

Derivations in the Predicate Calculus

The rules of proof for the full predicate calculus are the same as those for the monadic predicate calculus, with only the teeniest changes to accommodate the extra variables:

- PI At any stage of a derivation, you may write down a sentence ϕ with any set that contains ϕ as its premise set.
- TC If you have written down sentences $\psi_1, \psi_2, \dots, \psi_n$ in a derivation, and ϕ is a tautological consequence of $\{\psi_1, \psi_2, \dots, \psi_n\}$, then you may write down sentence ψ , taking the premise set to be the union of the premise sets of the ψ_i s. In particular, if ϕ is a tautology, we can write ϕ with the empty premise set.
- CP If you have derived ψ with premise set $\Gamma \cup \{\phi\}$, you may write $(\phi \rightarrow \psi)$ with premise set Γ .
- US If you've derived $(\forall v)\phi$, you may derive ϕ^v/c with the same premise set, for any variable v and individual constant c .
- UG For any variable v , if you've derived ϕ^v/c from Γ and if the individual constant c doesn't appear in ϕ or in any of the sentences in Γ , you may derive $(\forall v)\phi$ with premise set Γ .
- QE From $\neg(\forall v)\neg\phi$, you may infer $(\exists v)\phi$ with the same premise set, and *vice versa*, for each variable v .
From $(\forall v)\neg\phi$, you may infer $\neg(\exists v)\phi$ with the same premise set, and *vice versa*, for each variable v .
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From $(\forall v)\phi$, you may infer $\neg(\exists v)\neg\phi$ with the same premise set, and *vice versa*, for each variable v .
- EG If you have written ϕ^v/c , for any variable v and individual constant c , you may write $(\exists v)\phi$ with the same premise set.
- ES Suppose that you have derived $(\exists v)\phi$ with premise set Δ and that you have derived ψ with premise set $\Gamma \cup \{\phi^v/c\}$, for some individual constant c and variable v . Suppose further that the constant c does not appear in ϕ , in ψ , or in any member of Γ . Then you may derive ψ with premise set $\Delta \cup \Gamma$.

Our first two examples will be simple changes of variable. We'll show that " $(\forall x)Fx$ " implies " $(\forall y)Fy$ " and that " $(\exists x)Fx$ " implies " $(\exists y)Fy$ ":

1	1. $(\forall x)Fx$	PI
1	2. Fa	US,1
1	3. $(\forall y)Fy$	UG,2
1	1. $(\exists x)Fx$	PI
2	2. Fa	PI (for ES)
2	3. $(\exists y)Fy$	EG,2
1	4. $(\exists y)Fy$	ES,1,2,3

We'll now show that the sentence " $(\exists x)(\forall y)Fxy \rightarrow (\forall y)(\exists x)Fxy$ " ("If there is someone who is a friend of everyone, then everyone has a friend") is valid:

1	1. $(\exists x)(\forall y)Fxy$	PI
2	2. $(\forall y)Fay$	PI (for ES)
2	3. Fab	US,2
2	4. $(\exists x)Fxb$	EG,3
1	5. $(\exists x)Fxb$	ES,1,2,4
1	6. $(\forall y)(\exists x)Fxy$	UG,5
	7. $((\exists x)(\forall y)Fxy \rightarrow (\forall y)(\exists x)Fxy)$	CP,1,6

Next we formalize the argument:

All cows are animals.
Therefore all heads of cows are heads of animals.

The example is due to Augustus de Morgan,* who used it to illustrate the appalling weakness of the traditional Aristotelian

* To those of you of a literary bent, de Morgan will be best known as the author of these two poems:

Big fleas have little fleas
Upon their backs to bite 'em.
And little fleas have littler still,
And so, *ad infinitum*.

Big fleas have greater fleas,
Upon their backs to gnaw on.

logic. The inference is simple and obvious, yet to formalize it lies far beyond the capabilities of the traditional logic. For the predicate calculus, however, it's easy:

1	1. $(\forall x)(Cx \rightarrow Ax)$	PI
2	2. $(\exists y)(Cy \wedge Hay)$	PI
3	3. $(Cb \wedge Hab)$	PI (for ES)
3	4. Cb	TC,3
3	5. Hab	TC,3
1	6. $(Cb \rightarrow Ab)$	US,1
1,3	7. Ab	TC,4,6
1,3	8. $(Ab \wedge Hab)$	TC,5,7
1,3	9. $(\exists y)(Ay \wedge Hay)$	EG,8
1,2	10. $(\exists y)(Ay \wedge Hay)$	ES,2,3,9
1	11. $((\exists y)(Cy \wedge Hay) \rightarrow (\exists y)(Ay \wedge Hay))$	CP,2,10
1	12. $(\forall x)((\exists y)(Cy \wedge Hay) \rightarrow (\exists y)(Ay \wedge Hay))$	UG,11

Next we formalize the following argument:

Tereza caught a fish if any of the Brazilians did.
 A fish that was under the pier was caught by a Brazilian.
 Therefore, Tereza caught a fish.

1	1. $((\exists x)(Bx \wedge (\exists y)(Fy \wedge Cxy)) \rightarrow (\exists x)(Fx \wedge Ctx))$	PI
2	2. $(\exists x)((Fx \wedge Px) \wedge (\exists y)(By \wedge Cyx))$	PI
3	3. $((Fa \wedge Pa) \wedge (\exists y)(By \wedge Cya))$	PI (for ES)
3	4. $(Fa \wedge Pa)$	TC,3
3	5. Fa	TC,4
3	6. Pa	TC,4
3	7. $(\exists y)(By \wedge Cya)$	TC,3
8	8. $(Bb \wedge Cba)$	PI (for ES)
8	9. Bb	TC,8
8	10. Cba	TC,8
3,8	11. $(Fa \wedge Cba)$	TC,5,10
3,8	12. $(\exists y)(Fy \wedge Cby)$	EG,11

And great fleas have greater still,
 And so on, and so on.

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3,8	13. $(Bb \wedge (\exists y)(Fy \wedge Cby))$	TC,9,12
3,8	14. $(\exists x)(Bx \wedge (\exists y)(Fy \wedge Cxy))$	EG,13
3	15. $(\exists x)(Bx \wedge (\exists y)(Fy \wedge Cxy))$	ES,7,8,14
2	16. $(\exists x)(Bx \wedge (\exists y)(Fy \wedge Cxy))$	ES,2,3,15

Now let's symbolize the argument:

Anyone who flies is admired by everyone.
 Not everyone admires himself.
 So there is someone who doesn't fly.

1	1. $(\forall x)(Fx \rightarrow (\forall y)Ayx)$	PI
2	2. $\neg(\forall x)Axx$	PI
2	3. $(\exists x)\neg Axx$	QE,2
4	4. $\neg Aaa$	PI (for ES)
1	5. $(Fa \rightarrow (\forall y)Aya)$	US,1
4	6. $(\exists y)\neg Aya$	EG,4
4	7. $\neg(\forall y)Aya$	QE,6
1,4	8. $\neg Fa$	TC,5,7
1,4	9. $(\exists x)\neg Fx$	EG,8
1,2	10. $(\exists x)\neg Fx$	ES,3,4,9

We now prove that our two translations of "If any dog can fly, Tarmin can" are logically equivalent:

1	1. $((\exists x)(Dx \wedge Fx) \rightarrow Ft)$	PI
2	2. $(Da \wedge Fa)$	PI
2	3. $(\exists x)(Dx \wedge Fx)$	EG,2
1,2	4. Ft	TC,1,3
1	5. $((Da \wedge Fa) \rightarrow Ft)$	CP,2,4
1	6. $(\forall x)((Dx \wedge Fx) \rightarrow Ft)$	UG,5
	7. $((\exists x)(Dx \wedge Fx) \rightarrow Ft) \rightarrow (\forall x)((Dx \wedge Fx) \rightarrow Ft)$	CP,1,6
8	8. $(\forall x)((Dx \wedge Fx) \rightarrow Ft)$	PI
9	9. $(\exists x)(Dx \wedge Fx)$	PI
10	10. $(Db \wedge Fb)$	PI (for ES)
8	11. $((Db \wedge Fb) \rightarrow Ft)$	US,8
8,10	12. Ft	TC,10
8,9	13. Ft	ES,9,10,12
8	14. $((\exists x)(Dx \wedge Fx) \rightarrow Ft)$	CP,9,13
	15. $((\forall x)((Dx \wedge Fx) \rightarrow Ft) \rightarrow ((\exists x)(Dx \wedge Fx) \rightarrow Ft))$	CP,8,14

$$16. ((\exists x)(Dx \wedge Fx) \rightarrow Ft) \leftrightarrow (\forall x)((Dx \wedge Fx) \rightarrow Ft)$$

TC,7,15

Now we'll do the same thing for our two translations of "Any dog who chases any dog who chases any rabbit will be put in the pound":

1	1. $(\forall x)((Dx \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cxy)) \rightarrow Px)$	PI
2	2. $((Da \wedge (Db \wedge (Rc \wedge Cbc))) \wedge Cab)$	PI
2	3. $(Da \wedge (Db \wedge (Rc \wedge Cbc)))$	TC,2
2	4. Cab	TC,2
2	5. Da	TC,3
2	6. $(Db \wedge (Rc \wedge Cbc))$	TC,3
2	7. Db	TC,6
2	8. $(Rc \wedge Cbc)$	TC,6
2	9. Rc	TC,8
2	10. Cbc	TC,8
1	11. $((Da \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cay)) \rightarrow Pa)$	US,1
2	12. $(\exists z)(Rz \wedge Cbz)$	EG,8
2	13. $(Db \wedge (\exists z)(Rz \wedge Cbz))$	TC,7,12
2	14. $((Db \wedge (\exists z)(Rz \wedge Cbz)) \wedge Cab)$	TC,4,13
2	15. $(\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cay)$	EG,14
2	16. $(Da \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cay))$	TC,5,15
1,2	17. Pa	TC,11,16
1	18. $((Da \wedge (Db \wedge (Rc \wedge Cbc))) \wedge Cab) \rightarrow Pa)$	CP,2,17
1	19. $(\forall z)((Da \wedge (Db \wedge (Rz \wedge Cbz))) \wedge Cab) \rightarrow Pa)$	UG,18
1	20. $(\forall y)(\forall z)((Da \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge Cay) \rightarrow Pa)$	UG,19
1	21. $(\forall x)(\forall y)(\forall z)((Dx \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge Cxy) \rightarrow Px)$	UG,20
	22. $((\forall x)((Dx \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cxy)) \rightarrow Px) \rightarrow$ $(\forall x)(\forall y)(\forall z)((Dx \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge Cxy) \rightarrow$ $Px)$	CP,1,21
23	23. $(\forall x)(\forall y)(\forall z)((Dx \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge Cxy) \rightarrow Px)$	PI
24	24. $((Dd \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cdy))$	PI
24	25. Dd	TC,24
24	26. $(\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz)) \wedge Cdy)$	TC,24
27	27. $((De \wedge (\exists z)(Rz \wedge Cez)) \wedge Cde)$	PI (for ES)
27	28. $(De \wedge (\exists z)(Rz \wedge Cez))$	TC,27
27	29. Cde	TC,27
27	30. De	TC,28
27	31. $(\exists z)(Rz \wedge Cez)$	TC,28

32	32	$(Rf \wedge Cef)$	PI (for ES)
32	33.	Rf	TC, 32
32	34.	Cef	TC, 32
23	35.	$(\forall y)(\forall z)((Dd \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge Cdy) \rightarrow Pd)$	US, 23
23	36.	$(\forall z)((Dd \wedge (De \wedge (Rz \wedge Cez))) \wedge Cde) \rightarrow Pd)$	US, 35
23	37.	$((Dd \wedge (De \wedge (Rf \wedge Cef))) \wedge Cde) \rightarrow Pd)$	US, 36
32	38.	$(De \wedge (Rf \wedge Cef))$	TC, 30, 32
24, 32	39.	$(Dd \wedge (De \wedge (Rf \wedge Cef)))$	TC, 25, 32
24, 27, 32	40.	$((Dd \wedge (De \wedge (Rf \wedge Cef))) \wedge Cde)$	TC, 29, 39
23, 24, 27, 32	41.	Pd	TC, 37, 40
23, 24, 27	42.	Pd	ES, 31, 32, 41
23, 24	43.	Pd	ES, 26, 27, 42
23	44.	$((Dd \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz))) \wedge Cdy) \rightarrow Pd)$	CP, 24, 43
23	45.	$(\forall x)((Dx \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz))) \wedge Cxy) \rightarrow$	
Px)			UG, 44
	46.	$((\forall x)(\forall y)(\forall z)((Dx \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge Cxy) \rightarrow$	
		$Px) \rightarrow (\forall x)((Dx \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz))) \wedge$	
Cxy)		$\rightarrow Px))$	CP, 23, 45
	47.	$((\forall x)((Dx \wedge (\exists y)((Dy \wedge (\exists z)(Rz \wedge Cyz))) \wedge Cxy) \rightarrow$	
		$Px) \leftrightarrow (\forall x)(\forall y)(\forall z)((Dx \wedge (Dy \wedge (Rz \wedge Cyz))) \wedge$	
Cxy)		$\rightarrow Px))$	TC, 22, 46

The proof of the Soundness and Completeness Theorems is virtually unchanged from the MPC. To prove soundness, we verify that the roles preserve logical consequence by examining the rules, one by one, the same as before. We prove completeness by showing that every d-consistent set is consistent. We show this by demonstrating, just as before, that every d-consistent set is contained in a Henkin set. Then we show that, given a Henkin set Δ , we can construct a model \mathcal{A} in which all and only the members of Δ are true. \mathcal{A} is obtained as follows:

$$\begin{aligned}
 |\mathcal{A}| &= \{\text{natural numbers } 0, 1, 2, \dots\} \\
 \mathcal{A}(c_i) &= i \\
 \mathcal{A}(R) &= \{ \langle i_1, \dots, i_n \rangle : R c_{i_1}, \dots, c_{i_n} \in \Delta \}, \text{ for } R \text{ an } n\text{-place} \\
 &\quad \text{predicate}
 \end{aligned}$$

Once we have strong completeness, we can derive the Compactness Theorem and the Löwenheim-Skolem Theorem, just as before.