Identity

Quantifiers are so-called because they say how many. So far, we've only used the quantifiers to give the crudest possible answers to the question "How many dogs have fleas?": "All," "None," "Some," "Not all," "Some but not all." Now we are going to see how to use the quantifiers to express more detailed numerical information.

"At least one dog has fleas" is translated by an existential quantifier"

(1) $(\exists x)(Dx \wedge Fx)$

We are tempted to symbolize "At least two dogs have fleas" the same way:

(2) $(\exists x)(\exists y)((Dx \wedge Fx) \wedge (Dy \wedge Fy))$

but that would be a mistake, since (2) is logically equivalent to (1), because we haven't said that "x" and "y" refer to different dogs. To say "At least two dogs have fleas," we need the identity sign "=":

(3) $(\exists x)(\exists y)(\neg x{=}y \wedge ((Dx \wedge Fx) \wedge (Dy \wedge Fy)))$

A language for the *predicate calculus with identity* is a language for the predicate calculus, as we defined it before, that satisfies the further stipulation that, among the predicates of the language is the binary "=." An *interpretation* $\mathcal{A}$ of such a language is required to satisfy the condition:

$$\mathcal{A}(\text{“=”}) = \{<a,a>: a \in |\mathcal{A}|\}.$$

To strictly obey the definition of "formula," we should write the predicate "=" at the beginning, writing "=xy" rather than "x=y"; but doing so makes the formulas harder to read.

With "=" in our language, we can answer "How many?" questions much more precisely. "At most one dog has fleas" is the denial of (3):

(4) $\neg(\exists x)(\exists y)(\neg x{=}y \wedge ((Dx \wedge Fx) \wedge (Dy \wedge Fy)))$

To say "Exactly one dog has fleas" is to say that at least one dog has fleas and at most one dog has fleas," so we can symbolize it as the conjunction of (1) and (4):

(5) $((\exists x)(Dx \wedge Fx) \wedge \neg(\exists x)(\exists y)(\neg x{=}y \wedge ((Dx \wedge Fx) \wedge (Dy \wedge Fy))))$

or we can symbolize it more directly, as follows:

(6) $(\exists x)(\forall y)((Dy \wedge Fy) \leftrightarrow y{=}x)$

For larger numbers, we go on the same way. "At least three dogs have fleas" is this:

(7) $(\exists x)(\exists y)(\exists z)((\neg x{=}y \wedge (\neg x{=}z \wedge \neg y{=}z)) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge (Dz \wedge Fz))))$

"At most two dogs have fleas" is the denial of (7), namely:

(8) $\neg(\exists x)(\exists y)(\exists z)((\neg x{=}y \wedge (\neg x{=}z \wedge \neg y{=}z)) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge (Dz \wedge Fz))))$

"Exactly two dogs have fleas" can be written as the conjunction of (3) and (8), thus:

(9) $((\exists x)(\exists y)((Dx \wedge Fx) \wedge (Dy \wedge Fy)) \wedge \neg(\exists x)(\exists y)(\exists z)((\neg x{=}y \wedge (\neg x{=}z \wedge \neg y{=}z)) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge (Dz \wedge Fz)))))$

or we can write it more simply:

(10) $(\exists x)(\exists y)(\neg x{=}y \wedge (\forall z)((Dz \wedge Fz) \leftrightarrow (z{=}x \vee z{=}y)))$

"At least four dogs have fleas" is this:

(11) $(\exists x)(\exists y)(\exists z)(\exists w)((\neg x{=}y \wedge (\neg x{=}z \wedge (\neg x{=}w \wedge (\neg y{=}z \wedge (\neg y{=}w \wedge \neg z{=}w))))) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge ((Dz \wedge Fz) \wedge (Dw \wedge Fw)))))$

(How we group the various conjuncts is arbitrary.) "At most three dogs have fleas" is the denial of ((11), namely:

(12) $\neg(\exists x)(\exists y)(\exists z)(\exists w)((\neg x{=}y \wedge (\neg x{=}z \wedge (\neg x{=}w \wedge (\neg y{=}z \wedge (\neg y{=}w \wedge \neg z{=}w))))) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge ((Dz \wedge Fz) \wedge (Dw \wedge Fw)))))$

"Exactly three dogs have fleas" can be written as the conjunction of (7) and (12):

(13) $((\exists x)(\exists y)(\exists z)((\neg x{=}y \wedge (\neg x{=}z \wedge \neg y{=}z)) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge (Dz \wedge Fz)))) \wedge \neg(\exists x)(\exists y)(\exists z)(\exists w)((\neg x{=}y \wedge (\neg x{=}z \wedge (\neg x{=}w \wedge (\neg y{=}z \wedge (\neg y{=}w \wedge \neg z{=}w))))) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge ((Dz \wedge Fz) \wedge (Dw \wedge Fw))))))$

or more concisely:

(14) $(\exists x)(\exists y)(\exists z)((\neg x{=}y \wedge (\neg x{=}z \wedge \neg y{=}z)) \wedge (\forall w)((Dw \wedge Fw) \leftrightarrow (w{=}x \vee (w{=}y \vee w{=}z))))$

And so it goes for larger numbers. The same techniques will permit us to write "at least thirty" or "at most sixty-four" or "exactly five thousand two hundred eighty."

If we let Γ be the infinite set consisting of sentence (1) ("At least one dog has fleas"), sentence (3) ("At least two dogs have fleas"), sentence (7) ("At least three dogs have fleas"), sentence (11) ("At least four dogs have fleas"), and so on. Then for all the members of Γ to be true, there would have to be infinitely many dogs with fleas. So there is an infinite set of sentences that are jointly true in all and only the models in which there are infinitely many individuals that satisfy "(Dx ∧ Fx)."

By contrast, there is no set of sentences that are jointly true in all and only the models in which only finitely many individuals satisfy "(Dx ∧ Fx)." For suppose there were such a set Δ. Then Γ∪Δ would be inconsistent. By the Compactness Theorem, there is a finite subset Γ^f of Γ and a finite subset Δ^f of Δ such that $\Gamma^f \cup \Delta^f$ is inconsistent. There is a largest number *n* such that the sentence that formalizes "At least *n* dogs have fleas" occurs in Γ^f. Let $\mathcal{A}$ **be a model in which exactly *n* individuals satisfy "(Dx ∧ Fx)." Since at least *n* members of $|\mathcal{A}|$ satisfy "(Dx ∧ Fx)," all the members of Γ^f are true in $\mathcal{A}$. Since only finitely many members of $|\mathcal{A}|$ satisfy "(Dx ∧ Fx)," all the members of Δ are true in $\mathcal{A}$, hence all the members of Δ^f are true in $\mathcal{A}$. So $\Gamma^f \cup \Delta^f$ is consistent. Contradiction.**

There is no single sentence true in all and only the models in which infinitely many individuals satisfy "(Dx ∧ Fx)." If γ were such a sentence {¬γ} would be a set of sentences true in all and only the models in which only finitely many individuals satisfy "(Dx ∧ Fx)."

Our insistence that, within every interpretation, "=" stands for the identity relation on the domain of the interpretation validates some additional rules of inference, namely:

IR (Identity is reflexive.) You may write any sentence of the form c=c, with the empty set of premisses.

SI (Substitution of identicals) If you've written either c=d or d=c with premiss set Γ and you've written $\varphi^v/_c$ with premiss set Δ, you may write $\varphi^v/_d$ with premiss set Γ∪Δ.

To see that SI preserves logical consequence, suppose that $\mathcal{A}$ is an interpretation under which $\varphi^v/_c$ is true and under which either c=d or d=c is true. Let σ be a variable assignment for $\mathcal{A}$ with σ(v) = $\mathcal{A}$(c) = $\mathcal{A}$(d). Then, by the Substitution Principle, $\varphi^v/_c$ is true under $\mathcal{A}$ iff σ satisfies φ under $\mathcal{A}$ iff $\varphi^v/_c$ is true under $\mathcal{A}$.

A relation R is said to be *reflexive* just in case it satisfies the condition (∀x)Rxx. As our first example of a derivation, we show that identity is reflexive:

1 c=c	**IR**
2 (∀x)x=x	**UG, 1**

A relation R is *symmetric* iff it satisfies $(\forall x)(\forall y)(Rxy \rightarrow Rxy)$. We show identity is symmetric:

	1	$c=c$	IR
2	2	$c=d$	PI
2	3	$d=c$	SI, 1, 2
	4	$c=d \rightarrow d=c$	CP, 2, 3
	5	$(\forall y)(c=y \rightarrow y=c)$	UG, 4
	6	$(\forall x)(\forall y)(x=y \rightarrow y=x)$	UG, 5

In applying rule SI at line 3, we are taking our formula φ to be "y=c." Line 1 is $\varphi^{y}/_{c}$, and line 3 is $\varphi^{y}/_{d}$.

We now show that identity is transitive, that is, that it satisfies the condition $(\forall x)(\forall y)(\forall z)((Rxy \wedge Ryz) \rightarrow Rxz)$:

1	1	$(a=b \wedge b=c)$	PI
1	2	$a=b$	TC, 1
1	3	$b=c$	TC, 1
1	4	$a=c$	SI, 2, 3
	5	$((a=b \wedge b=c) \rightarrow a=c)$	CP, 1, 4
	6	$(\forall z)((a=b \wedge b=z) \rightarrow a=z)$	UG, 5
	7	$(\forall y)(\forall z)((a=y \wedge y=z) \rightarrow a=z)$	UG, 6
	8	$(\forall x)(\forall y)(\forall z)((x=y \wedge y=z) \rightarrow x=z)$	UG, 7

Next, let's derive "There are at least two dogs" from "Some, but not all, dogs have fleas":

1	1	$((\exists x)(Dx \wedge Fx) \wedge \neg(\forall x)(Dx \rightarrow Fx))$	PI
1	2	$(\exists x)(Dx \wedge Fx)$	TC, 1
3	3	$(Da \wedge Fa)$	PI (for ES)
3	4	Da	TC, 3
3	5	Fa	TC, 3
1	6	$\neg(\forall x)(Dx \rightarrow Fx)$	TC, 1
1	7	$(\exists x)\neg(Dx \rightarrow Fx)$	QE, 6
8	8	$\neg(Db \rightarrow Fb)$	PI (for ES)
8	9	Db	TC, 8
8	10	$\neg Fb$	TC, 8
11	11	$a=b$	PI
3,11	12	Fb	SI, 5, 11
3	13	$(a=b \rightarrow Fb)$	CP, 11, 12
3,8	14	$\neg a=b$	TC, 10, 13
3,8	15	$(\neg a=b \wedge (Da \wedge Db))$	TC, 4, 9, 14

3,8	16	$(\exists y)(\neg a=y \wedge (Da \wedge Dy))$	EG, 15
1,3	17	$(\exists y)(\neg a=y \wedge (Da \wedge Dy))$	ES, 7, 8, 16
1,3	18	$(\exists x)(\exists y)(\neg x=y \wedge (Dx \wedge Dy))$	EG, 17
1	19	$(\exists x)(\exists y)(\neg x=y \wedge (Dx \wedge Dy))$	ES, 2, 3, 18

We'll now use a derivation to show that our two symbolizations of "Exactly one dog has fleas" are equivalent. First, we'll show that (5) implies (6):

1	1	**$((\exists x)(Dx \wedge Fx) \wedge \neg(\exists x)(\exists y)(\neg x=y \wedge ((Dx \wedge Fx) \wedge (Dy \wedge Fy))))$**	PI
1	2	$(\exists x)(Dx \wedge Fx)$	TC, 1
1	3	**$\neg(\exists x)(\exists y)(\neg x=y \wedge ((Dx \wedge Fx) \wedge (Dy \wedge Fy)))$**	TC, 1
4	4	$(Da \wedge Fa)$	PI (for ES)
5	5	$(Db \wedge Fb)$	PI
1	6	$(\forall x)\neg(\exists y)(\neg x=y \wedge ((Dx \wedge Fx) \wedge (Dy \wedge Fy)))$	QE, 3
1	7	$\neg(\exists y)(\neg a=y \wedge ((Da \wedge Fa) \wedge (Dy \wedge Fy)))$	US, 6
1	8	$(\forall y)\neg(\neg a=y \wedge ((Da \wedge Fa) \wedge (Dy \wedge Fy)))$	QE, 7
1	9	$\neg(\neg a=b \wedge ((Da \wedge Fa) \wedge (Db \wedge Fb)))$	US, 8
1,4,5	10	$a=b$	TC, 4, 5, 9
1,4	11	$((Db \wedge Fb) \rightarrow a=b)$	CP, 5, 10
12	12	$a=b$	PI
4,12	13	$(Db \wedge Fb)$	SI, 4, 12
4	14	$(a=b \rightarrow (Db \wedge Fb))$	CP, 12, 13
1, 4	15	$((Db \wedge Fb) \leftrightarrow a=b)$	TC, 11, 14
1,4	16	$(\forall y)((Dy \wedge Fy) \leftrightarrow a=y)$	UG, 15
1,4	17	$(\exists x)(\forall y)((Dy \wedge Fy) \leftrightarrow x=y)$	EG, 16
1	18	$(\exists x)(\forall y)((Dy \wedge Fy) \leftrightarrow x=y)$	ES, 2, 4, 17

Now for the other direction:

1	1	$(\exists x)(\forall y)((Dy \wedge Fy) \leftrightarrow x=y)$	PI
2	2	$(\forall y)((Dy \wedge Fy) \leftrightarrow c=y)$	PI (for ES)
2	3	$((Dc \wedge Fc) \leftrightarrow c=c)$	US, 2
	4	$c=c$	IR
2	5	$(Dc \wedge Fc)$	TC, 3, 4
2	6	$(\exists x)(Dx \wedge Fx)$	EG, 5
1	7	$(\exists x)(Dx \wedge Fx)$	ES, 1, 2, 6
8	8	$((Dd \wedge Fd) \wedge (De \wedge Fe))$	PI
8	9	$(Dd \wedge Fd)$	TC, 8
8	10	$(De \wedge Fe)$	TC, 8
2	11	$((Dd \wedge Fd) \leftrightarrow c=d)$	US, 2
2,8	12	$c=d$	TC, 9, 11
2	13	$((De \wedge Fe) \leftrightarrow c=e)$	US, 2

2,8	14 c=e	TC, 10, 13
2,8	15 d=e	SI, 12, 14
2	16 (((Dd ∧ Fd) ∧ (De ∧ Fe)) → d=e)	CP, 8, 15
2	17 ¬(¬d=e ∧ ((Dd ∧ Fd) ∧ (De ∧ Fe)))	TC, 16
2	18 (∀y)¬(¬d=y ∧ ((Dd ∧ Fd) ∧ (Dy ∧ Fy)))	UG, 17
2	19 ¬(∃y)(¬d=y ∧ ((Dd ∧ Fd) ∧ (Dy ∧ Fy)))	QE, 18
2	20 (∀x)¬(∃y)(¬x=y ∧ ((Dx ∧ Fx) ∧ (Dy ∧ Fy)))	UG, 19
2	21 ¬(∃x)(∃y)(¬x=y ∧ ((Dx ∧ Fx) ∧ (Dy ∧ Fy)))	QE, 20
1	22 ¬(∃x)(∃y)(¬x=y ∧ ((Dx ∧ Fx) ∧ (Dy ∧ Fy)))	ES, 1, 2, 21
1	23 ((∃x)(Dx ∧ Fx) ∧ ¬(∃x)(∃y)(¬x=y ∧ ((Dx ∧ Fx) ∧ (Dy ∧ Fy)))	TC, 7, 22

Now let's derive "There are at least four dogs" from "There are at least two dogs with fleas" and "There are at least two dogs without fleas":

1	1 (∃x)(∃y)(¬x=y ∧ ((Dx ∧ Fx) ∧ (Dy ∧ Fy)))	PI
2	2 (∃x)(∃y)(¬x=y ∧ ((Dx ∧ ¬Fx) ∧ (Dy ∧ ¬Fy)))	PI
3	3 (∃y)(¬a=y ∧ ((Da ∧ Fa) ∧ (Dy ∧ Fy)))	PI (for ES)
4	4 (¬a=b ∧ ((Da ∧ Fa) ∧ (Db ∧ Fb)))	PI (for ES)
4	5 ¬a=b	TC, 4
4	6 Da	TC, 4
4	7 Fa	TC, 4
4	8 Db	TC, 4
4	9 Fb	TC, 4
10	10 (∃y)(¬c=y ∧ ((Dc ∧ ¬Fc) ∧ (Dy ∧ ¬Fy)))	PI (for ES)
11	11 (¬c=d ∧ ((Dc ∧ ¬Fc) ∧ (Dd ∧ ¬Fd)))	PI (for ES)
11	12 ¬c=d	TC, 11
11	13 Dc	TC, 11
11	14 ¬Fc	TC, 11
11	15 Dd	TC, 11
11	16 ¬Fd	TC, 11
17	17 a=c	PI
4,17	18 Fc	SI, 7, 17
4	19 (a=c → Fc)	CP, 17, 18
4,11	20 ¬a=c	TC, 14, 19
21	21 a=d	PI
4,21	22 Fd	SI, 7, 21
4	23 (a=d → Fd)	CP, 21, 22
4,11	24 ¬a=d	TC, 16, 23
25	25 b=c	PI
4,25	26 Fc	SI, 9, 25
4	27 (b=c → Fc)	CP, 25, 26
4,11	28 ¬b=c	TC, 14, 27
29	29 b=d	PI

4,29	30	Fd	SI, 9, 29
4	31	$(b=d \rightarrow Fd)$	CP, 29, 30
4,11	32	$\neg b=d$	TC, 16, 31
4,11	33	$((\neg a=b \wedge (\neg a=c \wedge (\neg a=d \wedge (\neg b=c \wedge (\neg b=d \wedge \neg c=d))))) \wedge (Da \wedge (Db \wedge (Dc \wedge Dd))))$	TC,5,6,8,12,13,15,20,24,28,32
4,11	34	$(\exists w)((\neg a=b \wedge (\neg a=c \wedge (\neg a=w \wedge (\neg b=c \wedge (\neg b=w \wedge \neg c=w))))) \wedge (Da \wedge (Db \wedge (Dc \wedge Dw))))$	EG, 33
4,10	35	$(\exists w)((\neg a=b \wedge (\neg a=c \wedge (\neg a=w \wedge (\neg b=c \wedge (\neg b=w \wedge \neg c=w))))) \wedge (Da \wedge (Db \wedge (Dc \wedge Dw))))$	ES, 10, 11, 34
4,10	36	$(\exists z)(\exists w)((\neg a=b \wedge (\neg a=z \wedge (\neg a=w \wedge (\neg b=z \wedge (\neg b=w \wedge \neg z=w))))) \wedge (Da \wedge (Db \wedge (Dz \wedge Dw))))$	EG, 35
2,4	37	$(\exists z)(\exists w)((\neg a=b \wedge (\neg a=z \wedge (\neg a=w \wedge (\neg b=z \wedge (\neg b=w \wedge \neg z=w))))) \wedge (Da \wedge (Db \wedge (Dz \wedge Dw))))$	ES, 2, 10, 36
2,4	38	$(\exists y)(\exists z)(\exists w)((\neg a=y \wedge (\neg a=z \wedge (\neg a=w \wedge (\neg y=z \wedge (\neg y=w \wedge \neg z=w))))) \wedge (Da \wedge (Dy \wedge (Dz \wedge Dw))))$	EG, 37
2,3	39	$(\exists y)(\exists z)(\exists w)((\neg a=y \wedge (\neg a=z \wedge (\neg a=w \wedge (\neg y=z \wedge (\neg y=w \wedge \neg z=w))))) \wedge (Da \wedge (Dy \wedge (Dz \wedge Dw))))$	ES, 3, 4, 38
2,3	40	$(\exists x)(\exists y)(\exists z)(\exists w)((\neg x=y \wedge (\neg x=z \wedge (\neg x=w \wedge (\neg y=z \wedge (\neg y=w \wedge \neg z=w))))) \wedge (Dx \wedge (Dy \wedge (Dz \wedge Dw))))$	EG, 39
1,2	41	$(\exists x)(\exists y)(\exists z)(\exists w)((\neg x=y \wedge (\neg x=z \wedge (\neg x=w \wedge (\neg y=z \wedge (\neg y=w \wedge \neg z=w))))) \wedge (Dx \wedge (Dy \wedge (Dz \wedge Dw))))$	ES, 1, 3, 40

We now show that our two symbolizations of "Exactly two dogs have fleas" are equivalent. First, we show that (9) implies (10):

1	1	$\boldsymbol{((\exists x)(\exists y)((Dx \wedge Fx) \wedge (Dy \wedge Fy)) \wedge \neg(\exists x)(\exists y)(\exists z)((\neg x=y \wedge (\neg x=z \wedge \neg y=z)) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge (Dz \wedge Fz)))))}$	PI
1	2	$\boldsymbol{(\exists x)(\exists y)((Dx \wedge Fx) \wedge (Dy \wedge Fy))}$	TC, 1
1	3	$\boldsymbol{\neg(\exists x)(\exists y)(\exists z)((\neg x=y \wedge (\neg x=z \wedge \neg y=z)) \wedge ((Dx \wedge Fx) \wedge ((Dy \wedge Fy) \wedge (Dz \wedge Fz)))}$	TC, 1