Completeness in the Monadic Predicate Calculus

We have a system of eight rules of proof. Let's list them:

PI At any stage of a derivation, you may write down a sentence φ with $\{\varphi\}$ as its premiss set.

TC If you have written down sentences ψ_1, ψ_2,..., ψ_n in a derivation, and φ is a tautological consequence of $\{\psi_1,\psi_2,...,\psi_n\}$, then you may write down sentence ψ, taking the premiss set to be the union of the premiss sets of the ψ_is. In particular, if φ is a tautology, we can write φ with the empty premiss set.

CP If you have derived ψ with premiss set $\Gamma \cup \{\varphi\}$, you may write $(\varphi \rightarrow \psi)$ with premiss set Γ.

US If you've derived $(\forall x)\varphi$, you may derive $\varphi^x/_c$ with the same premiss set, for any constant c.

UG If you've derived $\varphi^x/_c$ from Γ and if the constant c doesn't appear in φ or in any of the sentences in Γ, you may derive $(\forall x)\varphi$ with premiss set Γ.

QE From $\neg(\forall x)\neg\varphi$, you may infer $(\exists x)\varphi$ with the same premiss set, and *vice versa*.
From $(\forall x)\neg\varphi$, you may infer $\neg(\exists x)\varphi$ with the same premiss set, and *vice versa*.
From $\neg(\forall x)\varphi$, you may infer $(\exists x)\neg\varphi$ with the same premiss set, and *vice versa*.
From $(\forall x)\varphi$, you may infer $\neg(\exists x)\neg\varphi$ with the same premiss set, and *vice versa*.

EG If you have written $\varphi^x/_c$, for any constant c, you may write $(\exists x)\varphi$ with the same premiss set.

ES Suppose that you have derived $(\exists x)\varphi$ with premiss set Δ and that you have derived ψ with premiss set $\Gamma \cup \{\varphi^x/_c\}$, for some constant c. Suppose further that the constant c does not appear in φ, in ψ, or in any member of Γ. Then you may derive ψ with premiss set $\Delta \cup \Gamma$.

Something we were careful to check as we introduced each of the rules was that the rules were logical-consequence preserving, meaning that, if you use the rule to write a sentence, the sentence you write will be a logical consequence of its premiss set, provided that any earlier lines in the proof that you relied on in applying the rule were logical consequences of their premiss sets. It follows that, at each stage in a derivation according to the rules, the sentence you write down will be a logical consequence of its premiss set. Hence we have this:

> **Strong Soundness Theorem.** If you can derive a sentence φ from a premiss set that is included in the set Γ, then φ is a logical consequence of Γ.

Defining a *theorem of logic* to be a sentence which is derivable from the empty set, we have this:

> **Weak Soundness Theorem.** If a sentence is a theorem of logic, it's valid.

What we now want to see is that every valid sentence is a theorem of logic and that every logical consequence of Γ is derivable from Γ. This will tell us that we have enough rules to capture every logically correct argument; we don't need to add any more rules. Hence the results we're about to prove are called a *completeness theorem*. First we need a technical notion:

> **Definition.** A set of sentences Γ is *deductively consistent* (or *d-consistent*) iff there is no conjunction of members of Γ whose negation is a theorem of logic.

To say that a set of sentence is d-inconsistent means that it is inconsistent and, moreover, that you can show it is consistent by deriving an explicit contradiction. For example {"Fa," "$(\forall x)(Fx \to Gx)$," "$\neg Fb$," "$\neg Gb$," "$\neg Ga$"} is d-inconsistent, because the sentence "$\neg(Fa \wedge (\forall x)(Fx \to Gx) \wedge \neg Gb)$" is a theorem of logic.* We have this:

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* For the sake of readability, I've left off extra parentheses, writing $\neg(\varphi'\wedge'\psi \wedge \theta)$, instead of $\neg(\varphi \wedge (\psi \wedge \theta))$. To be strictly legal, the parentheses should be supplied. The derivation is as follows:

1	1. $(Fa \wedge (\forall x)(Fx \to Gx))$	PI
1	2. Fa	TC,1
1	3. $(\forall x)(Fx \to Gx)$	TC,1
1	4. $(Fa \to Ga)$	US,3
1	5. Ga	TC,2,4

Lemma. Every consistent set is d-consistent.

Proof: Let Γ be a consistent set. Then there is an interpretation $\mathcal{A}$ under which all the members of Γ are true. Now suppose that Γ is d-inconsistent. That mean that there are members γ_1, γ_2, ..., γ_n of Γ such that the sentence $\neg(\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n)$ is a theorem of logic. Since it's a theorem of logic, it's valid, so true in $\mathcal{A}$. But clearly we can't have γ_1, γ_2, ..., γ_n, and $\neg(\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n)$ all true in $\mathcal{A}$. Contradiction.☒

What we need to show is the converse this lemma, that every d-consistent set of sentences is consistent. This was first proven by Kurt Gödel in 1931 as his doctoral dissertation. It's not an easy proof.

Given a d-consistent set Γ, we want to construct a model in which all the members of Γ are true. Our plan is to start with Γ and build it up by adding more sentences until we get a set of sentences that completely describes an interpretation under which all the members of Γ are true. We'll deal with the quantifiers by constructing our model in such a way that every member of the universe of the interpretation is named by some constant. That means that an existential sentence will be true iff some instance is true, and a universal sentence is true iff every instance is true.

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6. $((Fa \wedge (\forall x)(Fx \rightarrow Gx)) \rightarrow Ga)$ CP,1,5
7. $\neg(Fa \wedge (\forall x)(Fx \rightarrow Gx) \wedge \neg Ga)$ TC,6

Definition. A *Henkin set is a set of sentences Δ with the following properties:**

(a) Δ is d-consistent.

(b) For each sentence φ, either φ ∈ Δ or ¬φ ∈ Δ.

(c) Whenever an existential sentence is in Δ, some instance of the sentence is in Δ.

Here are the basic facts about Henkin sets:

Fun Facts. Suppose Δ is a Henkin set. We have:

(i) A conjunction is in Δ iff both conjuncts are in Δ.

(ii) A disjunction is in Δ iff one or both disjuncts are in Δ.

(iii) A conditional is in Δ iff the consequent is in Δ or the antecedent is outside Δ.

(iv) A biconditional is in Δ iff either both components are in Δ or both are outside it.

(iv) A negation is in Δ iff the negatum is outside Δ.

(v) An existential sentence is in Δ iff some instance of it is in Δ.

(vi) A universal sentence is in Δ iff every instance of it is in Δ.

Proof: I'll only prove facts (i) and (v); the rest are similar.

(i) If (φ ∧ ψ) is in Δ but φ isn't in Δ, then, by (b), ¬φ is in Δ. But this is impossible. Since ¬((φ ∧ ψ) ∧ ¬φ) is a theorem (by TC), {(φ ∧ ψ), ¬φ} is a d-inconsistent, contrary to (a).

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* While it was Gödel who first proved a completeness theorem, the way we prove completeness today is to use a simpler technique discovered by Leon Henkin in 1949.

If $(\varphi \wedge \psi)$ is in Δ but ψ isn't in Δ, then, by (b), $\neg\psi$ is in Δ. But this contradicts (a), since $\{(\varphi \wedge \psi), \neg\psi\}$ is d-inconsistent.

If φ and ψ are both in Δ but $(\varphi \wedge \psi)$ isn't in Δ, then, by (b), $\neg(\varphi \wedge \psi)$ is in Δ. But this contradicts (a), since $\{\varphi, \psi, \neg(\varphi \wedge \psi)\}$ is d-inconsistent.

(v) It follows by clause (c) that, if an existential sentence is in Δ, then some instance of it is in Δ. For the converse, suppose that $\varphi^x/_c$ is in Δ but $(\exists x)\varphi$ isn't in Δ. Then, by (b), $\neg(\exists x)\varphi$ is in Δ But this contradicts (a), because $\{\varphi^x/_c, \neg(\exists x)\varphi\}$ is d-inconsistent, as the following derivation shows:

1	1. $\varphi^x/_c$	PI
1	2. $(\exists x)\varphi$	1,EG
	3. $(\varphi^x/_c \rightarrow (\exists x)\varphi)$	CP,1,2
	4. $\neg(\varphi^x/_c \wedge \neg(\exists x)\varphi)$	TC,3⊠

Our proof of the completeness theorem breaks into two parts:

> First Completeness Lemma. If Δ is a Henkin set, then there is an interpretation under which all and only the members of Δ are true.
>
> Second Completeness Lemma. Every d-consistent set is contained within some Henkin set.

The Henkin set obtained in the Second Completeness Lemma will not be a Henkin set for the original language, but a Henkin set for a language obtained from the original language by adding some new constants. A more precise version of the lemma is this:

> Given Δ, a d-consistent set of sentences in a language _, there is a language $\mathcal{M}$,obtained from _ by adding infinitely many individual constants, such that in $\mathcal{M}$ there is a Henkin set that contains Δ.

Clearly these two lemmas, taken together, will prove that every d-consistent set is consistent.

Proof of First Completeness Lemma: Enumerate the constants in the language as c_0, c_1, c_2, c_3,....* Define an interpretation $\mathcal{A}$, as follows:

$|\mathcal{A}|$ = {natural numbers 0,1,2,...}
$\mathcal{A}(c_i) = i$
$\mathcal{A}(R) = \{i: Rc_i \in \Delta\}$

We want to prove that, for each sentence φ, φ is true under $\mathcal{A}$ iff φ is an element of Δ. This follows immediately from the following:

Claim. For any natural number i and formula φ, i satisfies φ under $\mathcal{A}$ iff $\varphi^x/_{c_i}$ is an element of Δ.

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* Here I am assuming that there are infinitely many constants in the language. If there are only finitely many constants, the proof will be the same except that the universe of the interpretation will be a finite initial segment of the natural numbers, rather than the whole set of natural numbers. Note that the Henkin set we construct in the Second Completeness Lemma will contain infinitely many constants.

I am also assuming that the constants of the language can be arranged in a list c_0, c_1, c_2, c_3,.... For certain purely abstract formal languages that are considered in pure mathematics, this assumption won't be satisfied. The completeness theorems hold even without this assumption.

Proof: Let Σ be the set of formulas φ such that, for every number i, i satisfies φ under $\mathcal{A}$ iff $\varphi^x/_{c_i}$ is an element of Δ. To show that every formula is in Σ, it will be enough to show that Σ contains the atomic formulas and that it is closed under conjunction, disjunction, forming conditionals, forming biconditionals, negation, existential quantification, and universal quantification.

Atomic formulas are in Σ: If φ is an atomic formula of the form Rx, we have

i satisfies Rx under $\mathcal{A}$
iff i $\in$ $\mathcal{A}$(R)
iff $Rc_i \in \Delta$

If φ is an atomic formula of the form Rc_j, then $\varphi^x/_{c_i}$ is equal to φ, and we have

i satisfies Rc_j under $\mathcal{A}$
iff Rc_j is true under $\mathcal{A}$
iff $\mathcal{A}(c_j) \in \mathcal{A}(R)$
iff j $\in$ $\mathcal{A}$(R)
iff $Rc_j \in \Delta$
iff $Rc_j{}^x/_{c_i} \in \Delta$

Σ is closed under conjunction: Suppose φ and ψ are in Σ. We have

i satisfies $(\varphi \wedge \psi)$ under $\mathcal{A}$
iff i satisfies φ under $\mathcal{A}$ and i satisfies ψ under $\mathcal{A}$
iff $\varphi^x/_{c_i} \in \Delta$ and $\psi^x/_{c_i} \in \Delta$
iff $(\varphi^x/_{c_i} \wedge \psi^x/_{c_i}) \in \Delta$ (by Fun Fact (i))
iff $(\varphi \wedge \psi)^x/_{c_i} \in \Delta$.

Σ is closed under disjunction, forming conditionals, forming biconditionals, and negation. Similar.

Σ is closed under existential quantification. Suppose φ is in Σ. We have:

i satisfies $(\exists x)\varphi$ under $\mathcal{A}$
iff $(\exists x)\varphi$ is true under $\mathcal{A}$

iff, for some natural number j, j satisfies φ under $\mathcal{A}$
iff, for some natural number j, $\varphi^x/_{c_j} \in \Delta$
iff some instance of $(\exists x)\varphi$ is an element of Δ
iff $(\exists x)\varphi$ is an element of Δ
iff $((\exists x)\varphi)^x/_{c_i} \in \Delta$.

Σ is closed under universal quantification. Similar.☒

Proof of Second Completeness Lemma. We're going to form our Henkin set by starting out with Δ and forming larger and larger sets Γ_0, Γ_1, Γ_2, Γ_3,..., making sure that, at each stage, the set of sentences we construct is d-consistent. A couple of preliminaries. First, we add infinitely many individual constants to the language. Next, we enumerate the sentences of the expanded language as σ_0, σ_1, σ_2,....

Let $\Gamma_0 = \Gamma$. Then Γ_0 is d-consistent.
In forming Γ_1, there are three possibilities:

Case 1. $\Gamma_0 \cup \{\sigma_0\}$ is d-consistent and σ_0 isn't existential. In this case, we let $\Gamma_1 = \Gamma_0 \cup \{\sigma_0\}$.

Case 2. $\Gamma_0 \cup \{\sigma_0\}$ is d-consistent, and σ_0 has the form $(\exists x)\varphi$. Pick a constant that doesn't appear in $\Gamma_0 \cup \{\sigma_0\}$, and let $\Gamma_1 = \Gamma_0 \cup \{(\exists x)\varphi, \varphi^x/_c\}$. (The reason we added infinitely many constants was to make sure we could find a new constant here.). I claim that Γ_1 is d-consistent. For suppose not. Then there exist γ_1, γ_2,..., γ_k in Γ_0 such that the sentence $\neg(\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi \wedge \varphi^x/_c)$ is a theorem of logic. But we can extend a derivation from the empty set of $\neg(\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi \wedge \varphi^x/_c)$ to a derivation from the empty set of $\neg(\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi)$, as follows:

	1.	$\neg(\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi \wedge \varphi^x/_c)$	
2	2.	$(\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi)$	PI
2	3.	$\neg\varphi^x/_c$	TC,1,2
2	4.	$(\forall x)\neg\varphi$	UG,3
2	5.	$\neg(\exists x)\varphi$	QE,4
	6.	$((\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi) \rightarrow \neg(\exists x)\varphi)$	CP,2,5
	7.	$\neg(\gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_k \wedge (\exists x)\varphi)$	TC,6

But this is impossible, since $\Gamma_0 \cup (\exists x)\varphi\}$ is d-consistent.

Case 3. $\Gamma_0 \cup \{\sigma_n\}$ is d-inconsistent. In this case, we let Γ_1 be $\Gamma_0 \cup \{\neg\sigma_0\}$. I claim that, in this case too, Γ_1 is d-consistent. We know that, since $\Gamma_0 \cup \{\sigma_0\}$ is d-inconsistent, there exist γ_1, γ_2....., γ_k in Γ_0 such that the sentence $\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_k \wedge \sigma_0)$ is a theorem of logic. If Γ_1 were d-inconsistent, then there would exist sentences δ_1, δ_2,..., δ_m in Γ_0 such that the sentence $\neg(\delta_1 \wedge \delta_2 \wedge ... \wedge \delta_m \wedge \neg\sigma_0)$ is a theorem of logic. But then we could put these two derivations together, followed by a single application of TC to get a derivation from the empty set of $\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_k \wedge \delta_1 \wedge \delta_2 \wedge ... \wedge \delta_m)$. This contradicts the assumption that Γ_0 is d-consistent.

Thus we see that, in any case, Γ_1 is d-consistent.

We continue this process. We form Γ_2 from Γ_1 by adding either σ_1 or $\neg\sigma_1$ and also, if we add σ_1 and σ_1 is existential, by adding an instance of σ_1. Γ_2 will be d-consistent. We form Γ_3 from Γ_2 by adding either σ_2 or $\neg\sigma_2$ and also, if we add σ_2 and σ_2 is existential, by adding an instance of σ_2. Γ_3 will be d-consistent. And so on. At the n+1st step, having formed a d-consistent set Γ_n, we form Γ_{n+1} as follows:

> **If $\Gamma_n \cup \{\sigma_n\}$ is d-consistent and σ_n isn't existential, $\Gamma_{n+1} = \Gamma_n \cup \{\sigma_n\}$.**
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> **If $\Gamma_n \cup \{\sigma_n\}$ is d-consistent and σ_n has the form $(\exists x)\varphi$, pick a constant c that doesn't appear in any of the members of $\Gamma \cup \{_n\}$, and let Γ_{n+1} be $\Gamma_n \cup \{(\exists x)\varphi, \varphi^x/_c\}$.**
>
> **If $\Gamma_n \cup \{\sigma_n\}$ is d-inconsistent, let $\Gamma_{n+1} = \Gamma_n \cup \{\neg\sigma_n\}$.**

In any event Γ_{n+1} will be d-consistent. The proof is just as above.

Now let Γ_∞ be the union of all the Γ_ns. Γ_∞ is a Henkin set.☒

We now know that, every d-consistent set is consistent. This gives us the following:

> **Strong Completeness Theorem. If φ is a logical consequence of Γ, then there is a derivation of φ whose premiss set is included in Γ.**

Proof: If φ is a logical consequence of Γ, then $\Gamma \cup \{\neg\varphi\}$ is inconsistent. So it's d-inconsistent, which means that there are sentences γ_1, γ_2, ..., γ_n in Γ such that $\neg(\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n \wedge \neg\varphi)$ is derivable from the empty set. We can extend this derivation, as follows:

	1.	$\neg(\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n \wedge \neg\varphi)$	
2	**2.**	γ_1	**PI**
3	**3.**	γ_2	**PI**
....			
n+1	**n+1.**	γ_n	**PI**
2,3,...,n+1	**n+2.**	φ	**TC,1,2,...,n+1**☒

> **Weak Completeness Theorem. If φ is valid, it's a theorem of logic.**

Proof: If φ is valid, it's a logical consequence of the empty set, so that, by the Strong Completeness Theorem, it's derivable from the empty set.☒

> **Löwenheim-Skolem Theorem.* If a set of sentences Γ is consistent, there is a model whose universe is the natural numbers in which every member of Γ is true.**

Proof: If Γ is consistent, it's d-consistent. For each d-consistent set of sentences, our construction gives an interpretation with universe the set of natural numbers in which all members of the set are true.☒

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* Whereas the Strong Completeness Theorem doesn’t really depend upon the countability of the language, the Löwenheim-Skolem Theorem does.

Compactness Theorem. A set of sentences Γ is consistent iff every finite subset of Γ is consistent.

Proof: The left-to-right direction is trivial, since an interpretation under which all the members of Γ are true will be an interpretation under which all the members of each finite subset of Γ are true. To get the right-to-left direction, suppose that Γ is inconsistent. Then Γ is d-inconsistent, so that there exist γ_1, γ_2,..., γ_n in Γ such that the sentence $\neg(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n)$ is a theorem of logic. Then $\{\gamma_1, \gamma_2,..., \gamma_n\}$ is a finite, d-inconsistent subset of Γ. So it's a finite, inconsistent subset of Γ.☒

Corollary. If φ is a logical consequence of Γ, then it is a logical consequence of some finite subset of Γ.

Proof: If φ is a logical consequence of Γ, then there is a derivation of φ whose premiss set is a finite subset of Γ. φ is a logical consequence of this finite subset.☒