Solutions to Recitation 13 Problems

Before going into the problems, just for your information , we give a slightly different way of looking at the weight function and its equivalent IVP. One key property about the delta function is

$$\int_{-\epsilon}^{\epsilon} f(t)\delta(t)\,dt = f(0)$$

for any reasonable function f and for any ϵ. So to solve $m\dot{x} + bx = \delta(t)$, we integrate both sides, and we see

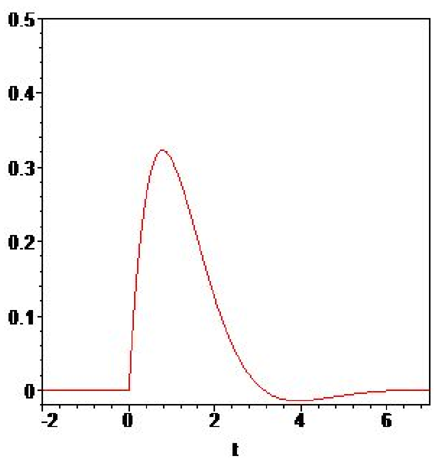
$$m[x(\epsilon) - x(-\epsilon)] + b\int_{-\epsilon}^{\epsilon} x = 1.$$

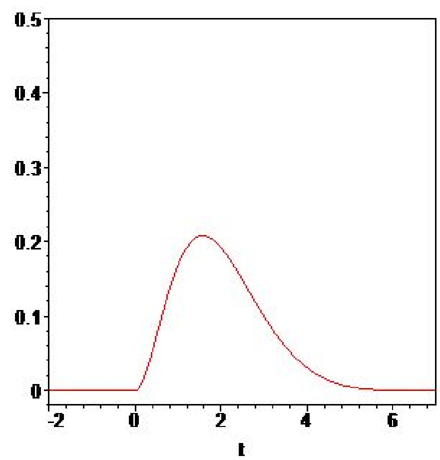
Because x is a bounded function, the second term on the left hand side must go to zero as ϵ goes to zero, which means that $x(0+)$ must be $\frac{1}{m}$ more than $x(0-) = 0$. So the weight function is the answer to $m\dot{x} + bx = 0$ with $x(0) = \frac{1}{m}$.
Similarly, if we have a second-order equation $m\ddot{x} + b\dot{x} + kx = \delta(t)$, then integrating, we see

$$m[\dot{x}(\epsilon) - \dot{x}(-\epsilon)] + b[x(\epsilon) - x(-\epsilon)] + k\int_{-\epsilon}^{\epsilon} x = 1.$$

Again, the last term goes to zero as we let $\epsilon \to 0$. We have a choice here, but we can let the first term contribute the 1, meaning that $x(0+) = x(0-) = 0$ but $\dot{x}(0+) = \dot{x}(0-) + \frac{1}{m} = \frac{1}{m}$. So to obtain the weight function, we want to solve the IVP with $x(0) = 0$ and $\dot{x}(0) = \frac{1}{m}$. Of course, in this second order case, the equation is a force equation, so we have a physical explanation: a unit impulse $1 = \int \delta(t)\,dt$ should give a unit change in the momentum $mv = m\dot{x}$, which is consistent with above.

1. The general solution is
$$C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t).$$
We want the solution with $x(0) = 0$ and $\dot{x}(0) = 1$, so we must have $C_1 = 0$ and $C_2 e^{-0}\cos(0) = 1$, so $C_2 = 1$. So the weight function is $u(t)e^{-t}\sin(t)$. It's an oscillating function with rapidly decreasing amplitude. The sketch is the left one of the below.





2. The answer is
$$f(t) * (e^{-t}\sin(t)) = \int_0^t f(u)e^{-(t-u)}\sin(t-u)\,du = \int_0^t f(t-u)e^{-u}\sin(u)\,du.$$

When $f(t) = e^{-t}$, we have

$$f(t) * (e^{-t}\sin(t)) = \int_0^t e^{u-t}e^{-u}\sin(u)\,du = e^{-t}\int_0^t \sin(u)\,du = e^{-t} - e^{-t}\cos(t).$$

Again, this is an oscillating function with decreasing amplitude (it always stays positive). The sketch is given above (the second one).

3. Because the system response is $\int f(u)w(t-u)du$, we see that this is also nonnegative for $t > 0$ (because we are starting in the rest initial condition, the system is zero for $t \le 0$).

4. Because $a \neq b$,
$$\int_0^t e^{au}e^{bt-bu}du = e^{bt}\frac{1}{a-b}\left(e^{at-bt}-1\right) = \frac{e^{at}-e^{bt}}{a-b}.$$

5. Now, we have
$$\int_0^t e^{au}e^{at-au}du = e^{at}\int_0^t 1\,du = te^{at}.$$

6. If the weight function is $u(t) = u(t)\cdot 1$, so the function 1 is the solution to the IVP with $m\dot{x} + bx = 0$ and $x(0) = \frac{1}{m}$. So $m = 1$ and $b = 0$. So the equation was $\dot{x} = 0$.

   For the next one, the given condition should have been weight function equaling to $u(t)2e^{-2t}$. In this case, $x(t) = 2e^{-2t}$ has the property that $x(0) = 2$, so $m = \frac{1}{2}$ (since this is nonzero, we can immediately tell that this weight function doesn't correspond to the second order equation). Now,
$$\frac{1}{2}\cdot(-4e^{-2t}) + 2be^{-2t} = 0$$
   implies $b = 1$, so the original equation was $\frac{1}{2}\dot{x} + x = 0$.

The below is the answer to I. 19 of Problem Set 4.

3D-1. Solve $\ddot{y} + 2\dot{y} + y = \delta(t) + u(t-1)$, $y(0-) = 0$, $\dot{y}(0-) = 1$.
Write $p(D) = D^2 + 2D + I$.
Our plan: find the solution to $p(D)x = \delta(t)$ with rest initial conditions; find the solution to $p(D)x = u(t-1)$ with rest initial conditions; find the solution to $p(D)x = 0$ with the given initial conditions; and add up the three functions. By superposition this gives the answer.
(i) Let's find the unit step response. This is the solution to $p(D)x = 1$ with rest initial conditions. $x_p = 1$. The characteristic polynomial has the repeated root -1, so $x_h = (c_1 t + c_2)e^{-t}$. $\dot{x}_h = (-c_1 t + (c_1 - c_2))e^{-t}$. $x_h(0) = c_2$, $\dot{x}_h(0) = c_1 - c_2$. We want $x_h(0) = -1$ and $\dot{x}_h(0) = 0$, so that with $x = x_p + x_h$, $x(0) = 0$ and $\dot{x}(0) = 0$. This gives $c_2 = -1$ and $c_1 = -1$, so for $t > 0$ the unit step response is $x(t) = u(t)(1 - (t+1)e^{-t})$. [Check this!]
(ii) The unit impulse response can be found in the same way, or by differentiating the step response: $w(t) = u(t)te^{-t}$. For $t > 0$ this is also the solution of $p(D)x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$. So the initial condition and the $\delta(t)$ in the signal contribute identical terms to the solution.
(iii) The solution to $p(D)x = u(t-1)$ with rest initial conditions is the step response delayed by one time unit (by time independence): substitute $(t-1)$ for t in the equation for the step response: $x(t) = u(t-1)(1 - te^{1-t})$.
(iv) For $t < 0$, we have the "initial" (maybe "terminal" would be a better term!) condition $y(0) = 0$, $\dot{y}(0) = 1$. The homogeneous solution with this initial condition is te^{-t}
(v) Putting all this together by superposition, we get

$$y = (u(t) + 1)te^{-t} + u(t-1)(1 - te^{1-t})$$

(vi) In case format this is

$$y = \begin{cases} te^{-t} & \text{if } t < 0 \\ 2te^{-t} & \text{if } 0 < t < 1 \\ 2te^{-t} + (1 - te^{1-t}) & \text{if } t > 1 \end{cases}$$

which coincides with the solution in the Exercises.

3D-2. Solve $\ddot{y} + y = r(t)$, $y(0-) = 0$, $\dot{y}(0-) = 1$, where $r(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \text{otherwise} \end{cases}$
This can be done in various ways.
(i) First solve $\ddot{x} + x = 1$ with initial condition $x(0) = 0$, $\dot{x}(0) = 1$. $x_p = 1$, so we want $x_h(0) = -1$ and $\dot{x}_h(0) = 1$. $x_h = -\cos t + \sin t$ works, so $x = 1 - \cos t + \sin t$.
(ii) For this function, $x(\pi) = 2$, and $\dot{x} = \sin t + \cos t$ so $\dot{x}(\pi) = -1$.
(iii) So now we want to solve $\ddot{x} + x = 0$, $x(\pi) = 2$, $\dot{x}(\pi) = -1$. $x = c_1 \cos t + c_2 \sin t$, $\dot{x} = -c_1 \sin t + c_2 \cos t$, $x(\pi) = -c_1$, $\dot{x}(\pi) = -c_2$, so we take $c_1 = -2$ and $c_2 = 1$: $x = -2\cos t + \sin t$.
(iv) Putting this together,

$$y = \begin{cases} \sin t & \text{if } t < 0 \\ 1 - \cos t + \sin t & \text{if } 0 < t < \pi \\ -2\cos t + \sin t & \text{if } t > \pi \end{cases}$$

again agreeing with the solution in the Exercises.