Going back: ("unreducing")

We chose to work in a frame with the center of mass fixed at the origin and axes such that the orbit lies in the x-y plane. We can "put back" these choices trivially. This is important for applications to distant star systems (double stars, extra-solar planetary systems, satellites, etc.). The relative orientation and velocity must be fit to observation.

Interesting effects:

projection of \( A \) (so that only bound \( m_1 + m_2 \))

motion of both bodies:

\[
\mathbf{r}_1' = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} + \frac{m_2}{m_1 + m_2} (\mathbf{r}_1 - \mathbf{r}_2) \]

\( \uparrow \)

fixed

orbit of 1: \( \frac{m_2}{m_1} \)

orbit of 2: \( \frac{m_2}{m_1} \)
Relative Motion: Recollection, and Getting to Rotation

\( \ddot{\mathbf{r}} = m \ddot{\mathbf{a}} \) in inertial frame. It is often natural, however, to use non-inertial frames - e.g., you may be interested in mechanics in a vehicle (elevator, train, ship, car, rocket) - or on surface of Earth!

It’s important to keep straight two cases:

i) moving origin (but fixed orientation)

\[
\begin{align*}
\mathbf{\ddot{R}} &= \mathbf{\ddot{r}} - \mathbf{\ddot{r}}_p \\
m\mathbf{\ddot{R}} &= m\mathbf{\ddot{r}} - m\mathbf{\ddot{r}}_p = \mathbf{\ddot{F}} - m\mathbf{\ddot{a}}_p
\end{align*}
\]

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example: a rising elevator \( \Rightarrow \) extra “fictitious” force downward
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\( \Rightarrow \) equivalence principle, ...

ii) rotating frame (but fixed origin)

\[
\mathbf{\ddot{r}}(t) = \mathbf{R}(t)^{-1} \mathbf{\ddot{r}}(t) \quad \text{so if} \quad \mathbf{\ddot{r}}(t) = \mathbf{R}(t) \mathbf{\ddot{r}}_0,
\]

\[
\text{"tacking out" rotation}
\]

\( \text{wince of rotation operator} \)
Small rotations around an axis

Consider rotation through $\Delta \phi$ around an axis $\hat{n}$. (The direction is fixed by right-hand rule.)

Then any vector $\vec{s}$ changes by

$$\vec{s} \rightarrow \vec{s} + \Delta \phi \, \hat{n} \times \vec{s} \quad (1)$$

This gives another nice interpretation of the $\times$ product.

Why? $\Delta \vec{s} \perp \vec{s}$ and $\perp \hat{n}$.

The projection $\hat{n}$ has no effect.

If $\vec{s} \perp \hat{n}$, rotation by $\Delta \phi$.

Example: $\hat{z}$-axis

$$\Delta \vec{s} = \Delta \phi \, \hat{z} \times (x, y, z) = \Delta \phi \, (-y, x, 0)$$

$$= \Delta \phi \, \begin{pmatrix} 0 & -\sin \beta & \cos \beta \end{pmatrix}^{\text{polar coordinates}}$$

$$= \Delta \phi \, \hat{\theta}$$

With $\hat{n} \Delta \phi = \vec{\omega} \, dt$, the angular velocity vector:

$$\vec{\omega} = \hat{n} \frac{\Delta \phi}{dt}$$

$$\Delta \vec{s} = \vec{\omega} \times \vec{s} \, dt$$
Master formula for dynamical (time-dependent) vectors in rotating frame:

\[
\left( \frac{d\vec{s}}{dt} \right)_{\text{initial}} = \left( \frac{d\vec{s}}{dt} \right)_{\text{rot.}} + \vec{\omega} \times \vec{s}
\]

At one level, very simple: vectors change from change measured within the rotating frame, and from change associated with motion of frame itself.

But puzzling, perhaps. How can \( \vec{s} \) have two different time derivatives? ??

Really:

\[ \vec{s}(t) = R(t) \vec{s}(t) \]

\[ \vec{s}(t + \Delta t) = R(t + \Delta t) \vec{s}(t) \]

\[ \vec{s}(t) + \frac{d\vec{s}}{dt} \Delta t = \left[ R(t + \Delta t) R^{-1}(t) R(t) \right] \left[ \vec{s}(t) + \frac{d\vec{s}}{dt} \Delta t \right] \]

\[ \simeq R(t) \vec{s}(t) + \left[ \vec{\omega} \times R \vec{s} + R \frac{d\vec{s}}{dt} \right] \Delta t \]
\[ \frac{\text{d}\vec{s}}{\text{d}t} = \vec{\omega} \times (\vec{R} \cdot \vec{s}) + \vec{R} \frac{\text{d}\vec{s}}{\text{d}t} \]

The "master formula" results by picking \( \vec{R}(t_0) = 1 \) at the moment of interest. The rotating frame is constantly "reinvented"!

49 I Master formula for dynamics

We need to apply this to the vectors of interest in dynamics, i.e., velocity and acceleration.

\[ \vec{v}_{\text{in.}} = \left( \frac{\text{d}\vec{r}}{\text{d}t} \right)_{\text{in.}} = \left( \frac{\text{d}\vec{r}}{\text{d}t} \right)_{\text{rot.}} + \vec{\omega} \times \vec{r} \quad (*) \]

\[ \vec{a}_{\text{in.}} = \left( \frac{\text{d}\vec{v}}{\text{d}t} \right)_{\text{in.}} = \left( \frac{\text{d}\vec{v}}{\text{d}t} \right)_{\text{rot.}} + \vec{\omega} \times \vec{v}_{\text{in.}} \]

\[ \begin{align*}
\text{Long (\star)} & \\
\left( \frac{\text{d}\vec{r}}{\text{d}t} \right)_{\text{rot.}} + \frac{\text{d}\vec{\omega}}{\text{d}t} \times \vec{r} + \vec{\omega} \times \left( \frac{\text{d}\vec{r}}{\text{d}t} \right)_{\text{rot.}} \\
& \quad + \vec{\omega} \times \left( \frac{\text{d}\vec{\omega}}{\text{d}t} \right)_{\text{rot.}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})
\end{align*} \]

[Note: \( \left( \frac{\text{d}\vec{\omega}}{\text{d}t} \right)_{\text{in.}} = \left( \frac{\text{d}\vec{\omega}}{\text{d}t} \right)_{\text{rot.}} \), since \( \vec{\omega} \times \vec{\omega} = 0 \).]
So, in rotating frame (dropping subscript)
\[ \frac{\mathbf{F}}{m} = \frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{\omega}}{dt} \times \mathbf{r} + 2 \mathbf{\omega} \times \mathbf{\omega} \times \mathbf{r} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]

\[ m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m \frac{d\mathbf{\omega}}{dt} \times \mathbf{r} - m \mathbf{2} \mathbf{\omega} \times \mathbf{\omega} - m \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

"gyro" \hspace{1cm} Coriolis \hspace{1cm} centrifugal

This contains a wealth of phenomena. Let's get oriented with semi-familiar cases:

1) \( \mathbf{\omega} = 0, \mathbf{\omega} = \text{const.} = (0,0,\omega) \)

\( \mathbf{\omega} \times \mathbf{r} = (0,0,\omega) \times (x,y,z) = (-\omega y, \omega x, 0) \)

\( \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) = (-\omega^2 x, -\omega^2 y, 0) \) gives non-vanishing term.

This is the "centripetal\footnote{State what is the centripetal force for constant rotation}" for constant rotation (that is, \( -m \omega^2 \mathbf{\omega} \times \mathbf{r} = m \omega^2 (x,y,0) \))

2) \( \frac{d\mathbf{\omega}}{dt} \neq 0 \), but still in \( \hat{z} \) direction \( \frac{\omega^2}{dt} = (0,0,\dot{\omega}) \) \( \hat{z} \) angular acceleration

New term \( -m \frac{d\omega}{dt} \times r = \hat{\theta} (m\dot{\omega}y, -m\dot{\omega}x) \)

\( = -m \dot{\theta} \hat{\theta} \)
just corrects for acceleration in \( \hat{a} \) direction.

[Note: \( r \)-dependent]

Note that if a body has large \( \vec{\alpha} \), changing it requires large forces in strange directions. This is the origin of gyroscopic phenomena. (due to “gyro” term.)