Intersection Theory on the Moduli Space of Holomorphic Curves with Lagrangian Boundary Conditions

by

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Abstract

We define a new family of open Gromov-Witten type invariants based on intersection theory on the moduli space of pseudoholomorphic curves of arbitrary genus with boundary in a Lagrangian submanifold. We assume the Lagrangian submanifold arises as the fixed points of an anti-symplectic involution and has dimension 2 or 3. In the strongly semi-positive genus 0 case, the new invariants coincide with Welschinger's invariant counts of real pseudoholomorphic curves.

Furthermore, we calculate the new invariant for the real quintic threefold in genus 0 and degree 1 to be 30. The techniques we introduce lay the groundwork for verifying predictions of mirror symmetry for the real quintic.

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Contents

Chapter 1

Introduction

1.1 The main idea

In 1985, Gromov initiated the study of pseudoholomorphic curves in symplectic geometry with his seminal paper [6]. Motivated by applications of Gromov's techniques in string theory, Witten developed a systematic way of organizing pseudoholomorphic curve information, later known as Gromov-Witten invariants [23, 24]. Over the following decade, mathematicians including Ruan-Tian [19], McDuff-Salamon [16], Li-Tian [14] and Fukaya-Ono [4], successfully established a rigorous foundation for Gromov-Witten invariants. Concurrently, Kontsevich [12, 11] initiated reseach that eventually succeeded in calculating the Gromov-Witten invariants in many situations. We briefly recall the definition of these invariants. Let (X, ω) be a symplectic manifold and denote by $\mathcal{J}_\omega$ the set of ω-tame almost complex structures on X. Fix a generic $J \in \mathcal{J}_\omega$. For $d \in H_2(X)$, let $\overline{\mathcal{M}}_{g,n}(X, d)$ denote the Gromov-compactification of the moduli space of J-holomorphic maps from a surface of genus g to X representing d together with a choice of n marked points on the domain. There exist canonical evaluation maps

$$ev_i : \overline{\mathcal{M}}_{g,n}(X, d) \to X.$$

Furthermore, denoting by $\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(\mathrm{pt}, 0)$ the Deligne-Mumford compactification of the moduli space of genus g curves with n marked points, there exists a canonical projection

$$\pi : \overline{\mathcal{M}}_{g,n}(X, d) \to \overline{\mathcal{M}}_{g,n}.$$

Let $A_i \in H^*(X)$ and $B \in H^*(\overline{\mathcal{M}}_{g,n})$. Choose differential forms $\alpha_i \in \Omega^*(X)$ and $\beta \in \Omega^*(\overline{\mathcal{M}}_{g,n})$ such that $[\alpha_i] = A_i$. The genus g Gromov-Witten invariant of X for cohomology classes A_i, B, takes the form of the integral

$$\int_{\overline{\mathcal{M}}_{g,n}(X,d)} ev_1^*(\alpha_1) \wedge \ldots \wedge ev_n^*(\alpha_n) \wedge \pi^*(\beta).$$

It follows from Stokes's Theorem and the fact that $\overline{\mathcal{M}}_{g,n}(X, d)$ is a closed orbifold that this integral does not depend on the choice of the forms α_i, β, or the choice of $J \in \mathcal{J}_\omega$. Hence, it is an invariant of the deformation class of ω parametrized by the cohomology classes A_i, B. Roughly speaking, the Gromov-Witten invariants of X count the number of J-holomorphic maps from a Riemann surface of fixed genus g to X representing d and intersecting fixed generic representatives of $PD(A_i)$. The class B can be used to fix the conformal structure on the domain Riemann surface or the relative position of the marked points.

Now, let $L \subset X$ be a Lagrangian submanifold, and let $(\Sigma, \partial\Sigma)$ be a Riemann surface with boundary. For some time, physicists [1, 13, 18, 25] have predicted the existence of "open" Gromov-Witten invariants counting pseudoholomorphic maps $(\Sigma, \partial\Sigma) \to (X, L)$ satisfying certain incidence conditions. These invariants would naturally generalize classical Gromov-Witten invariants to include maps from Riemann surfaces with boundary. Note, however, that according to [1] it may be necessary to specify some additional structure on L in order to uniquely determine the invariants. Katz and Liu took a first step toward the mathematical definition of open invariants given the additional structure of an S^1 action on the pair (X, L) [9, 15]. However, the existence of such an S^1 action is a rather restrictive condition.

Before entering a more detailed discussion, let us briefly establish some necessary notation. In the following, we denote by Σ a Riemann surface with boundary with fixed conformal structure. This avoids the issue of degenerations of Σ, which the author plans to treat in another paper in the near future. For $d \in H_2(X, L)$, let

$\mathcal{M}_{k,l}(L,\Sigma,d)$ be the moduli space of configurations of k distinct marked points in $\partial\Sigma$, l distinct marked points in Σ and J-holomorphic maps $u:(\Sigma,\partial\Sigma)\to(X,L)$ such that $u_*([\Sigma,\partial\Sigma])=d$. In this moduli space, points which are equivalent by automorphisms of Σ are identified. We denote by $\overline{\mathcal{M}}_{k,l}(L,\Sigma,d)$ the Gromov compactification of $\mathcal{M}_{k,l}(L,\Sigma,d)$. Finally, we denote by

$$evb_i:\overline{\mathcal{M}}_{k,l}(L,\Sigma,d)\to L,\qquad i=1\ldots k,$$
$$evi_j:\overline{\mathcal{M}}_{k,l}(L,\Sigma,d)\to X,\qquad j=1\ldots l,$$

the canonical evaluation maps at the marked points.

From a mathematical perspective, two main difficulties have obstructed progress on open invariants: orientation and bubbling in codimension one. Indeed, Fukaya et al. [5] showed that $\mathcal{M}_{k,l}(L,\Sigma,d)$ need not be orientable. In the same paper, they proved orientability if L is orientable and "relatively spin." However, in many interesting examples, i.e. $(X,L)=(\mathbb{C}P^2,\mathbb{R}P^2)$, L is not orientable and neither is $\mathcal{M}_{k,l}(L,\Sigma,d)$. In Theorem 1.1 we show that even if L is not orientable, under reasonable assumptions, the orientation bundle of L^k pulls-back to the orientation bundle of $\mathcal{M}_{k,l}(L,\Sigma,d)$ under the map $\prod_i ev_i$. This allows us to pull-back differential forms with values in the orientation bundle of L, wedge and integrate.

Considerably more troublesome is the problem of bubbling in codimension one. Put differently, $\overline{\mathcal{M}}_{k,l}(L,\Sigma,d)$ is an orbifold with corners. Intuitively, one should think of a manifold with many boundary components. The boundary consists of codimension one strata of the Gromov-compactification. This stands in contrast to the moduli space associated to a closed surface $\overline{\mathcal{M}}_{g,n}(X,d)$, which has no boundary since all strata of the Gromov compactification have codimension two or more. By analogy to the classical Gromov Witten invariants, we would like to define invariants parametrized by cohomology classes $A_i\in H^*(L)$ and $C_i\in H^*(X)$. Choose $\alpha_i\in\Omega^*(L)$ with $[\alpha_i]=A_i$ and $\gamma_j\in\Omega^*(X)$ with $[\gamma_j]=C_i$. The desired invariant should take the form

$$\int_{\overline{\mathcal{M}_{k,l}(L,\Sigma,d)}} evb_1^*(\alpha_1)\wedge\ldots\wedge evb_k^*(\alpha_k)\wedge evi_1^*(\gamma_1)\wedge\ldots\wedge evi_l^*(\gamma_l). \tag{1.1}$$

However, trouble arises in trying to prove independence of the choices of α_i, γ_j. For example, suppose α_1' also satisfies $[\alpha_1'] = A_1$. Then $\alpha_1 - \alpha_1' = d\delta$ for some δ and hence

$$evb_1^*(\alpha_1) \wedge \ldots \wedge evi_l^*(\gamma_l) - evb_1^*(\alpha_1') \wedge \ldots \wedge evi_l^*(\gamma_l) = \\ = d\left(evb_1^*(\delta) \wedge evb_2^*(\alpha_2) \wedge \ldots \wedge evi_l^*(\gamma_l)\right).$$

We would like to integrate the right-hand side of the above equation over $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)$ to obtain zero by Stokes's theorem. However, contributions from the integral of $ev_1^*(\delta) \wedge ev_2^*(\alpha_2) \wedge \ldots \wedge ev_l^*(\gamma_l)$ over the boundary of $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)$ may spoil this vanishing. So, the integral (1.1) may depend on the choice of α_i, γ_j.

Now, let us assume there exists an anti-symplectic involution

$$\phi : X \to X, \qquad \phi^*\omega = -\omega,$$

such that $L = Fix(\phi)$. We limit our discussion to the special case that $A_i \in H^{\dim L}(L)$, $C_j \in H^{\dim X}(X)$ and $\dim X \leq 6$. If L is not orientable, we assume $\dim L \leq 4$. Consequently, we can actually prove independence of (1.1) from the choice of α_i, γ_j.

We proceed to explain the idea of the proof. The extra structure ϕ enters the definition of the invariants through the almost complex structure. Indeed, we define

$$\mathcal{J}_{\omega,\phi} := \{J \in \mathcal{J}_\omega | \phi^* J = -J\}.$$

In the following we fix a generic $J \in \mathcal{J}_{\omega,\phi}$. Let $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)^{(1)}$ denote the union of the codimension one strata of $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)$, that is, strata consisting of two-component stable-maps. Think of $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)^{(1)}$ as the boundary of $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)$. Recall that it may have many connected components. We identify a subset of these components, the union of which we refer to as $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)^{(1a)}$, satisfying the following properties:

- There exists an orientation reversing involution $\widetilde{\phi}_2$ of $\overline{\mathcal{M}_{k,l}}(L, \Sigma, d)^{(1a)}$ that does not preserve any single connected component. Hence the quotient

$$\widehat{\mathcal{M}_{k,l}}(L, \Sigma, d) := \overline{\mathcal{M}_{k,l}}(L, \Sigma, d)/\widetilde{\phi}_2(x) \sim x$$

carries a natural orientation. Here, we use the ϕ invariance of J.

- The forms $evb_i^*(\alpha_i)$, $evi_j^*(\gamma_j)$ and $evb_1^*(\delta)$ descend naturally to

$$\widehat{\mathcal{M}_{k,l}}(L, \Sigma, d)$$

under the assumption that the γ_j are ϕ invariant.

- The differential form $evb_1^*(\delta) \wedge evb_2^*(\alpha_2) \wedge \ldots \wedge evi_l^*(\gamma_l)$ has support away from the boundary of $\widehat{\mathcal{M}_{k,l}}(L, \Sigma, d)$.

The independence of the integral (1.1) of the choices of β_i and γ_j follows immediately from Stokes's theorem: Indeed, we may replace the domain of integration in (1.1) with $\widehat{\mathcal{M}_{k,l}}(L, \Sigma, d)$. Since $evb_1^*(\delta) \wedge evb_2^*(\alpha_2) \wedge \ldots \wedge evi_l^*(\gamma_l)$ vanishes on the boundary of $\widehat{\mathcal{M}_{k,l}}(L, \Sigma, d)$,

$$\int_{\widehat{\mathcal{M}_{k,l}}(L,\Sigma,d)} d\left(evb_1^*(\delta) \wedge evb_2^*(\alpha_2) \wedge \ldots \wedge evi_l^*(\gamma_l)\right) = 0$$

as desired. Independence of the choice of $J \in \mathcal{J}_{\omega,\phi}$ follows by a similar argument. Note however, that a priori the invariant so obtained should depend on the choice of ϕ. We may interpret the choice of ϕ as the extra parameter involved in defining open invariants predicted by [1].

1.2 The definition

In the following, we denote by (X, ω) a symplectic manifold of dimension $2n$ and by $L \subset X$ a Lagrangian submanifold. Let $\mathcal{J}_\omega$ denote the set of ω-tame almost complex structures on X, and let $J \in \mathcal{J}_\omega$. Let $\mathcal{P}$ denote the set of J-anti-linear inhomogeneous perturbation terms generalizing those introduced by Ruan and Tian in [19], and let $\nu \in \mathcal{P}$. See Section 4 for more details. Fix a Riemann surface with boundary $(\Sigma, \partial\Sigma)$, let $\mathcal{M}_\Sigma$ denote the moduli space of conformal structures on $(\Sigma, \partial\Sigma)$, and fix $j \in \mathcal{M}_\Sigma$.

Suppose $\partial\Sigma = \coprod_{a=1}^{m}(\partial\Sigma)_a$, where $(\partial\Sigma)_a \simeq S^1$. Let

$$\mathbf{d} = (d, d_1, \ldots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},$$

let $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ and let $l \in \mathbb{N}$. By $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$, we denote the moduli space of (j, J, ν)-holomorphic maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ with k_a marked points on $(\partial\Sigma)_a$ and l marked points on Σ such that $u_*([\Sigma, \partial\Sigma]) = d$ and $u|_{(\partial\Sigma)_a *}([(\partial\Sigma)_a]) = d_a$. Let $\overline{\mathcal{M}_{\mathbf{k},l}}(L, \Sigma, \mathbf{d})$ denote its Gromov compactification. There exist natural evaluation maps

$$\begin{aligned} evb_{ai} &: \overline{\mathcal{M}_{\mathbf{k},l}}(L, \Sigma, \mathbf{d}) \to L, \qquad i = 1 \ldots k_a,\ a = 1 \ldots m, \\ evi_j &: \overline{\mathcal{M}_{\mathbf{k},l}}(L, \Sigma, \mathbf{d}) \to X, \qquad j = 1 \ldots l. \end{aligned}$$

We now digress for a moment to discuss the notion of a relatively $Pin^\pm$ Lagrangian submanifold. Let $V \to B$ be a vector bundle. Define the characteristic classes $p^\pm(V) \in H^2(B, \mathbb{Z}/2\mathbb{Z})$ by

$$p^+(V) = w_2(V), \qquad p^-(V) = w_2(V) + w_1(V)^2.$$

According to [10], $p^\pm(V)$ is the obstruction to the existence of a $Pin^\pm$ structure on V. See [10] for a detailed discussion of the definition of the groups $Pin^\pm$ and the notion of $Pin^\pm$ structures. Since it is not crucial for stating our result, we avoid discussing this at greater length here.

Now, suppose (X, ω) is a symplectic manifold and $L \subset X$ is a Lagrangian submanifold. Note that we do not assume L is the fixed points of an anti-symplectic involution yet. We say that L is relatively $Pin^\pm$ if

$$p^\pm(T_*L) \in \operatorname{Im}\left(i^* : H^2(X) \to H^2(L)\right).$$

and $Pin^\pm$ if $p^\pm(T_*L) = 0$. If L is $Pin^\pm$, we define a $Pin^\pm$ structure for L to be a $Pin^\pm$ structure for T_*L. If L is relatively $Pin^\pm$, a relative $Pin^\pm$ structure for L consists of the choice of a triangulation for the pair (X, L), an orientable vector bundle V over the three skeleton of X such that $w_2(V) = p^{\pm}(T_*L)$ and a $Pin^{\pm}$ structure on $T_*L|_{L^{(3)}} \oplus V|_L$. Note that by the Wu relations [17], if $n \leq 3$ then $p^-(T_*L) = 0$, so that L is always Pin^-. It follows that for the applications considered in this paper, we need only consider honest $Pin^{\pm}$ structures. However, we state Theorem 1.1 in full generality, since that requires little extra effort.

Theorem 1.1. *Assume L is relatively $Pin^{\pm}$ and fix a relative $Pin^{\pm}$ structure on (X, L). Assume $k_a \cong w_1(d_a) + 1 \mod 2$. Fix an orientation on L if it is orientable. Then, the choice of a relative $Pin^{\pm}$ structure for L canonically determines an isomorphism*

$$\det(T_*\overline{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})) \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^* \det(T_*L).$$

Remark 1.2. This theorem was proved in [5] in the special case that L is orientable.

Under the assumptions of Theorem 1.1, we define an invariant as follows. Let $H^*(L, \det(T_*L))$ denote the cohomology of L with coefficients in the flat line bundle $\det(T_*L)$. Poincare duality will hold whether or not L is orientable. Denote by $\Omega^*(L, \det(T_*L))$ the differential forms on L with values in $\det(T_*L)$, and denote by $\Omega^*(X)$ the ordinary differential forms on X. For $a = 1, \ldots m$, and $i = 1, \ldots, k_a$, let $\alpha_{ai} \in \Omega^n(L, \det(T_*L))$ represent the Poincare dual of a point in $H^n(L, \det(T_*L))$. Furthermore, for $j = 1, \ldots, l$, let $\gamma_j \in \Omega^{2n}(X)$ represent the Poincare dual of twice the point-class. We define

$$N_{\Sigma,\mathbf{d},\mathbf{k},l} := \int_{\overline{\mathcal{M}}_{\mathbf{k},l}(L,\Sigma,\mathbf{d})} evb_{11}^*\alpha_{11} \wedge \ldots \wedge evb_{mk_m}^*\alpha_{mk_m} \wedge evi_1^*\gamma_1 \wedge \ldots \wedge evi_m^*\gamma_m.$$

This integral makes sense because by Theorem 1.1, the integrand is a differential form taking values in the orientation line bundle of the moduli space over which it is to be integrated. Let $\mu : H_2(X, L) \to \mathbb{Z}$ denote the Maslov index as defined in [3]. Denote by g the genus of the closed Riemann surface $\Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ obtained by doubling Σ. Furthermore, we employ the shorthand $|\mathbf{k}| = k_1 + \ldots + k_m$. We note that by calculating the expected dimension of $\overline{\mathcal{M}}_{\mathbf{k},l}(L,\Sigma,\mathbf{d})$, it follows that unless

$$(n-1)(|\mathbf{k}|+2l) = n(1-g)+\mu(d)-\dim Aut(\Sigma) \tag{1.2}$$

the above integral must vanish.

Now, suppose there exists an anti-symplectic involution $\phi : X \to X$, such that $L = Fix(\phi)$. We define $\mathcal{J}_{\omega,\phi}$ to be the set of $J \in \mathcal{J}_\omega$ such that $\phi^* J = -J$. Define

$$\Omega^*_\phi(X) := \{\gamma \in \Omega^*(X) | \phi^*\gamma = \gamma\}.$$

Furthermore, define $\tilde{h} = h \circ r$ where $h : \pi_2(X) \to H_2(X)$ is the Hurewicz homomorphism and $r : H_2(X) \to H_2(X,L)$ is the natural homomorphism.

Assume that $\dim X \leq 6$, and if L is not orientable assume $\dim X \leq 4$ and $k_a \cong w_1(d_a)+1 \mod 2$. Note that these assumptions imply the hypothesis of Theorem 1.1. If $\Sigma = D^2$ and $k=0$ assume that

$$d \notin \operatorname{Im}\left(\tilde{h} : \pi_2(X) \to H_2(X,L)\right). \tag{1.3}$$

This is necessary to avoid a certain type of bubbling that requires taking into consideration real curves with empty real part.

Theorem 1.3. *The integers $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ do not depend on the choice of $J \in \mathcal{J}_{\omega,\phi}$, $\nu \in \mathcal{P}$, $j \in \mathcal{M}_\Sigma$, the choice of $\alpha_{ai} \in \Omega^n(L, \det(T_*L))$ or the choice of $\gamma_j \in \Omega^n_\phi(X)$. That is, the numbers $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ are invariants of the triple (X,ω,ϕ).*

Remark 1.4. The condition that $k_a \cong w_1(d_a)+1 \mod 2$ when L is not orientable is essentially redundant if $g=0$, as it can easily be derived from the dimension condition (1.2).

Remark 1.5. The definition of the integers $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ does not use ϕ or the condition that $\dim X \leq 6$ in an essential way. The author believes that there exist far more general conditions under which similarly defined integers are invariant.

We now present an example of a non-trivial calculation of these invariants. See also Section 1.3 where we develop the relationship with Welschinger invariants, for which many interesting calculations have already been carried out [8].

Example 1.6. Let (X, L) be the pair consisting of the quintic threefold and its real part. That is, $X := \{\sum_{i=0}^4 z_i^5 = 0\} \subset \mathbb{C}P^4$ equipped with the symplectic form coming from the restriction of the Fubini-Study metric, and $L := X \cap \mathbb{R}P^4$. Let $\ell \in H_2(X, L)$ denote the generator with positive symplectic area. It is not hard to see that ℓ satisfies condition (1.3). We calculate $N_{D^2,\ell,0,0} = 30$. This may be interpreted as the number of oriented lines in the real quintic. It is interesting to compare this with the classical computation of 2875 lines in the complex quintic.

1.3 The relationship with real algebraic geometry

Real algebraic geometry provides a rich source of examples of symplectic manifolds admitting anti-symplectic involutions. Indeed, given any smooth real projective algebraic variety, we can take X to be its complexification, ω to be the pull-back of the Fubini-Study metric and ϕ to be complex conjugation. For this reason, it makes sense to call triples (X, ω, ϕ) real symplectic manifolds. Fix an ω-compatible almost complex structure J such that $\phi^* J = -J$. Let Σ be a Riemann surface with an anti-holomorphic involution $c : \Sigma \to \Sigma$, and let ν be a $c - \phi$ equivariant inhomogeneous perturbation. We can define real (J, ν)-holomorphic curves to be (J, ν)-holomorphic maps $u : \Sigma \to X$ such that $\phi \circ u \circ c = u \circ a$ for some $a \in Aut(\Sigma, \nu)$. Note that a given Riemann surface may have several different anti-holomorphic involutions. So, when we need to specify that a curve is real with respect to a particular anti-holomorphic involution, we use the terminology c-real.

Now, suppose $(\Sigma', \partial\Sigma')$ is a Riemann surface with boundary such that $\Sigma \simeq \Sigma' \bigcup_{\partial\Sigma'} \overline{\Sigma'}$ and c acts by exchanging Σ' and $\overline{\Sigma'}$. Any c-real (J, ν)-holomorphic curve $u : \Sigma \to X$, must satisfy

$$u^{-1}(Fix(\phi)) = Fix(c) \simeq \partial\Sigma'.$$

Since $Fix(c)$ divides Σ into Σ' and $\overline{\Sigma'}$, restricting u to either Σ' or $\overline{\Sigma'}$ gives a (J,ν)-holomorphic curve with boundary in the Lagrangian submanifold $L = Fix(\phi)$. Conversely, given a (J,ν)-holomorphic map $u' : (\Sigma', \partial\Sigma') \to (X, L)$, we can construct a c-real (J,ν)-holomorphic map $\Sigma \to X$ by gluing u' and $\phi \circ u' : (\overline{\Sigma'}, \overline{\partial\Sigma'}) \to (X, L)$ along their common boundary $\partial\Sigma'$ by the Schwarz reflection principle.

Let us denote by $\overline{\mathcal{M}_n}(X, \Sigma, d)$ the Gromov-compactification of the space of (J,ν)-holomorphic maps $\Sigma \to X$ with n marked points and let $\overline{\mathbb{R}_c\mathcal{M}_n}(X, d)$ denote its c-real locus. Let $r : H_2(X) \to H_2(X, L)$. We have just shown there exists a canonical map

$$\coprod_{\substack{\mathbf{k},|k|=n \\ \mathbf{d}',2d'=r(d), \\ \sum_a d'_a=\partial d'}} \overline{\mathcal{M}_{\mathbf{k},0}}(L, \Sigma', \mathbf{d}') \to \overline{\mathbb{R}_c\mathcal{M}_n}(X, \Sigma, d).$$

If $(\Sigma', \nu|_{\Sigma'})$ is not biholomorphic to $(\overline{\Sigma'}, \nu|_{\overline{\Sigma'}})$, this map is $1:1$ on the open stratum. If $(\Sigma', \nu|_{\Sigma'})$ is biholomorphic to $(\overline{\Sigma'}, \nu|_{\overline{\Sigma'}})$, then restricted to the open stratum, this map is a $2:1$ covering map. As an immediate consequence, we have the following proposition:

Proposition 1.7. *If $(\Sigma', \nu|_{\Sigma'})$ is not biholomorphic to $(\overline{\Sigma'}, \nu|_{\overline{\Sigma'}})$, the number of c-real (J,ν)-holomorphic maps $\Sigma \to X$ intersecting n generic real points of X is bounded below by*

$$\sum_{\substack{\mathbf{k},|k|=n \\ \mathbf{d}',2d'=j_*d, \\ \sum_a d'_a=\partial d'}} N_{\Sigma',\mathbf{d}',\mathbf{k},l}. \tag{1.4}$$

If $(\Sigma', \nu|_{\Sigma'})$ is biholomorphic to $(\overline{\Sigma'}, \nu|_{\overline{\Sigma'}})$, then we should take one half of (1.4) as a lower bound instead.

In [20, 21, 22], for strongly semi-positive real symplectic manifolds, Welschinger defined invariants counting real rational J-holomorphic curves intersecting a generic ϕ-invariant collection of marked points. Unlike in the usual definition of Gromov-Witten invariants, which depends on intersection theory on the moduli space of J-holomorphic curves, Welschinger defined his curve count by assigning signs to individual curves based on certain geometric-topological criteria. However, it turns out that

Welschinger's invariants admit the following intersection theoretic interpretation:

Theorem 1.8. *Let X be a strongly semi-positive real symplectic manifold satisfying the assumptions of Theorem 1.3. Then the numbers $N_{D^2,d,k,l}$ are twice the corresponding Welschinger invariant.*

Chapter 2

Orienting Cauchy-Riemann operators

In this section we analyze the choices necessary to orient the determinant of a real-linear Cauchy-Riemann operator. In the following, we use the symbol Γ to denote an appropriate Banach space completion of the smooth sections of a vector bundle. The exact choice of completion will not be important. If $V \to B$ is a vector bundle, we denote by $\mathfrak{F}(V)$ the principal $O(n)$ bundle with fiber at $x \in B$ given by the set of orthonormal frames in V_x. We call $\mathfrak{F}(V)$ the frame-bundle of V.

Definition 2.1. A *Pin*$^{\pm}$ *structure* $\mathfrak{p} = (P, p)$ on a vector bundle $V \to B$ consists of principal *Pin*$^{\pm}$ bundle $P \to B$ and a *Pin*$^{\pm}(n)$-$O(n)$ equivariant bundle map

$$p : P \to \mathfrak{F}(V).$$

A morphism of vector bundles with *Pin* structure $\phi : V \to V'$ is said to preserve *Pin* structure if there exists a lifting $\tilde{\phi}$,

$$\begin{tikzcd} P \arrow[r, "\tilde{\phi}"] \arrow[d, "p"] & P' \arrow[d, "p'"] \\ \mathfrak{F}(V) \arrow[r, "\phi"] & \mathfrak{F}(V'). \end{tikzcd}$$

Definition 2.2. A *Cauchy-Riemann Pin boundary value problem*

$$\underline{D} = (\Sigma, E, F, \mathfrak{p}, D)$$

consists of

- A Riemann surface Σ with boundary $\partial\Sigma = \coprod_{a=1}^{m} (\partial\Sigma)_a, (\partial\Sigma)_a \simeq S^1$.
- A complex vector bundle $E \to \Sigma$.
- A totally real sub-bundle over the boundary

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ \partial\Sigma \arrow[r] & \Sigma. \end{tikzcd}$$

- A $Pin^{\pm}$ structure $\mathfrak{p}$ on F.
- An orientation of $F|_{(\partial\Sigma)_a}$ for each a such that $F|_{(\partial\Sigma)_a}$ is orientable.
- A differential operator

$$D : \Gamma\left((\Sigma, \partial), (E, F)\right) \to \Gamma\left(\Sigma, \Omega^{0,1}(E)\right),$$

  satisfying, for $\xi \in \Gamma\left((\Sigma, \partial), (E, F)\right)$ and $f \in C^{\infty}(\Sigma, \mathbb{R})$,

$$D(f\xi) = fD\xi + (\bar{\partial} f)\xi.$$

  Such a D is known as a *real-linear Cauchy-Riemann operator.*

When it does not cause confusion, we will refer to such a collection by the operator alone, i.e. D, leaving the domain and range implicit.

Definition 2.3. A *morphism of Cauchy-Riemann Pin boundary value problems* $\underline{\phi}$: $\underline{D} \to \underline{D}'$ consists of

- A biholomorphism $f : \Sigma \to \Sigma'$.

- A morphism of bundles $\phi : E \to E'$ covering f such that $\phi|_{\partial\Sigma}$ takes F to F' and $\phi \circ D = D' \circ \phi$.

Such a morphism is called an *isomorphism* if ϕ is an isomorphism of vector bundles preserving *Pin* structure and preserving orientation if F, F', are orientable. When it causes no confusion, we may refer to such a morphism by the bundle-morphism component alone, i.e. ϕ.

Definition 2.4. We define the *determinant line* of a Fredholm operator D by

$$\det(D) := \Lambda^{\max}(\ker D) \otimes \Lambda^{\max}(\operatorname{coker} D)^*.$$

If D is a family of Fredholm operators, then we denote by $\det(D)$ the corresponding line bundle with the natural topology, as explained in, for example, [16, appendix A.2].

We now briefly recall the definition of the Maslov index $\mu(E, F)$ of the vector bundle pair (E, F) appearing in the definition of a Cauchy-Riemann boundary value problem in the case that $\partial\Sigma \neq \emptyset$. Indeed, if $\partial\Sigma \neq \emptyset$, we may trivialize E over Σ. Writing

$$\partial\Sigma = \coprod_{a=1}^{m} (\partial\Sigma)_a, \qquad (\partial\Sigma)_a \simeq S^1,$$

the restriction of F to each boundary component $(\partial\Sigma)_a$ defines a loop of totally real subspaces of $\mathbb{C}^n$. The Maslov index μ_a of such a loop was defined in [2]. We define

$$\mu(E, F) = \sum_{a=1}^{m} \mu_a.$$

It is not hard to see that although μ_a may depend on the choice of trivialization of E, the sum $\mu(E, F)$ does not. On the other hand, μ_a is well defined $\pmod 2$, and coincides with the first Steifel-Whitney class $w_1\left(F|_{(\partial\Sigma)_a}\right)$. We will use the following topological classification of vector-bundle pairs (E, F).

Lemma 2.5. *Two vector bundle pairs (E, F) and (E', F') of the same dimension* *admit an isomorphism*

$$\begin{tikzcd} E \arrow[r, "\sim", "\phi"'] & E' \\ F \arrow[u] \arrow[r, "\sim", "\phi|_{\partial\Sigma}"'] & F' \arrow[u] \end{tikzcd}$$

if an only if

$$\mu(E, F) = \mu(E', F'), \qquad w_1(F) = w_1(F').$$

Now, we describe a canonical orientation for the determinant line of a number of special examples of Cauchy-Riemann operators. Let $\tau \to \mathbb{C}P^1$ denote the tautological bundle. Let $c' : \mathbb{C}P^1 \to \mathbb{C}P^1$ and $\tilde{c}' : \tau \to \tau$ denote the automorphisms induced by complex conjugation. The fixed points of c' are simply $\mathbb{R}P^1$, and they divide $\mathbb{C}P^1$ into two copies of D^2. We define $\tau_{\mathbb{R}}$ to be the fixed points of $\tilde{c}'$ on $\tau|_{\mathbb{R}P^1}$. We define basic Cauchy-Riemann Pin boundary value problems

$$\begin{aligned} \underline{D}(-1, n) &:= (D^2, \tau|_{D^2} \oplus \mathbb{C}^{n-1}, \tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}, D_{-1,n}, \mathfrak{p}_{-1}), \\ \underline{D}(0, n) &:= (D^2, \mathbb{C}^n, \mathbb{R}^n, D_{0,n}, \mathfrak{p}_0). \end{aligned}$$

Here, the $Pin^{\pm}$ structure $\mathfrak{p}_0$ is canonically induced by the splitting into line bundles. On the other hand, $\mathfrak{p}_{-1}$ is not canonical, but we fix once possible choice and remain with it for the rest of the paper. $D_{-1,n}$ and $D_{0,n}$ are taken to be the standard Cauchy-Riemann operators on these bundles. Since $D_{-1,n}$ and $D_{0,n}$ are surjective we have

$$\begin{aligned} \det(D_{-1,n}) &= \Lambda^{\max}(\ker(D_{-1,n})) = \Lambda^{\max}(\mathbb{R}^{n-1}), \\ \det(D_{0,n}) &= \Lambda^{\max}(\ker(D_{0,n})) = \Lambda^{\max}(\mathbb{R}^n). \end{aligned}$$

So, $\det(D_{-1,n})$ and $\det(D_{0,n})$ admit canonical orientations. Finally, in the case $n = 2$, we will need certain special automorphisms $Q(i)$ of the bundle pair $(\mathbb{C}^2, \mathbb{R}^2) \to (D^2, \partial D^2)$. We define the restriction of $Q(i)$ to $\mathbb{R}^2 \to \partial D^2$ to be given by a loop in $SO(2)$ with homotopy class $n \in \pi_1(O(2)) \simeq \mathbb{Z}$. Then, we extend this automorphism arbitrarily over the inside of D^2 using the fact that the inclusion $SO(2) \hookrightarrow U(2)$ induces the trivial map on the fundamental group.

Lemma 2.6. *If $n \geq 3$, automorphisms of the bundle pair*

$$(\mathbb{C}^n, \mathbb{R}^n) \to (D^2, \partial D^2)$$

preserving Pin structure and orientation are all homotopic to the identity. If $n = 2$, automorphisms preserving Pin structure and orientation are homotopic to $Q(2i)$, $i \in \mathbb{Z}$. For all n, there are two homotopy classes of automorphisms of

$$(\tau \oplus \mathbb{C}^{n-1}, \tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}) \to (D^2, \partial D^2)$$

preserving Pin structure. One is homotopic to the identity and the other is homotopic to $-\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^n}$.

Remark 2.7. The assertions of Lemma 2.6 are clearly true when $n = 1$ without any reference to *Pin* structure. This is not surprising because all automorphisms of a real line bundle preserve *Pin* structure.

Proof of Lemma 2.6. For the trivial bundle pair $(\mathbb{C}^n, \mathbb{R}^n)$, homotopy classes of automorphisms preserving orientation are classified by $\pi_2(U(n), SO(n))$, which is easily calculated from the homotopy long exact sequence of the pair. In the case $n \geq 3$, we have

$$\begin{array}{ccccccc} \pi_2(U(n)) & \longrightarrow & \pi_2(U(n), SO(n)) & \xrightarrow{\sim} & \pi_1(SO(n)) & \xrightarrow{0} & \pi_1(U(n)). \\ \| & & & & \| & & \\ 0 & & & & \mathbb{Z}/2\mathbb{Z} & & \end{array}$$

The automorphisms preserving *Pin* structure map to $0 \in \pi_1(SO(n))$ so they are all homotopic to the identity. In the case $n = 2$, we have the same exact sequence, but $\pi_1(SO(2)) \simeq \mathbb{Z}$ and hence $\pi_2(U(2), SO(2)) \simeq \mathbb{Z}$. The automorphisms preserving *Pin* structure map to the subgroup $2\mathbb{Z} \subset \mathbb{Z} \simeq \pi_1(SO(2))$. On the other hand, by definition, exactly one of the automorphisms $Q(2i)$ maps to each element of the subgroup $2\mathbb{Z}$, implying the claim.

In the case of a non-trivial boundary condition, we will have to make a more explicit argument. Most of the work will be devoted to showing that the claim of the lemma is true for homotopy classes of automorphisms of the boundary condition alone. To verify this, we construct a convenient model for the boundary condition $\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}$. Indeed, let $r_1 \in O(n)$ be the reflection that acts on $\mathbb{R}^n$ by

$$r(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

We identify

$$\begin{tikzcd} \tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1} \arrow[r, "\sim"] \arrow[d] & \mathbb{R}^n \times [0,1]/(x,0) \sim (r_1(x), 1) \arrow[d] \\ \partial D^2 \arrow[r] & {[0,1]/0 \sim 1.} \end{tikzcd}$$

Let $\pi : Pin(n) \to O(n)$ denote the covering map. Letting e_i denote the standard basis vectors in $\mathbb{R}^n$, and thinking of $Pin(n)$ as the group generated by the unit vectors in the Clifford algebra, we have $\pi(e_1) = r_1$. So, we may define a Pin structure on $\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}$ by Diagram 2-1. Here, $\cdot$ denotes Clifford multiplication. It follows that an

$$\begin{tikzcd} P \arrow[r, "\sim"] \arrow[d] & Pin(n) \times [0,1]/(p,0) \sim (e_1 \cdot p, 1) \arrow[d, "\pi \times \mathrm{Id}"] \\ \mathfrak{F}(\tau \oplus \mathbb{R}^{n-1}) \arrow[r, "\sim"] & O(n) \times [0,1]/(o,0) \sim (r_1 o, 1). \end{tikzcd}$$

Diagram 2-1

automorphism of P is given by a map

$$a : [0,1] \to Pin(n)$$

such that

$$e_1 \cdot a(0) = a(1) \cdot e_1. \tag{2.1}$$

In particular, we see that $\mathrm{Id}_{\tau_{\mathbb{R}}} \oplus \mathrm{Id}_{\mathbb{R}^{n-1}}$ lifts to the identity automorphism of P, and $-\mathrm{Id}_{\tau_{\mathbb{R}}} \oplus \mathrm{Id}_{\mathbb{R}^{n-1}}$ lifts to the automorphism of P given by $a(t) = e_1$.

We claim that up to homotopy, these are the only two possibilities. First we consider the case $n \geq 3$. Indeed, noting that $Pin(n)$ has two components, one containing $\mathrm{Id}_{Pin(n)}$ and the other containing e_1, it suffices to show that if automorphisms a and a' map to the same component of $Pin(n)$ then they are homotopic through automorphisms. Indeed, connect $a(0)$ to $a'(0)$ by an arbitrary path b. Connect $a(1)$ to $a'(1)$ by $\pm e_1 \cdot b \cdot e_1$, where the sign depends on whether we work in Pin^+ or Pin^-. The resulting loop is null homotopic by the simply-connectedness of each component of $Pin(n)$. Reparameterizing a null-homotopy, we obtain a family of automorphisms connecting a and a' as desired.

We turn to the case $n = 2$. Since $Pin(2)$ is not simply-connected, we must be more carefully. Indeed, topologically, $Pin(2) \simeq S^1 \coprod S^1$. One component consists of spinors of the form $cos(\theta) + sin(\theta)e_1 \cdot e_2$ and the other consist of spinors of the form $cos(\theta)e_1 + \sin(\theta)e_2$. So, if we think of S^1 as the complex numbers of unit length, conjugation by e_1 acts by complex conjugation on each component of $Pin(2)$. So, an automorphism of P is given by a path in one of the two copies of S^1 with complex conjugate endpoints. It suffices to show that any such path is homotopic through similar paths to the constant path at ± 1 or $\pm e_1$. Indeed, $\pi(\pm 1) = \mathrm{Id}_{\tau_{\mathbb{R}} \oplus \mathbb{R}}$ and $\pi(\pm e_1) = -\mathrm{Id}_{\tau_{\mathbb{R}}} \oplus \mathrm{Id}_{\mathbb{R}}$. So, consider the covering map $\mathbb{R} \to S^1$ given by $x \rightsquigarrow e^{2\pi i x}$. Given a path in S^1 with conjugate endpoints, we may lift it to path

$$x : [0, 1] \to \mathbb{R}$$

such that $x(0) \cong -x(1) \pmod 1$. Since either

$$\frac{x(0) + x(1)}{2} \cong 0 \pmod 1 \qquad \text{or} \qquad \frac{x(0) + x(1)}{2} \cong \frac{1}{2} \pmod 1,$$

linear interpolation between x and $\frac{x(0)+x(1)}{2}$ yields a homotopy x_t such that

$$x_t(0) \cong -x_t(1) \pmod 1, \qquad x_0(s) = x(s), \qquad x_1(s) \cong 0 \text{ or } \frac{1}{2} \pmod 1.$$

Then $\pi(x_t)$ yields the desired homotopy of a.

Now we extend our conclusion to automorphisms of the pair. Trivializing $\tau \oplus \mathbb{C}^{n-1}$

over D^2, we may identify an automorphism of $\tau \oplus \mathbb{C}^{n-1}$ with a map

$$A : D^2 \to U(n).$$

We are interested in A that preserve $\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}$ over ∂D^2, such that the induced automorphism of $\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}$ preserves *Pin* structure. By the preceding calculation, two examples are given by $\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^{n-1}}$ and $-\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^{n-1}}$. We claim that up to homotopy these are the only two examples. Indeed, given A, choose a lift of the induced automorphism on $\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}$ to P and denote it by a. As just proved, a is homotopic to either $\mathrm{Id}_{Pin(n)}$ or e_1. Denote the homotopy by

$$\tilde{B} : [0,1] \to Aut(P), \qquad \tilde{B}(0) = a, \qquad \tilde{B}(1) = \mathrm{Id}_{Pin(n)} \text{ or } e_1.$$

We will construct a homotopy from A to $A' : D^2 \to U(n)$, where A' correspond to the automorphism

$$\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^{n-1}} \text{ or } -\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^{n-1}}$$

as $\tilde{B}(1) = \mathrm{Id}_{Pin(n)}$ or $\tilde{B}(1) = e_1$ respectively. Indeed, $\tilde{B}$ defines a path

$$B : [0,1] \to Aut(\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}).$$

Denote by

$$i : Aut(\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}) \hookrightarrow Aut(\tau \oplus \mathbb{C}^{n-1}|_{\partial D^2})$$

the inclusion given by complexification. Restricting the previously mentioned trivialization of $\tau \oplus \mathbb{C}^{n-1}$ to ∂D^2, we may identify the path $i \circ B$ with a map

$$\hat{B} : [0,1] \times \partial D^2 \to U(n).$$

Capping off this cylinder with the disk A at one end and the disk A' at the other end, we obtain a map $S^2 \to U(n)$, which is well known to be null-homotopic. Reparameterizing a null-homotopy gives the required homotopy through automorphisms from A to A'. All these automorphisms preserve the boundary condition and its *Pin* structure by the construction of $\hat{B}$. $\square$

Proposition 2.8. *The determinant line of a real-linear Cauchy-Riemann Pin boundary value problem $\underline{D}$ admits a canonical orientation. If $\underline{\phi} : \underline{D} \to \underline{D}'$ is an isomorphism, then the induced morphism*

$$\phi : \det(D) \to \det(D')$$

preserves the canonical orientation. Furthermore, the canonical orientation varies continuously in a family of Cauchy-Riemann operators. That is, it defines a single component of the determinant line bundle over that family.

Proof. Near each boundary component $(\partial\Sigma)_a$ choose a closed curve γ_a homotopic to $(\partial\Sigma)_a$. Degenerate Σ by contracting the curves γ_a to points to obtain a nodal surface $\hat{\Sigma}$. $\hat{\Sigma}$ consists of one closed component $\tilde{\Sigma}$, a disk Δ_a corresponding to each boundary component $(\partial\Sigma)_a$ and a nodal point $\hat{\gamma}_a$ corresponding to each curve γ_a. There exists a continuous map $\pi : \Sigma \to \hat{\Sigma}$ which is a smooth diffeomorphism away from the nodal points $\hat{\gamma}_a$. So, we may define

$$\hat{F} = (\pi|_{(\partial\Sigma)_a})^{-1*}F.$$

At the same time, degenerate E to a vector bundle $\hat{E} \to \hat{\Sigma}$ such that

$$(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial\Delta_a}) \simeq \begin{cases} (\tau \oplus \mathbb{C}^{n-1}, \tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}) & \text{if } w_1(F|_{(\partial\Sigma)_a}) = 1 \\ (\mathbb{C}^n, \mathbb{R}^n) & \text{if } w_1(F|_{(\partial\Sigma)_a}) = 1 \end{cases} \tag{2.2}$$

We choose the isomorphism (2.2) to preserve *Pin* structure and to preserve orientation in the orientable case.

Equip $\hat{E}|_{\Delta_a}$ with the Cauchy-Riemann operator D_a induced by the isomorphism (2.2) from $D_{-1,n}$ (resp. $D_{0,n}$). The isomorphism induces an orientation on $\det(D_a)$ from the canonical orientation of $\det(D_{-1,n})$ (resp. $\det(D_{0,n})$). Choose a Cauchy-Riemann operator $\tilde{D}$ on $\hat{E}|_{\tilde{\Sigma}}$. Equip $\det(\tilde{D})$ with the canonical complex orientation.

Define an operator

$$d_{\hat{\gamma}_a} : \Gamma(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial\Delta_a}) \oplus \Gamma(\hat{E}|_{\tilde{\Sigma}}) \to E_{\hat{\gamma}_a}$$

by

$$d_{\hat{\gamma}_a}(\xi, \eta) = \xi(\hat{\gamma}_a) - \eta(\hat{\gamma}_a), \qquad \xi \in \Gamma(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial\Delta_a}), \qquad \eta \in \Gamma(\hat{E}|_{\tilde{\Sigma}}).$$

Gluing the D_a with $\tilde{D}$ at $\hat{\gamma}_a$ we obtain a Cauchy-Riemann operator $\#_a D_a \# \tilde{D}$ on E along with an isomorphism of virtual vector spaces

$$\operatorname{index}\left(\#_a D_a \# \tilde{D}\right) \simeq \operatorname{index}\left(\bigoplus_a D_a \oplus \tilde{D} \oplus \bigoplus_a d_{\hat{\gamma}_a}\right),$$

or equivalently, an isomorphism

$$\det(\#_a D_a \# \tilde{D}) \simeq \bigotimes_a \det(D_a) \otimes \det(\tilde{D}) \otimes \bigotimes_a \det(E_{\hat{\gamma}_a})^*. \tag{2.3}$$

Since the space of Cauchy-Riemann operators on E is contractible, choosing a one-parameter family $\underline{D}_t$ with $\underline{D}_0 = D$ and $\underline{D}_1 = \#_a D_a \# \tilde{D}$ and trivializing the line bundle $\det(\underline{D}_t)$ over the family induces a canonical orientation on $\det(D)$.

We claim that the orientation is independent of the choice of isomorphism (2.2), the choice of $\tilde{D}$, and the choice of $\underline{D}_t$. First we prove the independence of the choice of $\underline{D}_t$. Indeed, since the space of Cauchy-Riemann operators on E is contractible, given any two families $\underline{D}_t$ and $\underline{D}'_t$, we can construct a homotopy between them $\overline{\underline{D}}_{s,t}$, such that

$$\overline{\underline{D}}_{0,t} = \underline{D}_t, \qquad \overline{\underline{D}}_{1,t} = \underline{D}'_t.$$

Trivializing $\det(\overline{\underline{D}}_{t,s})$ over the homotopy proves that $\underline{D}_t$ and $\underline{D}'_t$ give the same answer.

Now we turn to proving independence of the choice of isomorphism (2.2) and the choice of $\tilde{D}$. Another choice of isomorphism (2.2) would induce a different operator D'_a in place of D_a. Also, let $\tilde{D}'$ be another Cauchy-Riemann operator on $\hat{E}|_{\tilde{\Sigma}}$. We prove that these new choices induce the same orientation on D. Choose homotopies

$\underline{D}_{a,t}$ and $\underline{\tilde{D}}_t$ such that

$$\underline{D}_{a,1} = D_a, \qquad \underline{D}_{a,\frac{1}{2}} = D'_a, \qquad \underline{\tilde{D}}_1 = \tilde{D}, \qquad \underline{\tilde{D}}_{\frac{1}{2}} = \tilde{D}'.$$

We choose the family $\underline{D}_t$ so that, as before, $\underline{D}_0 = D$ and $\underline{D}_1 = \#_a D_a \# \tilde{D}$, but we require also that

$$\underline{D}_t = \#_a \underline{D}_{a,t} \# \underline{\tilde{D}}'_t, \qquad t \in \left[\frac{1}{2}, 1\right].$$

Since this choice of $\underline{D}_t$ is as good as any other, it remains only to show that the orientation on $\det(D'_a)$ induced by the isomorphism $\det(D'_a) \xrightarrow{\sim} \det(D_{i,n})$, $i = -1$ or 0, agrees with the orientation induced from $\det(D_a)$ by trivializing $\det(\underline{D}_{a,t})$ over the interval $[\frac{1}{2}, 1]$. Similarly, we must show that the complex orientation on $\det(\tilde{D}')$ agrees with orientation induced from $\det(\tilde{D})$ by trivializing $\det(\underline{\tilde{D}}_t)$ over the interval $[\frac{1}{2}, 1]$. The latter agreement follows from the compatibility of the topology of the determinant bundle over a family with the canonical complex orientation. To see the former agreement, note that the isomorphism (2.2) is determined up to an automorphism preserving *Pin* structure of the right hand bundle pair. In the orientable case where $n \geq 3$, by Lemma 2.6, all such automorphisms are homotopic to the identity. So, we may assume that $\underline{D}_{a,t}$ is induced by a family of automorphisms. Then, it suffices to note the that the determinant bundle is tautologically trivial over a family of gauge-equivalent operators. The case $n = 2$ may be reduced to the higher dimensional case by stabilizing by a copy of the trivial bundle pair. Indeed, $Q(2i) \oplus \mathrm{Id}_{\mathbb{R}}$ is homotopic to $\mathrm{Id}_{\mathbb{R}^3}$. In the non-orientable case, we need to consider the additional possibility that the automorphism is homotopic to $-\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^{n-1}}$. But $-\mathrm{Id}_\tau \oplus \mathrm{Id}_{\mathbb{C}^{n-1}}$ clearly preserves the orientation of $\det(D_{-1})$, so this possibility does not effect the argument. The remaining claims of the lemma follow immediately from the construction. □

Lemma 2.9. *If $F|_{(\partial\Sigma)_a}$ is orientable, then changing the orientation on $F|_{(\partial\Sigma)_a}$ will change the canonical orientation on* $\det(D)$ *given in Proposition 2.8.*

Proof. This is an immediate consequence of the proof of Proposition 2.8. □

In the following lemmas, we use the fact that $H^1(B, \mathbb{Z}/2\mathbb{Z})$ acts naturally transitively on the set of *Pin* structures on a vector bundle $V \to B$. See [10].

Lemma 2.10. *Changing the Pin structure* $\mathfrak{p}$ *of a real-linear Cauchy-Riemann Pin boundary value problem by the action of the generator of*

$$H^1((\partial\Sigma)_a, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(\partial\Sigma, \mathbb{Z}/2\mathbb{Z})$$

reverses the canonical orientation of Proposition 2.8.

Proof. By the proof of Proposition 2.8 it suffices to consider the special cases $\underline{D}(-1,n)$ and $\underline{D}(0,n)$. For the case $\underline{D}(0,n)$, see [5, Remark 21.6]. For the case $\underline{D}(-1,n)$, it suffices to show that

$$A := \mathrm{Id}_\tau \oplus -\mathrm{Id}_\mathbb{C} \oplus \mathrm{Id}_{\mathbb{C}^{n-2}},$$

which clearly reverses the orientation of $\det(D_{-1})$, does not preserve *Pin* structure. We use the identification of Diagram 2-1 to show that that $A|_{\tau_\mathbb{R}\oplus\mathbb{R}^{n-1}}$ does not lift to an automorphism of P. Indeed, let $r_2 \in O(n)$ be the reflection that acts on $\mathbb{R}^n$ by the formula

$$r_2(x_1, x_2, x_3, \ldots, x_n) = (x_1, -x_2, x_3, \ldots, x_n).$$

and, as before, let $\pi : \mathit{Pin}(n) \to O(n)$ denote the canonical covering map. The automorphism $A|_{\tau_\mathbb{R}\oplus\mathbb{R}^{n-1}}$ acts on $\mathfrak{F}(\tau_\mathbb{R} \oplus \mathbb{R}^{n-1})$ by the explicit formula $(o,t) \rightsquigarrow (r_2 o, t)$. If this automorphism were to lift to P, it would be given left-multiplication by a where $a \in \mathit{Pin}(n)$ satisfied $\pi(a) = r_2$. So, thinking of $\mathit{Pin}(n)$ as the group generated by the unit vectors in the Clifford algebra, we would have $a = \pm e_2$. But this contradicts condition (2.1), since

$$e_1 \cdot e_2 = -e_2 \cdot e_1.$$

□

We now introduce a lemma concerning a particular class of Cauchy-Riemann *Pin* boundary value problem which will play an important role in understanding the significance of relative *Pin* structures.

Lemma 2.11. *Let $V \to \Sigma$ be a real vector bundle over a Riemann surface with boundary. Consider $\underline{D} = (\Sigma, V \otimes \mathbb{C}, V|_{\partial\Sigma}, \mathfrak{p}, D)$. The canonical orientation of* $\det(D)$ *is the same for any $\mathfrak{p}$ that arises by restricting a Pin structure for V over Σ to $\partial\Sigma$.*

Proof. Let $i : \partial\Sigma \to \Sigma$ denote the canonical inclusion. By Lemma 2.10, it suffices to show that any change of *Pin* structure over Σ would change the *Pin* structure over $\partial\Sigma$ by the action of the sum of the generators of $H^1((\partial\Sigma)_a)$ for an even number of components $(\partial\Sigma)_a$ of $\partial\Sigma$. Now, any two *Pin* structures of V over Σ may be related by the action of $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. So we may equivalently show that for all $\alpha \in H^1(\Sigma)$ we have $i^*\alpha(\partial\Sigma) = 0 \pmod 2$. But this follows immediately because with $\mathbb{Z}/2\mathbb{Z}$ coefficients i^* is the dual of i_*, and tautologically $i_*([\partial\Sigma]) = 0$. □

Now, we will calculate the sign of conjugation on the canonical orientation of the determinant line of a Cauchy-Riemann *Pin* boundary value problem. More precisely, given a Riemann surface Σ, let $\overline{\Sigma}$ denote the same topological surface with conjugate complex structure, and let

$$t : \Sigma \to \overline{\Sigma}$$

denote the tautological anti-holomorphic map. Similarly, let $(\overline{E}, \overline{F})$ denote the same real bundle pair with the opposite complex structure on E, and let

$$T : E \to \overline{E}$$

denote the tautological anti-complex-linear bundle map. Furthermore, a Cauchy-Riemann operator D on the bundle $E \to \Sigma$ is the same as a Cauchy-Riemann operator $\overline{D}$ on the bundle $\overline{E} \to \overline{\Sigma}$. So, given any Cauchy-Riemann *Pin* boundary problem $\underline{D}$, we may construct its conjugate $\overline{\underline{D}}$. Clearly, we have a tautological map of Cauchy-Riemann *Pin* boundary value problems,

$$\begin{tikzcd} \Gamma(\Omega^{0,1}(E)) \arrow[r, "t^{-1*}\otimes T"] & \Gamma(\Omega^{0,1}(\overline{E})) \\ \Gamma(E,F) \arrow[u, "D"] \arrow[r, "T"] & \Gamma(\overline{E},\overline{F}) \arrow[u, "\overline{D}"'] \end{tikzcd}$$

which we denote by

$$\underline{T} : \underline{D} \to \overline{\underline{D}}.$$

In the following proposition, we denote by g_0 the genus of $\Sigma/\partial\Sigma$ and we write $n = \dim_{\mathbb{C}}(E) = \dim_{\mathbb{R}}(F)$.

Proposition 2.12. *The sign of the induced isomorphism*

$$T : \det(D) \to \det(\overline{D})$$

relative to the canonical orientation is given by

$$\begin{aligned} \mathfrak{s}_T^+(\underline{D}) := & \frac{\mu(E,F)(\mu(E,F)+1)}{2} + (1-g_0)n + mn \\ & + \sum_{a<b} w_1(F)((\partial\Sigma)_a) w_1(F)((\partial\Sigma)_b) \mod 2, \end{aligned}$$

for Pin^+ structure and

$$\begin{aligned} \mathfrak{s}_T^-(\underline{D}) := & \frac{\mu(E,F)(\mu(E,F)+1)}{2} + (1-g_0)n + mn \\ & + \sum_{a<b} w_1(F)((\partial\Sigma)_a) w_1(F)((\partial\Sigma)_b) + w_1(F)(\partial\Sigma) \mod 2, \end{aligned}$$

for a Pin^- structure.

Remark 2.13. When $\Sigma = D^2$, since $\mu(E,F) \cong w_1(F)(\partial\Sigma) \pmod 2$, we have the relatively simple formula

$$\mathfrak{s}_T^{\pm}(\underline{D}) \cong \frac{\mu(E,F)(\mu(E,F) \pm 1)}{2} \pmod 2. \tag{2.4}$$

Before proving the Proposition, we will need the following lemma.

Lemma 2.14. *The map $\underline{T} : \underline{D}(-1,n) \to \overline{\underline{D}(-1,n)}$ preserves orientation if $\mathfrak{p}_{-1}$ is Pin^+, but not if $\mathfrak{p}_{-1}$ is Pin^-.*

Proof. In this proof, and later in this paper as well, we will need to make use of the anti-holomorphic involution,

$$c : \mathbb{C}P^1 \to \mathbb{C}P^1, \qquad [z_0 : z_1] \rightsquigarrow [\bar{z}_0 : -\bar{z}_1]$$

and the natural involution $\tilde{c}$ of the tautological bundle τ covering c. We note that c preserves the two hemispheres of $\mathbb{C}P^1$ which lie on either side of $\mathbb{R}P^1 \subset \mathbb{C}P^1$. So we may restrict c, $\tilde{c}$, to either of the hemispheres $D^2 \subset \mathbb{C}P^1$, and we denote the restriction as well by c, $\tilde{c}$. Furthermore, let C denote the bundle morphism of the trivial bundle $\mathbb{C} \to \mathbb{C}P^1$ covering c that acts on the fiber by complex conjugation. The lemma will follow immediately from Proposition 2.8 if we show that

$$\tilde{c} \oplus C^{\oplus n-1} : \overline{\underline{D}(-1, n)} \to \underline{D}(-1, n)$$

is an isomorphism of Cauchy-Riemann boundary value problems in the Pin^+ case whereas

$$\tilde{c} \oplus -C \oplus C^{\oplus n-2} : \overline{\underline{D}(-1, n)} \to \underline{D}(-1, n)$$

is an isomorphism in the Pin^- case. We treat only the Pin^- case since the Pin^+ case is very similar and not as interesting. The only property of being an isomorphism of Cauchy-Riemann boundary value problems which is not immediately evident is the preservation of Pin^- structure. To verify this, we again work with the explicit model of Diagram 2-1 for $\mathfrak{F}(\tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1})$ and P. In this model, it is not hard to see that at the level of the frame bundle,

$$\left(\tilde{c} \oplus -C \oplus C^{\oplus n-2}\right)(o, t) = (r_1 r_2 o, 1 - t).$$

So, a lifting of this map to P must act by

$$(p, t) \rightsquigarrow (a \cdot p, 1 - t)$$

where $a \in Pin^-(n)$ such that $\pi(a) = r_1 r_2$. It remains to check that this lifting respects the equivalence relation defining P. Indeed, $(p, 0)$ and $(e_1 \cdot p, 1)$ represent the same point in P, so we must have

$$(a \cdot p, 1) \sim (a \cdot e_1 \cdot p, 0).$$

But that is the same as

$$a = e_1 \cdot a \cdot e_1.$$

Choosing, for example, $a = e_1 \cdot e_2$, and using the Clifford multiplication of Pin^-, we verify

$$e_1 \cdot e_1 \cdot e_2 \cdot e_1 = -e_2 \cdot e_1 = e_1 \cdot e_2.$$

$\square$

Proof of Proposition 2.12. Trivialize E and let η be the number of boundary components $(\partial\Sigma_a)$ for which $w_1(F)((\partial\Sigma)_a) \simeq 1 \pmod 2$. Then the first Chern class of the bundle $\hat{E}|_{\tilde{\Sigma}}$ from the proof of Proposition 2.8 is given by

$$c_1\left(\hat{E}|_{\tilde{\Sigma}}\right) = \frac{\mu(E, F) + \eta}{2}.$$

Abbreviating $\mu = \mu(E, F)$, we calculate

$$\begin{aligned}
\frac{\mu(\mu+1)}{2} - \frac{\mu+\eta}{2} &\cong \frac{\mu^2}{2} - \frac{\eta}{2} \\
&\cong \frac{\eta^2 - \eta}{2} \\
&\cong \binom{\eta}{2} \\
&\cong \sum_{a<b} w_1(F)((\partial\Sigma)_a) w_1(F)((\partial\Sigma)_b) \pmod 2.
\end{aligned}$$

Here, the second congruence uses the fact that $\mu^2 \simeq \eta^2 \pmod 4$. So, with $\tilde{D}$ as in the proof of Proposition 2.8, the Riemann-Roch theorem gives

$$\begin{aligned}\operatorname{index}_{\mathbb{C}}(\tilde{D}) &= c_1\left(\hat{E}|_{\tilde{\Sigma}}\right) + n(1-g_0)\\ &\cong \frac{\mu(\mu+1)}{2} + n(1-g_0)\\ &\quad + \sum_{a<b} w_1(F)((\partial\Sigma)_a)w_1(F)((\partial\Sigma)_b) \pmod 2. \end{aligned} \tag{2.5}$$

We now combine the orientation changes under conjugation of the tensor factors on the right-hand side of equation (2.3). Note that conjugation on a complex virtual vector space leads to a sign change which is exactly its dimension $\pmod 2$. So, $\det(\tilde{D})$ changes orientation in accordance with the formula (2.5). Similarly, $\bigotimes_a \det(E_{\tilde{\gamma}_a})$ changes orientation by hn. The orientation change for $\det(D_a)$ was calculated in Lemma 2.14. This last factor accounts for the difference between $\mathfrak{s}_T^+$ and $\mathfrak{s}_T^-$. □

Definition 2.15. A *short exact sequence of families of Fredholm operators*

$$0 \to D' \to D \to D'' \to 0$$

consists of a parameter space B, short exact sequences of Banach space bundles over B,

$$0 \to X' \to X \to X'' \to 0,$$
$$0 \to Y' \to Y \to Y'' \to 0,$$

and Fredholm Banach space bundle morphisms

$$D : X \to Y, \qquad D' : X' \to Y', \qquad D'' : X'' \to Y'',$$

such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \\ 0 \arrow[r] & X' \arrow[r] \arrow[u, "D'"] & X \arrow[r] \arrow[u, "D"] & X'' \arrow[r] \arrow[u, "D''"] & 0 \end{tikzcd}$$

commutes.

Lemma 2.16. *A short exact sequence of families of Fredholm operators*

$$0 \to D' \to D \to D'' \to 0$$

induces an isomorphism

$$\det(D') \otimes \det(D'') \xrightarrow{\sim} \det(D).$$

Chapter 3

Orienting moduli spaces of open stable maps

First, we set the basic assumptions which will hold throughout this section. Let (X, ω) be a symplectic manifold with $\dim X = 2n$ and let $L \subset X$ be a Lagrangian submanifold. In the following, we assume L is relatively $Pin^{\pm}$ and fix a relative $Pin^{\pm}$ structure $\mathfrak{p}$ on L. Furthermore, if L is orientable, fix an orientation on L. Let $(\Sigma, \partial\Sigma)$ denote a Riemann surface with boundary and assume $\partial\Sigma = \coprod_{a=1}^{m} (\partial\Sigma)_a$, where $(\partial\Sigma)_a \simeq S^1$. Now, for

$$\mathbf{d} = (d, d_1, \ldots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},$$

define $B^{1,p}(L, \Sigma, \mathbf{d})$ to be the Banach manifold of $W^{1,p}$ maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ such that $u_*([\Sigma, \partial\Sigma]) = d$ and $u|_{(\partial\Sigma)_a *}([(\partial\Sigma)_a]) = d_a$. Furthermore, define

$$B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) := B^{1,p}(L, \Sigma, \mathbf{d}) \times \prod_a (\partial\Sigma)_a^{k_a} \times \Sigma^l \setminus \Delta.$$

Here Δ denotes the subset of the product in which two marked points coincide. We will use $\vec{z} = (z_{ai})$ and $\vec{w} = (w_j)$ to denote marked points in $\partial\Sigma$ and Σ respectively, and we use $\mathbf{u} = (u, \vec{z}, \vec{w})$ to denote elements of $B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$. Note that we may occasionally omit the L, $\Sigma, \mathbf{d}$, from the preceding notation when it is clear from the context. There exist canonical evaluation maps

$$evb_{ai} : \overline{\mathcal{M}_{\mathbf{k},l}}(L, \Sigma, \mathbf{d}) \to L, \qquad i = 1 \dots k_a,\ a = 1 \dots m,$$
$$evi_j : \overline{\mathcal{M}_{\mathbf{k},l}}(L, \Sigma, \mathbf{d}) \to X, \qquad j = 1 \dots l,$$

given by $evb_{ai}(\mathbf{u}) = u(z_a i)$ and $evi_j(\mathbf{u}) = u(w_j)$. We note that the above notation is also used for the evaluation maps from the moduli spaces of holomorphic curves, which are just restrictions of the maps above.

Define the Banach space bundle $\mathcal{E} \to B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ fiberwise by

$$\mathcal{E}_{\mathbf{u}} := L_p(\Sigma, \Omega^{0,1}(u^*TX))$$

for $\mathbf{u} \in B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$. Now, fix $J \in \mathcal{J}_\omega$, and $\nu \in \mathcal{P}$. Let

$$\overline{\partial}_{(J,\nu)} : B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \to \mathcal{E}$$

denote the section of $\mathcal{E}$ given by the ν-perturbed Cauchy Riemann operator. Using the canonical identification between the vertical tangent spaces of $\mathcal{E}$ and $\mathcal{E}$ itself we define

$$D := D\overline{\partial}_{(J,\nu)} : TB^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \to \mathcal{E}$$

to be the vertical component of the linearization of $\overline{\partial}_{(J,\nu)}$. We will denote linearization at any given $\mathbf{u} \in B^{1,p}_{\mathbf{k},l}$ by $D_{\mathbf{u}}$. Finally, define $\mathcal{L} \to B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ to be the determinant line bundle of the family of Fredholm operators D,

$$\mathcal{L} := \det(D).$$

The following proposition is a basic ingredient in the proof of Theorem 1.1. Suppose either L is orientable and provided with an orientation or $k_a \cong w_1(d_a) + 1 \mod 2$.

Proposition 3.1. *The combination of an orientation of L if L is orientable and the* *choice of relative $Pin^{\pm}$ structure $\mathfrak{p}$ on L canonically determine an isomorphism of line bundles*

$$\mathcal{L} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^{*} \det(TL).$$

Proof. Clearly, the proposition will follow immediately if we succeed in providing

$$\mathcal{L}' := \mathcal{L} \otimes \bigotimes_{a,i} evb_{ai}^{*} \det(TL)^{*}$$

with a canonical orientation depending only on the orientation of L and $\mathfrak{p}$. We observe that it suffices to canonically orient the fiber $\mathcal{L}'_{\mathbf{u}}$ over each $\mathbf{u} \in B_{\mathbf{k},l}^{1,p}$ individually in a way that varies continuously with $\mathbf{u}$.

Recall that the relative *Pin* structure of L specifies a triangulation of the pair (X, L). Using simplicial approximation, we homotope the map $u : (\Sigma, \partial\Sigma) \to (X, L)$ to a map $\hat{u} : (\Sigma, \partial\Sigma) \to (X^{(2)}, L^{(2)})$. Denote the homotopy by

$$\Phi : [0,1] \times (\Sigma, \partial\Sigma) \to (X, L).$$

We claim that the choice of Φ is unique up to homotopy. Indeed, suppose Φ' is another such homotopy. Concatenating Φ and Φ', we obtain a map

$$\Phi \# \Phi' : [-1,1] \times (\Sigma, \partial\Sigma) \to (X, L).$$

By simplicial approximation we may homotope $\Phi\#\Phi'$ to map into $(X^{(3)}, L^{(3)})$. Reparameterizing this homotopy, we obtain a homotopy from Φ to Φ'. We denote the homotopy from Φ to Φ' by

$$\Psi : [0,1]^2 \times (\Sigma, \partial\Sigma) \to (X, L),$$

such that,

$$\Psi(0,t) = \Phi(t), \qquad \Psi(1,t) = \Phi', \qquad \Psi(s,0) = u.$$

Now, define

$$B' := \left\{ (u, \vec{z}, \vec{w}) \in B^{1,p}_{\mathbf{k},l} \middle| u : (\Sigma, \partial\Sigma) \to (X^{(3)}, L^{(3)}) \right\}.$$

We now prove that the homotopy uniqueness of Φ implies that it suffices to orient $\mathcal{L}'|_{B'}$. Indeed, think of Φ (resp. Ψ) as a map from $[0,1]$ (resp. $[0,1]^2$) to $B^{1,p}_{\mathbf{k},l}$. Given an orientation on $\mathcal{L}'|_{B'}$, trivializing $\Phi^*\mathcal{L}'$ induces an orientation of $\mathcal{L}'_{\mathbf{u}}$. This orientation agrees with the orientation induced by any other homotopy Φ', because we may trivialize $\Psi^*\mathcal{L}'$. We note that the orientation on $\mathcal{L}'_{\mathbf{u}}$ thus induced varies continuously with $\mathbf{u}$. Indeed, given a one parameter family $\mathbf{u}_t \in B^{1,p}_{\mathbf{k},l}$ we may choose a homotopy of the one-parameter family Φ_t and trivialize $\Phi_t^*\mathcal{L}'$.

We turn to orienting $\mathcal{L}'|_{B'}$. The relative *Pin* structure of (X, L) provides a vector bundle $V \to X^{(3)}$ and a $Pin^{\pm}$ structure on $V|_{L^{(3)}} \oplus TL|_{L^{(3)}}$. We introduce the shorthand notation

$$V_{\mathbb{R}} := V|_{L^{(3)}} \oplus TL|_{L^{(3)}}, \qquad V_{\mathbb{C}} := V \otimes \mathbb{C}. \tag{3.1}$$

Again it suffices to canonically orient each individual line $\mathcal{L}'_{\mathbf{u}}$ for $\mathbf{u} \in B'$ in a way that varies continuously in families. Let D_0 be an arbitrary real Cauchy-Riemann operator on $u^*V \otimes \mathbb{C}$. We consider the operator $D_{\mathbf{u}} \oplus D_0$,

$$\begin{array}{ccc} TB^{1,p}_{\mathbf{k},l} \oplus W^{1,p}(u^*V_{\mathbb{C}}, u^*V_{\mathbb{R}}) & \xrightarrow{\quad D_{\mathbf{u}} \oplus D_0 \quad} & \mathcal{E} \oplus L^p(\Omega^{0,1}(u^*V_{\mathbb{C}})) \\ \| & & \| \\ W^{1,p}(u^*(TX \oplus V_{\mathbb{C}}), u|^*_{\partial\Sigma}(TL \oplus V_{\mathbb{R}})) \oplus \mathbb{R}^{|\mathbf{k}|} \oplus \mathbb{C}^l & & L^p(\Omega^{0,1}(u^*(TX \oplus V_{\mathbb{C}}))). \end{array}$$

Clearly, there exists a short exact sequence of Fredholm operators

$$0 \to D_{\mathbf{u}} \to D_{\mathbf{u}} \oplus D_0 \to D_0 \to 0.$$

So, by Lemma 2.16 there exists a natural isomorphism

$$\det(D_{\mathbf{u}}) \xrightarrow{\sim} \det(D_{\mathbf{u}} \oplus D_0) \otimes \det(D_0)^*,$$

and, after tensoring on both sides by $\bigotimes_{a,i}\left(evb_{ai}^*\det(TL)^*\right)_{\mathbf{u}}$,

$$\begin{aligned}\mathcal{L}'_{\mathbf{u}} &= \det(D_{\mathbf{u}}) \otimes \bigotimes_{a,i}\left(evb_{ai}^*\det(TL)^*\right)_{\mathbf{u}} \\ &\simeq \det(D_{\mathbf{u}} \oplus D_0) \otimes \det(D_0)^* \otimes \bigotimes_{a,i}\left(evb_{ai}^*\det(TL)^*\right)_{\mathbf{u}}.\end{aligned}$$

By Lemma 2.11, the Cauchy-Riemann *Pin* boundary value problem

$$\underline{D_0} = (\Sigma, u^*V \otimes \mathbb{C}, u^*V|_{\partial\Sigma}, \mathfrak{p}_0, D_0)$$

induces a canonical orientation on $\det(D_0)$ if we choose $\mathfrak{p}_0$ to be the restriction to $\partial\Sigma$ of a *Pin* structure on u^*V over Σ. So, it suffices to orient

$$\mathcal{L}'_{\mathbf{u}} \otimes \det(D_0) \simeq \det(D_{\mathbf{u}} \oplus D_0) \otimes \bigotimes_{a,i}\left(evb_{ai}^*\det(TL)^*\right)_{\mathbf{u}}. \tag{3.2}$$

Note that by pull back, the relative *Pin* structure on L gives a *Pin* structure on $u|_{\partial\Sigma}^*(TL \oplus V_{\mathbf{R}})$, the boundary condition for $D_{\mathbf{u}} \oplus D_0$. If L is orientable and given an orientation, since by definition V has an orientation, we have an induced orientation on $u|_{\partial\Sigma}^*(TL \oplus V_{\mathbf{R}})$. So, by Proposition 2.8, we have a canonical orientation on $\det(D_{\mathbf{u}} \oplus D_0)$. Since the orientation of L is equivalent to an orientation of $\det(TL)$, we have given everything on the right-hand side of equation (3.2) a canonical orientation.

If L is not orientable, choose an arbitrary orientation on $(evb_{a1}^*TL)_{\mathbf{u}}$ for each a such that $k_a \neq 0$. The complex structure on Σ induces a natural orientation on Σ and hence on $(\partial\Sigma)_a$ for each a. For each a and each $i \in [2, k_a]$, trivializing $u|_{\partial\Sigma}^*TL$ along the oriented line segment in $(\partial\Sigma)_a$ from z_{a1} to z_{ai} induces an orientation on $(evb_{ai}^*TL)_{\mathbf{u}}$. In the case that $u|_{(\partial\Sigma)_a}^*TL$ is orientable, the choice of orientation on $(evb_{a1}^*TL)_{\mathbf{u}}$ induces an orientation on $u|_{(\partial\Sigma)_a}^*TL$. By Proposition 2.8, an orientation on $u|_{(\partial\Sigma)_a}^*TL$ if orientable together with the chosen orientation on $V_{\mathbf{R}}$ and the previously mentioned *Pin* structure induces a canonical orientation on $\det(D_{\mathbf{u}} \oplus D_0)$ and hence the whole right-hand side of (3.2) is oriented after these choices.

Note that changing the orientation on $(evb_{a1}^*TL)_{\mathbf{u}}$ will change all the orientations it induces. By Lemma 2.9, the condition $k_a \cong w_1(d_a) + 1 \mod 2$ now implies that changing the orientation on any given $(evb_{a1}^* TL)_{\mathbf{u}}$ would make no difference because the total number of ensuing orientation changes would be even. Finally, the choice of D_0 is irrelevant because the space of real-linear Cauchy-Riemann operators on u^*V is contractible. Since the argument for orienting $\mathcal{L}'_{\mathbf{u}}$ applies word for word for a one-parameter family, we have indeed canonically oriented $\mathcal{L}'|_{B'}$. $\square$

At this point, we note that there is some freedom as to which isomorphism

$$\mathcal{L} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^* \det(TL)$$

we choose. Indeed, $B_{\mathbf{k},l}^{1,p}(L, \Sigma, \mathbf{d})$ consists of many connected components, at least one for each ordering of the marked points on their respective boundary components. It will turn out to be useful to define *the* canonical isomorphism to differ slightly from the one constructed in the proof of Proposition 3.1. Let

$$\varpi = (\varpi_1, \ldots, \varpi_m)$$

where ϖ_a is a permutation of the integers $1, \ldots, k_a$. Define

$$\operatorname{sign}(\varpi) := \prod_a \operatorname{sign}(\varpi_a).$$

Let

$$B_{\mathbf{k},l,\varpi}^{1,p}(L, \Sigma, \mathbf{d})$$

denote the component of $B_{\mathbf{k},l}^{1,p}(L, \Sigma, \mathbf{d})$ where the boundary marked points (z_{ai}) are ordered within $\partial\Sigma$ by the permutations ϖ.

Definition 3.2. When $\dim L \cong 0 \pmod 2$ we define *the canonical isomorphism*

$$\mathcal{L} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^* \det(TL)$$

to be the isomorphism constructed in the proof of Proposition 3.1 twisted by $(-1)^{\operatorname{sign}(\varpi)}$

over the component of the moduli space $B^{1,p}_{\mathbf{k},l,\varpi}(L,\Sigma,\mathbf{d})$. If $\dim L \cong 1 \pmod 2$, then we define the canonical isomorphism to be simply the isomorphism constructed in the proof of Proposition 3.1.

Now, we move on to orienting moduli spaces of stable maps. We will restrict attention to stable maps of two components, one of which is the original Riemann surface Σ, and the other of which is a disk bubble. This will suffice for the purposes of this paper. However, it is not hard to extend the results below to stable maps of arbitrarily many components of arbitrary topological type.

We consider the case that a disk bubbles off the boundary component $(\partial\Sigma)_b$ along with k'' of the marked points on $(\partial\Sigma)_b$ and l'' of the interior marked points. Let

$$k' := k_b - k'', \qquad \mathbf{k}' := (k_1, \ldots, k', \ldots, k_m).$$

Let $l' + l'' = l$. It will turn out to be convenient to keep track of exactly which marked points bubble off. So, let $\sigma \subset [1, k_b]$ denote the subset of boundary marked points that bubble off and let σ^c denote its complement. Furthermore, let $\varrho \subset [1, l]$ denote the set of interior marked points that bubble off and let ϱ^c denote its complement. Let

$$\mathbf{d}' = (d', d_1, \ldots, d'_b, \ldots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m}, \qquad d'' \in H_2(X, L),$$

satisfy

$$d' + d'' = d, \qquad d'_b + \partial d'' = d_b.$$

We will need to add an extra marked point to each of the two irreducible components of the stable map in order to impose the condition that the two components intersect. We denote by z'_0 the extra marked point on Σ and by z''_0 the extra marked point on D^2. We will use the notation

$$\mathbf{k}' + e_b = (k_1, \ldots, k' + 1, \ldots, k_m).$$

We define the space of $W^{1,p}$ stable-maps with this combinatorial data to be the fiber product

$$B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'') := B^{1,p}_{\mathbf{k}'+e_b,l'}(L,\Sigma,\mathbf{d}')\;{}_{evb_0'}\times_{evb_0''} B^{1,p}_{k''+1,l''}(L,D^2,d'').$$

Elements $\mathbf{u} \in B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'')$ take the form

$$\mathbf{u} = (\mathbf{u}',\mathbf{u}''), \qquad \mathbf{u}' \in B^{1,p}_{\mathbf{k}'+e_b,l'}(L,\Sigma,\mathbf{d}'), \qquad \mathbf{u}'' \in B^{1,p}_{k''+1,l''}(L,D^2,d''),$$
$$evb_0'(\mathbf{u}') = evb_0''(\mathbf{u}'').$$

Associated to each such $\mathbf{u}$ there is a nodal Riemann surface with boundary

$$\hat{\Sigma}_{\mathbf{u}} := \Sigma \cup D^2/z_0' \sim z_0''.$$

and a continuous map

$$u : (\hat{\Sigma}_{\mathbf{u}}, \partial\hat{\Sigma}_{\mathbf{u}}) \to (X,L)$$

given by u' on Σ and u'' on D^2. We denote the node of $\hat{\Sigma}_{\mathbf{u}}$ by z_0. Let

$$p' : B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'') \to B^{1,p}_{\mathbf{k}'+e_b,l'}(L,\Sigma,\mathbf{d}'),$$
$$p'' : B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'') \to B^{1,p}_{k''+1,l''}(L,D^2,d''),$$

denote the natural projections. Note that when various indices are clear from the context, we may abbreviate

$$B' = B^{1,p}_{\mathbf{k}'+e_b,l'}(L,\Sigma,\mathbf{d}'), \qquad B'' = B^{1,p}_{k''+1,l''}(L,D^2,d''),$$
$$B^{\#} := B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'').$$

Define the Banach space bundle $\mathcal{E}^{\#} \to B^{\#}$ by

$$\mathcal{E}^{\#} := p_1^*\mathcal{E}' \oplus p_2^*\mathcal{E}''.$$

Fiberwise, we have

$$\mathcal{E}^{\#}_{\mathbf{u}} := L_p(\Sigma, \Omega^{0,1}(u'^*TX) \oplus \Omega^{0,1}(u''^*TX))$$

for $\mathbf{u} \in B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')$. If $J \in \mathcal{J}_\omega$, and $\nu \in \mathcal{P}$, we let

$$\overline{\partial}^{\#}_{(J,\nu)} : B^{\#} \to \mathcal{E}^{\#}$$

denote the section of $\mathcal{E}$ given by the ν-perturbed Cauchy Riemann operator. Here, the natural ν-perturbed Cauchy-Riemann operator has a vanishing inhomogeneous term on the disk bubble. Using the canonical identification between the vertical tangent spaces of $\mathcal{E}^{\#}$ and $\mathcal{E}^{\#}$ itself we define

$$D^{\#} := D\overline{\partial}^{\#}_{(J,\nu)} : TB^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') \to \mathcal{E}^{\#}$$

to be the vertical component of the linearization of $\overline{\partial}^{\#}_{(J,\nu)}$. Finally, define $\mathcal{L}^{\#} \to B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ to be the determinant line bundle of the family of Fredholm operators $D^{\#}$,

$$\mathcal{L}^{\#} := \det(D^{\#}).$$

Again, suppose either L is orientable and provided with an orientation or $k_a \cong w_1(d_a) + 1 \mod 2$.

Proposition 3.3. *The combination of an orientation of L if L is orientable and the choice of relative Pin$^{\pm}$ structure $\mathfrak{p}$ on L canonically determine an isomorphism of line bundles*

$$\mathcal{L}^{\#} \xrightarrow{\sim} \bigotimes_{a,i} evb^*_{ai} \det(TL).$$

Proof. The proof is the same as the proof of Proposition 3.1 except for one extra subtlety, which is particularly important in the case that L is not orientable. That is, the Riemann surface underlying the Cauchy-Riemann *Pin* boundary problem associated to $D^{\#}$ is singular. As in equation (3.2) of the proof of Proposition 3.1, it suffices to orient

$$\mathcal{L}^{\#'}_{\mathbf{u}} \otimes \det(D_0^{\#}) \simeq \det(D^{\#}_{\mathbf{u}} \oplus D_0^{\#}) \otimes \bigotimes_{a,i} (evb^*_{ai} \det(TL)^*)_{\mathbf{u}} \tag{3.3}$$

for $\mathbf{u}$ such that

$$u : (\hat{\Sigma}_{\mathbf{u}}, \partial\hat{\Sigma}_{\mathbf{u}}) \to (X^{(3)}, L^{(3)}).$$

To this end, we need to describe $D_0^{\#}$ and $D^{\#}$ in greater detail. Using the notation (3.1), define

$$d^0_{00} : W^{1,p}(u'^* V_{\mathbb{C}}, u'^* V_{\mathbb{R}}) \oplus W^{1,p}(u''^* V_{\mathbb{C}}, u''^* V_{\mathbb{R}}) \to (ev^*_{b0} V_{\mathbb{R}})_{\mathbf{u}}$$

by

$$d^0_{00}(\xi', \xi'') = \xi'(z'_0) - \xi''(z''_0).$$

Denote,

$$W^{1,p}\left(u^* V_{\mathbb{C}}, u|^*_{\partial\hat{\Sigma}_{\mathbf{u}}} V_{\mathbb{R}}\right) := \ker(d^0_{00}).$$

At this point, we introduce abbreviated notation

$$W'_V := W^{1,p}(u'^* V_{\mathbb{C}}, u'^* V_{\mathbb{R}}), \qquad W''_V := W^{1,p}(u''^* V_{\mathbb{C}}, u''^* V_{\mathbb{R}}),$$
$$W_V := W^{1,p}\left(u^* V_{\mathbb{C}}, u|^*_{\partial\hat{\Sigma}_{\mathbf{u}}} V_{\mathbb{R}}\right),$$
$$Y'_V := L^p(u'^* V_{\mathbb{C}}), \qquad Y''_V := L^p(u''^* V_{\mathbb{C}}),$$
$$Y_V := L^p(u^* V_{\mathbb{C}}) = L^p(u'^* V_{\mathbb{C}}) + L^p(u'^* V_{\mathbb{C}}).$$

Then choose arbitrary Cauchy-Riemann operators

$$D'_0 : W'_V \to Y'_V, \qquad D''_0 : W''_V \to Y''_V,$$

and define

$$D_0^{\#} := (D'_0 \oplus D''_0)\,|_{W_V} : W_V \to Y_V.$$

Now, turning to $D^{\#}$, define

$$d_{00} : p'^{*}TB^{1,p}_{\mathbf{k}',l'} \oplus p''^{*}TB^{1,p}_{\mathbf{k}'',l''} \to evb_0^{*}TL$$

by

$$(\xi', \xi'') \rightsquigarrow devb_0'(\xi') - dev_0''(\xi'').$$

Note that

$$TB^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') = \ker(d_{00}).$$

So, we have a short exact sequence of families of Fredholm operators

$$\begin{tikzcd}
\mathcal{E}^{\#}_{\mathbf{u}} \oplus Y_V \arrow[r] & p'^{*}\mathcal{E}'_{\mathbf{u}} \oplus Y'_V \oplus p''^{*}\mathcal{E}''_{\mathbf{u}} \oplus Y''_V \arrow[r] & 0 \\
TB^{\#}_{\mathbf{u}} \oplus W_V \arrow[r] \arrow[u, "D^{\#}_{\mathbf{u}} \oplus D^{\#}_0"] & p'^{*}TB'_{\mathbf{u}} \oplus W'_V \oplus p''^{*}TB''_{\mathbf{u}} \oplus W''_V \arrow[r, "d_{00}"] \arrow[u, "D'_{\mathbf{u}} \oplus D'_0 \oplus D''_{\mathbf{u}} \oplus D''_0"] & evb_0^{*}\left(TL \oplus V_{\mathbf{R}}\right)_{\mathbf{u}}, \arrow[u]
\end{tikzcd}$$

and, hence, an isomorphism

$$\det(D^{\#}_{\mathbf{u}} \oplus D^{\#}_0) \xrightarrow{\sim} \det(D'_{\mathbf{u}} \oplus D'_0) \otimes \det(D''_{\mathbf{u}} \oplus D''_0) \otimes evb_0^{*}\det(TL \oplus V_{\mathbf{R}})^{*}_{\mathbf{u}}. \qquad (3.4)$$

Noting isomorphism (3.4), if L is oriented, the whole right-hand side of equation (3.3) is canonically oriented by arguing as in the proof of Proposition 3.1.

If L is not orientable, choose an arbitrary orientation on $(evb_{a1}^{*}TL)_{\mathbf{u}}$ for each a such that $k_a \neq 0$. The complex structure on Σ induces a natural orientation on Σ and hence on $(\partial\Sigma)_a$ for each a. Similarly, the complex structure on D^2 induces a natural orientation on ∂D^2. This said, any ordered pair of points (z, z'), $z \neq z_0$, in the same connected component of $\partial\hat{\Sigma}$ can be connected by a unique oriented line segment from the first to the second. For the non-singular boundary components of $\partial\hat{\Sigma}$, this is evident. For the singular boundary component,

$$(\partial\hat{\Sigma})_b := (\partial\Sigma)_b \cup \partial D^2 / z'_{b0} \sim z''_0,$$

we define the unique oriented line segment from z to z' as follows: For concreteness, assume that $z \in (\partial\Sigma)_b$. The same exact definition applies if $z \in \partial D^2$. Start from z and proceed in the direction of the orientation of $(\partial\Sigma)_b$ until reaching either z' or z'_0. If z' is reached first or if $z' = z'_0$, the path ends there. If z'_0 is reached first, then continue the path starting from $z''_0 \in \partial D^2$ and proceeding along ∂D^2 in the direction of the orientation. If z' belongs to ∂D^2 then the path ends when it reaches z'. Otherwise it continues around ∂D^2 back to z''_0 and then proceeds in the direction of the orientation from z'_0 along $(\partial\Sigma)_b$ until it reaches z'.

For each a and each $i \in [2, k_a]$, trivializing $u|^*_{\partial\hat{\Sigma}}TL$ along the oriented line segment in $(\partial\hat{\Sigma})_a$ from z_{a1} to z_{ai} induces an orientation on $(evb^*_{ai}TL)_{\mathbf{u}}$ as before. To orient the factor of $evb^*_0 \det(TL \oplus V_{\mathbf{R}})^*_{\mathbf{u}}$ appearing on the right-hand side of isomorphism (3.4), since $V_{\mathbf{R}}$ is equipped with a chosen orientation by the definition of a relative *Pin* structure, it suffices to orient $(evb^*_0TL)_{\mathbf{u}}$. For this purpose, we proceed as follows: If $k_b \neq 0$ then trivializing $u|^*_{\partial\hat{\Sigma}}TL$ along the oriented line segment from z_{b1} to z_0 induces an orientation on $evb^*_0TL_{\mathbf{u}}$. If $k_b = 0$ then choose an arbitrary orientation on $(evb^*_0TL)_{\mathbf{u}}$. For all $a \neq b$ if $u|^*_{(\partial\hat{\Sigma})_a}TL$ is orientable, since in this case $k_a \neq 0$ by the assumption $w_1(d_a) \cong k_a + 1 \pmod 2$, the choice of orientation on $(evb^*_{a1}TL)_{\mathbf{u}}$ induces an orientation on $u|^*_{(\partial\Sigma)_a}TL$. Furthermore, we may induce an orientation on either or both of $u|^*_{(\partial\Sigma)_b}TL$ and $u|^*_{\partial D^2}TL$ if orientable, from the orientation on $(evb^*_0TL)_{\mathbf{u}}$. By Proposition 2.8, the orientation on $u|^*_{(\partial\Sigma)_a}TL$ if orientable together with the chosen orientation on $V_{\mathbf{R}}$ and the previously mentioned *Pin* structure induce a canonical orientation on $\det(D'_{\mathbf{u}} \oplus D'_0)$. Similarly, the orientation on $u|^*_{D^2}TL$ if orientable together with the chosen orientation on $V_{\mathbf{R}}$ and the previously mentioned *Pin* structure induce a canonical orientation on $\det(D''_{\mathbf{u}} \oplus D''_0)$. Hence the whole right-hand side of (3.4) is oriented after these choices. Since we have also chosen an orientation on $(evb^*_{ai}TL)_{\mathbf{u}}$, it follows that the entire right-hand side of (3.3) is oriented.

Note that changing the orientation on $(evb^*_{a1}TL)_{\mathbf{u}}$ will change all the orientations it induces. Similarly, in the case $k_b = 0$, changing the chosen orientation on $(evb^*_0TL)_{\mathbf{u}}$ changes any orientation it induces. By Lemma 2.9, the condition $k_a \cong w_1(d_a) + 1 \mod 2$ now implies that changing the orientation on any given

$(evb_{a1}^*TL)_{\mathbf{u}}$ or $(evb_0^*TL)_{\mathbf{u}}$ would make no difference because the total number of ensuing orientation changes would be even. $\square$

Just as was the case with maps from curves with irreducible domain, so too with non-trivial stable-maps, there is some freedom in the choice of an isomorphism

$$\mathcal{L}^{\#} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^* \det(TL).$$

For this purpose, we note that there is a canonical ordering on the marked points in the boundary of the nodal curve $\hat{\Sigma}_{\mathbf{u}}$. Indeed, recall from the proof of Proposition 3.3 that given any pair of points $(z, z') \subset \partial\hat{\Sigma}$ such that $z \neq z_0$, there exists a unique oriented line segment from z to z'. We define z_{ai} to be ordered before $z_{ai'}$ if and only if the segment from z_{a1} to z_{ai} lies within the segment from z_{a1} to $z_{ai'}$. Again, we can divide the moduli space $B^{\#}$ into components $B^{\#}_{\varpi}$. By analogy with the irreducible case, we make the following definition.

Definition 3.4. If $\dim L = 0 \pmod 2$, we define *the canonical isomorphism*

$$\mathcal{L}^{\#} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^* \det(TL)$$

to be the isomorphism constructed in the proof of Proposition 3.3 twisted by $(-1)^{\mathrm{sign}(\varpi)}$ over the component $B^{\#}_{\varpi}$. If $\dim L = 1 \pmod 2$, we define the canonical isomorphism to be the isomorphism constructed in the proof of Proposition 3.3.

Chapter 4

The definition of the invariants revisited

At this point, we will rigorously define the integration carried out in defining the invariants in Section 1.2 by generalizing the techniques of Ruan and Tian [19].

First, we will carefully define the inhomogeneous perturbation to the Cauchy-Riemann equation relevant in the current situation. Let $\mathcal{C}$ be a parameter space to be specified later. Let π_i, $i = 1, 2$, denote the projection from $\Sigma \times X \times \mathcal{C}$ to the i^{th} factor and let π_i' denote the restriction of π_i to $\partial\Sigma \times L \times \mathcal{C}$. We define the set of inhomogeneous terms $\mathcal{P}$ to be the set of sections

$$\nu \in \Gamma\left(\Sigma \times X \times \mathcal{C}, Hom\left(\pi_1^* T\Sigma, \pi_2^* T\Sigma\right)\right)$$

such that

(i) ν is (j_Σ, J)-anti-linear, i.e. $\nu \circ j_\Sigma = -J \circ \nu$;

(ii) $\nu|_{\partial\Sigma \times L \times \mathcal{C}}$ carries the sub-bundle ${\pi_1'}^* T\partial\Sigma \subset {\pi_1'}^* T\Sigma$ to the sub-bundle

$${\pi_2'}^*(J\,TL) \subset {\pi_2'}^* TX.$$

For the time being, take $\mathcal{C} = B^{1,p}_{k,l}(L, \Sigma, \mathbf{d})$ and define the section $\bar{\partial}_{J,\nu}$ of $\mathcal{E}$ by

$$\bar{\partial}_{J,\nu} u := du \circ j_\Sigma + J \circ du - \nu(\cdot, u(\cdot), \mathbf{u}) \in L^p\left(\Omega^{0,1}(u^* TX)\right).$$

Lemma 4.1. *The operator $\bar{\partial}_{J,\nu}$ gives rise to a well-posed boundary value problem.*

Proof. Define $\widetilde{\Sigma} := \Sigma \cup_{\partial\Sigma} \overline{\Sigma}$. Clearly, we may extend any $\nu \in \mathcal{P}$ over $\widetilde{\Sigma}$ so that it continues to satisfy condition (i). Let J_ν be the automorphism of $T(\widetilde{\Sigma} \times X)$ given in matrix form by

$$J_\nu := \begin{pmatrix} j_\Sigma & 0 \\ \nu & J \end{pmatrix}.$$

It is not hard to check that condition (i) implies that J_ν is an almost complex structure on $\widetilde{\Sigma} \times X$. Condition (ii) implies that $\partial\Sigma \times L \subset \widetilde{\Sigma} \times X$ is a totally real submanifold. A map

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

satisfying $\bar{\partial}_{J,\nu} u = 0$ is equivalent to a standard J_ν-holomorphic map

$$\tilde{u} : (\Sigma, \partial\Sigma) \to (\widetilde{\Sigma} \times X, \partial\Sigma \times L)$$

satisfying $\pi_1 \circ \tilde{u} = \mathrm{Id}_\Sigma$. We conclude that $\bar{\partial}_{J,\nu}$ does indeed give rise to a well-posed boundary value problem. □

In light of Lemma 4.1, we define

$$\widetilde{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) := \bar{\partial}^{-1}_{J,\nu}(0) \subset B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}).$$

We call a map

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

ϕ-multiply covered if there does not exist $z \in \Sigma$ such that

$$du(z) \neq 0, \qquad u(z) \notin u(\Sigma \setminus \{z\}), \qquad u(z) \notin \operatorname{Im} \phi \circ u.$$

Such maps are also commonly termed *not ϕ-somewhere injective.* A standard argument shows that the moduli space of ϕ-somewhere injective maps has expected dimension for generic $J \in \mathcal{J}_{\omega,\phi}$ even when $\nu = 0$. See [5, Chapter 11]. However, if we take $\nu = 0$, then the moduli space $\widetilde{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ may be singular at ϕ-multiply covered maps even for a generic choice of J. Assume for a moment that $\Sigma = D^2$ and $\mu(d) > 2$. Then by the following Lemma, the image of ϕ-multiply covered maps under the evaluation map has codimension greater than or equal to 2.

Lemma 4.2. *Suppose $u : (D^2, \partial D^2) \to (X, L)$ is ϕ-multiply covered. Then, we may factor u as a composition $u = u' \circ \chi$, where u' is a real J-holomorphic map*

$$u' : \mathbb{C}P^1 \to X, \qquad \phi \circ u' \circ c' = u', \qquad \bar{\partial}_J u' = 0,$$

such that $u'|_{D^2}$ is ϕ-somewhere injective, and

$$\chi : (D^2, \partial D^2) \to (\mathbb{C}P^1, \mathbb{R}P^1)$$

is a holomorphic map of degree greater than or equal to 2.

Remark 4.3. Since we may extend χ to a holomorphic map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ by the Schwarz reflection principle, χ is actually given by polynomial. Thinking of D^2 as $H \cup \{\infty\}$ and thinking of $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$, the boundary condition on χ implies that χ is given by a real polynomial.

Proof of Lemma 4.2. Gluing together u and $\phi \circ u \circ c'$ we obtain a real J-holomorphic map

$$\tilde{u} : \mathbb{C}P^1 \to X, \qquad \phi \circ \tilde{u} \circ c' = \tilde{u}.$$

By a standard theorem [16, Chapter 2], we can factor $\tilde{u} = \tilde{\chi} \circ \tilde{u}'$ where $\tilde{u}'$ is somewhere injective. Another standard result [16, Theorem E.1.2] says that $\tilde{u}'$ is injective except at a finite number of points. Since the image of u' is clearly invariant under ϕ, ϕ induces an anti-holomorphic involution of $\mathbb{C}P^1$ away from a finite number of points. So, removing singularities, there exists an anti-holomorphic involution c'' on $\mathbb{C}P^1$ such that $\phi \circ \tilde{u}' \circ c'' = \tilde{u}'$. Moreover, c'' fixes the image of $\mathbb{R}P^1$ under χ. It follows that c'' is conjugate to c' by some biholomorphism $a \in PSL_2(\mathbb{C})$. So, we may define

$$u' = \tilde{u}' \circ a^{-1}, \qquad \chi = a \circ \tilde{\chi}|_{D^2}.$$

□

Consequently, under the previously mentioned conditions, ϕ-multiply covered maps are not important in the definition of intersection theory on the moduli space. So, even in the case $\nu = 0$, we may obtain a smooth moduli space by defining

$$\widetilde{\mathcal{M}}^*_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) := \widetilde{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \setminus \{\phi\text{-multiply covered maps}\}.$$

Then, to obtain an interesting intersection theory, we define

$$\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) := \widetilde{\mathcal{M}}^*_{\mathbf{k},l}(L, \Sigma, \mathbf{d})/PSL_2(\mathbb{R}), \tag{4.1}$$

where $PSL_2(\mathbb{R})$ acts by

$$(u, \vec{z}, \vec{w}) \rightsquigarrow (u \circ \varphi, (\varphi^{-1})^k(\vec{z}), (\varphi^{-1})^l(\vec{w})), \qquad \varphi \in PSL_2(\mathbb{R}).$$

This choice of action ensures that the evaluation maps descend to the quotient. Note that the action would not preserve the moduli space if we were to allow a general inhomogeneous term ν.

Equivalently, instead of quotienting by $PSL_2(\mathbb{R})$, we could consider an appropriate section of the group action. Fortunately, this approach, as observed in [19], generalizes to the situation where we allow a generic ν and so provides a definition of the moduli space that works even when (X, L) may admit holomorphic disks of Maslov index 0 or when $\Sigma \neq D^2$. For a detailed proof of the equivalence of the two approaches, see [16, Chapter 6].

So, we construct a section of the reparameterization group action in the following manner. Let

$$\pi_j : \widetilde{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \to \Sigma$$

be the projection sending $(u, \vec{z}, \vec{w}) \rightsquigarrow w_j$. First, suppose that $\Sigma \simeq D^2$. Choose an interior point $s_0 \in \Sigma$ and a line $\ell \subset D^2$ connecting s_0 to $\partial\Sigma$ such that for any pair of points $(w, w') \in \Sigma$ there exists a unique $\varphi \in Aut(\Sigma) \simeq PSL_2(\mathbb{R})$ that satisfies

$$\varphi(w) = s_0, \qquad \varphi(w') \in \ell.$$

For the time being, assume $l \geq 2$. We require that the dependence of ν on $\mathbf{u} \in B^{1,p}_{\mathbf{k},l}$ factors through $\pi_1 \times \pi_2$. Moreover, letting $d(\cdot, \cdot)$ denote the distance function on D^2, we impose on ν the condition

$$\nu(\cdot, \cdot, (w_1, w_2)) = \frac{d(w_1, w_2)}{d(w_1', w_2')} \nu(\cdot, \cdot, (w_1', w_2')). \tag{4.2}$$

In particular, ν vanishes uniformly in the limit $w_2 \to w_1$. We define

$$\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) := (\pi_1 \times \pi_2)^{-1}(s_0 \times \ell) \subset \widetilde{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}). \tag{4.3}$$

Standard arguments show that for a generic choice of ν satisfying (4.2), $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ will be a smooth manifold of expected dimension. We digress briefly to explain the significance of condition (4.2). To prove the invariance of $N_{\Sigma,\mathbf{d},\mathbf{k},l}$, we need to argue that stable maps in the Gromov compactification of $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ involving sphere bubbles occur only in codimension two. A sequence $\mathbf{u}^i \in \mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ such that $w_2^i \to w_1^i$ will Gromov converge to a stable map consisting of one disk component and one sphere component with both w_1 and w_2 on the sphere component. The nodal point where the sphere and disk are attached is fixed at s_0. However, there are no other fixed marked points on the disk component to compensate for the remaining T^1 symmetry. Condition (4.2) implies that the limit inhomogeneous term on the disk component is zero. So, we can simply quotient by the residual T^1 action. Although ϕ-multiply covered maps can arise, we may disregard them because by a standard argument, their image under the evaluation map has high codimension.

If, $\Sigma \simeq S^1 \times I$, we choose a line $\ell \subset \Sigma$ connecting the two boundary components of Σ such that for any $w \in \Sigma$ there exists a unique $\varphi \in Aut(\Sigma) \simeq T^1$ such that $\varphi(w) \in \ell$. We choose ν to be entirely independent of $\mathbf{u}$. Assuming for the time being that $l \geq 1$, define

$$\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) := \pi_1^{-1}(\ell) \subset \widetilde{\mathcal{M}}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}).$$

Again, for a generic choice of ν, standard arguments show that $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ will be a smooth manifold of expected dimension.

Now we turn to the case when $\Sigma \simeq D^2$ and $l = 0$. This case is of particular interest because it arises when X is a Calabi-Yau manifold and L is a special Lagrangian submanifold. The cases $\Sigma \simeq D^2$, $l = 1$, and $\Sigma \simeq S^1 \times I$, $l = 0$, use a very similar argument.

We start with the moduli space $\mathcal{M}_{\mathbf{k},2}(L, \Sigma, \mathbf{d})$ constructed above and proceed as follows. Choose smooth manifolds A, B, and maps

$$f : A \rightarrow X, \qquad g : B \rightarrow X$$

such that (A, f) and (B, g) define pseudo-cycles representing the Poincare-dual of any non-trivial ϕ-anti-invariant 2^{nd} cohomology class. The symplectic form ω provides at least one example, and for simplicity, we will continue with this example.

Lemma 4.4. *We can choose (A, f) and (B, f) that are ϕ-anti-invariant and do not intersect L.*

Proof. A straightforward transversality argument shows that we may assume (A, f) is transversal to L. If necessary, replacing A by $\frac{1}{2}A \coprod -\frac{1}{2}A$ and f by $f \coprod \phi \circ f$, we may assume that the pseudo-cycle (A, f) is ϕ-anti-invariant just like ω. Now, we consider a local model for (A, f) near L and show how to modify (A, f) near L to avoid intersecting L. Choose ϕ-invariant local symplectic coordinates $\Theta : X \rightarrow \mathbb{C}^n$ near L such that $\Theta(L) = \mathbb{R}^n$. We may assume that the image of f is given by the union of the vanishing sets of conjugate real-linear maps

$$\ell,\, \bar{\ell} : \mathbb{C}^n \longrightarrow \mathbb{R}^2.$$

More precisely, choose complex coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n$. Taking $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we write $z = x + iy$. So, we can decompose

$$\ell(z) = \ell^x(x) + \ell^y(y), \qquad \bar{\ell}(z) = \ell^x(x) - \ell^y(y).$$

Letting ℓ_i, $i = 1, 2$, denote the i^{th} component of ℓ, the equations for the image of f may be written as

$$0 = \ell_i \cdot \bar{\ell}_i = \ell_i^{x^2} - \ell_i^{y^2}, \quad i = 1, 2.$$

So, choosing small constants $\epsilon_i > 0$, $i = 1, 2$, we modify (A, f) so that locally the image of f satisfies equations

$$\ell_i^{x^2} - \ell_i^{y^2} = -\epsilon_i, \quad i = 1, 2.$$

Clearly, these equations have no real solutions. The same applies for B. $\square$

Furthermore, we may assume that $(A \times B, f \times g)$ is transversal to $evi_1 \times evi_2$ so that we may define

$$\mathcal{M}_{\mathbf{k},0}(L, \Sigma, \mathbf{d}) := \frac{1}{\omega(d)^2} \mathcal{M}_{\mathbf{k},2}(L, \Sigma, \mathbf{d}) \times_{X \times X} (A \times B)\,.$$

The factor $\frac{1}{\omega(d)^2}$ in front of the fiber-product means that each point in the moduli space should be counted with weight $\frac{1}{\omega(d)^2}$. This correction is designed to cancel the contribution of the divisors (A, f) and (B, g) as predicted by the divisor axiom of formal Gromov-Witten theory. Indeed a map $u : (\Sigma, \partial\Sigma) \longrightarrow (X, L)$ representing the class $d \in H_2(X, L)$ should intersect a pseudo-cycle Poincare dual to ω exactly $\omega(d)$ times.

Remark 4.5. Note that even when $l > 0$, we are free to fix the group action by adding divisor constraints just as in the case when $l = 0$.

Proof of Theorem 1.1. Given this definition of $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$, Proposition 3.1 immediately implies Theorem 1.1. We choose the isomorphism according to Definition 3.2. □

Now, for a sufficiently generic choice of points $\vec{x} = (x_{ai}), x_{ai} \in L$, and pairs of points $\vec{y} = (y_j)$,

$$y_j : \{0, 1\} \to X, \qquad y_j(1) = \phi \circ y_j(0),$$

the total evaluation map

$$\mathbf{ev} := \prod_{a,i} evb_{ai} \times \prod_{j} evi_j : \mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \longrightarrow L^{|\mathbf{k}|} \times X^l$$

will be transverse to

$$\prod_{a,i} x_{ai} \times \prod_{j} y_j \in L^{|\mathbf{k}|} \times X^l.$$

So, assuming the dimension condition (1.2) is satisfied, we may define

$$N_{\Sigma,\mathbf{d},\mathbf{k},l} := \#\mathbf{ev}^{-1}(\vec{x}, \vec{y}).$$

Here, $\#$ denotes the signed count with the sign of a given point $v \in \mathbf{ev}^{-1}(\vec{x}, \vec{y})$ depending on whether or not the isomorphism

$$d\mathbf{ev}_v : \det(T\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}))_v \xrightarrow{\sim} \mathbf{ev}^* \det \left(T \left(L^{|\mathbf{k}|} \times X^l\right)\right)_v$$

agrees with the isomorphism of Theorem 1.1 up to the action of the multiplicative group of positive real numbers. If the dimension condition (1.2) is not satisfied, we define $N_{\Sigma,\mathbf{d},\mathbf{k},l} := 0$.

Chapter 5

The sign of the conjugation on the moduli space

Now, suppose Σ is biholomorphic to $\overline{\Sigma}$. It follows that there exists a complex conjugation $c : \Sigma \to \Sigma$. Let ϕ be an anti-symplectic involution of X such that $Fix(\phi) = L$. Fix $J \in \mathcal{J}_{\omega,\phi}$, define $\mathcal{P}_{\phi,c}$ to be the set of $\nu \in \mathcal{P}$ such that $d\phi \circ \nu \circ c = \nu$ and let $\nu \in \mathcal{P}_{\phi,c}$. We define an involution

$$\phi_B : B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \to B^{1,p}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$$

by

$$(u, \vec{z}, \vec{w}) \rightsquigarrow \left(\phi \circ u \circ c, (c|_{\partial D^2})^{|k|}(\vec{z}), c^l(\vec{w})\right).$$

Furthermore, define an involution $\phi_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ covering ϕ_B by sending $\eta \in \mathcal{E}_u$ to $d\phi \circ \eta \circ dc \in \mathcal{E}_{\phi_B(u)}$. It is easy to see that $\overline{\partial}_{J,\eta}$ is $\phi_B - \phi_{\mathcal{E}}$ equivariant, and hence $\phi_{\mathcal{E}}$ induces an involution $\phi_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ covering ϕ_B. Now, the trivial bundle morphism of

$$\bigotimes_{a,i} evb^*_{ai} \det(TL)$$

covers ϕ_B. So, the involution $\phi_{\mathcal{L}}$ induces an involution $\phi'_{\mathcal{L}}$ of the bundle

$$\mathcal{L}' := Hom\left(\bigotimes_{a,i} evb_{ai}^* \det(TL), \mathcal{L}\right) \simeq \mathcal{L} \otimes \bigotimes_{a,i} evb_{ai}^* \det(TL).$$

covering ϕ_B. Now, the bundle $\mathcal{L}'$ has a canonical orientation as defined in Definition 3.2. So, the involution $\phi'_{\mathcal{L}}$ may either preserve the corresponding component of the complement of the zero section of $\mathcal{L}'$, exchange it with the opposite component or some combination of the two over different connected components of the base. We say that $\phi'_{\mathcal{L}}$ has sign 0 in the case it preserves this component, and sign 1 otherwise.

We would like to calculate the sign of ϕ_L. To properly formulate the result, we define a degree 0 homomorphism $\psi : H_*(X, L; \mathbb{Z}/2\mathbb{Z}) \to H_*(X; \mathbb{Z}/2\mathbb{Z})$ on the level of singular chains as follows: We implicitly use $\mathbb{Z}/2\mathbb{Z}$ coefficients everywhere. Suppose $\sigma \in C_*(X, L) := C_*(X)/C_*(L)$ is a relative singular chain. Let $\widetilde{\sigma} \in C_*(X)$ represent σ. Define

$$\widehat{\psi}(\sigma) := (\mathrm{Id} + \phi_*)\widetilde{\sigma}.$$

$\widehat{\psi}$ is well defined because if $\sigma = 0$ then $\widetilde{\sigma} \in C_*(L)$ and hence

$$(\mathrm{Id} + \phi_*)\widetilde{\sigma} = 2\widetilde{\sigma} \cong 0 \mod 2.$$

Since ϕ_* commutes with the boundary operator, so does $\widehat{\psi}$, so we can define $\psi := H(\widehat{\psi})$. Now, let g_0 denote the genus of $\Sigma/\partial\Sigma$.

Proposition 5.1. *Let $n = \dim L$. If $\mathfrak{p}$ is a relative Pin$^-$ structure then the involution $\phi'_{\mathcal{L}}$ has sign*

$$\begin{aligned}\mathfrak{s}^-(\mathbf{d}, \mathbf{k}, l) \cong{}& \frac{\mu(d)(\mu(d)+1)}{2} + (1 - g_0)n + mn + |\mathbf{k}| + l \\ &+ w_2(V)(\psi(d)) + w_1(\partial d) + \sum_{a<b} w_1(d_a)w_1(d_b) \\ &+ \sum_a w_1(d_a)(k_a - 1) + (n+1)\sum_a \frac{(k_a - 1)(k_a - 2)}{2} \mod 2.\end{aligned}$$

If $\mathfrak{p}$ is a relative Pin$^+$ structure, $\phi'_{\mathcal{L}}$ has sign

$$\begin{aligned}\mathfrak{s}^+(\mathbf{d},\mathbf{k},l) \cong{}& \frac{\mu(d)(\mu(d)+1)}{2} + (1-g_0)n + mn + |\mathbf{k}| + l \\ &+ w_2(V)(\psi(d)) + \sum_{a<b} w_1(d_a)w_1(d_b) \\ &+ \sum_a w_1(d_a)(k_a-1) + (n+1)\sum_a \frac{(k_a-1)(k_a-2)}{2} \mod 2.\end{aligned}$$

Remark 5.2. Suppose $\dim L \le 3$. Then L is *Pin*$^-$ by the Wu relations, so we can take $\mathfrak{p}$ to be a standard *Pin*$^-$ structure. In particular, $w_2(V) = 0$. Since $w_1(\partial d) \cong \mu(d)$ mod 2, we have

$$\begin{aligned}\mathfrak{s}^-(\mathbf{d},\mathbf{k},l) \cong{}& \frac{\mu(d)(\mu(d)-1)}{2} + (1-g_0)n + mn + |\mathbf{k}| + l \\ &+ \sum_{a<b} w_1(d_a)w_1(d_b) + \sum_a w_1(d_a)(k_a-1) \\ &+ (n+1)\sum_a \frac{(k_a-1)(k_a-2)}{2} \mod 2.\end{aligned}$$

In the particularly simple case that $\Sigma \simeq D^2$, we have

$$\mathfrak{s}^-(d,\mathbf{k},l) \cong \frac{\mu(d)(\mu(d)-1)}{2} + k + l + \mu(d)(k-1) + (n+1)\frac{(k-1)(k-2)}{2} \mod 2.$$

Now, recall from Section 3 that

$$B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'') := B^{1,p}_{\mathbf{k}'+e_b,l'}(L,\Sigma,\mathbf{d}')\,{}_{evb'_0}\!\times_{evb''_0} B^{1,p}_{k''+1,l''}(L,D^2,d'').$$

Note that $D^2 \simeq \overline{D^2}$ as demonstrated by the standard conjugation

$$c : D^2 \to D^2.$$

So, we have an involution $\phi_{B''}$ of the second factor of the fiber product. Since $L \subset Fix(\phi)$, the involution $\phi_{B''}$ of the second factor induces an involution $\phi_{B\#}$ of the whole fiber product. Similarly, the involution $\phi_{\mathcal{E}''}$ of the bundle $\mathcal{E}''$ over the second factor of the fiber product induces an involution $\phi_{\mathcal{E}^\#}$ on the bundle $\mathcal{E}^\# \longrightarrow B^\#$. Recall that the natural inhomogeneous perturbation ν for stable maps vanishes on bubble components. In particular, it is ϕ-invariant. So, $\bar{\partial}^\#_{J,\nu}$ defines a $\phi_{B^\#} - \phi_{\mathcal{E}^\#}$ invariant section of $\mathcal{E}^\#$, which in turn induces an involution $\phi_{\mathcal{L}^\#}$ of the determinant bundle $\mathcal{L}^\# \to B^\#$. As before, $\phi_{\mathcal{L}^\#}$ induces an involution $\phi'_{\mathcal{L}^\#}$ of

$$\mathcal{L}^{\#'} := Hom\left(\bigotimes_{a,i} evb_{ai}^* \det(TL), \mathcal{L}^\#\right) \simeq \mathcal{L}^\# \otimes \bigotimes_{a,i} evb_{ai}^* \det(TL).$$

Proving that the sign of $\phi'_{\mathcal{L}^\#}$ is well defined and calculating it will play a crucial role in the proof of the invariance of $N_{\Sigma,\mathbf{d},\mathbf{k},l}$. Before writing down the formula for the sign of $\phi_{\mathcal{L}^\#}$, let us introduce some new notation. We define

$$\Upsilon'(d'', k'') :\cong \mu(d'')k'' \cong w_1(\partial d'')k'' \pmod 2$$

and

$$\Upsilon''(d'_b, d'', k', k'') :\cong \begin{cases} 0, & w_1(d'_b) = w_1(\partial d'') = 0 \\ k', & w_1(d'_b) = w_1(\partial d'') = 1 \\ k'' - 1, & w_1(d'_b) = 1,\ w_1(\partial d'') = 0 \\ k_a - 1 = k'' + k' - 1, & w_1(d'_b) = 0,\ w_1(\partial d'') = 1. \end{cases}$$

Proposition 5.3. *Let $n = \dim L$. Suppose the marked point z_1 does not bubble off, i.e. $1 \notin \sigma$. Then the involution $\phi'_{\mathcal{L}^\#}$ of the line-bundle*

$$\mathcal{L}^{\#'} \to B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')$$

has sign

$$\begin{aligned} \mathfrak{s}^{\#'}_{\pm}(d'', k'', l'') :\cong \frac{\mu(d'')(\mu(d'') \pm 1)}{2} + w_2(\psi(d'')) + k'' + 1 + l'' \\ + \Upsilon'(d'', k'') + (n+1)\frac{k''(k''-1)}{2} \mod 2, \end{aligned} \tag{5.1}$$

with + in the Pin^+ *and − in the* Pin^- *case. On the other hand, suppose now that the marked point* z_1 *does bubble off, i.e.* $1 \in \sigma$*. Then the involution* $\phi'_{\mathcal{L}^\#}$ *has sign*

$$
\begin{aligned}
\mathfrak{s}^{\#''}_{\pm}(d'_b, d'', k', k'', l'') :\cong \frac{\mu(d'')(\mu(d'') \pm 1)}{2} &+ w_2(\psi(d'')) \\
&+ k'' + 1 + l'' + \Upsilon''(d'_b, d'', k', k'') + w_1(d'_b)w_1(\partial d'') \\
&+ (n+1)\left(\frac{(k''-1)(k''-2)}{2} + k_b k'' + k_b\right) \mod 2,
\end{aligned} \tag{5.2}
$$

with + in the Pin^+ *and − in the* Pin^- *case.*

Remark 5.4. Suppose L is orientable and $\dim L$ is odd. Then, using the fact that $\mu(d'')$ is even if L is orientable, we have

$$
\mathfrak{s}^{\#'} = \mathfrak{s}^{\#''} = \frac{\mu(d'')}{2} + w_2(\psi(d'')) + k'' + 1 + l''. \tag{5.3}
$$

Proof of Proposition 5.3: The first term in $\mathfrak{s}^{\#}$ comes from the formula (2.4). This accounts for the sign of conjugation on the moduli space of unmarked disks. The terms $k'' + l'' + 1$ account for conjugation on the configuration space of the marked points, adding one extra point for the incidence condition. Recall from the proof of Proposition 3.3 that the unique oriented path from $z \neq z_0$ to z' in the boundary of $\partial\hat{\Sigma}_{\mathbf{u}}$ played an important role in determining the canonical orientation of $\mathcal{L}^{\#'}$. This path depended on the orientation of $\partial\hat{\Sigma}$, which is reversed under conjugation. The terms Υ' (resp. $\Upsilon'' + w_1(d'_b)w_1(\partial d'')$) in $\mathfrak{s}^{\#'}$ (resp. $\mathfrak{s}^{\#''}$) account for this dependence. The remaining terms account for the reordering of the marked points under conjugation, which plays a role only in even dimensions according to Definition 3.4.

We now explain in more detail how the unique oriented path from $z \neq 0$ to z' changes under conjugation, and how that effects the orientation of $\mathcal{L}'^{\#}$. First, suppose $1 \notin \sigma$. The path from z_{b1} to z_{bi} for $i \in \sigma$ will change when the orientation of the boundary of the bubble ∂D^2 changes under conjugation. If $w_1(\partial d'') = 1$, this change of path changes the orientation of $(evb^*_{ai}TL)_{\mathbf{u}}$ for each $i \in \sigma$. Since, $|\sigma| = k''$, we obtain the total change of orientation given by Υ'. The explanation of Υ'' is similar.

The additional orientation change $w_1(d'_b)w_1(\partial d''_b)$ when $1 \in \sigma$ enters because then the path from z_{b1} to z_0 changes under conjugation. This path effects the orientation of $(evb_0^*TL)_{\mathbf{u}}$ when $w_1(\partial d_b'') = 1$. The orientation of $(evb_0^*TL)_{\mathbf{u}}$ enters twice into the orientation of $\mathcal{L}'^{\#}$ when $w_1(\partial d_b') = 0$. It determines the orientation of $evb_0^* \det(TL \oplus V_{\mathbb{R}})^*_{\mathbf{u}}$, and it determines the orientation of $u|^*_{(\partial\Sigma)_b}TL$, which in turn determines the orientation of $\det(D'_{\mathbf{u}} \oplus D'_0)$. Both of these determinants appear on the right-hand side of isomorphism (3.4). However, when $w_1(d_b') = 1$ we cannot orient $u|^*_{(\partial\Sigma)_b}TL$. So, the orientation of $(evb_0^*TL)_{\mathbf{u}}$ enters only once into the orientation of $\mathcal{L}'^{\#}$. So, there is an extra contribution to the orientation change of $\mathcal{L}'^{\#}$ exactly when

$$w_1(d_b')w_1(\partial d_b'') = 1.$$

$\square$

Chapter 6

Proof of invariance

Proof of Theorem 1.3. In order to prove independence from various choices, we construct cobordisms from parameterized moduli spaces. Complications arise in compactifying these cobordisms. For concreteness, we focus on independence of a variation of the constraints on marked points. The proof of independence of a variation of J, Σ or A, B, is very similar.

Recall that the definition of $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ depends on the choice of points

$$\vec{x} = (x_{ai}), \qquad x_{ai} \in L,$$

and pairs of points,

$$\vec{y} = (y_j), \qquad y_j : \{0,1\} \to X, \qquad y_j(1) = \phi(y_j(0)).$$

Suppose we choose different points $\vec{x}'$ and $\vec{y}'$ satisfying the same conditions. This corresponds to changing the forms α_{ai} and γ_j mentioned in Theorem 1.3. Let

$$\mathbf{x} : [0,1] \to L^{|\mathbf{k}|}, \qquad \mathbf{x}(0) = \vec{x}, \qquad \mathbf{x}(1) = \vec{x}',$$

$$\mathbf{y} : [0,1] \times \{0,1\}^l \to X^l, \qquad \mathbf{y}(t,1) = \phi(\mathbf{y}(t,0)),$$

$$\mathbf{y}(0,\star) = \vec{y}(\star), \qquad \mathbf{y}(1,\star) = \vec{y}'(\star).$$

If we choose $\mathbf{x}$ and $\mathbf{y}$ generically, they we will be transverse to the total evaluation map

$$\mathbf{ev} : \mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}) \to L^{|\mathbf{k}|} \times X^l.$$

So,

$$\mathcal{W} := \mathcal{W}(\mathbf{x}, \mathbf{y}) := \mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})\ {}_{\mathbf{ev}}\times_{(\mathbf{x}\times\mathbf{y})\circ\Delta} \left([0,1] \times \{0,1\}^l\right) \tag{6.1}$$

gives a smooth oriented cobordism between

$$\mathbf{ev}^{-1}(\vec{x}, \vec{y}) \quad \text{and} \quad \mathbf{ev}^{-1}(\vec{x}', \vec{y}').$$

However, $\mathcal{W}$ is generally not compact. So, in order to show the invariance of $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ we must study the non-trivial stable maps arising in the Gromov-compactification of $\mathcal{W}$, which we denote by $\partial_G \mathcal{W}$.

We now digress for a moment to describe $\partial_G \mathcal{W}$ more explicitly. We define

$$\widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') := \bar{\partial}^{\#-1}_{J,\nu}(0) \subset B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'').$$

Recall that the inhomogeneous perturbation ν vanishes on the bubble $D^2 \subset \hat{\Sigma}$. This means that the moduli space $\widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')$ may be singular at ϕ-multiply covered maps. As long as $\mu(d'') > 0$, this does not present a problem because Lemma 4.2 then shows that the image under the evaluation map of stable maps with ϕ-multiply covered bubbled components has codimension at least two. Also, constant holomorphic disks have expected dimension. On the other hand, in the case that (X, L) admits holomorphic disks of positive energy of Maslov index zero, we must take ϕ-multiply covered maps into consideration. We postpone the argument in this case to Section 7.

We continue now with the description of $\partial_G \mathcal{W}$. Since $\nu = 0$ on bubble components, we have an action of $PSL_2(\mathbb{R})$ on $\widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')$ given by

$$(\mathbf{u}', (u'', \vec{z}'', \vec{w}'')) \rightsquigarrow \left(\mathbf{u}', (u'' \circ \varphi, (\varphi^{-1})^{|\mathbf{k}|}(\vec{z}''), (\varphi^{-1})^l(\vec{w}''))\right), \quad \varphi \in PSL_2(\mathbb{R}).$$

On the other hand, a generic perturbation term ν will break the $Aut(\Sigma)$ invariance of $\bar{\partial}^{\#}_{J,\nu}$. So, we construct a section of the $Aut(\Sigma)$ action that would exist if ν vanished in the following manner. Let

$$\pi_j : \widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') \to \Sigma$$

be the projection sending $(u, \vec{z}, \vec{w}) \rightsquigarrow w_j$. In addition, we define

$$\pi_0' : \widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') \to \Sigma$$

by $\pi_0'(\mathbf{u}) = z_0'$, the point where the bubble attaches.

First, suppose that $\Sigma \simeq D^2$ and $l \geq 2$. Recall from Section 4 that in the construction of $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$, we chose an interior point $s_0 \in \Sigma$ and a line $\ell \subset D^2$ connecting s_0 to $\partial\Sigma$. We imposed the conditions $w_1 = s_0$ and $w_2 \in \ell$. Since s_0 is an interior point, w_1 cannot possibly bubble off onto a disk bubble, i.e. $1 \notin \varrho$. However, w_2 could bubble onto a disk bubble that bubbles off at $\ell \cap \partial\Sigma$. So, we consider the following two cases. If w_2 does not bubble off, i.e. $2 \notin \varrho$, define

$$\begin{aligned}\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') &:= (\pi_1 \times \pi_2)^{-1}(s_0 \times \ell)/PSL_2(\mathbb{R}) \\ &\subset \widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')/PSL_2(\mathbb{R}).\end{aligned}$$

If w_2 does bubble off, that is, $2 \in \varrho$, we define

$$\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') := (\pi_1 \times \pi_0')^{-1}(s_0 \times (\ell \cap \partial\Sigma))/PSL_2(\mathbb{R}).$$

Now, suppose $\Sigma \simeq S^1 \times I$ and $l \geq 1$. Recall from the construction of $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ that we chose a line $\ell \subset \Sigma$ connecting the two boundary components of Σ and imposed the condition $w_1 \in \ell$. So, w_1 could bubble off at a disk connecting to $\ell \cap \partial\Sigma$. So, we consider the following two cases. If w_1 does not bubble off, i.e. $1 \notin \varrho$, define

$$\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') := \pi_1^{-1}(\ell)/PSL_2(\mathbb{R}) \subset \widetilde{\mathcal{M}}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')/PSL_2(\mathbb{R}).$$

If w_1 does bubble off, i.e. $1 \in \varrho$, define

$$\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') := \pi_0'^{-1}(\ell \cap \partial\Sigma)/PSL_2(\mathbb{R}).$$

Now we turn to the case when $\Sigma \simeq D^2$ and $l = 0$. The cases $\Sigma \simeq D^2$, $l = 1$, and $\Sigma \simeq S^1 \times I$, $l = 0$, use a very similar argument, which we omit. Recall from the construction of $\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d})$ that we chose ϕ-anti-invariant pseudo-cycles (A, f) and (B, g) representing the Poincare dual of ω and satisfying various transversality conditions. Taking $\mathcal{M}_{\mathbf{k},\sigma,2,\varrho}(L, \Sigma, \mathbf{d}', d'')$ as defined above, we may perturb (A, f) and (B, g) slightly so that the evaluation map

$$ev_1 \times ev_2 : \mathcal{M}_{\mathbf{k},\sigma,2,\varrho}(L, \Sigma, \mathbf{d}', d'') \to X \times X$$

is transversal to $(A \times B, f \times g)$. So, we may define

$$\mathcal{M}_{\mathbf{k},\sigma,0,\varrho}(L, \Sigma, \mathbf{d}', d'') := \frac{1}{\omega(d)^2}\mathcal{M}_{\mathbf{k},\sigma,2,\varrho}(L, \Sigma, \mathbf{d}', d'') \times_{X \times X} (A \times B).$$

Perturbing $\mathbf{x}$ and $\mathbf{y}$ slightly assures they are transverse to the total evaluation map

$$\mathbf{ev} : \mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') \to X^l \times L^{|\mathbf{k}|}.$$

Each moduli space $\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}$ contributes a boundary stratum of the cobordism $\mathcal{W}$, which we denote by

$$\partial_G \mathcal{W}_{\sigma,\varrho} := \mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(\Sigma, L, \mathbf{d}', d'')\ {}_{\mathbf{ev}}\!\times_{(\mathbf{x}\times\mathbf{y})\circ\Delta} \left([0,1] \times \{0,1\}^l\right).$$

In total, the boundary of the Gromov compactification of $\mathcal{W}$ takes the form

$$\partial_G \mathcal{W} = \bigcup_{\substack{a \in [1,m],\, \sigma \subset [1,k_a] \\ \varrho \subset [1,l]}} \partial_G \mathcal{W}_{\sigma,\varrho}.$$

In general, one might expect an extra term in $\partial_G \mathcal{W}$ coming from sphere bubbles attached to a constant disk. If there are no marked points on the disk, this may happen in codimension 1. However, assumption (1.3) precludes this possibility.

Recall that $\mathbb{Z}/2\mathbb{Z}$ acts on $B_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'')$ by the involution $\phi_{B\#}$ that exchanges a disk bubble with its conjugate. Now, the boundary strata $\mathcal{W}_{\sigma,\varrho}$ are constructed from $B_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d''), X$ and L by considering the vanishing set of a $\mathbb{Z}/2\mathbb{Z}$ equivariant section and then taking various fiber products with respect to $\mathbb{Z}/2\mathbb{Z}$ equivariant maps. So, each stratum admits a canonical $\mathbb{Z}/2\mathbb{Z}$ action by an involution that we denote by $\phi_{\partial\mathcal{W}}$. We claim this action is fixed point free and orientation reversing. In other words,

$$\#\partial_G\mathcal{W} = 0,$$

so that

$$\begin{aligned} 0 = \#\partial\mathcal{W} &= \#(\mathbf{ev}^{-1}(\vec{x}',\vec{y}') - \mathbf{ev}^{-1}(\vec{x},\vec{y}) + \partial_G\mathcal{W}) \\ &= \#\mathbf{ev}^{-1}(\vec{x}',\vec{y}') - \#\mathbf{ev}^{-1}(\vec{x},\vec{y}). \end{aligned} \tag{6.2}$$

and $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ does not depend on the choice of $\vec{x}$, $\vec{y}$.

First, we show that $\phi_{\partial\mathcal{W}}$ is fixed point free. Indeed, as noted above, we are presently considering the case where we may assume there are no ϕ-multiply covered disks of positive energy. By definition, a ϕ-somewhere injective disk cannot be a fixed point of $\phi_{\partial\mathcal{W}}$. So, $\phi_{\partial\mathcal{W}}$ could only have a fixed point if a zero energy disk bubbled off. That would correspond to an interior marked point moving to the boundary. Clearly, marked points that are constrained away from L cannot possibly move to the boundary. This is where we use the fact that by Lemma 4.4 we have chosen (A,f) and (B,g) not to intersect L.

The following calculations show that $\phi_{\partial\mathcal{W}}$ reverses orientation. First, we consider the case that $\dim X = 6$ and L is orientable. Since $\dim L = 3$, by Wu's relations, $w_2(TL) = 0$. So, by formula (5.3), the sign of $\phi'_{\mathcal{L}\#}$ is

$$\frac{\mu(d'')}{2} + k'' + l'' + 1.$$

It follows that $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'')$ with this same sign. Indeed, in the case $l \leq 2$, we need to add marked points and fiber product with the divisors along the corresponding evaluation map. One of the extra marked points could be on the bubble component, so that $\mathbb{Z}/2\mathbb{Z}$ acts on it non-trivially. However, since the divisors are chosen to be ϕ anti-invariant, the total sign change induced on $\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}$ from the extra marked point will be zero. Also, we note at this point that the sign of the conjugation automorphism of $PSL_2(\mathbb{R})$ is zero.

Note that the sign of $\phi_{\partial\mathcal{W}}$ is independent of l''. Indeed, the action of ϕ on X is orientation reversing because $\phi^*\omega^3 = -\omega^3$. So, the sign of the $\mathbb{Z}/2\mathbb{Z}$ action on the fiber product of $\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'')$ with $[0,1]\times\{0,1\}^l$ over X^l where ϕ acts non-trivially on l'' of the factors of X is independent of l''. On the other hand, a straightforward virtual dimension calculation shows that $\partial_G\mathcal{W}_{\sigma,\varrho}$ must be empty unless

$$\mu(d'') = 2k'' + 4l''.$$

This in turn implies that $\mu(d'')/2 \cong k''$ (mod 2), or equivalently,

$$\frac{\mu(d'')}{2} + k'' + 1 \cong 1 \pmod{2}.$$

So, in this case, $\phi_{\partial\mathcal{W}}$ is orientation reversing.

Now we turn to the more difficult situation where $\dim X = 4$ and L may not be orientable. By the Wu relations, L is Pin^-. So, we assume that $\mathfrak{p}$ is given by a standard Pin^- structure. By the same argument as above, we conclude that $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(L,\Sigma,\mathbf{d}',d'')$ with sign given by formulas (5.1) or (5.2) depending on whether or not $1 \in \sigma$. Note that ϕ preserves the orientation of X because $\phi^*\omega^2 = \omega^2$. This implies that (5.1),(5.2) also give the right signs for the action of $\phi_{\partial\mathcal{W}}$. Next, observe that by virtual dimension counting, the stratum $\partial_G\mathcal{W}_{\sigma,\varrho}$ will be empty unless

$$\mu(d'') + r = k'' + 2l'' \tag{6.3}$$

for $r = 0$ or -1. We claim that this restriction implies that the signs (5.1) and (5.2) always simplify to exactly 1.

We first consider the case $1 \notin \sigma$. Using the restriction (6.3), we calculate

$$\begin{aligned}\frac{k''(k''-1)}{2} &= \frac{(\mu(d'')+r-2l'')(\mu(d'')+r-2l''-1)}{2} \\ &= \frac{\mu(d'')^2+2r\mu(d'')+r^2-4l''(\mu(d'')+r)+4l''^2-\mu(d'')-r-2l''}{2} \\ &\cong \frac{\mu(d'')(\mu(d'')-1)}{2}+\frac{r(r-1)}{2}+l''+r\mu(d''). \quad (\mathrm{mod}\ 2) \qquad (6.4)\end{aligned}$$

Again using the restriction (6.3), we calculate

$$\begin{aligned}\Upsilon'(d'',k'') \cong \mu(d'')k'' &\cong \mu(d'')^2+r\mu(d'')+2l''\mu(d'') \\ &\cong \mu(d'')+r\mu(d'') \quad (\mathrm{mod}\ 2). \qquad (6.5)\end{aligned}$$

Substituting equations (6.4) and (6.5) into (5.1), eliminating the remaining k'' by (6.3) and canceling expressions which occur in pairs yields

$$\mathfrak{s}_{\pm}^{\#'}(d'',k'',l'') \cong \frac{r(r+1)}{2}+1 \quad (\mathrm{mod}\ 2).$$

This is always exactly 1 since $r=0$ or -1.

We turn now to the case $1 \in \sigma$. Using the restriction (6.3), we calculate

$$\begin{aligned}\frac{(k''-1)(k''-2)}{2} &= \frac{(\mu(d'')+r-2l''-1)(\mu(d'')+r-2l''-2)}{2} \\ &= \frac{1}{2}\left[\mu(d'')^2+2r\mu(d'')+r^2-3\mu(d'')-3r\right. \\ &\qquad \left.+2+4l''^2+4l''(\mu(d'')+r)+6l''\right] \\ &\cong \frac{\mu(d'')(\mu(d'')+1)}{2}+\frac{r(r+1)}{2} \\ &\qquad +r\mu(d'')+l''+1 \quad (\mathrm{mod}\ 2). \qquad (6.6)\end{aligned}$$

Furthermore, using the condition $w_1(d_b) \cong k_b+1$ and (6.3),

$$k_b k''+k_b \cong (w_1(d_b)+1)(\mu(d'')+r+1) \quad (\mathrm{mod}\ 2). \qquad (6.7)$$

Substitute calculations (6.6) and (6.7) in sign formula (5.2) and use restriction (6.3). Cancelling pairs of similar terms, we obtain

$$
\begin{aligned}
\mathfrak{s}_{-}^{\#''} &\cong \frac{r(r+1)}{2} + r\mu(d'') + r \\
&\quad + (w_1(d_b)+1)(\mu(d'')+r+1) + \Upsilon'' + w_1(d_b')w_1(d'') \\
&\cong r\mu(d'') + r + (w_1(d_b)+1)(\mu(d'')+r+1) + \Upsilon'' + w_1(d_b')w_1(d''). \qquad (6.8)
\end{aligned}
$$

Here, the second congruence follows from the fact $r = 0$ or -1.

We now expand Υ'' to further analyze $\mathfrak{s}^{\#''}$. Using the fact that $w_1(d_b) = w_1(d_b') + w_1(\partial d'')$, it is easy to verify that

$$\Upsilon'' = w_1(d_b)(k''-1) + w_1(\partial d'')k'. \qquad (6.9)$$

By repeatedly applying restriction (6.3), the condition that $w_1(d_a) = 1 + k_a$ and the fact that $\mu(d'') \cong w_1(\partial d'')\pmod 2$, we calculate,

$$
\begin{aligned}
w_1(\partial d'')k' &\cong w_1(\partial d'')(k_b - k'') \\
&\cong w_1(\partial d'')(w_1(d_b) + 1 + \mu(d'') + r) \\
&\cong w_1(\partial d'')(w_1(d_b') + w_1(\partial d'') + 1 + \mu(d'') + r) \\
&\cong w_1(\partial d'')w_1(d_b') + \mu(d'')(\mu(d'') + r) \pmod 2.
\end{aligned}
$$

Substituting this calculation in formula (6.9), and using restriction (6.3) again, we obtain

$$\Upsilon'' \cong w_1(d_b)(\mu(d'')+r+1) + w_1(\partial d'')w_1(d_b') + \mu(d'')(\mu(d'')+r) \pmod 2.$$

Substituting this expression for Υ'' in formula (6.8) an cancelling all repeated terms leaves

$$\begin{aligned}
\mathfrak{s}_{-}^{\#''} &\cong r\mu(d'') + r + (1)(\mu(d'') + r + 1) + \mu(d'')(\mu(d'') + r) \\
&\cong r\mu(d'') + r + \mu(d'') + r + 1 + \mu(d'') + \mu(d'')r \\
&\cong 1 \pmod{2},
\end{aligned}$$

as desired.

Chapter 7

An equivariant Kuranishi structure

In this section, we complete the proof of invariance of $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ in the case when (X, L) may admit holomorphic disks of Maslov index 0. If $\dim L = 2$, the expected dimension of holomorphic disks with Maslov index 0 is negative. For generic J, by Lemma 4.2, such disks don't exist. So, we consider the case $\dim L = 3$. By assumption, L is orientable. The main tool we use to prove invariance in this case is the notion of a Kuranishi structure, introduced in [4] and extended in [5]. For a summary of relevant information on Kuranishi structures, see Appendix A.

Suppose $(X, \mathcal{K})$ is a space with Kuranishi structure $\mathcal{K} = (V_p, E_p, \Gamma_p, s_p, \psi_p)$. Let ι be an involution of X.

Definition 7.1. An *extension* $\tilde{\iota}$ of an involution ι to an involution of $\mathcal{K}$ consists of Γ_p-equivariant maps $\iota_p : V_p \to V_{\iota(p)}$ and $\hat{\iota}_p : E_p \to E_{\iota(p)}$ covering ι_p such that

(E1) $\iota_{\iota(p)} \circ \iota_p = \mathrm{Id}_{V_p}$.

(E2) $s_{\iota(p)} \circ \iota_p = \hat{\iota}_p \circ s_p$.

(E3) $\psi_{\iota(p)} \circ \iota_p|_{s_p^{-1}(0)} = \iota \circ \psi_p$.

(E4) ι_q maps $V_{pq} \subset V_q$ to $V_{\iota(p)\iota(q)} \subset V_{\iota(q)}$.

(E5) $\iota_p \circ \varphi_{pq} = \varphi_{\iota(p)\iota(q)} \circ \iota_q$ and $\hat{\iota}_p \circ \hat{\varphi}_{pq} = \hat{\varphi}_{\iota(p)\iota(q)} \circ \hat{\iota}_q$.

Note that ι_p, $\hat{\iota}_p$, induce bundle morphisms

$$\iota_p^T : \det(TV_p) \otimes \det(E_p) \to \det(TV_{\iota(p)}) \otimes \det(E_{\iota(p)})$$

covering ι_p. Now, suppose that $(X, \mathcal{K})$ has a tangent bundle given by Φ_{pq}. We say that $\tilde{\iota}$ acts smoothly on $(X, \mathcal{K})$ if

$$\Phi_{\iota(p)\iota(q)} \circ \hat{\iota}_q = \hat{\iota}_p \circ \Phi_{pq}.$$

If $\tilde{\iota}$ acts smoothly and $(X, \mathcal{K})$ is oriented, the bundle morphisms ι_p^T may either preserve or reverse the orientation of $\mathcal{K}$ over each connected component of X.

Now, we give the main idea of the proof. As we will explain, it is possible to construct an oriented Kuranishi structure with tangent bundle $\mathcal{K} = (V_p, E_p, \Gamma_p, s_p, \psi_p)$ on $\mathcal{W}$. Moreover, we may construct $\mathcal{K}$ so that the induced Kuranishi structure on $\partial_G\mathcal{W}$ admits a smooth orientation reversing involution $\tilde{\phi}_{\partial\mathcal{W}}$ extending $\phi_{\partial\mathcal{W}}$. There exists a good coordinate system $\mathcal{G} = (P, V_p', E_p', s_p', \psi_p')$ for $\mathcal{K}$ such that its restriction to $\partial_G\mathcal{W}$ is preserved by $\tilde{\phi}_{\partial\mathcal{W}}$. That is, $\phi_{\partial\mathcal{W}^p}$ preserves $V_p' \subset V_p$. Finally, we may choose $\phi_{\partial\mathcal{W}^p} - \hat{\phi}_{\partial\mathcal{W}^p}$ equivariant generic transverse multisections $s_{p,n}'$ approximating s_p' and coinciding exactly with s_p' away from $\psi_p^{-1\prime}(\partial_G\mathcal{W})$. Indeed, since we have avoided ϕ-multiply covered irreducible maps by means of an inhomogeneous perturbation ν, the unperturbed s_p' are already transverse away from $\psi_p^{-1}(\partial_G\mathcal{W})$.

Note that the charts of the induced Kuranishi structure on $\partial\mathcal{W}$ are just ∂V_p. We define

$$\partial_G V_p := \psi_p^{-1}(\partial_G\mathcal{W}) \subset \partial V_p.$$

The same applies for the charts of good-coordinate system and we use the analogous notation. The vanishing sets of the $s_{p,n}'$ define a 1-dimensional simplicial complex with boundary contained in the $\partial V_p'$. The boundary is simply a collection of points with signed rational weights. By the same reasoning as in equation (6.2), it suffices to show that part of this boundary contained in $\partial_G V_p'$ has total cardinality zero. Again, we use the sign reversing involutions $\phi_{\partial\mathcal{W}^p}$ to cancel the points in pairs. The only slight complication arises because $\phi_{\partial\mathcal{W}^p}$ may have fixed points. However, this is easily resolved by the observation that a point x of the vanishing set of $s'_{p,n}$ fixed by $\phi_{\partial\mathcal{W}^p}$ must have weight zero. Indeed, by definition of transversality for multisections, each branch $s^{i'}_{p,n}$ of $s'_{p,n}$ is transverse to zero. So, if $s^{i'}_{p,n}$ vanishes at x, the differential

$$ds^{i'}_{p,n} : T_x\partial V'_p \xrightarrow{\sim} (E'_p)_x$$

defines a non-zero element $\omega \in \det(TV'_p) \otimes \det(E'_p)$. Since $s'_{p,n}$ is $\phi_{\partial\mathcal{W}^p} - \hat{\phi}_{\partial\mathcal{W}^p}$ equivariant,

$$\hat{\phi}_{\partial\mathcal{W}^p} \circ s^{i'}_{p,n} \circ \phi_{\partial\mathcal{W}^p}$$

must also be a branch of $s'_{p,n}$, defining an element $\omega' \in \det(TV'_p) \otimes \det(E'_p)$. But since $\tilde{\phi}_{\partial\mathcal{W}}$ is orientation reversing, we know that ω and ω' belong to opposite components of $\det(TV'_p) \otimes \det(E'_p) \setminus \{0\}$. So, the branches of $s'_{p,n}$ that vanish at x come in pairs that induce opposite orientations on x. Therefore, the total weight of x is zero.

We turn our attention to the construction of $\mathcal{K}$. Recall from (6.1) that

$$\mathcal{W} = \mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \;{}_{\mathbf{ev}}\!\times_{(\mathbf{x}\times\mathbf{y})\circ\Delta} \left([0,1] \times \{0,1\}^l\right).$$

Let $\partial\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ denote the union of all the strata of the Gromov compactification of $\mathcal{M}_{\mathbf{k},l}$ except $\mathcal{M}_{\mathbf{k},l}$ itself. Let $\phi_{\partial\mathcal{M}}$ denote the involution of $\partial\mathcal{M}_{\mathbf{k},l}$ induced by $\phi_{B\#}$. As explained in [5, Appendix 2], a weakly submersive Kuranishi structure $\mathcal{K}_{\mathcal{M}}$ on $\overline{\mathcal{M}}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ naturally induces a Kuranishi structure $\mathcal{K}$ on the fiber product $\mathcal{W}$. If we let $\mathbb{Z}/2\mathbb{Z}$ act on X by ϕ and on $[0,1] \times \{0,1\}^l$ by exchanging 0 and 1, $\mathbf{x} \times \mathbf{y}$ is clearly $\mathbb{Z}/2\mathbb{Z}$ equivariant. So, if $\mathcal{K}_{\mathcal{M}}|_{\partial\mathcal{M}_{\mathbf{k},l}}$ admits an extension of $\phi_{\partial\mathcal{M}}$, then $\mathcal{K}|_{\partial_G\mathcal{W}}$ will admit an extension of $\phi_{\partial\mathcal{W}}$. So, we may narrow our focus to the construction of $\mathcal{K}_{\mathcal{M}}$.

We assume $\Sigma = D^2$. The other cases are similar but easier. Interpreting (4.3) as a fiber product, we have

$$\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) = \widetilde{\mathcal{M}}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \times_\Sigma (s_0 \times \ell).$$

Let $\partial\widetilde{\mathcal{M}}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ denote the union of all strata of the Gromov compactification of $\widetilde{\mathcal{M}}_{\mathbf{k},l}$ except $\widetilde{\mathcal{M}}_{\mathbf{k},l}$ itself. Let $\phi_{\partial\widetilde{\mathcal{M}}}$ denote the involution of $\partial\widetilde{\mathcal{M}}_{\mathbf{k},l}$ induced by $\phi_{B\#}$. Again, we reduce to constructing a weakly submersive Kuranishi structure $\mathcal{K}_{\widetilde{\mathcal{M}}}$ on $\overline{\widetilde{\mathcal{M}}}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ such that its restriction to $\partial\widetilde{\mathcal{M}}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ admits an extension of $\phi_{\partial\widetilde{\mathcal{M}}}$.

In [5], Fukaya et al. constructed a Kuranishi structure on the moduli space of J-holomorphic disks with Lagrangian boundary conditions. Away from $\partial\widetilde{\mathcal{M}}_{\mathbf{k},l}$, we use their construction without modification. Near $\partial\widetilde{\mathcal{M}}_{\mathbf{k},l}$ we need to modify the construction slightly to make sure we can find an extension of $\phi_{\partial\widetilde{\mathcal{M}}}$. So, we briefly recount the idea of the construction of the Kuranishi neighborhood of a point $p \in \partial\widetilde{\mathcal{M}}_{\mathbf{k},l}$ given in [5]. We assume that p is a stable map of two components. The construction for more components is similar. By definition, p is an equivalence class of quadruples $\mathbf{u} = (\hat{\Sigma}, u, \vec{z}, \vec{w}) \in B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'')$ such that $\bar{\partial}^{\#}_{J,\nu} u = 0$. The equivalence relation in the case of two components equates reparameterizations of the bubble component. When there are more than two components, the equivalence relation also takes into consideration automorphisms of the underlying tree of the stable map. We choose some $\mathbf{u}$ such that $[\mathbf{u}] = p$. Locally trivializing $\mathcal{E}^{\#}$ by parallel translation and projection to $\Lambda^{0,1}(TX)$, we define

$$D^{\#}_{\mathbf{u}} = D_{\mathbf{u}}\bar{\partial}^{\#}_{J,\nu} : T_{\mathbf{u}}B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(L, \Sigma, \mathbf{d}', d'') \longrightarrow \mathcal{E}^{\#}_{\mathbf{u}}$$

to be the linearization of $\bar{\partial}^{\#}_{J,\nu}$ at $\mathbf{u}$. Since the bubble component u'' of u may be ϕ-multiply covered, $D^{\#}_{\mathbf{u}}$ may not be surjective even for generic J and ν. However, since $D^{\#}_{\mathbf{u}}$ is Fredholm, we may choose a finite dimensional subspace $E_{\mathbf{u}} \subset \mathcal{E}^{\#}_{\mathbf{u}}$ such that if $\pi : \mathcal{E}^{\#}_{\mathbf{u}} \to \mathcal{E}^{\#}_{\mathbf{u}}/E_{\mathbf{u}}$ denotes the natural projection, then $\pi \circ D^{\#}_{\mathbf{u}}$ is surjective. Possibly enlarging $E_{\mathbf{u}}$, we may assume that the evaluation maps from $\ker \pi \circ D^{\#}_{\mathbf{u}}$ to $T_{u(z_i)}L$ and $T_{u(w_j)}X$ are surjective. This is necessary for the Kuranishi structure we are describing to be weakly submersive. By elliptic regularity, we may choose $E_{\mathbf{u}}$ to consist of smooth sections of $\Lambda^{0,1}(u^*TX)$. By the unique continuation theorem, we may assume these sections are compactly supported away from the singular point z_0.

Let $\mathbf{u}_\epsilon = (\hat{\Sigma}_\epsilon, u_\epsilon, \vec{z}_\epsilon, \vec{w}_\epsilon)$ be sufficiently close to $\mathbf{u}$. More precisely, choose a small $\delta >$

0. We allow $\hat{\Sigma}_\epsilon$ to differ from $\hat{\Sigma}$ in a δ neighborhood N_δ of the singular point z_0. Either z_0 may move slightly, or small neighborhoods of z_0 in each component of $\hat{\Sigma}$ may be removed and their boundaries glued to smooth the singularity. In particular, outside N_δ, there exists a canonical identification of $\hat{\Sigma}$ with $\hat{\Sigma}_\epsilon$. The pre-gluing construction explained in detail in [4, 16] gives a smooth map $\tilde{u} : (\hat{\Sigma}', \partial\hat{\Sigma}') \to (X, L)$ that agrees with u outside N_δ and stays very close to $u(z_0)$ within N_δ. We assume that u_ϵ is δ close to $\tilde{u}$ in the $W^{1,p}$ norm. Also, we assume that $\vec{z}_\epsilon, \vec{w}_\epsilon$, are δ close to $\vec{z}, \vec{w}$. Then, for δ sufficiently small, there exist unique shortest geodesics from $u(z)$ to $u_\epsilon(z)$ for each $z \in \hat{\Sigma} \setminus N_\delta$. For δ sufficiently small, we may assume N_δ is disjoint from the support of the sections of $E_{\mathbf{u}}$. So, we may parallel translate $E_{\mathbf{u}}$ along the geodesics and then project to $\Lambda^{0,1}(TX)$ to obtain a subspace of $\mathcal{E}^{(\#)}_{\mathbf{u}_\epsilon}$. Here, we parenthesize $\#$ because $\mathbf{u}_\epsilon$ may be an irreducible $W^{1,p}$ stable map. If δ is sufficiently small this subspace has constant dimension. We let $\underline{E}_{\mathbf{u}}$ denote the sub-bundle of $\mathcal{E}^{(\#)}$ so obtained. Similarly, we let $\underline{\pi} : \mathcal{E}^{(\#)} \to \mathcal{E}^{(\#)}/\underline{E}_{\mathbf{u}}$ define the projection to the quotient bundle.

According to [5, 4], we may essentially define V_p to be the set of $\mathbf{u}_\epsilon$ as above such that $\underline{\pi} \circ \bar{\partial}^{(\#)}_{J,\nu} u_\epsilon = 0$. Then, we define $E_p = \underline{E}_{\mathbf{u}}|_{V_p}$ and $s_p = \bar{\partial}^{(\#)}_{J,\nu}$. The definition of ψ_p is tautological. A great deal of hard analysis then shows that V_p is actually a smooth manifold with boundary modelled on

$$\ker\left(\pi \circ D^{\#}_{\mathbf{u}}\right) \times (R, \infty].$$

However, we do not need to know the details of this analysis at all for our purposes. Also, we note that it is necessary to further enlarge $E_{\mathbf{u}}$ in order to construct the maps $\varphi_{pq}, \hat{\varphi}_{pq}$. However, this step is essentially formal, and it is not hard to make it ϕ-equivariant. So, we do not discuss it here and instead refer the reader to [5, end of Chapter 18], or for more detail [4, Chapter 15]. Finally, we note that canonical orientation of $\det(D^{\#})$ induces an orientation of $\det(TV_p) \otimes \det(E_p)$. Indeed, note that there exist natural isomorphisms $\ker Ds_p \simeq \ker D\bar{\partial}_{J,\nu}$ and $\operatorname{coker} Ds_p \simeq \operatorname{coker} D\bar{\partial}_{J,\nu}$.

On the other hand, the exact sequence

$$0 \to \ker Ds_p \to TV_p \to E_p \to \operatorname{coker} Ds_p \to 0$$

induces a natural isomorphism

$$\det(TV_p) \otimes \det(E_p) \simeq \det(\ker Ds_p) \otimes \det(\operatorname{coker} Ds_p).$$

We list the modifications of the above construction necessary so that it will admit an extension of $\phi_{\partial\widetilde{\mathcal{M}}}$. Clearly, we must choose a ϕ-invariant metric on X for measuring all distances, constructing geodesics, and parallel transport. We consider two cases: First suppose $p_\phi := \phi_{\partial\widetilde{\mathcal{M}}}(p) \neq p$. We simultaneously construct Kuranishi neighborhoods of p and p_ϕ as well as the extension of $\phi_{\partial\widetilde{\mathcal{M}}}$. Indeed, we choose the representative $\mathbf{u}_\phi := \phi_{B\#}(\mathbf{u})$ of p_ϕ. Furthermore, we define $E_{\mathbf{u}_\phi} = \phi_{\mathcal{E}\#}(E_\mathbf{u})$. This is compatible with the construction above because of the ϕ-invariance of the metric. Again, by ϕ-invariance of the metric, it follows that $\phi_{\mathcal{E}\#}$ maps $\underline{E}_\mathbf{u}|_{B\#}$ to $\underline{E}_{\mathbf{u}_\phi}|_{B\#}$. Since $\bar{\partial}^\#_{J,\nu}$ is $\phi_{B\#} - \phi_{E\#}$ equivariant, it follows that $\phi_{B\#}$ maps ∂V_p to ∂V_{p_ϕ}. So, we define the extension of $\phi_{\partial\widetilde{\mathcal{M}}}$ by

$$\phi_{\partial\widetilde{\mathcal{M}}^p} := \phi_{B\#}|_{\partial V_p}, \qquad \hat{\phi}_{\partial\widetilde{\mathcal{M}}^p} := \phi_{E\#}|_{E_p}.$$

On the other hand, suppose that $\phi_{\partial\widetilde{\mathcal{M}}}(p) = p$. This may happen when u'' is a ϕ-multiply covered disk of even multiplicity. It is not hard to see that we may choose $\mathbf{u}$ representing p such that $\phi_{B\#}(\mathbf{u}) = \mathbf{u}$. Indeed, this follows from the fact that all anti-holomorphic involutions of D^2 are conjugate under the action of $PSL_2(\mathbb{R})$. This said, we may define the extension of $\phi_{\partial\widetilde{\mathcal{M}}}$ exactly as above. This completes the construction of $\mathcal{K}_{\widetilde{M}}$.

Now, a minor adaptation of the proof of [4, Lemma 6.3] gives the good cover $\mathcal{G}$. To obtain generic transverse multisections $s'_{p,n}$ such that $s'_{p,n}|_{\partial_G V'_p}$ is $\tilde{\phi}_{\partial W}$ equivariant, we use an argument from [5, Section 11]. That is, since $\mathcal{K}|_{\partial_G \mathcal{W}}$ admits an extension of $\phi_{\partial\mathcal{W}}$, it descends naturally to a Kuranishi structure on $\mathcal{W}/(\mathbb{Z}/2\mathbb{Z})$. Similarly, $\mathcal{G}|_{\partial_G \mathcal{W}}$

descends to a good coordinate system $\hat{\mathcal{G}} = (\hat{P}, \hat{V}'_p, \hat{E}'_p, \hat{s}'_p, \hat{\psi}'_p)$ on $\partial_G\mathcal{W}/(\mathbb{Z}/2\mathbb{Z})$. We denote by

$$\pi : \partial\mathcal{W} \to \partial\mathcal{W}/(\mathbb{Z}/2\mathbb{Z})$$

the quotient map. We can apply the standard machinery of Kuranishi structures developed in [4, Chapter 1], reviewed in Theorem A.4, to obtain generic transversal multi-sections $\hat{s}'_{p,n}$ on $\hat{V}'_p$. Pulling back $\hat{s}'_{p,n}$ under π, we obtain $\phi_{\partial\mathcal{W}^p} - \hat{\phi}_{\partial\mathcal{W}^p}$ equivariant transverse multisections over $\partial_G V'_p$. Since V'_p is a manifold with corners, it is not hard to extend a transverse section from $\partial_G V'_p$ to a neighborhood of $\partial_G V'_p$. In fact, away from corners, a neighborhood of $\partial_G V'_p$ is diffeomorphic to $[0,1) \times \partial_G V'_p$ and we can extend sections transverse to zero as constants over $[0,1)$. Since $\dim K = 1$, a transverse section cannot have zeros at corners of V'_p, so we can extend near corners arbitrarily. Since s'_p are already transverse away from $\partial_G V'_p$, it is not hard to patch them with the extensions we have just constructed while maintaining transversality everywhere. This completes the proof of Theorem 1.3. $\square$

Chapter 8

Calculations

In this section we prove Theorem 1.8 and Example 1.6. The main tool of the proofs is the notion of a short exact sequence of Cauchy-Riemann *Pin* boundary value problems. The first step in understanding short exact sequences of *Pin* boundary problems is to understand short exact sequences of bundles with *Pin* structure. Suppose

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

is a short exact sequence of real vector bundles over a base B.

Lemma 8.1. *Assume that at least one of V' and V'' is orientable. Then a Pin structure on any two of $V, V', V,''$ naturally induces a Pin structure on the third. If B is a one dimensional manifold then V' (resp. V'') automatically carries a Pin structure. If, on the other hand, V' (resp. V'') is one dimensional, the Pin structure on V' (resp. V'') may be chosen canonically.*

Proof. For the proof of this Lemma, we write $\dim V' = n$ and $\dim V'' = m$. Choosing a metric on V, we may identify $V \simeq V' \oplus V''$. By symmetry of the direct sum, we may assume that V' is orientable. Use the orientation of V' to reduce its structure group to $SO(n)$. So, the first claim of the Lemma follows from the existence of the commutative square of group homomorphisms

$$\begin{array}{ccc} Spin(n) \times Pin(m) & \longrightarrow & Pin(n+m) \\ \downarrow & & \downarrow \\ SO(n) \times O(n) & \longrightarrow & O(n+m). \end{array}$$

Indeed, we work on the level of transition functions which satisfy the co-cycle condition. The commutativity of the two factors of the product of groups ensures that the direct sum does not effect the cocyle condition.

The second claim of the Lemma follows because the obstruction to the existence of a *Pin* structure is a second cohomology class. The final claim follows when $B = \mathbb{R}P^1$ because, as noted in Remark 2.7, all automorphisms of a line bundle over $\mathbb{R}P^1$ preserve *Pin* structure and so we can induce a *Pin* structure canonically from a previously chosen one on $\tau_{\mathbb{R}} \to \mathbb{R}P^1$ or $\mathbb{R} \to \mathbb{R}P^1$. This extends to general B because a *Pin* structure on $V \to B$, if it exists, is determined by its restriction to loops in B. □

Now, let E, E', E'', be complex vector bundles over a Riemann surface with boundary Σ and let F, F', F'', be totally real subbundles of E, E', E'', respectively, over $\partial\Sigma$. Suppose further that we have an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

that restricts to an exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0. \tag{8.1}$$

We refer to such a short exact sequence as a short exact sequence of pairs of vector bundles. Let $\mathfrak{p}, \mathfrak{p}', \mathfrak{p}''$, be *Pin*-structures on F, F', F'', respectively. We say that $\mathfrak{p}$ is *compatible* with the short exact sequence (8.1) if $\mathfrak{p}$ agrees with the *Pin* structure naturally induced on F by $\mathfrak{p}'$ and $\mathfrak{p}''$ by Lemma 8.1. If F' or F'' is one dimensional, even if it does not come equipped with a *Pin* structure, we extend the notion of compatibility by equipping it with the canonical *Pin* structure of Lemma 8.1.

Definition 8.2. A *short exact sequence* of Cauchy-Riemann *Pin* boundary value problems

$$0 \longrightarrow \underline{D}' \longrightarrow \underline{D} \longrightarrow \underline{D}'' \longrightarrow 0$$

consists of

- An exact sequence of pairs of vector bundles

$$0 \longrightarrow (E', F') \longrightarrow (E, F) \longrightarrow (E'', F'') \longrightarrow 0 \tag{8.2}$$

  such that at least one of F' and F'' is orientable.

- Orientations on each of F, F', F'', which is orientable. If all three are orientable, we assume the orientation of F agrees with the orientation induced from F' and F''.

- *Pin* structures $\mathfrak{p}, \mathfrak{p}', \mathfrak{p}''$, on F, F', F'', respectively. If F' (resp. F'') has dimension 1, we do not require $\mathfrak{p}'$ (resp. $\mathfrak{p}''$) as part of the definition, since it may be chosen canonically by Lemma 8.1.

- Cauchy-Riemann operators

$$D : \Gamma(E, F) \longrightarrow \Gamma\left(\Omega^{0,1}(E)\right), \quad D' : \Gamma(E', F') \longrightarrow \Gamma\left(\Omega^{0,1}(E')\right),$$
$$D'' : \Gamma(E'', F'') \longrightarrow \Gamma\left(\Omega^{0,1}(E'')\right),$$

  such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma\left(\Omega^{0,1}(E')\right) \arrow[r] & \Gamma\left(\Omega^{0,1}(E)\right) \arrow[r] & \Gamma\left(\Omega^{0,1}(E'')\right) \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(E', F') \arrow[r] \arrow[u, "D'"] & \Gamma(E, F) \arrow[r] \arrow[u, "D"] & \Gamma(E'', F'') \arrow[r] \arrow[u, "D''"] & 0
\end{tikzcd}$$

  commutes.

Note that a short exact sequence of Cauchy-Riemann *Pin* boundary value problems is an example of a short exact sequence of Fredholm operators. See Definition

2.15.

Proposition 8.3. *Let*

$$0 \longrightarrow D' \longrightarrow D'' \longrightarrow D' \longrightarrow 0$$

be a short exact sequence of Cauchy-Riemann boundary value problems. Up to a universal sign, the isomorphism

$$\det(D') \otimes \det(D'') \xrightarrow{\sim} \det(D)$$

given by Lemma 2.16 respects the canonical orientations of Proposition 2.8 if and only if $\mathfrak{p}$ is compatible with the short exact sequence. The universal sign depends on the dimension of E, E', E'', the topology of Σ and the orientability of F, F', F'', restricted to each boundary component of Σ.

Proof. By a deformation argument, similar to the proof of Proposition 2.8, we would like to reduce the problem to a standard short exact sequence. Throughout the proof, we assume that $\mathfrak{p}$ is compatible with the short exact sequence. The other case follows from Lemma 2.10. As in the proof of Proposition 2.8, degenerate Σ along curves γ_a to a nodal Riemann surface $\hat{\Sigma}$ with nodal points $\hat{\gamma}_a$, and irreducible components $\Delta_a \simeq D^2$ and $\tilde{\Sigma} \simeq \Sigma/\partial\Sigma$. Simultaneously, degenerate E, E', E'', to complex vector bundles $\hat{E}, \hat{E}', \hat{E}''$, over $\hat{\Sigma}$ that all satisfy condition (2.2) for appropriate n. Now, by degenerate, we mean identify the fibers of $E|_{\gamma_a}$ (resp. $E'|_{\gamma_a}, E''|_{\gamma_a}$) with the single fiber $\hat{E}_{\hat{\gamma}_a}$ (resp. $E'_{\hat{\gamma}_a}, E''_{\hat{\gamma}_a}$). Such a degeneration satisfying condition (2.2) is unique up to homotopy. Furthermore, we may choose the degeneration of E to extend the degeneration of E'. These two degenerations induce a degeneration of E'' via the short exact sequence (8.2). So, we may assume that there exists a natural induced short exact sequence

$$0 \longrightarrow (\hat{E}', \hat{F}') \longrightarrow (\hat{E}, \hat{F}) \longrightarrow (\hat{E}'', \hat{F}'') \longrightarrow 0. \tag{8.3}$$

Choose a particular isomorphism (2.2) for $(\hat{E}'|_{\Delta_a}, \hat{F}'|_{\partial\Delta_a})$. Extend it to an isomorphism (2.2) for $(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial\Delta_a})$. Denote the canonical bundle pairs over $(D^2, \partial D^2)$

by

$$(E_{i,n}, F_{i,n}) := \begin{cases} (\tau \oplus \mathbb{C}^{n-1}, \tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}), & i = -1 \\ (\mathbb{C}^n, \mathbb{R}^n), & i = 0. \end{cases}$$

The following diagram shows that we have a naturally induced isomorphism (2.2) for $(\hat{E}''|_{\Delta_a}, \hat{F}''|_{\partial\Delta_a})$:

$$\begin{tikzcd} 0 \arrow[r] & (E_{i,n}, F_{i,n}) \arrow[r] & (E_{i+j,n+m}, F_{i+j,n+m}) \arrow[r] & (E_{j,m}, F_{j,m}) \arrow[r] & 0 \\ 0 \arrow[r] & (\hat{E}'|_{\Delta_a}, \hat{F}'|_{\partial\Delta_a}) \arrow[r] \arrow[u] & (\hat{E}|_{\Delta_a}, \hat{F}|_{\partial\Delta_a}) \arrow[r] \arrow[u] & (\hat{E}''|_{\Delta_a}, \hat{F}''|_{\partial\Delta_a}) \arrow[r] \arrow[u, dotted] & 0. \end{tikzcd}$$

Here, the top row makes sense because by assumption, either i or j or both are 0. So, the top row is tautological. The bottom row is just a restriction of short exact sequence (8.3). Since the morphisms in the top row commute with the canonical Cauchy-Riemann operators on the bundles $E_{i,n}$, the isomorphisms (2.2) just chosen induce Cauchy-Riemann operators D_a, D'_a and D''_a, on $\hat{E}|_{\Delta_a}, \hat{E}'|_{\Delta_a}$ and $\hat{E}''|_{\Delta_a}$ respectively, that commute with the morphisms of the short exact sequence (8.3). We claim that if $n + m \geq 3$, the preceding construction is unique up to homotopy. Indeed, the short exact sequences in the above diagram are split. So, the middle vertical morphism determines both of the other vertical morphisms. On the other hand, when $n \geq 3$, the middle vertical morphism is unique up to homotopy by Lemma 2.6. For $n = 2$, we use a stabilization argument as in the proof of Proposition 2.8 to reduce to the case $n = 3$.

Choose operators $\tilde{D}$, $\tilde{D}'$ and $\tilde{D}''$, on $\hat{E}|_{\tilde{\Sigma}}$, $E'|_{\tilde{\Sigma}}$ and $E''|_{\tilde{\Sigma}}$ compatible with the short exact sequence (8.3). Note that the induced isomorphism

$$\det(\tilde{D}') \otimes \det(\tilde{D}'') \xrightarrow{\sim} \det(\tilde{D}),$$

always preserves the canonical complex orientations of each side. Finally, choose homotopies of operators $\underline{D}_t, \underline{D}'_t$ and $\underline{D}''_t$, on E, E' and E'' respectively, compatible with the short exact sequence (8.3), such that

$$\underline{D}_0 = D, \qquad \underline{D}_1 = \#_a D_a \# \tilde{D}$$

and similarly for $\underline{D}'_t$ and $\underline{D}''_t$. Applying Lemma 2.16 to the short exact sequence of families of Fredholm operators,

$$0 \longrightarrow \underline{D}'_t \longrightarrow \underline{D}_t \longrightarrow \underline{D}''_t \longrightarrow 0$$

proves that the sign is universal, as claimed. □

We now prove a technical lemma that will be useful in the proof of Theorem 1.8. The idea of the proof is taken from [16, proof of Theorem C.1.10(iii)], which in turn follows the work of Hofer-Lizan-Sikorav [7].

Lemma 8.4. *Let $(E, F) \to (D^2, \partial D^2)$ be a vector bundle pair with $\dim_{\mathbb{C}} E = 1$, and denote its Maslov index by $\mu = \mu(E, F) \geq -1$. Let D be a real Cauchy-Riemann operator on E. Let $z_1, \ldots, z_k \in \partial D^2$, and $w_1, \ldots, w_l \in D^2$ be distinct marked points. Assume $l + 2k = \mu + 1$. Denote by*

$$ev : \ker(D) \to \mathbb{R}^k \oplus \mathbb{C}^l$$

the evaluation map defined by

$$\xi \rightsquigarrow (\xi(z_1), \ldots, \xi(z_k), \xi(w_1), \ldots, \xi(w_l)), \quad \xi \in \ker D.$$

Then ev is always surjective.

Proof. The Fredholm index of D is well known to be $\mu+1$. It follows that the Fredholm index of $D \oplus ev$ is 0. So, if ev is not surjective, there must exist some non-zero $\xi \in \ker(D)$ such that $\xi(z_i) = 0$, $i = 1, \ldots, k$, and $\xi(w_j) = 0$, $j = 1, \ldots, l$. According to [16, proof of Theorem C.1.10(iii)], there exists a complex linear Cauchy-Riemann operator D' on E and a function $u \in W^{1,p}(D^2, \mathbb{C})$, such that $D'(u\xi) = 0$. This leads to a contradiction since a holomorphic section of (E, F) may have at most μ zeros where interior zeros are counted twice. $\square$

Remark 8.5. Note that it is crucial for this argument that the underlying Riemann surface is a disk. Otherwise, the Fredholm index of D is not $\mu + 1$. This explains, at least in part, why Welschinger's counting scheme does not immediately extend to curves of higher genus.

Proof of Theorem 1.8. First, we treat the case $\dim L = 2$. Welschinger's invariants are defined only in the strongly semipositive case when $\Sigma = D^2$. So, we take $\nu = 0$. As explained in Section 4, we could consider the moduli space defined by quotienting by the action of $PSL_2(\mathbb{R})$, but for this proof it seems more natural to take a section of the $PSL_2(\mathbb{R})$. In particular, we add fixed marked points constrained to divisors, even when $l > 0$, as explained in Remark 4.5. As in Section 4, we denote by (A, f) and (B, g) the divisor constraint pseudo-cycles. In this proof, we will refer to the extra added marked points as z_{-1} and z_{-2}. As in Section 4, we fix z_{-1} to be at a point s_0 and we fix z_{-2} to lie on a line ℓ.

Recall from Section 4 that, by definition,

$$N_{D^2,d,k,l} := \#\mathbf{ev}^{-1}(\vec{x}, \vec{y}).$$

So, we are counting holomorphic curves through a generic collection of marked points, just like Welschinger. The sign of a given point $\mathbf{u} = (u, \vec{z}, \vec{w}) \in \mathbf{ev}^{-1}(\vec{x}, \vec{y})$ depends on whether or not the isomorphism

$$d\mathbf{ev}_{\mathbf{u}} : \det(T\mathcal{M}_{\mathbf{k},l}(L, \Sigma, \mathbf{d}))_{\mathbf{u}} \xrightarrow{\sim} \mathbf{ev}^* \det\left(T\left(L^{|\mathbf{k}|} \times X^l\right)\right)_{\mathbf{u}} \tag{8.4}$$

agrees with the isomorphism of Theorem 1.1 up to the action of the multiplicative group of positive real numbers. We would like to reduce this sign to the Welschinger sign associated with the curve $\mathbf{u}$, up to a universal correction factor. For this purpose, we apply Proposition 8.3 to a particular short exact sequence of Cauchy-Riemann boundary value problems. Indeed, we take the underlying short exact sequence of vector bundle pairs

$$0 \longrightarrow (TD^2, T\partial D^2) \xrightarrow{du} (u^*TX, u^*TL) \longrightarrow (N_u^X, N_u^L) \longrightarrow 0. \tag{8.5}$$

Pulling back the *Pin* structure on TL induces a *Pin* structure $\mathfrak{p}_u$ on u^*TL. We equip (u^*TX, u^*TL) with the linearized $\bar{\partial}_J$ operator $D_u = D\bar{\partial}_J|_u$ and we denote by D'_u and D''_u the natural operators it induces on the other terms of the short exact sequence. Note that $T\partial D^2$ is orientable and has a natural orientation, so we are in the situation of Definition 8.2. The natural orientation on $T\partial D^2$ also induces an orientation on $\ker D'_u$. In addition, note that D'_u is the Cauchy-Riemann operator induced by the complex structure of D^2 since u is J-holomorphic.

Before continuing, we introduce some notation for configuration space. Define the configuration space of k boundary points and l interior points of the disk to be

$$\mathcal{C}_{k,l} := (\partial D^2)^k \times (D^2)^l \setminus \Delta.$$

Thinking of $\mathcal{M}_{k,l}$ as the fiber product

$$\mathcal{M}_{k,l} := \widetilde{\mathcal{M}}_{k,l+2} \times_{X^2 \times (D^2)^2} (A \times B \times s_0 \times \ell),$$

we obtain Diagram 8-1. The central column of Diagram 8-1 is the short-exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T_u\mathcal{M}_{k,l} \arrow[r] \arrow[d] & T_u\mathcal{M}_{k,l} \arrow[d, dotted, "\wr"] \\
\begin{matrix} T_{z_{-1},z_{-2}}\mathcal{C}_{0,2} \oplus \ker D'_u \\ \oplus T\ell \oplus T(A \times B) \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} T_{\vec{z},\vec{w}'}\mathcal{C}_{k,l+2} \oplus \ker D'_u \\ \oplus T\ell \oplus T(A \times B) \end{matrix} \arrow[r] \arrow[d] & T_{\vec{z},\vec{w}}\mathcal{C}_{k,l} \oplus \ker D''_u \arrow[d] \\
T_{z_{-1},z_0}\mathcal{C}_{0,2} \oplus T(X \times X) \arrow[r] & T_{z_{-1},z_{-2}}\mathcal{C}_{0,2} \oplus T(X \times X) \arrow[r] & 0
\end{tikzcd}$$

Diagram 8-1

for the tangent space of the fiber product. The main content of the central row is the short exact sequence of solutions of the the short exact sequence of Cauchy-Riemann boundary value problems (8.5). This short exact sequence exists because by assumption the Cauchy-Riemann operator at each term of the sequence is surjective and we can apply the snake lemma from homological algebra. Diagram 8-1 shows how to construct a natural isomorphism

$$T\mathcal{M}_{k,l} \simeq T_{\vec{z},\vec{w}}\mathcal{C}_{k,l} \oplus \ker D_u''. \tag{8.6}$$

Choose an orientation ω on u^*TL if it is orientable. Since D_u is surjective by assumption, by Proposition 3.1, ω induces an orientation on $\ker(D_u) \simeq \det(D_u)$. The central row of Diagram 8-1 shows that the ω orientation on $\ker(D_u)$ induces an orientation of $T\mathcal{M}_{k,l} \simeq T_{\vec{z},\vec{w}}\mathcal{C}_{k,l} \oplus \ker D_u''$ as the quotient of an oriented vector space by an oriented vector space. Then, the isomorphism of Theorem 1.1 induces an orientation of $\mathbf{ev}^* \det \left(T\left(L^{|\mathbf{k}|} \times X^l\right)\right)_v$. The sign of $\mathbf{u}$ is now the sign of map (8.4) with respect to the orientations of the domain and range just outlined.

On the other hand, ω naturally induces an orientation ω' on N_u^L as a quotient bundle. Since D_u'' is surjective, invoking Proposition 3.1 again, ω' induces an orientation on $\ker(D_u'')$ and hence on $T\mathcal{M}_{k,l}$ by isomorphism (8.6). By Proposition 8.3, the ω' orientation on $T\mathcal{M}_{k,l}$ agrees with the ω orientation on $T\mathcal{M}_{k,l}$ if and only if $\mathfrak{p}_u$ is compatible with the short exact sequence (8.5).

Recall that Welschinger's invariant counts curves with sign determined by the parity of the isolated real double points. By the adjunction formula, a rational curve u of degree d in a symplectic four manifold has a topologically determined total number of double points

$$\delta(u) = \frac{d \cdot d - c_1(d) - 2}{2}.$$

Complex double points come in pairs. So, the parity of the real non-isolated double points is determined by the the parity of the real isolated double points. On the other hand, the parity of the real non-isolated double points determines the parity of the number of twists of the real part of the curve about itself. This exactly determines when the *Pin* structure $\mathfrak{p}_u$ is compatible with the short exact sequence (8.5).

Finally, we claim that the map (8.4) always preserves orientation if we consider the ω' orientation on $T\mathcal{M}_{k,l}$ together with the ω orientation on $\mathbf{ev}^* \det \left(T\left(L^{|\mathbf{k}|} \times X^l\right)\right)_v$.

Indeed, by Lemma 8.4, if we vary D'' to a standard complex linear Cauchy-Riemann operator on N_u^X and vary N_u^L to a standard boundary condition keeping the marked points distinct, the evaluation map will remain surjective during the whole variation. So, the sign is standard for a given ordering of the marked points. Then we use Definition 3.2 that twists the orientation of $T\mathcal{M}_{k,l}$ by $\operatorname{sign}(\varpi)$. Indeed, switching the order of the marked points induces a change of sign since each marked point is a codimension $n-1$ condition. The twisting cancels this sign change.

The case $n = 3$ is very similar. The same exact sequence (8.5) again plays a central role. Since N_u^X is now two dimensional, we need to define a *Pin* structure on N_u^L with which $\mathfrak{p}_u$ may or may not be compatible. In [21], Welschinger does exactly that using the splitting of N_u^X into holomorphic line bundles. It is important in that paper that X be convex so that J may be taken to be integrable. Then the Cauchy-Riemann operator D_u'' is the standard one, so the evaluation map is also standard and has a standard sign. If D_u'' were not standard, we could not apply Lemma 8.4 as before since N_u^X is no longer one dimensional. In [22], Welschinger modifies the definition of spinor states to take into account possible changes of orientation arising from walls where the evaluation map is not surjective. This allows him to extend the definition of his invariants to general strongly-semipositive real symplectic manifolds X. □

Now we turn to the proof of the calculation in Example 1.6. We generalize Kontsevich's idea for calculating the closed Gromov-Witten invariants of the quintic threefold [11] to deal with the open case as well. Extending previous notation, we denote by τ the tautological line bundle of $\mathbb{C}P^n$ equipped with its canonical complex structure and we denote by $\tau_{\mathbb{R}}$ the tautological line bundle of $\mathbb{R}P^n$. Furthermore, we let

$$c' : \mathbb{C}P^n \to \mathbb{C}P^n, \qquad \tilde{c}' : \tau \to \tau,$$

denote complex conjugation and the bundle-map of τ covering complex conjugation respectively. Let $s \in \Gamma(\tau^{*\otimes 5})$ be a real section, i.e.

$$\tilde{c}' \circ s \circ c' = s.$$

We take

$$X_s = \{s = 0\} \subset \mathbb{C}P^4, \qquad \omega_s = \omega_{FS}|_X, \qquad L_s = X_s \cap \mathbb{R}P^4.$$

That is, X_s is a quintic threefold equipped with the symplectic form induced by the restriction of the Fubini-Study form of $\mathbb{C}P^4$ and L_s is its real part. Choosing s generically, we may assume that X_s is a smooth manifold. When it does not lead to confusion, we may drop the subscript s.

We define a bundle $\mathcal{F}_d$ over $\mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, d)$ by specifying its fibers,

$$\mathcal{F}_{d_u} := \Gamma(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbf{R}}^{*\otimes 5}), \quad u \in \mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, d).$$

By restricting to the image of each curve, s induces a section $\hat{s}$ of $\mathcal{F}_d$ that vanishes exactly on those curves entirely contained in X_s.

Lemma 8.6. *The total space of $\mathcal{F}_d$ is orientable for each d. For d odd, $\mathcal{F}_d$ is an orientable vector bundle.*

Proof. Let $u \in \mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, d)$ and $\xi \in \mathcal{F}_u$. After choosing a connection on $\mathcal{F}_d$, there exists a canonical isomorphism

$$\begin{aligned} T_{(u,\xi)}\mathcal{F}_d &\simeq \Gamma\left(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbf{R}}^{*\otimes 5}\right) \oplus \Gamma\left(u^*T\mathbb{C}P^4, u^*T\mathbb{R}P^4\right) \\ &\simeq \Gamma\left(u^*\left(\tau^{*\otimes 5} \oplus T\mathbb{C}P^4\right), u^*\left(\tau_{\mathbf{R}}^{*\otimes 5} \oplus T\mathbb{R}P^4\right)\right). \end{aligned}$$

Since $\tau_{\mathbf{R}}^{*\otimes 5} \oplus T\mathbb{R}P^4$ is orientable, after choosing an orientation, Proposition 2.8 gives a canonical orientation on each tangent space. It is not hard to see that this orientation varies continuously with u and ξ.

When d is odd, we have

$$\mathcal{F}_{d_u} := \Gamma\left(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbb{R}}^{*\otimes 5}\right) \simeq \Gamma\left(\tau^{*\otimes 5d}|_{D^2}, \tau_{\mathbb{R}}^{*\otimes 5d}\right),$$

where we think of D^2 as one hemisphere of $\mathbb{C}P^1$ with boundary $\partial D^2 = \mathbb{R}P^1$. Since $5d$ is odd, again Proposition 2.8 gives each $\mathcal{F}_u$ a canonical orientation that varies continuously with u. $\square$

Let $V \to B$ be an orientable real vector bundle. We denote by $e(V)$ the Euler class of V.

Proposition 8.7. *Suppose $\hat{s}$ is transverse to the zero section of $\mathcal{F}$. Let d be odd. Then*

$$N_{D^2,d,0,0} = e(\mathcal{F}_d)$$

Remark 8.8. This proposition should still hold true when $\hat{s}$ is not transverse to the zero section of $\mathcal{F}$. However, the proof will be slightly more complicated. We leave it for a future paper.

Remark 8.9. If d is even, when the details of the necessary corrections from real curves with empty real part are worked out, an argument similar to the proof of Proposition 8.7 should show that $N_{D^2,d,0,0}$ is zero. Indeed, $N_{D^2,d,0,0}$ should be given by the self-intersection number of the zero section of $\mathcal{F}_d$. Since $\dim \mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, d) = 5d + 1$, which is odd when d is even, the self intersection number should be zero.

Proof of Proposition 8.7. Let N_{L_s} denote the normal bundle of L_s in $\mathbb{R}P^4$. By the adjunction formula, it is isomorphic to $\tau_{\mathbb{R}}^{*\otimes 5}|_{L_s}$. Since N_{L_s} is one dimensional, by Lemma 8.1, we may choose its Pin^+ structure canonically. Equip $\tau_{\mathbb{R}}^{*\otimes 5}$ with a Pin^+ structure corresponding to the Pin^+ structure of N_{L_s} under the isomorphism of the adjunction formula. Choose Pin^+ structures on TL_s and $T\mathbb{R}P^4$ compatible with the short exact sequence

$$0 \longrightarrow TL_s \longrightarrow T\mathbb{R}P^4 \longrightarrow N_{L_s} \longrightarrow 0.$$

As in isomorphism (8.6) of the proof of Theorem 1.8, we may identify

$$T_u M_{0,0}(L_s, D^2, d) \simeq \ker D''_u, \tag{8.7}$$

where D''_u is the naturally induced operator on the bundle pair $(N_u^{X_s}, N_u^{L_s})$. Equip $N_u^{L_s}$ with the Pin^+ structure induced by the short exact sequence

$$0 \longrightarrow T\partial D^2 \xrightarrow{du} u^*TL_s \longrightarrow N_u^{L_s} \longrightarrow 0.$$

Then, by Diagram 8-1 and Proposition 8.3, we may assume that isomorphism (8.7) is orientation preserving when $\ker D''_u$ is given the canonical orientation of Proposition 2.8. From Diagram 8-2 along with its conjugation invariant part, we deduce a short

$$\begin{tikzcd}
0 \arrow[r] & u^*N_X \arrow[r] & u^*N_X \\
TD^2 \arrow[r] \arrow[u] & u^*T\mathbb{C}P^4 \arrow[r] \arrow[u] & N_u^{\mathbb{C}P^4} \arrow[u] \\
TD^2 \arrow[r] \arrow[u] & u^*TX \arrow[r] \arrow[u] & N_u^X \arrow[u]
\end{tikzcd}$$

Diagram 8-2

exact sequence of Cauchy-Riemann boundary value problems

$$0 \longrightarrow (N_u^X, N_u^L) \longrightarrow (N_u^{\mathbb{C}P^4}, N_u^{\mathbb{R}P^4}) \longrightarrow (u^*N_X, u^*N_L) \longrightarrow 0. \tag{8.8}$$

The Cauchy-Riemann operators at each term of the sequence are induced by the rows of Diagram 8-2. The conjugation invariant parts of the rows of Diagram 8-2 induce Pin^+ structures on each of the boundary conditions in short exact sequence (8.8). The induced Pin^+ structures are compatible with the exact sequence because the already chosen Pin^+ structures on the conjugation invariant parts of the first two columns of Diagram 8-2 are compatible. Recall from the proof of the adjunction formula that the isomorphism $N_{X_s} \simeq \tau^{*\otimes 5}|_{X_s}$ is given by the differential ds. So, we have a diagram

$$
\begin{tikzcd}
 & & & \Gamma(u^*N_X, u^*N_L) \arrow[dd, "\wr\, ds"] \arrow[dr] & \\
0 \arrow[r] & \Gamma(N_u^X, N_u^L) \arrow[r] & \Gamma(N_u^{\mathbb{C}P^4}, N_u^{\mathbb{R}P^4}) \arrow[ur] \arrow[dr, "d\hat{s}"'] & & 0. \\
 & & & \mathcal{F}_u \arrow[ur] \arrow[d, equal] & \\
 & & & \Gamma(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbb{R}}^{*\otimes 5}) &
\end{tikzcd}
$$

Since $\mathbb{C}P^4$ is convex, $\Gamma(N_u^{\mathbb{C}P^4}, N_u^{\mathbb{R}P^4})$ has expected dimension. Since $\tau^{*\otimes 5}$ is a line bundle, its sections always have expected dimension. By assumption, $d\hat{s}$ is an isomorphism. So, by the snake lemma, $\Gamma(N_u^X, N_u^L)$ must have expected dimension, i.e., 0. So, its orientation is just a sign. Since all Pin^+ structures in the above diagram have been chosen compatibly, the sign is given exactly by the sign of $d\hat{s}$, as claimed. $\square$

Proof of Example 1.6. The section $s_F \in \Gamma(\tau^{*\otimes 5})$ defining the Fermat quintic does not satisfy the assumptions of Proposition 8.7 even in degree 1. However, an elementary transversality argument shows that we may choose a nearby section s which does, in degree 1. For such an s, we know that X_s is diffeomorphic to X_{s_F} and L_s is diffeomorphic to L_{s_F}. Note that if s is not sufficiently close to s_F, the topology of L_s could be different from that of L_{s_F}. However, since we choose s close to s_F, we may think of the deformation of s_F to s as a deformation of complex structure, which leaves the invariants unchanged. By Proposition 8.7, it suffices to calculate $e(\mathcal{F}_1)$. Let $G(k,n)$ denote the Grassmannian of real oriented k planes in n space and let τ_G denote its tautological bundle. It is not hard to see that

$$\mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, 1) \simeq G(2,5), \qquad \mathcal{F}_1 \simeq Sym^5(\tau_G).$$

Applying the splitting principle, we calculate the Pontryagin class

$$p_3(Sym^5(\tau_G)) = 225 p_1(\tau_G)^3. \tag{8.9}$$

Then, taking square roots, we have

$$e(Sym^5(\tau_G)) = 15e(\tau_G)^3 = 15e(\tau_G^{\oplus 3}).$$

Here, we have to include $G(2,5)$ into $G(2,n)$ for n sufficiently large so that both sides of equation (8.9) are not just zero. We use the unique factorization property of the polynomial ring $H^*(G(2,\infty))$ to justify taking square roots on both sides.

Finally, since there are two oriented 2-planes in the intersection of three generic hyperplanes in $\mathbb{R}^5$, we know that

$$\int_{G(2,5)} e(\tau_G^{\oplus 3}) = 2 \text{ or } 0. \tag{8.10}$$

To show the integral is actually 2, we proceed as follows. Let $\hat{G}(k,n)$ denote the Grassmannian of unoriented k-planes in n-space and let $\hat{\tau}_G$ denote its tautological bundle. Note that $\pi : G(2,5) \to \hat{G}(2,5)$ is the orientation cover. Moreover, $\pi^*\hat{\tau}_G \simeq \tau_G$ and $w_1(\hat{\tau}_G) = w_1(T\hat{G}(2,5))$. So, both points count the same and integral (8.10) is 2 as desired. $\square$

Appendix A

Kuranishi structures

In this appendix, we briefly review the definition of a Kuranishi structure, as introduced in [4] and extended in [5]. We essentially follow the conventions of [5, Appendix 2]. In the following discussion, we take X to be a compact metrizable space.

Definition A.1. A *Kuranishi structure with corners* on X of dimension d consists of the following data:

(1) For each point $p \in X$,

(1.1) A smooth manifold with corners V_p and a smooth vector bundle $E_p \to V_p$ such that $\dim V_p - \operatorname{rank} E_p = r$.

(1.2) A finite group Γ_p acting on $E_p \to V_p$.

(1.3) An Γ_p-equivariant smooth section s_p of E_p.

(1.4) A homeomorphism ψ_p from $s_p^{-1}(0)/\Gamma_p$ to a neighborhood of p in X.

(2) For each $p \in X$ and for each $q \in \operatorname{Im} \psi_p$,

(2.1) An open subset $V_{pq} \subset V_q$ containing $\psi_q^{-1}(q)$.

(2.2) A homomorphism $h_{pq} : \Gamma_q \to \Gamma_p$.

(2.3) An h_{pq}-equivariant embedding $\varphi_{pq} : V_{pq} \to V_p$ and an h_{pq}-equivariant injective bundle map $\hat{\varphi}_{pq} : E_q|_{V_{pq}} \to E_p$ covering φ_{pq}.

Furthermore, the above data should satisfy the following compatibility conditions:

(C1) $\hat{\varphi}_{pq} \circ s_q = s_p \circ \varphi_{pq}$.

(C2) $\psi_q = \psi_p \circ \varphi_{pq}$.

(C3) If $r \in \psi_q\left(s_q^{-1}(0) \cap V_{pq}\right)$, then in a sufficiently small neighborhood of r,

$$\hat{\varphi}_{pq} \circ \hat{\varphi}_{qr} = \hat{\varphi}_{pr}.$$

A crucial ingredient in the construction of the fundamental class of a Kuranishi structure is the notion of the tangent bundle of a Kuranishi structure. We take the following definition from [4, Section 5].

Definition A.2. A *tangent bundle* for a Kuranishi structure consists of a collection of vector bundle isomorphisms

$$\Phi_{pq} : N_{U_p}U_q \xrightarrow{\sim} E_p|_{V_{pq}}/E_q|_{V_{pq}}$$

covering the embeddings φ_{pq}. Furthermore, if $q \in \operatorname{Im}\psi_p$ and $r \in \psi_q\left(s_q^{-1}(0) \cap V_{pq}\right)$, then in a sufficiently small neighborhood of r, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{V_q}V_r & \longrightarrow & N_{V_p}V_r & \longrightarrow & N_{V_p}V_q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_{qr}} & & \downarrow{\scriptstyle \Phi_{pr}} & & \downarrow{\scriptstyle \Phi_{pq}} & & \\
0 & \longrightarrow & E_q/E_r & \longrightarrow & E_p/E_r & \longrightarrow & E_p/E_q & \longrightarrow & 0.
\end{array}$$

We also need the notion of an orientation for a Kuranishi structure, which again comes from [4, Section 5].

Definition A.3. An *orientation* of a Kuranishi structure with tangent bundle consists of a family of trivializations of $\det(TV_p) \otimes \det(E_p)$ compatible with the isomorphisms

$$\det(TV_q) \otimes \det(E_q)|_{V_{pq}} \xrightarrow{\sim} det(TV_p) \otimes \det(E_p)|_{V_{pq}}$$

induced by Φ_{pq}.

Without providing full detail, we remind the reader of certain definitions relating to multi-sections that are used in Section 7 of this paper. For details, see [4, Section 3]. In the following, for Z a space, we denote by $\mathcal{S}^\ell(Z)$ its ℓ^{th} symmetric power. That is

$$\mathcal{S}^\ell(Z) := Z^\ell/S_\ell,$$

where S_ℓ is the group of permutations of ℓ objects acting on Z^ℓ by permuting the factors. Let $E \to V$ be a vector bundle. If $U \subset V$ is sufficiently small, then $E|_U$ is trivial. So, if $\operatorname{rank} E = r$, a section of E over U may be specified a map $U \to \mathbb{R}^r$. After possibly shrinking U, a multi-section s is specified by a map $s_U : U \to \mathcal{S}^\ell(\mathbb{R}^r)$. Note that globally, the multiplicity ℓ can change. By definition, the multi-section s is said to be smooth if after possibly shrinking U again, there exists a smooth lifting of s_U to the Cartesian product,

$$\tilde{s}_U : U \to \left(\mathbb{R}^k\right)^\ell.$$

The components of this lifting locally define ℓ sections s_U^i of E. We call the s_U^i branches of s over U. If $E \to V$ is a Γ-equivariant vector bundle, then there is a natural notion of a Γ-equivariant multi-section coming from the induced action on the symmetric power. Note that for Γ-equivariant smooth sections, we do not require the local lifts $\tilde{s}_U$ to be Γ-equivariant. We call a smooth multi-section transverse if each branch of each local lifting is transverse. The vanishing set of a multi-section s is defined locally by

$$s^{-1}(0) \cap U = \bigcup_i \left(s_U^i\right)^{-1}(0).$$

If s is transverse and sufficiently generic, then $s^{-1}(0)$ admits a smooth triangulation. If we fix a trivialization of $\det(E) \otimes \det(V)$, then the vanishing set of any smooth section is oriented. So, $s^{-1}(0)$ actually defines a rational singular chain by weighting each simplex of its triangulation by the signed number of branches s_U^i that vanish on it, divided by ℓ.

Before we can define the fundamental chain, we need to recall the notion of a *good coordinate system* introduced in [4, Section 6]. Fix a Kuranishi structure on X. We denote the various parts of the Kuranishi structure by the same symbols as in Definition A.1. For $V'_p \subset V_p$, we denote by E'_p, ψ'_p, s'_p, etc. the restrictions of all the related parts of the Kuranishi structure. A good coordinate system specifies a finite ordered set $P \subset X$ and $V'_p \subset V_p$ for each $p \in P$ such that

$$X \subset \cup_{p \in P} \operatorname{Im} \psi'_p.$$

Furthermore, for $q, p \in P$ such that $q < p$, it specifies a neighborhood

$$V'_{pq} \supset \psi_q^{-1'}(\operatorname{Im} \psi'_p),$$

an embedding $\varphi'_{pq} : V'_{pq} \hookrightarrow V'_p$ and an injective bundle map

$$\hat{\varphi}_{pq} : E'_q|_{V'_{pq}} \to E'_p$$

covering φ'_{pq}. Of course, we must require $s'_p \circ \varphi'_{pq} = \hat{\varphi}'_{pq} \circ s'_q$. Also, φ'_{pq} (resp. $\hat{\varphi}'_{pq}$) must respect the actions of Γ_q and Γ_p in such a way as to define a map of the quotient orbifolds (resp. orbi-bundles). A few additional technical conditions ensure requisite compatibility.

The following is a restatement of [4, Theorem 6.4].

Theorem A.4. *Let $(P, V'_p, \psi'_p, s'_p, \varphi'_{pq}, \hat{\varphi}'_{pq})$ be a good coordinate system on a space X with Kuranishi structure. Suppose that X has a tangent bundle in the sense of Definition A.2. Then, for each $p \in P$, there exists a sequence of smooth Γ_p-equivariant multi-sections $s'_{p,n}$ such that*

(P1) $s'_{p,n} \circ \varphi'_{pq} = \hat{\varphi}'_{pq} \circ s'_{q,n}$.

(P2) $\lim_{n \to \infty} s'_{p,n} = s'_p$.

(P3) $s'_{p,n}$ *is transversal to* 0.

(P4) The restriction to $\operatorname{Im}\varphi'_{pq}$ *of differential of the composition of any branch of* $s'_{p,n}$ *and the projection* $E_p \to E_p/E_q$ *coincides with the isomorphism* $\Phi'_{pq} : N_{V'_p}V'_q \xrightarrow{\sim} E'_p/E'_q$.

Write $P = \{p_1, p_2, \ldots\}$. The proof of Theorem A.4 uses induction on the ordered set P. Assume the existence of perturbations $s'_{p_i,n}$ satisfying conditions (P1)-(P4) for $i < j$. The embeddings $\varphi'_{p_ip_j}$, the bundle maps $\hat{\varphi}'_{p_ip_j}$ and the isomorphisms $\Phi_{p_ip_j}$, allow the extension of $s'_{p_i,n}$ to a neighborhood of

$$\bigcup_{i<j} \operatorname{Im}\varphi'_{p_ip_j} \subset V_{p_j}.$$

A small perturbation of the extension produces $s'_{p_j,n}$ as desired. Full detail is given in [4, Section 6].

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