Monadic Predicate Calculus

To progress any further, we are going to need an analysis that goes deeper than looking at how complex sentences are formed out of simple sentence. We'll have to look at the internal structures of the simple sentences.

A good place to begin is the *Sophist*, where Plato gives an account of what makes makes the very simplist sentences true or false. Unlike the typical Platonic dialogue, where Socrates plsys the dominant role, the principal role is this dialogue is played by an unnamed stranger Elea. The interlocutor is the boy Theaetetus. Theaetetus will do on to distinguish himself as a courageous leader in battle and also as a geometer. It was Theaetetus who first discovered the five regular solids — polyhedra all of whose sides and angles are congruent — namely, the cube, the tetrahedron, the octahedron, the dedecahedron, and the icosahedron. But I digress. Here's a quote from Benjamin Jowett's translation:

> ***Stranger.* Then, as I was saying, let us first of all obtain a conception of language and opinion, in order that we may have clearer grounds for determining, whether not-being has any concern with them, or whether they are both always true, and neither of them ever false.**
>
> ***Theaetetus.* True.**
>
> ***Stranger.* Then, now, let us speak of names, as before we were speaking of ideas and letters; for that is the direction in which the answer may be expected.**

What they decided about ideas about ideas and about names was that some fit together and others don't. For example, you can't get a word by forming a string of consonants, but you can get a word by combining consonants and vowels in the right way.

> ***Theaetetus.* And what is the question at issue about names?**
>
> ***Stranger.* The question at issue is whether all names may be connected with one another, or none, or only some of them.**
>
> ***Theaetetus.* Clearly the last is true.**

***Stranger*. I understand you to say that words which have a meaning when in sequence may be connected, but that words which have no meaning when in sequence cannot be connected?**

***Theaetetus*. What are you saying?**

***Stranger*. What I thought that you intended when you gave your assent; for there are two sorts of intimation of being which are given by the voice.**

***Theaetetus*. What are they?**

***Stranger*. One of them is called nouns, and the other verbs.**

***Theaetetus*. Describe them.**

***Stranger*. That which denotes action we call a verb.**

***Theaetetus*. True.**

***Stranger*. And the other, which is an articulate mark set on those who do the actions, we call a noun.**

***Theaetetus*. Quite true.**

***Stranger*. A succession of nouns only is not a sentence any more than of verbs without nouns.**

***Theaetetus*. I do not understand you.**

***Stranger*. I see that when you gave your assent you had something else in your mind. But what I intended to say was, that a mere succession of nouns or of verbs is not discourse.**

***Theaetetus*. What do you mean?**

***Stranger*. I mean that words like "walks," "runs," "sleeps," or any other words which denote action,**

however many of them you string together, do not make discourse.

***Theaetetus*. How can they?**

***Stranger*. Or, again, when you say "lion," "stag," "horse," or any other words which denote agents — neither in this way of stringing words together do you attain to discourse; for there is no expression of action or inaction, or of the existence of existence or non-existence indicated by the sounds, until verbs are mingled with nouns; then the words fit, and the smallest combination of them forms language, and is the simplest and least form of discourse.**

***Theaetetus*. Again I ask, What do you mean?**

***Stranger*. When any one says "A man learns," should you not call this the simplest and least of sentences?**

***Theaetetus*. Yes.**

***Stranger*. Yes, for he now arrives at the point of giving an intimation about something which is, or is becoming, or has become, or will be. And he not only names, but he does something, by connecting verbs with nouns; and therefore we say that he discourses, and to this connection of words we give the name of discourse.**

***Theaetetus*. True.**

***Stranger*. And as there are some things which fit one another, and other things which do not fit, so there are some vocal signs which do, and others which do not, combine and form discourse.**

***Theaetetus*. Quite true.**

***Stranger*. There is another small matter.**

***Theaetetus*. What is it?**

***Stranger*. A sentence must and cannot help having a subject.**

***Theaetetus*. True.**

***Stranger*. And must be of a certain quality.**

***Theaetetus*. Certainly.**

***Stranger*. And now let us mind what we are about.**

***Theaetetus*. We must do so.**

***Stranger*. I will repeat a sentence to you in which a thing and an action are combined, by the help of a noun and a verb; and you shall tell me of whom the sentence speaks.**

***Theaetetus*. I will, to the best my power.**

***Stranger*. "Theaetetus sits" — not a very long sentence.**

***Theaetetus*. Not very.**

***Stranger*. Of whom does the sentence speak, and who is the subject that is what you have to tell.**

***Theaetetus*. Of me; I am the subject.**

***Stranger*. Or this sentence, again.**

***Theaetetus*. What sentence?**

***Stranger*. "Theaetetus, with whom I am now speaking, is flying."**

***Theaetetus*. That also is a sentence which will be admitted by every one to speak of me, and to apply to me.**

***Stranger*. We agreed that every sentence must necessarily have a certain quality.**

Theaetetus. Yes.

Stranger. And what is the quality of each of these two sentences?

Theaetetus. The one, as I imagine, is false, and the other true.

Stranger. The true says what is true about you?

Theaetetus. Yes.

Stranger. And the false says what is other than true?

Theaetetus. Yes.

Stranger. And therefore speaks of things which are not as if they were?

Theaetetus. True.

Stranger. And say that things are real of you which are not; for, as we were saying, in regard to each thing or person, there is much that is and much that is not.

Theaetetus. Quite true.

Stranger. The second of the two sentences which related to you was first of all an example of the shortest form consistent with our definition.

Theaetetus. Yes, this was implied in recent admission.

Stranger. And, in the second place, it related to a subject?

Theaetetus. Yes.

Stranger. Who must be you, and can be nobody else?

Theaetetus. Unquestionably.

> *Stranger*. And it would be no sentence at all if there were no subject, for, as we proved, a sentence which has no subject is impossible.
>
> *Theaetetus*. Quite true.
>
> *Stranger*. When other, then, is asserted of you as the same, and not-being as being, such a combination of nouns and verbs is really and truly false discourse.
>
> *Theaetetus*. Most true.

In our formal language, *individual constants*, usually lowercase letters from the early part of the alphabet, will play the role of names, and *predicates*, usually uppercase letters, will play the role of verb. Thus “t” will denote Theaetetus, and “S” and “F” will represent the actions of sitting and flying, respectively. “Theaetetus sits” will be symbolized “St,” and “Theaetetus flies” will be “Ft.” The sentence is true just in case the individual named by the name performs the action designated by the verb.

We want to start with Plato’s account and extend it, as far as we can, beyond the very simple sentences Plato considers. The first thing we notice is that simple sentences of the form

> name + copula + adjective

or

> name + copula + indefinite article + common noun

like

> Theaetetus is brave.

or

> Theaetetus is a Greek.

can be readily covered by Plato's account. Thus, we take a simple sentence to consist of a proper name, such as

"Theaetetus," and a *predicate*, such as "sits" or "is brave" or "is a Greek." The proper name designates an individual, and the predicate designates a property or action. The sentence is true just in case the individual has the property or performs the action. We'll symbolize "Theaetetus is brave" as "Bt," and we'll use "Gt" to symbolize "Theaetetus is a Greek."

We can combine the simple sentences by means of sentential connectives, so that "Theaetetus is a brave Greek" will be

$$(Bt \wedge Gt)$$

"Theaetetus either sits or flies" will be

$$(St \vee Tt)$$

"Theaetetus sits but he does not fly" is

$$(St \wedge \neg Ft)$$

"If Theaetetus is brave, so is Socrates" is

$$(Bt \rightarrow Bs)$$

It is tempting to try to treat "Something flies" as analogous to "Theaetetus flies." The temptation should be resisted. One way to see that there is a big difference between "Theaetetus flies" and "Something flies" is to observe that "Theaetetus flies" and "Theaetetus is a man" together imply "Theaetetus is a man who flies," whereas "Something flies" and "Something is a man" do not imply "Something is a man who flies."

The correct analysis, due to Frege, is this: Whereas "Theaetetus flies" and "Theaetetus is a man" are to be understood as attributing a property (flying; manhood) to an individual (Theaetetus), "Something flies" is to be understood as attributing a property to a property. Namely "Something flies" says about the property of flying that it is instantiated. Similarly, "Something is a man" says about the property of manhood that it is instantiated. We represent the property of flying in English by an open sentence "x

flies," and in the formal language by an open sentence "Fx." We indicate that something flies in the formal language by prefixing the *existential quantifier* "(∃x)" to the open sentence "Fx," getting "(∃x)Fx." "(∃x)" is read "for some x" or "there is an x such that," Similarly, the property of manhood is indicated in English by the open sentence "x is a man" and in the formal language by the open sentence "Mx." We indicate that something is a man by prefixing the existential quantifier to the open sentence "Mx," getting "(∃x)Mx." The property of being a man who flies is indicated in English by the open sentence "x is a man who flies" or "x is a man and x flies" and in the formal language by the open sentence "(Mx ∧ Fx)." We indicate that some men fly by prefixing the existential quantifier to the open sentence "(Mx ∧ Fx)," getting "(∃x)(Mx ∧ Fx)"

Similarly, it would be tempting to treat "Everything is a man" as analogous to "Theaetetus is a man." The resemblance between the two is superficial, however, as we can see from the following example: "Theaetetus is either a man or a woman" and "It is not the case that Theaetetus is a woman" together imply "Theaetetus is a man," whereas "Everything is either a man or a woman" and "It is not the case that everything is a woman" do not imply "Everything is a man." Whereas "Theaetetus is a man" indicates that a certain individual (Theaetetus) has a certain property (manhood), "Everything is a man" attributes a property to a property. Namely, "Everything is a man" tells us about the property of manhood that it is possessed by everything. We indicate that everything is a man by prefixing the *universal quantifier* "(∀x)" (read "for all x" or "for every x") to the open sentence "Mx," getting "(∀x)Mx." We indicate that everything flies by writing "(∀x)Fx." We indicate that all men fly by writing "(∀x)(Mx → Fx)," so that, for every x, either x is not a man or else x flies.

We can use Venn diagrams to illustrate quantified statements. "Everyone is a man or a woman" ["(∀x)(Mx ∨ Wx)"] is indicated by shading Cell 4 in Figure 1, to indicate that there's nothing in Cell 4. "All men fly" ["(x)(Mx → Fx)"]

is indicated by shading Cell 2 in Figure 2. "Everything that flies is a man" ["$(\forall x)(Fx \rightarrow Mx)$"] is indicated in Figure 3 by shading Cell 3. "Everything that flies is either a man or a woman" ["$(\forall x)(Fx \rightarrow (Mx \vee Wx))$"] is indicated by shading Cell 7 in Figure 4. "Everyone who is either a man or a woman flies" ["$(\forall x)((Mx \vee Wx) \rightarrow Fx)$"] is indicated by shading Cells 2, 4, and 6 in Figure 5. For "Everyone who is both a man and a woman flies" ["$(\forall x)((Mx \wedge Wx) \rightarrow Fx)$"] we shade Cell 2 in Figure 6, while for "Everyone who flies is both a man and a woman" ["$(\forall x)(Fx \rightarrow (Mx \wedge Wx))$"] we shade Cells 3, 5, and 7 in Figure 7.

How about sentences that begin with an existential quantifier? If we want to illustrate the sentence "Someone who is either a man or a woman flies" ["$(\exists x)((Mx \vee Wx) \wedge Fx)$"], we want to indicate that there is something in at least one of the three Cells 1, 3, and 5. We can do this by drawing a curve that passes through Cells 1, 3, and 5, as in Figure 8. You can think of the curve as like a train track; there is a locomotive somewhere along the track.

"There are some men who either sit or fly" ["$(\exists x)(Mx \wedge (Sx \vee Fx))$"] is indicated by a curve that passes through Cells 1, 2, and 3 in Figure 9. "There are some men who both sit and fly" ["$(\exists x)(Mx \wedge (Sx \wedge Fx))$"] is indicated by a curve that is contained entirely within Cell 1, as in Figure 10. "There are some men who fly, and there are some men who do not" [$((\exists x)(Mx \wedge Fx) \wedge (\exists x)(Mx \wedge \neg Fx))$"] is indicated in Figure 11 by having a curve that is contained entirely within Cell 1 and another curve that is entirely within Cell 2. "There are some men who sit, some men who fly, and some men who do neither" ["$(((\exists x)(Mx \wedge Sx) \wedge (\exists x)(Mx \wedge Fx) \wedge (\exists x)(Mx \wedge \neg(Sx \vee Fx)))$"] is indicated in Figure 12 by having a curve that passes through Cells 1 and 2, a second curve that passes through Cells 1 and 3, and yet another curve that is contained entirely within Cell 4.

Sentences that contain proper names are indicated the same way, except that now we label the curves. "Theaetetus is a man who either sits or flies" ["$(Mt \wedge (St \vee Ft))$"] is indicated in Figure 13 by having a curve marked "t" pass through Cells 1, 2, and 3. Theaetetus is a locomotive that is located somewhere along the track. "Theaetetus is a man who both sits and flies" ["$(Mt \wedge (St \wedge Ft))$"] is indicated in Figure 14 by a curve marked "t" contained entirely within Cell 1. "Rambo, who is a man, does not fly, but Dumbo, who is not a man, does fly" ["$((Mr \wedge \neg Fr) \wedge (\neg Md \wedge Fd))$"] is illustrated in Figure 15 by two curves, one, marked "r," contained within Cell 2, and the other, marked "d," contained within Cell 3.

We can use Venn diagrams to show that certain arguments are valid. For example, consider this argument:

> **All terriers are dogs.**
> **All dogs are mammals.**
> **Therefore all terriers are mammals.**

In symbols,

$$(\forall x)(Tx \rightarrow Dx)$$
$$(\forall x)(Dx \rightarrow Mx)$$
$$\therefore\ (\forall x)(Tx \rightarrow Mx)$$

We see whether it is possible to have the premise true and the conclusion false. The first premise is indicated in Figure 16 by shading Cells 3 and 4. The second premise is indicated by shading Cells 2 and 6. If the conclusion were false, there would be something either in Cell 2 or in Cell 4; we indicate this by a train track passing through Cells 2

and 4. But, while the train track would indicate that there is something either within Cell 2 or Cell 4, the fact that Cells 2 and 4 are both shaded indicates that there is nothing in either of those cells. So the attempt to diagram a situation in which the premises are true and the conclusion false ends up with an impossibility. So the argument must be valid.

Another example:

> All elephants are mammals.
>
> Some elephants can fly.
> Therefore some mammals can fly.

In symbols,

$$(\forall x)(Ex \rightarrow Mx)$$
$$(\exists x)(Ex \wedge Fx)$$
$$\therefore (\exists x)(Mx \wedge Fx).$$

In Figure 17, we try to diagram a situation in which the premises are true and the conclusion false. The first premise is indicated by shading Cells 3 and 4. The second premise is indicated by a train track passing through Cells 1 and 3. To say the conclusion is true is to say that there is something either in Cell 1 or in Cell 5. Thus, to indicate that the conclusion is false, we shade Cells 1 and 5. But this has the train track passing entirely through shaded territory, which is impossible. So the argument must be valid.

Here is an inference to consider:

> Dumbo is an elephant.
> Dumbo flies.

Therefore some elephants fly.

In symbols,

Ed
Fd
$\therefore (\exists x)(Ex \wedge Fx)$

To represent the first premise, we draw a train track marked "d" through Cells 1 and 2. We indicate the second premise by crossing out the part of this train track which lies outside circle "F." To indicate the falsity of the conclusion, we shade Cell 1. But this gives us a train track every part of which is either crossed out or shaded, which represents an impossible situation.

Now consider this inference:

Traveler is a horse.
All horses eat oats.
Therefore Traveler eats oats.

In symbols,

Ht
$(\forall x)(Hx \rightarrow Ox)$
$\therefore Ot.$

The first premise is indicated in Figure 19 by a curve marked "t" passing through Cells 1 and 2, and the second premise is indicated by shading Cell 2. We indicate the falsity of the conclusion by crossing out the part of curve "t" which lies inside circle "O." But this means the whole curve is either crossed out or shaded, which is impossible.

As a final example, consider

Everyone is either a man or a woman.

> **Not everyone is a man.**
> **Therefore someone is a woman.**

In symbols,

> **$(\forall x)(Mx \vee Wx)$**
> **$\neg(\forall x)Mx$**
> **$\therefore\ (\exists x)Wx$**

The first premise is indicated in Figure 19 by shading Cell 4. "$(\forall x)Mx$" says that there is nothing in either Cells 3 or Cell 4. "$\neg(\forall x)Mx$" denies this, so it says that there is something either in Cell 3 or in Cell 4, a fact we can indicate by drawing a curve passing through Cells 3 and 4. The conclusion says that there is someone either in Cell 1 or in Cell 3. So we can indicate the falsity of the conclusion by shading Cells 1 and 3. But this has the train track passing entirely through shaded territory, which is impossible.

We now turn to a more formal development. A *language for the monadic predicate calculus* (*MPC*) is given by specifying two kinds of things: *individual constants* (usually lowercase letters from the early part of the alphabet), which play the role of proper names, and *predicates* (usually uppercase letters), which play the roles of intransitive verbs, common nouns, and adjectives. An *atomic formula* consists either of a predicate followed by an individual constant or of a predicate followed by the *variable* "x." The *formulas* of the language constitute the smallest class of expressions which

> **contains the atomic formulas;**
> **contains $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ whenever it contains φ and ψ; and**
> **contain $\neg\varphi$, $(\forall x)\varphi$, and $(\exists x)\varphi$ whenever it contains φ.**

> **Unique Readability. A formula is built up from atomic formulas *in a unique way*.**

The *subformulas* of a particular formula are just the formulas that are contained within the given formula, where a formula is counted as a subformula of itself. If an occurrence of the letter "x" within a particular formula is contained within a subformula beginning with "(∀x)" or with "(∃x)," the occurrence is said to be *bound*. Otherwise it is said to be *free*. A formula with no free occurrences of "x" is a *sentence*. Where φ is a formula and c is a constant, we write $\varphi^{x}/_{c}$ for the sentence that results from replacing each free occurrence of "x" in φ by "c."

Examples: In "(Fx ∧ (∀x)(Gx ∧ ¬(∀x)Jx))," the first occurrence of "x" is free, and the other four are bound. "(Fx ∧ (∀x)(Gx ∧ ¬(∀x)Jx))"$^{x}/_{d}$ is the sentence "(Fd ∧ (∀x)(Gx ∧ ¬(∀x)Jx))."

In "((∀x)(Fx ↔ Gx) ∧ ((∃x)Fx ↔ (Hx ∧ Jc)))," the first five occurrences of "x" are bound and the remaining occurrence is free. "((∀x)(Fx ↔ Gx) ∧ ((∃x)Fx ↔ (Hx ∧ Jc)))"$^{x}/_{c}$ is "((∀x)(Fx ↔ Gx) ∧ ((∃x)Fx ↔ (Hc ∧ Jc)))," which is a sentence.

In "(((∀x)Fx ↔ Gx) ∧ (∃x)(Fx ↔ (Hx ∧ Jc)))," only the third occurrence of "x" is free; the other five are bound. "(((∀x)Fx ↔ Gx) ∧ (∃x)(Fx ↔ (Hx ∧ Jc)))"$^{x}/_{e}$ is the sentence "(((∀x)Fx ↔ Gx) ∧ (∃x)(Fx ↔ (He ∧ Jc)))."

"Fc" and "(∀x)Fx" are both sentences.

In general, if φ is a formula and c is a constant, $\varphi^{x}/_{c}$ is a sentence. Also, every formula which begins with either "(∀x)" or "(∃x)" is a sentence.

> **Definition. An *interpretation* (of a language of the MPC)is a function $\mathcal{A}$ defined on {"∀"} ∪ {individual constants of the language} ∪ {predicates of the language} that meets the following conditions:**
>
> > **$\mathcal{A}$("∀"), also written $|\mathcal{A}|$, is a nonempty set, called the *universe of discourse* or the *domain* of the interpretation.**
> > **If c is a constant, $\mathcal{A}$(c), also written $c^{\mathcal{A}}$, is an element of $|\mathcal{A}|$.**

> If R is a predicate, $\mathcal{A}$(R), also written $R^{\mathcal{A}}$, is a subset of $|\mathcal{A}|$.

The universe of discourse of a particular discussion consists of the things we are talking about within that discussion. When I say, sitting at the dinner table with the family, “Everybody who finishes her Brussel sprouts will get ice cream,” I’m not promising to reward everyone in the whole world who eats her Brussel sprouts, just everyone sitting there at the table. For any formula φ, there will be a set of members of the universe of $\mathcal{A}$ that satisfy φ in $\mathcal{A}$. If this set is nonempty, the sentence $(\exists x)\varphi$ will be true in $\mathcal{A}$. If every member of $|\mathcal{A}|$ satisfies φ in $\mathcal{A}$, $(\forall x)\varphi$ will be true in $\mathcal{A}$. If the member $c^{\mathcal{A}}$ of $|\mathcal{A}$ satisfies φ in $\mathcal{A}$, then the sentence $\varphi^{x}/_{c}$ will be true in $\mathcal{A}$. It makes no sense to talk about a sentence of the formal language being true or false absolutely. A sentence is true or false under an interpretation. Only a sentence can be either true or false under an intepretation; a formula with free variables cannot.

Intuitively, we have three fundamental semantic notions, truth, falsity, and satisfaction. A sentence expresses a thought that is either true or false, whereas a formula that is not a sentence represents a property, and the formula is satisfied by those elements of the universe that have the property. We shall simplify our treatment by departing from our intuitions a little bit, applying the notion of satisfaction to all formulas, whether or not the formulas contain free variables, stipulating that a true sentence is satisfied by every member of the universe of discourse, whereas a false sentence is satisfied by nothing. Specifically, we have the following:

> Given an interpretation $\mathcal{A}$,
> an atomic formula of the form Rx is satisfied by the members of $\mathcal{A}$(R);
> an atomic formula of the form Rc is satisfied by every member of the universe if $\mathcal{A}$(c) is an element of $\mathcal{A}$(R);
> otherwise, Rc is satisfied by nothing;
> a formula of the form $(\varphi \wedge \psi)$ is satisfied by those members of the universe of discourse which satisfy both φ and ψ;

a formula of the form $(\varphi \vee \psi)$ is satisfied by those members of the universe of discourse which satisfy either φ or ψ (or both);
a formula of the form $(\varphi \rightarrow \psi)$ is satisfied by those members of the universe of discourse which either satisfy ψ or fail to satisfy φ;
a formula of the form $(\varphi \leftrightarrow \psi)$ is satisfied by those members of the universe of discourse which satisfy both φ and ψ and also by those members of the domain which satisfy neither φ nor ψ;
a formula of the form $\neg\varphi$ is satisfied by those members of the universe of discourse which fail to satisfy φ;
if every member of the universe satisfies φ, then every member of the universe satisfies $(\forall x)\varphi$;
if some member of the universe fails to satisfy φ, nothing satisfies $(\forall x)\varphi$;
if some member of the universe satisfies φ, every member of the universe satisfies $(\exists x)\varphi$;
if no member of the universe satisfies φ, no member of the universe satisfies $(\exists x)\varphi$.

Example. As an example of an interpretation, let's let

$|\mathcal{A}|$ = {animals}
$\mathcal{A}$("b") = Bonzo the chimpanzee
$\mathcal{A}$("c") = Celia the canary
$\mathcal{A}$("r") = Reagan, the former president
$\mathcal{A}$("B") = {animals that bay at the moon}
$\mathcal{A}$("D") = {dogs}
$\mathcal{A}$("F") = {animals that fly}
$\mathcal{A}$("C") = {chipmunks}

Since Celia can fly, $\mathcal{A}$("c") is an element of $\mathcal{A}$("F"), and so every animal will satisfy "Fc." $\mathcal{A}$("r") $\notin$ $\mathcal{A}$("F"), since Reagan can't fly, so nothing will satisfy "Fr." "Bx" will be satisfied by the animals that bay at the moon and "Dx" will be satisfied by the dogs. "$(Dx \wedge Bx)$" will be satisfied by the dogs that bay at the moon. "$\neg Fx$" will

> **be satisfied by the animals that don't fly. "(Dx ∧ ¬Bx)" will be satisfied by the dogs that don"t bay at the moon. Since some dogs bay at the moon, every animal will satisfy "(∃x)(Dx ∧ Bx)." Since no dogs fly, nothing will satisfy "(∃x)(Dx ∧ Fx)." Nothing satisfies "(∀x)(Dx → Bx)," since not every dog bays at the moon. Since Reagan isn't a chipmunk, nothing satisfies "Cr." So every animal satisfies "^¬Cr." So every animal satisfies "(∀x)¬Cr."☒**

Let's introduce some technical jargon. A formula that begins with "(∀x)" is a *universal* formula. One that begins with "(∃x)" is an *existential* formula. Formulas that begin either with "(∀x)" or with "(∃x)" are said to be *initially quantified*. Conjunctions, disjunctions, negations, conditionals, and biconditionals are referred to as *molecular* formulas. Every formula which isn't either atomic or initially quantified is built up from atomic formulas and from initially quantified by means of the connectives "∧," "∨," "¬," "→," and "↔." We refer to those atomic and initially quantified sentences out of which a given sentence is built as its basic truth-functional components.

> **Definition. A sentence which is satisfied by every member of |′$\mathcal{A}$| under an interpretation $\mathcal{A}$ is said to be true under $\mathcal{A}$. A sentence which is satisfied by no member of |$\mathcal{A}$| under $\mathcal{A}$ is false under $\mathcal{A}$.**

> **Law of Bivalence. Given an interpretation $\mathcal{A}$, every sentence is either true under $\mathcal{A}$ or false under $\mathcal{A}$.**

Proof: Since every sentence is built up from atomic sentences and from initially quantified sentences by means of the sentential connectives, it will be enough to show that, given an interpretation $\mathcal{A}$, every atomic sentence and every initially quantified sentence is either true or false under $\mathcal{A}$ and that every sentence formed from sentences which are either true or false under $\mathcal{A}$ by means of the sentential connectives is either true of false under $\mathcal{A}$.

An atomic sentence takes the form "Fc." Such a sentence is true under $\mathcal{A}$ if $\mathcal{A}(c) \in \mathcal{A}(F)$ and false under $\mathcal{A}$ if $\mathcal{A}(c) \notin \mathcal{A}(F)$. A

universal sentence $(\forall x)\varphi$ is true under $\mathcal{A}$ if every member of $|\mathcal{A}|$ satisfies φ under $\mathcal{A}$, and it is false under $\mathcal{A}$ otherwise. An existential sentence $(\exists x)\varphi$ is true under $\mathcal{A}$ if at least one member of $|\mathcal{A}|$ satisfies φ under $\mathcal{A}$, and it is false under $\mathcal{A}$ otherwise.

A conjunction is true under $\mathcal{A}$ if both conjuncts are true under $\mathcal{A}$, and it is false under $\mathcal{A}$ if either conjunct is false under $\mathcal{A}$. A disjunction is true under $\mathcal{A}$ if either disjunct is true under $\mathcal{A}$, and it is false under $\mathcal{A}$ if both disjuncts are false under $\mathcal{A}$. A negation is true under $\mathcal{A}$ if the negatum is false under $\mathcal{A}$, and it is false under $\mathcal{A}$ if the negatum is true under $\mathcal{A}$. A conditional is true under $\mathcal{A}$ if the antecedent is false under $\mathcal{A}$ or the consequent is true under $\mathcal{A}$; if the antecedent is true under $\mathcal{A}$ and the consequent if false under $\mathcal{A}$, the conditional is false under $\mathcal{A}$. A biconditional is true under $\mathcal{A}$ if both components are true under $\mathcal{A}$ or both components are false under $\mathcal{A}$; if one component is true and the other is false, the biconditional is false under $\mathcal{A}$.⊠

> **Corollary. For any sentence φ, interpretation $\mathcal{A}$, and element a of $|\mathcal{A}|$, φ is true under $\mathcal{A}$ iff a satisfies φ under $\mathcal{A}$.**

Proof: If φ is true under $\mathcal{A}$, then, by definition of "true," every element of $|\mathcal{A}'|$ satisfies φ under $\mathcal{A}$. So in particular, a satisfies φ under $\mathcal{A}$. If, on the other hand, φ isn't true under $\mathcal{A}$, then, by bivalence, φ is false under $\mathcal{A}$, so that, by definition of "false," nothing satisfies φ under $\mathcal{A}$; so, in particular, a doesn't satisfy φ under $\mathcal{A}$.⊠

The following definition is taken over directly from the sentential calculus:

> **Definition. A *normal truth assignment (N.T.A.*) is a function which assigns a number, either 0 or 1, to each sentence, subject to the following conditions:**
>
> **A conjunction is assigned 1 iff both conjuncts are assigned 1.**

> **A disjunction is assigned 1 iff one or both disjuncts are assigned 1.**
>
> **A negation is assigned 1 iff the negatum is assigned 0.**
>
> **A conditional is assigned 1 iff the antecedent is assigned 0 or the consequent is assigned 1.**
>
> **A biconditional is assigned 1 iff both components are assigned the same value.**
>
> **Definition. A sentence is *tautological* iff it is assigned the value 1 by every N.T.A. A sentence is *valid* iff it is true under every N.T.A.**

For the sentential calculus, the words "tautological" and "valid" were different words for the same thing. Now that we've started on the predicate calculus, we need to distinguish them. Validity is the notion we're really interested in, but we need the notion of tautology as a technical notion.

> **Proposition. Every tautology is valid, but not vice versa.**

Proof: Suppose that θ is a tautology, and take an arbitrary interpretation $\mathcal{A}$. We get a normal truth assignment by stipulating that, for any φ,

$$\begin{aligned}\Im(\varphi) &= 1 \text{ if } \varphi \text{ is true under} \\ &= 0 \text{ otherwise}\end{aligned}$$

So $\Im(\theta) = 1$. Hence θ is true under $\mathcal{A}$. Since $\mathcal{A}$ was arbitrary, this shows that every tautological formula is valid. On the other hand, the tautological formula "$((\forall x)Fx \rightarrow Fc)$" is not tautological.$\boxtimes$

A tautological sentence is a valid sentence whose validity is determined by the sentence's truth functional structure. If, instead, the validity of a sentence depends upon the meaning of the quantifiers, the sentence won't be tautological.

We can test whether a sentence is tautological by the method of truth tables, examining each possible way to assign a truth value to the sentence's basic truth functional components. Alternatively, we can test the sentence by the search-for-counterexample method. For example, to show that "$(((\exists x)Fx \to (\forall x)Gx) \lor (\neg Hc \to (\exists x)Fx))$" is tautological, we have the following:

$$\underline{(((\exists x)Fx \to (\forall x)Gx) \lor (\neg Hc \to (\exists x)Fx))}$$

~~1 0 0 0 0 X~~

> Definition. A sentence φ is a *logical consequence* of a set of sentences Γ iff φ is true under every interpretation under which all the members of Γ are true. φ is a *tautological consequence* of a set of sentences Γ iff φ is assigned the value 1 by every N.T.A. which assigns the value 1 to every member of Γ.

The same reasoning which gave us the last proposition yields the following:

> Proposition. Every tautological consequence of a set of sentences is a logical consequence, but not vice versa.

The following definitions and theorems are lifted directly from the sentential calculus:

> Definitions. A sentence is *contradictory* (or *inconsistent*) iff it is false under every interpretation. A sentence is *indeterminate* iff it is true under some interpretations and false under others. A sentence φ *implies* (or *entails*) sentence ψ iff ψ is true under every interpretation under which φ is true. φ and ψ are *logically equivalent* iff they are true under precisely the same interpretations. An argument is *valid* iff the conclusion is true under every interpretation under which the premises are true. A set of sentences is *consistent* iff there is some interpretation under which all its members are true.

Theorems. A sentence is a valid iff its negation is contradictory.

A sentence is contradictory iff its negation is valid.

A sentence is indeterminate iff its negation is indeterminate.

A conjunction is valid iff both its conjuncts are valid.

If conjunction is contradictory if (but not necessarily only if) either of its conjuncts is.
A disjunction is valid if (but not only if) either disjunct is valid.

A disjunction is contradictory iff both disjuncts are contradictory.

A conditional is contradictory iff its antecedent is valid and its consequent is a contradiction.

Two sentences φ and ψ are logically equivalent iff the biconditional $(\varphi \leftrightarrow \psi)$ is valid.

$\neg(\varphi \vee \psi)$ is logically equivalent to $(\neg\varphi \wedge \neg\psi)$.

$\neg(\varphi \wedge \psi)$ is logically equivalent to $(\neg\varphi \vee \neg\psi)$.

φ implies ψ iff the conditional $(\varphi \rightarrow \psi)$ is valid.

A contradiction implies every sentence.

A valid sentence is implied by every sentence.

Two sentences are logically equivalent iff each implies the other.

An argument is valid iff the conjunction of the premises entails the conclusion.

An argument is valid iff the conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion is valid.

φ is a logical consequence of $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ if and only if the argument with $\gamma_1, \gamma_2, ..., \gamma_n$ as premises and with φ as conclusion is valid.

A sentence is a logical consequence of the empty set iff it's valid.

A sentence is a valid iff it is a logical consequence of every set of sentences.

Each member of a set of sentences is a logical consequence of that set of sentences.
If every member of Δ is a logical consequence of Γ and φ is a logical consequence of Δ, then φ is a logical consequence of Γ.

If Δ is a subset of Γ and φ is a logical consequence of Δ, then φ is a logical consequence of Γ.

For any sentence ψ and set of sentences Γ, ψ is a logical consequence of Γ if and only if Γ and $\Gamma \cup \{\psi\}$ have precisely the same logical consequences.

$(\varphi \wedge \psi)$ is a logical consequence of Γ iff φ and ψ are both logical consequences of Γ.

$(\varphi \rightarrow \psi)$ is a logical consequence of Γ iff ψ is a logical consequence of $\Gamma \cup \{\varphi\}$.

$\{\gamma_1, \gamma_2, ..., \gamma_n\}$ is inconsistent iff $(\gamma_1 \wedge (\gamma_2 \wedge ... \wedge \gamma_n)...)$ is an inconsistent sentence.

If Γ is an inconsistent set of sentences, then every sentence is a logical consequence of Γ.

A set of sentences Γ is inconsistent iff $(P \wedge \neg P)$ is a logical consequence of Γ.

A set of sentences Γ is inconsistent iff every sentence is a logical consequence of Γ.

If Δ is inconsistent and $\Delta \subseteq \Gamma$, then Γ is inconsistent.

φ is a logical consequence of Γ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

Substitution Principle. For any interpretation $\mathcal{A}$, individual constant c, and formula φ, $\varphi^x/_c$ is true under $\mathcal{A}$ iff $\mathcal{A}$(c) satisfies φ under $\mathcal{A}$.

Proof: I am going to write out this proof in excruciating detail, just so you'll see what one of these proofs looks like when written out in utter detail. I promise not to do it again.

Let $\mathcal{A}$ be an interpretation and c a constant, and let Σ be the set of formulas φ such that $\varphi^x/_c$ is true under $\mathcal{A}$ iff $\mathcal{A}$(c) satisfies φ under $\mathcal{A}$. Clearly {formulas} $\subseteq \Sigma$. But also, since {formulas} is the smallest class of expressions which contains the atomic formulas and which is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification, if we can show that Σ is a class of expressions which contains the atomic formulas and which is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification, this will tell us that {formulas} $\subseteq \Sigma$. This will tell us that {formulas} = Σ, which is what we want.

Atomic formulas are in Σ. If φ is an atomic formula, then either it has the form Fx or it has the form Fd. If φ has the from Fx, then $\varphi^x/_c$ is Fc. We have

$\varphi^x/_c$ is true under $\mathcal{A}$
iff Fc is true under $\mathcal{A}$
iff $\mathcal{A}(c) \in \mathcal{A}(F)$
iff $\mathcal{A}$(c) satisfies Fx under $\mathcal{A}$.

If φ has the form Fd, then $\varphi^x/_c = \varphi$. We have

$\varphi^x/_c$ is true under $\mathcal{A}$
iff φ is true under $\mathcal{A}$
iff $\mathcal{A}$(c) satisfies φ under $\mathcal{A}$ [by the corollary to the principle of bivalence].

Σ is closed under conjunction. Suppose that φ and ψ are both in Σ. Then $(\varphi'\wedge'\psi)^x/_c$ is equal to $(\varphi^x/_c \wedge \psi^x/_c)$, and we have:

$(\varphi \wedge \psi)^x/_c$ is true under
iff $(\varphi^x/_c \wedge \psi^x/_c)$ is true under
iff both $\varphi^x/_c$ and $\psi^x/_c$ is true under
iff $\mathcal{A}$(c) satisfies φ under $\mathcal{A}$ and (c) satisfies ψ under [because φ and ψ are both in Σ]
iff $\mathcal{A}$(c) satisfies $(\varphi \wedge \psi)$ under $\mathcal{A}$.

So $(\varphi \wedge \psi)$ is in Σ.

Σ is closed under disjunction. Suppose that φ and ψ are both in Σ. Then $(\varphi'\vee'\psi)^x/_c$ is equal to $(\varphi^x/_c \vee \psi^x/_c)$, and we have:

$(\varphi \vee \psi)^x/_c$ is true under $\mathcal{A}$
iff $(\varphi^x/_c \vee \psi^x/_c)$ is true under $\mathcal{A}$
iff either $\varphi^x/_c$ or $\psi^x/_c$ is true under $\mathcal{A}$
iff either $\mathcal{A}$(c) satisfies φ under $\mathcal{A}$ or $\mathcal{A}$(c) satisfies ψ under [because φ and ψ are both in Σ]
iff $\mathcal{A}$(c) satisfies $(\varphi \vee \psi)$ under $\mathcal{A}$.

So $(\varphi \vee \psi)$ is in Σ.

Σ is closed under the formation of conditionals. Suppose that φ and ψ are both in Σ. Then $(\varphi'\rightarrow'\psi)^x/_c$ is equal to $(\varphi^x/_c \rightarrow \psi^x/_c)$, and we have:

$(\varphi \rightarrow \psi)^x/_c$ is true under $\mathcal{A}$
iff $(\varphi^x/_c \rightarrow \psi^x/_c)$ is true under $\mathcal{A}$
iff either $\varphi^x/_c$ isn't true under $\mathcal{A}$ or $\psi^x/_c$ is true under $\mathcal{A}$
iff either $\mathcal{A}$(c) doesn't satisfy φ under $\mathcal{A}$ or $\mathcal{A}$(c) does satisfies ψ under $\mathcal{A}$ [because φ and ψ are both in Σ]

iff $\mathcal{A}$(c) satisfies $(\varphi \rightarrow \psi)$ under $\mathcal{A}$.

So $(\varphi \rightarrow \psi)$ is in Σ.

Σ is closed under the formation of biconditionals. Suppose that φ and ψ are both in Σ. Then $(\varphi'\leftrightarrow'\psi)^x/_c$ is equal to $(\varphi^x/_c \leftrightarrow \psi^x/_c)$, and we have:

$(\varphi \leftrightarrow \psi)^x/_c$ is true under $\mathcal{A}$
iff $(\varphi^x/_c \leftrightarrow \psi^x/_c)$ is true under $\mathcal{A}$
iff $\varphi^x/_c$ and $\psi^x/_c$ are either both true under $\mathcal{A}$ or both false under $\mathcal{A}$
iff either $\mathcal{A}$(c) satisfies both φ and ψ under $\mathcal{A}$ or $\mathcal{A}$(c) satisfies neither φ nor ψ under $\mathcal{A}$ [because φ and ψ are both in Σ]
iff $\mathcal{A}$(c) satisfies $(\varphi \leftrightarrow \psi)$ under $\mathcal{A}$.

So $(\varphi \leftrightarrow \psi)$ is in Σ.

Σ is closed under negation. Suppose that φ is in Σ. Then $(\neg\varphi)^x/_c$ is equal to $\neg(\varphi^x/_c)$, and we have:

$(\neg\varphi)^x/_c$ is true under $\mathcal{A}$
iff $\neg(\varphi^x/_c)$ is true under $\mathcal{A}$
iff $\varphi^x/_c$ isn't true under $\mathcal{A}$
iff $\mathcal{A}$(c) doesn't satisfies φ under $\mathcal{A}$ [because φ and ψ are both in Σ]
iff $\mathcal{A}$(c) satisfies $\neg\varphi$ under $\mathcal{A}$.

So $\neg\varphi$ is in Σ.

Σ is closed under universal quantification. Suppose φ is in Σ. $((\forall x)\varphi)^x/_c$ is equal to $(\forall x)\varphi$, and we have

$((\forall x)\varphi)^x/_c$ is true under $\mathcal{A}$
iff $(\forall x)\varphi$ is true under $\mathcal{A}$
iff $\mathcal{A}$(c) satisfies $(\forall x)\varphi$ under $\mathcal{A}$ [by the corollary to the principle of bivalence].

So $(\forall x)\varphi$ is in Σ.

Σ is closed under existential quantification. Suppose φ is in Σ. $((\exists x)\varphi)^{x}/_{c}$ is equal to $(\exists x)\varphi$, and we have

> $((\exists x)\varphi)^{x}/_{c}$ is true under $\mathcal{A}$
> iff $(\exists x)\varphi$ is true under $\mathcal{A}$
> iff $\mathcal{A}$(c) satisfies $(\exists x)\varphi$ under $\mathcal{A}$ [by the corollary to the principle of bivalence].

So $(\exists x)\varphi$ is in Σ.$\boxtimes$

If our language has just three predicates, "F," "G," and "H," then any interpretation of the language divides the universe into 8 cells, numbered 1 through 8 in the figure (where some of the cells may be empty). If two members of the universe lie in the same cell, they satisfy all the same formulas. This observation is perfectly general:

> **Indiscernibility Principle.** Given an interpretation $\mathcal{A}$. Any two members of $|\mathcal{A}|$ which satisfy precisely the same atomic formulas under $\mathcal{A}$ satisfy all the same formulas under $\mathcal{A}$.

Proof: Suppose that a and b satisfy precisely the same atomic formulas under $\mathcal{A}'$. Let Σ be the set of formulas φ such that a satisfies φ under $\mathcal{A}$ iff b satisfies φ under $\mathcal{A}$. We want to see that Σ is equal to the set of all formulas. To show this, we need to show that Σ contains the atomic formulas and that it is closed under conjunction, disjunction, formation of conditionals, biconditionals, negation, universal quantification, and existential quantification.

Atomic formulas are in Σ. Given.

Σ is closed under conjunction. Suppose that φ and ψ are both in Σ. We have

> a satisfies $(\varphi \wedge \psi)$ under $\mathcal{A}$
> iff a satisfies both φ and ψ under $\mathcal{A}$
> iff b satisfies both φ and ψ under $\mathcal{A}$ [because φ and ψ are both in Σ]

iff b satisfies $(\varphi \wedge \psi)$ in Σ.

So $(\varphi \wedge \psi)$ is in Σ.

Σ is closed under disjunction, formation of conditional, and formation of biconditionals. Similar.

Σ is closed under universal quantification. Suppose that φ is in Σ. We have

a satisfies $(\forall x)\varphi$ under $\mathcal{A}$
iff $(\forall x)\varphi$ is true under $\mathcal{A}$ [by the corollary to biva lence]
iff b satisfies $(\forall x)\varphi$ under $\mathcal{A}$ [by the corollary to bivalence again].

So $(\forall x)\varphi$ is in Σ.

Σ is closed under existential quantification. Similar.☒

To see whether a sentence is true under an interpretation, you have to see what the universe of the interpretation is, and you have to see what values the interpretation assigns to the constants and predicates that appear within that sentence. That's all you have to look at. You don't have to look at the values the interpretation assigns to the constants and predicates that don't even occur within the sentence. The following theorem makes this observation precise:

> Locality Principle. Let $\mathcal{A}$ and $\mathcal{B}$ be two interpretations with the same universe of discourse that assign the same values to all the constants and predicates that occur in the formula φ. Then precisely the same individuals satisfy φ under $\mathcal{A}$ and under $\mathcal{B}$.

Proof: Given interpretations $\mathcal{A}$ and $\mathcal{B}$ with the same universe of discourse, let Σ = {formulas φ: if $\mathcal{A}$ and $\mathcal{B}$ assign the same values to all the constants and predicates that occur in φ, then the same individuals satisfy φ under $\mathcal{A}$ and under $\mathcal{B}$}. We want to show that Σ is the set of all formulas. To show this, it will be enough to shoe that Σ contains the atomic formulas and that it is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification.

Σ contains the atomic formulas. Let φ be an atomic formula such that any constant or predicate that appears in φ is assigned the sa: me value by $\mathcal{A}$ and by $\mathcal{B}$. Take a $\in |\mathcal{A}|$. Either φ has the form Fx or else it has the form Fc.

If φ has the form Fx, we have

a satisfies φ under $\mathcal{A}$
iff a $\in \mathcal{A}$(F)
iff a $\in \mathcal{B}$(F)
iff a satisfies φ under $\mathcal{B}$.

If φ has the form Fc, we have

a satisfies φ under $\mathcal{A}$
iff $\mathcal{A}$(c) $\in \mathcal{A}$(F)
iff $\mathcal{B}$(c) $\in \mathcal{B}$(F)
iff a satisfies φ under $\mathcal{B}$.

Σ is closed under conjunction. Suppose that φ and ψ are both in Σ, and take a $\in |\mathcal{A}|$. Suppose that any constant or predicate that occurs in $(\varphi \wedge \psi)$ is assigned the same value by $\mathcal{A}$ and by $\mathcal{B}$. Then every constant or predicate that occurs in φ is assigned the same value by $\mathcal{A}$ and by $\mathcal{B}$, so that, since φ is in Σ, a satisfies φ under $\mathcal{A}$ iff a satisfies φ under $\mathcal{B}$. Similarly for ψ. Hence

a satisfies $(\varphi \wedge \psi)$ under $\mathcal{A}$
iff a satisfies both φ and ψ under $\mathcal{A}$
iff a satisfies both φ and ψ under $\mathcal{B}$
iff a satisfies $(\varphi \wedge \psi)$ under $\mathcal{B}$

Σ is closed under disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification. Similar.☒

Let _ be the language whose predicates are "M," "W," and "F" and whose only individual constant is "t," and let be the interpretation of _ given by:

|$\mathcal{A}$| = {animals}
$\mathcal{A}$("M") = {mammals}
$\mathcal{A}$("W") = {warm-blooded animals}
$\mathcal{A}$("F"} = {animals that fly}
$\mathcal{A}$("t") = Tarmin the dog

This structure can be represented by a Venn diagram that partitions the universe into eight cells. There are animals in Cell 1, bats, for instance. Tarmin, among others, is in Cell 2. There aren't any animals in Cells 3 and 4, because all mammals are warm-blooded. Canaries are in Cell 5, and penguins in Cell 6. We have our butterflies in Cell 7 and our banana slugs in Cell 8. Thus the only empty cells are 3 and 4.

We now want to create a second structure $\mathcal{B}$ with the same structural features as $\mathcal{A}$ whose only elements are numbers. To represent the fact that in $\mathcal{A}$ there are animals is Cells 1, 2, 5, 6, 7, and 8, the universe of $\mathcal{B}$ will consist of the numbers 1, 2, 5, 6, 7, and 8. Specifically,

|$\mathcal{B}$| = {1,2,5,6,7,8}
$\mathcal{B}$("M") = {1,2}
$\mathcal{B}$("W") = {1,2,5,6}
$\mathcal{B}$("F") = {1,5,7}
$\mathcal{B}$("t") = 2

We set $\mathcal{B}$("t") equal to 2 to represent the fact that Tarmin, who is $\mathcal{A}$("t"), is in Cell 2. It is not hard to convince ourselves* that, if a is an animal in the kth cell, then, for any formula φ, a satisfies φ under $\mathcal{A}$ iff k satisfies φ under $\mathcal{B}$. In particular, a sentence is true under $\mathcal{A}$ iff it's true under $\mathcal{B}$.

---

* A formal proof consists of setting Σ equal to the set of formulas φ such that, for any animal a, if a is in the kth cell, then a satisfies φ in $\mathcal{A}$ **iff k satisfies φ in $\mathcal{B}$. Then show Σ contains the atomic formulas and that it is closed under conjunction, disjunction, formation of conditionals, formation of biconditionals, negation, universal quantification, and existential quantification.**

The procedure we used here is perfectly general. An interpretation $\mathcal{A}$ of a language with n predicates partitions the universe of $\mathcal{A}$ into 2^n cells, which we can number 1 through 2^n. (Some of the cells may be empty.) Form an interpretation $\mathcal{B}$ as follows:

> $|\mathcal{B}|$ = {numbers k: under $\mathcal{A}$, the kth cell is nonempty}.
> $\mathcal{B}$(F) = {numbers k: the kth cell is a nonempty part of $\mathcal{A}$(F)}, for F a predicate.
> $\mathcal{B}$(c) = the number k such that $\mathcal{A}$(c) is in the kth cell, for c an individual constant

For any element a of $|\mathcal{A}|$, if a is in the kth cell, then, for any formula φ, a satisfies φ under $\mathcal{A}$ iff k satisfies φ under $\mathcal{B}$. In particular, if φ is a sentence, φ will be true under $\mathcal{A}$ iff it's true under $\mathcal{B}$. Let us call the model $\mathcal{B}$ obtained in this way the ***canonical model*** associated with $\mathcal{A}$. Since the universe of the canonical model is a nonempty subset of $\{1,2,3,...,2^n\}$, we have the following:

> Theorem. A sentence containing n predicates is valid iff it is true under every interpretation whose universe is contained in the set $\{1,2,3,...,2^n\}$.

Given a language _ with *n* predicates and *m* constants, we can determine a canonical model of the language by deciding which elements of $\{1,2,3,...,2^n\}$ are to be elements of the universe of the model and by deciding which element of the universe of the model each of the *m* constants is to denote. Thus the total number of canonical models for _ will be:

$$\sum_{i=1}^{2^n}$$

> Corollary. There is an algorithm — that is, a mechanical procedure — for testing whether a sentence is valid.

The algorithm just described isn't at all practical, for the interpretations are far too numerous for it to be feasible to

examine them all. The theoretical possibility of testing a sentence φ for validity by examining all the models of φ whose universe is contained within $\{1,2,3,...,2^n\}$ remains only that, a theoretical possibility.

In the next chapter, we are going to learn a more practical method for showing valid sentences valid.