Substitutions

A *substitution* is a function s associating SC sentences with SC sentences that meets the following conditions:

$$s((\varphi \vee \psi)) = (s(\varphi) \vee s(\psi))$$
$$s((\varphi \wedge \psi)) = (s(\varphi) \wedge s(\psi))$$
$$s((\varphi \rightarrow \psi)) = (s(\varphi) \rightarrow s(\psi))$$
$$s((\varphi \leftrightarrow \psi)) = (s(\varphi) \leftrightarrow s(\psi))$$
$$s(\neg\varphi) = \neg s(\varphi)$$

For example, if s(“A”) = “(C → D)” and s(“B”) = “(D ↔ ¬E), then s(“(A ∧ ¬B)”) = “((C → D) ∧ ¬(D ↔ ¬E)).”

If φ is a sentence and s is a substitution, then s(φ) is said to be a *substitution instance* of φ.

If s is a substitution and $\Im$ is a N.T.A., let $\Im$°s be the N.T.A. given by

$$\Im{\circ}s(\varphi) = \Im(s(\varphi)),$$

for every atomic sentence φ. It’s easy to convince ourselves that the equation

$$\Im{\circ}s(\varphi) = \Im(s(\varphi))$$

holds for all sentences, complex as well as simple.

Substitution Theorem 1. Any substitution instance of a tautology is a tautology. Any substitution instance of a contradiction is a contradiction.

Proof: Suppose that φ is a tautology and s is a substitution. Take any N.T.A. $\Im$. Because φ is a tautology and $\Im$°s is a N.T.A., $\Im$°s(φ) = 1. So s(φ) is true under $\Im$. Since $\Im$ was arbitrary, we conclude that s(φ) is true under every N.T.A., and hence that φ is a tautology. The argument for contradictions is similar.X

Substitution Theorem 2. Let s be a substitution. If φ implies ψ, then s(φ) implies s(ψ). If φ and ψ are logically equivalent, s(φ) and s(ψ) are logically equivalent. If φ is a logical consequence of Γ, then s(φ) is a logical consequence of {s(γ): γ ∈ Γ}.

Proof: Similar.X

In analogy with the theorem before last, you might expect that every substitution instance of a consistent sentence is consistent. But that’s not true. A counterexample is the inconsistent sentence "$((Q \wedge \neg Q) \wedge P)$," which is a substitution instance of the consistent sentence "$(A \wedge B)$." What we have instead is this:

> **Substitution Theorem 3.** A sentence φ is consistent if and only if some substitution instance of φ is tautological.

Proof: (⇒) Let $\Im$ be a N.T.A. under which φ is true. Define a substitution s by:

$$s(\psi) = \text{"}(P \vee \neg P)\text{" if } \psi \text{ is an atomic sentence that is true under } \Im$$
$$= \text{"}(P \wedge \neg P)\text{" if } \psi \text{ is an atomic sentence that is false under } \Im$$

It is easy to convince ourselves that, for any sentence θ, if θ is true under $\Im$, then $s(\theta)$ is a tautology, whereas if θ is false under $\Im$,$s(\theta)$ is a contradiction. To show this in detail, we'd give a proof by reductio ad absurdum: Assume that the thing you're trying to prove is false, then show that this assumption leads to a contradiction. So assume that there a sentence θ such that either $\Im(\theta)$= 1 but $s(\theta)$ isn't tautological or $\Im(\theta) = 0$ even though $s(\theta)$ isn't contradictory. Let θ be a simplest such sentence. The proof the breaks down into six cases, depending on whether θ is atomic, a disjunction, a conjunction, a conditional, a biconditional, or a negation. I won't go through the details.

Since $\Im(\varphi)$= 1, $s(\varphi)$ is a tautological substitution instance of φ.

(⇐) If φ is inconsistent, then every substitution instance of φ is inconsistent. So no substitution instance of φ is tautological.X

> **Substitution Theorem 4.** A sentence φ is tautological iff every substitution instance of φ is tautological iff every substitution instance of φ is consistent. A sentence ψ is contradictory iff every substitution instance of ψ is contradictory iff every substitution instance of ψ is invalid.

Proof: Let **(a)** be "φ is tautological," **(b)** be "Every substitution instance of φ is tautological," and **(c)** be "Every substitution instance of φ is consistent. We show, first, that **(a)** implies **(b)**, next that **(b)** implies **(c)**, and finally that **(c)** implies **(a)**.

(a) ⇒ (b): Substitution Theorem 1.

(b) ⇒ (c): Immediate.

(c) ⇒ (a): What we'll actually prove is that the negation of **(a)** implies the negation of **(c)**, which comes to the same thing. If φ isn't tautological, then $\neg\varphi$ is consistent. So, by Substtution Theorem 3, there is a substitution s such that $s(\neg\varphi)$ is tautological. Since the negation of $s(\varphi)$ is tautological, $s(\varphi)$ is contradictory. So φ has a substitution instance that is inconsistent.

We could prove the second part of Substitution Theorem 4 the same way, but a quicker proof appeals to the first part of Substitution Theorem 4, thus:

Ψ is contradictory
iff ¬ψ is tautological
iff every substitution instance of ¬ψ is tautological
[because **(a)** is equivalent to **(b)**}
iff every substitution instance of ψ is contradictory
iff every substitution instance of ¬ψ is consistent
[because **(b)** is equivalent to **(c)**]
iff every substitution instance of ψ is invalid.X̲

Let φ be a sentence whose only connectives are "∧," "∨," and "¬." Let φ^{Dual} be the sentence obtained from φ by exchanging "∧"s and "∨"s everywhere. Let *d* be the substitution that replaces each atomic sentence by its negation. It's easy to convince ourselves, using de Morgan's laws, that φ^{Dual} is logically equivalent to the negation of *d*(φ). Hence:

Substitution Theorem 5. Let φ and ψ be sentences whose only connectives are "∧," "∨," and "¬." Then if φ implies ψ, ψ^{Dual} implies φ^{Dual}. If φ is logically equivalent to ψ, φ^{Dual} is logically equivalent to ψ^{Dual}.

Proof: If φ implies ψ then, by Substitution Theorem 2, *d*(φ) implies *d*(ψ). So the negation of φ^{Dual} implies the negation of ψ^{Dual}. So there is no N.T.A. under which the negation of φ^{Dual} is true and the negation of ψ^{Dual} is false. Hence there is no N.T.A. under which ψ^{Dual} is true and φ^{Dual} is false; that is, ψ^{Dual} implies φ^{Dual}.

The second part of Substitution Theorem 5 appeals to the first. If φ is logically equivalent to ψ, then φ implies ψ and ψ implies φ. It follows by the first part of the theorem that ψ^{Dual} implies φ^{Dual} and φ^{Dual} implies ψ^{Dual}. Consequently, φ^{Dual} is logically equivalent to ψ^{Dual}.X̲