Derivations in the Monadic Predicate Calculus

In the last chapter, we described a procedure for testing whether an MPC sentence is valid. Unfortunately, the method is not very practically useful, simply because it takes too long to check all the canonical models. We shall no learn another method for showing a valid sentence valid that is more useful in practice. If an MPC sentence is valid, this procedure will show what it is valid, and it will usually do so fairly efficiently. However, unlike the earlier method, the new method won't given us any way to show an invalid sentence to be invalid. If a sentence if valid, the method will enable us to prove it, but, if we have been unable to construct a proof of a sentence, we won’t have any way of knowing whether the reason is because the sentence is invalid or just because we haven't been clever enough to come up with a proof yet. The old method was a *decision procedure*: It enables us to test whether a given sentence is valid. The new method is only a *proof procedure*: If a sentence is valid, the method will enable to show that it is valid, but the method won’t provide us any way of showing that an invalid sentence is invalid.

Our real reason for learning the new method won’t appear until we turn to the study of the full predicate calculus (as opposed to the monadic predicate calculus), where we talk about relations among individuals as well as properties of individuals. The new method generalizes to the full predicate calculus. The old method does not.

There are a great number of systems of proof in use. The particular system we shall study here was developed by Benson Mates; see his *Elementary Logic* (2nd ed. New York: Oxford University Press, 1972).

A *proof* or *derivation* consists of a consecutively numbered sequence of sentences. The number of a sentence, which is written directly to its left, is its *line number*. To the left of the line number for the nth line is a (possibly empty) sequence of numbers ≤ n. These are the *premiss numbers* of the nth line. The sentences whose line numbers are the premiss numbers of the nth line constitute the *premiss set* of the nth line. To the right of the nth sentence, a rule is cited which justifies our writing the sentence. Here is an example of a derivation; explanations will come later:

1	1. $(\forall x)(Gx \to Hx)$	PI
2	2. $(\forall x)(Hx \to Mx)$	PI
3	3. Ga	PI
1	4. $(Ga \to Ha)$	US, 1

1,3	5.	Ha	TC, 3,4
2	6.	(Ha → Ma)	US, 2
1,2,3	7.	Ma	TC, 5,6
1,2	8.	(Ga → Ma)	CP, 3,7
1,2	9	(∀x)(Gx → Mx)	UG, 8

Here the premiss set of line 1 is {"(∀x)(Gx → Hx)"}; line 1 is gotten by rule P (what rule P is will appear presently). The premiss set of line 7 is {"(∀x)(Gx → Hx)," "(∀x)(Hx → Mx)," "Ga"}; line 7 is gotten from lines 5 and 6 by rule TC. The premiss set of line 9 is {"(∀x)(Gx → Hx)," "(∀x)(Hx → Mx)"}; line 9 is gotten from line 8 by rule UG.

The rules we shall develop are intended to guarantee that each sentence we write down is a logical consequence of its premiss set. Thus the derivation above is a demonstration that "All Greeks are mortal" ["(∀x)(Gx → Mx)"] is a logical consequence of "All Greeks are human beings" ["(∀x)(Gx → Hx)"] and "All human beings are mortal" ["(∀x)(Hx → Mx)"]. (The derivation is longer than it needed to be.)

The main thing we want to make sure of in introducing our rules of derivation is that each sentence we write down at each stage of a derivation is a logical consequence of its premiss set. If we introduce a new rule, what we have to make sure of is this: If we use the new rule to add a new line to a derivation that up till now has had the property that each line is a logical consequence of its premiss set, the new line will be a logical consequence of its premiss set. If all our rules have this property, we can be confident that any sentence we write down at any stage of a derivation will be a logical consequence of its premiss set.

The first three rules we learn will only involve ideas from the sentential calculus. Afterward we shall learn rules involving the quantifiers.

> **Premiss introduction rule (PI).** At any stage of a derivation, you may write down a sentence φ with any set that contains φ as its premiss set.

Obviously, anything you write down by rule PI will be a logical consequence of its premiss set, since φ is a logical consequence of any set that includes it. One use of rule PI is simple to let us write down the premisses of our argument. Other, more technical uses, will emerge.

Our next rule incorporates the entire sentential calculus at one fell swoop:

> **Tautological consequence rule (TC).** If you have written down sentences ψ_1, ψ_2,..., ψ_n in a derivation, and φ is a tautological consequence of $\{\psi_1,\psi_2,...,\psi_n\}$, then you may write down sentence ψ, taking the premiss set to be the union of the premiss sets of the ψ_is. In particular, if φ is a tautology, we can write φ with the empty premiss set.

For example, using "Wx" for "x went up the hill," "a" for "Jack," "b" for "Jill," and "c" for "Clarissa," here is a simple derivation of "Either Jill or Clarissa went up the hill" ["(Wb ∨ Wc)"] from "Either Jack or Jill went up the hill" ["(Wa ∨ Wb)"] and "If Jack went up the hill, so did Clarissa" ["(Wa → Wc)"]:

1	1. (Wa ∨ Wb)	PI
2	2. (Wa → Wc)	PI
1,2	3. (Wb ∨ Wc)	TC, 1, 2

We can check that line 3 is truly a logical consequence of lines 1 and 2 by applying the search-for-counterexample method, putting a "1" under the main connective of each premiss, and a “0” under the main connective of the conclusion:

(Wa ∨ Wb)	(Wa → Wc)	∴(Wb ∨ Wc)
~~1 1 0~~	~~X 1 0~~	~~0 0 0~~

The fact that there is a mechanical procedure for testing whether a sentence is a tautological consequence of a set of sentences is important. In order for our derivations to have any probative value, we have to be able to recognize when a sequence of sentences really is a proof, which means that we need an algorithm for checking when a rule has been properly applied. The rule "You may write down a sentence whenever that sentence is a logical consequence of its premiss set" is an unacceptable rule, just because we have no way of recognizing when the rule has been successfully followed.

We want to see that rule TC is logical-consequence preserving, that is, we want to see that, if φ is gotten from $\psi_1, \psi_2, ..., \psi_n$ by rule TC and each of the ψ_is is a logical consequence of its premiss set, then φ is a logical consequence of its premiss set. In showing this, we'll make use of the following three facts:

> If φ is a tautological consequence of Γ, then φ is a logical consequence of Γ.
>
> If ψ is a logical consequence of Δ and Δ is a subset of Ω, then ψ is a logical consequence of Ω.
>
> If φ is a logical consequence of Γ and every member of Γ is a logical consequence of Ω, then φ is a logical consequence of Ω.

Let the premiss set of ψ_i be Δ_i. If φ is a tautological consequence of the $\{\psi_1, \psi_2, ..., \psi_n\}$, then φ is a logical consequence of $\{\psi_1, \psi_2, ..., \psi_n\}$. If each ψ_k is a logical consequence of Δ_k, then ψ_k is a logical consequence of the union of the Δ_is. So φ is a logical consequence of the union of the Δ_is.

TC is an immensely powerful rule, for it incorporates the entire sentential calculus at one fell swoop. For example, consider the following argument:

> Either Preston or Quincy is a member of the Logic Club. If either Quincy or Rudolf is a member, Stuart is not. Unless Stuart is a member, Trumbull is a member and Rudolf is not. But Preston is not a member. Consequently, Quincy and Trumbull are both members.

which we can symbolize as follows:

$(Mp \lor Mq)$
$((Mq \lor Mr) \to \neg Ms)$
$(\neg Ms \to (Mt \land \neg Mr))$
$\neg Mp$
$\therefore (Mq \land Mt)$

The conclusion of this argument is a tautological consequence of the premisses, so we can derive the conclusion from the premisses, as follows:

1	1.	(Mp ∨ Mq)	PI
2	2.	((Mq ∨ Mr) → ¬Ms)	PI
3	3.	(¬Ms → (Mt ∧ ¬Mr))	PI
4	4.	¬Mp	PI
1,2,3,4	5.	(Mq ∧ Mt)	TC,1,2,3,4

This is a perfectly good derivation, as far as the rules go, since the conclusion is a tautological consequence of the premisses. But most of us aren't good enough at the sentential calculus to see at a glance that the conclusion is a tautological consequence of the premisses. The rest of us will have to break the proof down into simpler parts.

First of all, we notice that the sentence we are trying to prove is a conjunction; the way to prove a conjunction is to prove each of the conjuncts, so we want to prove "Mq" and "Mt." But notice that lines 1 and 4 already give us "Mq"; so we're half done. We write the following:

1	1.	(Mp ∨ Mq)	PI
2	2.	((Mq ∨ Mr) → ¬Ms)	PI
3	3.	(¬Ms → (Mt ∧ ¬Mr))	PI
4	4.	¬Mp	PI
1,4	5.	Mq	TC,1,4

Now we have "Mq." "Mq" gives us "(Mq ∨ Mr)," which gives us "¬Ms" by premiss 2:

1	1.	(Mp ∨ Mq)	PI
2	2.	((Mq ∨ Mr) → ¬Ms)	PI
3	3.	(¬Ms → (Mt ∧ ¬Mr))	PI
4	4.	¬Mp	PI
1,4	5.	Mq	TC,1,4
1,4	6.	(Mq ∨ Mr)	TC,5
1,2,4	7.	¬Ms	TC,2,6

We have "¬Ms," which is the antecedent of line 3, so we can derive its consequent:

1	1.	(Mp ∨ Mq)	PI
2	2.	((Mq ∨ Mr) → ¬Ms)	PI

3	3. (¬Ms → (Mt ∧ ¬Mr))	PI
4	4. ¬Mp	PI
1,4	5. Mq	TC,1,4
1,4	6. (Mq ∨ Mr)	TC,5
1,2,4	7. ¬Ms	TC,2,6
1,2,3,4	8. (Mt ∧ ¬Mr)	TC,3,7

We have a conjunction, so we can derive both its conjuncts. In general, it's a good idea to do so, because the conjuncts, being simpler, are easier to deal with than the conjunction:

1	1. (Mp ∨ Mq)	PI
2	2. ((Mq ∨ Mr) → ¬Ms)	PI
3	3. (¬Ms → (Mt ∧ ¬Mr))	PI
4	4. ¬Mp	PI
1,4	5. Mq	TC,1,4
1,4	6. (Mq ∨ Mr)	TC,5
1,2,4	7. ¬Ms	TC,2,6
1,2,3,4	8. (Mt ∧ ¬Mr)	TC,3,7
1,2,3,4	9. Mt	TC,8
1,2,3,4	10. ¬Mr	TC,8

Now we have "Mt" as well as "Mq," which is what we wanted:

1	1. (Mp ∨ Mq)	PI
2	2. ((Mq ∨ Mr) → ¬Ms)	PI
3	3. (¬Ms → (Mt ∧ ¬Mr))	PI
4	4. ¬Mp	PI
1,4	5. Mq	TC,1,4
1,4	6. (Mq ∨ Mr)	TC,5
1,2,4	7. ¬Ms	TC,2,6
1,2,3,4	8. (Mt ∧ ¬Mr)	TC,3,7
1,2,3,4	9. Mt	TC,8
1,2,3,4	10. ¬Mr	TC,8
1,2,3,4	11. (Mq ∧ Mt)	TC,5,10

And we are done.

A little later, we'll talk about some general strategies to use in developing these proofs. But first, we'll get some more rules.

Our next rule gives us a method for proving a conditional: To prove a conditional, assume the antecedent as a premiss, then try to derive the consequent. This will be the central strategy for almost all our proofs.

> **Conditional proof rule (CP).** If you have derived ψ with premiss set $\Gamma \cup \{\varphi\}$, you may write $(\varphi \to \psi)$ with premiss set Γ.

Rule CP is logical-consequence preserving, since, if ψ is a logical consequence of $\Gamma \cup \{\varphi\}$, then $(\varphi \to \psi)$ is a logical consequence of Γ.

As an example, let's derive "(Fa → (Fb → Fc))" from "((Fa ∧ Fb) → Fc)." We want to fill in the blank in this derivation:

1	1.	((Fa ∧ Fb) → Fc)	PI
1		(Fa → (Fb → Fc))	

The sentence we want to prove is a conditional, so we assume the antecedent as a premiss and try to derive the consequent:

1	1.	((Fa ∧ Fb) → Fc)	PI
2	2.	Fa	PI
1,2		(Fb → Fc)	
1		(Fa → (Fb → Fc))	CP,2,

Once we've derived "(Fb → Fc)" with lines 1 and 2 as premiss set, we can use rule CP to derive "(Fa → (Fb → Fc))" with line 1 as premiss set.

Now again the sentence we want to prove is a conditional, so, once again, we assume the antecedent and try to derive the consequent:

1	1.	((Fa ∧ Fb) → Fc)	PI
2	2.	Fa	PI
3	3.	Fb	PI
1,2,3		Fc	

1,2	$(Fb \rightarrow Fc)$	CP,3,
1	$(Fa \rightarrow (Fb \rightarrow Fc))$	CP,2,

Once we've derived "Fc" with lines 1, 2, and 3 as premiss set, we can use rule CP to derive "$(Fb \rightarrow Fc)$" with lines 1 and 2 as premiss set.

We want to get "Fc." Because of line 1, to get "Fc" it will be enough to get "$(Fa \wedge Fb)$." But we can get "$(Fa \wedge Fb)$" because we have its conjuncts on lines 2 and 3; so we're able to complete the proof:

1	1. $((Fa \wedge Fb) \rightarrow Fc)$	PI
2	2. Fa	PI
3	3. Fb	PI
2,3	4. $(Fa \wedge Fb)$	TC,2,3
1,2,3	5. Fc	TC,1,4
1,2	6. $(Fb \rightarrow Fc)$	CP,3,5
1	7. $(Fa \rightarrow (Fb \rightarrow Fc))$	CP,2,6

So far, all the rules we've introduced have just used ideas we've taken over from the sentential calculus. Now we're going to learn some rules that describe the operation of the quantifiers.

> **Universal specification rule (US).** If you've derived $(\forall x)\varphi$, you may derive $\varphi^x/_c$ with the same premiss set, for any constant c.

For example, we derive "Ms" ("Socrates is mortal") from "Gs" ("Socrates is a Greek") and "$(\forall x)(Gx \rightarrow Mx)$" ("All Greeks are mortal"):

1	1. Gs	PI
2	2. $(\forall x)(Gx \rightarrow Mx)$	PI
2	3. $(Gs \rightarrow Ms)$	US,2
1,2	4. Ms	TC,1,3

It's clear that rule US is logical-consequence preserving, because $\varphi^x/_c$ is a logical consequence of $(\forall x)\varphi$. If $(\forall x)\varphi$ is true under $\mathcal{A}$, **every member of ′ satisfies φ under $\mathcal{A}$. So, in particular, $\mathcal{A}$(c) satisfies φ under $\mathcal{A}$, so that, by the Substitution Principle, $\varphi^x/_c$ is true under $\mathcal{A}$.**

> **Universal generalization rule (UG). If you've derived $\varphi^x/_c$ from Γ and if the constant c doesn't**

appear in φ or in any of the sentences in Γ, you may derive (∀x)φ with premiss set Γ.

For example, we derive "(∀x)(Gx → Mx)" ("All Greeks are mortal") from "(∀x)(Gx → Hx)" ("All Greeks are human beings") and "(∀x)(Hx → Mx)" ("All human beings are mortal"):

1	**1. (∀x)(Gx → Hx)**	**PI**
2	**2. (∀x)(Hx → Mx)**	**PI**
3	**3. Ga**	**PI**
1	**4. (Ga → Ha)**	**US,1**
1,3	**5. Ha**	**TC,3,4**
2	**6. (Ha → Ma)**	**US,2**
1,2,3	**7. Ma**	**TC,5,6**
1,2	**8. (Ga → Ma)**	**CP,3,7**
1,2	**9. (∀x)(Gx → Mx)**	**UG,8**

Here, line 8 is gotten from "(Gx → Mx)," by substituting "a" for free occurrences of "x," while line nine is gotten from the same formula by prefixing the universal quantifier "(∀x)." Since the constant "a" doesn't appear in the formula "(Gx → Mx)" and it doesn't appear in the premisses of line 8, we can derive line 9 from line 8 by rule UG.

The idea behind rule UG is that, if you have derived $\varphi^x/_c$ from Γ, where c doesn't appear in Γ or in φ, then the reason you know $\varphi^x/_c$ is true if all the members of Γ are true can't have anything special to do with the particular individual named by c, because you don't know anything about the particular individual named by c; that individual isn't even mentioned in the premiss set. Whatever reasons you have for believing that the individual named by c satisfies φ are reasons that would apply just as well to any other element of the universe. So every member of the universe satisfies φ. So (∀x)φ is true.

More formally, we show that rule UG is logical-consequence preserving, as follows: Suppose that $\varphi^x/_c$ is a logical consequence of Γ and that the constant c doesn't appear in φ or in any of the members of Γ. We want to see that (∀x)φ is a logical consequence of Γ.

Take an interpretation $\mathcal{A}$ under which all the members of Γ are true. We want to see that $(\forall x)\varphi$ is true under $\mathcal{A}$. Take a member a of $|\mathcal{A}|$. We want to show that a satisfies φ under $\mathcal{A}$. Since a was chosen arbitrarily, this will tell us that every member of $|\mathcal{A}|$ satisfies φ under $\mathcal{A}$, so that $(\forall x)\varphi$ is true under $\mathcal{A}$.

Let $\mathcal{B}$ be an interpretation which is just like $\mathcal{A}$ except that $\mathcal{B}$(c) = a. It follows from the Locality Principle that all the members of Γ are true under $\mathcal{B}$. Hence $\varphi^{x}/_{c}$ is true under $\mathcal{B}$. It follows by the Substitution Principle that $\mathcal{B}$(c), which is a, satisfies φ under $\mathcal{B}$. Using the Locality Principle again, we know that a satisfies φ under $\mathcal{A}$, which is what we wanted to show.

As another example, let's derive "$(\forall x)(Cx \vee Ax)$" ("Everyone is either a child or an adult") from "$(\forall x)(Cx \vee (Mx \vee Wx))$," "$(\forall x)(Mx \rightarrow Ax)$," and "$(\forall x)(Wx \rightarrow Ax)$"; we want to fill in the blank in this:

1	**1. $(\forall x)(Cx \vee (Mx \vee Wx))$**	**PI**
2	**2. $(\forall x)(Mx \rightarrow Ax)$**	**PI**
3	**3. $(\forall x)(Wx \rightarrow Ax)$**	**PI**
1,2,3	**$(\forall x)(Cx \vee Ax)$**	

The sentence we're trying to prove is universal. There is a general strategy for proving universal sentences: To prove $(\forall x)\varphi$, pick a new constant c that doesn't appear anywhere else in the proof, and try to prove $\varphi^{x}/_{c}$ with the same premisses; then use UG. Thus we try to prove "$(Cc \vee Ac)$":

1	1. (∀x)(Cx ∨ (Mx ∨ Wx))	PI
2	2. (∀x)(Mx → Ax)	PI
3	3. (∀x)(Wx → Ax)	PI
1,2,3	(Cc ∨ Ac)	
1,2,3	(∀x)(Cx ∨ Ax)	UG,

We're now trying to prove something about the individual named by "c." What we know is a bunch of universal statements. From those universal statements, we are trying to derive conclusions about the individual named by "c," and the obvious way to do this is to use rule US:

1	1. (∀x)(Cx ∨ (Mx ∨ Wx))	PI
2	2. (∀x)(Mx → Ax)	PI
3	3. (∀x)(Wx → Ax)	PI
4	4. (Cc ∨ (Mc ∨ Wc))	US,1
5	5. (Mc → Ac)	US,2
6	6. (Wc → Ac)	US,3
1,2,3	(Cc ∨ Ac)	
1,2,3	(∀x)(Cx ∨ Ax)	UG,

Now the sentence we're trying to prove is a disjunction. We don't have any general strategy for proving a disjunction, but we do have a strategy for proving conditionals: assume the antecedent and try to prove the consequent. So what we want to do is to convert the sentence we are trying to prove, "(Cc ∨ Ac)" into the tautologically equivalent conditional "(¬Cc → Ac)," then to assume "¬Cc" as a premiss and try to derive "Ac":

1	1. (∀x)(Cx ∨ (Mx ∨ Wx))	PI
2	2. (∀x)(Mx → Ax)	PI
3	3. (∀x)(Wx → Ax)	PI
4	4. (Cc ∨ (Mc ∨ Wc))	US,1
5	5. (Mc → Ac)	US,2
6	6. (Wc → Ac)	US,3

7	**7. ¬Cc**	**PI**

1,2,3,7	**Ac**	
1,2,3	**(¬Cc → Ac)**	**CP,7,**
1,2,3	**(Cc ∨ Ac)**	**TC,**
1,2,3	**(∀x)(Cx ∨ Ax)**	**UG,**

Lines 4 and 7 together tautologically imply "(Mx ∨ Wc)"; one way to see this is to rewrite line 4 as "(¬Cc → (Mc ∨ Wc))" and to apply *modus ponens*:*

1	**1. (∀x)(Cx ∨ (Mx ∨ Wx))**	**PI**
2	**2. (∀x)(Mx → Ax)**	**PI**
3	**3. (∀x)(Wx → Ax)**	**PI**
1	**4. (Cc ∨ (Mc ∨ Wc))**	**US,1**
2	**5. (Mc → Ac)**	**US,2**
3	**6. (Wc → Ac)**	**US,3**
7	**7. ¬Cc**	**PI**
1,7	**8. (Mc ∨ Wc)**	**TC,4,7**

1,2,3,7	**Ac**	
1,2,3	**(¬Cc → Ac)**	**CP,7,**
1,2,3	**(Cc ∨ Ac)**	**TC,**
1,2,3	**(∀x)(Cx ∨ Ax)**	**UG,**

Lines 8, 5, and 6 tautologically imply "Ac," which is what we want:

1	**1. (∀x)(Cx ∨ (Mx ∨ Wx))**	**PI**
2	**2. (∀x)(Mx → Ax)**	**PI**
3	**3. (∀x)(Wx → Ax)**	**PI**
1	**4. (Cc ∨ (Mc ∨ Wc))**	**US,1**
2	**5. (Mc → Ac)**	**US,2**
3	**6. (Wc → Ac)**	**US,3**

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* "*Modus ponens*" is a medieval name for the inference from (φ → ψ) and φ to ψ. *Modus tollens* is the inference from (φ → ψ) and ¬ψ to ¬φ.

7	7. ¬Cc	PI
1,7	8. (Mc ∨ Wc)	TC,4,7
1,2,3,7	9. Ac	TC,5,6,8
1,2,3	10. (¬Cc → Ac)	CP,7,9
1,2,3	11. (Cc ∨ Ac)	TC,10
1,2,3	12. (∀x)(Cx ∨ Ax)	UG,11

The last move we made in joining the two ends of our proof was an instance of a general strategy to use when one of the things you know is a disjunction. If you have (φ ∨ ψ) and you're trying to prove θ, try to prove (φ → θ) and (ψ → θ). Then you can put the pieces together by rule TC.

Whenever you derive a sentence φ with the premiss set Γ, you'll know that φ is a logical consequence of Γ. In particular, if you derive φ from the empty set of premisses, you can conclude that φ is a logical consequence of the empty set, that is, you can conclude that φ is valid.

As an example, let's derive "((∀x)(Fx ∧ Gx) ↔ ((∀x)Fx ∧ (∀x)Gx))" from the empty set. The way to derive a biconditional is to break it up into two conditionals, deriving the two directions separately, then using TC to put the parts together. So we want to fill in the blanks in this:

((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))

(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))
((∀x)(Fx ∧ Gx) ↔ ((∀x)Fx ∧ (∀x)Gx)) TC,

To fill in the first blank, we assume the antecedent and try to derive the
consequent:

1	1. (∀x)(Fx ∧ Gx)	PI

1 ((∀x)Fx ∧ (∀x)Gx)
((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx)) CP,1

(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))
((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx)) TC,

To prove a conjunction, try to prove both conjuncts:

1 1. (∀x)(Fx ∧ Gx) PI

1 (∀x)Fx

1 (∀x)Gx
1 ((∀x)Fx ∧ (∀x)Gx) TC
((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx)) CP,1

(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))
((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx)) TC,

To prove a universal statement, prove an instance* of it with a new constant. We use this strategy to prove both "(∀x)Fx" and "(∀x)Gx":

1 1. (∀x)(Fx ∧ Gx) PI
1 2. (Fa ∧ Ga) US,1
1 3. Fa TC,2

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* An *instance* of an initially quantified sentence is a sentence you get from the given sentence by dropping the initial quantifier and replacing all free occurrences of "x" by a constant. "Fa" is an instance of "(∀x)Fx."

1	4. (∀x)Fx	UG,3
1	5. (Fb ∧ Gb)	US,1
1	6. Gb	TC,5
1	7. (∀x)Gx	UG,6
1	8. ((∀x)Fx ∧ (∀x)Gx)	TC,4,7
	9. ((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	CP,1,8

	(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))	
	((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	TC,

To fill in the remaining blank, we assume the antecedent and try to derive the consequent.

1	1. (∀x)(Fx ∧ Gx)	PI
1	2. (Fa ∧ Ga)	US,1
1	3. Fa	TC,2
1	4. (∀x)Fx	UG,3
1	5. (Fb ∧ Gb)	US,1
1	6. Gb	TC,5
1	7. (∀x)Gx	UG,6
1	8. ((∀x)Fx ∧ (∀x)Gx)	TC,4,7
	9. ((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	CP,1,8
10	10. ((∀x)Fx ∧ (∀x)Gx)	PI

10	(∀x)(Fx ∧ Gx)	
	(((∀x)Fx ∧ (∀x)Gx) → (∀x)(Fx ∧ Gx))	CP,10,
	((∀x)(Fx ∧ Gx) → ((∀x)Fx ∧ (∀x)Gx))	TC,

What we have to work from is a conjunction, "((∀x)Fx ∧ (∀x)Gx)"; we simplify this by writing its two conjuncts on separate lines. What we're trying to prove is a universal sentence, "(∀x)(Fx ∧ Gx)," which we prove by first proving "(Fc ∧ Gc), intending to apply rule UG:

1	1. (∀x)(Fx ∧ Gx)	PI

1	2. $(Fa \wedge Ga)$	US,1
1	3. Fa	TC,2
1	4. $(\forall x)Fx$	UG,3
1	5. $(Fb \wedge Gb)$	US,1
1	6. Gb	TC,5
1	7. $(\forall x)Gx$	UG,6
1	8. $((\forall x)Fx \wedge (\forall x)Gx)$	TC,4,7
	9. $((\forall x)(Fx \wedge Gx) \rightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	CP,1,8
10	10. $((\forall x)Fx \wedge (\forall x)Gx)$	PI
10	11. $(\forall x)Fx$	TC,10
10	12. $(\forall x)Gx$	TC,10
10	13. Fc	US,11
10	14. Gc	US,12
10	15. $(Fc \wedge Gc)$	TC,13,14
10	16. $(\forall x)(Fx \wedge Gx)$	UG,15
	17. $(((\forall x)Fx \wedge (\forall x)Gx) \rightarrow (\forall x)(Fx \wedge Gx))$	CP,10,16
	18. $((\forall x)(Fx \wedge Gx) \rightarrow ((\forall x)Fx \wedge (\forall x)Gx))$	TC,9,17

As another example, let's derive "$(Fa \rightarrow (\forall x)(Gx \rightarrow Fa))$" from the empty set:

1	1. Fa	PI
1	2. $(Gb \rightarrow Fa)$	TC,1
1	3. $(\forall x)(Gx \rightarrow Fa)$	UG,2
	4. $(Fa \rightarrow (\forall x)(Gx \rightarrow Fa))$	CP,1,3

Let's derive "$((\forall x)(Fx \rightarrow Gx) \rightarrow ((\forall x)Fx \rightarrow (\forall x)Gx))$" from the empty set:

1	1. $(\forall x)(Fx \rightarrow Gx)$	PI
2	2. $(\forall x)Fx$	PI
1	3. $(Fa \rightarrow Ga)$	US,1
2	4. Fa	US,2
1,2	5. Ga	TC,3,4
1,2	6. $(\forall x)Gx$	UG,5
1	7. $((\forall x)Fx \rightarrow (\forall x)Gx)$	CP,2,6
	8. $((\forall x)(Fx \rightarrow Gx) \rightarrow ((\forall x)Fx \rightarrow (\forall x)Gx))$	CP,1,7

Remember the restriction on rule UG. If you use rule UG to derive $(\forall x)\varphi$ from $\varphi^{x}/_{c}$, the constant c shouldn't appear either in

φ or in any members of the premiss set of $\varphi^x/_c$. Otherwise, you'll get poppycock, like the following bad derivation from the empty set of "(Ws → (∀x)Wx)" ("If Socrates is wise, everyone is wise"):

1	**1. Ws**	**PI**
1	**2. (∀x)Wx**	**Bad use of UG,1**
	3. (Ws → (∀x)Wx)	**CP,1,2**

We now have all the rules we need for dealing with universal quantifiers. Let's get some rules for dealing with existential quantifiers:

Quantifier exchange rule (QE).
From ¬(∀x)¬φ, you may infer (∃x)φ with the same premiss set, and *vice versa*.
From (∀x)¬φ, you may infer ¬(∃x)φ with the same premiss set, and *vice versa*.
From ¬(∀x)φ, you may infer (∃x)¬φ with the same premiss set, and *vice versa*.
From (∀x)φ, you may infer ¬(∃x)¬φ with the same premiss set, and *vice versa*.

It's easy to see that rule QE is logical-consequence preserving. ¬(∀x)¬φ is logically equivalent to (∃x)φ. Similarly for the other clauses.

As an illustration, let's derive "¬(∃x)(Mx ∧ Wx)" ("No one is both a man and a woman") from "(∀x)(Mx → Bx)" ("All men have beards") and "¬(∃x)(Wx ∧ Bx)" ("No women have beards"):

1	**1. (∀x)(Mx → Bx)**	**PI**
2	**2. ¬(∃x)(Wx ∧ Bx)**	**PI**
2	**3. (∀x)¬(Wx ∧ Bx)**	**QE,2**
1	**4. (Ma → Ba)**	**US,1**
2	**5. ¬(Wa ∧ Ba)**	**US,3**

1,2	**¬(Ma ∧ Wa)**	
1,2	**(∀x)¬(Mx ∧ Wx)**	**US,**

1,2 ¬(∃x)(Mx ∧ Wx) QE,

We've used rule QE to turn negated existentials into universals. Now the sentence we're trying to prove a negated conjunction. How do we prove a negated conjunction? Turn it into a conditional, since we know how to prove conditionals. So we rewrite "¬(Ma ∧ Wa)" as "(Ma → ¬Wa)." While we're about it, let's rewrite "¬(Wa ∧ Ba)" as "(Wa → ¬Ba)," on the theory that conditionals are more familiar, and hence easier to use, than negated conjunctions:

1	**1. (∀x)(Mx → Bx)**	**PI**
2	**2. ¬(∃x)(Wx ∧ Bx)**	**PI**
2	**3. (∀x)¬(Wx ∧ Bx)**	**QE,2**
1	**4. (Ma → Ba)**	**US,1**
2	**5. ¬(Wa ∧ Ba)**	**US,3**
2	**6. (Wa → ¬Ba)**	**TC,5**
7	**7. Ma**	**PI**
1,7	**8. Ba**	**TC,4,7**
1,2,7	**9. ¬Wa**	**TC,6,8**
1,2	**10. (Ma → ¬Wa)**	**CP,7,9**
1,2	**11. ¬(Ma ∧ Wa)**	**TC,10**
1,2	**12. (∀x)¬(Mx ∧ Wx)**	**US,11**
1,2	**13. ¬(∃x)(Mx ∧ Wx)**	**QE,12**

As another example, let's derive "(∃x)(Gx ∧ Wx)" ("Some Greeks are wise") from "Gs" ("Socrates is a Greek"), "Ps" ("Socrates is a philosopher"), and "(∀x)(Px → Wx)" ("All philosophers are wise"):

1	**1. Gs**	**PI**
2	**2. Ps**	**PI**
3	**3. (∀x)(Px → Wx)**	**PI**
3	**4. (Ps → Ws)**	**US,3**
2,3	**5. Ws**	**TC,2,4**

1,2,3 ¬(∀x)¬(Gx ∧ Wx)

1,2,3 (∃x)(Gx ∧ Wx) QE,

Now we're stuck, so let me give you a general strategy for what to do if you get stuck. The method is a formalized version

of *reductio ad absurdum*. You assume the opposite of what you're trying to prove, and you show that this assumption leads to an absurdity. By "assume the opposite of what you're trying to prove," what I mean is this: if you're trying to prove a negation, assume the negatum as a new premiss; otherwise, assume the negation of what you're trying to prove. By showing that this assumption "leads to an absurdity," I mean deriving a sentence that is truth functionally inconsistent with one of the things you've already assumed or derived. Thus, in this case, we assume "$(\forall x)\neg(Gx \wedge Wx)$" and go for an absurdity:

1	1. Gs	PI
2	2. Ps	PI
3	3. $(\forall x)(Px \rightarrow Wx)$	PI
3	4. $(Ps \rightarrow Ws)$	US,3
2,3	5. Ws	TC,2,4
6	6. $(\forall x)\neg(Gx \wedge Wx)$	PI
6	7. $\neg(Gs \wedge Ws)$	US,6
6	8. $(Gs \rightarrow \neg Ws)$	TC,7
1,6	9. $\neg Ws$	TC,1,8
1,2,3	$\neg(\forall x)\neg(Gx \wedge Wx)$	
1,2,3	$(\exists x)(Gx \wedge Wx)$	QE,

Now we've gotten our absurdity. We've derived "¬Ws," but we know from line 5 that "Ws" is true. To complete the proof, we use rule CP to derive the conditional whose antecedent is the assumption we made that was the opposite of what we were trying to prove and whose consequent is the absurdity we have derived. Then we use rule TC to complete the proof:

1	1. Gs	PI
2	2. Ps	PI
3	3. $(\forall x)(Px \rightarrow Wx)$	PI
3	4. $(Ps \rightarrow Ws)$	US,3
2,3	5. Ws	TC,2,4
6	6. $(\forall x)\neg(Gx \wedge Wx)$	PI
6	7. $\neg(Gs \wedge Ws)$	US,6
6	8. $(Gs \rightarrow \neg Ws)$	TC,7
1,6	9. $\neg Ws$	TC,1,8

1	10. $((\forall x)\neg(Gx \wedge Wx) \rightarrow \neg Ws)$	CP,6,9
1,2,3	11. $\neg(\forall x)\neg(Gx \wedge Wx)$	TC,5,10
1,2,3	12. $(\exists x)(Gx \wedge Wx)$	QE,11

Such a simple argument oughtn't require such a complicated derivation. We adopt a new rule that will make such arguments easier. The new rule is, strictly speaking superfluous, in that anything that we can prove with the rule we can also prove, in a more complicate fashion, without it. But there's no real harm in having extra rules.

> Existential generalization rule (EG). If you have written $\varphi^{x}/_{c}$, for any constant c, you may write $(\exists x)\varphi$ with the same premiss set.

It's easy to see that this new rule is logical-consequence preserving, since $(\exists x)\varphi$ is a logical consequence of $\varphi^{x}/_{c}$.

Let's see how this new rule makes our previous deduction easier:

1	1. Gs	PI
2	2. Ps	PI
3	3. $(\forall x)(Px \rightarrow Wx)$	PI
3	4. $(Ps \rightarrow Ws)$	US,3
2,3	5. Ws	TC,2,4
1,2,3	6. $(Gs \wedge Ws)$	TC,1,5
1,2,3	7. $(\exists x)(Gx \wedge Wx)$	EG,6

We now have a rule for what to do with an universal sentence up top, among the things we are assumed to know; namely, the rule US. We have a rule telling us what to do when we are trying to prove a universal sentence, namely UG. We have a rule about what to do to prove an existential sentence, EG. What we still need is a rule that tell us what to do with an existential sentence up top. The good news is that this is the last of our rules. The bad news is that it's pretty complicated.

> Existential specification (ES). Suppose that you have derived $(\exists x)\varphi$ with premiss set Δ and that you have derived ψ with premiss set $\Gamma \cup \{\varphi^{x}/_{c}\}$, for some constant c. Suppose further that the constant

c does not appear in φ, in ψ, or in any member of Γ. Then you may derive ψ with premiss set $\Delta \cup \Gamma$.

How we use this complicated new rule is this: Suppose we are trying to prove ψ and one of the things we have up top, among the things we are assumed to know, is an existential sentence $(\exists x)\varphi$. The way we use the existential sentence is this: We take an instance $\varphi^{x}/_{c}$ of the existential sentence as a new premiss, where c is a constant that hasn't yet appeared in the proof. Then we derive ψ from the instance, together with whatever other premisses we have available. Once we have done this, we use rule ES to upgrade our premiss set, replacing $\varphi^{x}/_{c}$ in the premiss set with the premiss set of $(\exists x)\psi$.

Here's an example: We derive "(∃x)(Mx ∧ Fx)" ("Some mammals fly") from "(∃x)(Sx ∧ Fx)" ("Some squirrels fly") and "(∀x)(Sx → Mx)" ("All squirrels are mammals"):

1	1. (∃x)(Sx ∧ Fx)	PI
2	2. (∀x)(Sx → Mx)	PI
3	3. (Sa ∧ Fa)	PI
3	4. Sa	TC,3
3	5. Fa	TC,3
2	6. (Sa → Ma)	US,2
2,3	7. Ma	TC,4,6
2,3	8. (Ma ∧ Fa)	TC,5,7
2,3	9. (∃x)(Mx ∧ Fx)	EG,8
1,2	10. (∃x)(Mx ∧ Fx)	ES,1,3,9

Here we were trying to prove "(∃x)(Mx ∧ Fx)," and one of the premisses we had available to use was the existential sentence "(∃x)(Sx ∧ Fx)." The way we used this existential premiss was to take an instance of the existential premiss as a new premiss at line 3. Then, at line 9, we used this new premiss to prove "(∃x)(Mx ∧ Fx)." At line 10, we used rule ES to upgrade our premisses, replacing "(Sa ∧ Fa)" in the premiss set of "(∃x)(Mx ∧ Fx)" with the premiss set of "(∃x)(Sx ∧ Fx)."

We know that some squirrels fly. Take a flying squirrel; call him "a." From the assumption that a is a flying squirrel, we

are able to conclude that some mammals fly. So this conclusion must already be entailed by "Some squirrels fly."

Notice that what rule ES permits you to do is merely to repeat a sentence you've already written earlier. What's changed is your premiss set.

As another example, let's derive "(∃x)(Fx ∧ Gx)" from "(∃x)Fx" and "(∀x)Gx":

1	**1. (∃x)Fx**	**PI**
2	**2. (∀x)Gx**	**PI**
3	**3. Fa**	**PI**
2	**4. Ga**	**US,2**
2,3	**5. (Fa ∧ Ga)**	**TC,3,4**
2.3	**6. (∃x)(Fx ∧ Gx)**	**EG,5**
1,2	**7. (∃x)(Fx ∧ Gx)**	**ES,1,3,6**

We had "(∃x)Fx" as a premiss. We made use of it by taking "Fa" as a premiss, then using rule ES to upgrade the premiss set of our conclusion.

Now let's derive "(∃x)(Fx ∧ Gx)" from the premisses "(∃x)(Ax ∧ Bx)," "(∀x)(Ax → Fx)," and "(∀x)(Bx → Gx)":

1	**1. (∃x)(Ax ∧ Bx)**	**PI**
2	**2. (∀x)(Ax → Fx)**	**PI**
3	**3. (∀x)(Bx → Gx)**	**PI**
4	**4. (Aa ∧ Ba)**	**PI**
4	**5. Aa**	**TC,4**
4	**6. Ba**	**TC,4**
2	**7. (Aa → Fa)**	**US,2**
2,4	**8. Fa**	**TC,5,7**
3	**9. (Ba → Ga)**	**US,3**
3,4	**10. Ga**	**TC,6,9**
2,3,4	**11. (Fa ∧ Ga)**	**TC,8,10**
2,3,4	**12. (∃x)(Fx ∧ Gx)**	**EG,11**
1,2,3	**13. (∃x)(Fx ∧ Gx)**	**ES,1,4,12**

We now want to see that rule ES is superfluous, in that anything that you can derive with the rule you can also derive

more circuitously without it. Since we already know that the other rules are all logical-consequence preserving, this will show that rule ES is logical-consequence preserving. So suppose that we have a derivation which contains these lines:

Premiss set	Number	Sentence	Justification
Δ	i	$(\exists x)\varphi$	
$\varphi^{x}/_{c}$	ii	$\varphi^{x}/_{c}$	PI
$\Gamma \cup \{\varphi^{x}/_{c}\}$	iii	ψ	
$\Delta \cup \Gamma$	iv	ψ	ES,i,ii,iii

where c doesn't appear in φ, in ψ, or in any member of Γ. We may replace this derivation with the following:

Premiss set	Number	Sentence	Justification
Δ	i	$(\exists x)\varphi$	
$\varphi^{x}/_{c}$	ii	$\varphi^{x}/_{c}$	PI
$\Gamma \cup \{\varphi^{x}/_{c}\}$	iii	ψ	
Γ	iv	$(\varphi^{x}/_{c} \rightarrow \psi)$	CP,2,3
$\{\neg\psi\}$	v	$\neg\psi$	PI
$\Gamma \cup \{\neg\psi\}$	vi	$\neg\varphi^{x}/_{c}$	TC,iv,v
$\Gamma \cup \{\neg\psi\}$	vii	$(\forall x)\neg\varphi$	UG,vi
$\Gamma \cup \{\neg\psi\}$	viii	$\neg(\exists x)\varphi$	QE,vii
Γ	ix	$(\neg\psi \rightarrow \neg(\exists x)\varphi)$	CP,v,viii
$\Delta \cup \Gamma$	x	ψ	TC,i,ix

A different way we might have proceed would have been to take US, UG, ES, and EG as our axioms for the quantifiers, show those axioms to be sound, and then obtain QE as a derived rule. Or we could have started with ES, EG, and QE, and derived US and UG.

Now that we have the rules, let me tell you some strategies for applying the rules. These strategies aren't part of the rules; they're techniques for using the rules efficiently. To have a correct derivation, you have to follow the rules, but you don't have to follow the strategies if you don't want to. The strategies don't always work, but they usually do.

The basic plan is always to work from two ends toward the middle. Thus, at each stage of a derivation, there will be a set of sentences you are assumed to know, and there are one or more

sentences you are trying to prove. The strategies are techniques for breaking up the sentences you are working with into sentences that are simpler and therefore, hopefully, easier to work with.

There are two groups of strategies, one having to do with simplifying the sentence you're trying to prove, the other with simplifying the sentences you are assumed to know. Here is the main strategy for simplifying the sentence you're trying to prove:

If you're trying to prove a conditional, assume the antecedent as a premiss and try to derive the consequent. Then apply rule CP.

Here are more bottom-up strategies:

If you're trying to prove a disjunction ($\varphi \vee \psi$), prove the conditional ($\neg\varphi \rightarrow \psi$), then apply rule TC. (Occasionally, you can simply prove one of the disjuncts.)

If you're trying to prove a biconditional ($\varphi \leftrightarrow \psi$), prove the two conditionals ($\varphi \rightarrow \psi$) and ($\psi \rightarrow \varphi$), then apply TC.

If you're trying to prove a conjunction, prove each of the conjuncts, then apply rule TC.

If you're trying to prove a universal sentence, try to prove an instance of it the sentence with a new constant. Then apply UG.

If you're trying to prove an existential sentence, try to prove an instance of it. Then apply EG.

If you're trying to prove a negated conditional $\neg(\varphi \rightarrow \psi)$, prove φ and $\neg\psi$, then apply TC.

If you're trying to prove a negated disjunction $\neg(\varphi \vee \psi)$, prove $\neg\varphi$ and $\neg\psi$, then apply TC.

If you're trying to prove a negated biconditional $\neg(\varphi \leftrightarrow \psi)$, try to prove ($\varphi \leftrightarrow \neg\psi$), then apply TC.

If you're trying to prove negated conjunction, $\neg(\varphi \wedge \psi)$, try to prove ($\varphi \rightarrow \neg\psi$), then apply TC.

If you're trying to prove a negated universal sentence $\neg(\forall x)\varphi$, prove $(\exists x)\neg\varphi$, then apply QE.

If you're trying to prove a negated existential sentence $\neg(\exists x)\varphi$, prove $(\forall x)\neg\varphi$, then apply QE.

If you're trying to prove a negated negation $\neg\neg\varphi$, prove φ, then apply TC.

Now we turn to the top-down strategies for simplifying the premisses and sentences you have derived from the premisses. Let me refer to the premisses and the things you have derived from the premisses as your "assumptions"; here are the strategies for simplifying assumptions:

If one of your assumptions is a conjunction, write the two conjuncts on separate lines, using TC.

If one of your assumptions is a disjunction $(\varphi \vee \psi)$ and you're trying to prove θ, prove the two conditionals $(\varphi \rightarrow \theta)$ and $(\psi \rightarrow \theta)$, then apply TC. Whenever you apply this strategy, you're sure to wind up with a pretty long proof, so use this strategy only as a last resort. Something to try first is to rewrite the disjunction as a conditional $(\neg\varphi \rightarrow \psi)$ (using TC), then to see if you can apply *modus ponens* or *modus tollens*.

If one of your assumptions is a conditional $(\varphi \rightarrow \psi)$, see if you can apply *modus ponens* or *modus tollens*. If not, rewrite the conditional as $(\neg\varphi \vee \psi)$.

If one of your assumptions is a biconditional $(\varphi \leftrightarrow \psi)$, see if you know how to prove one component, in which case you can derive the other by TC. See if you know how to prove the negation of one component, in which case you can derive the negation of the other. Otherwise, rewrite the biconditional as $((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$, using TC

If one of your assumptions is an existential sentence $(\exists x)\varphi$, pick a new constant c and assume $\varphi^{x}/_{c}$ as a new premiss. Once you've proven what you're trying to prove, use ES to upgrade your premiss set, replacing $\{\varphi^{x}/_{c}\}$ by the premiss set of $(\exists x)\varphi$.

If one of your assumptions is a universal sentence ($(\forall x)\varphi$), deduce $\varphi^{x}/_{c}$ for each constant c that appears in the proof.*

If one of your assumptions is a negated conjunctions $\neg(\varphi \wedge \psi$_), rewrite it as $(\neg\varphi \vee \neg\psi)$, using TC.

If one of your assumptions is a negated disjunction, $\neg(\varphi \vee \psi)$, rewrite it as $(\neg\varphi \wedge \neg\psi)$, using TC.

If one of you assumptions is a negated conditional $\neg(\varphi \rightarrow \psi)$, rewrite it as $(\varphi \wedge \neg\psi)$, using TC.

If one of your assumptions is a negated biconditional $\neg(\varphi \leftrightarrow \psi)$, rewrite it as $(\varphi \leftrightarrow \neg\psi)$, using TC.

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* It generally only helps to derive $\varphi^{x}/_{c}$ for constants c that appear elsewhere in the proof. It usually does no good to instantiate with a brand new constant. The only exceptions that I know of occur when there haven't been any constants in the proof so far. An example is the derivation from the empty set of $((\forall x)Fx \rightarrow (\exists x)Fx)$:

1	1. $(\forall x)Fx$	PI
1	2. Fa	US,1
1	3. $(\exists x)Fx$	(EG),2
	4 $((\forall x)Fx \rightarrow (\exists x)Fx)$	CP,1,3

If one of your assumptions is a negated existential sentence $\neg(\exists x)\varphi$, rewrite it as $(\forall x)\neg\varphi$, using QE.

If one of your assumptions is a negated universal sentence $\neg(\forall x)\varphi$, rewrite it as $(\exists x)\neg\varphi$, using QE.

If one of your assumptions is a negated negation $\neg\neg\varphi$, rewrite it as φ, using TC.

One final bottom-up rule:

If all else fails, assume the negation of what you're trying to prove and try to derive an absurdity. If you're trying to ψ, use PI to assume $\neg\psi$. If you are able to prove $\neg\varphi$, where φ is one of your assumptions, you can use CP to get $(\neg\psi \rightarrow \neg\varphi)$, then use TC to get ψ.

Let's do some examples. Let's derive "$(\exists x)(Fx \vee Hx)$" from {"$((\exists x)Fx \vee (\exists x)Gx)$," "$(\forall x)(Gx \rightarrow Hx)$"}. We follow the strategy for using a disjunctive assumption, first proving "$((\exists x)Fx \rightarrow (\exists x)(Fx \vee Hx))$" and "$((\exists x)Gx \rightarrow (\exists x)(Fx \vee Hx))$," then applying TC:

1	1. $((\exists x)Fx \vee (\exists x)Gx))$	PI
2	2. $(\forall x)(Gx \rightarrow Hx)$	PI
3	3. $(\exists x)Fx$	PI
4	4. Fa	PI (for ES)
4	5. $(Fa \vee Ha)$	TC,4
4	6. $(\exists x)(Fx \vee Hx)$	EG,5
3	7. $(\exists x)(Fx \vee Hx)$	ES,3,4,6
	8. $((\exists x)Fx \rightarrow (\exists x)(Fx \vee Hx))$	CP,3,7
9	9. $(\exists x)Gx$	PI
10	10. Gb	PI (for ES)
2	11. $(Gb \rightarrow Hb)$	US,2
2,10	12. Hb	TC,10,11
2,10	13. $(Fb \vee Hb)$	TC,12
2,10	14. $(\exists x)(Fx \vee Hx)$	EG,13
2.9	15. $(\exists x)(Fx \vee Hx)$	ES,9,10,14
2	16. $((\exists x)Gx \rightarrow (\exists x)(Fx \vee Hx))$	CP,9,15
1,2	17. $(\exists x)(Fx \vee Hx)$	TC,1,8,16

As another example, let's derive "(((∃x)Ax ∨ (∃x)Bx) ↔ (∃x)(Ax ∨ Bx))" from the empty set. We prove the two directions separately. We prove the left-to-right direction by proving "((∃x)Ax → (∃x)(Ax ∨ Bx))" and "((∃x)Bx → (∃x)(Ax ∨ Bx))," then using TC; this is our general strategy for working with disjunctive assumptions. We then prove the right-to-left direction by converting the disjunction we're trying to prove to a conditional:

1	**1. ((∃x)Ax ∨ (∃x)Bx)**	**PI**
2	**2. (∃x)Ax**	**PI**
3	**3. Aa**	**PI (for ES)**
3	**4. (Aa ∨ Ba)**	**TC,3**
3	**5. (∃x)(Ax ∨ Bx)**	**EG,4**
2	**6. (∃x)(Ax ∨ Bx)**	**ES,2,3,5**
	7. ((∃x)Ax → (∃x)(Ax ∨ Bx))	**CP,2,6**
8	**8. (∃x)Bx**	**PI**
9	**9. Bb**	**P (for ES)**
9	**10. (Ab ∨ Bb)**	**TC,9**
9	**11. (∃x)(Ax ∨ Bx)**	**EG,10**
8	**12. (∃x)(Ax ∨ Bx)**	**ES,8,9,11**
	13. ((∃x)Bx) → (∃x)(Ax ∨ Bx))	**CP,8,12**
1	**14. (∃x)(Ax ∨ Bx)**	**TC,1,7,13**
	15, (((∃x)Ax ∨ (∃x)Bx) → (∃x)(Ax ∨ Bx))	**CP,1,14**
16	**16. (∃x)(Ax ∨ Bx)**	**PI**
17	**17. (Ac ∨ Bc)**	**PI (for ES)**
18	**18. ¬(∃x)Ax**	**PI**
18	**19. (∀x)¬Ax**	**QE,18**
18	**20. ¬Ac**	**US,19**
17	**21. (¬Ac → Bc)**	**TC,17**
17,18	**22. Bc**	**TC,20,21**
17,18	**23. (∃x)Bx**	**EG,22**
16,18	**24. (∃x)Bx**	**ES,16,17.23**
16	**25. (¬(∃x)Ax → (∃x)Bx)**	**CP,18,24**
16	**26. ((∃x)Ax ∨ Bx)**	**TC,25**
	27. ((∃x)(Ax ∨ Bx) → ((∃x)Ax ∨ (∃x)Bx))	**CP,16,26**
	28. (((∃x)Ax ∨ (∃x)Bx) ↔ (∃x)(Ax ∨ Bx))	**TC,15,27**

Now we derive "((∃x)Fx → Ga)" from {"(∀x)(Fx → Ga)"}:

1	1.	$(\forall x)(Fx \rightarrow Ga)$	PI
2	2.	$(\exists x)Fx$	PI
3	3.	Fb	PI (for ES)
1	4.	$(Fb \rightarrow Ga)$	US,1
1,3	5.	Ga	TC,3,4
1,2	6.	Ga	ES,2,3,5
1	7.	$((\exists x)Fx \rightarrow Ga)$	CP,2,6

We now do the converse proof, deriving "$(\forall x)(Fx \rightarrow Ga)$" from {"$((\exists x)Fx \rightarrow Ga)$"}:

1	1.	$((\exists x)Fx \rightarrow Ga)$	PI
2	2.	Fb	PI
2	3.	$(\exists x)Fx$	EG,2
1,2	4.	Ga	TC,1,3
1	5.	$(Fb \rightarrow Ga)$	CP,2,4
1	6.	$(\forall x)(Fx \rightarrow Ga)$	UG,5

Next we formalize an argument from Lewis Carroll's *Symbolic Logic*:

> **No one who really appreciates Beethoven fails to keep silence while the *Moonlight Sonata* is being played.**
> **Guinea pigs are hopelessly ignorant of music.**
> **No one who is hopelessly ignorant of music ever keeps silence while the *Moonlight Sonata* is being played.**
> **Therefore, guinea pigs never really appreciate Beethoven.**

In symbols:

$\neg(\exists x)(Bx \land \neg Sx)$

$(\forall x)(Gx \rightarrow Ix)$

$\neg(\exists x)(Ix \land Sx)$

$\therefore \neg(\exists x)(Gx \land Bx)$

We derive the translated conclusion from the translated premisses, thus showing the English argument is valid:

1	1.	$\neg(\exists x)(Bx \land \neg Sx)$	PI
2	2.	$(\forall x)(Gx \rightarrow Ix)$	PI
3	3.	$\neg(\exists x)(Ix \land Sx)$	PI
4	4.	Ga	PI

1	5. (∀x)¬(Bx ∧ ¬Sx)	QE,1
3	6. (∀x)¬(Ix ∧ Sx)	QE,3
2	7. (Ga → Ia)	US,2
1	8. ¬(Ba ∧ ¬Sa)	US,5
3	9. ¬(Ia ∧ Sa)	US,6
2,4	10. Ia	TC,4,7
3	11. (Ia → ¬Sa)	TC,9
2,3,4	12. ¬Sa	TC,10,11
1	13. (Ba → Sa)	TC,8
1,2,3,4	14. ¬Ba	TC,12,13
1,2,3	15. (Ga → ¬Ba)	CP,4,14
1,2,3	16. ¬(Ga ∧ Ba)	TC,15
1,2,3	17. (∀x)¬(Gx ∧ Bx)	UG,16
1,2,3	18. ¬(∃x)(Gx ∧ Bx)	QE,17

Here is a more complicated example, again from Lewis Carroll:

> Animals are always mortally offended them if I fail to notice them.
> The only animals that belong to me are in that field.
> No animal can guess a conundrum unless it has been properly trained in a Board-school.
> All badgers are animals.
> None of the animals in that field are badgers.
> When a animal is mortally offended, it always rushes about wildly and howls.
> I never notice any animals unless it belongs to me.
> No animal that has been properly trained in a Board-school ever rushes about wildly and howls.
> Therefore, no badger can guess a conundrum.

In symbols:

> (∀x)((Ax ∧ ¬Nx) → Ox)
> (∀x)((Ax ∧ Mx) → Fx)
> ¬(∃x)((Ax ∧ Gx) ∧ ¬Tx)
> (∀x)(Bx → Ax)
> ¬(∃x)((Ax ∧ Fx) ∧ Bx)
> (∀x)((Ax ∧ Ox) → (Rx ∧ Hx))
> ¬(∃x)((Ax ∧ Nx) ∧ ¬Mx)

$\neg(\exists x)((Ax \wedge Tx) \wedge (Rx \wedge Hx))$
$\therefore\ \neg(\exists x)(Bx \wedge Gx)$

Now we derive the translated conclusion from the translated premisses:

1	1. $(\forall x)((Ax \wedge \neg Nx) \rightarrow Ox)$	PI
2	2. $(\forall x)((Ax \wedge Mx) \rightarrow Fx)$	PI
3	3. $\neg(\exists x)((Ax \wedge Gx) \wedge \neg Tx)$	PI
4	4. $(\forall x)(Bx \rightarrow Ax)$	PI
5	5. $\neg(\exists x)((Ax \wedge Fx) \wedge Bx)$	PI
6	6. $(\forall x)((Ax \wedge Ox) \rightarrow (Rx \wedge Hx))$	PI
7	7. $\neg(\exists x)((Ax \wedge Nx) \wedge \neg Mx)$	PI
8	8. $\neg(\exists x)((Ax \wedge Tx) \wedge (Rx \wedge Hx))$	PI
3	9. $(\forall x)\neg((Ax \wedge Gx) \wedge \neg Tx)$	QE,3
5	10. $(\forall x)\neg((Ax \wedge Fx) \wedge Bx)$	QE,5
7	11. $(\forall x)\neg((Ax \wedge Nx) \wedge \neg Mx)$	QE,7
8	12. $(\forall x)\neg((Ax \wedge Tx) \wedge (Rx \wedge Hx))$	QE,8
13	13. Ba	PI
1	14. $((Aa \wedge \neg Na) \rightarrow Oa)$	US,1
2	15. $((Aa \wedge Ma) \rightarrow Fa)$	US,2
4	16. $(Ba \rightarrow Aa)$	US,4
6	17. $((Aa \wedge Oa) \rightarrow (Ra \wedge Ha))$	US,6
3	18. $\neg((Aa \wedge Ga) \wedge \neg Ta)$	US,9
5	19. $\neg((Aa \wedge Fa) \wedge Ba)$	US,10
7	20. $\neg((Aa \wedge Na) \wedge \neg Ma)$	US,11
8	21. $\neg((Aa \wedge Ta) \wedge (Ra \wedge Ha))$	US,12
4,13	22. Aa	TC,13,16
5	23. $((Aa \wedge Fa) \rightarrow \neg Ba)$	TC,19
5,13	24. $\neg(Aa \wedge Fa)$	TC,13,23
5,13	25. $(Aa \rightarrow \neg Fa)$	TC,24
4,5,13	26. $\neg Fa$	TC,22,24
2,4,5,13	27. $\neg(Aa \wedge Ma)$	TC,15,26
2,4,5,13	28. $(Aa \rightarrow \neg Ma)$	TC,27
2,4,5,13	29. $\neg Ma$	TC,22,28
7	30. $((Aa \wedge Na) \rightarrow Ma)$	TC,20
2,4,5,7,13	31. $\neg(Aa \wedge Na)$	TC,29,30
2,4,5,7,13	32. $(Aa \rightarrow \neg Na)$	TC,31
2,4,5,7,13	33. $\neg Na$	TC,22,32
2,4,5,7,13	34. $(Aa \wedge \neg Na)$	TC,22,32

1,2,4,5,7,13	35. Oa	TC,14,34
1,2,4,5,7,13	36. (Aa ∧ Oa)	TC,22,35
1,2,4,5,6,7,13	37. (Ra ∧ Ha)	TC,17,36
8	38. ((Aa ∧ Ta) → ¬(Ra ∧ Ha))	TC,21
1,2,4,5,6,7,8,13	39. ¬(Aa ∧ Ta)	TC,37,38
1,2,4,5,6,7,8,13	40. (Aa → ¬Ta)	TC,39
1,2,4,5,6,7,8,13	41. ¬Ta	TC,22,40
3	42. ((Aa ∧ Ga) → Ta)	TC,18
1,2,3,4,5,6,7,8,13	43. ¬(Aa ∧ Ga)	TC,41,42
1,2,3,4,5,6,7,8,13	44. (Aa → ¬Ga)	TC,43
1,2,3,4,5,6,7,8,13	45. ¬Ga	TC,22,44
1,2,3,4,5,6,7,8	46. (Ba → ¬Ga)	CP,13,45
1,2,3,4,5,6,7,8	47. ¬(Ba ∧ Ga)	TC,46
1,2,3,4,5,6,7,8	48. (∀x)¬(Bx ∧ Gx)	UG,47
1,2,3,4,5,6,7,8	49. ¬(∃x)(Bx ∧ Gx)	QE,48