

Compactness Theorem

There is only one big theorem in the sentential calculus. This is it:

Compactness Theorem. A set Γ of SC sentences is consistent if and only if every finite subset of Γ is consistent.

If we say that a set of English sentences is *compossible* if it is possible for them all to be true, then an analogue for English of the Compactness Theorem would be the statement that a set Γ of English sentences is compossible if and only if every finite subset of Γ is compossible. The analogue is false, as we can see by considering the following set of sentences:

There are finitely many Starbucks.
There is at least one Starbucks.
There are at least two Starbucks.
There are at least three Starbucks.
.....

Every finite set of this set of sentences is compossible, since we can make any finite subset true by building a sufficiently large finite set of subsets. However, the whole set is not compossible. So, if we identify consistency with compossibility, the Compactness Theorem will fail for English.

Before beginning the proof, let me describe the connection between this theorem and the notion of compactness employed by topology. Let us stipulate that a set S of N.T.A.s is *closed* just in case there is a set of sentences Γ such that $S =$ the set of N.T.A.s under which every member of Γ is true. It is straightforward to verify that this defines a topology on the set of N.T.A.s. What the Compactness Theorem tells us is that this topology is compact.

Proof: The left-to-right direction is obvious: Any N.T.A. under which every member of Γ is true will be a N.T.A. under which every member of each finite subset of Γ is true. The other direction is harder.

Assume that every finite subset of Γ is consistent. This means that, for each finite subset of Γ , we can find a N.T.A. under which every member of that finite subset is true. *Prima facie*, this would allow that, even though for each finite subset of Γ there is a N.T.A., you have to pick different N.T.A.s for different finite subsets. What the Compactness Theorem tells us is that, in fact, we can pick a single N.T.A. that works for every member of Γ . It's a one-size-fits-all theorem.

It is a little easier to work with sets of sentences than with truth assignments. For that reason, we make the following:

Definition. A set of sentences Ω is a *complete story* just in case it satisfies the following five conditions, for any ϕ and ψ :

- a) $(\phi \wedge \psi) \in \Omega$ iff $\phi \in \Omega$ and $\psi \in \Omega$.
- b) $(\phi \vee \psi) \in \Omega$ iff $\phi \in \Omega$ or $\psi \in \Omega$ (or both).

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- c) $(\varphi \rightarrow \psi) \in \Omega$ iff $\varphi \notin \Omega$ or $\psi \in \Omega$ (or both).
- d) $(\varphi \leftrightarrow \psi) \in \Omega$ iff φ and ψ are both in Ω or neither of them is.
- e) $\neg\Phi \in \Omega$ iff $\Phi \notin \Omega$.

Clearly, a set of sentences Ω is a complete story if and only if there is a N.T.A. under which all and only the members of Ω are true. Thus, in order to show that Γ is consistent, it will be enough to show that there's a complete story that contains Γ . We'll construct our complete story in stages, starting with Γ and adding sentences to it until we get a complete story.

Lemma. A set of sentences Ω is a complete story iff it satisfies these two conditions:

- 1) Every finite subset of Ω is consistent.
- 2) For each sentence φ , either φ or $\neg\varphi$ is an element of Ω .

Proof: (\Rightarrow) If Ω is a complete story, then there is a N.T.A. under which all the members of Ω are true. So Ω is consistent. So every finite subset of Ω is consistent. This proves 1). 2) follows immediately from e).

(\Leftarrow) Assuming that Ω satisfies 1) and 2), we verify that it satisfies a)-e).

a) (\Rightarrow) Assume, for *reductio ad absurdum*, that $(\varphi \wedge \psi)$ is in Ω but φ and ψ aren't both in Ω . If φ isn't in Ω , then by 2), $\neg\varphi$ is in Ω . But this means that $\{(\varphi \wedge \psi), \neg\varphi\}$ is a finite, inconsistent subset of Ω , contrary to 1). On the other hand, if ψ isn't in Ω , then, by 2), $\neg\psi$ is in Ω , so that $\{(\varphi \wedge \psi), \neg\psi\}$ is a finite, inconsistent subset of Ω , again contradicting 1). Either way, we get a contradiction.

(\Leftarrow) Suppose that both φ and ψ are in Ω but $(\varphi \wedge \psi)$ isn't in Ω . Then, by 2), $\neg(\varphi \wedge \psi)$ is in Ω . But this means that $\{\varphi, \psi, \neg(\varphi \wedge \psi)\}$ is a finite, inconsistent subset of Ω , contrary to 1).

b)-e) are similar. X

Our plan is to start with Γ , which satisfies 1), then to march through the sentences one by one, at each stage adding either the sentence or its negation, making sure that at the end of each stage we still have a set that satisfies 1). The end of the process will satisfy 1) and 2).

The key fact we need is the following:

Lemma. Suppose that every finite subset of Δ is consistent. Take a sentence ψ . Then either every finite subset of $\Delta \cup \{\psi\}$ is consistent or every finite subset of $\Delta \cup \{\neg\psi\}$ is consistent.

Proof: Suppose not. Then there is a finite subset Π of Δ such that $\Pi \cup \{\psi\}$ is inconsistent, and there is a finite subset Σ of Δ such that $\Sigma \cup \{\neg\psi\}$ is inconsistent. $\Pi \cup \Sigma$ is a finite subset of Δ , so

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it's consistent, so there's a N.T.A. \mathfrak{S} under which all the members of $\Pi \cup \Sigma$ are true. If ψ is true under \mathfrak{S} , then \mathfrak{S} is a N.T.A. under which all the members of $\Pi \cup \{\psi\}$ are true, contradicting the fact that $\Pi \cup \{\psi\}$ is inconsistent. On the other hand, if ψ is false under \mathfrak{S} , then \mathfrak{S} is a N.T.A. under which all the members of $\Sigma \cup \{\neg\psi\}$ are true, which contradicts the fact that $\Sigma \cup \{\neg\psi\}$ is inconsistent. Either way, we get a contradiction. X

Now let's enumerate all the sentences of the language, in some sort of alphabetical order, as $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \dots$ and so on. Form an infinite sequence $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \dots$ of sets of sentences, as follows:

$\Gamma_0 = \Gamma$. By hypothesis, every finite subset of Γ_0 is consistent.

If every finite subset of $\Gamma_0 \cup \{\zeta_0\}$ is consistent, let $\Gamma_1 = \Gamma_0 \cup \{\zeta_0\}$.
Otherwise, let $\Gamma_1 = \Gamma_0 \cup \{\neg\zeta_0\}$. The lemma assures us that every finite subset of Γ_1 is consistent.

If every finite subset of $\Gamma_1 \cup \{\zeta_1\}$ is consistent, let $\Gamma_2 = \Gamma_1 \cup \{\zeta_1\}$.
Otherwise, let $\Gamma_2 = \Gamma_1 \cup \{\neg\zeta_1\}$. The lemma assures us that every finite subset of Γ_2 is consistent.

If every finite subset of $\Gamma_2 \cup \{\zeta_2\}$ is consistent, let $\Gamma_3 = \Gamma_2 \cup \{\zeta_2\}$.
Otherwise, let $\Gamma_3 = \Gamma_2 \cup \{\neg\zeta_2\}$. The lemma assures us that every finite subset of Γ_3 is consistent.

And so on. Given Γ_n so that every finite subset of Γ_n is consistent,

If every finite subset of $\Gamma_n \cup \{\zeta_n\}$ is consistent, let $\Gamma_{n+1} = \Gamma_n \cup \{\zeta_n\}$.
Otherwise, let $\Gamma_{n+1} = \Gamma_n \cup \{\neg\zeta_n\}$. The lemma assures us that every finite subset of Γ_{n+1} is consistent.

Finally, let Γ_∞ be the union of the Γ_n s. Clearly, Γ_∞ satisfies conditions 1) and 2), which is what we need. X

Aside (this will only make sense if you know about transfinite numbers): In proving the theorem, we have assumed that the language we're talking about is countable, that is, that it is possible to arrange the sentences of the language in an infinite list, $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \dots$. The way we've defined "language for the sentential calculus," there are mathematically possible languages that aren't countable. The theorem would still be true even if the language were uncountable, but the proof would be a bit more difficult. The same holds for most of the other theorems we'll prove.

Corollary. If ϕ is a logical consequence of a set of sentences Δ , then ϕ is a logical consequence of some finite subset of Δ .

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Proof: If φ is a logical consequence of Δ , then $\Delta \cup \{\neg\varphi\}$ is inconsistent. So, by the Compactness Theorem, some finite subset of $\Delta \cup \{\neg\varphi\}$ is inconsistent. So there is a finite subset Σ of Δ such that $\Sigma \cup \{\neg\varphi\}$ is inconsistent. φ is a logical consequence of Σ . X