Solutions to Recitation 6 Problems

1. $(e^{rt})' = re^{rt}$ and $(e^{rt})'' = r^2e^{rt}$, so

$$\ddot{x} + 4\dot{x} + kx = (r^2 + 4r + k)e^{rt}$$

when $x = e^{rt}$. So for $x = e^{rt}$ to be a solution, we must have $r^2+4r+k = 0$. When $k = 3$, this polynomial factors as $(r+3)(r+1)$, so $x = e^{-3t}$ and $x = e^{-t}$. When $k = 4$, this polynomial factors as $(r+2)^2$, so $x = e^{-2t}$ works. When $k = 8$, using the quadratic formula, the roots are $r = -2 \pm \sqrt{2^2 - 8} = -2 \pm 2i$, so $x = e^{(-2\pm 2i)t}$ are solutions.

2. $e^{(-2+2i)t} = e^{-2t}e^{2ti} = e^{-2t}(\cos(2t)+i\sin(2t))$, so the real part is $e^{-2t}\cos(2t)$ and the imaginary part is $e^{-2t}\sin(2t)$. Similarly, the real part of $e^{(-2-2i)t}$ is $e^{-2t}\cos(-2t) = e^{-2t}\cos(2t)$ and the imaginary part is $e^{-2t}\sin(-2t) = -e^{-2t}\sin(2t)$. So the real parts are the same, and the imaginary parts are negative of each other, so taking real/imaginary parts of one suffices.

$$\begin{aligned}
(e^{-2t}\cos(2t))' &= (-2\cos(2t) - 2\sin(2t))e^{-2t}, \\
(e^{-2t}\cos(2t))'' &= \big(4\sin(2t) - 4\cos(2t) + (-2)(-2\cos(2t) - 2\sin(2t))\big)e^{-2t} \\
&= 8e^{-2t}\sin(2t), \qquad \text{so} \\
(e^{-2t}\cos(2t))'' &+ 4(e^{-2t}\cos(2t))' + 8(e^{-2t}\cos(2t)) \\
&= \big(8\sin(2t) + 4(-2\cos(2t) - 2\sin(2t)) + 8\cos(2t)\big)e^{-2t} = 0.
\end{aligned}$$

The computation is similar for $e^{-2t}\sin(2t)$.

3. A time-translate of a solution is still a solution, and letting $t = s + \pi/12$, we see that

$$e^{-3(s+\pi/12)}\cos(2(s+\pi/12) - \pi/6) = e^{-\pi/4}e^{-3s}\cos(2s)$$

is a solution. Scaling down, we see that $e^{-3s}\cos(2s)$ is a solution. This is the real part of $e^{(-3+2i)s}$, so $-3+2i$ must be a solution to the characteristic polynomial. The other solution is the conjugate $-3-2i$, so $b = 6$ and $(b/2)^2 - k = -4$ implies $k = 13$.

Similarly, if $e^t\sin(4t)$ is a solution, then $1\pm 4i$ are solutions to the characteristic polynomial. So $b = -2$, and $(b/2)^2 - k = -16$ implies $k = 17$.

If $1 = e^{0t}$ and e^{-t} are solutions, then 0 and -1 are the solutions to the characteristic polynomial, so $s^2 + bs + k = (s-0)(s-(-1))$ shows that $b = 1$ and $k = 0$.