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INTRINSIC NILPOTENT APPROXIMATION

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Abstract

This report is a preliminary version of work on an intrinsic approximation process arising in the context of a non-isotropic perturbation theory for certain classes of linear differential and pseudodifferential operators P on a manifold M. A basic issue is that the structure of P itself determines the minimal information that the initial approximation must contain. This may vary from point to point, and requires corresponding approximate state spaces or phase spaces.

This approximation process is most naturally viewed from a seemingly abstract algebraic context, namely the approximation of certain infinite-dimensional filtered Lie algebras L by (finite-dimensional) graded nilpotent Lie algebras g_x, or $g_{(x,\xi)}$, where $x \in M$, $(x,\xi) \in T^*M/0$. It requires the notion of "weak homomorphism". A distinguishing feature of this approach is the intrinsic nature of the approximation process, in particular the minimality of the approximating Lie algebras. The process is closely linked to "localization", associated to an appropriate module structure on L.

The analysis of the approximating operators involves the unitary representation theory of the corresponding Lie groups. These representations are for the most part infinite-dimensional, and so involve a kind of "quantization". Not all the representations enter. The filtered Lie algebra L leads to an "approximate Hamiltonian action" of $G_{(x,\xi)}$, the group associated to $g_{(x,\xi)}$, and thus induces (via an adaptation of a construction of Helffer and Nourrigat) an intrinsically defined "asymptotic

moment-map" with image in $g^*_{(x,\xi)}$. The relevant representations are those associated to this image by the Kirillov correspondence.

The genesis of this work has been in the context of linear partial differential operators, in particular the question of hypoellipticity. For example, our framework leads to a natural hypoellipticity conjecture enlarging on that of Helffer and Nourrigat. We believe, however, that the approximation process is likely to have broader applicability, particularly in those contexts where the process can be extended to filtrations with an L^0 term. This yields not simply a graded nilpotent algebra, but a semi-direct sum with a graded nilpotent. As we show, one such context arises in the approximation of non-linear control systems.

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§0. Introduction

This report is a preliminary version of work to date on an approximation process arising in the context of constructing an appropriate non-isotropic "perturbation" theory for certain classes of naturally arising linear differential operators P. This requires the construction of approximate state spaces or phase spaces. These will depend on the structure of P itself, and may vary locally, i.e., from point to point of the base manifold M, or microlocally, i.e. from point to point of the cotangent space. A basic issue is that the structure of P itself determines the minimal amount of information that the initial approximation must contain, and this may vary from point to point.

It is a remarkable fact that this approximation process is most naturally viewed from a seemingly abstract algebraic context, namely the "approximation" of certain infinite-dimensional filtered Lie algebras L (of vector fields or of pseudo-differential operators) by finite-dimensional graded Lie algebras g_{x_0}, or $g_{(x_0,\xi_0)}$, where $x_0 \varepsilon M$, $(x_0,\xi_0)\varepsilon T^*M/0$. The algebras g_{x_0} (or $g_{(x_0,\xi_0)}$) are not determined purely by the abstract structure of L as a Lie algebra over $\mathbb{R}$, but also depend on the module structure of L over an $\mathbb{R}$-algebra F on which L acts as a Lie algebra of derivations. In the local case we take F to be $C^\infty(M)$, and in the microlocal case essentially the algebra of zero-order pseudo-differential operators with real principal symbol. The algebra F is essential in obtaining the correct "localization". Here "approximation" is closely linked to "localization", this being either at the level of the base manifold or at the level of the cotangent space. Roughly speaking, one treats P as an element of the "enveloping algebra" of the filtered Lie algebra, and approximates it by an element in the enveloping algebra of the finite-dimensional graded Lie algebra g. A distinguishing feature of this approach is the intrinsic nature of the approximation process (i.e., coordinate-independence and functoriality), in particular the minimality of the approximating Lie algebras.

The analysis of the approximating operator leads naturally into the unitary representation theory (i.e., "Fourier analysis") of the simply-connected Lie group G corresponding to the finite-dimensional graded Lie algebra g. These representations are, for the most part, infinite-dimensional, and so involve a kind of "quantization". The decomposition into irreducible representations may be viewed as a finer subdivision of the approximating state space or phase space.

Not all the irreducible representations enter into the approximation. Which ones do appears to be determined by the original filtered Lie algebra. This is discussed most naturally at the level of the cotangent or phase space, with its associated Poisson bracket structure. According to the theory of Kirillov [26], Kostant [27] and others, the irreducible unitary representations of G are intimately related to the orbits in g^*, the dual space of g, under the coadjoint action of G. If one has a Hamiltonian action of G on the symplectic manifold N one gets an intrinsically defined moment-map $\Phi: N \to g^*$ which is equivariant with respect to the G-actions. As a heuristic principle one expects the irreducible representations which enter into the "quantization" (if it exists) of the G-action on N to be those associated to the coadjoint orbits lying in the image of Φ. (In case G and

N are compact this is given precise realization in recent work of Guillemin and Sternberg [14]). In our context the original infinite-dimensional filtered Lie algebra L leads to an "approximate" Hamiltonian action of $G_{(x_0,\xi_0)}$. This allows us, adapting a construction of Helffer and Nourrigat ([19], [21], [32], [33]), to intrinsically define an "asymptotic" moment-mapping, with image in $g^*_{(x_0,\xi_0)}$. This image determines the relevant representations.

As indicated above, the genesis of this work has been in the context of linear partial differential operators, particularly the question of hypoellipticity, and, to a lesser extent, local solvability and construction of parametrices, i.e., approximate inverses. In this context (aside from the metaplectic group, which enters in the study of second order operators) the Lie algebras which arise are graded nilpotent. We believe, however, that the approximation process is likely to have rather broader applicability than to questions of hypoellipticity, or, for that matter, the study of linear P.D.E.'s. For example, under appropriate conditions the approximation process can be extended to the case where the filtration contains an L^0 term. Now the procedure now longer yields only a graded nilpotent Lie algebra, but a semi-direct sum $g^0 Ⓐ g$, where g^0 is "arbitrary" and g is graded nilpotent as before. In a series of papers (see for example [4]) Crouch has shown that in the context of approximation of non-linear control systems by means of Volterra series certain solvable Lie algebras, of the form $\mathbb{R}$ Ⓐ g, with g graded nilpotent, naturally arise. Starting with a filtered Lie algebra L suggested by [4], one finds that the resulting Lie algebra coming from the approximation process is of the correct type. It appears quite likely that this process can be brought to bear on the questions treated by Crouch.

The organization of this report is as follows: In §1 we construct the local approximation process and examine its properties. In §2 we show how to carry out a version of the group-level lifting process of Rothschild-Stein [37] and the corresponding homogeneous-space approximation process of Helffer and Nourrigat [20] in the more general context of §1. We in addition illustrate the connection of these results with questions of hypoellipticity.

In §3 we shall treat the microlocal version of the approximation process, including a discussion of the asymptotic moment-map. In this context we can frame a natural hypoellipticity conjecture enlarging on that of Helffer and Nourrigat ([19], [21], [32], [33]).

In §4 we shall extend the approximation process (both local and microlocal) to the case when the filtration has an L^0 term. We shall also briefly examine the connection with the work of Crouch.

We shall conclude in §5 with a summary of the main directions for further work.

In the remainder of this Introduction we shall go into more detail on the motivation and background of this work.

The initial idea of using graded nilpotent Lie algebras for local (i.e., on the base manifold) approximation (akin to normal coordinates) seems to be due to Stein [38]. The aim was to develop a generalized Calderón-Zygmund theory of singular integral operators in a non-abelian, non-isotropic context, i.e. with certain directions weighted differently from others. (This is how the nilpotent groups arose. The only Lie groups with dilations are nilpotent, though not all nilpotent groups have dilations). The analysis of the resulting non-Euclidean balls is fundamental to the theory.

The approximation process appears as follows. One begins with a hypoelliptic operator P on M, constructed as a polynomial with C^∞ coefficients in the vector fields $X_1,\dots,X_k$ satisfying the Hörmander spanning conditions ([23]), i.e., the iterated commutators span the tangent space at each point of M. Corresponding to these vector fields one introduces the free nilpotent Lie algebra g on k generators of step r, r being the order of iterated commutators of the X_i's needed to span (in the nbhd of a point $x_0 \varepsilon M$). Let G be the corresponding group. Notice that in general dim G > dim M. Because the spanning condition is satisfied it is possible to "lift" the vector fields $X_1,\dots,X_k$, in a nbhd of x_0, to vector fields $\tilde{X}_1,\dots,\tilde{X}_k$ on a manifold $\tilde{M}$ of dimension equal to dim G, and so that the $\tilde{X}_i$ are free up to step r at each point in a nbhd of $\tilde{x}_0 \varepsilon \tilde{M}$, i.e., the commutators up to step r satisfy no inessential linear relations at $\tilde{x}_0$. At each point $\tilde{x}$ in a nbhd of $\tilde{x}_0$ $\tilde{M}$ can be locally identified with a nbhd of the identity in G, and the $\tilde{X}_i$ can be approximated by $\overset{\bullet}{X}_i$, the generators of g, viewed as left-invariant vector fields. This is an approximation in the following sense: The dilations on g (and hence on G) introduce a natural notion of "local order" at a point for functions or vector fields via, for example, Taylor series with non-isotropically weighted variables. Then $\tilde{X}_i$ differs from $\overset{\bullet}{X}_i$ by a term of lower order in this sense. (This is a more stringent requirement than lower order in the classical sense. A vector field may be of lower order classically, but of comparable, or higher, order in this sense, and hence not negligible). One then approximates $\tilde{P}$, the lift of P, at $\tilde{x}$ by $\hat{P}_{\tilde{x}}$ a (homogeneous) left-invariant differential operator on G. In the particular context considered by Rothschild and Stein it is seen that the $\hat{P}_{\tilde{x}}$ are also hypoelliptic, and hence have fundamental solutions $\hat{E}_{\tilde{x}}$ of special type (i.e., homogeneous distributions). One glues together the $\hat{E}_{\tilde{x}}$ to construct a parametrix $\tilde{E}$ for $\tilde{P}$, and pushes this down to get a parametrix E for P. An important point here is that the $\hat{E}_{\tilde{x}}$ vary smoothly with $\tilde{x}$. Later Metivier [30] showed that, under an appropriate constancy of rank condition for the $X_1,\ldots,X_k$ one could use groups G_x of the same dimension as M; however, these groups would in general vary with the point $x \in M$.

The main concern in this work was not with deriving hypoellipticity criteria, but rather in constructing parametrices and obtaining <u>sharp</u> a priori estimates for operators known to be hypoelliptic, primarily the fundamental sum-of-squares of vector fields operators of Hörmander [23]. The primary emphasis was on the <u>structure</u> theory rather than the representation theory of the nilpotent Lie groups involved.

When considering primarily such sum-of-squares operators the <u>representation</u> theory of the groups G_x can be disregarded, since the representation theoretic criteria for hypoellipticity are automatically satisfied. However, hypoellipticity is not restricted to second-order operators, and does not inhere specifically in the spanning condition. Rather, the spanning condition (more precisely, the rank needed for spanning) determines which group to use as a local model, and then the hypoellipticity of the given operator is studied via the unitary representation theory of that particular group. The importance of representation theoretic conditions, as distinct from spanning conditions, for hypoellipticity was first emphasized, I believe, in my own work [35]. Here a general representation theoretic criterion was formulated for homogeneous left-invariant operators on nilpotent Lie groups, and shown to hold for the Heisenberg group, the prototype (and simplest) non-abelian nilpotent Lie group. Interestingly, all the unitary irreducible representations, including the "degenerate" ones not appearing in the Plancherel decomposition, play a role. The criterion was later shown to be valid for arbitrary graded nilpotent Lie groups by Helffer and Nourrigat ([17], [18]). The issue motivating the work in [35] was not, however, local approximation by nilpotent Lie groups, but a seemingly unrelated question, namely, to better understand a mysterious quantization process arising in the microlocal analysis of certain degenerate-elliptic operators.

From the mid 1960's onward the emphasis in the study of linear P.D.E.'s was on the use of phase space (i.e., cotangent rather than base space) methods. This included both sophisticated phase space decompositions (going back at least as far as Hörmander's partition of unity in his analysis of subelliptic estimates [22]) and the use of symplectic geometry. One studied Hamiltonian mechanical systems on phase space, the Hamiltonians coming essentially from the principal symbols of the operators being considered. The connection between these classical systems and the original operators was basically made via a kind of geometrical optics or W.K.B. type of relationship.

In the context of degenerate-elliptic operators, again going back at least to Hörmander's test-operators in [22], and to the work of Grušin [12], certain "intermediate" P.D.O.'s (partial differential operators) arose, with polynomial coefficients constructed out of the "total" symbol of the original operator. The analysis of the original operator required the study of these intermediate P.D.O.'s, acting on certain intermediate Hilbert spaces. If the original phase space methods are viewed as a 1st quantization, then the above context is reminiscent of a 2nd quantization process.

In the particular context of my notes [34] a "test-operator" (i.e. unitary equivalence class of intermediate P.D.O.'s) is introduced for each point $(x,\xi) \in \Sigma$, the characteristic variety (i.e., zero-set of the principal symbol), assumed to be symplectic, of the original degenerate-elliptic operator P. The intermediate Hilbert space at (x,ξ) is $L^2(\mathbb{R}^k)$, where $2k$ = codim Σ in $T^*M/0$. In fact $L^2(\mathbb{R}^k)$ is the Hilbert space associated to a polarization of the (necessarily symplectic) normal space $N(\Sigma)_{(x,\xi)}$ to Σ at (x,ξ). The striking similarity was noted in [34] between (1) this 2nd quantization process on the one hand, and (2) the "coadjoint-orbit" method of Kirillov [26] for obtaining the unitary irreducible representations of a nilpotent Lie group G by polarizing all the coadjoint orbits of G in g^*. The work in [35] was undertaken with the hope of elucidating this analogy with the Kirillov theory. One explicit link was the following. Returning to the context of [34], it was shown that $N(\Sigma)_{(x,\xi)} \times \mathbb{R}$ could be naturally identified with h_k, the Lie algebra of the Heisenberg group H_k. For this group the generic representations (equivalently, coadjoint orbits) are parameterized by one parameter, Planck's "constant". It was shown that the test-operators associated with the ray through (x,ξ) correspond to the images, under the representations with positive Planck's constant, of a homogeneous left-invariant operator $\hat{P}$ on H_k. This observation reinforced the expectation expressed in [35] that one could eventually use nilpotent groups for <u>microlocal</u> approximation. In particular, in conjunction with the conjectured representation theoretic hypoellipticity criteria for these groups, this could lead to hypoellipticity results for more general operators P, and, in fact, could lead to the formulation of natural hypoellipticity criteria which might have no simple explicit expression in terms of the classical (total) symbol of P, and hence be totally overlooked.

The preceding analogy, arising as it does in the specific context of a symplectic characteristic variety Σ, needs certain important refinements in order to give the correct intuitions more generally: (1) In the symplectic case the group, H_k, which arises does not vary with the point (x,ξ) of Σ. (2) There are only two classes of representations, the generic ones associated with non-zero Planck's constant, and the 1-dimensional ones associated with zero Planck's constant. The former are, essentially, in one-to-one correspondence with the characteristic variety Σ, and the latter are controlled via a kind of transverse (to Σ) ellipticity condition. In particular, Σ is singled out as special in various ways.

In more general contexts, even in essentially "rank 2" contexts as treated in Boutet-Grigis-Helffer [3], and as applied by Helffer to the group theoretic context in [16], there are more than two classes of representations: in particular, to each point $(x,\xi) \in \Sigma$ there may correspond a whole family of representations.

In our context of microlocal nilpotent approximation to a filtered Lie algebra $\{L^i\}$ the above points are easy to discuss. Given P, there may be one, several, or no $\{L^i\}$ pertinent to its analysis. The characteristic variety $\sum$ of P will influence this determination of $\{L^i\}$, but does not play a really decisive role. We then obtain a graded nilpotent Lie algebra $g_{(x_0,\xi_0)}$ at every point of $T^*M/0$, not just at points of $\sum$;i.e., we deal with the total phase space. However the algebras, and their ranks, will vary from point to point, and, generally, at points not on $\sum$, $g_{(x_0,\xi_0)}$ is trivial, i.e., of rank 1. To each (x_0,ξ_0) there is associated a family of unitary irreducible representations of $G_{(x_0,\xi_0)}$, namely those associated to the coadjoint orbits in $g^*_{(x_0,\xi_0)}$ which are in the image of the asymptotic moment-map at (x_0,ξ_0). This may be viewed as a refined "phase space decomposition" determined by the filtered Lie algebra $\{L^i\}$: To each point (x_0,ξ_0) of the phase space $T^*M/0$ we associate a subset of the "irreducible phase spaces" in $T^*(G_{(x_0,\xi_0)})$, namely those in the image of the asymptotic moment-map. This setting is itself suggestive of an infinite-dimensional Kirillov theory, or, better, the approximation of an infinite dimensional Kirillov theory by finite-dimensional Kirillov theory.

Both to aid the reader in understanding the viewpoint and results presented here, and to give proper acknowledgement, we would like to make clearer the relation to other work in this general area, in particular the microlocal work of Helffer and Nourrigat ([21], [32], [33]). The idea of introducing filtered Lie algebras in the context of hypoellipticity and of thereby obtaining intrinsically defined nilpotent approximations seems to be new. The local construction of nilpotent Lie algebras by Stein, Folland, Rothschild, Goodman, Helffer, Nourrigat, and others (including Crouch, in the context of control) is not intrinsic, in that a Lie algebra is introduced externally, for example a free nilpotent on an appropriate number of generators, and of appropriate rank). The same is true of the microlocal construction of Helffer and Nourrigat to be discussed below. The local construction of Metivier [30], under the correct constancy of rank conditions, does not introduce the nilpotent Lie algebras externally, but is also not intrinsic; it involves an explicit choice of vector fields $X_1,\ldots,X_k$.

What are the advantages of an intrinsic construction of the nilpotent g_{x_0} (or $g_{(x_0,\xi_0)}$), and the introduction of filtered Lie algebras $\{L^i\}$? For one thing, of course, an intrinsic construction leads to functoriality properties. Moreover, by insuring that the g_x's (or $g_{(x,\xi)}$'s) are intrinsic we can view them as a family (as x varies in a nbhd of x_0, or (x,ξ) in a nbhd of (x_0,ξ_0)) of local invariants of the initial data (i.e., the filtered Lie algebra) somewhat reminiscent of the local ring of a singularity [13]. Under appropriate "stability" conditions on these invariants, one can hope to obtain local (or microlocal) canonical form results for the initial data (e.g., akin to the canonical form results in Trèves [39], Chapter 9). Also, the significance of the "spanning" condition is more sharply brought out; in the intrinsic construction, unlike the external construction, something like a spanning condition is needed to even construct the Lie algebra g_{x_0}. (Without such a condition the construction yields an infinite graded Lie algebra).

The introduction of the filtered algebra $\{L^i\}$ is extremely natural. It defines the class of operators we are examining, namely the "enveloping" algebra "U(L)" of $\{L^i\}$ (not to be confused with the usual enveloping algebra; in the local case this consists of the differential operators on M constructed out of (non-commutative) polynomials in the vector fields in L, with coefficients in $C^{\infty}_{\mathbb{C}}(M)$, and in the microlocal case polynomials in the ψDO's (pseudo-differential operators) in L, with coefficients 0-order ψDO's on M). At the same time it defines sharp form of hypoellipticity, L-hypoellipticity, which, for P ε "U(L)", depends only on the leading part of P with respect to the filtration. The same operator P could be viewed as lying in "U(L)" for various filtered Lie algebras L, L', and satisfy the criteria for L-hypoellipticity, but not L'-hypoellipticity (or satisfy them for no filtration, as for example if P is not hypoelliptic). The filtration also suggests a notion of L-wave-front set associated with the phase space decomposition discussed earlier, coinciding with the standard notion of WF-set in the case of the natural rank 1 filtered algebra L.

The intrinsic construction does not require that the generators of L all be of degree 1, but works equally well in general. For example, the analogue of Metivier's approximation result holds in this more general setting, and hence, apparently, so do the corresponding hypoellipticity results of Rothschild [36]. The fact that L need not be generated by L^1 is of interest particularly in the microlocal context. In this context the setting is often "geometrical", the operator P (and associated symbol calculi) under investigation being characterized, for example, in terms of the symplectic geometry of various varieties associated to the total symbol, and not in terms of an explicitly associated set of first order pseudo-differential operators (the analogue of vector fields). This is the case, for example, for the operator class $L^{m,k}$ studied, for example, in Boutet de Monvel [2], Helffer [15], Boutet-Grigis-Helffer [3], Grigis ([9], [10], [11]), both in connection with hypoellipticity and propagation of singularities. One can, as shown in [15], choose associated 1st-order φDO's but the choice is not unique. It turns out, however, that there is an intrinsically associated filtered Lie algebra $\{L^i\}$ of rank 2, not necessarily generated by L^1, so that $L^{m,k}$ is, essentially, "$U^k(L)$", and so that the notion of hypoellipticity studied in the above papers is, essentially, L-hypoellipticity. Of course, for most purposes, one can undoubtedly use an ad hoc extension of the "external" method in order to handle the case where the generators are not all of degree 1; however, the free nilpotent algebras thus introduced are much larger than necessary, and one thereby loses a good deal of naturality.

In a noted dated Nov. 22, 1981, and privately circulated, we sketched out a program of microlocal nilpotent approximation in the context of a filtered Lie algebra L of 1st order φDO's. We formulated a microlocal "spanning" condition at $(x,\xi) \varepsilon T^*M/0$, and determined a process for intrinsically associating to $(L,(x,\xi))$, where (x,ξ) is of finite rank r, a pair $(g_{(x,\xi)},\eta)$, where $g_{(x,\xi)}$ is a graded nilpotent Lie algebra of rank r, and $\eta \varepsilon g^*_{(x,\xi)}/0$. The aim was to associate to each P ε "$U^m(L)$", in an intrinsic way, $\hat{P} \varepsilon U^m_{hom}(g_{(x,\xi)})$ so that L-hypoellipticity of P at (x,ξ) would be equivalent to hypoellipticity of $\hat{P}$ at η, with respect to the natural filtration. The latter was to have a representation-theoretic criterion, but involving only a subset of the representations of $G_{(x,\xi)}$

(equivalently, only a subset $\Gamma_\eta \subset g^*_{(x,\xi)}$ of coadjoint obits) depending on L but not on P. A provisional suggestion for Γ_η was made, inspired by the results in [3], [16] which, as indicated above, we viewed as corresponding to L of rank 2.

The construction of $g_{(x,\xi)}$, while intrinsic, seemed hard to work with, and not amenable to computation. In particular, it was not clear how to relate $g_{(x,\xi)}$ analytically to L as a genuine approximation. More recently we discovered a more explicit variant of our construction which circumvents this difficulty. In contrast with the externally introduced free nilpotents, the $g_{(x,\xi)}$ do not come equipped with partial homomorphisms into L. However, one can prove that any "cross-section" β from $g_{(x,\xi)}$ to L provides a "weak-homomorphism", which can be used to prove (in the local context) variants of "lifting"-theorems, and, in general, seem to provide adequate substitutes for partial homomorphisms.

The provisional ideas about L-hypoellipticity also need to be modified in two essential and related points, both involving $\Gamma_{(x,\xi)}$ (the image in $g^*_{(x,\xi)}$ of the "asymptotic" moment-map, to be discussed below). To begin with, although there is no difficulty in making an intrinsic association to $P \in U^m_{\mathbb{C}}(L)$ (the ordinary enveloping algebra) of $\hat{P} \in U^m_{hom}(g_{(x,\xi)})$, this is not necessarily possible for $P \in$ "$U^m(L)$". However, it can be shown (modulo details we have not carried out; see Note 3.24.3) that $P \to \pi(\hat{P})$ is well-defined for those π associated to orbits in $\Gamma_{(x,\xi)}$. Also, in general, Γ_η is very likely larger than necessary for L-hypoellipticity of P, only $\Gamma_{(x,\xi)}$ being required.

Independently of our own work Helffer and Nourrigat were investigating related questions, as part of their study of "maximal hypoellipticity", growing out of their earlier work on the representation theoretic hypoellipticity criterion for nilpotent Lie groups. This was primarily in the context of differential (or, later, pseudo-differential) operators constructed out of explicitly given vector fields or 1st-order ψDO's satisfying the spanning condition, corresponding, in my framework, to L being generated by L^1. They externally introduced a nilpotent Lie algebra (a free nilpotent), and wished to characterize the maximal hypoellipticity at (x,ξ) in terms of a <u>subset</u>, $\Gamma_{(x,\xi)}$, of representations. They succeeded in obtaining a precise determination of $\Gamma_{(x,\xi)}$, and in formulating a precise conjecture. They have made substantial analytic progress, proving sufficiency of the representation theoretic condition in a variety of cases, and recently ([32], [33]) necessity in general. As they point out, this conjecture, if true, would, in particular, subsume many of the known regularity results for linear P.D.E.'s under a single broad rubric. Among the tools used are the microlocal techniques of Hormander ([24], [25]) and Egorov [5] for the study of subelliptic estimates. In particular Nourrigat ([32], [33]), generalizing techniques of Hormander, derives a kind of substitute for the lifting theorems, by showing how, in a precise sense, the generating ψDO's are approximated at (x,ξ) by the representations in $\Gamma_{(x,\xi)}$. One no longer approximates by the regular representation, as in the lifting theorem, but by a subset of irreducibles. We first learned of the set $\Gamma_{(x,\xi)}$ and the microlocal approximation result from Nourrigat at the Boulder conference on P.D.E.'s of July 1983.

The construction of $\Gamma_{(x,\xi)}$ is made in terms of an explicitly chosen partial homomorphism from the free nilpotent to the generating ψDO's. An analogous set $\Gamma_{(x,\xi)} \subset g^*_{(x,\xi)}$ can be introduced in our context, and shown to be an invariant of L. This is done by choosing an arbitrary cross-section β from $g_{(x,\xi)}$ to L, defining $\Gamma^{\beta}_{(x,\xi)}$, and proving it independent of the choice of β. Using this set $\Gamma_{(x,\xi)}$ we can formulate a sharp form of our earlier conjecture for L-hypoellipticity naturally incorporating that of Helffer and Nourrigat for maximal hypoellipticity. If we regard $\Gamma_{(x,\xi)}$ as the image of an asymptotic moment-map, which we shall see is quite reasonable, then in view of the result of Guillemin-Sternberg [14] mentioned earlier, the conjecture seems extremely natural.

Although we present some analytic applications, our main contribution here is the formulation and construction of the approximation process. Various of the techniques (and results) of Helffer and Nourrigat can, with modification, doubtless be carried over to our more general context. For example, as we shall indicate, a modified version of Nourrigat's proof of the approximation result appears to carry over, and this, basically, is what is needed to prove the necessity portion of his maximal hypoellipticity criterion. We do not pursue this line, however, since we feel that a more natural and fruitful approach would be based more squarely on the invariants of L, in particular on the "phase space" decomposition determined by L. This remains a program for the future.

§1. Local Nilpotent Approximation

The initial context for this work is a family $X_1,\dots,X_k$ of C^∞ vector fields defined in a neighborhood of $x_0 \in M$, and satisfying the Hörmander spanning condition of rank r. That is, the X_i's, along with their iterated commutators of length $\leq r$ span $T_{x_0}M$, the tangent space at x_0.

We have already indicated in the Introduction how this setting leads to the introduction of nilpotent Lie groups for the purpose of approximating differential operators P expressible as polynomials in the vector fields $X_1,\dots,X_k$.

In the initial context P was "the" $\Box_b$-Laplacian on the boundary M of a strongly pseudoconvex domain D. If one used a generalized upper half-plane $\hat{D}$ to geometrically approximate D at x_0, then it was natural to use the boundary of $\hat{D}$ to approximate M. But this boundary turns out to be the Heisenberg group H_n, the most elementary (and also most fundamental) non-abelian nilpotent Lie group.

The later work of Stein and collaborators relied less on this type of geometric "normal coordinates" approximation, and more on the algebraic structure of $X_1,\dots,X_k$.

Let $g_{k,s}$ denote the free nilpotent Lie algebra on k generators $\hat{X}_1,\dots,\hat{X}_k$, and of step s. Then there is a unique <u>partial</u> homomorphism $\lambda: g_{k,s} \to$ vector fields on M in a neighborhood of x_0 such that $\lambda(\hat{X}_i) = X_i$ for $i=1,\dots,k$.

Write $g_{k,s} = g_1 \oplus \dots \oplus g_s$. To say that λ is a partial homomorphism means that

(1.1) λ is linear in $\mathbb{R}$

(1.2) $\lambda([Y_i,Y_j]) = [\lambda(Y_i), \lambda(Y_j)]$ for every $Y_i \varepsilon g_i$, $Y_j \varepsilon g_j$ with $i+j \leq s$

The pertinent $g_{k,s}$ is the one with s=r, where r comes from the spanning condition. This leads to a "lifting" process, since dim $g_{k,r}$ may be greater than dim M. When the rank of the X_i's is not constant near x_0 this extra dimensionality may be unavoidable. Under a constancy of rank condition Metivier, in a paper [30] applying nilpotent approximation in the context of spectral theory, was able to construct approximating nilpotent Lie algebras g_x with dim g_x = dim M, but with g_x necessarily varying (smoothly) with x near x_0.

The construction of g as given by Stein, Rothschild and others is not intrinsic, in that g is introduced externally and, at least a priori, depends on the explicit choice of vector fields $X_1,\ldots,X_k$. (What happens, for example, if we take instead $Y_1,\ldots,Y_k$, some "invertible" linear combination of the $X_1,\ldots,X_k$?) The construction of Metivier, under the constant rank assumption, does not introduce g externally, but it is also not intrinsic.

In what follows we show how to make an intrinsic construction of a graded nilpotent Lie algebra g_{x_0} as an invariant attached to a filtered Lie algebra L at x_0. In a sense made precise by our version of the lifting theorem, g_{x_0} is an approximation to L at x_0. The algebra g_{x_0} depends not just on $\{L^i\}$ as an abstract Lie algebra over $\mathbb{R}$, but also on its $C^\infty(M)$ module structure.

This intrinsic construction has a number of advantages.

(1) It leads to natural functoriality results.

(2) It handles at the same time the case where the generators of L are not all of rank 1.

(3) It recovers the Metivier approximation in an intrinsic fashion, and extends it to the more general context (2).

(4) It generalizes to other contexts, such as the microlocal and non-nilpotent local contexts, which we shall treat in §3 and §4.

One basic distinction between the intrinsic and the external constructions is that the former does not come equipped with a partial homomorphism. In fact, since the Lie algebras will vary with the point x_0, one cannot expect to have available a partial homomorphism. However, a less stringent substitute notion is available, namely that of <u>weak</u> homomorphism. In the context of the intrinsic construction this notion is extremely natural. Much of the technical difficulty of carrying over to the intrinsic context results like the lifting theorem comes from having only weak homomorphisms to work with.

A further distinction between the intrinsic construction and the external construction is worth noting. In the external construction, as we saw, no spanning condition is needed in order to construct the nilpotent Lie algebra or the partial homomorphism (though such a condition is needed to construct the lifting). In the intrinsic construction something like the spanning condition is needed to even construct the nilpotent Lie algebra

g_{x_0}. (Without such a condition the construction yields an <u>infinite</u> graded Lie algebra, with finite-dimensional terms of each degree.) The precise conditions, as we shall see, is that $\dot{L}_x^r = \dot{L}_x^{r+1} = \ldots = \dot{L}_x^{r+i}$ for all $i \geq 0$, i.e., the sequence stablizes. This is automatically the case if "spanning" holds. In light of Frobenius' theorem (or better, Nagano's theorem in the real-analytic case) the above condition (modulo a constancy of rank assumption in the C^∞ case) is like a spanning condition on an integral submanifold.

We begin with some preliminaries. We shall work primarily in the C^∞ category, and deal with modules L of C^∞ vector fields on M, a smooth, paracompact, manifold. That is, modules over the ring $C^\infty(M)$ of real-valued C^∞ functions. At times we shall only want to take M an open neighborhood of x_0, and generally we shall work with germs. In the C^∞ category partitions of unity are available.

We will have occasion to work with the formal power series or real-analytic categories. In the real-analytic context we do not have partitions of unity, so we should, strictly speaking, probably work at the level of sheaves of modules rather than modules, but we shall forego this degree of precision.

<u>Notation:</u>

(1.3) $\dot{C}_x^\infty$, $\dot{\mathcal{O}}_x$ denote germs at x of real-valued C^∞, respectively real-analytic functions; $\mathcal{F}_x$ denotes the ring of formal power series at x.

(1.4) If L is a $C^\infty(M)$-module of vector fields, $\dot{L}_x$ denotes the $\dot{C}^\infty_x$-module of germs at x of vector fields in L. Similarly in the real-analytic case. If we pass to formal power series instead of germs we obtain an $\mathcal{F}_x$-module.

<u>Remarks 1.1:</u> 1) Each of the three rings in (1.3) is a local ring with identity. The unique maximal ideal $\dot{m}_x$ consists of the germs (resp., formal power series) vanishing at x. Moreover, the composition

$$\mathbb{R} \to \dot{C}^\infty_x \to \frac{\dot{C}^\infty_x}{\dot{m}_x}$$

is bijective. Similarly in the remaining two cases. (See, for example, Malgrange [29]).

2) $\dot{\mathcal{O}}_x$ and $\mathcal{F}_x$ are Noetherian. (Malgrange [29]).

3) If L is the module of <u>all</u> vector fields in the C^∞ or real-analytic context, then $\dot{L}_x$ is finitely generated over $\dot{C}^\infty_x$, $\dot{\mathcal{O}}_x$ (and $\mathcal{F}_x$, if we pass to formal power series). In fact, choose local coordinates, and take $\partial/\partial x_1, \ldots, \partial/\partial x_n$ (more precisely, their germs) as generators.

4) As a corollary of 2) and 3) we get: Let B be any

submodule over $\dot{\mathcal{O}}_x$ of germs of real-analytic vector fields at x (resp., any submodule over $\mathcal{F}_x$ of formal vector fields). Then B is finitely generated.

Def. 1.2: Let M be a C^∞ manifold, $F = C^\infty(M)$, and $x \in M$.
A filtered Lie algebra L at x of C^∞ vector fields (with increasing filtration), is a, generally infinite dimensional, Lie algebra over $\mathbb{R}$ of vector fields on M, together with a sequence of subspaces L^i i=1,2,..., such that

(1) $L^1 \subset L^2 \subset L^3 \subset \ldots$

(2) $[L^i, L^j] \subset L^{i+j} \quad \forall\, i,j$

(3) $L = \bigcup_{j=1}^{\infty} L^j$

(4) Each L^i is an F-module, i.e., $FL^i \subset L^i$, where FL^i refers to multiplication of vector fields by C^∞ functions.

(5) As an $\dot{F}_x$ (i.e., $\dot{C}^\infty_x$) module $\dot{L}^i_x$ is finitely generated for each i.

Remarks 1.3:

1) In view of remark 3) above, if we assume the spanning condition (see below) at x, then $\dot{L}^r_x$ is automatically finitely generated.

2) In practice one often uses a stronger condition than (5), namely (5 local): There exists an open neighborhood U of x such that for every i $L_i(U)$ is finitely generated as a $C^\infty(U)$-module.

Then (5 local) of course implies (5) for every $x \in U$. In fact it implies a slightly stronger, and useful consequence:

Suppose (5 local) holds for the neighborhood U. Let $X_1,\dots,X_j$ be elements of $L^i(U)$ such that the germs $X_{1x},\dots,X_{jx}$ generate $\dot{L}^i_x$. (Such generators exist since (5) holds at x). Then there exists a nbhd V of x such that $X_1 \restriction V,\dots,X_j \restriction V$ generate $L^i(V)$ as a $C^\infty(V)$-module.

We omit the simple proof. A somewhat more carefully worded variant of these remarks holds in the real-analytic case.

Examples 1.4:

1) Take $X_1,\dots,X_k$ vector fields in a nbhd of x, and take L^1 = all C^∞ linear combinations of $X_1,\dots,X_k$; $L^2 = L^1 + [L^1,L^1]$; ... $L^{j+1} = L^j + [L^1,L^j]$.

That each L^j is a C^∞-module follows from the identity $[fX,Y] = f[X,Y] + [f,Y]X$. Finite generation is obvious.

2) Take $X_1,\dots,X_k$, $Y_1,\dots,Y_\ell$ vector fields. Set L^1 = all C^∞ linear combinations of $X_1,\dots,X_k$; L^2 = (all C^∞ linear combinations of $Y_1,\dots,Y_\ell$) + $L^1 + [L^1,L^1]$; and set

$$L^j = \sum_{\substack{j_1,\ldots,j_k \in \{1,2\} \\ j_1+\ldots+j_k \le j \\ 1 \le k}} [[\ldots[L^{j_1}, L^{j_2}], L^{j_3}]\ldots L^{j_k}].$$

3) If $L^1 \subset L^2 \subset \ldots$ is a filtered Lie algebra at x which is not of finite rank (see below), we can "embed" it in a filtered Lie algebra of arbitrary rank r, as follows. Define the filtered Lie algebra

$$\hat{L}^i = \begin{cases} L^i \text{ if } i<r \\ \text{The module of all } C^\infty \text{ vector fields} \\ \qquad\qquad \text{if } i \ge r. \end{cases}$$

(By remark 1.13 we maintain the finite generation condition.)

<u>Notation:</u>

(1.5) For vector fields in a nbhd of x, let α_x:vector fields $\to T_xM$ be the $\mathbb{R}$-linear map which is evaluation at x. Clearly α_x depends only on the germ of vector field at x.

<u>Def. 1.5</u>: The filtered Lie algebra L is of <u>finite rank</u> at x if there exists r such that $\alpha_x:\dot{L}^r_x \to T_xM$ is surjective. The smallest such r is called the <u>rank</u> of L at x.

<u>Notes 1.6</u>:

1) In the case of the first example above the finite rank condition is just the standard spanning condition.

2) If $\alpha_x:\dot{L}_x^r \to T_xM$ is surjective, so is $\alpha_y:\dot{L}_y^r \to T_yM$ for all y sufficiently close to x. Thus, if x is of finite rank r, then y is of finite rank $\leq$ r for all y sufficiently close to x. Moreover, $\dot{L}_y^r$ = set of germs of all C^∞ vector fields at y for y sufficiently close to x. In particular $\dot{L}_y^r = \dot{L}_x^s \quad \forall s \geq r$.

Pf: Let $e_1,\ldots,e_n$ be a basis for T_xM.. Choose germs $X_{1x},\ldots, X_{nx} \in \dot{L}_x^r$ such that $X_i(x) = e_i$. Then these germs, in a nbhd of x, form frames for the tangent bundle.

Prop. 1.7: Let $\{L^i\}$, x be a filtered Lie algebra of finite rank, r, at x. Let

$$g_x^i = \frac{\dot{L}_x^i}{\dot{L}_x^{i-1}+\dot{m}_x\dot{L}_x^i} \qquad \text{(where } \dot{L}_x^0 = 0) \ .$$

Then

(1) For $i>r$, $g_x^i=0$.

(2) For $i\leq r$, g_x^i is a finite-dimensional vector space over $\mathbb{R}$.

(3) Let $\pi_i:L^i \to g_x^i$ be the canonical $\mathbb{R}$-linear projection. Define $g_x=g_x^1 \oplus \ldots \oplus g_x^r$. Then via the π_i's g_x inherits canonically the structure of graded (nilpotent) Lie algebra over $\mathbb{R}$.

(4) $\pi_i(fX) = f(x)\pi_i(X)$ for $X \in L^i$ and $f \in C^\infty$.

<u>Remark 1.8</u>: g_x is graded of rank r, but it may be of "step" < r. That is, fewer than r commutators may yield 0. For example, $\mathbb{R}^n$ with non-standard dilations can be graded of rank > 1, though it is of step 1.

<u>Pf:</u>

(1) By Note 1.6.2 $\dot{L}_x^r = \dot{L}_x^s \quad \forall s \geq r$.

(2),(4) $\dot{L}_x^i$ is a module over $\dot{C}_x^\infty$, and by definition g_x^i inherits the structure of module over $\dot{C}_x^\infty/\dot{m}_x = \mathbb{R}$. By hypothesis $\dot{L}_x^i$ is finitely generated over $\dot{C}_x^\infty$, and so g_x^i is finite-dimensional over $\mathbb{R}$.

(3) Define a bracket $[\ \ ]: g_x^i \times g_x^j \to g_x^{i+j}$ as follows:

For X^i, $X^j \in g_x^i$, g_x^j, respectively, choose Y^i, $Y^j \in L^i$, L^j s.t.

$\pi_i(Y^i) = X^i$, $\pi_j(Y^j) = X^j$.

Let $[X^i, X^j] = \pi_{i+j}[Y^i, Y^j]$.

To prove that this bracket is well-defined it suffice to show:

$[\dot{L}_x^{i-1} + \dot{m}_x\dot{L}_x^i, \dot{L}_x^j] \subset \dot{L}_x^{i+j-1} + \dot{m}_x\dot{L}_x^{i+j}$.

But $[\dot{L}_x^{i-1}, \dot{L}_x^j] \subset \dot{L}_x^{i+j-1}$ by (2) of Def. 1.2

and $[\dot{m}_x\dot{L}_x^i, \dot{L}_x^j] \subset \dot{m}_x[\dot{L}_x^i, \dot{L}_x^j] + [\dot{m}_x, \dot{L}_x^j]\dot{L}_x^i$.

Again by (2), the first term is in $\dot{m}_x\dot{L}_x^{i+j}$.

Of the second term all one can say is that it is in

$\dot{C}_x^\infty\dot{L}_x^i = \dot{L}_x^i$.

<u>Now we use the fact that our filtration begins with an L^1-term.</u> Thus, $i+j-1 = i+(j-1) \geq i$. So, since the filtration

is increasing, $\dot{L}_x^i \subset \dot{L}_x^{i+j-1}$.

This proves that our bracket is well-defined; and from the definition it is clear that it satisfies the conditions for a Lie bracket.

<u>Note 1.9</u>: Even if $\{L^i\}$ is not of finite rank at x, each g_x^i is finite dimensional; only now $g_x = g_x^1 \oplus g_x^2 \oplus \ldots$ is an infinite direct sum. It is still a Lie algebra over $\mathbb{R}$ with respect to the above-defined bracket.

The graded Lie algebra $g_x = g_x^1 \oplus \ldots \oplus g_x^r$ is clearly intrinsically associated to $\{L^i\}$, x. We begin our examination of g_x by asking how much "collapsing" has taken place in its construction. We shall need an elementary but basic tool which we shall also use later for other purposes, and which arises because we are dealing with local rings.

<u>Prop. 1.10: Nakayama's Lemma</u> (see [13]). Let A be a commutative local ring with unit, and M a <u>finitely generated</u> A-module such that M = mM, where m is the unique maximal ideal in A. Then M = {0}.

<u>Cor. 1.11</u>: Let M′ be a submodule of M such that M = M′ + mM. Then M=M′.

<u>Pf:</u> Let N = M/M′. Then N is still a finitely generated A-module. But N/mN = M/mM + M′ = {0} by hypothesis, so N = mN, and N = {0} by Nakayama.

<u>Cor. 1.12</u>: M/mM is a finite-dimensional vector space over the field A/m.

Let $\phi: M \to M/mM$ be the natural projection map and $v_1, \ldots, v_n$ a basis for this vector space. Choose $e_1, \ldots, e_n$ in M such that $\phi(e_i) = v_i$. Then $e_1, \ldots, e_n$ form a set of generators for M over A.

<u>Pf:</u> Since M is finitely generated as an A-module, M/mM is finitely generated as an A/m module. But m being a maximal ideal, A/m is a field, and so M/mM is a finite-dimensional vector space.

The converse is harder, and uses Nakayama. Choose a basis $v_1, \ldots, v_n$ for M/mM, and preimages $e_1, \ldots, e_n$ under ϕ. Let M' be the submodule of M spanned by $e_1, \ldots, e_n$. It follows immediately that $M = M' + mM$. So, by Cor. 1.11., $M = M'$.

In our context we take $A = \dot{C}^\infty_x$ and $m = \dot{m}_x$. The first consequence is

<u>Lemma 1.13:</u> $g^i_x = 0 \Leftrightarrow \dot{L}^{i-1}_x = \dot{L}^i_x$. (The non-trivial direction is $\Rightarrow$). In particular, if r is the rank of $\{L^i\}$, x then $g^r_x \neq 0$. That is, g_x cannot be "small" unless $\{L^i\}$, x is "small".

<u>Pf:</u> Cor. 1.11.

<u>Remark 1.14</u>: Note here in passing that

$$g^i_x \equiv \frac{\dot{L}^i_x}{\dot{L}^{i-1}_x + \dot{m}_x \dot{L}^i_x} \cong \frac{\dot{L}^i_x/\dot{L}^{i-1}_x}{\dot{m}_x(\dot{L}^i_x/\dot{L}^{i-1}_x)} \text{ as } \frac{\dot{C}^\infty_x}{\dot{m}_x} \text{ modules (i.e., as vector spaces over } \mathbb{R})$$

This follows immediately from the fact that

$$\dot{L}_x^i \rightarrow \frac{\dot{L}_x^i/\dot{L}_x^{i-1}}{\dot{m}_x(\dot{L}_x^i/\dot{L}_x^{i-1})}$$

is onto, with kernel $\dot{L}_x^{i-1} + \dot{m}_x\dot{L}_x^i$.

We next examine the functoriality properties of g_x.

Def. 1.15: Let $\{L^i\},x$ be a filtered Lie algebra, and $h = h_1 \oplus \ldots \oplus h_s$ a graded (nilpotent) Lie algebra. A weak homomorphism γ (at x) from h into L consists of an R-linear map γ such that

(1) $\gamma: h_i \rightarrow L^i$

(2) For any Y_i, Y_j ε h_i, h_j, respectively,

$$\gamma([Y_i, Y_j]) - [\gamma(Y_i), \gamma(Y_j)] \;\varepsilon\; \dot{L}_x^{i+j-1} + \dot{m}_x\dot{L}_x^{i+j}$$ (after passing to germs at x)

Remark 1.16: Suppose $\{L^i\}$, x is of rank r at x, and $h = h_1 \oplus \ldots \oplus h_s$, with $s \geq r$. Then if $\lambda: h \rightarrow L$ is a partial homomorphism then λ is also a weak homomorphism. By a partial homomorpism we mean an $\mathbb{R}$-linear map such that

(1) $\lambda: h_i \rightarrow L^i$

(2) For any Y_i, Y_j ε h_i, h_j, respectively, with $i+j \leq s$,

$$\lambda([Y_i, Y_j]) = [\lambda(Y_i), \lambda(Y_j)] \ .$$

Generally, we also wish to assume

(3) The image $\lambda(h_i)$, i.e., this finite-dimensional vector space,

generates L^i as a module over C^∞, modulo L^{i-1}, for $i \leq r$. (This condition certainly holds if we take L as in Example 1.4.1 and use the original definition of partial homomorphism).

Pf: (of remark). It suffices to show that λ satisfies (2) of Def. 1.15 for $i+j>s$. But $i+j>s \Rightarrow i+j>r$, so $i+j-1 \geq r$. Hence $\dot{L}_x^{i+j-1}$ consists of all C^∞ vector fields at x, so done.

Notice that although the notion of weak homomorphism is referred to a point x, via the appearance of $\dot{m}_x$, that of partial homomorphism is not; that is, the latter assumes the L^i are all defined in some fixed (though arbitrarily small) nbhd of x, and the homomorphism is viewed as holding in this nbhd. In particular, the preceding proof shows that if λ is a partial homomorphism near x then it is a weak homomorphism at y for all y in a nbhd of x. This is one reason why the notion of partial homomorphism is too stringent in general.

Def. 1.17: Let $\{L^i\},x$ be a filtered Lie algebra, with g_x canonically associated to it. Let $\pi_i : L^i \to g_x^i$ be the canonical projection. A cross-section of π is an $\mathbb{R}$-linear map β such that

(1) $\beta : g_x^i \to L^i$

(2) $\pi_i \circ \beta = \mathrm{Id}$ for every i.

Clearly, since π_i is surjective, such cross-sections exist; one takes a basis for g_x^i and maps to preimages under π_i.

<u>Prop. 1.18</u>: Let $\beta: g_x \to L$ be a cross-section. Then

(1) $\beta([Y_i, Y_j]) - [\beta(Y_i), \beta(Y_j)] \in \dot{L}_x^{i+j-1} + \dot{m}_x\dot{L}_x^{i+j}$ (after passing to germs at x)

for every Y_i, $Y_j \in g_x^i$, g_x^j, respectively. That is, β is a weak homomorphism.

(2) For every $X_i \in L^i$, $X_i - \beta \circ \pi_i(X_i) \in \dot{L}_x^{i-1} + \dot{m}_x\dot{L}_x^i$

(3) For any cross-sections β, β', $\beta(Y_i) - \beta'(Y_i) \in \dot{L}_x^{i-1} + \dot{m}_x L_x^i$ $\forall\, Y_i \in g_x^i$.

<u>Pf:</u>

(1) Suffices to show $\pi_{i+j}(\beta[Y_i,Y_j] - [\beta(Y_i), \beta(Y_j)]) = 0$. But this equals $\pi_{i+j}(\beta[Y_i,Y_j]) - \pi_{i+j}[\beta(Y_i), \beta(Y_j)]$. The first term equals $[Y_i, Y_j]$, by definition of cross-section. But the second term equals $[Y_i, Y_j]$ by the definition of Lie bracket for g_x.

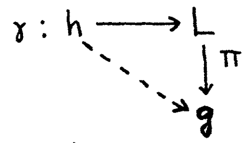
(2) Suffices to show $\pi_i(X_i - \beta \circ \pi_i(X_i)) = 0$. But this equals $\pi_i(X_i) - (\pi_i \circ \beta)(\pi_i(X_i)) = 0$, by (2) of definition of cross section.

(3) Follows from (2), together with (2) of definition of cross-section, by taking $X_i = \beta(Y_i)$ and replacing β by β' in (2).

We next prove the "universal" property of g_x.

<u>Prop. 1.19:</u> Let $\{L^i\},x$ be of rank r at x, and let $h_1 \oplus \cdots \oplus h_s$ be a

graded Lie algebra, with $\gamma: h \to L$ a weak homomorphism. Then the map $\pi \circ \gamma$ (i.e., $\pi_i \circ \gamma_i : h_i \to g_x^i$) is a homorphism of graded Lie algebras.



Moreover, if, in particular, $\gamma(h_i)$ generates L^i over C^∞ modulo L^{i-1} for every $i \leq r$, the $\pi \circ \gamma$ is surjective.

Pf: Clearly $\pi \circ \gamma$ is $\mathbb{R}$-linear. Need to show

$$\pi_{i+j} \circ \gamma_{i+j}[Y_i, Y_j] = [\pi_i \circ \gamma_i(Y_i), \pi_j \circ \gamma_j(Y_j)] .$$

By definition of [,] in g_x, the right-hand side equals $\pi_{i+j}[\gamma_i(Y_j), \gamma_j(Y_j)]$. But, by definition of weak homomorphism, $\pi_{i+j}(\gamma_{i+j}[Y_i, Y_j] - [\gamma_i(Y_i), \gamma_j(Y_j)]) = 0$.

Surjectivity, under the hypotheses of the Proposition follows immediately from the definition of g_x.

Cor. 1.20: If $h = h_1 \oplus \dots \oplus h_s$ where $s \geq r$=rank at x of $\{L^i\}$, and if $\lambda: h \to L$ is a partial homomorphism (in a nbhd of x), then there are corresponding homomorphisms $\pi_y \circ \lambda \to g_y$ for all y in a (smaller) nbhd of x. Moreover, if λ satisfies (3) of Remark 1.16, then for all y in a possibly smaller nbhd of x, $\pi_y \circ \lambda : h \to g_y$ is surjective. That is, for all y in a nbhd of x, g_y is a quotient of h.

Pf:

Follows from above Prop. and from Remark 1.16, if we recall that rank $\{L^i\} \leq r$ at all y in a nbhd of x.

It is not clear what is the most natural notion of weak morphism between two filtered Lie algebras $\{L^i\},x$, $x\varepsilon M$ and $\{K^i\},y$, $y\varepsilon N$. We give one variant at the germ level.

<u>Def. 1.21</u>: A <u>weak morphism</u> between $\{L^i\},x$ and $\{K^i\},y$ consists of a sequence of R-linear maps $W_i:\dot{L}^i_x \to \dot{K}^i_y$, together with an $\mathbb{R}$-linear map $\phi:\dot{C}^\infty_x(M) \to \dot{C}^\infty_y(N)$ such that

(1) $\phi:\dot{m}_x(M) \to \dot{m}_y(N)$

(2) $W_i(fX) = \phi(f)W_i(X) \quad \forall\ f \ \varepsilon\ \dot{C}^\infty_x(M),\ X\ \varepsilon\ \dot{L}^i_x$

(3) $W_i:\dot{L}^{i-1}_x \to \dot{K}^{i-1}_y$

(4) $[W_i(X_i),\ W_j(X_j)] - W_{i+j}([X_i,X_j])\ \varepsilon\ \dot{K}^{i+j-1}_y + \dot{m}_y\dot{K}^{i+j}_y$
for X_i, X_j ε $\dot{L}^i_x$, $\dot{L}^j{}_x$, respectively.

<u>Notes 1.22:</u>

1) Of course an interesting special case occurs when (4) is replaced by the stronger assumption (4'): $[W_i(X_i),\ W_j(X_j)] = W_{i+j}[X_i,X_j]$.

2) We do not assume that the W_i piece together to form a single R-linear map $W:\dot{L}_x \to \dot{K}_y$ such that $W_i = W|L^i$. We instead assume the weaker consistency condition (3), which is all that can be expected in various examples (such as 3) below).

<u>Examples 1.23:</u>

1) Let $\phi:M \to N$ be a diffeomorphism, and suppose that $K^i = \phi_*(L^i)$, where ϕ_* denotes push-forward of vector fields. Then for every $x\varepsilon M$ ϕ defines a weak morphism between $\{L^i\},x$ and $\{K^i\}$, $\phi(x)$, where $\phi:C_x^\infty(M) \to C_y^\infty(N)$ is given by $f \to f\circ\phi^{-1}$. Of course here (4') is satisfied, as well as the strong consistency condition in Note 1.22.2.

2) $M=N$, ϕ=identity, $L^i \subset K^i$, and $W^i:L^i \to K^i$ the inclusion map. Note that the induced morphism of graded Lie algebras (see Prop. 1.24) is not necessarily injective. In fact rank L_x may certainly be greater than rank K_x. As an illustration take K^i as in Example 1.26.1 below, and L^i arbitrary.

3) Let $\{L^i\},x$ and $\{K^i\},y$ both be of finite rank, with associated graded nilpotents g_x, h_y ,respectively. Suppose there is a morphism $\lambda:g_x \to h_y$ of graded Lie algebras, and let β be an arbitrary cross-section for h_y. Define $\phi:\dot{C}_x^\infty(M) \to \dot{C}_y^\infty(N)$ by $\phi:f \longmapsto$ the constant function $f(x)$; define $W^i:\dot{L}^i \to \dot{K}^i$ by $W^i = \beta_i\circ\lambda_i\circ\pi_i$. Then W^i is a weak morphism.

<u>Prop. 1.24:</u>

(a) The composition of weak morphisms is again a weak morphism.

(b) A weak morphism $\Phi:\{L^i\},x \to \{K^i\},y$ induces canonically a morphism of graded (nilpotent, if x,y are of finite rank) Lie algebras $\hat{\Phi}:g_x \to h_y$.

(c) $(\Phi\circ\Psi)^\wedge = \hat{\Phi}\circ\hat{\Psi}$.

<u>Pf:</u>

(a) This is obvious except for (4) which involves a small calculation requiring application of (1), (2), (3).

(b) Define $\hat{\Phi}_i:g_x^i \to h_y^i$ as follows. Choose an arbitrary cross-section β_i^L for g_x. Let $\hat{\Phi}_i = \pi_i^K\circ\Phi_i\circ\beta_i^L$. Clearly $\hat{\Phi}_i$ is $\mathbb{R}$-linear.

<u>Claim</u>:

(1) $[\hat{\Phi}_i(Y^i), \hat{\Phi}_j(Y_j)] = \hat{\Phi}_{i+j}([Y_i,Y_j])$ for $Y_i,Y_j \in g_x^i, g_x^j$, respectively.

(2) $\hat{\Phi}_i$ is well-defined independently of the choice of cross section β_i.

In fact (1) follows from the definition of [,] in g_x, h_y, from Prop. 1.18. (1), and an argument analogous to that in (a) above. Statement (2) follows from Prop. 1.18.(3).

(c) By Prop. 1.18.(2), $\beta_i^L\circ\pi_i^L = \mathrm{Id}$ mod terms in $\dot{L}_x^{i-1} + \hat{m}_x\dot{L}_x^i$, and by (1)-(3) of the definition of weak morphism. Φ_i sends the "error" terms into the kind of kernel of π_i^K.

<u>Cor. 1.15:</u> (see Example 1.23.1). The isomorphism class of the graded Lie

algebra g_x attached to $\{L^i\}, x$ is invariant under diffeomorphisms applied to $\{L^i\}, x$.

We next illustrate the computation of g_{x_0} in a number of cases (in each of which the rank is finite). We retain the notation α_{x_0} for evaluation of vector fields (see (1.5)).

Examples 1.26:

1) Let L^i, i=1,2,... consist of all C^∞ vector fields in a nbhd of x_0. As in Remark 1.1.3 we see that L^i is locally finitely generated by taking as generators $\partial/\partial x_1, \ldots, \partial/\partial x_n$ where $x_1, \ldots, x_n$ are local coordinates. Clearly x_0 is of rank r=1, and so $g_{x_0} = g^1_{x_0}$, abelian, with standard isotropic dilations. Claim: $g^1_{x_0}$ is canonically isomorphic to $T_{x_0}M$ viewed as a vector space with standard dilations. In fact, the map $\alpha_{x_0} : \dot{L}^1_x \to T_{x_0}M$ is surjective and factors through $\dot{m}_{x_0}\dot{L}^1_{x_0}$ to give a map

$$\rho : g^1_{x_0} = \frac{\dot{L}^1_{x_0}}{\dot{m}_{x_0}\dot{L}^1_{x_0}} \to T_{x_0}M \to 0 \; .$$

Using the generators $\partial/\partial x_1, \ldots, \partial/\partial x_n$ as local frames we see that ρ is also injective, and hence bijective. Notice that this example is the general case of r=1.

2) If $\{L^i\}$ is generated by L^1, as in Example 1.4.1, then g_{x_0} is generated by $g^1_{x_0}$, i.e.,

$$g^i_{x_0} = \underbrace{[g^1_{x_0}[\ldots[g^1_{x_0}, g^1_{x_0}]\ldots]}_{i \text{ factors}} .$$

This follows from the definition of Lie bracket in g_{x_0} and the fact that $\pi_i : L^i \to g^i_{x_0}$ is surjective.

3) Let $g = g_1 \oplus \ldots \oplus g_r$ be a graded nilpotent Lie algebra, and G a corresponding Lie group (uniquely determined in a nbhd of e). View the elements of g as left-invariant vector fields on G, and let

$$L^i = \sum_{j=1}^{i} C^\infty(G) \otimes_{\mathbb{R}} g_j \quad ,$$

i.e., take $C^\infty(G)$ linear combinations of the left-invariant vector fields.

Claim At any $x \varepsilon G$, $\{L^i\}$ has rank r, and $g_x \cong g$ canonically.

Pf: Basis elements of $g_1 \oplus \ldots \oplus g_i$ form frames for L^i (i.e., are everywhere linearly independent and spanning). This shows in particular that rank = r. Let $\tau_x : L^i \to g_i$ be defined by applying α_x, identifying T_xG canonically with g, and projecting onto the i-th component. Passing to the germ level we see just as in Example 1) that τ_x factors through $\dot{L}^{i-1}_x + m_x\dot{L}^i_x$ to give a bijection between g^i_x and g_i. The definition of Lie bracket in g_x shows that this is a Lie algebra isomorphism.

4) Let H be a Lie group with Lie algebra h. Choose an increasing filtration by finite-dimensional subspaces $h_1 \subsetneq h_2 \subsetneq \ldots \subsetneq h_n = h$ such that $[h_i, h_j] \subset h_{i+j}$. View the elements of h as left-invariant vector fields on H, and define L^i to be $C^\infty(H) \otimes_{\mathbb{R}} h_i$. Then, just as in example 3), one sees that for any $x \varepsilon H$, g_x is of rank r and

$$g_x = h_1 \oplus \frac{h_2}{h_1} \oplus \ldots \oplus \frac{h_r}{h_{r-1}} \quad ,$$

the graded Lie algebra associated to the filtered Lie algebra h. (Notice that h defines a filtration on the vector space T_xH, and g_x defines the corresponding grading.)

5) Let M = V, a finite-dimensional graded vector space, i.e., $V = V_1 \oplus \ldots \oplus V_r$, on which we define stardard dilations δ_t (t>0) by $\delta_t \restriction V_i = t^i$. Using the dilations δ_t one can intrinsically define (without explicit choice of coordinates) the notion of homogeneity (in a nbhd of 0) for a C^∞ function and a C^∞ vector field. (The vector field is homogeneous of degree k if, applied to functions, it lowers homogeneity by degree exactly k; since the function is C^∞ this implies that the derivative $\equiv 0$ if the degree of homogeneity of the function is less than k.)

Choose a basis for V consisting of bases for the V_i. If u_{jk} is one of the standard coordinate functions for V_k, then u_{jk} is homogeneous of degree k, and so is $\partial/\partial u_{jk}$. Say that a vector field is of <u>local order</u> $\lesssim$ i at 0 if the coefficient $f_{jk}(u)$ of $\partial/\partial u_{jk}$ has its Taylor series at 0 begin with terms of order $\geq$ k-i, where order is determined as above. (This can be formulated more intrisically in terms of δ_t).

Since the highest grading occurring in V is r, it follows that

(1) Every homogeneous vector field is of degree $\leq r$.

(2) Every vector field is of local order $\leq r$.

For X of local order $\leq i$ there is an intrinsically defined "leading" term $\hat{X}$, namely the unique vector field homogeneous of degree i such that $X-\hat{X}$ is of local order $\leq i-1$. In local coordinates $\hat{X}$ is the sum of the terms $\hat{f}_{jk}(u)\partial/\partial u_{jk}$ where $\hat{f}_{jk}(u)$ is the sum of the terms of order k-i in the Taylor expansion of $f_{jk}(u)$. (Of course we can then continue and define the component homogeneous of degree i-1, etc.)

Let g^i, i=1,...,r be the space of vector fields homogeneous of degree i. Then $[g^i, g^j] \subset g^{i+j}$. Hence $g = g^1 \oplus \ldots \oplus g^r$ is a clearly finite-dimensional and, hence, nilpotent subalgebra of the vector fields on V.

Let L^i = all vector fields on V of local order $\leq i$ at 0. It is easy to check that

1) L^i is a $C^\infty(V)$ - module.

2) $[L^i, L^j] \subset L^{i+j}$

3) $L^1 \subset L^2 \subset \ldots \subset L^r = L^{r+1} \ldots$ = all C^∞ vector fields on V near 0.

4) $\alpha_0(L^r) = T_0(V)$ (and $\alpha_0(L^{r-1}) \neq T_0(V)$ unless $V_r = \{0\}$).

(Statement 4), and thus 3), follows from (2) above. Statement 2) implies that the leading term of the commutator is the commutator of the leading terms.) To show that L^i, $i \geq 1$, is finitely generated it suffices, since $L^i = L^0 \oplus g^1 \oplus \ldots \oplus g^i$ to show that L^0 is finitely generated. But one (non-minimal) set of generators is given by all vector fields in L^0 with coefficients polynomials of degree $\leq r$ in the classical, isotropic, sense. To see this recall that any C^∞ function vanishing at 0 of order $\geq r$ in the classical sense is a C^∞ linear combination of monomials of degree r; and any vector field with coefficients monomials of degree r in the classical sense is in L^0.

Let $g_0 = g_0^1 \oplus \ldots \oplus g_0^r$ be the graded nilpotent associated to $\{L^i\},0$. Let $\hat{g}_0^i = g_0^i$ except for i=1. Define g_0^1 as

$$\hat{g}_0^1 = \frac{\dot{L}_0^1}{\dot{L}_0^0 + \dot{m}_0\dot{L}_0^1} \qquad \text{(in contrast to } g_0^1 = \frac{\dot{L}_0^1}{\dot{m}_0\dot{L}_0^1}\text{)}.$$

The proof of Prop. 1.7 goes through unchanged to show that $\hat{g}_0$ inherits the structure of graded nilpotent Lie algebra. Since $[L^0, L^j] \subset L^j$ it follows that $\hat{g}_0 \equiv \hat{g}_0^1 \oplus \ldots \oplus \hat{g}_0^r$ is (canonically) the quotient of g_0 by the ideal

$$\frac{\dot{L}_0^0 + \dot{m}_0\dot{L}_0^1}{\dot{m}_0\dot{L}_0^1}$$

(lying in g_0^1), which is in the center of g_0. (With a bit more work this ideal could be naturally identified with an explicit subspace of L^0 consisting of polynomial-coefficient vector fields).

<u>Claim:</u> There is a natural isomorphism $\hat{g}_0 \cong g$.

<u>Pf:</u> Define the $\mathbb{R}$-linear map $\gamma: \dot{L}_0^i \to g^i$ by $X \to \hat{X}$, the leading term, of degree i. Clearly, γ is surjective and factors through $\dot{L}_0^{i-1} + \dot{m}_0 \dot{L}^i$. In fact, $\dot{m}_0 \dot{L}_0^i \subset \dot{L}_0^{i-1}$ and kernel $\gamma = \dot{L}_0^{i-1}$, so γ determines a vector space isomorphism between $\hat{g}_0^i$ and g^i. It is easily seen that the Lie algebra structure is preserved.

6) Let M and g be as in Example 5). Let $h = h_1 \oplus \ldots \oplus h_r$ be a graded subalgebra of g such that $\alpha_0(h) = T_0 V$.

Let L^i consist of all $C^\infty(V)$ linear combinations of vector fields in h_j, $j \leq i$. Then $\{L^i\},0$ is a filtered Lie algebra of rank r. Let g_0 be the associated graded nilpotent. Let γ be the restriction to L^i of the corresponding map in Example 5). Choosing some representation

$$X = \sum_{\substack{k \leq i \\ j}} f_{jk}(u) Y_{jk} \quad ,$$

where Y_{jk} is a basis for h_k over R, we see that

$$\gamma(X) = \sum_j f_{ji}(0) Y_{jk} \quad .$$

An easy argument then shows that γ induces a natural Lie algebra isomorphism $g_0 \cong h$.

Let H be the simply connected nilpotent Lie group with Lie algebra h. Then by the same argument as in Folland [7], using the existence of the dilations, one can show that the infinitesimal action of h on V can be exponentiated to give a transitive right action of H on V. In other words, letting G_0 denote the simply connected group associated to g_0, V is a right homogeneous space of G_0 (and T_0V is g_0/k, where k is the graded subalgebra of all vector fields in g_0 vanishing at 0.) This is interesting to compare with the homogeneous space lifting theorem of §2.

7) Special case of Example 6). Let $X_1,\ldots,X_k$ be homogeneous (of degree 1) vector fields on V, h_1 the vector space over $\mathbb{R}$ spanned by the X_i's, and h the Lie algebra generated by h_1.

8) Same setting as Example 7). Let $Y_1,\ldots,Y_k$ be vector fields of local order $\lesssim 1$ at 0 whose iterated commutators of order $\lesssim$ r span T_0V. Let $X_1,\ldots,X_k$, homogeneous of degree 1, be the corresponding leading terms, h_1 the vector space they span, and h the graded Lie algebra generated by h_1. Let L^1 be the space of all C^∞ linear combinations of $Y_1,\ldots,Y_k$, and L the filtered Lie algebra generated by L^1. Let g_0 be the graded nilpotent associated to L,0.

We wish to examine the relationship of g_0 to h. We begin with a special case. Take $V = V_1 \oplus V_2 = \underset{x}{\mathbb{R}} \oplus \underset{t}{\mathbb{R}}$, $Y_1 = \partial/\partial x$, $Y_2 = \partial/\partial x + x^2\, \partial/\partial t$. Then $X_1 = X_2 = \partial/\partial x$, so $h = h_1 = 1$ - dim space spanned by $\partial/\partial x$. (In particular, $\alpha_0(h) \neq T_0V$.) However, $g_0 = g^1 \oplus g^2 \oplus g^3$, the rank 3 nilpotent Lie algebra generated by the Y_1, Y_2 themselves. Probably the easiest way to see this is to put a different grading on V; namely, make the t component of order 3. Then Y_1, Y_2 are both homogeneous (of degree 1) and we can apply the result of Example 6. (This is legitimate since no dilation structure on M is involved in the definition of g_x.)

The above illustrates that g_0 need not equal h in general. However,

Claim: There is a canonical surjective morphism of graded Lie algebras $g_0 \to h \to 0$. (Thus h is the quotient of g_0 by a graded ideal.)

Pf. Define γ as in Example 6. Noting, as in Example 5, that the leading term of a commutator is the commutator of the corresponding leading terms we see that γ maps L^i into and onto h^i. Clearly, γ factors through to induce a map $g_0^i \to h^i \to 0$. The above remark about the leading term of a commutator shows that this map is a Lie algebra homomorphism.

We give two cases in which the above map $g_0 \to h$ is an isomorphism (as shown by dimension arguments).

Case 1. As a Lie algebra over $\mathbb{R}$, h is the free nilpotent on k generators of rank r. (This does not imply that as a Lie algebra of vector fields on V h is free of rank r at 0.

Example: $h_1 = \frac{\partial}{\partial x}, x\frac{\partial}{\partial t}$; $h_2 = \frac{\partial}{\partial t}$. $V = \underset{x}{V_1} \oplus \underset{t}{V_2}$.)

Pf: g_0 is generated by g_0^1, which has dim $\leq$ k since L^1 has k generators. Also, g_0 has rank = r. Hence dim $g_0 \leq$ dim h, and so $g_0 \to h$ is an isomorphism.

Case 2. $Y_1,\ldots,Y_k$ satisfy the Metivier condition (to be discussed below) and, in addition, the vector fields in h span T_0V. (I don't know whether this latter assumption in fact follows from the Metivier condition.)

Pf: Metivier condition $\Rightarrow$ dim g_0 = dim V; spanning condition on h $\Rightarrow$ dim h $\geq$ dim V.

Def. 1.27: The filtered Lie algebra $\{L^i\},x_0$, of finite rank r at x_0, satisfies the Metivier condition if there is a nbhd U of x_0 such that for every i=1,...,r dim $\alpha_x(L^i)$ is independent of $x \in U$. This is a transposition to the general context of Metivier's [30] condition in the context where L^1 generates L.

The following proposition, in conjunction with the lifting theorem, leads to hypoellipticity results. It shows, in particular, that the nilpotent Lie algebras arising in Metivier [30] can be given an intrinsic formulation and generalize to the context where L^1 need not generate L.

Prop. 1.28: Suppose that $\{L^i\},x_0$ satisfies the Metivier condition. Then

(1) For all x in a nbhd of x_0, dim g_x = dim M. Furthermore $g_x = g_x^1 \oplus g_x^2 \oplus \ldots \oplus g_x^r$ where $n_x^i \equiv \dim g_x^i$ is independent of x.

(2) The g_x vary "smoothly" for x in a nbhd of x_0, and the smoothness is compatible with the projection operators $\pi_x^i : L^i \to g_x^i$. More precisely, choosing bases we can identify each g_x, as a graded vector space, with $\mathbb{R}^{n^1} \oplus \ldots \oplus \mathbb{R}^{n^r}$ in such a way that the Lie algebra operations are C^∞ with respect to x. That is, we can

regard $x \mapsto g_x$ as defining a smoothly varying family of Lie algebra structures on $\mathbb{R}^{n^1} \oplus \ldots \oplus \mathbb{R}^{n^r}$. Moreover, for any $X \varepsilon L^i$ the map $x \mapsto \pi_x^i(X) \varepsilon g_x^i$ is C^∞.

The proof will be given following some preliminaries needed as well for the lifting theorem.

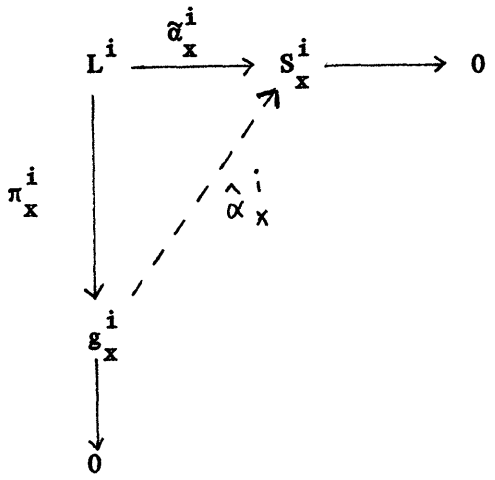
Note 1.29: Although, under the Metivier hypothesis, the g_x vary smoothly with x, the g_x need not be isomorphic as Lie algebras. In this case Cor. 1.20 shows there is no partial homomorphism from g_{x_0} to $\{L^i\}$ in a nbhd of x_0. For, by that corollary, such a homomorphism would induce Lie algebra homomorphisms, and hence isomorphisms, by equality of dimension, from g_{x_0} onto g_x for all x in a nbhd of x_0.

Let $\{L^i\},x_0$ be of rank r. Then $\alpha_x(L^1) \subset \alpha_x(L^2) \subset \ldots \subset \alpha_x(L^r) = T_xM$ forms a filtration of T_xM. Let

$$S_x^i = \frac{\alpha_x(L^i)}{\alpha_x(L^{i-1})} \qquad (1.6)$$

so $S_x = \bigoplus_{i=1}^{r} S_x^i$ defines the associated graded vector space.

Let $L^i \xrightarrow{\tilde{\alpha}_x^i} S_x^i \longrightarrow 0$ be the natural quotient of the evaluation map α_x. Clearly $\tilde{\alpha}_x^i(L^{i-1} + m_xL^i) = 0$. Passing to the germ level, $\tilde{\alpha}_x^i$ factors through to give a map $\hat{\alpha}_x^i$ filling in the diagram below:



(1.7)

<u>Def. 1.30:</u> Let $h_x^i = \ker \hat{\alpha}_x^i$, and $h_x = h_x^1 \oplus \cdots \oplus h_x^r$.

<u>Lemma 1.31</u>:

(a) $\dim g_x^i/h_x^i = \dim S_x^i$, so $\dim g_x/h_x = \dim M$.

(b) h_x is a graded subalgebra of g_x, i.e., $[h_x^i, h_x^j] \subset h_x^{i+j}$.

(c) If the Metivier condition holds at x_0, then $h_x = 0$; in fact $h_x = 0$ for all x in a nbhd of x_0. That is, $\hat{\alpha}_x^i$ is an isomorphism for all x in a nbhd of x_0.

This result in effect is an extension to the general context of a result of Helffer-Nourrigat [20]. Because of the intrinsic, minimal, nature of g_x, part (c) is sharper than their corresponding result. (They can only show h_x is an ideal).

<u>Pf:</u>

(a) obvious.

(b) The basic point is that if two vector fields both vanish at x_0, so does their commutator. Let $Y^i \in h^i_{x_0}$, $Y^j \in h^j_{x_0}$, with preimages (under π) X^i, X^j. By definition of h_{x_0}, $\exists\, X^{i-1}$, X^{j-1} in L^{i-1}, L^{j-1}, respectively, such that $\alpha_{x_0}(X^i - X^{i-1}) = 0$ and $\alpha_{x_0}(X^j - X^{j-1}) = 0$. Hence $\alpha_{x_0}([X^i - X^{i-1}, X^j - X^{j-1}]) = 0$. But the Lie bracket $= [X^i, X^j]$ + element in L^{i+j-1}. Hence $[Y^i, Y^j] = \pi^{i+j}_{x_0}([X^i, X^j]) \in h^{i+j}_{x_0}$.

(c) For $1 \leq i \leq r$ let $k_i = \dim S^i_{x_0}$. Choose vectors $e^i_j \in T_{x_0}(M)$, $1 \leq j \leq k_i$ such that $e^1_1, \ldots, e^1_k$ is a basis for $\alpha_{x_0}(L^1)$, $e^1_1, \ldots, e^1_k$; $e^2_1, \ldots, e^2_k$ is a basis for $\alpha_{x_0}(L^2)$, etc. Choose $X^i_j \in L^i$ such that $\alpha_{x_0}(X^i_j) = e^i_j$. For x sufficiently close to x_0, the vectors $\alpha_x(X^i_j) \in T_x M$ are linearly independent, since the $\alpha_{x_0}(X^i_j)$ are. This, together with the Metivier hypothesis, implies that the $\alpha_x(X^i_j)$ with $i \leq i_0$ form a <u>basis</u> for $\alpha_x(L^i)$ for all $i_0 \leq r$ (for x sufficiently close to x_0), which varies smoothly with x. In particular, in a nbhd U of x_0, L^i is the space of sections of a vector bundle with fiber $\alpha_x(L^i)$. Thus, in this nbhd, any $X \in L^i$ can be written as

$$X = \sum_{i' \leq i} f^{i'}_j X^{i'}_j, \quad \text{with uniquely determined } f^{i'}_j \in C^\infty \tag{1.8}$$

By definition of X^i_j, $\alpha_{x_0}(X) \in \alpha_{x_0}(L^{i-1}) \Longleftrightarrow f^i_j(x_0) = 0\ \forall j$. This says simply that $f^i_j \in m_{x_0}$, i.e., that $X \in L^{i-1} + m_{x_0} L^i$. That is, $h_{x_0} = 0$. The same argument shows $h_x = 0$ for any $x \in U$.

We can now prove Prop. 1.28.

<u>Pf:</u>

(1) Clear from (a) and (c) of preceding lemma, and fact that, by

Metivier hypothesis, dim S_x^i is constant for $x \in U$.

(2) Maintain the notation in (c) of preceding proof. For $x \in U$ let $Y_j^i(x) = \pi_x^i(X_j^i)$. By definition of the X_j^i and preceding argument, the $\tilde{\alpha}_x^i(X_j^i)$ form a basis for S_x^i, for each $x \in U$ and each i. Hence, since $\hat{\alpha}_x^i$ is an isomorphism, the $Y_j^i(x)$ form a basis for g_x^i. We shall show that the Lie bracket of the $Y_j^i(x)$ is smooth in x. Fix i_1, i_2, j_1, j_2. By definition of the bracket, $[Y_{j_1}^{i_1}(x), Y_{j_2}^{i_2}(x)] = \pi_x^{i_1+i_2}([X_{j_1}^{i_1}, X_{j_2}^{i_2}])$. By (1.8)

$$[X_{j_1}^{i_1}, X_{j_2}^{i_2}] = \sum_{i \leq i_1+i_2} f_j^i X_j^i ,$$

where the f_j^i are smooth.

Fix x. Then f_j^i = the constant function $f_j^i(x)$ + element in m_x. So

$$\pi_x^{i_1+i_2}([X_{j_1}^{i_1}, X_{j_2}^{i_2}]) = \sum_j f_j^{i_1+i_2}(x) Y_j^{i_1+i_2}(x) .$$

So, since f_j^i is smooth, the bracket is smooth.

Similarly, using (1.8) we see that for any $X \in L^i$, $\pi_x^i(X) = \sum f_j^i(x) Y_j^i(x)$. So, $x \mapsto \pi_x^i(X)$ is smooth with respect to the given bases.

Lemma 1.32: Let β be an arbitrary cross-section of $\{L^i\},x$. Then for every $i \geq 0$, $\alpha_x(L^i) = \alpha_x(\beta(g_x^1 \oplus \ldots \oplus g_x^i))$. $(L^0 = \{0\})$.

This follows immediately from the stronger Lemma 1.35 proved below. A simple direct argument also follows from diagram (1.7).

Our definition of h_x has been completely intrinsic. It is useful to have the following characterization, more in line with the analogous construction of Helffer-Nourrigat.

Cor. 1.33: Let β be an arbitrary cross-section of $\{L^i\},x$. Then $h_x^i = (\alpha_x \circ \beta_i)^{-1}(\alpha_x(\beta(g_x^1 \oplus \dots \oplus g_x^{i-1})))$. ($h_x^1 = (\alpha_x \circ \beta_1)^{-1}(0)$.)

Pf. Follows immediately from preceding lemma and diagram (1.7).

Cor. 1.34:

(a) $\alpha_x(\beta(g_x)) = T_xM$.

(b) The mapping $u \to e^{\beta(u)}x$ is locally a submersion from a nbhd of 0 in g_x to a nbhd of x in M.

Pf:

(a) Follows immediately from Lemma 1.32 and the spanning condition.

(b) Follows from (a) and the fact that the differential:$T_0(g_x) \to T_xM$ of the above map is given by $v \mapsto \alpha_x(\beta(v))$; here we identify g_x with its tangent space at 0.

The following consequence (Cor. 1.36) of Nakayama's Lemma will be a basic tool in our proof of the lifting theorem in §2 (and, in its microlocal variant, in our discussion of the asymptotic moment-map in §3). It is our substitute for an explicit a priori relation between g_x and a set of generators for $\{L^i\},x$. The result holds at the germ level, and this is sufficient in practice for handling the local level, especially in view of Remark 1.3.2.

Lemma 1.35: Let $\{L^i\},x$ be a filtered Lie algebra of finite rank, r. Let d = dim g_x. Let $\{Y_\alpha\}$, $1 \le \alpha \le d$ be an arbitrary graded basis for g_x i.e., a basis such that each $Y_\alpha \in g_x^i$ for some i, which we denote by $|\alpha|$. Let β be an arbitrary cross-section. Then for each $i \le r$ the germs of vector fields $\{\beta(Y_\alpha)\}_{|\alpha| \le i}$ span $\dot{L}_x^i$ as a $\dot{C}_x^\infty$ module.

Pf: By induction on i. Suppose true for i-1. Let $\{Y_\alpha\}$, $|\alpha| = i$, be a basis for

$$g_x^i = \frac{\dot{L}_i}{\dot{L}_{i-1} + \dot{m}_x \dot{L}_i} .$$

Given the cross-section β_i, let $[\beta_i(Y_\alpha)]$ denote the image in $\dot{L}_i/\dot{L}_{i-1}$ of $\beta(Y_\alpha)$. Now, by Cor. 1.12 together with Remark 1.14, it follows that $\{[\beta_i(Y_\alpha)]\}$ generates $\dot{L}_i/\dot{L}_{i-1}$ as a $\dot{C}_x^\infty$ -module. So, $\dot{L}^i$ = (span of $\{\beta(Y_\alpha)\}_{|\alpha|=i}) + \dot{L}^{i-1}$ (this not being a direct sum). But by the induction hypothesis, $\dot{L}^{i-1}$ = span of $\{\beta(Y_\alpha)\}_{|\alpha| \le i-1}$. It remains only to treat the case of i=1. But for i=1, $\dot{L}_{i-1} = 0$, and

$$\frac{\dot{L}_i}{\dot{L}_{i-1}+\dot{m}_x\dot{L}_i} = \frac{\dot{L}_1}{\dot{m}_x\dot{L}_1} ,$$

and the result follows from Cor. 1.12.

Of course, when i=r the statement follows directly from the spanning hypothesis.

Cor. 1.36: Let $X \in \dot{L}^i_x$. Then

$$X = \sum_{|\alpha|=i} c_\alpha \beta(Y_\alpha) + \sum_{|\alpha|\leq i} f_\alpha \beta(Y_\alpha) + \sum_{|\alpha|=i} g_\alpha \beta(Y_\alpha) \tag{1.8}$$

where the $c_\alpha \in R$ are uniquely determined by the equation

$$\pi_i(X) = \sum_{|\alpha|=i} c_\alpha Y_\alpha , \tag{1.9}$$

where the f_α are in $\dot{C}^\infty_x$, and the g_α in $\dot{m}_x$. (The f_α and g_α are not necessarily uniquely determined).

Pf. By the Lemma,

$$X = \sum_{|\alpha|<i} f_\alpha \beta(Y_\alpha) + \sum_{|\alpha|=i} h_\alpha \beta(Y_\alpha),$$

with f_α, h_α in $\dot{C}^\infty_x$. For $|\alpha| = i$, let $c_\alpha = h_\alpha(x)$ and $g_\alpha = h_\alpha - c_\alpha$. This gives a representation (1.8). Applying π_i to both sides of (1.8) yields (1.9).

<u>Remarks 1.37</u>:

1) Most of the work in this section has been based on elementary local algebra considerations, and so (see Remark 1.1.1) carries over to the real-analytic and formal power series contexts. In particular, we can define $g_x^{analytic}$, g_x^{formal}, and the corresponding h_x's. If L^{formal} is the "formalization" of the C^∞ filtered algebra L, or if L is the "C^∞-version" of the real-analytic filtered algebra $L^{analytic}$, then variants of Prop. 1.19 and 1.24 show that there are, respectively, canonically defined surjective Lie algebra morphisms

$$g_x \to g_x^{formal} \to 0 \tag{1.10}$$

$$g_x^{analytic} \to g_x \to 0$$

mapping h_x onto h_x^{formal}, $h_x^{analytic}$ onto h_x, and, hence, canonical isomorphisms of homogeneous spaces

$$\frac{g_x^{\text{analytic}}}{h_x^{\text{analytic}}} \cong \frac{g_x}{h_x} \cong \frac{g_x^{\text{formal}}}{h_x^{\text{formal}}}, \quad \text{all of dimension} = \dim M. \tag{1.11}$$

I do not know in general when the maps in (1.10) are isomorphisms; however, this is easily seen to hold if L satisfies the Metivier condition.

2) Suppose that the strong finite generation condition (5 local) of Remark 1.3.2 is satisfied at x_0. Then Remark 1.3.2, together with Lemma 1.35, shows that any cross-section β at x_0 determines, for each x sufficiently close to x_0, a graded $\mathbb{R}$-linear surjective map $g_{x_0} \to g_x$ via $Y \varepsilon g_{x_0}^i \longmapsto \pi_x^i(\beta(Y))$. This map is in general not a morphism of Lie algebras (since $Y \mapsto \beta(Y)_x$ is not necessarily a weak homomorphism except when $x=x_0$). However, as x approaches x_0 the "deviation" from a Lie algebra morphism approaches 0. (Compare with the proof of Prop. 1.28).

§2. The Local Lifting Theorem

In this section we prove a version of the Rothschild-Stein lifting theorem, based on the treatment of Goodman [8], and follow this by a proof of the corresponding homogeneous space (rather than group) version of Helffer-Nourrigat [20].

Goodman has observed that, simply for purposes of lifting, it is not necessary to insist on a free nilpotent group. We carry this idea further in showing that such a lifting can be carried out in the general context of filtered Lie algebras via the intrinsically associated nilpotent Lie algebras. The lifting results give a precise sense in which these nilpotent Lie algebras "approximate" the original filtered Lie algebras. This is of interest since, in view of Prop. 1.19, these algebras are in some sense "minimal" approximants.

Direct applications are to hypoellipticity, as we shall show in this section, and, possibly, to approximation of control systems, as we shall indicate in §4.

One significant fact is that weak (vs. partial) homomorphisms, which are all that we have available, are sufficient. One consequence is that we can do a direct lifting in the Metivier case.

Although the main line of the argument is very close to that of Goodman, there are differences due to dealing with weak homomorphisms, among them an increased complexity of "bookkeeping". To save space we shall not give full details.

Let $g = g_1 \oplus \dots \oplus g_r$ be a graded nilpotent Lie algebra. Then the natural dilations δ_t $(t>0)$ given by $\delta_t \restriction g_i = t^i$ are Lie algebra automorphisms, and determine associated Lie group automorphisms. The discussion of homogeneity with respect to dilations and of local order $\leq i$ in Example 1.26.5 carries over fully to the present context.

For $Y \varepsilon g$, let $\tilde{Y}$ denote the pull-back via the exponential map $\exp: g \to G$ (the associated simply-connected nilpotent Lie group) of the left-invariant vector field on G associated to Y. (More loosely, $\tilde{Y}$ is the left-invariant vector field associated to Y, written in exponential coordinates).

$$\tilde{Y}(f \circ \exp)(u) = \frac{d}{dt}\Big|_{t=0} f(\exp u \exp tY), \qquad f \varepsilon C^{\infty}(G) \tag{2.1}$$

If $Y \varepsilon g_i$ then $\tilde{Y}$, viewed as a vector field on g, is homogeneous of degree i. That is, homogeneity as an element of the Lie algebra g, or, more generally, as an element of U(g), the enveloping algebra, is consistent with the notion of homogeneity as a differential operator on a graded vector space.

<u>Notation 2.1:</u> $C_m^{\infty}(U)$ is the set of C^{∞} functions vanishing of order $\geq m$ at $0 \varepsilon g$, in the sense of Example 1.26.5. (U is a nbhd of 0 in g). $C_m^{\infty}(U) = C^{\infty}(U)$ if $m \leq 0$.

Note that

$$C_m^{\infty} \cdot C_m^{\infty} \subset C_{m+n}^{\infty} \tag{2.2}$$

If $f \varepsilon C_m^\infty$ and X is a vector field of local order $\leq$ j at 0, then fX is of local order $\leq$ j-m. (2.3)

(Of course this statement is useful only when $m \geq 0$.)

$$m_0 = C_1^\infty; \text{ more generally, } C_{rN}^\infty \subset m_0^N \subset C_N^\infty \quad . \tag{2.4}$$

(We often use the inclusion $m_0 \subset C_1^\infty$.)

Let $\{L^i\}, x_0$ be a filtered Lie algebra of rank r on the manifold M. Let β be an arbitrary cross-section. In analogy with Goodman, define a map $W: C_{x_0}^\infty(M) \to C_0^\infty(g_{x_0})$ (really from a nbhd of x_0 in M to a nbhd of 0 in g_{x_0}) via

$$(Wf)(u) = f(e^{\beta(u)}x_0) \quad . \tag{2.5}$$

Notice that for any vector field X on M and $f \varepsilon C_{x_0}^\infty(M)$

$$W(fX) = W(f)\cdot WX \; ; \quad \text{if } f \varepsilon m_{x_0}^k \text{ then } W(f) \varepsilon m_0^k \subset C_k^\infty \quad . \tag{2.6}$$

The theorem below states that W is a "weak intertwining" between the elements of $\{L^i\}$ and the elements of g_{x_0}, viewed as left-invariant vector fields.

<u>Theorem 2.2:</u> (Lifting Theorem). Let $Y \in g^i_{x_0}$ and let $X = \beta(Y)$. Then $WX = (\tilde{Y}+R)W$, where R is a C^∞ vector field in a nbhd of 0 in g_{x_0} which is of local order $\lesssim$ i-1 at 0.

<u>Remarks 2.3</u>:

1) Since in general the map $u \rightarrow e^{\beta(u)}x_0$ is not a diffeomorphism, but only a submersion (see Cor. 1.34),the vector fields R are not uniquely determined.

2) Each R can be expressed as a C^∞ linear combination of the frames $\tilde{Y}$. It then follows directly from the homogeneity degrees of the R's that the span at 0 of the vector fields $\tilde{Y}$+R is the same as that of the $\tilde{Y}$, i.e., all of $T_0 g_{x_0}$.

3) Although g_{x_0} is in some sense the minimal algebra to which one can lift, the same proof holds if we replace g_{x_0} by any graded nilpotent g with a weak homomorphism γ from g to L at x_0 such that the associated Lie algebra homomorphism $\pi \;\; \gamma: g \rightarrow g_{x_0}$ (see Prop. 1.19) is surjective; (alternately, for any graded nilpotent g together with a surjective homomorphism to g_{x_0}). This follows from the fact that Lemma 1.35 and its corollary, which are basic ingredients in the proof of the lifting theorem, hold with β replaced by γ. This is seen from a trivial argument with the diagram in Prop. 1.19.

The lifting theorem has the following corollary.

<u>Cor. 2.4:</u> Let $X \in L^i$. Then $WX = (\pi_i(X) + S)W$ where S is of local order $\lesssim$ i-1 at 0.

Notice that although W depends on β the element $\pi_i(X) \in g^i_{x_0}$ is intrinsic.

<u>Pf:</u> Apply the lifting theorem, and then use Cor. 1.36, together with (2.6).

We pass next to the proof of the lifting theorem, following Goodman. One begins with the following identities between formal series in an associative algebra (X,Y,Z being elements of the algebra, and D_X = ad X: $Y \mapsto XY-YX$)

$$\left.\frac{d}{dt}\right|_{t=0} e^{X+tY} = e^X E(X)Y \tag{2.7}$$

$$e^X Y = \left.\frac{d}{dt}\right|_{t=0} e^{X+tB(X)Y} \tag{2.8}$$

$$\left.\frac{d}{dt}\right|_{t=0} e^{X+tY+tZ} = \left.\frac{d}{dt}\right|_{t=0} e^{X+tY} + \left.\frac{d}{dt}\right|_{t=0} e^{X+tZ} \tag{2.9}$$

where

$$E(X) = \frac{1-e^{-D_X}}{D_X} = \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} D_X^k \tag{2.10}$$

$$B(X) = \frac{D_X}{1-e^{-D_X}} = \sum_{k\geq 0} \frac{1}{k!} b_k D_X^k \quad (b_k = \text{k-th Bernoulli number}). \tag{2.11}$$

The identities (2.7)-(2.9) are to be interpreted in finite terms, as graded identities, via the symmetrization operator σ, given by

$$\sigma(X^nY) = \frac{1}{n+1}(X^nY + X^{n-1}YX + \ldots + YX^n)$$

For example, (2.7) is equivalent to

$$\frac{1}{n!}\sigma(X^nY) = \sum_{k+m=n} \frac{(-1)^k}{(k+1)!m!} X^m D_X^k(Y), \quad n=0,1,2,\ldots \tag{2.7g}$$

We will be applying these in the context of two associative algebras, that determined by the Lie algebra of vector fields on M in a nbhd of x_0, and that determined by the Lie algebra of vector fields on g_{x_o} in a nbhd of 0.

From (2.8) it follows that, for $Y\varepsilon g_{x_o}$, the curves in G_{x_o} $t \mapsto \exp u \exp tY$ and $t \mapsto \exp(u+tB(u)Y)$ have the same tangent vector at t=0, so

$$\tilde{Y}f(u) = \frac{d}{dt}\bigg|_{t=0} f(u+tB(u)Y). \tag{2.12}$$

That is, at u $\tilde{Y}$ is the directional derivative in the direction B(u)Y. (Since g_{x_0} is nilpotent the series for B(u) terminates after finitely many terms, so that B(u)Y is polynomial in u.)

Next one works at the level of <u>formal power series at u=0</u>. That is, one constructs R such that Thm. 2.2 (1) holds as an equality of Taylor series at u=0.

<u>Notation:</u> For $\phi \in C^\infty(g_{x_0})$ defined in a nbhd of 0, and $\phi_n \in C_n^\infty$

$$\phi \sim \sum_n \phi_n \text{ means that } \phi - \sum_{k\leq m} \phi_k \in C_{m+1}^\infty \quad \text{for every m.} \tag{2.13}$$

Since β is $\mathbb{R}$-linear it follows that

$$Wf(u) \sim \sum_{n\geq 0} \frac{1}{n!} (\beta(u)^n f)(x_0), \quad \text{for any } f \in C_{x_0}^\infty(M) . \tag{2.14}$$

We may express this by saying that, <u>formally</u>, $W = e^{\beta(u)}$. (As in Lemma 1.35 let $\{Y_\alpha\}$, $1\leq\alpha\leq d$ be a graded basis for g_{x_0}, with $\{u_\alpha\}$ the corresponding dual basis. Then, by the $\mathbb{R}$-linearity of β, (2.14) gives the Taylor coefficients at 0 of Wf, with respect to the coordinates u_α, in terms of the Taylor coefficients of f at x_0.)

Then, by (2.8), for every $1 \leq \alpha \leq d$

$$W\beta(Y_\alpha) = \frac{d}{dt}\bigg|_{t=0} e^{\beta(u)+tB(\beta(u))\beta(Y_\alpha)} \tag{2.15}$$

Examine the right-hand side of (2.15). <u>If β were a homomorphism of Lie algebras over $\mathbb{R}$</u> and, hence, of associative algebras, then as formal series $\beta(u) + tB(\beta(u))\beta(Y_\alpha)$ would equal $\beta(u+tB(u)Y_\alpha)$. But replacing f by Wf in (2.12) and using (2.14) we get

$$(\tilde{Y}Wf)(u) \sim \frac{d}{dt}\bigg|_{t=0} (e^{\beta(u+tB(u)Y)}f)\Big|_{x_0}, \quad \text{where } \sim \text{ denotes equality of Taylor series at } u=0. \tag{2.16}$$

Thus, if β were a Lie algebra homomorphism, we would have $W\beta(Y_\alpha) = \tilde{Y}_\alpha W$ formally.

The crux of our work then consists of showing that with β "close enough" to a homomorphism we can get good control of the difference between $\beta(B(u)Y_\alpha)$ and $B(\beta(u))\beta(Y_\alpha)$.

We start with the basic weak homomorphism equation from Def. 1.15 which we write in the more convenient form

$$[\beta(Y_\gamma),\ \beta(Y_\alpha)] = \beta([Y_\gamma,\ Y_\alpha]) + \dot{L}^{|\gamma|+|\alpha|-1} + \dot{m}_{x_0} L^{|\gamma|+|\alpha|} \tag{2.17}$$

for Y_γ, Y_α ε $g_{x_0}^{|\gamma|}$, $g_{x_0}^{|\alpha|}$, respectively.

where $|\alpha|$ (as in Lemma 1.35) denotes the weight of Y_α.

Write

$$u = \sum_{|\gamma|\leq r} u_\gamma Y_\gamma \ .$$

Following Goodman, let $K = \{k(\gamma)\}_{|\gamma|\leq r}$ denote a multi-exponent, $|K| = \sum k(\gamma)$, the usual length of K, and $w(K) = \sum k(\gamma)|\gamma|$ the weight of K.

Let

$$D_\gamma = \text{ad } Y_\gamma \qquad u^K = \prod_\gamma u_\gamma^{k(\gamma)}$$

$$\hat{D}_\gamma = \text{ad } \beta(Y_\gamma) \qquad D^K = \prod_\gamma D_\gamma^{k(\gamma)} \qquad ,\ b_K = \frac{1}{K!} b_{|K|},$$

$$\hat{D}^K = \prod_\gamma \hat{D}_\gamma^{k(\gamma)}$$

and let σ, as before, be the symmetrization operator. Note that

$$u^K \varepsilon C^\infty_{W(K)} \tag{2.18}$$

By definition of B, and $\mathbb{R}$-linearity of β we have

$$B(u) = \sum_{|K|\geq 0} b_K u^K \sigma(D^K)$$

$$B(\beta(u)) = \sum_{|K|\geq 0} b_K u^K \sigma(\hat{D}^K) \tag{2.19}$$

An induction on $|K|$, starting from the equation (2.17), proves

<u>Lemma 2.5:</u>

$$\hat{D}^K(\beta(Y_\alpha)) = \beta(D^K(Y_\alpha)) + \dot{L}^{|\alpha|+w(K)-1} + \dot{m}_{x_0}\dot{L}^{|\alpha|+w(K)}$$

(Of course, for K=0 we don't need the two error terms on the right-hand side).

<u>Cor. 2.6:</u> $B(\beta(u))\beta(Y_\alpha) \sim \beta(B(u)Y_\alpha) + T_\alpha(u) + S_\alpha(u)$, where

$$T_\alpha(u) \sim \sum_{|K|\geq 1} b_K u^K \dot{L}^{|\alpha|+w(K)-1}$$

$$S_\alpha(u) \sim \sum_{|K|\geq 1} b_K u^K (\dot{m}_{x_0} \dot{L}^{|\alpha|+w(K)})$$

(Notice, $|K|\geq 1$ for all the above terms.)

Substituting this into (2.15), and using (2.9) we get

$$W\beta(Y_\alpha) = \frac{d}{dt}\bigg|_{t=0} e^{\beta(u)+t\beta(B(u)Y_\alpha)} + \frac{d}{dt}\bigg|_{t=0} e^{\beta(u)+tT_\alpha(u)} + \frac{d}{dt}\bigg|_{t=0} e^{\beta(u)+tS_\alpha(u)} \tag{2.20}$$

We saw in (2.16) that the first term is $\tilde{Y}_\alpha W$. Using (2.7) we see that the remaining two terms are given by

$$\frac{d}{dt}\bigg|_{t=0} e^{\beta(u)+tT_\alpha(u)} = WE(\beta(u))T_\alpha(u)$$
$$\frac{d}{dt}\bigg|_{t=0} e^{\beta(u)+tS_\alpha(u)} = WE(\beta(u))S_\alpha(u) \tag{2.21}$$

As in (2.19) we can express $E(\beta(u))$ in terms of the dual basis $\{u_\gamma\}$.

$$E(\beta(u)) = \sum_{|K|\geq 0} \frac{(-1)^K}{(K+1)!} u^K \sigma(D^K) \tag{2.22}$$

But $\beta(Y_\gamma) \varepsilon L^{|\gamma|}$, and each $\beta(Y_\gamma)$ has a coefficient of u_γ to accompany it in (2.22). Thus a simple computation shows that

$$\begin{aligned} E(\beta(u))T_\alpha(u) &\sim \sum_{|K|\geq 1} c_K u^K \dot{L}^{|\alpha|+w(K)-1} \\ E(\beta(u))S_\alpha(u) &\sim \sum_{|K|\geq 1} a_K u^K \dot{L}^{|\alpha|+w(K)-1} + \sum_{|K|\geq 1} d_K u^K (\dot{m}_{x_0} \dot{L}^{|\alpha|+w(K)}), \end{aligned} \tag{2.23}$$

where the c_K, a_K, d_K are (universal) constants.

Now use Lemma 1.35, together with the fact that, since x_0 is of rank r, $\dot{L}^r_{x_0} = \dot{L}^{r+1}_{x_0} = \ldots$. We express this as follows:

Let $X \varepsilon \dot{L}^i_{x_0}$.

$$\begin{aligned} &\text{If } i<r \quad X = \sum_{|\alpha|\leq i} f_\alpha \beta(Y_\alpha) \quad , \quad f_\alpha \varepsilon \dot{C}^\infty_{x_0}(M) \\ &\text{If } i\geq r \quad X = \sum_{|\alpha|\leq r} f_\alpha \beta(Y_\alpha) \quad , \quad f_\alpha \varepsilon \dot{C}^\infty_{x_0}(M) \end{aligned} \tag{2.24}$$

In the summations in (2.23) it will be more convenient to use w(K) instead of $|K|$ as the index of summation. (Note that $w(K) \geq 1 \Longleftrightarrow |K| \geq 1$). Now use (2.24) in conjunction with (2.23). The first summation gives

$$E(\beta(u))T_\alpha(u) \sim \sum_{w(K)=1}^{r-|\alpha|} u^K\Big(\sum_{|\gamma|\le|\alpha|+w(K)-1} f_{\alpha\gamma K}\beta(Y_\gamma)\Big) + \sum_{u(K)\ge r-|\alpha|+1} u^K\Big(\sum_{|\gamma|\le r} g_{\alpha\gamma K}\beta(Y_\gamma)\Big) \tag{2.25}$$

where $f_{\alpha\gamma K}$, $g_{\alpha\gamma K}$ lie in $\dot{C}^\infty_{x_0}$.

Next apply W. Letting

$$\theta^{(n)}_{\alpha\gamma} = \begin{cases} \sum_{w(K)=n} u^K W(f_{\alpha\gamma K}) & \text{if } 1\le n\le r-|\alpha| \\ \sum_{w(K)=n} u^K W(g_{\alpha\gamma K}) & \text{if } n>r-|\alpha| \end{cases} \tag{2.26}$$

we get

$$WE(\beta(u))T_\alpha(u) \sim \sum_{\substack{|\gamma|\le r \\ n\ge 1}} \theta^{(n)}_{\alpha\gamma} W\beta(Y_\gamma) \tag{2.27}$$

Since $u^K \varepsilon C^\infty_{w(K)}$ it follows that

$$\theta^{(n)}_{\alpha\gamma} \varepsilon C^{\infty}_{n} \quad \text{for every } n \geq 1. \tag{2.28}$$

Also, a close examination of the indices appearing in the preceding derivation gives

$$\theta^{(n)}_{\alpha\gamma} = 0 \text{ unless} \begin{cases} 1 \leq n \leq r-|\alpha| \text{ and } |\gamma| \leq |\alpha|+n-1 \\ \\ \text{or } n > r-|\alpha| \end{cases} \tag{2.29}$$

Since $|\gamma| \leq r$, in either alternative in (2.29) $n \geq |\gamma|-|\alpha|+1$. We conclude from (2.28) and (2.29) that

$$\theta^{(n)}_{\alpha\gamma} \varepsilon C^{\infty}_{n} \quad C^{\infty}_{|\gamma|-|\alpha|+1} \quad \text{for all } \alpha, \gamma, \text{ and all } n \geq 1. \tag{2.30}$$

A similar analysis is done with the second sum in the second identity in (2.23). Using the fact, noted in (2.6), that if $f \varepsilon \mathring{m}_{x_0}$ then $W(f) \varepsilon \mathring{m}_0 \subset \dot{C}^{\infty}_{1}$, we find that

$$W\Big(\sum_{w(K)\geq 1} d_K u^K(\mathring{m}_{x_0} L^{|\alpha|+w(K)})\Big) = \sum_{\substack{|\gamma| \leq r \\ n \geq 1}} \Phi^{(n)}_{\alpha\gamma} W\beta(Y_\alpha) \tag{2.31}$$

where

$$\Phi_{\alpha\gamma}^{(n)} \ \varepsilon\ C_{n+1}^{\infty} \quad C_{|\gamma|-|\alpha|+1}^{\infty} \qquad \text{for all } \alpha,\ \gamma, \text{ and all } n \geq 1. \tag{2.32}$$

Let

$$\phi_{\alpha\gamma}^{(n)} = \theta_{\alpha\gamma}^{(n)} + \Phi_{\alpha\gamma}^{(n)} \ .$$

Adding the results of (2.27) and (2.31) we obtain from (2.20) and (2.21) that

$$W\beta(Y_\alpha) \sim \tilde{Y}_\alpha W + \sum_{\substack{n\geq 1 \\ |\gamma|\leq r}} \phi_{\alpha\gamma}^{(n)} W\beta(Y_\gamma), \quad \text{where } \phi_{\alpha\gamma}^{(n)} \ \varepsilon\ C_n^{\infty} \quad C_{|\gamma|-|\alpha|+1}^{\infty} \tag{2.33}$$

The identity (2.33) is of the basic form introduced by Goodman, but the condition on $\phi_{\alpha\gamma}^{(n)}$ is more delicate than that arising in his treatment. Let ϕ_n denote the matrix $(\phi_{\alpha\gamma}^{(n)})$, and let $\tilde{Y}$, $\beta(\tilde{Y})$, $W\beta(\tilde{Y})$ denote the respective column vectors $(\tilde{Y}_\alpha)$, $(\beta(Y_\alpha))$, $(W\beta(Y_\alpha))$.

Since ϕ_n has its entries in C_n^{∞}, i.e. , of successively higher degree, the formal series $S = \sum \phi_n$ converges asymptotically. Since $n \geq 1$, S vanishes to order ≥ 1 at u=0, so the geometric series $T = \sum_{n\geq 1} S^n$ converges asymptotically (and vanishes to order ≥ 1 at u=0). Next use the more delicate condition on $\phi_{\alpha\gamma}^{(n)}$. This implies that $S_{\alpha\gamma}$ is in $C_{|\gamma|-|\alpha|+1}^{\infty}$, and hence that $S_{\alpha\gamma}^n$, the $\alpha\gamma$ entry of S^n, lies in $C_{|\gamma|-|\alpha|+n}^{\infty}$.

Thus

$$T_{\alpha\gamma} \in C^{\infty}_{|\gamma|-|\alpha|+1} \tag{2.34}$$

and, since I+T is the formal inverse of (I−S), we obtain from (2.33) that

$$W\beta(Y) = (I+T)\tilde{Y}W, \quad \text{as formal series} . \tag{2.35}$$

Notice that (2.34) and (2.35) prove the formal series version of the lifting theorem.

Remarks 2.7:

1) Thus far we have not really needed the spanning condition $\alpha_{x_0}(L^r) = T_{x_0}M$, but only the stability condition $\dot{L}^r_{x_0} = \dot{L}^{r+1}_{x_0} = \ldots$. It is only when we pass to the C^∞ rather than the formal level that we need the stronger condition, so that we can apply the implicit function theorem.

2) This work at the formal level should not be confused with working with $g^{\text{formal}}_{x_0}$ (see Remark 1.37), which can be a strictly smaller-dimensional Lie algebra than g_{x_0}. For example, two elements in L^i formally equivalent at x_0 may nevertheless have distinct projections in $g^i_{x_0}$ and hence, by Cor. 2.4, distinct lifts. For some purposes we may wish to lift to $g^{\text{formal}}_{x_0}$ (see for example Remark 4.3.3). However, for most purposes of analysis we must retain information at the germ rather than formal series level. For example, we may need to lift at all points x in a nbhd of x_0 while maintaining smoothness in x. As another example, $\Gamma_{(x,\xi)}$, to be discussed in §3 (and its local analogue Γ_x) do not appear definable at the purely formal

series level.

The passage from the formal series level to the C^∞ level is now exactly the same as in Goodman: First use Borel's theorem to find a matrix, also denoted T, of genuine C^∞ functions in a nbhd of u=0 having the original matrix T as its formal power series expansion. Thus (2.35) is replaced by the corresponding C^∞ equality, but with an error term in $C^\infty_\infty = \bigcap_{n\geq 0} C^\infty_n$. (Notice that C^∞_∞ is invariant under arbitrary diffeomorphisms, as needed for the remainder of the argument.). By the spanning condition (see Cor. 1.34) it follows that the map $u \to e^{\beta(u)}x_0$ is a submersion in a nbhd of u=0, so one can apply the C^∞ implicit function theorem to find local coordinates $t_1,\dots,t_d$ in a nbhd of u=0 such that $t_1,\dots,t_m$ are local coordinates in a nbhd of x_0, and such that the above map takes the form of projection $(t_1,\dots,t_d) \to (t_1,\dots,t_m)$. Thus $(Wf)(t_1,\dots,t_d) = f(t_1,\dots,t_m)$. (This, coincidentally, gives additional sense to the term "lifting".). From this one easily sees that the error term can be written as QW, where Q is a column of vector fields of local order $-\infty$. Taking R = TY+Q proves the theorem.

<u>Remark 2.8:</u> A corresponding lifting theorem holds in the real-analytic context. Start from the equation $WX = (\tilde{Y}+R)W$ which holds in the C^∞ sense. But now W and X are real-analytic; and $\tilde{Y}$, being a left-invariant vector field on g_{x_0} is real-analytic (in fact, polynomial, as we saw in (2.12), since g_{x_0} is nilpotent). Thus RW is real-analytic, even though R is only C^∞. Using the real-analytic version of the implicit function theorem to

express W as a projection, and using Taylor series truncation, it is easy to find R' real analytic such that $R'W = RW$, and with the correct local order at $u=0$.

We next introduce the "enveloping" algebra "U(L)".

Def. 2.9: Let m be a non-negative integer. Then "$U^m(L)$" is the vector space of all differential operators of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) X_{\alpha_1} \cdots X_{\alpha_j} ,$$

where $X_{\alpha_i} \in L^{\alpha_i}$, $a_\alpha(x) \in C^\infty_{\mathbb{C}}(M)$ (the complex-valued C^∞ functions), and $|\alpha| = \alpha_1 + \ldots + \alpha_j$.

Notes 2.10:

1) "U(L)" is not the same as the enveloping algebra $U_{\mathbb{C}}(L)$ in the algebraic sense; it is, rather, the image of $C^\infty_{\mathbb{C}}(M) \otimes_{\mathbb{R}} U(L)$ under the natural map into the differential operators on M.

2) The representation of P in the above form is not unique.

For g graded nilpotent let $U_m(g)$ denote the elements in $U_{\mathbb{C}}(g)$ homogeneous of degree m. Given $P \in$ "$U^m(L)$" and $x_0 \in M$ we would like to be able to intrinsically assign to P the element $\hat{P}_{x_0} \in U_m(g_{x_0})$ given by

$$\hat{P}_{x_0} = \sum_{|\alpha|=m} a_\alpha(x_0)\hat{X}_{\alpha_1}\cdots\hat{X}_{\alpha_j}, \quad \text{where } \hat{X}_{\alpha_i} = \pi_{x_0}^{\alpha_i}(X_{\alpha_i}) \in g_{x_0}^{\alpha_i} \quad . \tag{2.36}$$

In general, in view of the non-uniqueness of representation, $\hat{P}_{x_0}$ is not well-defined as the example below shows. (This is quite natural, since in general dim M < dim g_{x_0}). However, as we shall discuss in §3, $\pi(\hat{P}_{x_0})$ is well-defined, where π is any unitary irreducible representation of G_{x_0} associated to a coadjoint orbit in Γ_{x_0}.

<u>Example 2.11:</u> Let

$$M = \mathbb{R}^1_t + \mathbb{R}^2_{x_1,x_x} \quad , \quad X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial x_1}, \quad X_3 = t\frac{\partial}{\partial x_2}, \quad X_4 = \frac{\partial}{\partial x_1}, \quad X_5 = \frac{\partial}{\partial x_2} \quad .$$

Let h_1 be the span of X_1, X_2, X_3, and h_2 the span of X_4, X_5. Then the graded Lie algebra $h = h_1 + h_2$ determines a filtered Lie algebra as in Example 1.26.6. Note that $X_2X_5 - X_3X_4 = 0$ as an element of "$U^3(L)$", though $\neq 0$ as an element of $U_3(h)$.

If the Metivier condition of Def. 1.27 holds then since dim g_x = dim M, the map $u \to \exp\beta(u)x$ from g_x to M is a local diffeomorphism, and not just a submersion; so the associated map W is just pull-back with respect to this diffeomorphism. In particular it follows from Cor. 2.4 that $WPW^{-1} = \tilde{\hat{P}}_{x_0} + S$, where S is a differential operator of local order $\leq$ m-1. Thus $\hat{P}_{x_0}$ is well-defined.

Maintain the Metivier condition. We want to complete part (2) of Prop.

1.28 by showing that this local diffeomorphism can be arranged to vary smoothly with x.

Lemma 2.12: Suppose the filtered Lie algebra $\{L^i\},x_0$ satisfies the Metivier condition. Then for x in a nbhd of x_0 it is possible to choose the cross-section $\beta_x:g_x \to L$ in such a way that the map $(x,u) \mapsto \exp \beta_x(u)x$ is smooth simultaneously in x and u.

Pf. We keep the notation from the proof of Prop. 1.28.(2). Since the $Y^i_j(x)$ form a basis for g^i_x, we can define a cross-section by $\beta_x(Y^i_j(x)) = X^i_j$. Thus the map $(x,u) \mapsto \exp \beta_x(u)x$ becomes simply the map $(x,u) \mapsto \exp(\sum u_{ij}X^i_j(x))x$. But the vector fields X^i_j are C^∞, so we are done.

Cor. 2.13: Choose β as above, and W correspondingly. Then the "remainder" terms R and S in Thm. 2.2 and Cor. 2.3 vary smoothly with x in a nbhd of x_0 (simultaneously with smoothness in u).

Pf: (In the case of Thm. 2.2 we are, of course, assuming Y is chosen to vary smoothly with x). $W_x\beta(Y_x)W_x^{-1} = \tilde{Y}_x + R$, and $W_xXW_x^{-1} = \widetilde{\pi^i_x(X)} + S$. Since everything else is smooth, so are R and S.

The type of smoothness in x occurring in Prop. 1.28 and in the above results seems essential for applying the techniques of Rothschild-Stein [37]. It is possible that in our context, by elaborating on the observation in Remark 1.37.2 one may avoid lifting to a free nilpotent, but lift instead to a Metivier context. (In any case, one expects all hypoellipticity information to involve only the Γ_x; in particular, only the g_x.)

In the context of a filtered Lie algebra L one can define a natural notion of hypoellipticity (which we shall also wish to use in §3):

<u>Def. 2.14:</u> Let P be as element of "$U^m(L)$". Then P is <u>L-hypoelliptic</u> at x_0 if there is an open nbhd U of x_0 such that for every Q ε "$U^m(L)$" there exists a constant $C_Q > 0$ such that

$$||Qf||^2_{L^2(U)} \leq C_Q(||Pf||^2_{L^2(U)} + ||f||^2_{L^2(U)}) \qquad \forall f \in C_0^\infty(U) . \qquad (2.37)$$

This is, of course, just the analogue of the maximal hypoellipticity notion of Helffer-Nourrigat ([19], [21]); in fact, from one vantage point it is simply maximal hypoellipticity in the context where there can be generators of degree not equal to 1. We feel, however, that this notion is viewed most naturally in the context of the filtration L.

<u>Remarks 2.15:</u>

1) In the preceding definition we assume to be on the safe side, that the strong finite generation condition (5 local) of Remark 1.3.2 is satisfied at x_0.

2) We are assuming also that x_0 is of finite rank r. Also, if L^1 does not generate L, we assume that m is an integer multiple of the least common multiple of 1,2,...,r. Then there seems to be no obstacle to extending the

arguments in [17], [19] (based on Thm. 17 and Lemma 18.2 of [37]) to show that L-hypoellipticity at $x_0 \rightarrow$ hypoellipticity in a nbhd of x_0. (The condition on m, while necessary in general, is quite harmless.)

Rothschild ([36]), using the nilpotent algebras constructed by Metivier in [30], derives a sufficiency criterion for the hypoellipticity of differential operators constructed from vector fields. (The necessity of her condition, for maximal hypoellipticity, follows from Helffer-Nourrigat [20]). Using our Prop. 1.28 and Cor. 2.13 in place of the Metivier construction, the proof seems to carry over, essentially unmodified, to the context where L^1 need not generate. We state this as

<u>Prop. 2.16:</u> Let $\{L^i\}, x_0$ satisfy the Metivier condition (and the conditions in Remarks 2.15). Then

$P \in$ "$U^m(L)$" is L-hypoelliptic at $x_0 \longleftrightarrow \pi(\hat{P}_{x_0})$ is left-invertible for every non-trivial unitary irreducible representation π of G_x .

Helffer and Nourrigat [20], motivated in part by Folland [27] (see also Example 1.26.6) prove that in Goodman's original lifting context (L^1 generating, and with a partial homomorphism) one can obtain an actual local diffeomorphism rather than simply a lifting, by passing to a suitable right homogeneous space of the group and the corresponding induced (by the identity) representation instead of the right regular representation (as in the lifting theorem).

The corresponding result holds in our context, starting from our version of the lifting theorem. The homogeneous space in question is that associated to the intrinsically constructed graded subalgebra $h_{x_0} = h^1_{x_0} \oplus .. \oplus h^r_{x_0}$, $h^i_{x_0}$ being ker $\hat{\alpha}^i_{x_0}$, of Def. 1.30. Aside from technical modifications of the type needed in our proof of the lifting theorem, the argument is essentially that of Helffer and Nourrigat. We shall therefore limit ourselves to a statement of the result, and omit the proof.

Let G_{x_0}/H_{x_0} denote the right cosets of H_{x_0}. Then right translation by G_{x_0} determines a representation of G_{x_0} on $L^2(G_{x_0}/H_{x_0})$ which, at the Lie algebra level, maps $Y \varepsilon g$ to its push-forward vector field (well-defined) under the canonical projection $\pi: G_{x_0} \rightarrow G_{x_0}/H_{x_0}$ (equivalently, to the vector field on G_{x_0}/H_{x_0} associated to the right action of G_{x_0} on G_{x_0}/H_{x_0}). This turns out to be $\pi_{(0,h_{x_0})}$, the unitary representation of G_{x_0} induced by the 1-dimensional identity representation of H_{x_0} . (In terms of our earlier notation, $\tilde{Y} = \pi_{(0,0)}(Y)$.)

Following Helffer-Nourrigat, we introduce a concrete realization of $\pi_{(0,h_{x_0})}$. Choose a supplement $V^i_{x_0}$ to $h^i_{x_0}$ in $g^i_{x_0}$, and let

$$V_{x_0} = \bigoplus_{i=1}^{r} V^i_{x_0} .$$

By Lemma 1.31 dim V_{x_0} = dim M. For $u \varepsilon V_{x_0}$ let u_i, $1 \leq i \leq r$, denote its projection in $V^i_{x_0}$. Then there exists a map $\gamma: V_{x_0} \rightarrow g_{x_0}$ defined by

$$e^{u_r}e^{u_{r-1}}\ldots e^{u_1} = e^{\gamma(u)}, \qquad u\varepsilon V_x \tag{2.38}$$

Define $h(u,a):V_{x_0} \times g_{x_0} \to h_{x_0}$ and $\sigma(u,a):V_{x_0} \times g_{x_0} \to V_{x_0}$ by

$$e^{\gamma(u)}e^{a} = e^{h(u,a)}e^{\gamma(\sigma(u,a))} \tag{2.39}$$

Since V_{x_0} is a graded subspace of g_{x_0}, the natural dilations on g_{x_0} induce dilations on V_{x_0}. The map γ clearly commutes with dilations, and hence so do h and σ. The induced representation $\pi_{(0,h_{x_0})}$ is realized on $L^2(V_{x_0})$ via

$$\pi(0,h_{x_0})(e^a)f(u) = f(\sigma(u,a)), \qquad a \varepsilon g_{x_0}. \tag{2.40}$$

Thus

$$\pi(0,h_{x_0})(a)f(u) = \frac{d}{dt}\bigg|_{t=0} f(\sigma(u,ta)) \tag{2.41}$$

In particular, since σ commutes with dilations,

$$\pi(0,h_{x_0})(Y) \text{ is a vector field on } V_{x_0} \text{ homogeneous of degree } i \text{ if } Y \varepsilon g^i_{x_0}. \tag{2.42}$$

(This is consistent with the intrinsic realization of $\pi_{(0,h_{x_0})}(Y)$ as the push-forward of $\widetilde{Y}$ under π.)

Let $\beta:g_{x_0} \to L$ be a cross-section, and define θ_{x_0} from a (sufficiently small) nbhd of 0 in V_{x_0} to a nbhd of x_0 in M via

$$\theta_{x_0}(u) = e^{\beta(\gamma(u))}x_0 \tag{2.43}$$

One sees easily from Cor. 1.34 that θ_{x_0} is a local diffeomorphism. Let $\theta^*_{x_0}$ denote the pull-back of vector fields with respect to θ_{x_0}.

We can now state the theorem.

<u>Theorem 2.17:</u> Let $Y \varepsilon g^i_{x_0}$. Then $\theta^*_{x_0}\beta(Y) = \pi_{(0,h_{x_0})}(Y) + R$, where R is a C^∞ vector field in a nbhd of 0 in V_{x_0} of local order $\lesssim$ i-1 at 0.

Of course, since $\theta^*_{x_0}$ is a diffeomorphism, the vector fields $\pi_{(0,h_{x_0})}(Y) + R$ span at 0.

<u>Cor. 2.18:</u> Let $X \varepsilon L^i$. Then $\theta^*_{x_0}X = \pi_{(0,h_{x_0})}(\pi^i_{x_0}(X)) + S$, where S is of local order $\lesssim$ i-1 at 0.

§3. Microlocal Nilpotent Approximation

The correct formulation in the microlocal context is suggested by the motivating problem, that of microlocal hypoellipticity. Since microlocal hypoellipticity should be invariant under Fourier integral conjugation, one takes the local context and conjugates by FIO's (Fourier integral operators). Vector fields become 1st-order ψDO's (with pure imaginary principal symbols); C^∞ functions become 0-order ψDO's. The microlocal analogue of the spanning condition (alternately viewed, a microlocal controllability condition) is at the outset more problematical. Once one realizes that we are allowing the cotangent vector (x,ξ) at x to vary, and that the approximation should depend on ξ as well as x, one sees that spanning is too strong a criterion. We shall discuss the correct condition below.

It suffices (and is probably most natural) to carry out the approximation process at the principal symbol level. We shall assume our ψDO's are "classical", i.e., that their total symbols have positive-homogeneous asymptotic expansions, in particular, positive-homogeneous principal symbols. With some minor modification, as we shall indicate, our work can probably be carried out in the context of the larger symbol classes, $S^j_{1,0}$ of Hörmander. (Since the principal symbol of $\frac{\partial}{\partial x_j} = i\xi_j$, we shall, in order to deal with real principal symbols, find it more convenient to work with $\frac{1}{i}$ · principal symbol.)

Various of the constructions (and results) are quite analogous to those in §1, so we can give here a somewhat terser exposition. (For a specific result of §1, the corresponding microlocal correlate will be denoted by the suffix "m"; e.g., Prop. 1.7 m.)

Let M be a paracompact C^∞ manifold, and let S^j_{hom} be the vector space over $\mathbb{R}$ of functions $p:T^*M/0 \to \mathbb{R}$ such that

$$p(x,\lambda\xi) = \lambda^j p(x,\xi) \qquad \text{for } \lambda>0 \tag{3.1}$$

As is well-known

(a) $S^i_{hom} \cdot S^j_{hom} \subset S^{i+j}_{hom}$ (3.2)

(b) $\{S^i_{hom}, S^j_{hom}\} \subset S^{i+j-1}_{hom}$, where $\cdot$ denotes multiplication, and $\{,\}$ denotes Poisson bracket.

(c) $\{f,gh\} = g\cdot\{f,h\} + \{f,g\}\cdot h$, for any $f,g,h \in C^\infty(T^*M/0)$

Specializing to the case where j=0 or 1, we get

(a) S^0_{hom} is an $\mathbb{R}$-algebra under multiplication. (3.3)

(b) S^1_{hom} is a Lie algebra over $\mathbb{R}$ with respect to Poisson bracket.

(c) S^1_{hom} is an S^0_{hom}-module under multiplication.

(d) S^1_{hom} acts, via Poisson bracket, as a Lie algebra of derivations of S^0_{hom}; moreover, the actions are consistent, i.e., (3.2)(c) holds for $f,h \in S^1_{hom}$ and $g \in S^0_{hom}$.

As in §1, let $\cdot$ denote germs, but now in a conic nbhd. For example, $\dot{S}^1_{hom(x_0,\xi_0)}$ denotes germs in a conic nbhd of (x_0, ξ_0).

The following result is simply the conic version of Remark 1.1.1, with M replaced by $S^*M/0$, the unit-sphere bundle in $T^*M/0$.

$\dot{S}^0_{hom}(x_0,\xi_0)$ is a local ring with identity, with maximal ideal $\dot{m}(x_0,\xi_0)$ consisting of all germs equal to 0 on (the ray through) (x_0,ξ_0). Moreover, the map (3.4)

$$\mathbb{R} \longrightarrow \dot{S}^0_{hom}(x_0,\xi_0) \longrightarrow \frac{\dot{S}^0_{hom(x_0,\xi_0)}}{\dot{m}(x_0,\xi_0)} \quad \text{is bijective.}$$

Def. 3.1: A filtered Lie algebra L at (x_0,ξ_0) of homogeneous symbols is a Lie subalgebra over R, generally infinite dimensional, of S^1_{hom}, together with a sequence of subspaces L^i $i=1,2,\dots,$ such that

(1) $L^1 \subset L^2 \subset L^3 \subset \dots$

(2) $[L^i, L^j] \subset L^{i+j} \quad \forall\, i,j$

(3) $L = \bigcup_{j=1}^{\infty} L^j$

(4) Each L^i is an S^0_{hom}-module under multiplication

(5) As an $\dot{S}^0_{hom}(x_0,\xi_0)$-module $\dot{L}^i_{(x_0,\xi_0)}$ is finitely generated for each i.

(For our purposes all points $(x_0,\lambda\xi_0)$, $\lambda>0$, i.e., the ray through (x_0,ξ_0), are essentially equivalent.)

In the local case the spanning condition, of rank r, is equivalent by Note 1.6.2 to the condition that $\dot{L}^r_{x_0}$ = all germs of C^∞ vector fields at x_0. We make the analogous definition here.

Def. 3.2: The filtered Lie algebra L is of finite-rank at (x_0, ξ_0) if there exists r such that $\dot{L}^r_{(x_0,\xi_0)} = \dot{S}^1_{hom(x_0,\xi_0)}$. The smallest such r is called the rank of L at (x_0,ξ_0).

Notes 3.3:

1) This is, in fact, an ellipticity condition. More precisely, L is of rank r at $(x_0,\xi_0) \leftrightarrow \exists f \varepsilon \dot{L}^r_{(x_0,\xi_0)}$ s.t. $f(x_0,\xi_0) \neq 0$, and r is the smallest such integer.

Pf:

$(\Rightarrow)$ obvious.

$(\Leftarrow)$ Since $f(x_0, \xi_0) \neq 0$, $1/f \; \varepsilon \; \dot{S}^{-1}_{hom(x_0,\xi_0)}$, and so for any g ε $\dot{S}^1_{hom(x_0,\xi_0)}$ $1/f \cdot g \; \varepsilon \; \dot{S}^0_{hom(x_0,\xi_0)}$. Since f ε $\dot{L}^r_{(x_0,\xi_0)}$ and $\dot{L}^r_{(x_0,\xi_0)}$ is an $\dot{S}^0_{hom(x\ ,\xi\ )}$ module it follows that g ε $\dot{L}^r_{(x_0,\xi_0)}$.

2) We shall see below that, just as in the local case, to construct $g_{(x_0,\xi_0)}$ we do not need the full strength of the finite-rank condition, but merely the stabilization condition $\dot{L}^r_{(x_0,\xi_0)} = \dot{L}^{r+1}_{(x_0,\xi_0)} = \ldots$.

3) The closest analogue to the map α_x and the diagram (1.7) seems to be the following. Let α be the canonical 1-form on $T^*M/0$ s.t. $d\alpha = \omega$, the symplectic form. (In local coordinates $\alpha = \sum \xi_i dx_i$.) By Euler's theorem it follows immediately that $\alpha_{(x_0,\xi_0)}(H_f) = f(x_0,\xi_0)$ if f is positive-homogeneous of degree 1, and $\alpha_{(x_0,\xi_0)}(H_g) = 0$ if g is positive-homogeneous of degree 0. In particular, since $H_{gf} = gH_f + fH_g$, $\alpha_{(x_0,\xi_0)}(H_f) = 0$ if f ε $\dot{m}_{(x_0,\xi_0)}\dot{L}^i_{(x_0,\xi_0)}$, for any i.

4) If $f = \tilde{X}$, the symbol of a vector field X on M, then, of course, $f(x_0,\xi_0) = \tilde{X}(x_0,\xi_0) = \langle X_{x_0},\xi_0\rangle = \langle \alpha_{x_0}(X),\ \xi_0\rangle$.

Let $\{L^i\}$ be a filtered Lie algebra of vector fields. Since $X \mapsto \tilde{X}$ is a Lie algebra isomorphism we obtain a filtered Lie algebra of symbols $\{\tilde{L}^i\}$ by letting $\tilde{L}^i$ be all S^0_{hom} linear combinations of symbols in L^i. The above shows that $\{L^i\}$ is of rank r at $x_0 \longleftrightarrow \forall\xi \ \varepsilon\ T^*_{x_0}M/0$, $\{\tilde{L}^i\}_{(x_0,\xi)}$ is of rank $\leq$ r. (This rank can vary with ξ.)

5) Recall that $S^j_{1,0}$ is defined to be the set of all C^∞ functions p on $T^*M/0$ (real-valued for our purposes) such that for each compact set $K \subset M$ $|\partial^\alpha_\xi\partial^\beta_x p(x,\xi)| \leq C_K(1+|\xi|)^{j-|\alpha|}$. In this context two functions and, hence, germs, are identified if they agree for $|\xi|$ sufficiently large. The statements (3.2) and (3.3) hold with $S^j_{1,0}$ replacing S^j_{hom}, j = 0,1. An element $f \ \varepsilon\ S^1_{1,0}$ is "elliptic" in a conic nbhd of (x_0,ξ_0) provided $\liminf_{\lambda\to\infty} |f(x_0,\ \lambda\xi_0)| > 0$. Thus, Def. 3.1 could naturally be extended to this context provided one can find an appropriate localization for $S^0_{1,0}$ as a substitute for (3.4). At the moment it is not clear how best to do this.

6) It follows from 1) that if $(x_0,\ \xi_0)$ is of finite rank, r, then (x,ξ) is of finite rank $\leq$ r for all (x,ξ) in a conic nbhd of (x_0,ξ_0).

<u>Prop. 3.4:</u> Let $\{L^i\}$, (x_0,ξ_0) be a filtered Lie algebra of finite rank, r. Then there is a canonically associated pair $g_{(x_0,\xi_0)}$, η, where $g_{(x_0,\xi_0)} = g^1_{(x_0,\xi_0)} \oplus \cdots \oplus g^r_{(x_0,\xi_0)}$ is a graded nilpotent Lie algebra over $\mathbb{R}$, and

$\eta \in g^{*}_{(x_0,\xi_0)}/0$. (In fact, $\eta \in g^{r}_{(x_0,\xi_0)}{}^{*}$.)

Pf: Define $g^{i}_{(x_0,\xi_0)}$ by

$$g^{i}_{(x_0,\xi_0)} = \frac{\dot{L}^{i}_{(x_0,\xi_0)}}{\dot{L}^{i-1}_{(x_0,\xi_0)} + \dot{m}_{(x_0,\xi_0)}\dot{L}^{i}_{(x_0,\xi_0)}} \tag{3.5}$$

Then exactly the same proof as for Prop. 1.7 shows that the corresponding statements (1)-(4) hold.

Next define η as follows:

$$\text{For } X \in g^{i}_{(x_0,\xi_0)},\quad \langle X,\eta\rangle = \check{X}(x_0,\xi_0),\ \text{where } \check{X} \in \dot{L}^{i}_{(x_0,\xi_0)} \tag{3.6}$$

$$\text{is any element such that } \pi_i(\check{X}) = X.$$

Since r is the smallest integer i s.t. $\dot{L}^{i}_{(x\ \ ,\xi\ \ )}$ contains an elliptic element, it follows immediately that η is well-defined and satisfies the asserted properties.

Notes 3.5:

1) The proof of Note 3.3.1 shows that $\dot{L}^{r}_{(x_0,\xi_0)}$ can be generated as an S^{0}_{hom} module by a single generator, namely any elliptic element. It follows that $g^{r}_{(x_0,\xi_0)}$ is a 1-dimensional vector space.

2) The analogues of Lemma 1.13 and Remark 1.14 hold, with the same

proofs as in the local case.

The notions of weak homorphism, partial homomorphism, and cross-section carry over to the microlocal context, as do Prop. 1.18, 1.19, and Cor. 1.20. In particular, $g_{(x_0,\xi_0)}$ enjoys the "universal" property analogous to that of g_{x_0}. This shows that $g(x_0,\xi_0)$ is in some sense the minimal nilpotent approximation to L at (x_0,ξ_0). We shall later in this section give a more precise sense to this notion of approximation.

The definition of weak morphism carries over, as does the functoriality result, Prop. 1.24. (To obtain functoriality also at the level of the canonically determined $\eta \in g^*_{(x_0,\xi_0)}$ appears to require additional structure, which is present in the following basic example.)

Example 3.6:

1) Let $\phi: T^*M/0 \to T^*N/0$ be a homogeneous canonical transformation mapping (x_0, ξ_0) to (x_0', ξ_0'). (It suffices that ϕ be defined in a conic nbhd of (x_0,ξ_0).) Given $\{L^i\}$, (x_0, ξ_0) define $K^i = \{f \quad \phi^{-1} | f \in L^i\}$. Then ϕ determines a weak morphism from $\{L^i\}, (x_0,\xi_0)$ to $\{K^i\}$, (x_0',ξ_0') via $f \to f \circ \phi^{-1}$, and hence an associated morphism $\hat{\phi}$ of graded nilpotents. As in Cor. 1.25, rank is preserved and $\hat{\phi}$ is an isomorphism. Also, the associated dual map $\hat{\phi}^*$ takes $\eta' \in g^*_{(x_0',\xi_0')}$ to the corresponding $\eta \in g^*_{(x_0,\xi_0)}$. Of course, at the operator level, this example corresponds to invariance of $g^*(x_0,\xi_0)$ under FIO conjugation.

2) Special case of the preceding: M=N and ϕ leaves L^i invariant, i.e.,

$\dot{L}_{\phi(x,\xi)} = \{f\circ\phi^{-1} | f \in \dot{L}^i_{(x,\xi)}\}$ for every (x,ξ) in a conic nbhd of (x_0,ξ_0).

3) The "identity" map $s_t:\dot{L}^i_{(x,\xi)} \to \dot{L}^i_{(x,t\xi)}$ ($t>0$, fixed) induces the isomorphism $\hat{s}_t:g_{(x,\xi)} \to g_{(x,t\xi)}$. An immediate computation shows $s_t^*(\eta_t) = t\eta$, where η_t is the corresponding dual element.

We next determine $g_{(x_0,\xi_0)}$ for a few examples.

Example 3.7: Let $g = g_1 \oplus \dots \oplus g_r$ be a graded nilpotent Lie algebra, and G the corresponding (simply-connected) Lie group. Let $\eta \in g^*/0$, and view η as an element (e,η) of $T_e^*G/0$. For $Y \in g$ the associated left-invariant vector field $\tilde{Y}$ determines a symbol, also denoted $\tilde{Y}$, in S^1_{hom} via $(x,\xi) \in T^*G/0 \mapsto \tilde{Y}(x,\xi)$. The injection $g \to S^1_{hom}$ given by $Y \mapsto \tilde{Y}$ is a Lie algebra homomorphism. Let L^i consist of all S^0_{hom} linear combinations of symbols $\tilde{Y}$ such that $Y \in g_1 + \dots + g_i$. Clearly $\{L^i\}$ is a filtered Lie algebra. Let k be the smallest integer such that $\eta \restriction g_k \neq 0$. Since g is graded, $V_k \oplus g_{k+1} \oplus \dots \oplus g_r$ is an ideal for any subspace V_k of g_k, in particular for $V_k = \ker(\eta\restriction g_k)$. Thus, $g_1 \oplus \dots \oplus g_{k-1} + \dfrac{g_k}{\ker(\eta\restriction g_k)}$ is a graded Lie algebra.

Claim: $\{L^i\}$ is of rank k at (e,η), and $g_{(e,\eta)}$ is canonically isomorphic to $g_1 \oplus \dots \oplus g_{k-1} + \dfrac{g_k}{\ker(\eta\restriction g_k)}$; the associated element of $g^*_{(e,\eta)}$ is the element of $\left(\dfrac{g_k}{\ker(\eta\restriction g_k)}\right)^*$ determined by η.

The statement regarding rank is obvious. Next, since the homomorphism $g \to S^1_{hom}$ is, in particular, a weak homomorphism $g \to L$, it induces, by Prop. 1.19m, a graded homomorphism from g onto $g_{(e,\eta)}$. It remains only to determine the kernel of this map $g_i \to g^i_{(e,\eta)}$ for each $i=1,\ldots,r$. This is clearly all of g_i for $i>k$. We know $g^k_{(e,\eta)}$ is one dimensional. Hence, using the map $\alpha_{(x_0,\xi_0)}$ of Note 3.3.3 and the associated diagram (1.7m), we see that for $i=k$ the kernel is $\ker(\eta \restriction g_k)$. (This also proves the last statement of the claim.) The following lemma completes the argument by showing that for $i<k$ the kernel is 0.

<u>Lemma 3.8.</u> Let $Y_1,\ldots,Y_j \in g_1 \oplus \ldots \oplus g_{k-1}$ be linearly independent. Then for any $a_1,\ldots,a_j \in S^0_{hom}$ such that $\sum_{i=1}^{j} a_i(x,\xi)\tilde{Y}_i(x,\xi) \equiv 0$ in a conic nbhd of (e,η), $a_i(e,\eta) = 0$ $\forall i=1,\ldots,j$.

<u>Pf:</u>

Fix i, and choose $\varphi \in (g_1 + \ldots + g_{k-1})^*$ such that $\langle\varphi,Y_i\rangle = 1$ and $\langle\varphi,Y_\ell\rangle = 0$ for $\ell \neq i$. For $\varepsilon>0$ sufficiently small $(e, \varepsilon\varphi+\eta)$ lies in the given conic nbhd. Since η annihilates $g_1 \oplus \ldots \oplus g_{k-1}$, $\tilde{Y}_\ell(e, \varepsilon\varphi+\eta) = \langle\varepsilon\varphi+\eta,Y_\ell\rangle = \varepsilon\langle\varphi,Y_\ell\rangle = \varepsilon\delta_i$. Thus, $a_i(e, \varepsilon\varphi+\eta) = 0$. Let $\varepsilon \to 0$.

Our computation shows, in particular, that even when L comes from g the associated approximation $g_{(e,\eta)}$ depends on η itself and <u>not</u> just on its coadjoint orbit in g^*. This is as it should be. For example, the "characteristic variety" is in general not invariant under the coadjoint action.

Example 3.9: Let $\{L^i\},x_0$ be a filtered Lie algebra of vector fields; and for any $(x_0,\xi_0) \in T^*_{x_0}M/0$ let $\{\tilde{L}^i\},(x_0,\xi_0)$ be the associated filtered Lie algebra of symbols, as in Note 3.3.4. Then there is a canonical surjective homomorphism of graded Lie algebras $g_x \longrightarrow g_{(x\ ,\xi\ )} \longrightarrow 0$. (As observed in Note 3.3.4 rank $(x_0,\xi_0) \leq$ rank x_0). In fact, let β be a cross section of $\{L^i\},x_0$. Then β is a weak homomorphism and so, since $X \to \tilde{X}$ is a Lie algebra isomorphism, determines a weak homorphism $g_{x_0} \to \tilde{L},(x_0,\xi_0)$. By Prop. 1.19m this determines a surjective homomorphism of graded Lie algebras $g_{x_0} \longrightarrow g_{(x_0,\xi_0)} \to 0$. Finally, Prop. 1.18 (3) shows that this homomorphism is independent of the choice of β. Heuristically, if we focus attention at (x_0,ξ_0) only a part of the information (i.e., representation theory) in g_{x_0} is needed, namely representations lifted from $g_{(x_0,\xi_0)}$; of these, only the ones in $\Gamma_{(x_0,\xi_0)}$ are needed.

Example 3.10: Suppose $\{L^i\}$, (x_0,ξ_0) is of rank 1. Then we know $g_{(x_0,\xi_0)} \cong \mathbb{R}$. In fact, $\{L^i\}$, $_{(x_0,\xi_0)}$ is the filtered Lie algebra of symbols associated to the rank one filtered Lie algebra of vector fields of Example 1.26.1, with $g_{x_0} \cong T_{x_0}M$. So, by Example 3.9, $g_{(x_0,\xi_0)}$ is naturally identified with $\frac{T_{x_0}M}{\xi_0^\perp}$, and $g^*_{(x_0,\xi_0)}$ is thus naturally identified with the line through (x_0,ξ_0) in T^*M. Under this identification η corresponds to (x_0,ξ_0).

Example 3.11: The next example defines filtered Lie algebras related to the operator classes $L^{m,k}$ of Boutet de Monvel [2], as mentioned in the Introduction. Let Σ be a smooth conic submanifold of $T^*M/0$. Let $L^1 = \{u \in S^1_{hom} | u=0 \text{ on } \Sigma\}$ and let $L^2 = L^3 = \ldots = S^1_{hom}$. Then $\{L^i\}$ is of rank 1 at any point $(x,\xi) \varepsilon \sum$, and of rank 2 at any $(x_0,\xi_0) \varepsilon \sum$. In fact, since $\sum$ is smooth, we can find local defining functions $u_1,\ldots,u_k \varepsilon S^1_{hom}$ (where $k =$ codim $\sum$) that that $du_1,\ldots,du_k$ are linearly independent at (x_0,ξ_0), (and hence at all nearby points). The $u_1,\ldots,u_k$ are generators for $\dot{L}^1_{(x,\xi)}$ for all $(x,\xi) \varepsilon \sum$ near (x_0,ξ_0). Let $N(\sum)_{(x,\xi)}$ denote the conormal space to $\sum$, and define a graded Lie algebra structure on $N(\sum)_{(x,\xi)} \oplus \mathbb{R}$ via $[(df_1,r_1),(df_2,r_2)] = (0, \omega(df_1,df_2)|_{(x,\xi)}) = (0, \{f_1,f_2\}|_{(x,\xi)})$. Note that these Lie algebras are not isomorphic unless the rank of $\omega|\sum$ is constant in a nbhd of (x_0,ξ_0) in $\sum$.

<u>Claim:</u> For $(x,\xi) \varepsilon \sum$ near (x_0,ξ_0), $N(\sum)_{(x,\xi)} \oplus \mathbb{R} \cong g_{(x,\xi)}$. This follows from the next lemma and the definition of Lie bracket in $g_{(x,\xi)}$.

<u>Lemma 3.12:</u> Suppose $du_1,\ldots,du_k$ are linearly independent at the point $(x,\xi) \varepsilon \sum$. For any $a_1,\ldots,a_k \varepsilon S^0_{hom}$ such that $\sum a_i(x',\xi')u_i(x',\xi') = 0$ in a conic nbhd of (x,ξ), $a_1(x,\xi) = \ldots = a_k(x,\xi) = 0$.

<u>Pf:</u>

$$d(a_iu_i)|_{(x,\xi)} = a_i(x,\xi)du_i|_{(x,\xi)} + u_i(x,\xi)da_i|_{(x,\xi)}. \text{ But } u_i(x,\xi) = 0.$$

<u>Remark:</u> Any homogeneous canonical transformation ϕ mapping $\sum$ into $\sum$ induces an isomorphism between $g_{(x,\xi)}$ and $g_{\phi(x,\xi)}$. (See Example 3.6.2).

We next show how to construct in our context the analogue of the set

$\Gamma_{(x_0,\xi_0)}$ of Helffer-Nourrigat ([19], [21], [32], [33]). Because of the minimality of $g_{(x_0,\xi_0)}$ the construction is particularly natural in our general context. When L is generated by L^1, so that the Helffer-Nourrigat construction is defined, the relation to the construction below can be stated precisely. (see Cor. 3.19).

As in our proof of the lifting theorem, the main tools will be Lemma 1.35 and Cor. 1.36m.

Let $\{L^i\}$, (x_0,ξ_0) be a filtered Lie algebra, of rank r, with $g_{(x_0,\xi_0)}$, η the associated graded nilpotent and dual vector. Let $\{\delta_t\}$ denote the standard dilations on $g_{(x_0,\xi_0)}$, defined as multiplication by t^i on $g^i_{(x_0,\xi_0)}$.

<u>Def. 3.13:</u>

1) A <u>sequence</u> is a sequence $\{t_n, (x_n,\xi_n)\}$ with $t_n \in \mathbb{R}^+$, $(x_n,\xi_n) \in T^*M/0$, such that $x_n \to x_0$, $|\xi_n| \to \infty$, and $\frac{\xi_n}{|\xi_n|} \to \frac{\xi_0}{|\xi_0|}$.

2) Let β be a cross-section. The sequence $\{t_n, (x_n,\xi_n)\}$ is <u>β-admissible</u> if there exists $\ell \in g^*_{(x,\xi)}$ such that $\lim \beta(\delta_t Y)(x_n,\xi_n)$ exists and equals $\langle \ell, Y\rangle$ $\forall Y \in g_{(x,\xi)}$.

<u>Notes 3.14:</u>

1) Of course, if the limit exists $\forall Y$ it is linear in Y, and so determines $\ell \in g^*_{(x,\xi)}$.

2) In view of 3) below, the definition depends only on the "germ of β" at (x_0,ξ_0), i.e., the image in $\dot{L}$.

3) Since $L \subset S^1_{hom}$ and β is $\mathbb{R}$-linear

$$\beta(\delta_{t_n} Y)(x_n,\xi_n) = t_n^i|\xi_n|\beta(Y)(x_n, \frac{\xi_n}{|\xi_n|}) \qquad \forall Y \in g^i_{(x_0,\xi_0)}. \tag{3.7}$$

But

$$\beta(Y)(x_n, \frac{\xi_n}{|\xi_n|}) \to \beta(Y)(x_0, \frac{\xi_0}{|\xi_0|}) .$$

Since

$$\beta(Y)(x_0, \frac{\xi_0}{|\xi_0|}) = 0 \text{ if } i<r \quad (\text{and} \neq 0 \text{ if } i=r \text{ and } Y\neq 0)$$

it follows that for any β-admissible sequence:

$$\text{for } i<r, \ \mathcal{L}\upharpoonright g^i_{(x_0,\xi_0)} = 0 \text{ unless } t_n^i|\xi_n| \to \infty . \tag{3.8}$$

$$t_n^r|\xi_n| \text{ converges; in particular } t_n^r|\xi_n| \text{ is bounded.} \tag{3.9}$$

(more particularly, $t_n \to 0$.)

<u>Prop. 3.15:</u> Let β_1, β_2 both be cross-sections. Then the sequence $\{t_n, (x_n,\xi_n)\}$ is β_1-admissible $\Longleftrightarrow$ it is β_2-admissible. Moreover, the limit $\mathcal{L} \in g^*_{(x\ ,\xi\ )}$ determined by this sequence is independent of the choice of β.

Pf:

Let $\{t_n, (x_n,\xi_n)\}$ be β_1-admissible, with associated limit $\ell \in g^{*}_{(x_0,\xi_0)}$. Let $Y \in g^i_{(x_0,\xi_0)}$. We must show $\lim\ \beta_2(\delta_t\ Y)(x_n,\xi_n) = \langle \ell, Y\rangle$.

Since $\beta_2(Y) \in L^i$, and since $\pi^i_{(x_0,\xi_0)}(\beta_2(Y)) = Y$, by definition of cross-section, it follows from Cor. 1.36m that, at the germ level,

$$\beta_2(Y) = \beta_1(Y) + \sum_{|\alpha|<i} f_\alpha \beta_1(Y_\alpha) + \sum_{|\alpha|=i} g_\alpha \beta_1(Y_\alpha) ,$$

where the f_α are in S^0_{hom} and the g_α are in $m_{(x_0,\xi_0)}$.

Applying (3.7) we see it suffices to prove that

$$\lim_{n\to\infty} t^i_n |\xi_n| \Big(\sum_{|\alpha|<i} f_\alpha(x_n, \frac{\xi_n}{|\xi_n|})\beta_1(Y_\alpha)(x_n, \frac{\xi_n}{|\xi_n|})$$

$$+ \sum_{|\alpha|=i} g_\alpha(x_n, \frac{\xi_n}{|\xi_n|})\beta_1(Y_\alpha)(x_n, \frac{\xi_n}{|\xi_n|})\Big) \text{ exists, and } =0.$$

But $f_\alpha(x_n, \frac{\xi_n}{|\xi_n|})$ converges as $n \to \infty$, to $f_\alpha(x_0, \frac{\xi_0}{|\xi_0|})$; hence $t_n^{i-|\alpha|} f_\alpha(x_n, \frac{\xi_n}{|\xi_n|}) \to 0$, since $|\alpha|<i$. Also, $g_\alpha(x_n, \frac{\xi_n}{|\xi_n|}) \to 0$. But since is the β_1-limit of the sequence, it follows from (3.7) that $t_n^{\alpha}|\xi_n|\beta_1(Y_\alpha)(x_n, \frac{\xi_n}{|\xi_n|})$ converges, to $\langle \ell, Y_\alpha\rangle$. Writing $t^i_n|\xi_n| = t_n^{i-\alpha}(t^\alpha_n|\xi_n|)$ concludes the proof.

We can thus speak of admissible sequences, without reference to a particular cross-section.

<u>Def. 3.15:</u> Let $\mathcal{A}_{(x_0,\xi_0)}$ denote the set of admissible sequences at (x_0,ξ_0). The <u>asymptotic moment-map</u> is the map $\Phi_{(x_0,\xi_0)}: \mathcal{A}_{(x_0,\xi_0)} \to g^*_{(x_0,\xi_0)}$ defined by $\{t_n, (x_n,\xi_n)\} \mapsto \ell$. Let $\Gamma_{(x_0,\xi_0)}$ denote the image of $\Phi_{(x_0,\xi_0)}$.

<u>Prop. 3.16:</u>

1) There are two natural R^+ actions on $\mathcal{A}_{(x_0,\xi_0)}$, namely $\{t_n,(x_n,\xi_n)\} \mapsto \{st_n,(x_n,\xi_n)\}$, and $\{t_n,(x_n,\xi_n)\} \mapsto \{t_n,(x_n,s\xi)\}$. The first passes under $\Phi_{(x_0,\xi_0)}$ to the dilation $\ell \to \delta_s^*\ell$. The second passes to scalar multiplication $\ell \to s\ell$. In particular, $\Gamma_{(x_0,\xi_0)}$ is invariant under both operations.

2) $\Gamma_{(x_0,\xi_0)}$ is closed in $g^*_{(x_0,\xi_0)}$.

3) $\eta \in \Gamma_{(x_0,\xi_0)}$.

<u>Pf:</u>

1) is immediate.

2) follows, exactly as in Helffer-Nourrigat [21], by taking a subsequence of a double sequence.

3) Choose $t_n \to 0$, and take $(x_n, \xi_n) = (x_0, \frac{1}{t_n^r}\xi_0)$. The result then follows since (x_0,ξ_0) is of rank r.

The weak homomorphism $\beta: g_{(x_0,\xi_0)} \to L$ can be exponentiated to give a "weak action" of $G_{(x_0,\xi_0)}$, the corresponding simply-connected Lie group, on $\mathcal{A}_{(x_0,\xi_0)}$, which goes over, via $\Phi_{(x_0,\xi_0)}$, to the genuine coadjoint action of $G_{(x_0,\xi_0)}$ on $g^*_{(x_0,\xi_0)}$. This is one justification for the term "asymptotic moment map".

<u>Prop. 3.17:</u> Let $\{t_n,(x_n,\xi_n)\}$ be admissible, with associated $\ell \in g^*_{(x_0,\xi_0)}$. Let $z = \exp Z$. Then $\{t_n, \exp H_{\beta(\delta_{t_n} Z)}(x_n,\xi_n)\}$ is also admissible, and is mapped to $Ad^* z(\ell)$ under $\Phi_{(x_0,\xi_0)}$. In particular, $\Gamma_{(x_0,\xi_0)}$ is invariant under the coadjoint action of $G_{(x_0,\xi_0)}$.

<u>Pf:</u>

The proof is analogous to the corresponding argument of Nourrigat [33] but with additional work needed since β is only a weak rather than partial homomorphism. We shall only give a sketch.

Here $\exp H_{\beta(\delta_{t_n} Z)}(x_n,\xi_n)$ denotes the endpoint at time=1 of the flow of the indicated Hamiltonian vector field starting at (x_n,ξ_n) at time=0.

Since $t_n \to 0$ and $(x_n, \frac{\xi_n}{|\xi_n|}) \to (x_0, \frac{\xi_0}{|\xi_0|})$ this is well-defined for n sufficiently large if we replace ξ_n by $\frac{\xi_n}{|\xi_n|}$. But $\beta(\delta_{t_n} Z) \in S^1_{hom}$, so

$$(x_n', \xi_n') \equiv \exp H_{\beta(\delta_{t_n} Z)}(x_n,\xi_n) = |\xi_n| \exp H_{\beta(\delta_{t} Z)}(x_n, \frac{\xi_n}{|\xi_n|}). \tag{3.10}$$

Since

$$\beta(\delta_{t_n} Z) = \sum_{i=1}^{r} t_n^i \beta(Z_i), \text{ where } Z = Z_1 + \ldots + Z_r \tag{3.11}$$

is the graded decomposition of Z ,

it follows in particular from (3.10) that $|\xi_n'| \to \infty$ and $(x_n', \frac{\xi_n'}{|\xi_n'|}) \to (x_0, \frac{\xi_0}{|\xi_0|})$.

Let $Y^i \in g^i_{(x_0,\xi_0)}$. It follows from (3.10), (3.11), and Taylor's theorem that

$$|\beta(\delta_{t_n} Y^i)(x_n', \xi_n') - \sum_{j=0}^{r} \frac{1}{j!} (H^j_{\beta(\delta_{t_n} Z)}(\beta(\delta_{t_n} Y^i)))(x_n,\xi_n)| \leq C\, t_n^{r+1} |\xi_n| \tag{3.12}$$

for some constant C.

Since $t_n^r|\xi_n|$ is bounded, $t_n^{r+1}|\xi_n| \to 0$. It follows from (3.12) that to prove the proposition we need only show that the finite sum occurring in (3.12) converges to $\langle Ad_z^*(\ell), Y^i\rangle$.

Using (3.11) together with the weak homomorphism property, Prop. 1.18.1m, and the fact the $H_f(g) = \{f,g\}$, we find that the finite sum in (3.12) is equal to $\beta(\sum_{j=0}^{r} \frac{1}{j!} (ad\ \delta_{t_n} Z)^j(\delta_{t_n} Y^i))(x_n,\xi_n)$ + terms of the form $(t_n^k\, m_{(x_0,\xi_0)} L^k + t_n^{k+1} L^k)(x_n,\xi_n)$. Exactly as in Prop. 3.15 we see that these error terms $\to 0$. Since $g_{(x_0,\xi_0)}$ is of rank r, the main term is just $\beta(\delta_{t_n}(Ad\ z(Y^i))(x_n,\xi_n)$, which converges to $\langle \ell, Ad\ z(Y^i)\rangle$.

Let g be a graded Lie algebra, of rank s, with $\gamma: g \to L$ a weak homomorphism at (x_0,ξ_0). Let $\hat{\gamma} = \pi \circ \gamma : g \to g_{(x_0,\xi_0)}$ be the corresponding homomorphism of graded Lie algebras (see Prop. 1.19m). Then, just as in Def. 3.13, we can define the notion of γ-admissible sequence, and a corresponding set $\Gamma^{\gamma}_{(x_0,\xi_0)} \subset g^*$. Of course, this set depends on γ, in general. (We should also assume that $t_n^k|\xi_n|$ is bounded for some k so that the proof of Prop. 3.17 is valid in this context.)

Prop. 3.18:

(a) $\Gamma^{\gamma}_{(x_0,\xi_0)} \supset \hat{\gamma}^*(\Gamma_{(x_0,\xi_0)})$, where $\hat{\gamma}^*: g^*_{(x_0,\xi_0)} \to g^*$ is the dual of $\hat{\gamma}$.

(b) If $\hat{\gamma}$ is surjective then $\Gamma^{\gamma}_{(x_0,\xi_0)} = \hat{\gamma}^*(\Gamma_{(x_0,\xi_0)})$.

Note that if $\hat{\gamma}$ is surjective, $\hat{\gamma}^*$ is injective. Statement (b) implies that the unitary representations associated to $\Gamma^{\gamma}_{(x_0,\xi_0)}$ by Kirillov theory are precisely the lifts of the representations associated to $\Gamma_{(x_0,\xi_0)}$.

Pf:

(a) Let β be a cross-section for $g_{(x_0,\xi_0)}$. By Prop. 1.18.3m for $X_i \in g^i$, $\beta(\hat{\gamma}(X_i)) - \gamma(X_i) \in \dot{L}^{i-1}_{(x_0,\xi_0)} + \dot{m}_{(x_0,\xi_0)}\dot{L}^{i}_{(x_0,\xi_0)}$. Hence, since $\hat{\gamma}$ is graded, the same argument as in Prop. 3.15 shows that if $\{t_n, (x_n,\xi_n)\}$ is β-admissible, with associated ℓ, then $\gamma(\delta_{t_n}X)(x_n,\xi_n) \to \langle \ell, \hat{\gamma}(X)\rangle$.

(b) If $\hat{\gamma}$ is surjective then it follows from Lemma 1.35m that $\gamma(g^1 \oplus \dots \oplus g^i)$ generates $\dot{L}^{i}_{(x_0,\xi_0)}$ as an $\dot{S}^{0}_{(x_0,\xi_0)}$ module. The same identity as in (a) can then be used to prove the reverse inclusion.

The set $\Gamma_{(x_0,\xi_0)}$ of Helffer and Nourrigat, defined in the setting equivalent to L being generated by L^1, is the subset $\Gamma^{\lambda}_{(x_0,\xi_0)}$ of g^*, where g is a free nilpotent with a partial homomorphism $\lambda: g \to L$. Then Cor. 1.20m and Prop. 3.18b show:

Cor. 3.19: The Helffer-Nourrigat set $\Gamma_{(x_0,\xi_0)}$ is the image of our set $\Gamma_{(x_0,\xi_0)}$ under the injection $\hat{\lambda}^*$.

Remarks 3.20:

1) It follows in particular from the above corollary that the computations ([21]) of Helffer-Nourrigat of $\Gamma_{(x_0,\xi_0)}$ in a variety of examples furnish the corresponding information for our $\Gamma_{(x_0,\xi_0)}$, once $\hat{\lambda}$ is computed.

2) A trivial but interesting observation: in Example 3.10, under the given identification of $g^*_{(x_0,\xi_0)}$ with the line through (x_0,ξ_0), $\Gamma_{(x_0,\xi_0)}$ is the half-line through (x_0,ξ_0).

3) Let G be a nilpotent Lie group with graded Lie algebra $g = g_1 \oplus \dots \oplus g_r$. Suppose there is a genuine conic Hamiltonian action of G on $T^*M/0$, with $\gamma: g \to S^1_{hom}$ the corresponding Lie algebra homomorphism. Let $\Phi: T^*M/0 \to g^*$ be the genuine moment-map, defined by $\langle\Phi(x,\xi),X\rangle = \gamma(X)(x,\xi)$. Then $\gamma(\delta_{t_n}X)(x_n,\xi_n) = \langle\delta^*_{t_n}\Phi(x_n,\xi_n),X\rangle$; hence admissiblity, modulo (3.9), corresponds to the condition that $\lim_{n\to\infty} \delta^*_{t_n} \Phi(x_n,\xi_n)$ exists, with the "asymptotic moment map" giving the limit.

4) In the case of Example 3.7 an easy argument which we omit (and requiring only admissible sequences with x_n=e) shows that $\Gamma_{(e,\eta)}$ is as large as possible; i.e., $\Gamma_{(e,\eta)} = \{\ell \in g^*_{(e,\eta)} \,|\, \ell|g^k_{(e,\eta)} = \lambda\eta$, for some $\lambda \geq 0\}$.

We can define "$U^m(L)$" in analogy with Definition 2.9 (but using the ψDO principal symbol in preference to the geometric principal symbol).

<u>Def. 3.21:</u> "$U^m(L)$", for m a non-negative integer, is the vector space of all ψDO's of the form $P = \sum_{|\alpha|\leq m} A_\alpha(x,D)X_{\alpha_1}\dots X_{\alpha_j}$, where X_{α_i} is a first-order ψDO defined in a conic nbhd of (x_0,ξ_0) with real principal symbol (in the ψDO sense) $\tilde{X}_{\alpha_i} \in L^{\alpha_i}$, and where $A_\alpha(x,D)$ is a 0-order ψDO (with principal symbol not necessarily real). Notice that choosing different X_α's with the

same principal symbol, i.e., replacing X_α by $X_\alpha + R_\alpha$, where R_α is 0-order, is tantamount to replacing P by P+Q, where Q ε "$U^{m-1}(L)$".

As in the local case, we would like to be able to intrinsically assign to P ε "$U^m(L)$" the element $\hat{P}_{(x_0,\xi_0)}$ ε $U_m(g_{(x_0,\xi_0)})$ given by

$$\hat{P}_{(x_0,\xi_0)} = \sum_{|\alpha|=m} \tilde{A}_\alpha(x_0,\xi_0)\hat{X}_{\alpha_1}\ldots\hat{X}_{\alpha_j}, \text{ where } \hat{X}_{\alpha_i} = \frac{1}{\sqrt{-1}}\pi^{\alpha_i}_{(x_0,\xi_0)}(\tilde{X}_\alpha) \quad (3.13)$$

(Here $\tilde{A}_\alpha$ is the principal symbol of A_α; note also the extra factor of $\frac{1}{\sqrt{-1}}$ in contrast with (2.36).)

Just as in the local case, $\hat{P}_{(x_0,\xi_0)}$ is not necessarily well-defined. However it follows, in particular, from Nourrigat's approximation theorem (to be discussed below) that $\pi(\hat{P}_{(x_0,\xi_0)})$ is well-defined, where π is any unitary irreducible representation of $G_{(x_0,\xi_0)}$ associated to a coadjoint orbit in $\Gamma_{(x_0,\xi_0)}$. (This is definitely the case when L^1 generates; the same appears to work in general. See Notes 3.24 .)

We can give a definition of L-hypoellipticity at (x_0,ξ_0), the microlocal variant of Def. 2.14 as follows. (We retain the caveats of Remarks 2.15). We simply replace the estimate (2.37) by

$$||BQf||^2_{L^2(U)} \leq C_Q(||Pf||^2_{L^2(U)} + ||f||^2_{L^2(U)}) \quad \forall f \in C_0^\infty(U) \quad (3.14)$$

Here B is a 0-order φDO elliptic at (x_0,ξ_0), independent of Q.

An alternate version is given by

$$\mu \in \mathcal{D}' \text{ and } P\mu \in H^s_{(x_0,\xi_0)} \Rightarrow Q\mu \in H^s_{(x_0,\xi_0)} \tag{3.15}$$

Here $H^s_{(x_0,\xi_0)}$ is the standard microlocalized Sobolev space. In [21] Helffer and Nourrigat discuss various versions of maximal hypoellipticity (equivalent to L-hypoellipticity in the context of L^1 generating). Using a microlocal result of Bolley-Camus-Nourrigat ([1]) as a substitute for the local results of ([37]), they show that if (x_0,ξ_0) is of finite rank then (3.14) and (3.15) are equivalent (to each other and) to a priori stronger hypoellipticity conditions. In particular, (3.14) implies hypoellipticity. It is likely, though I cannot say for a fact, that the corresponding statement holds when L^1 does not generate. (I have not tried to extend the B-C-N result to this setting.)

We conjecture the following L-hypoellipticity variant of the Helffer-Nourrigat maximal hypoellipticity conjecture. In view of Cor. 3.19, they are equivalent when L^1 generates.

L-hypoellipticity conjecture: (3.16)

For $\{L^i\}$, (x_0,ξ_0) of finite rank, $P \in$ "U(L)" is L-hypoelliptic at $(x_0,\xi_0) \longleftrightarrow$ $\pi(P)$ is left-invertible for every unitary irreducible representation π of $G_{(x_0,\xi_0)}$ associated to an orbit in $\Gamma_{(x_0,\xi_0)}$ (other than $\{0\}$).

The local analogue is stated with $\Gamma_{(x_0,\xi_0)}$ replaced by

$$\Gamma_{x_0} = \bigcup_{\xi \in T^*_{x_0}M/0} \hat{\Gamma}_{(x_0,\xi)}\ ,$$ where $\hat{\Gamma}_{(x_0,\xi)}$ is the image in $g^*_{x_0}$ of $\Gamma_{(x_0,\xi)}$ under the injection $g^*_{(x_0,\xi)} \to g^*_{x_0}$. (See Example 3.9)

Note 3.22. For simplicity, since we will not be pursuing the analytic questions here, we do not elaborate on some important, but more technical aspects: (1) the precise nature of the left-invertibility; (2) an additional conjectured equivalent condition relating estimates at the level of M with families of estimates, with uniform constant, at the level of the representation spaces; this is analogous to the method of "compactification of estimates" used by Helffer-Nourrigat in their proof of the representation theoretic hypoellipticity criterion for nilpotent Lie groups.

As we stated in the Introduction, Helffer and Nourrigat have proved the sufficiency of their conjectured maximal hypoellipticity criterion in a number of cases, and recently Nourrigat ([32], [33]) has proved the necessity in general. His main tool is an approximation theorem, (based on a generalization of methods of Hörmander [24], [25]) which serves as a type of microlocal substitute for the lifting theorem. We present a version below. In our terminology the context is that of L generated by L^1, with (x_0,ξ_0) a point of finite rank; g denotes a free nilpotent, with associated group G; λ is a partial homomorphism into L; d=dim M.

Theorem 3.23: (Nourrigat) Let $\{t_n,(x_n,\xi_n)\}$ be a λ-admissible sequence with associated $\ell \in \overset{\lambda}{\Gamma}_{(x_0,\xi_0)} \subset g^*$, with corresponding unitary irreducible representation π . Then there exist

1) An induced representation π of G, acting on $L^2(R^k)$ for some $0 \leq k \leq d$, and having π_ℓ in its spectrum.

2) A sequence $\chi_n : V_n \rightarrow W_n$ of symplectic transformations, where the V_n form an exhaustive system of nbhds of (0,0) in R^{2d}, where W_n is a nbhd of (x_n,ξ_n), and where $\chi_n(0,0) = (x_n,\xi_n)$ such that, for some subsequence, $\lambda(\delta_{t_n}X) \circ \chi_n \rightarrow \frac{1}{\sqrt{-1}}\pi(X)$ in $C^\infty(R^{2d})$, uniformly on compact subsets, for all $X \in g$.

Notes 3.24:

1) Nourrigat passes from this to a corresponding operator version on $L^2(R^d)$, which is his basic tool.

2) I believe, but have not absolutely convinced myself, that the nbhds W_n become "small", so that they converge to the ray through (x_0,ξ_0).

3) Although I have not fully carried out the details, it is clear that by an argument closely akin to Nourrigat's necessity proof, using the operator version of the theorem and ([21], Prop. 2.2.1 of Chap. II), one can prove that $\pi_\ell(\hat{P}_{(x_0,\xi_0)})$ is well-defined, as claimed earlier. We probably need to use 2) above to handle the $A_\alpha(x,D)$ terms. (We also use Cor. 1.36m.)

Granting that 2) holds, it appears that the corresponding theorem is valid in general, (i.e., without the restriction that L^1 generates L, and with g replaced by $g_{(x_0,\xi_0)}$, and "λ-admissible" by "admissible"). This basically involves verifying that Nourrigat's proof can be modified so as to work with a weak homomorphism replacing the partial homomorphism. This seems to follow from the same sort of argument as in Props. 3.15 and 3.17.

As stated in the Introduction, we shall not carry out the details here.

The setting of filtered Lie algebra $\{L^i\},(x_0,\xi_0)$ suggests a notion of L-wave-front set $\subset \Gamma_{(x_0,\xi_0)}/0$. We conclude this section with a provisional version of this idea.

Def. 3.25: Let $\{L^i\},(x_0,\xi_0)$ be of finite rank, and let $u \in \mathcal{D}'(M)$. Then $WF^L_{(x_0,\xi_0)}(u) = \bigcap \gamma(P)_{(x_0,\xi_0)}$, where the intersection is over all $P \in$ "U(L)" such that $Pu \in C^\infty$, and where $\gamma(P)_{(x_0,\xi_0)} = \{\ell \in \Gamma_{(x_0,\xi_0)}/0 \,|\, \pi_\ell(\hat{P}_{(x_0,\xi_0)})$ is not left-invertible$\}$.

Remarks 3.26.

1) If we take $\{L^i\}$ the standard rank 1 algebra, i.e., $L^1 = S^1_{hom}$, then WF^L corresponds to the standard WF-set. That is,

$$WF^L_{(x_0,\xi_0)}(u) = \begin{cases} \emptyset \text{ if } (x_0,\xi_0) \notin WF(u) \\ \\ \text{the ray through } (x_0,\xi_0) \text{ if } (x_0,\xi_0) \in WF(u) \end{cases}$$

(see Example 3.10 and Remark 3.20.2).

2) If we take $\{L^i\}$ as in Example 3.11, with additional hypotheses (e.g., on the rank of ω), then WF^L seems closely related to the quasi-homogeneous WF-set introduced by Lascar [28], and also used by Grigis [9], [10], [11], for the study of propagation of singularities. For a related construction within the group context itself see Miller ([31]). It would be very interesting if one could find examples (other than those already

treated in [9], [10], [11]) of propagation of WF^L along "microbicharacteristics".

3) One expects, in view of 1) (especially the corresponding local version, Example 1.26.1) and the references cited in 2), that there should be a close relationship between $WF^L_{(x_0,\xi_0)}(u)$ and the rate of decay, along δ_t-homogeneous cones in $g^*_{(x_0,\xi_0)}$ of the abelian Fourier transform (from $g_{(x_0,\xi_0)}$ to $g^*_{(x_0,\xi_0)}$) of appropriate "liftings" of u to $g_{(x_0,\xi_0)}$. However, at the moment, I am unable to make this any more precise, in general.

4) Since $\hat{P}_{(x_0,\xi_0)}$ is homogeneous wrt δ_t, $WF^L_{(x_0,\xi_0)}$ is closed under dilations. Also, $WF^L_{(x_0,\xi_0)}$ is clearly invariant under the coadjoint action of $G_{(x_0,\xi_0)}$. Moreover, it is easy to see that $WF^L_{(x_0,\xi_0)}$ is empty if $(x_0,\xi_0) \notin WF(u)$. That is, under the natural projection of $\bigcup \Gamma_{(x,\xi)}$ onto $T^*M/0$, $WF^L(u)$ projects into $WF(u)$.

5) A much harder question, which I cannot answer at present, and which accounts for the provisional nature of the definition, is whether $WF^L(u)$ projects <u>onto</u> $WF(u)$. To appreciate the difficulty, observe that if the definition did not involve P varying with $\ell \in \Gamma_{(x_0,\xi_0)}/0$, but only a fixed P, then the surjectivity of projection would be equivalent to the sufficiency part of the L-hypoellipticity conjecture. Even granting the L-hypoellipticity conjecture, additional work will be needed: in particular one will need to show that $WF^L_{(x_0,\xi_0)}$ is closed in $g^*_{(x_0,\xi_0)}/0$ (and hence its complement open), in order that, just as in the proof that the standard WF set projects onto the singular support, one can reduce to the consideration of a single operator P not varying with ℓ. These questions certainly involve the delicate considerations mentioned in Note 3.22.

§4. Filtrations with L^0-Term

The construction of the graded nilpotent g_{x_0} in §1 was carried out in the context of a filtered Lie algebra $\{L^i\}$ beginning with an L^1-term. Under certain natural conditions it is possible to extend the construction to the case where there is an L^0-term. One now obtains a semidirect sum $g^0_{x_0} \circledS g_{x_0}$, where $g^0_{x_0}$ is an "arbitrary" Lie algebra acting as graded derivations on the graded nilpotent g_{x_0}.

There are a number of possible variants of the construction, it not being clear as yet which is the most useful. Because of this provisional nature of the construction we shall not attempt a systematic treatment, but instead shall only make a brief series of remarks.

Let $\{L^i\}$, $i=1,2,\ldots$ be a filtered Lie algebra of vector fields at x_0, as in Def. 1.2. In addition, let L^0 be a possibly infinite-dimensional subspace (over $\mathbb{R}$) of vector fields on M such that

$$[L^0, L^0] \subset L^0 \quad \text{, i.e., } L^0 \text{ is a Lie algebra over R.} \tag{4.1}$$

$$[L^0, L^i] \subset L^i \qquad \text{for } i \geq 1 \tag{4.2}$$

In addition, we assume <u>either</u> (4.3) or (4.4) below.

(a) L^0 is an F-module, and at the germ level is finitely generated, i.e., as an $\dot{F}_{x_0}$-module $\dot{L}^0_{x_0}$ is finitely-generated. (4.3)

(b) $L^0 \subset L^1$.

L^0, as a vector space over $\mathbb{R}$, is finite-dimensional. (4.4)

Of course, (4.1)-(4.3) simply extends Def. 1.2 to include an L^0-term.

The basic condition that we need (suggested by the corresponding hypothesis in Crouch [4]) is

$$\alpha_{x_0}(L^0) = \{0\}, \quad \text{i.e., the vector fields in } L^0 \text{ all vanish at } x_0 \tag{4.5}$$

We use this in the form

$$L^0(C^\infty(M)) \subset m_{x_0}; \text{ in particular, } L^0(m_{x_0}) \subset m_{x_0}. \tag{4.5'}$$

We assume that $\{L^i\}$, $i \geq 1$ is of finite rank, r, at x_0. We treat separately the two cases (4.4) and (4.3):

If (4.4) holds we take as our graded nilpotent the Lie algebra $g_{x_0} = g^1_{x_0} \oplus \ldots \oplus g^r_{x_0}$ constructed in §1, and take $g^0_{x_0} = L^0$.

If (4.3) holds we take as our graded nilpotent $\hat{g}_{x_0}$, defined below, and as g^0_x we take $\dfrac{\dot{L}^0_{x_0}}{\dot{m}_{x_0}\dot{L}^0_{x_0}}$.

<u>Def. 4.1.</u> Assuming (4.3) holds, we define $\hat{g}^i_{x_0} = \dfrac{\dot{L}^i_{x_0}}{\dot{L}^{i-1}_{x_0} + \dot{m}_{x_0}\dot{L}^i_{x_0}}$, $i=1,\ldots,r$. Of course, $\hat{g}^i_{x_0} = g^i_x$ except for $i=1$. The difference for $i=1$ is due to the fact that since L^0 was taken as 0 in §1, $g^1_{x_0} = \dfrac{\dot{L}^1_{x_0}}{\dot{m}_{x_0}\dot{L}^1_{x_0}}$. The proof of Prop. 1.7

goes through unchanged to show that $\hat{g}_{x_0}$ inherits the structure of a graded nilpotent Lie algebra. It follows from (4.2) that $\hat{g}_{x_0}$ is (canonically) the quotient of g_{x_0} by an ideal (lying in $g^1_{x_0}$) contained in the center of g_{x_0}.

<u>Lemma 4.2:</u>

(1) $g^0_{x_0}$ is a finite-dimensional vector space, and, under the canonical $\mathbb{R}$-linear projection $\pi_0 : L^0 \to g^0_{x_0}$, inherits the structure of Lie algebra over $\mathbb{R}$.

(2) Via the canonical $\mathbb{R}$-linear projection $\pi_i : L^i \to g^i_x$ (resp., $\hat{g}^i_{x_0}$), $g^0_{x_0}$ acts as a Lie algebra of graded derivations on g_{x_0} (resp., $\hat{g}_{x_0}$), under hypothesis (4.4) (resp., (4.3)), (where, by graded, we mean preserving gradation).

In particular, we have a naturally defined semi-direct sum $g^0_{x_0} Ⓐ g_{x_0}$ (resp., $g^0_{x_0} Ⓐ \hat{g}_{x_0}$).

<u>Pf:</u>

(1) Finite dimensionality is clear. We only need to show, in case (4.3) that the induced Lie bracket is well-defined, just as in Prop. 1.7. It suffices to show $[\dot{m}_{x_0} \dot{L}^0_{x_0}, \dot{L}^0_{x_0}] \subset \dot{m}_{x_0} \dot{L}_{x_0}$. But this follows from (4.1) and (4.5′).

(2) Just as in the proof of Prop. 1.7, it suffices to show, for $j \geq 1$ that $[\dot{L}^0_{x_0}, \dot{L}^{j-1}_{x_0} + \dot{m}_{x_0} \dot{L}^j_{x_0}] \subset \dot{L}^{j-1}_{x_0} + \dot{m}_{x_0} \dot{L}^j_{x_0}$ and, in case of (4.3), that $[\dot{m}_{x_0} \dot{L}^0_{x_0}, \dot{L}^j_{x_0}] \subset \dot{L}^{j-1}_{x_0} + \dot{m}_{x_0} \dot{L}^j_{x_0}$. We must be careful here: if $j=1$ then $\dot{L}^{j-1}_{x_0}$ is taken as 0 in case of (4.4), and as $\dot{L}^0_{x_0}$ in case of (4.3). In either case

the first inclusion holds, as follows from (4.2) and (4.5'). In case of (4.3) the second inclusion holds, by (4.2) and (4.3b), (when $j \geq 2$). The case of $j=1$ also follows since $\dot{L}_{x_0}^{j-1}$ is $\dot{L}_{x_0}^{0}$ rather than 0. This explains why we must work with $\hat{g}_{x_0}$ rather than g_{x_0} in case of (4.3).

Remarks 4.3:

(1) Let h_{x_0} be the subalgebra of g_{x_0} given by Def. 1.30. It follows easily from (4.5') that $g_{x_0}^{0}$ maps h_{x_0} into itself.

(2) The corresponding microlocal construction, of $g^{0}_{(x_0,\xi_0)} \circledS g(x_0,\xi_0)$ (or $g^{0}_{(x_0,\xi_0)} \circledS \hat{g}_{(x_0,\xi_0)}$), carries through if we replace condition (4.5) by the condition

$$H_f\big|_{(x_0,\xi_0)} = 0 \quad \text{for every } f \in L^0 \tag{4.6}$$

This insures the condition corresponding to (4.5').

(3) In practice the condition (4.2) may make it difficult for the $\dot{L}_{x_0}^{i}$ to be finitely generated as $\dot{C}_{x_0}^{\infty}$ modules. However, if the vector fields are real-analytic, or if we pass to their formal power series at x_0 then we have finite generation over $\dot{\mathcal{O}}_{x_0}$, $\mathcal{F}_{x_0}$, respectively, by Remark 1.1.4.

(4) In view of 3) lifting is still possible: we can lift to $g_{x_0}^{\text{analytic}}$, or lift to $g_{x_0}^{\text{formal}}$ at the formal power series level and then pass to the C^{∞} level, as in the discussion following Remark 2.7.2. See also Remark 2.7.2 and Remark 2.8. In this we assume that (4.4) holds, so that our graded nilpotent is the same as in §1 and §2, and so that the results there apply directly. Presumably, in the context of (4.3) analogous results

can be derived for $\hat{g}_{x_0}$. However, we have not attempted to cary these out. (In particular, we would need to work with a variant of Lemma 1.35 involving the L^0 term).

Example 4.4: (suggested by Crouch [4]). Let X_0, $X_1,\ldots,X_k$ be vector-fields, with $\alpha_{x_0}(X_0) = 0$. Let L^0 be the 1-dim vector space spanned by X_0. Let L^1 be the F-module generated by the vector-fields of the form $ad^j X_0(X_i)$, $j \geq 0$, $1 \leq i \leq k$;

$$L^2 = L^1 + [L^1,L^1]; \ \ldots \ L^{s+1} = L^s + [L^1,L^s]. \quad \text{(Compare Example 1.4)}$$

We assume $\alpha_{x_0}(L^r) = T_{x_0}M$. Then, modulo the finite-generation question of Remark 4.3.3, we obtain, since $g^0_{x_0} = L^0 \cong \mathbb{R}$, that $g^0_{x_0} Ⓐ\ g_{x_0} \cong \mathbb{R}\, Ⓐ\ g_{x_0}$, which is clearly solvable, since g_{x_0} is nilpotent. (A further example is provided by Example 1.26.5, where we now retain the L^0 term).

In view of the preceding we can apply the homogeneous space lifting theorem, Thm. 2.17, and its corollary, though we may need to pass via g_x^{formal} rather than g_x. (However, see (1.11)). According to these results the vector fields in L^i, $i \geq 1$, have a convenient realization on the graded vector space V_{x_0} via the local diffeomorphism θ_{x_0}. It turns out that the vector fields in L^0 also behave well under this diffeomorphism. Starting from the given realization of the L^i, $i \geq 1$, and using the facts that $[L^0,L^i] \subset L^i$ and $\alpha_{x_0}(L^0) = \{0\}$ one can show, by a short argument which we omit, that

$$\theta^*_{x_0} X \text{ is of local order } \leq 0 \qquad \text{for every } X \in L^0. \tag{4.7}$$

For $Y \in g^i_{x_0}$ let $\sigma(Y) = \begin{cases} \pi_{(0,h_{x_0})}(Y) & \text{if } i \geq 1 \\ \text{the principal part (i.e., homogenous of degree 0)} & \\ \text{of } \theta^*_{x_0} Y & \text{if } i=0. \end{cases}$

It follows from Cor. 2.18 that σ is a Lie algebra homomorphism from $g^0_{x_0} \textcircled{A}\, g_{x_0}$ onto the Lie algebra of principal parts of the vector fields $\theta^*_{x_0} X$, $X \in L^i$, $i \geq 0$.

The result is quite similar in character to the solvable approximations to control systems derived by Crouch ([4]). Of course there are notable differences: He deals with input-output systems, and, moreover, derives an approximation for the truncated Volterra series of each order ($\geq$ the minimal order needed for controllability). Nevertheless, it is reasonable to expect that with further work the methods of this report may be brought to bear on the types of question he considers.

5. Conclusion

The approximation process introduced here raises many questions, and suggests a variety of directions for further investigation. We feel there are two main, related, lines of inquiry:

(1) To construct an appropriate Fourier analysis associated to the "phase-space decomposition" determined by the filtered Lie algebra $\{L^i\}$ on $T^*M/0$.

(2) To systematically investigate the properties of the asymptotic moment map and its connections with quantization, as has been done by a number of workers in the context of the genuine moment mapping, where there is an exact rather than approximate symmetry group. In this regard we mention again the striking similarity between the L-hypoellipticity conjecture and the result of Guillemin-Sternberg ([14]) on the irreducible representations entering into the quantization of a compact Hamiltonian G-action. The study of the asymptotic moment map should be extended to the case where there is an L^0-term, so that the associated Lie algebra is not purely nilpotent, but a semi-direct sum with a graded nilpotent.

A particularly intriguing question bearing on (1) is to elucidate the relationship of the "phase-space decomposition" determined by $\{L^i\}$ to the phase-space decomposition associated to a single operator by Fefferman and Phong ([6]).

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