18. Convolution

18.1. **Superposition of infinitesimals: the convolution integral.** The system response of an LTI system to a general signal can be reconstructed explicitly from the unit impulse response.

To see how this works, start with an LTI system represented by a linear differential operator L with constant coefficients. The system response to a signal $f(t)$ is the solution to $Lx = f(t)$, subject to some specified initial conditions. To make things uniform it is common to specify "rest" initial conditions: $x(t) = 0$ for $t < 0$.

We will approach this general problem by decomposing the signal into small packets. This means we partition time into intervals of length say Δt: $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t$, and generally $t_k = k\Delta T$. The kth packet is the null signal (i.e. has value zero) except between $t = t_k$ and $t = t_{k+1}$, where it coincides with $f(t)$. Write $f_k(t)$ for the kth packet. Then $f(t)$ is the sum of the $f_k(t)$'s.

Now by superposition the system response (with rest initial conditions) to $f(t)$ is the sum of the system responses to the $f_k(t)$'s separately.

The next step is to estimate the system response to a single packet, say $f_k(t)$. Since $f_k(t)$ is concentrated entirely in a small neighborhood of t_k, it is well approximated as a rate by a multiple of the delta function concentrated at t_k, $\delta(t - t_k)$. The multiple should be chosen so that the cumulative totals match up; that is, it should be the integral under the graph of $f_k(t)$, which is itself well approximated by $f(t_k)\Delta t$. Thus we replace $f_k(t)$ by

$$f(t_k)(\Delta t)\delta(t - t_k).$$

The system response to this signal, a multiple of a shift of the unit impulse, is the same multiple of the same shift of the weight function (= unit impulse response):

$$f(t_k)(\Delta t)w(t - t_k).$$

By superposition, adding up these packet responses over the packets which occur before the given time t gives the system response to the signal $f(t)$ at time t. As $\Delta t \to 0$ this sum approximates an integral taken over time between time zero and time t. Since the symbol t is already in use, we need to use a different symbol for the variable in the integral; let's use the Greek equivalent of t, τ ("tau"). The t_k's get replaced by τ in the integral, and Δt by $d\tau$:

(1)
$$\boxed{x(t) = \int_0^t f(\tau)w(t-\tau)\,d\tau}$$

This is a really wonderful formula. Edwards and Penney call it "Duhamel's principle," but they seem somewhat isolated in this. Perhaps a better name would be the "superposition integral," since it is no more and no less than an integral expression of the principle of superposition. It is commonly called the **convolution integral**. It describes the solution to a general LTI equation $Lx = f(t)$ subject to rest initial conditions, in terms of the unit impulse response $w(t)$. Note that in evaluating this integral τ is always less than t, so we never encounter the part of $w(t)$ where it is zero.

18.2. **Example: the build up of a pollutant in a lake.** Every good formula deserves a particularly illuminating example, and perhaps the following will serve for the convolution integral. We have a lake, and a pollutant is being dumped into it, at a certain variable rate $f(t)$. This pollutant degrades over time, exponentially. If the lake begins at time zero with no pollutant, how much is in the lake at time $t > 0$?

The exponential decay is modeled by taking a quantity p of pollutant at time τ, and multiplying it by $e^{-a(t-\tau)}$. The number a is the decay constant, and $t-\tau$ is the time elapsed. We apply this formula to the small drip of pollutant added between time τ and time $\tau + \Delta\tau$. The quantity is $p = f(\tau)\Delta\tau$ (remember, $f(t)$ is a *rate*; to get a quantity you must multiply by time), so at time t the this drip has been reduced to the quantity

$$e^{-a(t-\tau)}f(\tau)\Delta\tau$$

(assuming $t > \tau$; if $t < \tau$, this particular drip contributed zero). Now we add them up, starting at the initial time $\tau = 0$, and get the convolution integral (1), which here is

(2)
$$x(t) = \int_0^t f(\tau)e^{-a(t-\tau)}\,d\tau.$$

We found our way straight to the convolution integral, without ever mentioning differential equations. But we can also solve this problem by setting up a differential equation for $x(t)$. The amount of this chemical in the lake at time $t+\Delta t$ is the amount at time t, minus the fraction that decayed, plus the amount newly added:

$$x(t+\Delta t) = x(t) - ax(t)\Delta t + f(t)\Delta t$$

Forming the limit as $\Delta t \to 0$, we obtain

$$\dot{x} + ax = f(t), \qquad x(0) = 0. \tag{3}$$

We conclude that (2) gives us the solution with rest initial conditions.

In this first order case we can recover this expression by using the method of variation of parameter, or, equivalently, of integrating factors. (3) can be solved using the integrating factor e^{at}:

$$x(t) = e^{-at} \int e^{at} f(t)\, dt.$$

If we reexpress this using a definite integral, we have to introduce a new variable to run from 0 to t—call it τ:

$$x(t) = e^{-at} \int_0^t e^{a\tau} f(\tau)\, d\tau$$

Choosing 0 as the lower limit of integration enforces the initial condition $x(0) = 0$. Now t is constant as far as the integral is concerned, so we can bring the factor e^{-at} inside the integral. Using the laws of exponentials, we find

$$x(t) = \int_0^t e^{-a(t-\tau)} f(\tau)\, d\tau,$$

which is (2).

An interesting case occurs if $a = 0$. Then the pollutant doesn't decay at all, and so it just builds up in the lake. At time t the total amount in the lake is just the total amount dumped in up to that time, namely

$$\int_0^t f(t)\, dt,$$

which is consistent with (2).

18.3. **Convolution as a product.** The integral (1) is called the *convolution* of $w(t)$ and $f(t)$, and written using an asterisk:

$$w(t) * f(t) = \int_0^t w(t-\tau) f(\tau)\, d\tau, \qquad t > 0. \tag{4}$$

Thus:

Theorem. The solution to an LTI equation $Lx = f(t)$, of any order, with rest initial conditions, is given by

$$x(t) = w(t) * f(t),$$

where $w(t)$ is the unit impulse response.