18.03 Class 25, April 7, 2004

The Pole Diagram of the Laplace Transform and the End of Time;
Numerical Methods.

1. The LT method first gives $X(s)$, rather easily. The hard part of the method is deducing $x(t)$ from $X(s)$.

The pole diagram of $X(s)$ is a crude and easily visualized bit of information about $X(s)$.

Examples:

f(t)	Poles of F(s)
1	$s = 0$
e^{at}	$s = a$
$\cos(\omega t)$, $\sin(\omega t)$	$s = \pm i \omega$
$e^{at} \cos(\omega t)$	$s = a \pm i \omega$

General comments: (1) If $f(t)$ is real-valued, then the pole diagram is symmetric about the real axis.

(2) If $F(s)$ and $G(s)$ don't share any poles, then the pole diagram of $F(s) + G(s)$ is formed by combining the two diagrams.

So for example the the pole diagram for $\sin(t) + e^{-t}$ is $\{i,-i,-1\}$.

It turns out that the pole diagram of $X(s)$ tells you:

. nothing at all about $x(t)$ for small t , but
. very precise information about $x(t)$ as $t \to \infty$.

Note two facts:

(1) The pole diagram of the LT of $f_a(t)$ is the same as that of $f(t)$,
despite the fact that $f_a(t) = 0$ for $t < a$ no matter what $f(t)$ is.
This comes from the t-shift rule: $f_a(t) \longrightarrow e^{-as}F(s)$.
The exponential function e^{-as} is never zero and has no poles.
So multiplying by it doesn't cancel any poles of $F(s)$ and doesn't introduce
any new ones either.

(2) Poles arise from failure of convergence of the improper integral.
Thus if $f(t) = 0$ for $t \gg 0$, then $F(s)$ has no poles.

General rule: Poles at $a \pm i\omega$ contribute behavior to $f(t)$
for t large which is like $A e^{at} \cos(\omega t - \phi)$.

The larger the real part, the faster the growth. So:

The right-most poles of $F(s)$ control the dominant behavior of $f(t)$
for t large.

All the poles have negative real part exactly when
$f(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

When there are several rightmost poles on the same vertical, the behavior
will be a mix of different oscillations and will look complex.
"PolesandVibes" shows this behavior pretty well.

2. Euler's method

The study of differential equations rests on three legs:

. Analytic, exact, symbolic methods

. Quantitative methods (critical points, for example)

. Numerical methods.

Even if we can solve exactly, the question of computing values remains.
The number e is by definition the value $x(1)$ of the solution to
$x' = x$ with $x(0) = 1$. But how do you find that in fact

$e = 2.718282828459045....$? The answer is: numerical methods.

As an example, take the first order linear ODE $y' = t - y^2 = F(t,y)$ with initial condition $y(0) = 1$. Question: what is $y(1)$?

Here's an approach: use the tangent line approximation.

I opened the Euler Mathlet to see this.

The direction field has value $F(0,1) = -1$, so the first approximation to $y(1)$ is 0.

Better: go to $t = .5$ along the tangent line and look at the value of the direction field there. You are at $(.5,.5)$, and $F(.5,.5) = .25$, so now you should head off with slope .25, and arrive at $(1,.625)$: so the next approximation is .625.

We can do this with more breaks, too. Let's set up the general picture with some notation. We have the ODE $x' = F(t,x)$, with initial condition $x(a) = x_0$. Suppose we want to compute x on the interval $[a,b]$, or even just find $x(b)$, with good accuracy. Divide the interval into some number, N, of equal pieces. Each will have width $h = (b-a)/N$. This is called the "stepsize." The endpoints of these pieces are

$t_0 = a, t_1 = a + h, t_2 = a + 2h, \ldots , t_n = a + nh, \ldots t_N = b.$

Make a table with the following entries:

n	tn	xn	F(tn,xn)
0	$t_0=a$	x_0	$F(t_0,x_0)$
1	$t_1=a+h$	x_1	$F(t_1,x_1)$
2	$t_2=a+2h$	x_2	$F(t_2,x_2)$
...			

In the line $n = 1$, $x_1 = x_0 + h F(t_0,x_0)$

In the line $n = 2$, $x_2 = x_1 + h F(t_1,x_1)$

and in general $x_{n+1} = x_n + h F(t_n,x_n)$.

This is "Euler's method."