**State Descriptions, Disjunctive Normal Form,
and Expressive Completeness**

We have already learned how to find the truth table for a given SC sentence. We now want to attack the opposite problem: Given a truth table, how can we find a sentence that has that truth table?

Under any N.T.A., exactly one of the following eight SC sentences is true: "(A ∧ (B ∧ C))," "(A ∧ (B ∧ ¬C))," "(A ∧ (¬B ∧ C))," "(A ∧ (¬B ∧ ¬C))," "(¬A ∧ (B ∧ C))," "(¬A ∧ (B ∧ ¬C))," "(¬A ∧ (¬B ∧ C))," and "(¬A ∧ (¬B ∧ ¬C))." These eight sentences are said to be the *state descriptions* for "A," "B," and "C," since they completely describe the state of the world with respect to these three sentences. Each of the state descriptions is associated with a line of the truth table, as follows:

A	B	C	Associated state description
1	1	1	(A ∧ (B ∧ C))
1	1	0	(A ∧ (B ∧ ¬C))
1	0	1	(A ∧ (¬B ∧ C))
1	0	0	(A ∧ (¬B ∧ ¬C))
0	1	1	(¬A ∧ (B ∧ C))
0	1	0	(¬A ∧ (B ∧ ¬C))
0	0	1	(¬A ∧ (¬B ∧ C))
0	0	0	(¬A ∧ (¬B ∧ ¬C))

A state description is true at the line of the truth table it's associated with, and nowhere else.

With the notion of state description in hand, the solution to our problem of getting a sentence with a given truth table is simple: We take our sentence to be the disjunction whose disjuncts are the state descriptions associated with those lines of the truth table where "1"s appear.

Example: Find an SC sentence whose truth table is this:

A	B	C	
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

Solution: "1" appears in the truth table at line two, three, and six, with which the associated state descriptions are "(A ∧ (B ∧ ¬C))," "(A ∧ (¬B ∧ C))," and "(¬A ∧ (B ∧ ¬C))," respectively. So the sentence we want is the disjunction of those three state descriptions, namely: "((A ∧ (B ∧ ¬C)) ∨ ((A ∧ (¬B ∧ C)) ∨ (¬A ∧ (B ∧ ¬C))))."

Example: Find an SC sentence whose truth table is this:

A	B	C	
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Solution: Here there is only one "1," so we don't need to form a disjunction. We just take the single state description corresponding to the line where "1" appears, namely, "(A ∧ (¬B ∧ ¬C))."

Exception: Our procedure tells us to form the disjunction of the state descriptions associated with lines of the truth table where "1"s appear. But what do we do if there aren't any "1"s? How do we find a sentence with the truth table

A	B	C	
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Solution: This is easy. We just use "(A ∧ ¬A)."

The procedure we have been developing is perfectly general. Given atomic sentences $\alpha_1, \alpha_2, ..., \alpha_n$, a truth table for $\alpha_1, \alpha_2, ..., \alpha_n$ will have 2^n rows, one row for each possible assignment of "1"s and "0"s to $\alpha_1, \alpha_2, ..., \alpha_n$. Given a column of 2^n "0"s and "1"s, we want to find a sentence that has that column as its truth table. With each line of the truth table, we associate a state description, a conjunction of 2^n conjuncts whose i^{th} conjunct is either α_i or $\neg\alpha_i$, depending on whether "1" or "0" appears under α_i in that line. As our sentence

whose truth table is the given column, we take the disjunction of the state descriptions of the state descriptions associated with positions in which "1" appears, assuming there is more than one such position. If "1" appears in only one position, we take our sentence to be the state description associated with that position. If there are no "1"s at all, we take our sentence to be ($\alpha_1 \wedge \neg\alpha_1$).

In every case, the sentence we obtain by our procedure has a special form. The sentence we get is a disjunction of one or more sentences each of which is a conjunction of one or more atomic sentences or negated atomic sentences. A sentence that has this form is said to be in disjunctive normal form. Thus we have shown that, for any given truth table, we can find a sentence in disjunctive normal form that has that truth table.

Given any sentence, we can find a logically equivalent sentence in disjunctive normal form by writing out the truth table for the given sentence, then using the procedure given above to find a sentence in disjunctive normal form that has that truth table.

A sentence in disjunctive normal form contains only the connectives "$\vee$," "$\wedge$," and "$\neg$." Thus we see that, for any given truth table, we can find a sentence that contains only the connectives "$\vee$," "$\wedge$," and "$\neg$" that has that truth table. A set S of sentential connectives is said to be *expressively complete* iff, for any given truth table, you can find a sentence containing only connectives from S that has that truth table. Thus we see that {"$\vee$,""$\wedge$,""$\neg$"} is expressively complete.

Now "$\vee$" can be defined in terms of "$\wedge$" and "$\neg$," since ($\varphi \vee \psi$) is logically equivalent to $\neg(\neg\varphi \wedge \neg\psi)$. This tells us that, for any given SC sentence, we can find a logically equivalent sentence whose only connectives are "$\wedge$" and "$\neg$" by the following procedure: First find a sentence in disjunctive normal form which is logically equivalent to the given sentence. Next look within the sentence in disjunctive normal form for a subsentence of the form ($\varphi \vee \psi$); replace this subsentence with $\neg(\neg \varphi \wedge \neg\psi)$. Continue eliminating "$\vee$"s until they're all gone. This gives us a sentence whose only connectives are "$\wedge$" and "$\neg$." So {"$\wedge$," "$\neg$"} is expressively complete.

Similarly, for any given truth table, we can find a sentence with that truth table which contains only the connectives "$\vee$" and "$\neg$," by first finding a sentence in disjunctive normal form that has that truth table, then replacing each subsentence of the form ($\chi \wedge \theta$) by $\neg(\neg\chi \vee \neg\theta)$. Hence {"$\vee$,""$\neg$"} is expressively complete.

{"$\rightarrow$," "$\neg$"} is expressively complete, since we can write ($\varphi \vee \psi$) as ($\neg\varphi \rightarrow \psi$), and we can write ($\chi \wedge \theta$) as $\neg(\chi \rightarrow \neg\theta)$.

By contrast, {"$\leftrightarrow$," "$\neg$"} isn't expressively complete, since any four-line truth table constructed out of "$\leftrightarrow$" and "$\neg$" will have "1"s on an even number of lines.

Similarly, the set {"∨,""∧,""→,""↔"} is not expressively complete, as we can see by observing that any sentence constructed out of these connectives is sure to have "1" at the top line of its truth table.