

18.03 Class 24, April 5, 2004

Laplace Transform IV: Weight and transfer functions, $|W(s)|$, convolution

1. We saw that $\delta(t) \rightarrow 1$.

We can put this to work solving differential equations. Suppose we have a spring/mass/dashpot system and bang it with a unit impulse: say

$x'' + 2x' + 5x = \delta(t)$, with rest initial conditions

or more generally suppose we have any LTI operator $p(D)$ and look at its unit impulse response. We called the unit impulse response the "weight function," and wrote $w(t)$ for it:

$p(D)w = \delta(t)$, rest initial conditions.

Apply LT:

$p(s)W = 1$, or $W(s) = 1/p(s)$.

$W(s)$ is called the "transfer function," and we have found (again) that:

LT transforms the weight function to the transfer function,

$w(t) \rightarrow W(s) = 1/p(s)$

The weight function is a specific free (= homogeneous) system response, with certain initial conditions. It alone determines the characteristic polynomial, and hence the response of the system to any input signal.

We can find the weight function this way. In our example, $w(t)$ is the inverse LT of

$1/(s^2+2s+5) = 1/((s+1)^2+4)$

and using the s-shift rule and the formula for the LT of $\sin(\omega t)$, this is

$(1/2) e^{-t} \sin(2t)$

or, better

$$w(t) = (1/2) u(t) e^{-t} \sin(2t)$$

Laplace transform is a GOOD WAY to find the weight function.
The $x_p + x_h$ method is not well suited to rest initial conditions.

2. The pole diagram of the Laplace transform

The poles of $W(s)$ are the roots of $p(s)$.

In our example, the poles of $W(s)$ occur at $-1 \pm 2i$. The "pole diagram" of $W(s)$ is the complex plane with those two points marked.

The function $W(s)$ can be visualized, in part, by thinking of its absolute value. Think of the complex plane as horizontal. The graph of $|W(s)|$ is a surface lying above the complex plane. It sweeps up to infinity above the poles, and falls off to just above the plane as you get away from the origin.

The Weight function enters into the Exponential Response Formula,

$z_p = W(s) e^{st}$ is a solution to $p(D)x = e^{st}$ for (complex) constant s .

The poles of $W(s)$ are the "resonant exponents":
the complex numbers r for which the Exponential Response Formula fails and
no multiple of e^{rt} is a solution.

Solutions of the real equation

$$p(D)x = \cos(\omega t)$$

are given by the real part of solutions of

$$p(D)z = e^{i \omega t}$$

The sinusoidal solution is

$$H(\omega) \cos(\omega t - \phi)$$

where $H(\omega) = |W(i \omega)| = 1/|p(i \omega)|$

and $\phi = -\arg(W(i \omega)) = \text{Arg}(p(i \omega))$.

In our example, $p(i\omega) = (5 - \omega^2) + 2i\omega$

$$H(\omega) = 1 / \sqrt{(5-\omega)^2 + 4\omega^2}$$

The graph of this starts (at $\omega = 0$) at height 1 and rises to a resonant peak, and then falls off towards zero. We can think of it for ω negative as well; it is an even function.

What we are seeing is the curve where the graph of $|W(s)|$ cuts the plane rising vertically above the imaginary axis.

The amplitude response can be understood, roughly, from the pole diagram.

3. Convolution.

Convolution provides the mechanism for computing the solution to $p(D)x = f(t)$ (with rest initial conditions) from the weight function:

$$x(t) = f(t) * w(t) := \int_0^t f(u) w(t-u) du$$

This fits very nicely with Laplace transform, since we just decided that

$$X(s) = F(s) W(s)$$

The conclusion is that

$$f(t) * g(t) \xrightarrow{\text{LT}} F(s) G(s)$$

--- LT carries convolution to ordinary multiplication of functions.

Since that ordinary multiplication of functions is commutative and associative, we conclude that the same is true of $*$:

$$f(t) * g(t) = g(t) * f(t)$$

$$f(t) * (g(t) * h(t)) = (f(t) * g(t)) * h(t)$$

Also, since $\delta(t) \xrightarrow{\text{LT}} 1$, we conclude that $\delta(t)$ is a "unit" for $*$:

$$\delta(t) * f(t) = f(t) = f(t) * \delta(t)$$