9e_{12} + e_3e_4e_{10}e_{11}$
$-e_1e_8^2e_{11} - 2e_1e_3e_4e_{20} - e_1e_9^3 - e_2e_{12}e_{14} + 2e_1e_3e_9e_{15} + 2e_1e_6e_{21}$
$+e_3^2e_5e_{17} + 2e_2e_3e_8e_{15} + e_1e_{13}e_{14} - 3e_5e_6e_7e_{10} + 2e_1e_2e_4e_{21}$
$+2e_1e_2e_9e_{16} - e_{10}e_{18} - e_3e_5e_{10}^2 + e_4e_6e_9^2 + 2e_3e_7e_9^2 + e_2e_8e_9^2$
$-2e_1e_6e_{10}e_{11} + 3e_{12}e_{16} - 2e_1e_2e_6e_{19} - 2e_3e_5e_6e_{14} + 4e_3e_6e_7e_{12}$
$-e_4^2e_5e_{15} + e_3e_5^2e_{15} - e_1^3e_{25} - 2e_1e_2e_3e_{22} - 2e_3e_4e_5e_{16} + 3e_1e_3e_{11}e_{13}$
$-2e_2e_3e_6e_{17} + 2e_1e_4e_6e_{17} - 3e_1e_2e_7e_{18} - e_6^2e_{16} - 2e_2e_3e_{10}e_{13}$
$-2e_5e_8e_{15} - 2e_2e_3e_9e_{14} + 2e_1e_4e_9e_{14} + e_2e_5e_9e_{12} + 2e_2e_{11}e_{15}$
$-e_2e_8^2e_{10} + e_1^2e_{11}e_{15} + 2e_3e_{10}e_{15} - 2e_1e_{12}e_{15} - e_5e_6^2e_{11} - e_4^2e_6e_{14}$
$-2e_3e_4e_8e_{13} + e_3e_6e_8e_{11} - 2e_4e_5e_8e_{11} - e_4e_{10}e_{14} + 2e_1e_6e_9e_{12}$
$+e_1e_5e_7e_{15} - 2e_2e_4e_8e_{14} - e_4e_6e_8e_{10} + 3e_2e_6^2e_{14} - e_3e_8^2e_9$
$+3e_8e_{20} + e_2e_4e_{11}^2 + 2e_2e_9e_{17} - e_1e_4^2e_{19} + 2e_1e_{10}e_{17} + e_3e_7e_8e_{10}$
$+2e_2e_4e_9e_{13} - 5e_1e_5e_9e_{13} + 2e_1e_5e_8e_{14} + e_1e_5e_{11}^2 - e_2^2e_9e_{15}$
$-e_4e_6^2e_{12} - e_2^2e_5e_{19} + 3e_2^2e_{10}e_{14} + e_3^2e_4e_{18} + e_1^2e_6e_{20} - 2e_2e_3e_5e_{18}$
$+e_1e_3e_7e_{17} + 2e_1^2e_7e_{19} + e_9^2e_{10} - e_2e_{26} + e_3e_5e_7e_{13} - 3e_2e_6e_7e_{13}$

$+3e_4^2e_8e_{12} - e_2^2e_4e_{20} - 4e_3e_7^2e_{11} - 2e_1e_8e_{19} + e_4e_5e_9e_{10} - 2e_3e_9e_{16}$

8 $e_4^2e_{11}e_{13} + 3e_3e_7e_{11}^2 + 2e_1e_7e_{24} - e_3^2e_{12}e_{14} + e_3e_{13}e_{16} + 3e_5^2e_9e_{13}$

$+3e_3e_7^2e_{15} - 2e_6e_9e_{17} + e_4e_5e_{11}e_{12} + e_5e_7e_8e_{12} - e_3^2e_9e_{17} - e_{10}e_{11}^2$

$+2e_2e_4e_{26} - e_1^2e_{14}e_{16} + e_8e_{11}e_{13} + 2e_6e_{12}e_{14} + 2e_1e_2e_{11}e_{18} + e_2^2e_{13}e_{15}$

$+2e_9e_{11}e_{12} + 2e_1e_2e_6e_{23} - e_3e_5e_{11}e_{13} - e_1^2e_{11}e_{19} - 2e_6e_{11}e_{15} + 3e_2e_{14}e_{16}$

$-2e_1e_4e_6e_{21} - 2e_3e_{10}e_{19} - 4e_4e_8^2e_{12} - 2e_5e_9e_{18} - 2e_3e_6e_7e_{16}$

$+e_1e_{15}e_{16} - 2e_2e_{11}e_{19} - 2e_3e_{11}e_{18} - 2e_1e_2e_{29} + 3e_2e_4e_{12}e_{14} - 3e_2e_7e_8e_{15}$

$+e_5e_8e_{19} - e_2e_5^2e_{20} + 2e_1e_{12}e_{19} - 2e_3e_6e_{23} - 2e_3e_4e_6e_{19} - e_1e_9e_{11}^2$

$+2e_4e_{10}e_{18} - e_1^2e_{12}e_{18} + e_2^2e_7e_{21} - 2e_1e_5e_{26} + e_3e_4e_8e_{17}$

$+2e_1e_6e_{11}e_{14} + 2e_2e_{12}e_{18} - 2e_2e_4e_9e_{17} - 2e_5e_{10}e_{17} - e_1^2e_7e_{23}$

$-2e_3e_7e_{10}e_{12} - 2e_5e_6e_{21} - 2e_3e_6e_{11}e_{12} - e_1^2e_4e_{26} - e_2e_4e_{13}^2 + e_1^3e_{29}$

$+e_6e_{26} + 2e_2e_5e_6e_{19} + 2e_2e_6e_{11}e_{13} - 2e_1e_5e_{12}e_{14} - 2e_1e_{13}e_{18} - 2e_2e_7e_{23}$

$+e_1e_6^2e_{19} + 2e_1e_7e_{10}e_{14} + e_2^2e_{11}e_{17} + 3e_1e_9^2e_{13} + 2e_3e_4e_9e_{16}$

$+e_{10}^2e_{12} - 2e_3e_5e_9e_{15} - 2e_4^2e_8e_{16} - 2e_1e_2e_{13}e_{16} + 6e_1e_5e_9e_{17}$

$+e_1e_8e_{23} - 3e_4^2e_{24} - 2e_2e_7e_{11}e_{12} + 4e_3e_8e_{21} - 3e_1e_2e_8e_{21} - 2e_4e_6e_9e_{13}$

$+2e_4e_{11}e_{17} + 2e_1e_2e_3e_{26} + e_{13}e_{19} + 2e_4e_5e_7e_{16} + 2e_3e_{12}e_{17} - 2e_1e_7e_{11}e_{13}$

$-2e_5e_{13}e_{14} - 3e_2^2e_{10}e_{18} - 2e_2e_{13}e_{17} + e_1e_2e_{14}e_{15} + e_1e_4^2e_{23}$

$-2e_2e_3e_4e_{23} + 3e_1^2e_{13}e_{17} + e_2e_5e_{11}e_{14} + e_1e_2^2e_{27} - 2e_1e_2e_4e_{25}$

$+2e_2e_5e_7e_{18} + 2e_3e_4e_{25} + e_2^2e_{28} - e_{16}^2 - 2e_1e_{14}e_{17} + 2e_4e_8e_{10}^2$

$+e_1e_8e_{11}e_{12} + 2e_1e_3e_{10}e_{18} - e_2^2e_{14}^2 - 4e_8^2e_{16} - 2e_7e_{10}e_{15} + 2e_7e_{12}e_{13}$

$-e_1^2e_3e_{27} + e_2^2e_9e_{19} + e_4e_{14}^2 - 2e_4e_5e_9e_{14} - 3e_1e_6e_8e_{17} - 2e_2e_4e_7e_{19}$

$-e_3^2e_4e_{22} - e_3e_7e_9e_{13} + e_9e_{23} + 2e_3e_5e_{10}e_{14} - 2e_5e_6e_9e_{12} + e_2e_4^2e_{22}$

$+e_1e_{10}^2e_{11} + e_2^2e_4e_{24} + e_{14}e_{18} - 2e_4e_5e_6e_{17} - e_4^3e_{20} + e_5^3e_{17}$

$-e_9^2e_{14} + e_3e_5e_8e_{16} - 2e_3e_4e_{10}e_{15} + e_1e_4e_{12}e_{15} + e_2e_3e_{13}e_{14}$

$+2e_3e_6e_9e_{14} - e_6e_7^2e_{12} - e_5^2e_6e_{16} + e_4e_5e_8e_{15} - 2e_2e_3e_{27} + e_3e_4^2e_{21}$

$+2e_1e_3e_4e_{24} - 3e_2e_8e_9e_{13} - e_1^2e_6e_{24} + e_3e_6e_{10}e_{13} + e_3e_6^2e_{17}$

$+2e_2e_3e_{10}e_{17} + 2e_1e_4e_{27} + e_6^2e_{20} + 2e_3e_4e_5e_{20} + e_2^2e_{12}e_{16} - e_4^2e_{12}^2$

$-e_4e_5^2e_{18} + e_4^2e_6e_{18} + e_7^3e_{11} + 2e_2e_4e_6e_{20} - 2e_8e_{24} - 2e_1e_4e_{13}e_{14}$

$-e_1^2e_2e_{28} - e_3^2e_{26} + e_2e_7e_9e_{14} + 2e_2e_7e_{10}e_{13} - e_3^2e_6e_{20} + 2e_1e_3e_{28}$

$+e_2e_6e_8e_{16} - e_2e_9^2e_{12} - e_6^3e_{14} + 2e_2e_6e_7e_{17} + e_1e_{31} - e_{32} - 2e_1e_4e_{10}e_{17}$

$+2e_3e_5e_6e_{18} + e_6^2e_9e_{11} - e_6e_{13}^2 + e_{11}e_{21} - 2e_8e_{12}^2 - e_5e_7e_9e_{11}$

$+e_4e_6^2e_{16} - 3e_1e_8e_9e_{14} + 3e_3^2e_{11}e_{15} + 2e_2e_5e_{10}e_{15} - 2e_1e_5e_6e_{20}$

$+2e_1e_6e_7e_{18} - 2e_1e_4e_9e_{18} - 3e_3e_7e_8e_{14} + e_5e_6^2e_{15} + 2e_2e_3e_9e_{18}$

$+e_2^2e_5e_{23} + 2e_4e_5e_{23} - 2e_1e_{10}e_{21} + e_4^2e_5e_{19} + 4e_1e_7e_8e_{16} - 3e_{12}e_{20}$

$-3e_5e_8e_9e_{10} + e_7e_9e_{16} - e_5^2e_{22} - 3e_2e_6^2e_{18} - 2e_1e_5e_7e_{19} + 2e_1e_3e_{12}e_{16}$

$+e_1e_6e_{10}e_{15} + e_2e_4e_8e_{18} + e_{15}e_{17} + e_2e_{30} + e_5e_{27} + e_2e_8e_{10}e_{12}$

$-2e_1e_3e_{11}e_{17} + 2e_2e_3e_5e_{22} + e_5e_6e_{10}e_{11} + 2e_4e_9e_{19} + 2e_5e_{12}e_{15}$

$+2e_4e_7e_{21} + e_{10}e_{22} - 2e_1e_4e_5e_{22} - 2e_1e_9e_{10}e_{12} + 4e_1e_4e_8e_{19} + e_1e_3e_{14}^2$

$+e_3e_9e_{10}^2 - 2e_1e_3e_7e_{21} - 2e_1e_2e_{12}e_{17} - e_5^2e_{10}e_{12} + 4e_3e_8e_9e_{12}$

$+2e_6^2e_8e_{12} + e_4^2e_{10}e_{14} + 3e_3^2e_7e_{19} + 3e_4e_6e_{10}e_{12} - e_3^2e_5e_{21}$

$+3e_5e_8^2e_{11} - 2e_3e_5e_7e_{17} - e_6^2e_{10}^2 + 2e_3e_5e_{24} - e_3e_5^2e_{19} - 2e_2e_3e_7e_{20}$

$-2e_2e_5e_9e_{16} - 2e_3e_4e_{11}e_{14} - 3e_2^2e_6e_{22} - e_1^2e_{30} - 2e_4e_7e_{10}e_{11}$

$$
\begin{aligned}
&+e_6^2e_7e_{13}+2e_5e_7e_{20}+3e_1^2e_5e_{25}-2e_2e_5e_{25}-2e_3e_{14}e_{15}+4e_8e_9e_{15}\\
&+3e_1e_5^2e_{21}+4e_6e_8e_{18}+2e_2e_6e_9e_{15}-e_1e_7e_9e_{15}+2e_1e_4e_7e_{20}\\
&-2e_1e_3e_5e_{23}-e_4e_6e_{11}^2+e_4e_8e_9e_{11}-2e_2e_4e_5e_{21}-2e_1e_6e_{25}\\
&-e_2e_{15}^2+2e_2e_4e_{10}e_{16}+e_8e_{10}e_{14}-5e_2e_6e_{10}e_{14}+2e_4e_6e_{22}-2e_4e_5e_{10}e_{13}\\
&+e_3e_4e_{12}e_{13}-e_4e_9^2e_{10}+3e_1e_5e_{13}^2+e_2^2e_3e_{25}+e_4^2e_7e_{17}+e_5e_9^3\\
&+e_5e_7e_{10}^2-3e_3e_8e_{10}e_{11}-2e_6e_7e_{19}-e_3^2e_{10}e_{16}-2e_9e_{10}e_{13}-2e_3e_7e_{22}\\
&+e_7e_{25}-2e_2e_5e_{12}e_{13}-e_3e_9^2e_{11}+3e_6e_{10}e_{16}-3e_1e_5e_8e_{18}-3e_5e_6e_8e_{13}\\
&+e_4e_6e_8e_{14}+e_4^2e_9e_{15}+2e_2^2e_8e_{20}+2e_2e_9e_{10}e_{11}-7e_4e_8e_{20}\\
&-e_1e_3e_{13}e_{15}-2e_1e_3e_9e_{19}+e_2e_8e_{22}-2e_2e_3e_{12}e_{15}-2e_3e_4e_7e_{18}\\
&-2e_2e_4e_{11}e_{15}-2e_1e_2e_5e_{24}+e_3^3e_{23}-2e_4e_6e_7e_{15}-3e_6e_7e_8e_{11}\\
&-2e_1e_6e_{12}e_{13}-e_7^2e_{18}+2e_3e_9e_{20}-2e_1e_9e_{22}-e_1e_3^2e_{25}+3e_1^2e_9e_{21}\\
&-2e_2e_9e_{21}-e_1^2e_{10}e_{20}-e_5e_7^2e_{13}+2e_2e_{10}e_{20}-e_2e_7^2e_{16}+2e_1e_2e_7e_{22}\\
&+e_5e_{11}e_{16}-e_2e_3^2e_{24}+2e_1e_{11}e_{20}-2e_2e_3e_{11}e_{16}+e_6e_7e_9e_{10}\\
&-e_1e_5e_{11}e_{15}-e_2^3e_{26}+e_1e_3e_8e_{20}-2e_1e_6e_9e_{16}-2e_7e_{11}e_{14}\\
&-2e_1e_5e_{10}e_{16}+2e_1e_3e_6e_{22}-e_5^2e_7e_{15}-e_2e_{10}^3+e_3e_5e_{12}^2\\
&+2e_1e_4e_{11}e_{16}-e_1e_7^2e_{17}-2e_1e_2e_9e_{20}+2e_4e_{13}e_{15}-3e_4e_{28}+2e_2e_6e_{24}\\
&+e_4e_7e_9e_{12}+e_3e_{29}+2e_1e_2e_{10}e_{19}-5e_4e_{12}e_{16}+2e_5e_6e_7e_{14}+3e_2e_8^2e_{14}\\
&+e_2e_6e_{12}^2+4e_4e_7e_8e_{13}-e_4e_7^2e_{14}+e_1e_7e_{12}^2+e_7e_8e_{17}\\
&-3e_2e_3e_8e_{19}+2e_2e_3e_6e_{21}
\end{aligned}
$$

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